1 Stepwise Automata and Visibly Pushdown Automata

Theorem 1.1. *Let $L \subseteq \Sigma^*$. If L_S is a visibly pushdown language then $c_{tree}(L_S)$ is recognised by stepwise automata, where $L_S = \{c_s W r_s : W \in L\}$.*

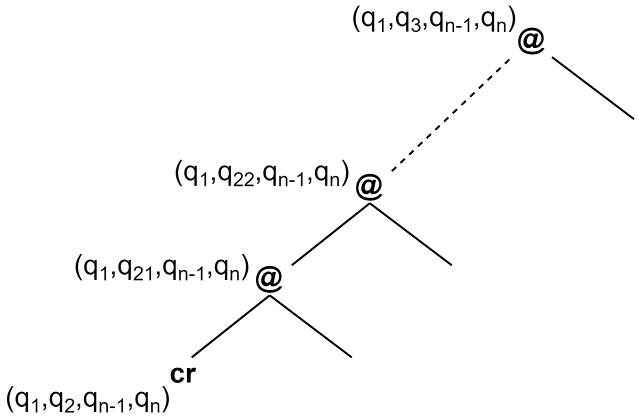
Proof. Let L_S is visibly pushdown language, then there exist a deterministic visibly pushdown automaton $A = (Q, q_0, Q_f, \Gamma, \delta)$ such that $L(A) = L_S$. Construct the stepwise automaton $B = (Q', Q'_f, rules)$, where

- $Q' = Q \times Q \times Q \times Q$
- $Q'_f = \{(q_0, q, q, q_f) : q_f \in Q_f, q \in Q\}$
- $rules$ consists of the followings:
  - ⋆ $cr \to (a_1, a_2, b_1, b_2)$ if $a_1 \xrightarrow{c|x} a_2$ and $b_1 \xrightarrow{r|x} b_2 \in \delta$
  - ⋆ $i \to (a_1, a_2, a_2, a_2)$ if $a_1 \xrightarrow{i} a_2 \in \delta$
  - ⋆ $(a_1, a_2, b_1, b_2)@(a_2, a_3, a_3, a_4) \to (a_1, a_4, b_1, b_2)$

We will prove the using induction on order of the word W, that is total number of internal and return symbols in that word that for any word cW, where W is a well-nested word, there exist runs

$$q_1 \xrightarrow{c|x} q_2 \xrightarrow{W} q_3 \text{ and } q_{n-1} \xrightarrow{r|x} q_n \text{ in } A$$

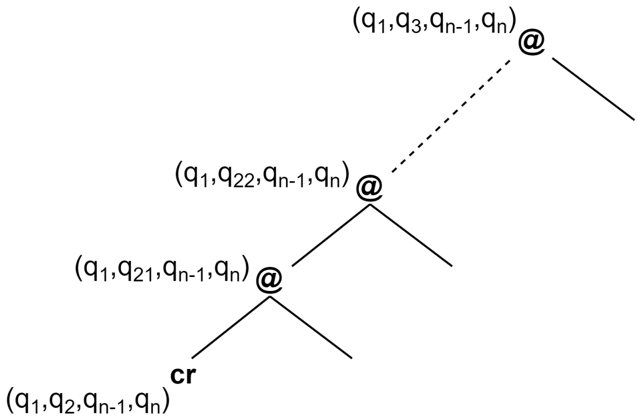
iff there exist the following run on $c_{tree}(cWr)$ in B.



