$P\left(\frac{Z}{n} > p+w\right)$ $\quad w = \frac{t}{n}$

$= P(Z > E[Z]+t) \le e^{-nf(w)} \le e^{\frac{-2t^2}{n}}$

$f(w) = KL(p+w \| p) \ge 2w^2$

$Z = \sum_{i=1}^{n} X_i \quad X_i \sim Ber(p_i),\ \mu = E[Z]$

$P(Z > (1+\varepsilon)\mu) \le \left\{\frac{e^{\varepsilon}}{(1+\varepsilon)^{1+\varepsilon}}\right\}^{\mu} \le e^{\frac{-\varepsilon^2 \mu}{3}}$ $\quad 0<\varepsilon<1$

$P(Z < (1-\delta)\mu) \le \left(\frac{e^{-\varepsilon}}{(1-\varepsilon)^{(1-\varepsilon)}}\right)^{\mu} \le e^{\frac{-\varepsilon^2}{2}\mu}$

$\mu_L \le \mu \le \mu_H$

a) $\forall t>0$ $\quad P(Z > \mu_H + t), P(Z < \mu_L - t) \le e^{\frac{-2t^2}{n}}$

b) $0<\varepsilon<1$ $\quad P(Z > (1+\varepsilon)\mu_H) \le \exp\left(\frac{-\varepsilon^2}{3}\mu_H\right)$

$P(Z < (1-\varepsilon)\mu_L) \le \exp\left(\frac{-\varepsilon^2}{2}\mu_L\right)$

$P(\text{error}) = P(Y=0)P(\hat{y}=1|Y=0) + P(Y=1)P(\hat{y}=0|Y=1)$

$P(S_n - E[S_n|Y=1] > -E[S_n|Y=1] \mid Y=1)$

H_0, H_1 Hypothesis

Write H_0/H_1 example

$P(\hat{y}=0|Y=1) = P(S_n > 0 | Y=1)$

$S_n(X_1^n) = \sum_{i=1}^{n} \log \frac{P_0(X_i)}{P_1(X_i)}$

$E[S_n|Y=1] = n\left[p_1 \log \frac{p_0}{p_1} + (1-p_1)\log\frac{1-p_0}{1-p_1}\right]$

If $X \sim Poi(\lambda)$ $\quad E[e^{sX}] = e^{\lambda(e^{s}-1)}$

4/1/2023.

WLLN $\quad X_i \sim$ iid

$$Z = \frac{\sum_{i=1}^{n} X_i}{n}$$

$$\forall \varepsilon > 0 \lim_{n \to \infty} P(|Z - E[Z]| > \varepsilon) = 0$$

Markov Inequality $\quad$ Non-ve RV X, $E[X] < \infty$, $\alpha > 0$

$$P(X > \alpha) \le \frac{E[X]}{\alpha}$$

Can be improved by assuming higher moments of X exist

Chebyshev $\quad$ RV X

$$P(|X - \mu| > \alpha) \le \frac{\sigma^2}{\alpha^2}$$

For lower tail apply negative exponential

Chernoff Bound $\quad P(X>a) = P(f(X) > f(a))$ if $f(\cdot)$ is increasing.

Further if $f(\cdot) > 0$ apply markov

$f(t) = \exp(st)$ $\quad s>0$

$$P(X>t) = P(e^{sX} > e^{st}) \le e^{-st} E[e^{sX}]$$

Log MGF $\quad \Psi_X(s) = \log(E(e^{sX}))$ $\quad s>0$

$$\Psi_X^*(a) = \sup_{s>0} (sa - \Psi_X(s))$$

Chernoff Bound $\quad P(X > a) \le e^{-\Psi_X^*(a)}$

Chernoff Bound For sum of i.i.d R.V. $\quad \Psi_Z^*(na) = \Psi_{X_1}^*(a) \times n$, $Z = \sum_{i=1}^{n} X_i$

If $X_i \sim Ber(p)$ $\quad \Psi_X^*(a) = KL(Ber(a) \| Ber(p))$

CLT $\quad X_1, X_2, \dots, X_n$ i.i.d. finite mean μ, Variance σ^2

$S_n = X_1 + \dots + X_n$ $\quad$ variance $n\sigma^2$

$\frac{S_n}{\sqrt{n}}$ $\quad$ variance σ^2

$$Z_n = \frac{S_n - n\mu}{\sqrt{n}\,\sigma} \xrightarrow{D} N(0,1)$$

Cramer-Chernoff Bound. $\quad Z = N(0,1)$ $\quad P(Z \ge u) \le \frac{1}{u\sqrt{2\pi}} \int_u^{\infty} x \exp\left(\frac{-x^2}{2}\right) dx$

$$\le \sqrt{\frac{1}{2\pi u^2}} \exp\left(-\frac{u^2}{2}\right)$$

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$$P(\hat{\mu} \ge \mu + \varepsilon) = P\left(Z_n \ge \varepsilon\sqrt{\frac{n}{\sigma^2}}\right) \approx P\left(Z \ge \varepsilon\sqrt{\frac{n}{\sigma^2}}\right)$$

- If $X_1 \sim Ber(p)$ $\quad \Psi_{X_1}^*(a) = KL(Ber(a) \| Ber(p)) = a\log\frac{a}{p} + (1-a)\log\frac{1-a}{1-p}$

$Z = \sum_{i=1}^{n} X_i \sim Bin(n,p)$ $\quad P\left(\frac{Z}{n} > a\right) \le \exp(-n\Psi_{X_1}^*(a))$

$P(X_1 > a) \le \exp(-\Psi_{X_1}^*(a))$

For $x>0$ $\quad \ln(1+x) \ge \frac{x}{1+\frac{x}{2}}$ $\quad 0<\delta<1$ $\quad \ln(1-\delta) \ge -\delta + \frac{\delta^2}{2}$

The Taylor Remainder. $\quad f(x) = f(a) + f'(a)(x-a) + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$

$$R_n = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad \text{for some } c \in (a,x)$$

If $X \sim N(\mu, \sigma^2)$ $\quad E[e^{sX}] = e^{\mu s + \frac{\sigma^2 s^2}{2}}$

If $f(w) = (p+w)\log\frac{p+w}{p} + (q-w)\log\frac{q-w}{q}$, $p+q=1$

$f(w) \ge 2w^2$ $\quad$ Proof using Taylor & AM-GM

Hoeffding lemma $\quad E[Y]=0$, $Y \in [a,b]$ $\quad E[e^{\theta Y}] \le \exp\left(\frac{\theta^2(b-a)^2}{8}\right)$

