Assignment VI

Mathematical and Computational Toolkit
PHY-1110-1

Spring 2022

Neutrino Oscillation

In the last discussion session, we solved the problem of Neutrino oscillation. As promised, the remaining simplification and plotting exercise are listed as an assignment problem below.

A brief summary of what we did – we started with the set of three differential equations governing the amplitude of finding three different flavors of neutrino.

$$
\begin{aligned}
i\hbar\frac{d\psi_e}{dt} &= E_0\psi_e(t) - A\psi_\mu(t) \\
i\hbar\frac{d\psi_\mu}{dt} &= -A\psi_e(t) + E_0\psi_\mu(t) - B\psi_\tau(t) \\
i\hbar\frac{d\psi_\tau}{dt} &= -B\psi_\mu(t) + E_0\psi_\tau(t)
\end{aligned}
$$

where, ψ_e, ψ_μ, and ψ_τ are the amplitude of finding electron neutrino, muon neutrino, and tau neutrino respectively. The coefficients E_0, A, and B are real constants.

We began by writing the above sets of equations as a single matrix equation:

$$
\boldsymbol{\Psi}: \quad i\hbar\begin{pmatrix} \frac{d\psi_e}{dt} \\ \frac{d\psi_\mu}{dt} \\ \frac{d\psi_\tau}{dt} \end{pmatrix} = \begin{pmatrix} E_0 & -A & 0 \\ -A & E_0 & -B \\ 0 & -B & E_0 \end{pmatrix} \begin{pmatrix} \psi_e(t) \\ \psi_\mu(t) \\ \psi_\tau(t) \end{pmatrix}
$$

Without realizing, we have choosen a basis to represent the equations in a matrix form. We call this the **physical basis** Ψ. The underlying physics is clear in this basis; however, the problem is not solvable. As discussed, we need to take this to the **eigen basis** Φ.

- **Exercise 1:** Use the sympy package in python to diagonalize the above matrix. In other words, find the matrix representation of the above map in eigen basis (which is essentially a diagonal matrix with three eigenvalues as the non-zero elements). Also, find the basis transformation matrix $\boldsymbol{P}$ that takes from eigen basis to the physical basis (which is essentially the three eigenvectors arranged as columns). **[5]**

We saw that in the eigen basis, the matrix equation takes the form:

$$\mathbf{\Phi}: \quad i\hbar \begin{pmatrix} \frac{d\phi_e}{dt} \\ \frac{d\phi_\mu}{dt} \\ \frac{d\phi_\tau}{dt} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} \phi_e(t) \\ \phi_\mu(t) \\ \phi_\tau(t) \end{pmatrix}$$

This decomposes to a set of three uncoupled differential equations that can be easily solved.

- **Exercise 2:** Show that the solution to the above sets of differential equation is:

$$\begin{pmatrix} \phi_e(t) \\ \phi_\mu(t) \\ \phi_\tau(t) \end{pmatrix} = \begin{pmatrix} \exp\left(-\frac{i}{\hbar}\lambda_1 t\right) & 0 & 0 \\ 0 & \exp\left(-\frac{i}{\hbar}\lambda_2 t\right) & 0 \\ 0 & 0 & \exp\left(-\frac{i}{\hbar}\lambda_3 t\right) \end{pmatrix} \begin{pmatrix} \phi_e(0) \\ \phi_\mu(0) \\ \phi_\tau(0) \end{pmatrix} = U \begin{pmatrix} \phi_e(0) \\ \phi_\mu(0) \\ \phi_\tau(0) \end{pmatrix}$$

  You can take any one of the three equations and show that the solution takes the given form. [**5**]

We now have the solution in the eigen basis. However, the underlying physics is clear in physical basis. Thus, we will use the basis transformation matrix calculated earlier to express the solution in physical basis.

$$P \begin{pmatrix} \phi_e(t) \\ \phi_\mu(t) \\ \phi_\tau(t) \end{pmatrix} = PU \begin{pmatrix} \phi_e(0) \\ \phi_\mu(0) \\ \phi_\tau(0) \end{pmatrix}$$

The left hand side is simply the required solution vector in physical basis. On the right hand side, we can use the inverse of the basis transformation matrix to express the initial condition in physical basis.

$$\begin{pmatrix} \psi_e(t) \\ \psi_\mu(t) \\ \psi_\tau(t) \end{pmatrix} = PUP^{-1} \begin{pmatrix} \psi_e(0) \\ \psi_\mu(0) \\ \psi_\tau(0) \end{pmatrix}$$

- **Exercise 3:** Our problem is now reduced to calculating the PUP^{-1} matrix. Calculate it explicitly using the sympy package (or, use pen and paper if you are up for an adventure). [**5**]

- **Exercise 4:** Let us take an initial state where all the neutrinos are electron neutrinos: $\psi_e(0) = 1$, $\psi_\mu(0) = 0$, and $\psi_\tau(0) = 0$. Use this initial condition to compute $\psi(t)$ explicitly. [**5**]

- **Exercise 5:** Suppose now that $A = B$. Show that the state of the system reduces to:

$$\psi(t) = \begin{pmatrix} \psi_e(t) \\ \psi_\mu(t) \\ \psi_\tau(t) \end{pmatrix} = \frac{1}{2} e^{-\frac{i}{\hbar}E_0 t} \begin{pmatrix} \cos\kappa t + 1 \\ i\sqrt{2}\sin\kappa t \\ \cos\kappa t - 1 \end{pmatrix}$$

  where κ is the quantity you need to determine. [**5**]

- **Exercise 6:** As mentioned in the discussion session, the probability that a detected neutrino will be an electron neutrino, a muon neutrino, or a tau neutrino is given by the absolute value squared of their components $|\psi_e(t)|^2$, $|\psi_\mu(t)|^2$, and $|\psi_\tau(t)|^2$ respectively. On the same graph, plot all three of these values as a function of the dimensionless quantity κt, and comment on your results. Are there any values of κt at which you are guaranteed to detect:

  (i) an electron neutrino?

