Quantum Light-Matter Interactions: Superradiance in the Dicke Framework

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1 Abstract

The Dicke model is one of the fundamental models of quantum optics. It studies the light-matter interaction and the Superradiant phase transition that this interaction can lead to. The Dicke model is described as an ensemble of N two-level atoms that are coupled to an optical cavity mode. Above a certain critical coupling, the cavity becomes macroscopically occupied and leads to the Superradiant phase. The Dicke model falls under the category of dissipative-phase transition and is also important when studying quantum information and quantum sensing. In this report, I will attempt to understand the superradiant phase of the Dicke Model using the Lindblad formalism and derive the critical coupling. Furthermore, I will compare the results from theory and numerical simulation and then finally visualize it using Wigner Function on python.

2 Dicke Model

Dicke model is an ensemble of N two level systems coupled to a bosonic mode (quantized mode of an optical cavity). The Hamiltonian of this system is given as follows [KEB13]:

$$H = \hbar\omega_c c^\dagger c + \hbar\omega_a J_z + \frac{\hbar\lambda}{\sqrt{N}}(c^\dagger + c)(J_+ + J_-) \tag{2.1}$$

Here, ω_c is the cavity frequency, ω_a determines the energy splitting between the two-level atoms, N is the number of atoms, λ denotes the coupling between the cavity and the atoms, and $c^\dagger, c$ are the creation and annihilation operators of the cavity. Similarly, $J_{\pm}$ are raising and lowering operator. It is important to note that we are considering the ensemble of N two level atoms as a spin of length $j = N/2$ and from here J_z is defined for measuring collective angular momentum in the z-direction. J_+ and J_- are defined as collective excitation and de-excitation.

For this hybrid quantum system, we can understand the motivation behind using this hamiltonian by breaking it down in three separate term. The first term is (i) Cavity term, it defines the energy of the cavity field and each photon ($c^\dagger c$) contributes $\hbar\omega_c$ energy to the system. The second term is (ii) Atomic Term, this can be understood by considering only a single atom first. For one atom we can represent the splitting of the energy in ground and excited states by just using a Pauli z operator, σ_z. The diagonal term represents the two energy levels. Each level will contribute to an energy of $\pm\hbar\omega_a/2$. But, now that we have an ensemble of atoms we take the Pauli z-operator for each and sum them, this summation is represented as J_z the collective angular momentum. Finally, the third term (iii) Interaction term, this couples the cavity mode ($c^\dagger + c$) to the atomic excitations ($J_+ + J_-$).

With this motivation of the hamiltonian we go ahead and derive the dissipative part of our system. As we are not isolated from the environment, we need a model for the loss/decay due to environmental interaction. Using the Lindblad formalism we can define a cavity decay rate, κ, this means our hybrid quantum system is now an open hybrid quantum system. The Lindblad equation equation for this system will look as follows:

$$\frac{d}{dt}\rho(t) = -\frac{i}{\hbar}[H, \rho(t)] + \kappa\mathfrak{D}[c]\rho(t) \tag{2.2}$$

ρ is our density matrix, $[H, \rho(t)]$ is the commutation relation and $\mathfrak{D}[c]$ is the dissipator of the system:

$$\mathfrak{D}[c]\rho(t) = c\rho(t)c^\dagger - \frac{1}{2}[c^\dagger c, \rho(t)]_- \tag{2.3}$$

Here, $[c^\dagger c, \rho(t)]_-$ represents the anticommutator. Now, the aim is to derive the critical coupling that takes the system from a normal phase to the superradiant phase. To do this we take the Hamiltoninan in the thermodynamic limit of $N \to \infty$, but before that we want to rewrite the system which will describe N two-level atoms by a single bosonic mode. This is called the Holstein-Primakof transformation, it is done for ease of calculation. The transformation is as followed [EB03]:

$$\begin{aligned} &J_+ = a^\dagger\sqrt{2j - a^\dagger a}, \qquad J_- = \sqrt{2j - a^\dagger a}\; a \\ &J_z = (a^\dagger a - j) \end{aligned} \tag{2.4}$$

Here, a and $a^\dagger$ are the ladder operators for the atomic mode. If we substitute the values from equation 2.4 to the Hamiltonian in equation 2.1:

$$H = \omega_a(a^\dagger a - j) + \omega_c c^\dagger c + \lambda(c^\dagger + c)\left(a^\dagger\sqrt{1 - \frac{a^\dagger a}{2j}} + \sqrt{1 - \frac{a^\dagger a}{2j}}\,a\right) \tag{2.5}$$

For simplicity I have taken $\hbar = 1$. The substitution is simple, I made two simplification in there, first I replaced $N = 2j$ and second, in the interaction term N or $2j$ is taken common and canceled by the $2j$ or N outside. This is our Hamiltonian under Holstein-Primakof approximation and we have to solve this for critical coupling.