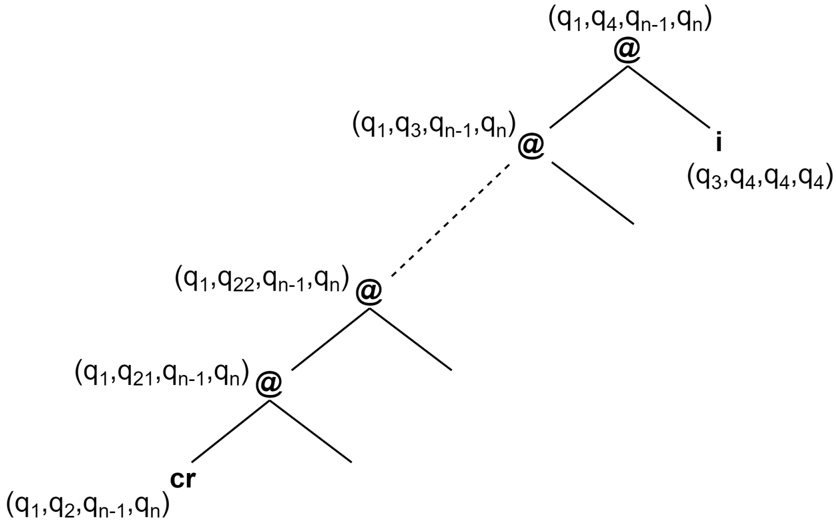
Base Case: Let order$(W) = 0$, then $cW = c$ and there exist runs $q_1 \xrightarrow{c|x} q_2$ and $q_{n-1} \xrightarrow{r|x} q_n$ in A iff $cr \to (q_1, q_2, q_{n-1}, q_n) \in rules$ by construction.

Inductive Case: Assume it holds for all words of order less than k. Let order$(W) = k$, then $W = W_1 i$ or $W = W_1 c_2 W_2 r_2$ for some well-nested words W_1 and W_2 and for some $c_2, r_2 \in \Sigma$.

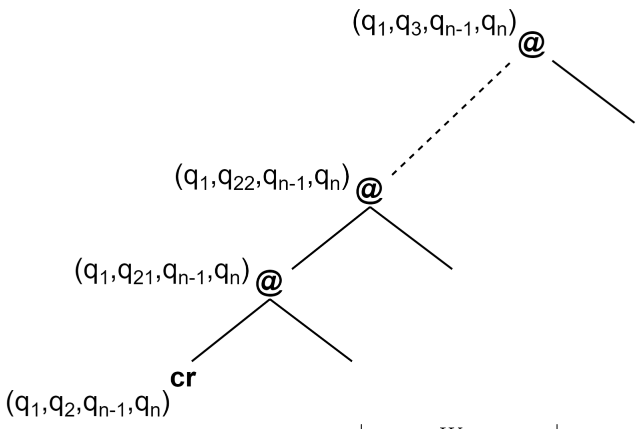
- Let $W = W_1 i$ for some well-nested word W_1, then since order$(W_1) = k - 1 < k$, by induction hypothesis there exist runs $q_1 \xrightarrow{c|x} q_2 \xrightarrow{W_1} q_3$ and $q_{n-1} \xrightarrow{r|x} q_n$ in A iff there exist the following run on $c_{tree}(cW_1 r)$ in B.

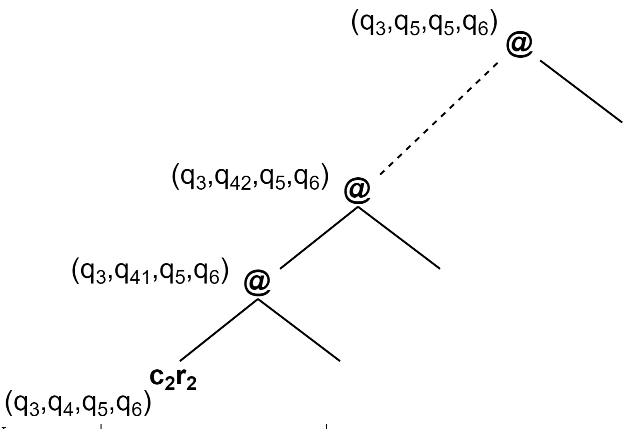


Hence there exist runs $q_1 \xrightarrow{c|x} q_2 \xrightarrow{W_1} q_3 \xrightarrow{i} q_4$ and $q_{n-1} \xrightarrow{r|x} q_n$ in A iff there exist the following run on $\mathrm{c}_{tree}(cW_1ir)$ in B, since $(q_1, q_3, q_{n-1}, q_n)@(q_3, q_4, q_4, q_4) \to (q_1, q_4, q_{n-1}, q_n) \in rules$.