  (ii) a muon-neutrino?

  (iii) a tau-neutrino?

  If so, what are these values of κt? **Hint:** You should not need the numerical values of A, E_0, or $\hbar$ to plot these graphs. [**10**]

Coupled Harmonic Oscillator

In the discussion session, we also briefly discussed about the problem of coupled harmonic oscillator. You will now solve this problem on your own with the skills equipped from the previous exercises.

Uncoupled Oscillator

Let us start off easy: consider the simple system given in figure 1.

Figure 1: A system of two uncoupled spring-mass system.

- **Exercise 1:** Using the Newton's law of motion and Hooke's law, show that the system satisfies the differential equations: [**5**]

$$m_1\ddot{x}_1 = -k_1 x_1$$

$$m_2\ddot{x}_2 = -k_1 x_2$$

- **Exercise 2:** Show that the equations can be reduced to a matrix equation: [**5**]

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

As before, we have chosen the 'physical' basis to express the set of equations. However, notice that the matrix is already diagonalized. We are already in the state to solve the differential equation.

- **Exercise 3:** Take the ansatz $x(t) = A_\alpha e^{i\alpha t}$ and plug in to the set of differential equations individually. Determine the possible values that α can take. [**5**]

Given that they are second order differential equations, we will get two linearly independent solutions. We will also need two initial conditions for each equation to fully specify the problem.

- **Exercise 4:** Use the initial conditions $x_1(0) = 1$, $\dot{x_1}(0) = 0$, $x_2(0) = 0$, and $\dot{x_2}(0) = 1$ to write the full solution. [**5**]

Coupled Oscillator

We will now add a third spring with different spring constant in the middle to couple the individual masses.

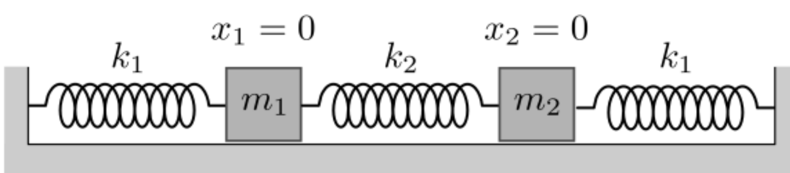

Figure 2: A system of two coupled spring-mass system.

Let us take a simpler case when $m_1 = m_2 = m$.

- **Exercise 1:** Show that the differential equations are given by :

$$m\ddot{x}_1 = -k_1 x_1 + k_2(x_2 - x_1) = -(k_1 + k_2)x_1 + k_2 x_2$$
$$m\ddot{x}_2 = -k_1 x_2 - k_2(x_2 - x_1) = k_2 x_1 - (k_1 + k_2)x_2$$

  Hint: Displace m_1 by x_1 and m_2 by x_2 such that $x_2 > x_1$. Carefully consider the direction in which springs are stretched or compressed and the direction of the restoring force. [**5**]

- **Exercise 2:** Write the above sets of equation in a matrix form in physical basis:

$$\boldsymbol{X} : \ddot{\boldsymbol{X}} = -\Lambda \boldsymbol{X}$$

  where, Λ is a 2×2 map. [**5**]

- **Exercise 3:** Guess a solution of the form $\boldsymbol{C_\alpha} k_\alpha e^{i\alpha t}$, where $\boldsymbol{C_\alpha}$ is a column vector and k_α is a real number. Plug in the matrix equation and find the possible values of α^2. (**Hint**: These are just the eigen values of Λ.) You will later learn that these are the **normal mode frequencies**. [**5**]

- **Exercise 4:** Find the column-vectors $\boldsymbol{C_\alpha}$ that correspond to each eigenvalue α^2. These are the **normal modes**. (**Hint:** These are just the eigenvectors corresponding to each eigenvalue.) [**5**]

- **Exercise 5:** Write the full solution of the form:

$$\boldsymbol{X} = \boldsymbol{C_\alpha} k_\alpha e^{i\alpha t}$$

  You need to write the solution as two independent equations (not in matrix form). (**Hint:** There are two possible values of α^2 which means that there will be four possible values of α. Similarly, each value of α^2 will have a corresponding eigenvector column $\boldsymbol{C_\alpha}$. The constants k_α are determined by the initial conditions.) [**5**]

You will study more about the **normal modes** and **normal mode frequencies** in your *Oscillations, Waves, and Optics* course. In the meantime, ensure that you are fairly comfortable with the idea of eigenvalues and eigenvectors and how to calculate them using pen/paper and a computer.

References

[1] The questions in this problem set are taken and modified from the examination and assignment from *Mathematical and Computational Toolkit* course offered in Spring 2019. The image and question credit goes to Prof. Vikram Vyas and Philip Cherian.