STOCHASTIC VOLATILITY MODELS IN FINANCE

A Project Report Submitted

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MA498 Project I

by

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CERTIFICATE

This is to certify that the work contained in this project report entitled "Stochastic Volatility Models in Finance" submitted by Dhoolam Saichandan and Satyadev Badireddi (Roll No.: 180123011 and 180123041) to the Department of Mathematics, Indian Institute of Technology Guwahati towards partial requirement of Bachelor of Technology in Mathematics and Computing has been carried out by them under my supervision.

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ABSTRACT

Option pricing is a major issue of importance in mathematical finance. To fit the market data better than the Black-Scholes Model, two other models are introduced. The purpose of this study is to present the models and the explicit formulae used for pricing under the Heston and Double Heston Model and discuss the calibration process.

Contents

Chapter 1

Introduction

Financial institutions, like banks and insurance companies, are responsible for carefully managing the vast amounts of assets received from various sources. Despite the fact that different financial institutions have diverse investing strategies, they are all exposed to the risk of many unforeseeable future occurrences, such as the financial crisis. The solution is the financial derivatives, which can be considered as the insurance for investment strategies that seek to mitigate a variety of risks.

The three major categories of financial instruments are derivatives, equities, and debt. In finance, the term "derivative" refers to a product whose value is determined by the value of an underlying S. There are many possibilities of S. The underlying can be of a single stock or a group of stocks, other types of securities, commodities, or of foreign exchange etc.

In our discussion of pricing, we refer to derivative securities. In the financial industry, quick and accurate pricing models are highly desired. On the one hand, the market's most basic requirement is to be provided with the fair price of the derivative every second; otherwise, hedging activities cannot be carried out. On the other hand, The Greeks, or the sensitivity of a derivative's price which play an important role in risk management by providing traders with a window through which they can monitor the risk that their investment strategies are taking, can be calculated from the pricing model.

1.1 Structure of the Project

Pricing financial derivatives is not easy, and Stochastic calculus is an indispensable tool in this field. When the volatility term is a constant σ times the Brownian motion, as it is in the Black-Scholes pricing model, it is easier to use Ito-calculus to price Euopean options.

We emphasise the case in which σ is replaced by a random variable v in our project. Chapter 2 will introduce the Heston Model and discuss about the pricing of options under the model. As the Heston model does not fit the market implied volatility well, Chapter 3 will discuss the double Heston model which is a well–known practical pricing model that used to calculate the price of a variety of derivatives, including European options. It is an extension of Heston model and has a better fit to the implied volatility surface.

Chapter 2

Heston Model

In the wake of the October market crash in 1987, traders noticed that volatility smiles started appearing in option prices. Thereafter, the possibility for extreme events needed to be factored into options pricing.
Plotting the Implied volatility for European options against different strike prices gives a U-shaped curve called the Volatility smile which can be observed in figure (2.1).
A major drawback of the Black-Scholes Model is that the volatility is not arbitrary but fixed.

2.1 The Model

Heston model was constructed by factoring in the randomness of volatility into the Classical Black-Scholes Model. Thus, there is an extra Brownian motion W_{t2}.
Under some Probability measure μ_p, it can be described as following:

