Introduction to homotopy type theory: re-exam

DAT235/DIT577/PhD reading course

2024, March 13

- Grade scale:

Fraction of points	≥ 0	$\geq 2/5$	$\geq 3/5$	$\geq 4/5$
Grade	U	3	4	5

- Time: 4 hours
- No aids allowed.
- You may use familiar facts from the course book or our discussions without justification, provided they do not already include the statement to be proven or depend on it.
- The axioms of function extensionality and univalence may only be used where stated.

1. **[4 points]** Consider a type A. The equality type of A has an induction principle involving

$$\mathsf{ind\text{-}eq}_{a,P} : P(a, \mathsf{refl}_a) \to \prod_{x:A} \prod_{p:a=x} P(x,p)$$

for $a : A$ and a family of types $P(x,p)$ indexed by $x : A$ and $p : a = x$.

Define composition of identifications in A (you can choose its with judgmental behaviour). Explicitly state the parameter P when you use $\mathsf{ind\text{-}eq}$.

2. **[4 points]** Consider a type A and a family B of types over A. State the axiom of extensionality for dependent functions from $a : A$ to $B(a)$. You may use the notion of equivalence without explanation, but everything else needs to be defined.

3. **[4 points]** Let $f : A \to B$ be a map such that:

   (1) f has a section,

   (2) for $x, y : A$, the map $\mathsf{ap}_f : (x =_A y) \to (f(x) =_B f(y))$ has a section.

   Prove that f is an equivalence (bi-invertible).

4. **[4 points]** Consider sets A and B. Show that the coproduct $A + B$ is again a set. You may use the characterization of identifications in $A + B$ from the course.

5. **[4 points]** Consider a type A and a univalent universe $\mathcal{U}$ containing the identity types of A. Consider the function $v : A \to \mathcal{U}^A$ sending x to $\lambda y.\, y =_A x$. Show that the action of v on identifications has a section. You may use function extensionality.

6. **[4 points]** Let $\mathbb{F}$ be the univalent universe of finite types. Construct an equivalence

$$\mathbb{F} \simeq \sum_{X:\mathbb{F}} X.$$

You may use function extensionality.