Like any other hamiltonian system, if we know some of the underlying physics then we can assume a solution for the system and proceed with that. So, this hamiltonian, as it transitions to the superradiant phase it will show a macroscopic occupation ($|\alpha|$, $|\beta|$) and some quantum fluctuation around it. So, we can take an ansatz for the ladder operators that are like displacement and then some fluctuations around the displacements:

$$\begin{aligned} a = d - \sqrt{\beta}, \quad & a^\dagger = d^\dagger - \sqrt{\beta^*} \\ c = e + \sqrt{\alpha}, \quad & c^\dagger = e^\dagger + \sqrt{\alpha^*} \end{aligned} \tag{2.6}$$

Now, we can input these ansatz in the hamiltonian derived through Holstein-Primakof approximation (2.4) and the dissipator (2.3). As we are working in the thermodynamic limit, we expand the square root of 2.4 and then neglect terms with power of j in the denominator. Right now we will just substitute the ansatz:

$$H = \omega_c[e^\dagger e + e\sqrt{\alpha^*} + e^\dagger\sqrt{\alpha} + \alpha] + \omega_a[\zeta - j] + \lambda(e^\dagger + e + \sqrt{\alpha} + \sqrt{\alpha^*})\left(d^\dagger\sqrt{1-\zeta} + \sqrt{1-\zeta}\,d - 2\sqrt{\beta(1-\zeta)}\right) \tag{2.7}$$

Here, I have assumed that β is real and have substituted the values of ansatz. ζ is defined as:

$$\zeta = \frac{d^\dagger d + \beta - \sqrt{\beta}(d + d^\dagger)}{2j}$$

Using this we can write another term of the form:

$$\sqrt{\xi} = \sqrt{1 - \frac{d^\dagger d - \sqrt{\beta}(d^\dagger + d)}{k}} \tag{2.8}$$

Here $k = 2j - \beta$, now we will rewrite our Hamiltonian in terms of k and j:

$$H = \omega_c[e^\dagger e + \sqrt{\alpha^*}e + \sqrt{\alpha}e^\dagger + \alpha] + \omega_a[d^\dagger d - \sqrt{\beta}(d + d^\dagger) + \beta - j] + \lambda\sqrt{\frac{k}{2j}}(e^\dagger + e + \sqrt{\alpha} + \sqrt{\alpha^*})(d^\dagger\sqrt{\xi} + \sqrt{\xi}d - 2\sqrt{\beta\xi}) \tag{2.9}$$

Now, we can finally do the expansion of the square root, we will expand it till second order:

$$\sqrt{\xi} = \sqrt{1 - \frac{d^\dagger d - \sqrt{\beta}(d^\dagger + d)}{k}} \approx 1 - \frac{d^\dagger d - \sqrt{\beta}(d^\dagger + d)}{2k} - \frac{(d^\dagger d - \sqrt{\beta}d^\dagger + d)^2}{8k^2} \tag{2.10}$$

In this expansion any ladder operator of third order will be ignored and we will keep terms only till second order. So, after doing that and simplifying we get:

$$\sqrt{\xi} \approx 1 - \frac{d^\dagger d}{2k} + \frac{\sqrt{\beta}(d^\dagger + d)}{2k} - \frac{\beta(d^{\dagger 2} + d^2)}{8k^2} - \frac{\beta d^\dagger d}{4k^2} \tag{2.11}$$

Now we can first start working with only d ladder operator terms in the Hamiltonian that is $d^\dagger\sqrt{\xi} + \sqrt{\xi}d - 2\sqrt{\beta\xi}$. Even in this we for now only deal with the first two terms, and as before we ignore all the terms of order higher than 2 [Bra+24]:

$$d^\dagger\sqrt{\xi} + \sqrt{\xi}d \approx d^\dagger + d + \frac{\sqrt{\beta}d^\dagger(d^\dagger + d)}{2k} + \frac{\sqrt{\beta}(d^\dagger + d)d}{2k} \tag{2.12}$$