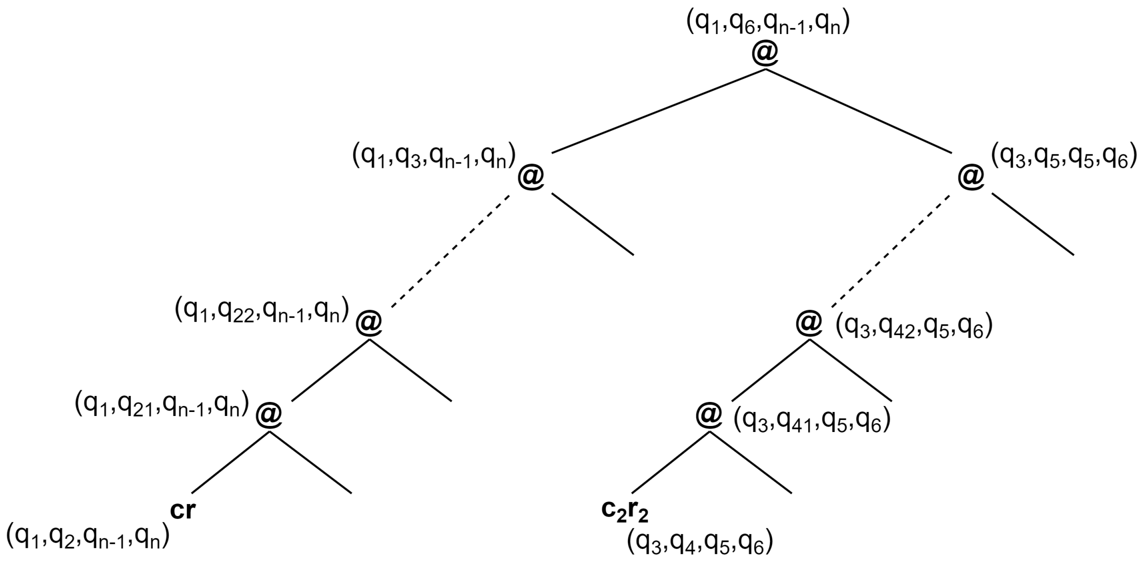


- If $W = W_1c_2W_2r_2$ for some well nested words W_1 and W_2, then since order of both W_1 and W_2 is less than k, by induction hypothesis there exist runs $q_1 \xrightarrow{c|x} q_2 \xrightarrow{W_1} q_3$, $q_{n-1} \xrightarrow{r|x} q_n$ and $q_3 \xrightarrow{c_2|x_2} q_4 \xrightarrow{W_2} q_5$, $q_5 \xrightarrow{r_2|x_2} q_6$ in A iff there exist the following runs on $\mathrm{c}_{tree}(cW_1r)$ and $\mathrm{c}_{tree}(c_2W_2r_2)$ respectively in B

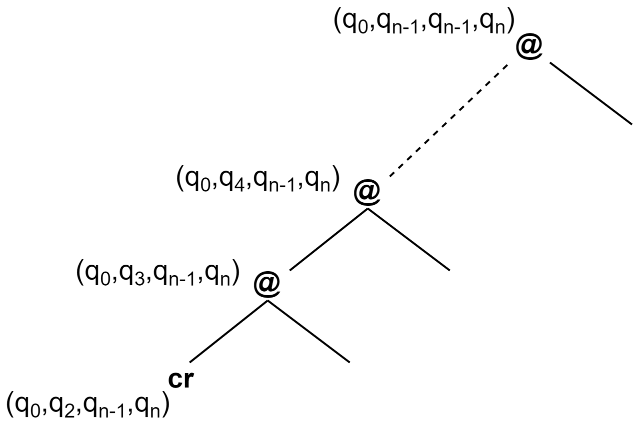




Hence, there exist runs $q_1 \xrightarrow{c|x} q_2 \xrightarrow{W_1} q_3 \xrightarrow{c_2|x_2} q_4 \xrightarrow{W_2} q_5 \xrightarrow{r_2|x_2} q_6$ and $q_{n-1} \xrightarrow{r|x} q_n$ in A iff there exist the following run on $\mathrm{c}_{tree}(cW_1c_2W_2r_2r)$ in B, since $(q_1, q_3, q_{n-1}, q_n)@(q_3, q_5, q_5, q_6) \to (q_1, q_6, q_{n-1}, q_n) \in rules$.



