Introduction to homotopy type theory: exam

DAT235/DIT577/PhD reading course

2024, January 12

- Grade scale:

Fraction of points	≥ 0	$\geq 2/5$	$\geq 3/5$	$\geq 4/5$
Grade	U	3	4	5

- Time: 4 hours
- No aids allowed.
- You may use familiar facts from the course book or our discussions without justification, provided they do not already include the statement to be proven or depend on it.
- The axioms of function extensionality and univalence may only be used where stated.

1. **[4 points]** Fix a type A. The equality type of A has an induction principle involving

$$\mathsf{ind\text{-}eq}_{a,P} : P(a, \mathsf{refl}_a) \to \prod_{x:A} \prod_{p:a=x} P(x,p)$$

for $a : A$ and a family of types $P(x,p)$ indexed by $x : A$ and $p : a = x$.

Construct the following elements. Explicitly state the parameter P when you use $\mathsf{ind\text{-}eq}$.

(a) $\mathsf{f} : \prod_{x,y:A} x = y \to y = x$,

(b) $\mathsf{g} : \prod_{x,y:A} \prod_{p:x=y} f(y, x, f(x, y, p)) = p$.

Recall that the eliminator satisfies the judgmental equality $\mathsf{ind\text{-}eq}_{a,P}(d, a, \mathsf{refl}_a) \doteq d$. There are several ways to define f and g, here is an easy one.

For (a), we can take

$$f(x) : \prod_{y:A} \prod_{p:x=y} y = x,$$
$$f(x) := \mathsf{ind\text{-}eq}_{x,P_x}(\mathsf{refl}_x)$$

where $P_x(y,p) := (y = x)$. Note that

$$f(x, x, \mathsf{refl}_x) \doteq \mathsf{ind\text{-}eq}_{x,P}(\mathsf{refl}_x, x, \mathsf{refl}_x) \doteq \mathsf{refl}_x.$$

For (b), we take

$$g(x) : \prod_{y:A} \prod_{p:x=y} f(y, x, f(x, y, p)) = p,$$
$$g(x) := \mathsf{ind\text{-}eq}_{x,Q_x}(d_x, y, p)$$

where $Q_x(y,p) := (f(y, x, f(x, y, p)) = p)$. It remains to define

$$d_x : f(x, x, f(x, x, \mathsf{refl}_x)) = \mathsf{refl}_x.$$

But

$$f(x, x, f(x, x, \mathsf{refl}_x)) \doteq f(x, x, \mathsf{refl}_x) \doteq \mathsf{refl}_x,$$

so we can set $d_x := \mathsf{refl}_{\mathsf{refl}_x}$.

2. **[4 points]** Given a type A, define what it means:

(a) for A to be contractible,

(b) for A to have truncation level n where $n : \mathbb{Z}_{\geq -2}$.

For (a), contractibility of A is defined as the type

$$\sum_{a:A} \prod_{x:A} x = a.$$

For (b), we use induction on n. In the base case, A is defined to have truncation level -2 if it is contractible. In the induction step, A is defined to have truncation level $n+1$ if, for all $x, y : A$, the equality type $x =_A y$ has truncation level n.

3. **[4 points]** The type of *fixpoints* of a function $u : X \to X$ is defined as

$$\mathsf{fix}(u) := \sum_{x:X} u(x) = x.$$

Given $f : A \to B$ and $g : B \to A$, show that $\mathsf{fix}(g \circ f) \simeq \mathsf{fix}(f \circ g)$.

Here is an elegant way to construct this equivalence. There are other, more concrete ways.

Consider the type

$$T := \sum_{a:A} \sum_{b:B} f(a) = b \times g(b) = a.$$

Since equivalences are composable and invertible, it suffices to show that T is equivalent to $\mathsf{fix}(g \circ f)$ and $\mathsf{fix}(f \circ g)$. We only show $T \simeq \mathsf{fix}(g \circ f)$, the other case is dual.

