Introduction to homotopy type theory: practice exam

DAT235/DIT577/PhD reading course

2023, December 23

- Grade scale:

Fraction of points	≥ 0	$\geq 2/5$	$\geq 3/5$	$\geq 4/5$
Grade	U	3	4	5

- Time: 4 hours
- No aids allowed.
- You may use familiar facts from the course book or our discussions without justification, provided they do not already include the statement to be proven or depend on it.
- The axioms of function extensionality and univalence may only be used where stated.

1. **[4 points]** Recall that the type $\mathbb{N}$ of natural numbers is inductively generated by $0 : \mathbb{N}$ and $S(n) : \mathbb{N}$ for $n : \mathbb{N}$. Its eliminator (induction) takes a family P over $\mathbb{N}$ with

   - $z : P(0)$,
   - $s(n, x) : P(S(n))$ for $n : \mathbb{N}$ and $x : P(n)$,

   and gives $\mathsf{ind}_{P,z,s}(n) : P(n)$ for $n : \mathbb{N}$.

   Justify the definition of $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by pattern matching

$$\begin{aligned} f(0, b) &= S(b) \\ f(S(a), 0) &= f(a, S(0)) \\ f(S(a), S(b)) &= f(a, f(S(a), b)) \end{aligned}$$

   by translating it a definition using the eliminator instead.

2. **[4 points]** Let A be a type with an element a. Consider the following type:

$$\sum_{x:A} \sum_{p:a=x} \sum_{q:x=a} p \cdot q = \mathsf{refl}_a$$

   Is this type always contractible? If yes, prove so; if no, provide a counterexample.

3. **[4 points]** A type A is a *retract* of a type B if we have $s : A \to B$ and $r : B \to A$ with $rs \sim \mathsf{id}_A$. Prove that any retract of a contractible type is again contractible.

4. **[4 points]** Consider maps

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D.$$

   Assume that $g \circ f$ and $h \circ g$ are bi-invertible. Show that f, g, h are all bi-invertible.

5. **[8 points in total]** One of the most technical parts in the setup of homotopy type theory is the proof that bi-invertible maps are contractible. In this problem, we will work through an alternative approach from discussions with David Wärn.

   We start with maps $g : A \to B$ and $f, h : B \to A$ with $p : g \circ f \sim \mathsf{id}_A$ and $q : h \circ g \sim \mathsf{id}_B$. Prove the following sequence of claims. You may use all previous problems and parts.

   (a) **[2 points]** The following maps are bi-invertible:

   - $\mathsf{ap}_{g \circ f}^{a_0, a_1} : (a_0 = a_1) \to (g(f(a_0)) = g(f(a_1)))$ for $a_0, a_1 : A$,
   - $\mathsf{ap}_{h \circ g}^{b_0, b_1} : (b_0 = b_1) \to (h(g(b_0)) = h(g(b_1)))$ for $b_0, b_1 : B$.

   You may use function extensionality here.

   (b) **[2 points]** The map $\mathsf{ap}_f^{a_0, a_1} : (a_0 = a_1) \to (f(a_0) = f(a_1))$ is bi-invertible for $a_0, a_1 : A$.

   (c) **[2 points]** For $a_0 : A$, the type $\sum_{a_1 : A} f(a_0) = f(a_1)$ is contractible.

   (d) **[2 points]** The map f is contractible.

6. **[4 points]** Define what it means for a universe $\mathcal{U}$ to be univalent.