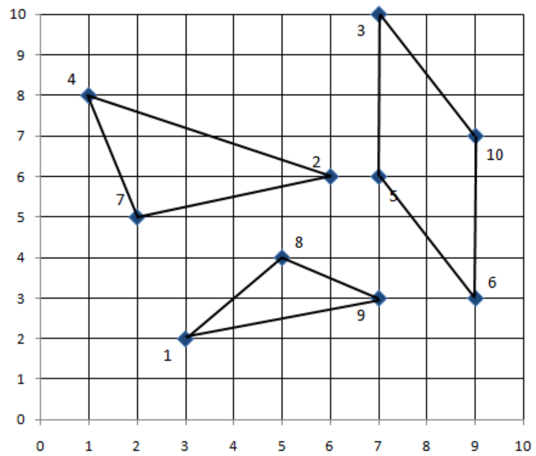
Optimization of Logistics Systems / Logistics Systems Planning I
5. Exercise

Exercise 5.1 (Subtour elimination constraints)

1. Formulate the 2-matching model. Relax integrality and subtour elimination constraints.
2. Consider the relaxed solution of a 10-city TSP shown in the following figure. Formulate the necessary subtour elimination constraint explicitly, so that the subtours of this relaxed solution are prevented.



Solution:
SECs:

$$\sum_{\{i,j\}\in E(S)} x_{ij} \leq |S| - 1 \quad \text{for all } S \subset V, 3 \leq |S| \leq |V| - 3$$

Example: $S = \{1, 2, 3, 4, 5\}$. In a solution, at most 4 edges between these 5 nodes must exist $\Rightarrow$ no subtour can connect these 5 nodes. For subsets of these 5 nodes, however, subtours would still be possible $\Rightarrow$ formulate additional SECs for each subset

Problem: Exponentially many SECs $\Rightarrow$ usually the respective optimization model cannot be written down explicitly.

1. Solve the relaxed model (without SECs and integrality):

$$z_{STSP} = \min \sum_{\{i,j\}\in E} d_{ij} x_{ij} \tag{1}$$

$$\text{s.t.} \quad \sum_{\{i,j\}\in \delta(i)} x_{ij} = 2 \quad \forall i \in V \tag{2}$$

$$0 \leq x_{ij} \leq 1 \quad \forall \{i,j\} \in E \tag{3}$$

2. Check whether there are subtours in the solution,
   if no $\Rightarrow$ solution is found;
   if yes $\Rightarrow$ formulate SECs to prevent the subtours and add the SECs to the model
   repeat until no subtours exist.

   $\Rightarrow$ In the given relaxed solution, there are 3 subtours:
   1. $S_1 = \{1, 8, 9\} \Rightarrow x_{18} + x_{19} + x_{89} \leq 2$
   2. $S_2 = \{2, 4, 7\} \Rightarrow x_{24} + x_{27} + x_{47} \leq 2$
   3. $S_3 = \{3, 5, 6, 10\} \Rightarrow x_{35} + x_{56} + x_{6,10} + x_{3,10} + x_{36} + x_{5,10} \leq 3$

Exercise 5.2 (Related problems of the TSP and transformations)

Model each of the following problems as STSP on a weighted graph $G = (V, E, c_{ij})$ or as ATSP on a digraph $D = (V, A, c_{ij})$

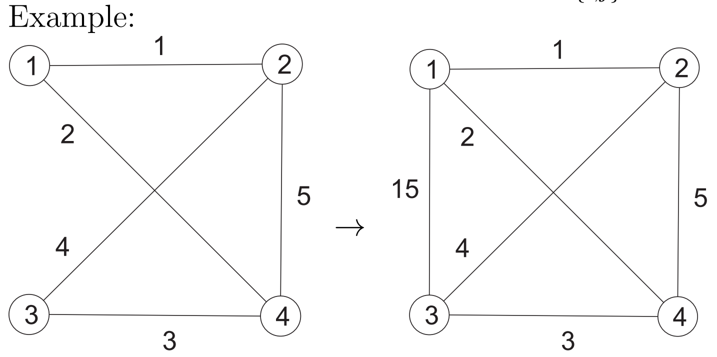
1. In general, the TSP is defined on a complete graph G, i.e., all possible edges $E_{full} := \{\{i, j\} : i, j \in V, i < j\}$ are included in the edge set E. How can the case $E \subsetneqq E_{full}$ be modeled as STSP defined on the complete graph (V, E_{full}) by adding the missing edges and setting the respective costs?

2. The MAX-TSP searches for a round trip with maximum traveled distance. Model both variants—the MAX-TSP on an undirected graph and on a digraph—as STSP and ATSP, respectively.

3. How can the solution of an ATSP be achieved by the solution of an STSP that is defined in appropriate fashion? (Hint: Model every node of the given ATSP by two corresponding nodes in the STSP. Which edges do you then include in your new model and what are the weights of these edges?)

Solutions:

1. Idea: Put very large weights M on the "missing" edges in the complete graph, so that the edges with weight M will never exist in an optimal solution of the TSP.
   Choose for example the value of $M := \sum_{\{i,j\} \in E} |c_{ij}|$
   Example:

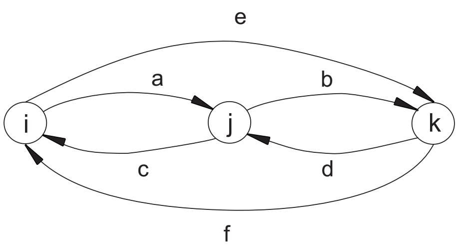


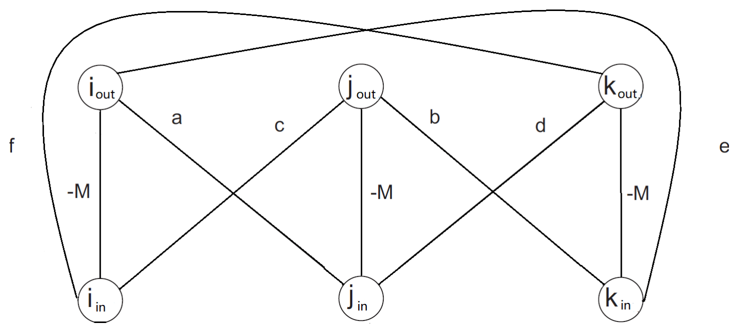
$$\begin{pmatrix} - & 1 & - & 2 \\ & - & 4 & 5 \\ & & - & 3 \\ & & & - \end{pmatrix} \rightarrow \begin{pmatrix} - & 1 & 15 & 2 \\ & - & 4 & 5 \\ & & - & 3 \\ & & & - \end{pmatrix}$$

2. Idea: $\boxed{-\max -f(x) = \min f(x)}$
   Multiply all weights by -1: $c'_{ij} = -c_{ij}$.
   Then model the problem on new graph (V, E, c'), or digraph (V, A, c').

3. Idea: Replace one node i of the ASTP with two nodes i_{in}, i_{out} in the STSP.

   - i_{out} represents the possibility of leaving node i
   - i_{in} represents the possibility of entering node i
   - Each of the edges (i, j) of the old graph are replaced on the new graph by arc from i_{out} to j_{in} with the same weights
   - Additionally there are new edges from i_{in} to i_{out} for all nodes i with the weight $-M$





all edges not shown have weight $+M$