Hoeffding $\quad \{X_1, X_2, \dots, X_n\}$ support $X_i \in [a_i, b_i]$, $S_n = X_1 + X_2 + \dots + X_n$

$$\forall t>0 \quad P(S_n - E[S_n] \ge t) \le \exp\left(\frac{-2t^2}{\sum_{i=1}^{n}(b_i - a_i)^2}\right)$$

or $\le -t$

$$D_{KL}(P\|Q) = \sum_{x \in \mathcal{X}} P(x) \log\frac{P(x)}{Q(x)}$$

For $\theta \mapsto f(x|\theta)$ (i/p θ, o/p)

$$\hat{\theta}_{MLE}(x) = \arg\max_{\theta} f(x|\theta)$$

$$\hat{\theta}_{MAP}(x) = \arg\max_{\theta} f(\theta|x) = \arg\max_{\theta} \frac{f(x|\theta)g(\theta)}{\int_{\vartheta} f(x|\vartheta)g(\vartheta)d\vartheta}$$

g: PDF of θ

$$\hat{\theta}_{MAP}(x) = \arg\max_{\theta} f(x|\theta)g(\theta)$$

If $X_1 \sim Ber(p)$, $\Psi_{X_1}^*(a) = a\log\frac{a}{p} + (1-a)\log\frac{1-a}{1-p}$

Lemma: If $Y \sim Ber(p)$ $\quad M_Y(s) = E[e^{sY}] \le e^{p(e^{s}-1)}$

Used in proving. $\quad P(Z > (1+\varepsilon)E[Z]) \le \left[\frac{e^{\varepsilon}}{(1+\varepsilon)^{(1+\varepsilon)}}\right]^{E[Z]}$

$Z = \sum X_i$ $\quad X_i \sim Ber(p_i)$

Bennett's Inequality. independent $E[X_i]=0$, $|X_i| \le c$, $\sigma_{X_i}^2$. $S_n = \sum X_i$

$$P(S_n \ge t) \le \exp\left(\frac{-n\sigma^2}{c^2} h\left(\frac{ct}{n\sigma^2}\right)\right)$$

$n\sigma^2 = \sum_{i=1}^{n} \sigma_{X_i}^2$ $\quad h(u) = (1+u)\log(1+u) - u$,

$P(|X_i| \le c) = 1$ $\quad E[X_i] = \mu$, for any $\varepsilon > 0$

Bernstein's Inequality

$$\cancel{P(S_n > n\varepsilon)}\ P(|S_n - n\mu| > n\varepsilon) \le 2\exp\left(\frac{-n\varepsilon^2}{2\sigma^2 + \frac{2c\varepsilon}{3}}\right)$$

(a) $E[I\{\dots\}|\mathcal{F}_n] = P(\dots|\mathcal{F}_n)$

(b) If RV Y is f^n of $X_m, m \le n$, $E[ZY|\mathcal{F}_n] = Y E[Z|\mathcal{F}_n]$

(c) $n \ge m \ge 0$ $\quad E[E[X|\mathcal{F}_n]|\mathcal{F}_m] = E[X|\mathcal{F}_m]$

Summary If X_i's independent $S_n = \sum_{i=1}^{n} X_i$ then

1. $E[e^{\theta X_i}]$ is known: Chernoff.
2. $a_i \le X_i \le b_i$ Hoeffding.
3. $|X_i| \le c$, $\sigma_{X_i}^2$ known: Bennett/Bernstein

Negative Association

A set of RV $X_1, X_2, \cdots, X_n$ are said to be Negatively Associated (NA) if for 2 disjoint index sets $I, J \subseteq [n]$ and $I \cap J = \phi$ and $\forall\, f: \mathbb{R}^{|I|} \to \mathbb{R}$, $g: \mathbb{R}^{|J|} \to \mathbb{R}$. $\forall f, g$ which are either both monotone increasing or decreasing

$$E[f(X_i : i \in I)\, g(X_j : j \in J)] \le E[f(X_i : i \in I)]\, E[g(X_j : j \in J)]$$

Properties

1. $E\left[\prod_{i \in [n]} f_i(X_i)\right] \le \prod_{i \in [n]} E[f_i(X_i)]$
2. $E\left[\prod_{i \in [n]} X_i\right] \le \prod_{i \in [n]} E[X_i]$
3. $P(X_i \ge t_i, i \in I \mid X_j \ge t_j, j \in J) \le P(X_i \ge t_i, i \in I)$
4. $P(X_i \ge t_i, i \in [n]) \le \prod_{i \in [n]} P(X_i \ge t_i)$
5. $P\left(\sum_{i=1}^n X_i > t\right) \le e^{-\theta t} \prod_{i=1}^n E[e^{\theta X_i}]$
6. i.i.d. Chernoff Bound holds for NA X_i's.

The Zero-One Principle Let X_i's be 0-1 R.V. such that $\sum_i X_i \le 1$ always. Then $X_1, X_2, \cdots, X_n$ are NA

$D_i =$

Closure Properties

i) Union of independent sets of NA RV are NA, i.e. if $X_1, \cdots, X_n$ are NA, $Y_1, Y_2, \cdots, Y_n$ are NA and $\{X_i\}_i$ are independent of $\{Y_i\}_i$, then $X_1, X_2, \cdots, X_n, Y_1, Y_2, \cdots, Y_m$ are NA

ii) Concordant monotone functions defined on disjoint subsets of a set of NA R.V. are N.A.

k-wise independence A set of R.V. is k-wise independent if for $I \subseteq [n]$ s.t. $|I| \le k$

$$P\left(\bigcap_{i \in I} \{X_i = x_i\}\right) = \prod_{i \in I} P(X_i = x_i)$$

Consequence $\quad E\left[\prod_{i \in I} X_i\right] = \prod_{i \in I} E[X_i] \qquad |I| \le k$

$a, b \ge 0$, $0 \le \theta \le 1$: General AM-GM inequality (Jensen for $-\log x$)

$$a^\theta b^{1-\theta} \le \theta a + (1-\theta) b$$

$\theta = \frac{1}{2}$ AM-GM

Holder's Inequality $\vec{x} \cdot \vec{y} \le \|x\|_p \|y\|_q$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$

Martingale

A sequence of R.V. $Z_0, Z_1, \cdots$ is a martingale wrt $X_0, X_1, X_2, \cdots$ (sequence of RV) if (i) $Z_n = f(X_0, X_1, \cdots, X_n)$ (ii) $E[|Z_n|] < \infty$