Now we can see that, $cWr \in L(A)$ iff there exist runs $q_0 \xrightarrow{c|x} q_1 \xrightarrow{W} q_{n-1} \xrightarrow{r|x} q_n$ for some $q_n \in Q_f$ iff there exist the following run on $\mathrm{c}_{tree}(cWr)$ in B



iff $\mathrm{c}_{tree}(cWr) \in L(B)$. Note that both A and B accepts only well nested words of the form cWr by construction. Hence $L(A) = L(B.)$ □

Corollary 1.1.1. *If L is nested word language which is recognised by visibly pushdown automata then $\mathrm{c}_{tree}(L_S)$ is recognised by stepwise automata.*

Proof. If L is recognised by a VPA A, then it is easy to construct another VPA from A recognising L_S. □

Theorem 1.2. *Let $L \subseteq \Sigma^*$. If $\mathrm{c}_{tree}(L_S)$ is recognised by stepwise automata then L_S is a visibly pushdown language, where $L_S = \{c_s W r_s : W \in L\}$.*

Proof. Let $\mathrm{c}_{tree}(L_S)$ is recognised by a stepwise automaton $B = (Q, Q_f, rules)$. Construct the visibly pushdown automaton $A = (Q', q_0, Q'_f, \Gamma, \delta)$, where

- $Q' = Q \cup \{q_0, q_f\}$, where $q_0,\ q_f \notin Q$.
- $Q'_f = \{q_f\}$
- $\Gamma = Q' \times Q'$
- δ consists of the followings:

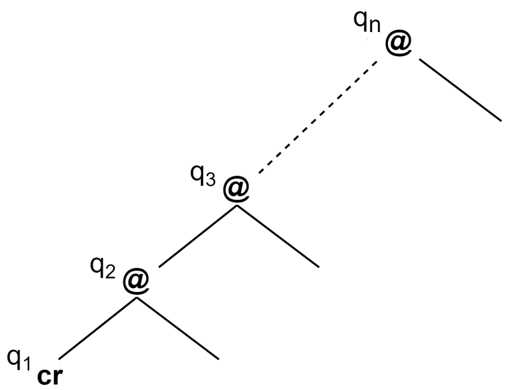
- $\star$ $q \xrightarrow{c_1|(q,q_1)} q_1$ if $q \in Q'$ and $c_1 r_1 \rightarrow q_1 \in rules$
- $\star$ $q \xrightarrow{i} q_1$ if $i \rightarrow q_i$ and $q@q_i \rightarrow q_1$ are in $rules$.
- $\star$ $q_2 \xrightarrow{r_1|(q_4,q_1)} q_3$ if $c_1 r_1 \rightarrow q_1$ and $q_4@q_2 \rightarrow q_3$ are in $rules$.
- $\star$ $q \xrightarrow{r_s|(q_0,q_s)} q_f$ if $q \in Q_f$ and $c_s r_s \rightarrow q_s \in rules$.

We will prove using induction on order of the word W (as defined earlier) that for any word cW, where W is a well-nested word, there exist runs on cW

$$q \xrightarrow{c|(q,q_1)} q_1 \xrightarrow{W} q_n \text{ in } A$$

(note that the stack symbol for c has to be (q, q_1) by construction)

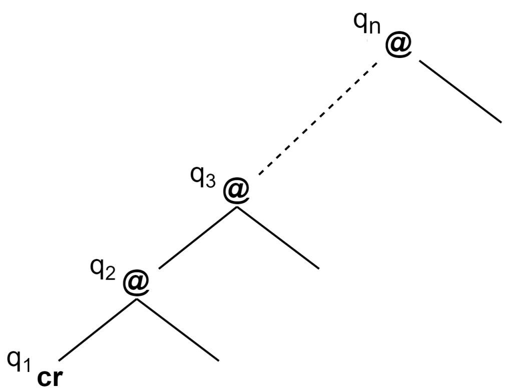
iff there exist the following run on $\mathrm{c}_{tree}(cWr)$ in B for some $r \in \Sigma$.