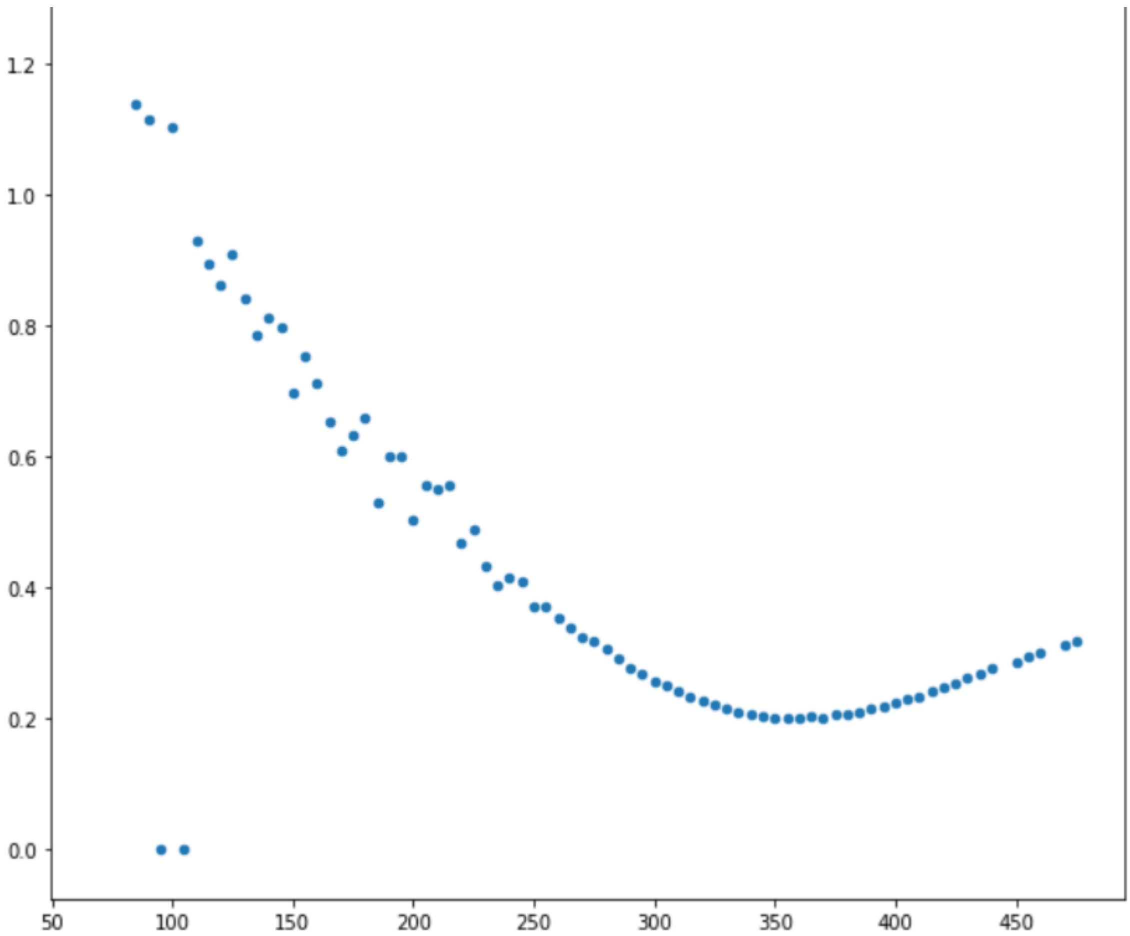

Figure 2.1: Market Implied Volatility plotted against the strike price

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_{t1} \tag{2.1}$$

$$dv_t = k(v_t)dt + v_t dW_{t2} \tag{2.2}$$

$$\mathbb{E}_p[dW_{t1} dW_{t2}] = \rho dt \tag{2.3}$$

μ : Drift of the process for the stock

θ : Mean reversion level for the variance

k : Mean reversion rate for variance

$$\sigma: \text{ Volatility of the variance}$$

$$\rho: \text{ the correlation between the Brownian motions } W_{t1} \text{ and } W_{t2}$$

Applying Ito's lemma:

$$dS_t e^{-rt} = e^{-rt} dS_t - re^{-rt} S_t dt = e^{-rt}[(\mu - r)S_t dt + \sqrt{v_t} S_t dW_{t1}] \tag{2.4}$$