(iii) $E[Z_{n+1} \mid X_0, X_1, \cdots, X_n] = Z_n$

Lemma $\quad E[Z_k] = E[Z_0] \qquad k$: constant

Stopping Time A random variable $T \in \{0, 1, \cdots, \infty\}$ is a stopping time if for $0 \le n \le \infty$ $\{T_n \le n\} \in \mathcal{F}_n$, equivalently $\{T = n\} \in \mathcal{F}_n$

Martingale Stopping Theorem

If T is a stopping time, then $E[Z_T] = E[Z_0]$ whenever atleast one of (i) $|Z_i| \le C_1$ (ii) $E[|Z_{i+1} - Z_i| \mid \mathcal{F}_i] \le C_2$ (iii) $P(T < \infty) = 1$ holds

Azuma Hoeffding

Let $\{X_0, X_1, \cdots, X_n\}$ be a martingale such that $a_k \le (X_k - X_{k-1}) \le b_k$. Then $\forall t \ge 0$

$$\max\left(P\{X_n - X_0 > t\}, P(X_n - X_0 < -t)\right) \le \exp\left(\frac{-2t^2}{\sum_{i=1}^n (b_i - a_i)^2}\right)$$

Doob's Martingale

Def: $f: \mathbb{R}^n \to \mathbb{R}$ $\quad Z = f(X_1, X_2, \cdots, X_n)$. Doob's Martingale associated with X_i

$Y_k = E[Z \mid X_1, X_2, \cdots, X_k]$

$|E[Z \mid X_1, X_2, \cdots, X_i] - E[Z \mid X_1, \cdots, X_{i-1}]| \le C_i$

Bounded Martingale difference condition for Y

Average Lipschitz Condition

$$\left|E[\underbrace{f(X_1, \cdots, X_n)}_{Z} \mid X_1, \cdots, X_{i-1}, X_i = a] - E[f(X_1, \cdots, X_n) \mid X_1, \cdots, X_{i-1}, X_i = a']\right| \le C_i \quad \forall i = 1, \cdots, n$$

then $P(|Z - E[Z]| > t) \le 2e^{-2t^2/c}$ $\quad$ ~~$c = \sum_{i=1}^n (c_i - \dots)$~~ $\quad c = \sum_{i=1}^n (c_i - (-c_i))^2$

Bounded Difference Condition Let $f: \mathcal{X}^n \to \mathbb{R}$ for some space $\mathcal{X}$. Then f satisfies bounded differences with constants c_i if for each $i \in \{1, \cdots, n\}$ all $x_1^n \in \mathcal{X}^n$ & $x_i' \in \mathcal{X}$ we have

$$|f(x_1^{i-1}, x_i, x_{i+1}^n) - f(x_1^{i-1}, x_i', x_{i+1}^n)| \le c_i$$

BDC + independence $\Rightarrow$ ALC $\not\Rightarrow$ BMDC

Mc Diarmid's inequality

Bounded Differences Inequality

Let $f: \mathcal{X}^n \to \mathbb{R}$ satisfy bounded differences with constants c_i, and let X_i be independent R.V.

$$P(f(X_1^n) - E[f(X_1^n)] \geq t) \vee P(f(X_1^n) - E[f(X_1^n)] \leq -t) \leq \exp\left(\frac{-2t^2}{\sum_{i=1}^n c_i^2}\right)$$

Application
① Bounding probability of deviation of chromatic number χ of Erdos-Renyi $G_{n,p}$
② Stochastic Bin Packing.
③ Stochastic TSP

BDC $d_i \geq 1$
Proof Sketch $P(d_H(x,A) - \mu \leq -t) \leq e^{\frac{-2t^2}{n}}$

Blowing-Up Lemma.

$\{X_i\}$ ind R.V. $\quad d_H(\vec{x}, \vec{y}) = \sum_{i=1}^n \mathbb{1}_{\{x_i \neq y_i\}}$

$X_k \in \Omega_k \quad \Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_n$

Then $\forall t \geq 0$ & every set $A \subseteq \Omega$

$$P(\vec{X} \in A) P(d_H(\vec{x}, A) \geq t) \leq \exp\left(\frac{-t^2}{2n}\right)$$

$$d_H(\vec{x}, A) = \min_{y \in A} d_H(\vec{x}, \vec{y})$$

$e^x \leq 1 + x + x^2 \quad \forall |x| \leq 1$
$1 + x \leq e^x$
$\Rightarrow 1 + x^2 \leq e^{x^2}$
Proof uses

Generalised Blowing up Lemma

$\vec{\alpha} = \{\alpha_i\}_{i=1}^N \geq 0$

$$d_{\vec{\alpha}}(\vec{x}, \vec{y}) = \sum_{i=1}^n \alpha_i \mathbb{1}_{\{x_i \neq y_i\}}$$

Theorem

Let $\vec{x} = (x_1, x_2, \cdots, x_n)$ such that x_i's are independent

$\vec{\alpha}$: non-ve unit norm vector

$$P(\vec{x} \in A) P(d_{\vec{\alpha}}(\vec{x}, \vec{y}) \geq t) \leq \exp\left(\frac{-t^2}{2}\right)$$

Application: Median Concentration

$Z = f(X_1, X_2, \cdots, X_n)$ satisfies BDC with $(c_1, c_2, \cdots, c_n)$

$m = \text{Median}(Z) \quad P(Z \geq m) \geq \frac{1}{2}$ && $P(Z \leq m) \leq \frac{1}{2}$

$$P(Z - m \geq t) \vee P(Z - m \leq -t) \leq 2\exp\left(\frac{-t^2}{2c^2}\right)$$

$$c^2 = c_1^2 + c_2^2 + \cdots + c_n^2$$

$S_a = \{\vec{y} : f(\vec{y}) \leq a\}$ Sublevel set at a.