Base Case: If order$(W) = 0$, then for any $c \in \Sigma$, there exist a run $q \xrightarrow{c|(q,q_1)} q_1$ in A iff $cr \rightarrow q_1 \in rules$ for some $r \in \Sigma$.

Inductive Case: Assume it holds for all words of order less than k. Let order$(W) = k$, then $W = W_1 i$ or $W = W_1 c_2 W_2 r_2$ for some well-nested words W_1 and W_2 and for some $c_2, r_2 \in \Sigma$.

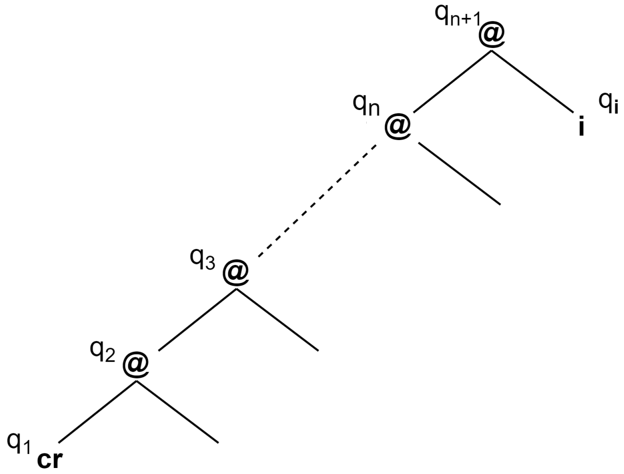
- Let $W = W_1 i$ for some well-nested word W_1, then since order$(W_1) = k - 1 < k$, by induction hypothesis there exist runs $q \xrightarrow{c|(q,q_1)} q_1 \xrightarrow{W_1} q_n$ in A iff there exist the following run on $\mathrm{c}_{tree}(cW_1 r)$ in B for some $r \in \Sigma$.



Also there exist $q_n \xrightarrow{i} q_{n+1}$ in δ iff $i \rightarrow q_i$ and $q_n@q_i \rightarrow q_{n+1} \in rules$ some $q_i \in Q$. Hence there exist runs on cW_1i

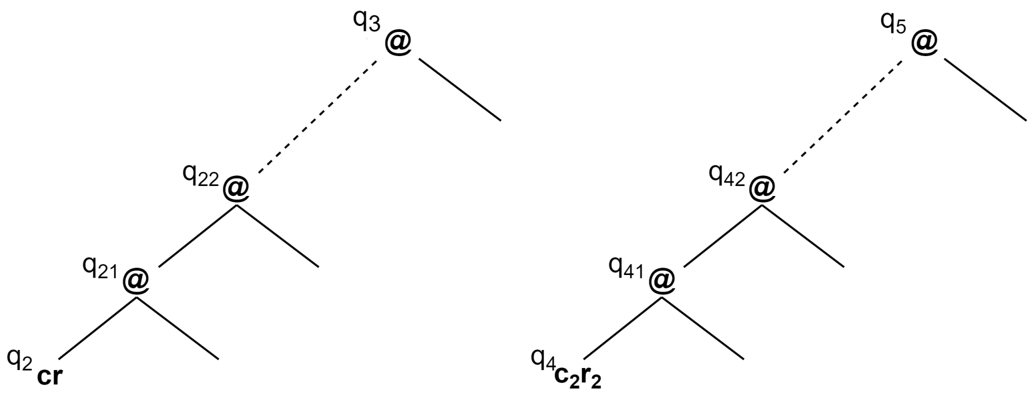
$$q \xrightarrow{c|(q,q_1)} q_1 \xrightarrow{W_1} q_n \xrightarrow{i} q_{n+1}$$

iff there exist the following run on $\mathrm{c}_{tree}(cW_1ir)$ in B for some $r \in Sigma$.



- Let $W = W_1c_2W_2r_2$ for some well nested words W_1 and W_2. We have $q_5 \xrightarrow{r_2|(q_3,q_4)} q_6 \in \delta$ iff $c_2r_2 \rightarrow q_4$ and $q_3@q_5 \rightarrow q_6$ are in $rules$.