We can simplify this further into:

$$d^\dagger\sqrt{\xi}+\sqrt{\xi}d \approx d^\dagger + d + \frac{\sqrt{\beta}(d^\dagger+d)^2}{2k} \tag{2.13}$$

Now, we can add the third β term also, doing that and simplifying will give us:

$$d^\dagger\sqrt{\xi}+\sqrt{\xi}d - 2\sqrt{\beta\xi} \approx (1-\frac{\beta}{k})(d^\dagger+d) + \frac{\sqrt{\beta}(2k+\beta)}{4k^2}(d^\dagger+d)^2 + \frac{\sqrt{\beta}}{k}d^\dagger d - 2\sqrt{\beta} \tag{2.14}$$

We finally have the interaction term and now we can add it with the hamiltonian and collect like terms. So, first doing it for $(d^\dagger+d)^2$:

$$\text{Collecting } (d^\dagger+d)^2 \text{ terms}: (\sqrt{\alpha}+\sqrt{\alpha^*})\lambda\sqrt{\frac{k}{2j}}\frac{\sqrt{\beta}(2k+\beta)}{4k^2}$$

$$\text{Collecting } (e^\dagger+e)(d^\dagger+d) \text{ terms}: \lambda\sqrt{\frac{k}{2j}}\Big(1-\frac{\beta}{k}\Big)$$

$$\text{Collecting } (d^\dagger d) \text{ terms}: (\sqrt{\alpha}+\sqrt{\alpha^*})\lambda\sqrt{\frac{k}{2j}}\frac{\sqrt{\beta}}{k}+\omega_a$$

$$\text{Collecting } (e^\dagger e) \text{ terms}: \omega_c$$

Now the rest of the terms can form another hamiltonian which contain all the linear terms:

$$H_s^1 = -2\lambda\sqrt{\frac{k}{2j}}\sqrt{\beta}(e^\dagger+e)+\omega_c(\sqrt{\alpha}e^\dagger+\sqrt{\alpha^*}e)+\lambda\sqrt{\frac{k}{2j}}(\sqrt{\alpha}+\sqrt{\alpha^*})(1-\frac{\beta}{k})(d^\dagger+d)-\omega_a\sqrt{\beta}(d^\dagger+d) \tag{2.15}$$

Another hamiltonian with quadratic ladder operators:

$$H_s^2 = \Omega_0 d^\dagger d + \omega_c e^\dagger e + \Lambda(e^\dagger+e)(d^\dagger+d) + M(d^\dagger+d)^2 \tag{2.16}$$

The coefficients are the same as the collected coefficients above. Now, we can finally write our Lindblad equation, so substituting the ansatz 2.6 in the Dissipator 2.3, we get the following Lindblad equation after some simplification:

$$\frac{d}{dt}\rho(t) = -i[H_s^1 + i\sqrt{\alpha^*}\frac{\kappa}{2}e - i\sqrt{\alpha}\frac{\kappa}{2}e^\dagger - i[H_s^2,\rho(t)] - \frac{\kappa}{2}(e^\dagger e\rho(t)+\rho(t)e^\dagger e - 2e\rho(t)e^\dagger) \tag{2.17}$$

Now, that we have this we can try to understand what the above equation is actually representing. It is important to realize that the H_s^1 Hamiltonian has all the linear terms in it and that is why we added the other linear term from the dissipator to that Hamiltonian's commutation relation as well. These terms are proportional to $\sqrt{N}$ and dominate in the thermodynamic limit. The H_s^2 are quadratic term which are like fluctuations around the macroscopic occupation $(|\alpha|,|\beta|)$. Now, for the macroscopic occupation to establish the superradiant phase this linear hamiltonian terms should vanish in the thermodynamic limit $N\to\infty$, this means[HRP18]:

$$H_s^1 + i\sqrt{\alpha^*}\frac{\kappa}{2}e - i\sqrt{\alpha}\frac{\kappa}{2}e^\dagger = 0 \tag{2.18}$$

We can input H_s^1 and then solve independently for each of the macroscopic occupation α and β. This can be done by isolating d, $d^\dagger$, e and $e^\dagger$ and solving for the equations independently.