$P(f(\vec{x}) \leq a) P(f(\vec{x}) \geq a + t) \leq \exp\left(-\frac{t^2}{2c^2}\right)$

$f(\vec{x}) \geq a + t \Rightarrow \frac{d_{\vec{\alpha}}(\vec{x}, S_a)}{c} \geq \frac{t}{c}$

Lemma: Let a R.V. Y have mean μ & median m, $a, b \geq 0$

If $P(Y - m \geq t) \leq a e^{-t^2/b}$ $\forall t \geq 0$ then $\mu - m \leq \frac{\sqrt{\pi}\, ab}{2}$ | If $P(Y - m \leq -t) \leq a e^{-t^2/b}$ then $m - \mu \leq \frac{\sqrt{\pi}\, ab}{2}$

both hold than $|\mu - m| \leq \frac{\sqrt{\pi}\, ab}{2}$

→ BDC outside a bad set Let f be a bounded f^n of $X_1, X_2, \cdots, X_n$ with $m \leq f(X_1, X_2, \cdots, X_n) \leq M$

$\boxed{B^c : \text{satisfies ALC}}$, B: Bad Event

$$|E[Z | \bar{X}_{i-1}, X_i, B^c] - E[Z | \bar{X}_{i-1}, X_i', B^c]| \leq c_i$$

$c = \sum_{i=1}^n c_i^2 \quad P(Z > E[Z] + t) = P(Z > E[Z] + t | B^c) P(B^c) + P(Z > E[Z] + t) P(B)$

$E[Z|B^c]$

$$P(|Z - E[Z]| > t + (M - m)P(B)) \leq \left\{e^{-\frac{2t^2}{c}} + P(B)\right\} 2$$

More general version $P(Z > E[Z] + t) \wedge P(Z < E[Z] - t) \leq \exp\left(\frac{-2t^2}{c}\right) + P(B)$

$|E[Z] - E[Z|B^c]| \leq P(B)(M - m) \qquad E[Z] - E[Z|B^c] = P(B)(E[Z|B] - E[Z|B^c])$

Variance Bound For Martingales.

Let Z_i be martingale wrt X_i with

① $|Z_i - Z_{i-1}| \leq c_i$ ② $v_i = \sup_{F_{i-1}} \text{var}(Z_i - Z_{i-1} | F_{i-1})$

Let $V = \sum V_i$

Then $\begin{matrix} P(Z_n > Z_0 + t) \\ P(Z_n < Z_0 - t) \end{matrix} \leq \exp\left(\frac{-t^2}{4V}\right)$ for $0 \leq t \leq \frac{2V}{\max_{i \in [n]} c_i}$

Talagrand's Inequality.

$\sum \alpha_i \mathbb{1}_{(x_i \neq y_i)}$

Talagrand's Convex distance $d_T(\vec{x}, \vec{y}) = \sup_{\{\vec{\alpha} : \|\alpha\|_2 = 1\}} d_{\vec{\alpha}}(\vec{x}, \vec{y})$

If $\vec{X} = (X_1, X_2, \cdots, X_n)$ are independent R.V., $A \subseteq \Omega$, $\forall t \geq 0$

$$P(\vec{X} \in A) P(d_T(\vec{x}, A) \geq t) \leq \exp\left(\frac{-t^2}{4}\right)$$

$$d_T(\vec{x}, A) = \sup_{\alpha | \|\alpha\| = 1} d_{\vec{\alpha}}(\vec{x}, A) = \sup_{\alpha : \|\alpha\|_2 \leq 1} \min_{\vec{y} \in A} d_\alpha(\vec{x}, \vec{y})$$

can be f of $\vec{x}$

$$d_T(\vec{x}, \vec{y}) \geq d_{\vec{\alpha}(\vec{x})}(\vec{x}, \vec{y})$$

Configuration F^n $\quad f: \mathbb{R}^n \to \mathbb{R}$ is c-config if $\forall x, y \in \Omega$

$f(\vec{x}) \leq f(\vec{y}) + \sqrt{c f(\vec{x})}\, d_{\vec{\alpha}}(\vec{x}, \vec{y})$ for some $\vec{\alpha}(\vec{x})$ non-ve unit norm.

ex. Longest increasing subsequence

Max. Length common subsequence.

$E_1 \Rightarrow E_2$ then $P(E_1) \leq P(E_2)$

For c-config. f^n Talagrand gives MEDIAN concentration.

Theorem Let f be a c-config f^n of $\vec{X} = (X_1, X_2, \cdots, X_n)$ where X_i's are independent then $\forall t \geq 0$

m: median.

$$P(f(\vec{x}) \geq m + t) \leq 2\exp\left(\frac{-t^2}{4c(m+t)}\right)$$

$$P(f(\vec{x}) \leq m - t) \leq 2\exp\left(\frac{-t^2}{4cm}\right)$$

Theorem Suppose for each $\vec{x}$, $\exists \vec{\alpha}(\vec{x})$ such that

$$f(\vec{x}) \leq f(\vec{y}) + c\, d_{\vec{\alpha}}(\vec{x}, \vec{y}) \quad \forall \vec{y} \in \Omega.$$

X_i independent

$$P(|f(\vec{x}) - m| \geq t) \leq 4\exp\left(\frac{-t^2}{4c^2}\right)$$

Precursor to Efron-Stein

$Z = f(X_1, X_2, \cdots, X_n)$ X_i: independent from same alphabet

$E_i[Z] = E[Z \mid X_1^{i-1}, X_{i+1}^n] = E[Z \mid X^{(i)}]$

Precursor $\operatorname{Var}(Z) \leq \sum_{i=1}^n E[(Z - E_i[Z])^2]$

Efron-Stein

$$\operatorname{Var}(Z) \leq \frac{1}{2}\sum_{i=1}^n E[(Z - Z_i')^2]$$

$Z_i' = f(X_1^{i-1}, X_i', X_{i+1}^n)$ & X_i' is an independent copy of X_i

Using BDC on f if applicable. $\operatorname{Var}(Z) \leq \sum c_i^2/2$

Proof Sketch $V = Z - E[Z]$

$V_i = E[Z|F_i] - E[Z|F_{i-1}]$ $\quad V = \sum_{i=1}^n V_i$ $\quad \operatorname{Var}(Z) = E[\sum_{i=1}^n V_i^2] + 2\sum_{i<j} \underbrace{E[V_iV_j]}_{0}$

$V_i^2 = (E[Z|F_i] - E[Z|F_{i-1}])^2$

$= (E[E[Z|F_n]|F_i] - E[E_i[Z]|F_{i-1}])^2$ $\quad F_{i-1} \to F_i$

Now conditional Jensen.

