Introduction to homotopy type theory: exam

DAT235/DIT577/PhD reading course

2024, January 12

- Grade scale:

Fraction of points	≥ 0	$\geq 2/5$	$\geq 3/5$	$\geq 4/5$
Grade	U	3	4	5

- Time: 4 hours
- No aids allowed.
- You may use familiar facts from the course book or our discussions without justification, provided they do not already include the statement to be proven or depend on it.
- The axioms of function extensionality and univalence may only be used where stated.

1. **[4 points]** Fix a type A. The equality type of A has an induction principle involving

$$\mathsf{ind\text{-}eq}_{a,P} : P(a, \mathsf{refl}_a) \to \prod_{x:A} \prod_{p:a=x} P(x,p)$$

for $a : A$ and a family of types $P(x,p)$ indexed by $x : A$ and $p : a = x$.

Construct the following elements. Explicitly state the parameter P when you use $\mathsf{ind\text{-}eq}$.

(a) $\mathsf{f} : \prod_{x,y:A} x = y \to y = x$,

(b) $\mathsf{g} : \prod_{x,y:A} \prod_{p:x=y} f(y,x,f(x,y,p)) = p$.

2. **[4 points]** Given a type A, define what it means:

(a) for A to be contractible,

(b) for A to have truncation level n where $n : \mathbb{Z}_{\geq -2}$.

3. **[4 points]** The type of *fixpoints* of a function $u : X \to X$ is defined as

$$\mathsf{fix}(u) := \sum_{x:X} u(x) = x.$$

Given $f : A \to B$ and $g : B \to A$, show that $\mathsf{fix}(g \circ f) \simeq \mathsf{fix}(f \circ g)$.

4. **[4 points]** Given a type A, show that the following are logically equivalent for $x, y : A$:

(1) the propositional truncation $\|x = y\|$,

(2) $Q(x) \simeq Q(y)$ for all families Q of propositions indexed by A.

5. **[4 points]** Let $\mathcal{U}$ be a univalent universe with $I : \mathcal{U}$. We have a function

$$h : \mathcal{U}^I \to \sum_{X:\mathcal{U}} I^X$$

sending $Y : I \to U$ to $(\sum_{i:I} Y(i), \mathsf{pr}_1)$. Define a function k in the opposite direction with $k \circ h \sim \mathsf{id}$. You may use function extensionality.

6. **[4 points]** Consider $f : S^1 \to S^1$ with $H : f \circ f \sim f$. Show that

$$\mathsf{is\text{-}equiv}(f) + \|\mathsf{is\text{-}constant}(f)\|.$$

You may use function extensionality and univalence.