$$-2\lambda\sqrt{\frac{k}{2j}}\sqrt{\beta}(e^{\dagger}+e)+\omega_c(\sqrt{\alpha}e^{\dagger}+\sqrt{\alpha^*}e)+\lambda\sqrt{\frac{k}{2j}}(\sqrt{\alpha}+\sqrt{\alpha^*})(1-\frac{\beta}{k})(d^{\dagger}+d)-\omega_a\sqrt{\beta}(d^{\dagger}+d)-i\frac{\kappa}{2}\sqrt{\alpha}e^{\dagger}+i\frac{\kappa}{2}\sqrt{\alpha^*}e=0 \tag{2.19}$$

Now, writing out the terms for each ladder operator separately:

$$e^{\dagger}\left(\omega_c\sqrt{\alpha}-2\lambda\sqrt{\beta}\sqrt{\frac{k}{2j}}-i\frac{\sqrt{\alpha}\kappa}{2}\right)=0 \tag{2.20}$$

$$e\left(\omega_c\sqrt{\alpha}-2\lambda\sqrt{\beta}\sqrt{\frac{k}{2j}}+i\frac{\sqrt{\alpha}\kappa}{2}\right)=0 \tag{2.21}$$

$$d^{\dagger}\left((\sqrt{\alpha}+\sqrt{\alpha^*})\lambda\sqrt{\frac{k}{2j}}(1-\beta/k)-\sqrt{\beta}\omega_a\right)=0 \tag{2.22}$$

Equations for $d^{\dagger}$ and d are the same. Now solving the equations will give us:

$$\sqrt{\alpha}=\frac{2\lambda\sqrt{1-\frac{\beta}{2j}}\sqrt{\beta}}{\omega_c-i\frac{\kappa}{2}} \tag{2.23}$$

$$\sqrt{\alpha^*}=\frac{2\lambda\sqrt{1-\frac{\beta}{2j}}\sqrt{\beta}}{\omega_c+i\frac{\kappa}{2}} \tag{2.24}$$

$$\beta=j\left(1-\frac{\kappa^2\omega_a}{16\lambda^2\omega_c}-\frac{\omega_c\omega_a}{4\lambda^2}\right)=j\left[1-\frac{\lambda_2^2}{\lambda^2}\right] \tag{2.25}$$

Another trivial solution for β is 0, but we will use the non-trivial one and substitute in α:

$$\alpha=\pm\frac{2\lambda\sqrt{2j(1-\frac{\lambda_2^4}{\lambda^4})}}{2\omega_c-i\kappa} \tag{2.26}$$

These are the expressions for the macroscopic occupation which establishes the superradiant phase. λ_2^2 is defined as:

$$\lambda_2^2=\frac{(\kappa^2+4\omega_c^2)\omega_a}{16\omega_c} \tag{2.27}$$

Now, the decay rate has come in play. For $\lambda>\lambda_2$ α and β the macroscopic occupation, will have non-zero values. So we have the critical coupling above which superradiance transition happens.

$$\boxed{\lambda_c^2=\frac{(\kappa^2+4\omega_c^2)\omega_a}{16\omega_c}} \tag{2.28}$$

Using the Lindblad equation, I have derived the critical coupling required to achieve the superradiant phase. The decay rate, κ, plays a crucial role in sustaining a steady state, which shifts toward macroscopic occupation only at the critical coupling. The inclusion of the decay constant arises because, in the thermodynamic limit, the linear terms must vanish. This is due to the construction of the dissipator, which introduces a linear term that combines with the commutation relation of the Hamiltonian's first term. The decay rate strikes a balance between the normal and superradiant phase and accurately captures the critical coupling for the Dicke model[Kir+19].

3 Using QuTip to study the Superradiant Phase Transition

I derived the critical coupling using the Lindblad equation. We can also solve the hamiltonian in equation 2.1 computationally and visualize the superradiant phase transition. To do that I am using QuTip software that can model open quantum and composite quantum systems. Here, we will solve the the lindblad equation and calculate ρ then calculate the occupation of the cavity [Joh24].