$\operatorname{Var}(Z) \leq \sum_{i=1}^n E[(Z - E_i[Z])^2]$

$= \sum E[E_i(Z - E_i[Z])^2]$

$\leq \sum E[\frac{1}{2} E_i[(Z - Z_i')^2]]$

First Passage Percolation Time $\quad p^* \in \arg\min_{p \in P} \sum_{e_j \in p} X_j$

$\operatorname{Var}(Z) \leq \frac{1}{2}\sum_{i=1}^n E[(Z - Z_i')^2]$

$\{Z_i' > Z\} \Rightarrow \{X_i' > X_i\} \& \{e_i \in p^*\}$

$= \sum_{i=1}^n E[(Z - Z_i')^2 \mathbb{1}_{\{Z_i' < Z_i\}}]$

$\leq \sum_{i=1}^n E[(X_i' - X_i)^2 \mathbb{1}_{\{e_i \in p^*\}}]$

Ex1 Bin Packing $\sum_{i \in a\, bin} x_i \leq 1$

$E.S \leq \frac{1}{2}\sum_{i=1}^n E[(Z - Z_i')^2] \leq \frac{n}{2}$

Self Bounding Function.

$g: \mathcal{X}^n \to \mathbb{R}$ is self bounding if $g(\cdot) \geq 0$ and $\exists \{g_i, i = 1, 2, \cdots, n\}$ such that

ⓐ $0 \leq g(x_1^n) - g_i(x_1^{i-1}, x_{i+1}^n) \leq 1$

ⓑ $\sum_{i=1}^n \{g(x_1^n) - g_i(x_1^{i-1}, x_{i+1}^n)\} \leq g(x_1^n)$

Corollary Let $Z = g(X_1^n)$, where g is a self-bounding function, then $\operatorname{var}(Z) \leq E[Z]$

Proof $0 \leq Z - Z_i \leq 1$

$(Z - Z_i)^2 \leq (Z - Z_i)$

sum over all i use ⓑ

Z_i here is different from that used in Efron-Stein

X: Real R.V. on $(\Omega, \mathcal{F}, P)$ with $E[X^2] < \infty$ & $\mathcal{G} \subset \mathcal{F}$

$$E[(X - E[X|\mathcal{G}])^2 \mid \mathcal{G}] \leq E[(X - Z)^2 \mid \mathcal{G}]$$

$\forall$ $\mathcal{G}$-measurable Z

$Z_i = g_i(X_1^{i-1}, X_{i+1}^n)$

$$\operatorname{Var}(Z) \leq \sum_{i=1}^n E[(Z - E_i[Z])^2] = \sum_{i=1}^n E[E_i[(Z - E_i[Z])^2]]$$

$$\leq \sum_{i=1}^n E[E_i[(Z - Z_i)^2]] = \sum_{i=1}^n E[(Z - Z_i)^2]$$

Hereditary Property P is defined over $\mathcal{X}^n$ as:

① $P = \{P_1, P_2, \cdots, P_n\}$ is a sequence of sets with $P_k \subseteq \mathcal{X}^k$, $k = 1, 2, \cdots, n$

② $(x_1, x_2, \cdots, x_m) \in P_m \Rightarrow$ any subsequence $(x_{i_1}, x_{i_2}, \cdots, x_{i_\ell}) \in P_\ell$ for any $\ell \leq m$ & for any $m = 1, 2, \cdots, n$.

~~Example~~

Configuration Function Given a sequence $(x_1, x_2, \cdots, x_n) \in \mathcal{X}^n$ the config f^n wrt a hereditary property P is the cardinality of largest sub-sequence of $\vec{x}$ in P

Config f^n are Self-Bounding

$t = \varepsilon\sqrt{n}$

STSP $\quad P(T_n - E[T_n] > t) \leq e^{\frac{-t^2}{8n}}$ using ALC $2\sqrt{2}$ & inde.

$T_n(i) \leq T_n \leq T_n(i) + 2Z_i$ $\quad \leq e^{\frac{-2t^2}{8 + c^2\log n}}\ e^{\frac{-n}{\log n}}$ BMDC

$Z_i = \min_{j>i} d(P_i, P_j)$ $\quad \leq e^{-n}$ using Talagrand.

$E[T_n(i)|F_i] = E[T_n(i)|F_{i-1}]$

$|E[T_n|F_i] - E[T_n|F_{i-1}]| \leq 2\max(\overset{a}{E[Z_i|F_{i-1}]}, \overset{b}{E[Z_i|F_i]})$

$a, b \leq \frac{c}{\sqrt{n-i}} \wedge 2\sqrt{2}$

$P_{X|Y=0} = \begin{cases} p_0 & H \\ 1-p_0 & T \end{cases}$ $\quad 0 \le \alpha \le p_i(x) \le \beta \le 1$

$P_{X|Y=1} = \begin{cases} p_1 & H \\ 1-p_1 & T \end{cases}$

$S_n(X_1^n) = \sum_{i=1}^n \log \frac{p_0(X_i)}{p_1(X_i)}$

$P(\hat{y}=0 \mid Y=1) = P(S_n > c \mid Y=1)$

$E[S_n \mid Y=1] = n\left[p_1 \log \frac{p_0}{p_1} + (1-p_1)\log\frac{1-p_0}{1-p_1}\right]$

$= -n\, KL(p_1 \| p_0)$

$P(S_n - E[S_n|Y=1] > -E[S_n|Y=1] \mid Y=1)$

$P(\hat{y}=0 \mid Y=1) \le \exp\left(\frac{-2n^2 D^2(p_1\|p_0)}{n\,(\log\beta - \log\alpha)^2}\right)$

Bound $\mu = 0$, $|X_i| \le c$, σ^2 known

$E[e^{\theta X_i}] = 1 + 0 + \sum_{r=2}^{\infty} \frac{\theta^r E[X_i^r]}{r!}$

$= 1 + 0 + \theta^2\sigma^2 \sum_{r=2} \frac{\theta^{r-2} E[X^2 X^{r-2}]}{r!\, \sigma^2}$

$\le 1 + 0 + \theta^2\sigma^2 \sum_{r=2} \frac{\theta^{r-2} c^{r-2} \sigma^2}{r!\, \sigma^2}$

$\frac{\theta\sigma^2}{\theta^2 c^2} \sum \frac{\theta^r c^r}{r!}$

$1 + \frac{\sigma^2}{c^2}(e^{\theta c} - 1 - \theta c)$

$\{X_i^n\}$ 0/1 k-wise independent bound $P(\sum_{i=1}^n X_i > t)$

$G_k(X_1^n) = \sum_{\substack{I \subseteq [n] \\ |I| = k}} \prod_{i \in I} X_i$ for $m \ge 0$, $m \in \mathbb{Z}^+$