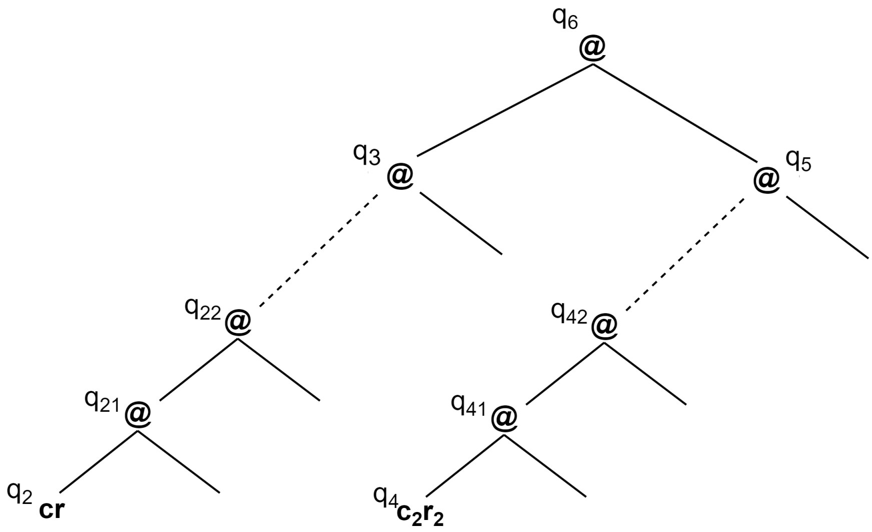
  Also since order of both W_1 and W_2 is less than k, by induction hypothesis there exist runss $q_1 \xrightarrow{c|(q_1,q_2)} q_2 \xrightarrow{W_1} q_3$ and $q_3 \xrightarrow{c_2|(q_3,q_4)} q_4 \xrightarrow{W_2} q_5 \xrightarrow{r_2|(q_3,q_4)} q_6$ in A iff there exist the following runs on $\mathrm{c}_{tree}(cW_1r)$ and $\mathrm{c}_{tree}(c_2W_2r_2)$ respectively in B for some $r \in \Sigma$.



Hence there exist runs on $cW_1c_2W_2r_2$

$$q_1 \xrightarrow{c|(q_1,q_2)} q_2 \xrightarrow{W_1} q_3 \xrightarrow{c_2|(q_3,q_4)} q_4 \xrightarrow{W_2} q_5 \xrightarrow{r_2|(q_3,q_4)} q_6 \text{ in } A$$

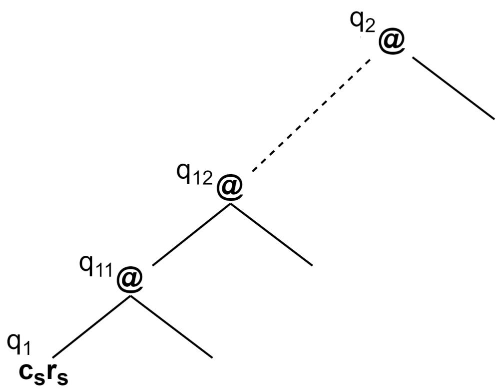
iff there exist the following run on $\mathrm{c}_{tree}(cW_1c_2W_2r_2r)$ in B.



Now we can see that, the only transitions in δ to the final state q_f in A are from Q_f through r_s, so clearly A accepts only words of the form c_sWr_s (also B accepts only trees of the form $\mathrm{c}_{tree}(c_sWr_s)$ by definition). And also $c_sWr_s \in L(A)$ iff for some $q_2 \in Q_f$, there exist runs on c_sW

$$q_0 \xrightarrow{c_s|(q_0,q_1)} q_1 \xrightarrow{W} q_2 \text{ in } A$$

iff for some $q_2 \in Q_f$ there exist the following run on $\mathrm{c}_{tree(c_sWr_s)}$ in B



iff $\mathrm{c}_{tree(cW_sr_s)} \in L(B)$. Hence $L(A) = L(B)$. $\square$