Reassociating dependent sums, we have

$$T \simeq \sum_{a:A} \sum_{(b,p):\sum_{b:B} f(a)=b} g(b) = a.$$

Note that $\sum_{b:B} f(a) = b$ is contractible with center $(f(a), \mathsf{refl}_{f(a)})$. The map

$$\mathbf{1} \to \sum_{b:B} f(a) = b$$

selecting the center is an equivalence. Since dependent sums preserve equivalences, we have

$$T \simeq \sum_{a:A} \sum_{1} g(f(a)) = a \simeq \sum_{a:A} g(f(a)) = a.$$

Dually, we have

4. **[4 points]** Given a type A, show that the following are logically equivalent for $x, y : A$:

   (1) the propositional truncation $\|x = y\|$,

   (2) $Q(x) \simeq Q(y)$ for all families Q of propositions indexed by A.

Let (1) be given. Consider a family of propositions Q indexed by A. To construct an equivalence between $Q(x)$ and $Q(y)$, it suffices to construct maps in both directions since both sides are propositions. So given $Q(x)$, let us show $Q(y)$ (the other case is dual). Given $x = y$, we have $Q(y)$ by transporting the given element of $Q(x)$. So using the universal property of the propositional truncation ($Q(y)$ is proposition), given $\|x = y\|$, we have $Q(y)$. But $\|x = y\|$ holds by (1).

Now let (2) be given. We set $Q(a) := \|x = a\|$, a proposition. We have $\eta(\mathsf{refl}_x) : Q(x)$. By (2), we then have $Q(y)$, which is (1).

5. **[4 points]** Let $\mathcal{U}$ be a univalent universe with $I : \mathcal{U}$. We have a function

$$h : \mathcal{U}^I \to \sum_{X:\mathcal{U}} I^X$$

sending $Y : I \to U$ to $(\sum_{i:I} Y(i), \mathsf{pr}_1)$. Define a function k in the opposite direction with $k \circ h \sim \mathsf{id}$. You may use function extensionality.

Given $X : \mathcal{U}$ with $f : X \to I$, we define $k(X, f) : I \to \mathcal{U}$ by sending i to $\mathsf{fib}_f(i)$. Unfolded:

$$f(X, f)(i) := \sum_{x:X} f(x) = i.$$

Given $Y : I \to U$, we need to show $k(h(Y)) =_{U^I} Y$. Using function extensionality, it suffices to show $k(h(Y))(i) =_U Y(i)$ given $i : I$. This we get from univalence of $\mathcal{U}$ and the equivalence

$$\begin{aligned} k(h(Y))(i) &\simeq \sum_{(i',y):\sum_{i:I} Y(i)} i' = i \\ &\simeq \sum_{(i',p):\sum_{i':I} i'=i} Y(i') \\ &\simeq Y(i). \end{aligned}$$

The second line reassociates dependent sums. The third line uses that $\sum_{i':I} i' = i$ is contractible with center (i, refl_i) (together with dependent sums preserving equivalences).

6. **[4 points]** Consider $f : S^1 \to S^1$ with $H : f \circ f \sim f$. Show that

$$\mathsf{is\text{-}equiv}(f) + \|\mathsf{is\text{-}constant}(f)\|.$$

You may use function extensionality and univalence.

Note that f cannot be constant and an equivalence at the same time (2-out-of-6 would imply S^1 is contractible, but S^1 is not a set). Therefore, the goal (a binary sum of propositions) is still a proposition. Since S^1 is connected, we have $\|f(\mathsf{base}) = \mathsf{base}\|$. Since our goal is a proposition, we can thus assume $p : f(\mathsf{base}) = \mathsf{base}$.