$\sum_{i=1}^n X_i = m \Leftrightarrow G_k(X_1^n) = \binom{m}{k}$

$P(\sum_i X_i > t) = P(G_k(X_1^n) > \binom{t}{k}) \le \frac{E[G_k(X_1^n)]}{{}^tC_k}$

$\frac{\binom{n}{k} p^k}{\binom{t}{k}} = \frac{\sum_{|I|=k} \prod_{i\in I} E[X_i]}{{}^tC_k}$

ALC $\Rightarrow$ BMDC $\left| E[f(X_1^n) \mid X_1^{i-1}, X_i = a_i] - E[f(X_1^n) \mid X_1^{i-1}, X_i = a_i'] \right| \le c_i$

$|E[Z|F_i] - E[Z|F_{i-1}]| = \left|\sum_a P(X_i = a \mid F_{i-1}) E[Z|F_i] - \sum_a E[Z \mid F_{i-1}, X_i = a] P(X_i = a \mid F_{i-1})\right|$

$\le \sum_a (P(X_i = a \mid F_{i-1})) |E[Z|F_i] - E[Z \mid F_{i-1}, X_i = a]|$

$\le c_i$

any f^n of independent r.v.

(1) Mean concentration McDiarmid.

(2) Median conc^n Blowing up Lemma.

TSP $|T_n(P_1^{i-1}, P_i, P_{i+1}^n) - T_n(P_1^{i-1}, P_i', P_{i+1}^n)| \le 2\sqrt{2}$

$P(T_n - E_n[T_n] > t) \le e^{-2t^2/8n}$.

BMDC $|E[T_n \mid P_1^k] - E[T_n \mid P_1^{k-1}]| \le c_{n,k}$

$T_n(i) \le T_n \le T_n(i) + 2Z_i$

$Z_i = \min_{j > i} d(P_i, P_j)$ $\quad Z_i \le 2\sqrt{2}$

$E[T_n|F_i] \le E[T_n(i)|F_i] + 2E[Z_i|F_i]$

$E[T_n|F_{i-1}] \le E[T_n(i)|F_{i-1}] + 2E[Z_i|F_{i-1}]$

can change to F_i

BMDC $|E[T_n|F_i] - E[T_n|F_{i-1}]| \le 2\max(E[Z_i|F_i], E[Z_i|F_{i-1}])$

$E[Z_i|F_i], E[Z_i|F_{i-1}] \le \min\left(2\sqrt{2}, \frac{c}{\sqrt{n-i}}\right)$

$P(T_n - E[T_n] > t) \le \exp\left(\frac{-2t^2}{8 + c^2 \log n}\right) \sim \exp\left(\frac{-n}{\log n}\right)$

Mean close to med $t = \beta\sqrt{n}$

$\mu - m = E[Y] - m$ $\quad Z = -Y$

$= E[Y - m]$

$\le E[(Y-m)^+]$ if $P(Y - m \ge t) \le a e^{-t^2/b}$

$\le \int_0^\infty a e^{-t^2/b}\, dt$

Ex1 Randomised Algo Q

$P(Z \le \mu_Z - t) \le \exp\left(\frac{-2t^2}{n}\right)$

Ex2 Max balls in bins BC

$P(Z \ge (1+\epsilon)E[Z]) \le \left[\frac{e^{\epsilon}}{(1+\epsilon)^{1+\epsilon}}\right]^{\mu}$

$\max_i B_i \ge c \Rightarrow \frac{n e^{c-1}}{c^c}$

$c \sim \frac{3\log n}{\log\log n}$.

Ex3 m balls n bins $B_i = \sum_{k=1}^m B_{i,k}$

Conc^n Result $\quad$ Ball k in bin i

no. of non empty bins

$X_i = \mathbb{1}_{\{B_i > 0\}}$ Bernoulli

$p = 1 - (1 - \frac{1}{n})^m$ $\quad$ $p(\text{ball})$

Claim $\{B_i\}$ NA

$P(\sum X_i > \theta) \le e^{-n\Psi_X^*(\theta)}$

Proving NA for $\sum_i X_i = 1$ $\{0,1\}$

$\tilde{f}(X_i; i \in I) = f(X_i; i \in I) - f(\vec{0}_{|I| \times 1})$

$\tilde{g}(X_i; i \in J) = g(X_i; i \in J) - g(\vec{0}_{|J| \times 1})$

$E[\tilde{f}\tilde{g}] \le E[\tilde{f}]E[\tilde{g}]$

Let change

$E[(\tilde{f} + f(0))(\tilde{g} + g(0))] = E[\tilde{f} + f(0)]E[\tilde{g} + g(0)]$

Bins: Files

Balls: Sequential Requests

$|E[X \mid F_{k-1}, E_k = e_k] - E[X \mid F_{k-1}, E_k = e_k']|$

$= \frac{{}^{m-1}C_{s-k}}{{}^{m}C_{s-k}} \sum_{i \in V} \prod_{j=1}^{k-1} I(i \notin E_j) |I(i \in e_k) - I(i \in e_k')|$

$\le 4k \le 4 = c_k$

Steiner tree containing all pts in $\vec{y}$

$E = \text{edges}(S(\vec{y})) \cup$ edges in $T(\vec{x})$ such that atleast 1 end pt lies in $\vec{x} \setminus \vec{y}$

$B_k = |e_{k,1}| + |e_{k,2}|$

length of ingoing edge to k $\quad$ length of outgoing edge from k

$\text{mst}(\vec{x}) \le$ Sum of length of edges in E $\le \text{mst}(\vec{y}) + d_B(\vec{x}, \vec{y})$

Random Minimum Spanning Tree

$P(\text{shortest edge} > b) = (1-b)^{n-1}$

$E[\text{shortest edge}] = 1/n$

$\lim_{n\to\infty} \text{mst}(\vec{x}) \to \sum_j \frac{1}{j^3} = \gamma$

Theorem $\forall t > 0$, $\exists \delta(t) > 0$

$P(|\text{mst}(\vec{x}) - \gamma| \ge t) \le \exp(-\delta(t) n)$

S1] $Y_e \to X_e = \min\{Y_e, b\}$ $\quad$ unif(0,1)

