# Graph Theory Assignment 3


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1. (a) Let $e = (x, y) \in T - T'$. Since $T$ is a spanning tree, $T - e$ has two (connected) components, say $C_x$ containing $x$ and $C_y$ containing $y$. Now consider the path $P$ from $x$ to $y$ in $T'$. Since $x \in C_x$ and $y \in C_y$, there exist an edge $e' = (x', y')$ in path $P$ such that $x' \in C_x$ and $y' \in C_y$. Note that $e' \notin T$ since $C_x$ and $C_y$ aren't connected in $T - e$. Hence $T - e + e'$ has one component ($C_x$ and $C_y$ are connected by $e'$) and number of edges in $T - e + e'$ is same as number of edges in $T$, so $T - e + e'$ is a spanning tree where $e' \in T' - T$.

   (b) Let $e \in T - T'$. Since $T'$ is a spanning tree, $T' + e$ contains a cycle, say $C$ (containing the edge $e$). If all the edges in $C - e$ are also in $T$, then $T$ contains the cycle $C$, a contradiction. Hence, there exist an edge $e' = (x', y') \in C - e - T \implies e' \in T' - T$ since all edges in $C - e$ are in $T'$. Note that $T + e - e'$ is connected (since there exist a path between $x'$ and $y'$ in $C - e$) and has same number of edges as $T'$, hence $T' + e - e'$ is a spanning tree.

2. Let $G = K_{n_1,n_2} = (A, B, E)$ such that $|A| = n_1$ and $|B| = n_2$. Let $A = \{v_1, v_2, \cdots, v_{n_1}\}$ and $B = \{v_{n_1+1}, \cdots, v_{n_1+n_2}\}$. Then the adjacency matrix of $G$ is $A = \begin{pmatrix} 0 & J_{n_1\times n_2} \\ J_{n_2\times n_1} & 0 \end{pmatrix}$, and the laplacian of $G$ is

$$L = \operatorname{diag}(d_1, d_2, \cdots, d_{n_1+n_2}) - A = \begin{pmatrix} n_2 I_{n_1} & -J_{n_1\times n_2} \\ -J_{n_2\times n_1} & n_1 I_{n_2} \end{pmatrix}.$$

First we will evalute determinant of the matrix $(mI_n + J_n)$ to use it later. By using row reduction methods (adding all the columns to the first column gives a column where all its entries equal to $m + n$ and then substracting the first row from the other rows), we obtain

$$\begin{aligned} \det(mI_n + J_n) &= \det \begin{pmatrix} (m+1) & 1 & \cdots & 1 \\ 1 & m+1 & \cdots & 1 \\ & \cdot & & \\ & \cdot & & \\ 1 & 1 & \cdots & m+1 \end{pmatrix} \\ &= \det \begin{pmatrix} (m+n) & 1 & \cdots & 1 \\ 0 & m & \cdots & 0 \\ & \cdot & & \\ & \cdot & & \\ 0 & 0 & \cdots & m \end{pmatrix} = m^{n-1}(m+n). \end{aligned} \tag{1}$$

As we saw in class, number of spanning tress of $G$ is

$$\begin{aligned}
\tau(G) = \frac{\det(J+L)}{(n_1+n_2)^2} &= \frac{\det\begin{pmatrix} n_2 I_{n_1} + J_{n_1} & 0 \\ 0 & n_1 I_{n_2} + J_{n_2} \end{pmatrix}}{(n_1+n_2)^2} \\
&= \frac{\det(n_2 I_{n_1} + J_{n_1}) \cdot \det(n_1 I_{n_2} + J_{n_2})}{(n_1+n_2)^2} \\
&= \frac{n_2^{n_1-1}(n_2+n_1) \cdot n_1^{n_2-1}(n_1+n_2)}{(n_1+n_2)^2} = \boxed{n_2^{n_1-1} \cdot n_1^{n_2-1}}.
\end{aligned} \tag{2}$$