First step is to define our parameters that we are working with, the oscillation frequencies, dissipation rate and other such properties;

```
# Defining oscillation frequncies and dissipation rate

w_cavity=1 #frequency of the quantized electromagnetic field mode (cavity mode) with which the atoms interact.

w_split=1 #represents the energy splitting between the two levels of each atom in the ensemble. It is the characteristic frequency of the two-level atomic system.

kappa_cavity=0.1 #or can take 1 cavity dissipation rate

kappa_spin=0 #spin dissipation rates here it is set to 0 but can be initialized if needed

M = 16 # number of cavity states considered, dimension of the cavity Hilbert space
N = 4 #number of spin 1/2 particles, number of two level systems
j = N / 2 #total angular momentum quantum number
n = 2 * j + 1 #dimension of Hilbert space
```

Once our parameters are defined, we want to define some operators that will do our job. For example, annihilation and creation operator, and angular momentum operator such as J_+, J_- and J_z. Once we have that we define the hamiltonian, one normal coherent part and the other with coupling. The next step after this would be to describe the lindbladian for our system using annihilation operator and J_-

```
a = tensor(destroy(M), qeye(int(n))) # annihilation operator for M-dimensional cavity state
J_plus = tensor(qeye(M), jmat(j, "+")) #J_+
J_minus = tensor(qeye(M), jmat(j, "-")) #J_-
J_z = tensor(qeye(M), jmat(j, "z")) #J_z

H_coherent = w_split*a.dag()*a +w_cavity * J_z
H_coupling = 1/np.sqrt(N)*(a + a.dag()) * (J_plus + J_minus)
Lindblad =[np.sqrt(kappa_cavity) * a, np.sqrt(kappa_spin) * J_minus]
```

Now, for a range of coupling values we evolve our system and calculate its density matrix and from that density matrix we will calculate the photon occupation in the cavity:

```
%%time

#range of interaction strengths from 0.01 to 1.0, evenly spaced
↪ with 20 points.
coupling_list = np.linspace(0.01, 1.0, 20)

#calculating rho (solving lindblad equation)
rho_1 = [steadystate(H_coherent + coupling_strength * H_coupling,
↪ Lindblad) for coupling_strength in coupling_list]

#number of photon in the cavity
photon_cavity_1 = expect(a.dag() * a, rho_1)
```

If we plot our result for varying coupling strength and cavity occupation we obtain the following result.

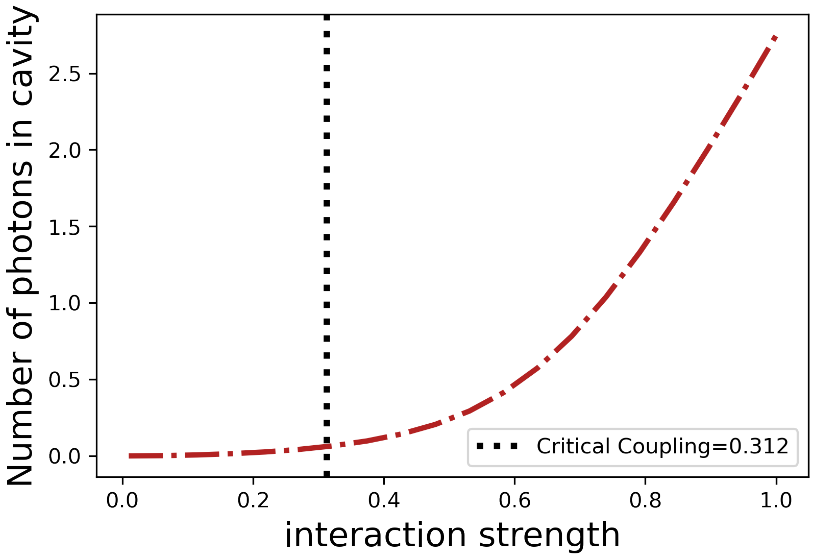

Figure 3.1: The cavity occupation can be seen to rise drastically after a critical coupling point. This is the indication that the Dicke model has reached superradiant phase. The critical coupling line there is from the theory.

We can also visualize this using the Wigner function. Wigner function is used to represent the quantum state of any system in a phase space defined by position and momentum. It is a type of quasi-probability distribution, because it can take positive and negative values, reflecting the inherently quantum nature of the system. It connects operators with phase-space functions, and we will use this to visualize what happens to the Wigner function in superradiant phase.