The discounted price of the stock has to be a martingale under the risk neutral probability measure.
So, the risk neutral measure (μ_R) is given by

$$dW_{t1}^R = dW_{t1} + \frac{\mu - r}{\sqrt{v_t}} dt \tag{2.5}$$

The stock equation under the measure μ_R will be

$$dS_t = rS_t dt + \sqrt{v_t} S_t dW_{t1}^R \tag{2.6}$$

2.2 Pricing

Under the risk neutral probability measure μ_R, the discounted price of the stock has to be a martingale.
Hence equation(2.1) turns out to

$$dS_t = rS_t dt + \sqrt{v_t} S_t dW_{t1}^{RN} \tag{2.7}$$

where

$$dW_{t1}^{RN} = dW_{t1} + \frac{\mu - r}{\sqrt{v_t}} dt \tag{2.8}$$

Using Ito's formula on price of the option, and the term as a risk premium that offsets the effect from the risk caused by the uncertainty about v_t.

$$dv_t = k^*(\theta^* - v_t)dt + \sigma v_t dW_{t2}^{RN} \tag{2.9}$$

$$E^{RN}[dW_{t1}^{RN} dW_{t2}^{RN}] = \rho dt \tag{2.10}$$

where

$$dW_{t2}^{RN} = dW_{t2} + \lambda v_t \sigma dt \quad , k^* = k + \lambda \quad and \quad \theta^* = \frac{k\theta}{k + \lambda} \tag{2.11}$$

Equating the value of $C(S, v, t)$, price of call option, to the discounted value of the expectation of the final payoff under the risk neutral measure, we get:

$$C(S, v, t) = S_t P_1 K e^{-r(T-t)} P_2 \tag{2.12}$$

where $P_1 = P^Q(lnS_t > lnK)$ is the probability that the call option expires in-the-money under the measure $\mu^Q(where \quad \frac{d^{RN}}{d^Q} = \frac{e^{r(T-t)}}{\frac{S_T}{S_t}})$
and $P_2 = P^R N(lnS_t > lnK)$ is the probability that the call option expires in-the-money under the risk neutral measure μ^{RN}.

To solve P_1 and P_2, we will first derive the PDE's for P_1 and P_2. To find the PDEs, we need to construct a hedging portfolio.

Let us consider a portfolio M, which includes one option $V = V(S, v, t)$, δ_1 units of the stock, and δ_2 units of another option $U(S, v, t)$. Then,

$$dM = dV + \delta_1 dS + \delta_2 dU \tag{2.13}$$

Solving above equation using Ito's formula, we get

$$\frac{\frac{\partial V}{\partial t}+\frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2}+\rho\sigma vS\frac{\partial^2 V}{\partial S\partial v}+\frac{1}{2}v\sigma^2\frac{\partial^2 V}{\partial v^2}+rS\frac{\partial V}{\partial S}-rV}{\frac{\partial V}{\partial v}}=f(S,v,t) \tag{2.14}$$

Hesen set this function as,

$$f(S,v,t)=k(\theta-v)+(S,v,t) \tag{2.15}$$

To solve this setup, we need to construct another portfolio. Let us consider a portfolio H, which longs one share of call option $V(S,v,t)$ and shorts Δ shares of S.
Hedging the above portfolio under risk neutral measure ^{RN}, we get
$E^{RN}(dH-rHdt)=0$ implies the equation (2.15)

Using the same setup, Black and Scholes demonstrated that the value of any asset $C(S,v,t)$ must satisfy the following PDE:

$$\frac{1}{2}vS^2\frac{\partial^2 C}{\partial S^2}+\rho\sigma vS\frac{\partial^2 C}{\partial S\partial v}+\frac{1}{2}v\sigma^2\frac{\partial^2 C}{\partial v^2}+rS\frac{\partial C}{\partial S}+[k(\theta-v)+(S,v,t)]\frac{\partial C}{\partial v}-rC+\frac{\partial C}{\partial t}=0 \tag{2.16}$$