Recall the characterization of the loop space of S^1. It tells us that the function $\mathbb{Z} \to \mathsf{base} = \mathsf{base}$ sending k to loop^k is an equivalence. That means there is unique $k : \mathbb{Z}$ such that $p^{-1} \cdot \mathsf{ap}_f(\mathsf{loop}) \cdot p = \mathsf{loop}^k$. So we have

$$\mathsf{ap}_f(\mathsf{loop}) = p \cdot \mathsf{loop}^k \cdot p^{-1}.$$

Recall that $f \sim g$ requires $\alpha : f(\mathsf{base}) = g(\mathsf{base})$ making the following square commute:

$$\begin{tikzcd} f(\mathsf{base}) \arrow[r, equal, "\alpha"] \arrow[d, equal, "\mathsf{ap}_f(\mathsf{loop})"] & g(\mathsf{base}) \arrow[d, equal, "\mathsf{ap}_g(\mathsf{loop})"] \\ f(\mathsf{base}) \arrow[r, equal, "\alpha"] & g(\mathsf{base}). \end{tikzcd}$$

We will show $f \sim \mathsf{id}$ or $f \sim \mathsf{const}_{\mathsf{base}}$. In both cases, we set $\alpha := p$. For $g := \mathsf{id}$, the condition is $\mathsf{loop}^k = \mathsf{loop}$. For $g := \mathsf{const}_{\mathsf{base}}$, it is $\mathsf{loop}^k = \mathsf{refl}_{\mathsf{base}}$. So it suffices to show $k = 1$ or $k = 0$.

We compute

$$\begin{aligned} \mathsf{ap}_{f \circ f}(\mathsf{loop})) &= \mathsf{ap}_f(\mathsf{ap}_f(\mathsf{loop})) \\ &= \mathsf{ap}_f(p \cdot \mathsf{loop}^k \cdot p^{-1}) \\ &= \mathsf{ap}_f(p) \cdot \mathsf{ap}_f(\mathsf{loop})^k \cdot \mathsf{ap}_f(p)^{-1} \\ &= \mathsf{ap}_f(p) \cdot (p \cdot \mathsf{loop}^k \cdot p^{-1})^k \cdot \mathsf{ap}_f(p)^{-1} \\ &= (\mathsf{ap}_f(p) \cdot p) \cdot \mathsf{loop}^{k^2} \cdot (\mathsf{ap}_f(p) \cdot p)^{-1}. \end{aligned}$$

We plug the formulas for $\mathsf{ap}_{f\circ f}(\mathsf{loop}))$ and $\mathsf{ap}_f(\mathsf{loop})$ into naturality of H at loop:

$$\begin{tikzcd}
\mathsf{base} \arrow[r, equal, "(\mathsf{ap}_f(p)\cdot p)^{-1}"] \arrow[d, equal, "\mathsf{loop}^{k^2}"'] & f(f(\mathsf{base})) \arrow[r, equal, "H_{\mathsf{base}}"] \arrow[d, equal, "\mathsf{ap}_{f\circ f}(\mathsf{loop})"'] & f(\mathsf{base}) \arrow[r, equal, "p"] \arrow[d, equal, "\mathsf{ap}_f(\mathsf{loop})"] & \mathsf{base} \arrow[d, equal, "\mathsf{loop}^k"] \\
\mathsf{base} \arrow[r, equal, "(\mathsf{ap}_f(p)\cdot p)^{-1}"'] & f(f(\mathsf{base})) \arrow[r, equal, "H_{\mathsf{base}}"'] & f(\mathsf{base}) \arrow[r, equal, "p"'] & \mathsf{base}.
\end{tikzcd}$$

The function $\mathsf{loop}^{(-)}$ sends addition in $\mathbb{Z}$ to composition of loops in S^1 at base. Since addition is commutative, so is composition of loops at base. That is, $p \cdot q = q \cdot p$ for $p, q : \mathsf{base} = \mathsf{base}$. With this, the outer rectangle reduces to $\mathsf{loop}^{k^2} = \mathsf{loop}^k$, that is, $k^2 = k$, so $k = 1$ or $k = 0$.