$L_n := \text{mst}(\vec{q})$ $\quad L_n^{(b)} = \text{mst}(\vec{x})$

$P(L_n - L_n^{(b)} \ge t) \le \exp(-\delta(t) n)$

S2] $\text{mst}(\vec{q}) \le \sum_{i \in E(\vec{p})} q_i$

$= \sum_{i \in E(\vec{p})} q_i + (p_i - q_i)^+$

$\le \sum_{i \in E(\vec{p})} p_i + b\mathbb{1}_{\{p_i \ne q_i\}}$

$= \text{mst}(\vec{p}) + d_{\vec{x}}(\vec{p}, \vec{q})$

$\le \text{mst}(\vec{p}) + b\sqrt{n}\, d_Z(\vec{x}, \vec{y})$

$P(|\text{mst}(\vec{x}) - m| \ge t) \le 4\exp\left(\frac{-t^2}{4n}\right)$

$P(|\text{mst}(\vec{y}) - m| \ge 2t)$

$= P(|\underbrace{\text{mst}(\vec{y}) - \text{mst}(\vec{x})}_{R} + \underbrace{\text{mst}(\vec{x}) - m}_{S}| \ge 2t)$

$= P(|R + S| \ge 2t)$

$\le P(|S| \ge t) + P(|R| \ge t)$

$\le e^{-\delta(t) n} + 4\exp\left(\frac{-t^2 n}{4c^2}\right)$

S3 Show $|E[L_n] - m|$ small

condition on $|L_n - m| \ge t/4$

$|E[L_n] - m| \le E[|L_n - m|]$

$\le t/4 + (n-1)e^{-bn}$

$K(\vec{x}) = \{k = \{i_1, i_2, \dots, i_{|k|}\} \subseteq \{1, 2, \dots, n\}$ s.t. $i_1 < i_2 < \dots < i_{|k|}$
and $x_{i_1} < x_{i_2} < \dots < x_{i_{|k|}}\}$

$inc(\vec{x}) = \sup_{k \in K(\vec{x})} |k|$ → 1 config f^n

$inc(\vec{y}) \geq |\{i \in k(\vec{x}) : y_i = x_i\}| = \sum_{i \in k(\vec{x})} \mathbb{1}\{x_i = y_i\} =$

$\geq |k(\vec{x})| - \sum_{i \in k(\vec{x})} \mathbb{1}\{x_i \neq y_i\}$, $\alpha_i(\vec{x}) = \frac{1}{\sqrt{|k(\vec{x})|}} \mathbb{1}\{i \in k(\vec{x})\}$

$inc(\vec{y}) \geq inc(\vec{x}) - \sqrt{inc(\vec{x})}\, d_{\vec{\alpha}(\vec{x})}(\vec{x}, \vec{y})$

Max length common subsequence. $com(\vec{x}, \vec{y})$: 2 config f^n

$\vec{z} = \vec{x}^{n_1} : \vec{y}^{n_2}$ let $com(\vec{x}, \vec{y}) = l$ $k_1 \subseteq [n_1]$ $k_2 \subseteq [n_2]$
s.t. $|k_1| = |k_2| = l$

$\vec{w} = \vec{a} : \vec{b}$ $k_1 = \{i_r, r = [l]\}$ $i_1 < i_2 < \dots < i_l$ $x_{i_r} = y_{j_r}$
$k_2 = \{j_r, r = [l]\}$ $j_1 < \dots < j_l$

$com(\vec{a}, \vec{b}) \geq |\{r \in [l] : a_{i_r} = x_{i_r},\ b_{j_r} = y_{j_r}\}|$
$= l - |\{r \in [l] : a_{i_r} \neq x_{i_r}$ or $b_{j_r} \neq y_{j_r}\}|$
$\geq com(\vec{x}, \vec{y}) - \sum_{r=1}^{l} \mathbb{1}\{a_{i_r} \neq x_{i_r}\} - \sum_{r=1}^{l} \mathbb{1}\{b_{j_r} \neq y_{j_r}\}$

$\alpha_s = \frac{1}{\sqrt{2l}} \mathbb{1}\{s \in k_1\} \cup \{s - n \in k_2\}$

Median Concentration Results: Let $f(\vec{x})$ be a c-config f^n $\vec{x} = (x_1, \dots, x_n)$ ind. R.V.

$P(f(\vec{x}) \geq m+t) \leq 2\exp\left(\frac{-t^2}{4c(m+t)}\right)$ $P(f(\vec{x}) \leq m-t) \leq 2\exp\left(-\frac{t^2}{4cm}\right)$ $m = \text{median}(f(\vec{x}))$.

f is a c-config $\forall \vec{x}, \vec{y} \in \Omega$, $\exists$ a unit norm$(\vec{\alpha})$ $f(\vec{x}) \leq f(\vec{y}) + \sqrt{cf(\vec{x})}\, d_{\vec{\alpha}}(\vec{x}, \vec{y})$

$S_a = \{y : f(y) \leq a\}$

$f(\vec{x}) \leq \cancel{f(\vec{y})}\ a + \sqrt{cf(\vec{x})} \inf_{y \in S_a} d_{\vec{\alpha}}(\vec{x}, \vec{y})$ $f(\vec{x}) \leq a + \sqrt{cf(\vec{x})}\, d_T(\vec{x}, S_a)$

$d_T(\vec{x}, S_a) \geq \frac{f(\vec{x}) - a}{\sqrt{cf(\vec{x})}}$ RHS inc in $f(\vec{x})$

$\leq a + \sqrt{cf(\vec{x})} \sup_{\vec{\alpha} : \|\alpha\| = 1} \inf_{y \in S_a} d_{\vec{\alpha}}(\vec{x}, \vec{y})$

$f(\vec{x}) \geq a+t \Rightarrow d_T(\vec{x}, A) \geq \frac{t}{\sqrt{c(a+t)}}$ $A \Rightarrow B$ $P(A) \leq P(B)$

Putting $A = S_a$ in Talagrand

$P(\vec{X} \in S_a)\, P\left(d_T(\vec{X}, S_a) \geq t/\sqrt{c(a+t)}\right) \leq e^{-t^2/4c(a+t)}$

$P(f(\vec{X}) \leq a)\, P(f(\vec{X}) \geq a+t) \leq e^{-t^2/4c(a+t)}$ $a = m$ $a = m-t$ $P(f(\vec{X}) \geq m) \geq \frac{1}{2}$

Theorem Suppose for each $\vec{x}$, $\exists \vec{\alpha}(\vec{x})$ such that $f(\vec{x}) \leq f(\vec{y}) + c\, d_{\vec{\alpha}}(\vec{x}, \vec{y})\ \forall \vec{y} \in \Omega$

x_i independent $P(|f(\vec{x}) - m| \geq t) \leq 4\exp\left(-\frac{t^2}{4c^2}\right)$

$P(\vec{X} \in A)\, P(d_T(\vec{X}, A) \geq t) \leq \exp\left(-\frac{t^2}{4}\right)$

$d_T(\vec{x}, S_a) \geq \frac{f(\vec{x}) - a}{c}$

$P(tsp(\vec{x}) - \beta\sqrt{n} > t\sqrt{n}) = \Theta(1)$ using ALC + ind
$\sim e^{-n/\log n}$ using BMDC
$\sim e^{-n}$ by Talagrand's.