Setting $x=ln(S)$ and replacing C by equation (2.12), we get

$$\frac{1}{2}v\frac{\partial^2 P_j}{\partial x^2}+\rho\sigma v\frac{\partial^2 P_j}{\partial x\partial v}+\frac{1}{2}v\sigma^2\frac{\partial^2 P_j}{\partial v^2}+(r+u_jv)\frac{\partial P_j}{\partial x}+(a-b_jv)\frac{\partial P_j}{\partial v}+\frac{\partial P_j}{\partial t}=0 \tag{2.17}$$

for $j=1,2$, where $u_1=\frac{1}{2},u_2=-\frac{1}{2},a=k\theta,b_1=k+-\rho\sigma,b_2=k+\lambda$

Instead of solving P_1 and P_2 directly, we consider the characteristic function. And the following theorem shows us a way to find P_1 and P_2 when the characteristic functions are given.

Theorem 2.2.1. *Inversion Theorem*

$$Pr(x > k) = \frac{1}{2} + \frac{1}{\pi} \int_{-\infty}^{\infty} Re(\frac{e^{-iuk} f(u)}{iu})\, du \tag{2.18}$$

where f(u) is the characteristic function of the random variable x.

$$P_j = Pr^j(lnS_T > lnk) = \frac{1}{2} + \frac{1}{\pi} \int_{0}^{\infty} Re(\frac{e^{-i\phi lnk} f_j(\phi, x, v, T)}{i\phi})\, d\phi \tag{2.19}$$

Here $f_j(\phi, x, v, T)$ is the characteristic function of $X_T = lnS_T$.
Note that $f_j(\phi) = E(e^{i\phi lnS_T}) = E(e^{i\phi lnS_T}|F_t)$. The Feynman-Kac Theorem gives us the PDEs that f_j satisfies.

Theorem 2.2.2. *Feynman-Kac Theorem*

Suppose that the vector $\bar{x}_t = \begin{pmatrix} x_1(t) \\ . \\ . \\ x_n(t) \end{pmatrix}$ *follows the n-dimensional stochastic process*

$$d\bar{x}_t = \begin{pmatrix} \mu_1(\bar{x}_t, t) \\ . \\ . \\ \mu_n(\bar{x}_t, t) \end{pmatrix} dt + \begin{pmatrix} \sigma_1 1(\bar{x}_t, t) & . & . & \sigma_m 1(\bar{x}_t, t) \\ . & . & . & . \\ . & . & . & . \\ \sigma_n 1(\bar{x}_t, t) & . & . & \sigma_n m(\bar{x}_t, t) \end{pmatrix} \begin{pmatrix} dw_1^Q(t) \\ . \\ . \\ dw_m^Q(t) \end{pmatrix} \tag{2.20}$$

$\bar{W_1^Q} = \begin{pmatrix} w_1^Q(t) \\ . \\ . \\ w_m^Q(t) \end{pmatrix}$ *is a vector of m-dimensional independent Q-Brownian motion.*

$\bar{\sigma}(\bar{x_t}, t)$ *is the volatility matrix of size n by m and* $\bar{\mu}(\bar{x_t}, t)$ *is n-dimensional. Define the Heston generator* Λ *as*

$$\Lambda = \sum_{i=1}^{n} \mu_i \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (\bar{\sigma}\bar{\sigma^T})_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \tag{2.21}$$

The Feynman-Kac theorem states that the partial differential equation of a function $u(\bar{x_t}, t)$ *given by*

$$\frac{\partial u}{\partial t} - r(\bar{x_t}, t)u + \Lambda u = 0 \tag{2.22}$$

and with the terminal condition $u(\bar{x_T}, T) = f(\bar{x_T}, T)$*, has the solution*

$$u(\bar{x_T}, T) = E^Q[e^{-\int_t^T r(\bar{x_s}, s)ds} f(\bar{x_T}, T)|F_t] \tag{2.23}$$

