Graph Theory - Assignment 6

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Exercise (a)

Proof. For $S = \phi$, we have

$$0 = |S| \geq o(G - S) = o(G) \implies o(G) = 0.$$

So all the components of G are even, hence number of vertices, n is also even. $\square$

Exercise (b)

Proof. Suppose $T = \phi$ (such that $(\star)$ is tight for T and T is maximal w.r.t that property). Let x be a vertex of G. Taking $S = \{x\}$, we have

$$1 = |S| \geq o(G - S) = o(G - x).$$

Suppose $o(G - S) = 0$ then number of vertices in $G - S$ is even, but $|V(G - S)| = n - 1$ is odd. A contradiction. Hence $o(G - S) = 1$. But then $S = \{x\}$ is tight for $(\star)$ which contradicts the maximality of $T = \phi$. Therefore, $T \neq \phi$. $\square$

Exercise (c)

Proof. Since $(\star)$ is tight for T, we have $|T| = o(G - T)$. Suppose C is an even component of $G - T$. Considering the subgraph C, by exercise (b) there exists a vertex subset $T_c \neq \phi$ of C which satisfies $|T_c| = o(C - T_c)$. Now taking $S = T \cup T_c$, since all components of $G - T$ except C are also components of $G - S$ and C is an even component, we have

$$o(G - S) = o(G - T) + o(C - T_c) = |T| + |T_c| = |S|.$$

Hence, $(\star)$ is tight for S, which contradicts the maximality of T. Therefore, every components of $G - T$ is an odd component. $\square$

Exercise (d)

Proof. Let S_D be a vertex subset of D. Taking $S = T \cup \{x\} \cup S_D$, since all components of $G - T$ except C are also components of $G - S$, we have

$$o(G - S) = (o(G - T) - 1) + o(D - S_D) = |T| - 1 + o(D - S_D).$$

Also by maximality of T, we have

$$o(G - S) < |S| = |T| + 1 + |S_D| \implies o(D - S_D) < |S_D| + 2 \implies |S_D| + 1 \geq o(D - S_D).$$

Claim: $|S_D| \geq o(D - S_D)$.
Suppose not. Then $|S_D| + 1 = o(D - S_D)$. Note that $|V(D)| = |V(C)| - 1$ is even since C is an odd component by exercise (c). Since $|V(D)| - |S_D|$ is the total number of vertices in the components of $D - S_D$, we have

$$\begin{aligned} |V(D)| &\equiv |S_D| + o(D - S_D) \pmod 2 \\ &\equiv |S_D| + (|S_D| + 1) \pmod 2 \\ &\equiv 1 \pmod 2. \end{aligned} \tag{1}$$

A contradiction. Hence, $|S_D| \geq o(D - S_D)$. Since S_D is arbirary, D satisfies $(\star)$ and hence D has a perfect matching by induction hypothesis. □

Exercise (e)

Proof. Suppose U be a subset of $X = T$ such that $|U| > |N(U)|$, where $N(U)$ is the set of neighbours of the vertices of U (in the graph B). So the (odd) components in $Y - N(U)$ don't have any edge with the vertices of U. Hence, the (odd) components in $Y - N(U)$ are also odd components in the graph $G - (T - U)$. Taking $S = T - U$, since $|Y| = |X| = |T|$ and $|U| > |N(U)|$, we have

$$o(G - S) > |Y - N(U)| = |Y| - |N(U)| > |T| - |U| = |T - U| = |S|$$

But it contradicts Tutte condition. Therefore, $|U| \leq |N(U)|$ for any subset U of X. So B satisfies Hall's bipartite matching condition and hence has a perfect matching, say M_B.

Now consider a matching M as constructed below: For every vertex v in T, there exist an edge from v to a vertex x_c in the (odd) component $C = M_B(v)$, set $M(v) = x_c$. Now by exercise $(c), (d)$; for any component C of $G - T$, it is an odd componet and hence there exist a perfect matching M_c in the graph $C - x_c$; for any vertex v in $C - x_c$, set $M(v) = M_c(v)$. Clearly, M is a perfect matching for G. Hence, G has a 1-factor. □