We first reduce the size of matrix we are using and take every 5th value from the density matrix to make a smaller matrix. This smaller matrix is reduced to the cavity density matrix by taking a partial trace over the smaller matrix (part of the complete system) and then for each cavity matrix a Wigner function is plotted:

```python
# Select a subset of density matrices for plotting (every 5th matrix from rho_1)
rho_sub = rho_1[::5]  # Subsampling for visualization purposes

# Define the range for the phase space (spanning -4 to 4 with 500 points)
phase_space = np.linspace(-4, 4, 500)


plt.figure(figsize=(10, 10))

# Loop through the selected density matrices to compute and plot the Wigner function
for idx, rho_ss in enumerate(rho_sub):
    # Partial trace: Extract the cavity subsystem from the full density matrix
    rho_cavity = ptrace(rho_ss, 0)

    # Compute the Wigner function over the defined phase space
    W = wigner(rho_cavity, phase_space, phase_space)

    # Create a subplot for each Wigner function
    plt.subplot(2, 2, idx + 1)
    plt.contourf(phase_space, phase_space, W, 100, cmap="RdBu_r")
    plt.title(f"Wigner Function {idx + 1}", fontsize=16)
    plt.xlabel("x", fontsize=16)
    plt.ylabel("p", fontsize=16)

# Adjust subplot spacing for better layout
plt.tight_layout()

# Display the plot
plt.show()
```

The result of this plot is as follows:

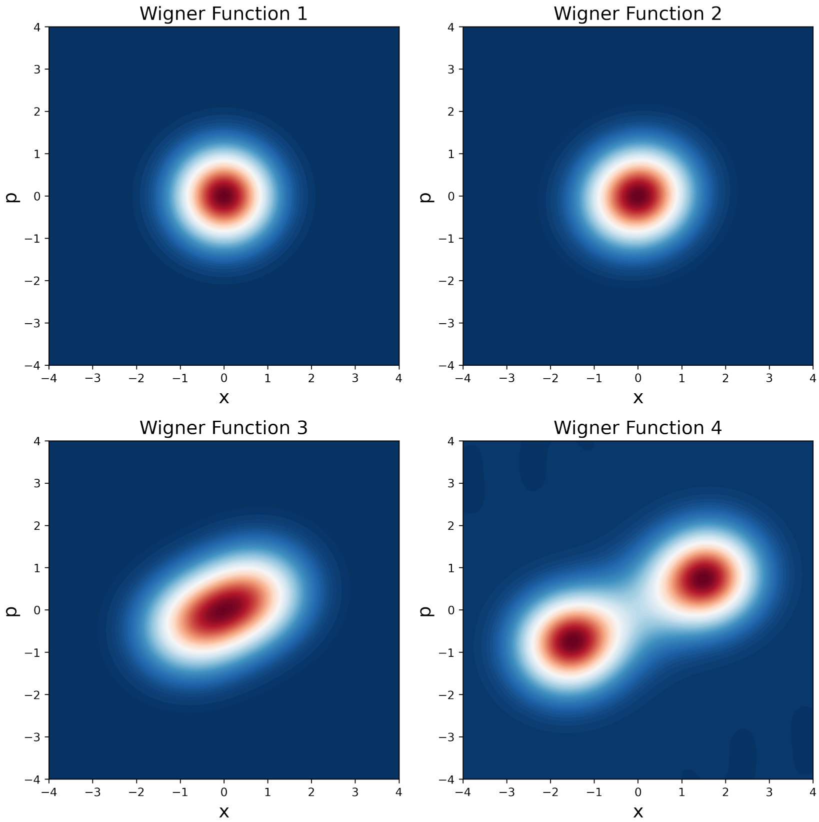

Figure 3.2: The Wigner function in the normal phase is just one blob but as it transitions to the superradiant phase it splits in two blobs. This splitting is the representation of the superradaint phase transition.

The reason that the Wigner function is splitting in two blob can be thought of as symmetry breaking during the phase transition. Our Hamiltonian also had this symmetry that if the angular momentum and ladder operator were switched with a negative sign then the Hamiltonian remains unchanged. So at the phase transition point in the superradiant phase we can see the splitting happening towards negative and positive sides. Using Wigner function I have showed that the superradiant phase can be visualized and identified quite accurately.

4 Conclusion

The Dicke model is an important quantum optics model used to understand light-matter interaction, especially dissipative phase transition. In this report, I have shown that the critical coupling can be derived for the system using the lindbladian formalism and this can be cross checked through simulating the system. The critical coupling from theory and simulation show very close match with each other as can be seen in Fig (3.1). Furthermore, we also visualized the Dicke model superradiant phase using Wigner function which shows a distinct splitting that is a direct indication of the phase transition happening.

References

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