Hence, choosing $\tau = T - t$, the PDE for the characteristic function is

$$\begin{aligned} \frac{1}{2}v\frac{\partial^2 f_j}{\partial x^2} + \rho\sigma v\frac{\partial^2 f_j}{\partial x \partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 f_j}{\partial v^2} + (r + u_j v)\frac{\partial f_j}{\partial x} \\ + (a - b_j v)\frac{\partial f_j}{\partial v} - \frac{\partial f_j}{\partial \tau} = 0 \end{aligned} \tag{2.24}$$

subject to

$$f_j(\phi, x, v, T) = e^{i\phi ln S_T} \tag{2.25}$$

2.3 Solving the Characteristic Function

Let us assume that the characteristic function has the following form

$$f_j(\phi, x, v, T) = exp(C_j(\tau, \phi) + D_j(\tau, \phi)v + i\phi x) \tag{2.26}$$

Substituting f_j in equation(2.24) by above equation and dropping the f_j terms, we get an equation which is equivalent to following two ODEs

$$\frac{\partial D_j}{\partial \tau} = \rho\sigma i\phi D_j - \frac{1}{2}\phi^2 + \frac{1}{2}\sigma^2 D_j^2 + u_j i\phi - b_j D_j \tag{2.27}$$

$$\frac{\partial C_j}{\partial \tau} = ri\phi + aD_j \tag{2.28}$$

and by solving above ODEs we get the solution of D_j as

$$D_j(\tau, \phi) = \frac{b_j - \rho\sigma i\phi + d_j}{\sigma^2}\left(\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}}\right) \tag{2.29}$$

where

$$d_j = \sqrt{(\rho\sigma i\phi - b_j)^2 - \sigma^2(2u_j i\phi\phi^2)}, \quad g_j = \frac{b_j - \rho\sigma i\phi + d_j}{b_j - \rho\sigma i\phi - d_j} \tag{2.30}$$

$$u_1 = \tfrac{1}{2}, u_2 = -\tfrac{1}{2}, a = k\theta, b_1 = k + -\rho\sigma, b_2 = k + \lambda$$

Integrating the equation(2.28) on both sides, we get

$$C_j(\tau, \phi) = ri\phi\tau + \frac{a}{\sigma^2}\left[(b_j - \rho\sigma i\phi + d_j)\tau - 2ln\left(\frac{1 - g_j e^{d_j\tau}}{1 - g_j}\right)\right] \tag{2.31}$$

All parameters have to satisfy the Feller's condition in order to ensure that v_t will always be strictly positive.

Theorem 2.3.1. *Feller's condition*

If the parameters obey the following condition, then the process v_t is strictly *positive:*

$$2k\theta > \sigma^2 \tag{2.32}$$

We use Albrecher's form for D_j as it creates less error for numerical integration and is stable than Heston's under the full parameter space. Albrecher modified Heston form of D_j by multiplying both numerator and denominator with $exp(-d_j\tau)$.
We get,

$$D_j(\tau,\phi) = \frac{b_j - \rho\sigma i\phi + d_j}{\sigma^2}\left(\frac{1-e^{-d_j\tau}}{\tau_1 - m_j e^{-d_j\tau}}\right), \quad m_j = \frac{1}{g_j} \tag{2.33}$$

$$C_j(\tau,\phi) = ri\phi\tau + \frac{k\theta}{\sigma^2}\left[(b_j - \rho\sigma i\phi - d_j)\tau - 2ln\left(\frac{1-m_j e^{-d_j\tau}}{1-m_j}\right)\right] \tag{2.34}$$

In equation(2.19), it can be observed that the integrand dampens to zero as ϕ increases. For the integrand to decay fast, we have to choose a large upper bound which will increase the computation.