$e_{i,1}$ incoming edge to x_i
$e_{i,2}$ outgoing

Fact for an optimal tour
$\exists\ 0 < c < \infty$ such that $\forall n \geq 1$
$\sum_i |e_{i,1}|^2 = \sum_i |e_{i,2}|^2 \leq c$ → ind of n & how coordinates picked

$\beta_k = |e_{k,1}| + |e_{k,2}|$ $\sum \beta_k^2 \leq 4c$

$\vec{\alpha}(\vec{x}) = \left(\frac{\beta_1}{\|\beta\|_2}, \frac{\beta_2}{\|\beta\|_2}, \dots, \frac{\beta_n}{\|\beta\|_2}\right)$

Claim for any $\vec{y} \in \Omega$
$tsp(\vec{x}) \leq tsp(\vec{y}) + d_{\vec{\beta}}(\vec{x}, \vec{y})$
$\leq tsp(\vec{y}) + 2\sqrt{c}\, d_{\vec{\alpha}}(\vec{x}, \vec{y})$

$P(|tsp(\vec{x}) - m| \geq t) \leq 4\exp\left(\frac{-t^2}{16c}\right)$

Precursor to Efron Stein $Z = f(X_1^n)$, X_i: independent & same alphabet

Theorem $Var(Z) \leq \sum_{i=1}^{n} E\left[(Z - E_i[Z])^2\right]$, $E_i[Z] = E[Z \mid X_1^{i-1}, X_{i+1}^n]$

Efron-Stein Inequality $Var(Z) \leq \frac{1}{2} \sum_{i=1}^{n} E\left[(Z - Z_i')^2\right]$, $Z_i' = f(X_1^{i-1}, X_i', X_{i+1}^n)$
X_i' is ind copy of X_i

Proof of Precursor $V = Z - E[Z]$
$V_i = E[Z \mid X_1^i] - E[Z \mid X_1^{i-1}]$
$V = \sum_i V_i$ $E[V_i E[V_j \mid \mathcal{F}_i]]$ $= 0$
$Var(Z) = E[(Z - E[Z])^2]$
$= E[(\sum_i V_i)^2]$
$= E[\sum V_i^2] + 2\sum_{i<j} E[V_i V_j]$ → 0

Claim $V_i^2 = \left(E[Z \mid \mathcal{F}_i] - E[Z \mid \mathcal{F}_{i-1}]\right)^2$ Claim $V_i^2 \leq E[(Z - E_i[Z])^2 \mid \mathcal{F}_i]$
$= \left(\cancel{E[E[Z \mid \mathcal{F}_n] - E[Z \mid X_1^{i-1}, X_{i+1}^n]]}\right)$
$= \left(E[E[Z \mid \mathcal{F}_n] \mid \mathcal{F}_i] - E[E[Z \mid X_1^{i-1}, X_{i+1}^n] \mid \mathcal{F}_i]\right)^2$
$= \left(E\left[E[Z \mid \mathcal{F}_n] - E_i[Z] \mid \mathcal{F}_i\right]\right)^2$ $f(E[X]) \leq E[f(X)]$
$\leq E\left[(Z - E_i[Z])^2 \mid \mathcal{F}_i\right]$
$\Rightarrow E[V_i^2] \leq E[(Z - E_i[Z])^2]$

Proof of Efron Stein
$E_i[(Z - E_i[Z])^2] = \frac{1}{2} E_i[(Z - Z_i')^2]$
LHS $E_i[Z^2] - E_i^2[Z]$ = RHS

Something different than Efron-Stein
$E[(X - E[X \mid Y])^2] \leq E[(X - h(Y))^2 \mid Y]$

Precursor to ES
$Var(Z) \leq \sum_i E[(Z - E_i[Z])^2]$
$= \sum_i E[E_i(Z - E_i[Z])^2]$
$\leq \sum_i E[E_i[(Z - Z_i)^2]]$
here $Z_i = g_i(X_1^{i-1}, X_{i+1}^n)$
$= \sum_i E[(Z - Z_i)^2]$

Self-Bounding f^n: A f^n $g : \mathcal{X}^n \to \mathbb{R}$ is self bounding if $g(\cdot) \geq 0$ &
$\exists g_i$ $i \in [n]$ such that
(i) $0 \leq g(x_1^n) - g_i(x_1^{i-1}, x_{i+1}^n) \leq 1$
(ii) $\sum_{i=1}^{n} \{g(x_1^n) - g_i(x_1^{i-1}, x_{i+1}^n)\} \leq g(x_1^n)$

Corollary Let $Z = g(X_1^n)$ where g is self bounding
then $Var(Z) \leq E[Z]$
$(Z - Z_i)^2 \leq (Z - Z_i)$
$\sum_i (Z - Z_i)^2 \leq Z$
$Var(Z) \leq \sum_{i=1}^{n} E[(Z - Z_i)^2]$
$\leq E[Z]$

$Z_i' > Z \Leftrightarrow X_i' > X_i$ $c_i \in f^n$

$Var(Z)$

Ex1
Distinctness is a hereditary property.
$P_1 = \mathcal{X}$
$P_2 = \{(x_1, x_2) \in \mathcal{X}^2 : x_1 \neq x_2\}$
$P_3 = \{(x_1, x_2, x_3) \in \mathcal{X}^3 : x_1 \neq x_2 \neq x_3 \neq x_1\}$

Ex2 Increasingness is a hereditary
subsequence of an increasing sequence

Config F^n

$\mathcal{X} = 1, 2, \dots, 9$ $\vec{x} = (1, 2, 1, 3, 4, 5)$
wrt hereditary property of increasing sequence, config f^n is 5

Config f^n are self bounding proof

$Z = g^{(n)}(X_1, \dots, X_n)$
$Z_i = g^{(n-1)}(X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n)$
→ same config f^n over $(n-1)$ length sequence.