Introduction to homotopy type theory: re-exam

DAT235/DIT577/PhD reading course

2024, March 13

- Grade scale:

Fraction of points	≥ 0	$\geq 2/5$	$\geq 3/5$	$\geq 4/5$
Grade	U	3	4	5

- Time: 4 hours
- No aids allowed.
- You may use familiar facts from the course book or our discussions without justification, provided they do not already include the statement to be proven or depend on it.
- The axioms of function extensionality and univalence may only be used where stated.

1. [**4 points**] Consider a type A. The equality type of A has an induction principle involving

$$\mathsf{ind\text{-}eq}_{a,P} : P(a, \mathsf{refl}_a) \to \prod_{x:A} \prod_{p:a=x} P(x,p)$$

for $a : A$ and a family of types $P(x,p)$ indexed by $x : A$ and $p : a = x$.

Define composition of identifications in A (you can choose its with judgmental behaviour). Explicitly state the parameter P when you use $\mathsf{ind\text{-}eq}$.

For example, we define

$$\mathsf{concat} : \prod_{x,y,z:A} (x = y) \to (y = z) \to (x = z)$$
$$\mathsf{concat}(x, y, z, p, q) = J_{P,x}(y, \lambda z', q'.\, q', y, p, z, q)$$

where

$$P(y', p') = \prod_{z':A} \prod_{q':y'=z'} (y' = z') \to (x = z').$$

This uses the book convention of applying an iterated function to a tuple of arguments.

2. [**4 points**] Consider a type A and a family B of types over A. State the axiom of extensionality for dependent functions from $a : A$ to $B(a)$. You may use the notion of equivalence without explanation, but everything else needs to be defined.

Let f and g be dependent functions from $a : A$ to $B(a)$. We have a function

$$h_{f,g} : f = g \to \prod_{a:A} f(a) = g(a)$$

defined by identification induction via

$$h_{f,f}(\mathsf{refl}_f)(a) = \mathsf{refl}_{f(a)}.$$

The axiom of function extensionality states that $h_{f,g}$ is an equivalence.

3. [**4 points**] Let $f : A \to B$ be a map such that:

   (1) f has a section,

   (2) for $x, y : A$, the map $\mathsf{ap}_f : (x =_A y) \to (f(x) =_B f(y))$ has a section.

   Prove that f is an equivalence (bi-invertible).

   By condition (1), f has a section $s : B \to A$. We show that s is also a retraction of f. This means $s(f(a)) = a$ for $a : A$. Using condition (2), it suffices to show $f(s(f(a))) = f(a)$. But this holds since $f(s(b)) = b$ for $b : B$.

4. [**4 points**] Consider sets A and B. Show that the coproduct $A + B$ is again a set. You may use the characterization of identifications in $A + B$ from the course.

   Given $x_0, x_1 : A + B$, we have to show that $x_0 = x_1$ is a proposition. By equality induction it suffices to show that $x = x$ is a proposition for $x : A + B$. We use coproduct elimination on x.

- For $x \doteq \mathsf{inl}(a)$ with $a : A$, we have that $x = x$ is equivalent to $a = a$. This is a proposition since A is a set.
- For $x \doteq \mathsf{inr}(b)$ with $b : B$, we have that $x = x$ is equivalent to $b = b$. This is a proposition since B is a set.

We finish by recalling that being a proposition is invariant under equivalence.

5. [**4 points**] Consider a type A and a univalent universe $\mathcal{U}$ containing the identity types of A. Consider the function $v : A \to \mathcal{U}^A$ sending x to $\lambda y.\, y =_A x$. Show that the action of v on identifications has a section. You may use function extensionality.

Given $x_0, x_1 : A$, we must show that the action

$$\mathsf{ap}_v : x_0 = x_1 \longrightarrow v(x_0) =_{\mathcal{U}^A} v(x_1)$$

of v on identifications has a retraction. We will more generally show that it is invertible.

By 2-out-of-3, it suffices to show that the composition with the function extensionality equivalence

$$h_{v(x_0),v(x_1)} : v(x_0) =_{\mathcal{U}^A} v(x_1) \longrightarrow \prod_{y:A} (y = x_0) =_U (y = x_1)$$

from Problem 2 is invertible. We further compose with the equivalence

$$(y = x_0) =_U (y = x_1) \longrightarrow (y = x_0) \simeq (y = x_1)$$

from univalence. It remains to show that the map

$$x_0 = x_1 \longrightarrow \prod_{y:A} (y = x_0) \simeq (y = x_1)$$

sending reflexivity to the family of identity equivalences is invertible. By 2-out-of-3, it sufices to show that the map

$$\prod_{y:A} (y = x_0) \simeq (y = x_1) \longrightarrow x_0 = x_1$$

evaluating at x_0 and reflexivity is invertible.

Every family of maps $e_y : (y = x_0) \to (y = x_1)$ for $y : A$ is a family of equivalences. To see this, we check that the induced map

$$\sum_{y:A} (y = x_0) \to \sum_{y:A} (y = x_1)$$

on total spaces is invertible. But both sides here are contractible singletons.

It remains to show that the map

$$\prod_{y:A} ((y = x_0) \to (y = x_1)) \longrightarrow x_0 = x_1$$

evaluating at x_0 and reflexivity is invertible. This is equivalently

$$\prod_{(y,p):\sum_{y:A} y=x_0} y = x_1 \longrightarrow x_0 = x_1$$

evaluating at (x_0, refl). This is invertible because $\sum_{y:A} y = x_0$ is a contractible singleton with center (x_0, refl).

6. **[4 points]** Let $\mathbb{F}$ be the univalent universe of finite types. Construct an equivalence

$$\mathbb{F} \simeq \sum_{X:\mathbb{F}} X.$$

You may use function extensionality.

In the forward direction, we define $f : \mathbb{F} \to \sum_{x:\mathbb{F}} X$ by sending A to $(\mathsf{1} + A, \mathsf{inl}(\star))$. In the reverse direction, we define $g : \sum_{x:\mathbb{F}} X \to \mathbb{F}$ by sending $X : \mathbb{F}$ with $x_0 : X$ to the type $\sum_{x:X} x \neq x_0$. Recall that finite types are sets with decidable equality. Therefore, $g(X, x_0)$ is a decidable subtype of X, which is again finite.

We now check that these maps are inverse to each other. Given $A : \mathbb{F}$, using univalence of $\mathbb{F}$, we need to show that $\sum_{x:\mathsf{1}+A} x \neq \mathsf{inl}(\star)$ is equivalent to A. Distributing the sum over the coproduct and using the characterization of identifications in coproducts, this is the coproduct of $\sum_{z:\mathsf{1}} z \neq \star$ and $\sum_{a:A} \neg\emptyset$. Simplifying further, this is coproduct of the empty type and A, or just A.

Given $X : \mathbb{F}$ with $x_0 : X$, using univalence and the characterization of identifications in dependent sums, we need to construct an equivalence $e : \star + \sum_{x:X} x \neq x_0 \simeq X$ such that $e(\mathsf{inl}(\star)) = x_0$. This suggests defining $e(\mathsf{inl}(\star)) \doteq x_0$ and $e(\mathsf{inr}(x, p)) = x$.