Hence, we use Attari's formulation for the call price under the Heston Model.

$$C(K) = S12Ke^{-r\tau} - \frac{Ke^{-r\tau}}{\pi}\int_0^\infty A(u)du \tag{2.35}$$

where

$$A(u) = \frac{(R_2(u) + \frac{I_2(u)}{u})cos(uh) + (I_2(u)\frac{R_2(u)}{u})sin(uh)}{1+u^2} \tag{2.36}$$

$$\phi(u) = f_j exp(-iu(lnS + r\tau)) = R_2(u) + iI_2(u), \quad h = \frac{Ke^{-r\tau}}{S} \tag{2.37}$$

The major drawback of the Heston Model is that Heston implied volatility fails to fit the market implied volatility at short maturities.

Chapter 3

Double Heston Model

The Heston model does not fit the volatility effectively although it does considerably well under longer maturities.
For this reason, the Double Heston model was introduced.

3.1 The Model

Under some probability measure μ_p, the Heston model can be described as following:

$$dS = rSdt + \sqrt{v_1}SdW_1 + \sqrt{v_2}SdW_2 \tag{3.1}$$

$$dv_1 = \kappa_1(\theta_1 - v_1)dt + \sigma_1\sqrt{v_1}dW_3 \tag{3.2}$$

$$dv_2 = \kappa_2(\theta_2 - v_2)dt + \sigma_2\sqrt{v_2}dW_4 \tag{3.3}$$

Here, the wiener processes W_1, W_3 has zero covariance with W_2 and W_4 and vice versa.

$$E[dW_1dW_3] = \rho_1 dt \tag{3.4}$$

$$E[dW_2 dW_4] = \rho_2 dt \tag{3.5}$$

$$E[dW_1 dW_2] = E[dW_3 dW_4] = E[dW_1 dW_4] = E[dW_2 dW_3] = 0 \tag{3.6}$$

3.2 Pricing

We make an equivalent partial differential equation for the Double Heston Model using the multi-dimensional Feynman-Kac Theorem.

Setting $x = lnS$, we have

$$dx = (r - \frac{1}{2}(v_1 + v_2))dt + \sqrt{v_1}dW_1 + \sqrt{v_2}dW_2 \tag{3.7}$$

Set $\bar{x_t} = (x, v_1, v_2)$, we get volatility matrix

$$\bar{\sigma}(\bar{x_t}, t) = \begin{pmatrix} \sqrt{v_1} & \sqrt{v_2} & 0 & 0 \\ \sigma_1\sqrt{v_1}\rho_1 & 0 & \sigma_1\sqrt{v_1(1-\rho_1^2} & 0 \\ 0 & \sigma_2\sqrt{v_2}\rho_2 & 0 & \sigma_2\sqrt{v_2(1-\rho_2^2} \end{pmatrix} \tag{3.8}$$

and the drift μ is

$$\mu = \begin{pmatrix} r - \frac{1}{2}(v_1 + v_2) \\ \kappa_1(\theta_1 - v_1) \\ \kappa_2(\theta_2 - v_2) \end{pmatrix} \tag{3.9}$$

Hence, the generator for the double Heston Model is

$$\begin{aligned} \Lambda = [r - \frac{1}{2}(v_1 + v_2)]\frac{\partial}{\partial x} + \kappa_1(\theta_1 - v_1)\frac{\partial}{\partial v_1} + \kappa_2(\theta_2 - v_2)\frac{\partial}{\partial v_2} + \frac{1}{2}(v_1 + v_2)\frac{\partial^2}{\partial x^2} \\ + \rho_1\sigma_1 v_1\frac{\partial^2}{\partial x \partial v_1} + \rho_2\sigma_2 v_2\frac{\partial^2}{\partial x \partial v_2} + \frac{1}{2}\sigma_1^2 v_1\frac{\partial^2}{\partial v_1^2} + \frac{1}{2}\sigma_2^2 v_2\frac{\partial^2}{\partial v_2^2} \end{aligned} \tag{3.10}$$

And the characteristic function for $(x_T, v_{1,T}, v_{2,T})$ has the following form

$$\begin{aligned} f(\phi, \phi_1, \phi_2, x_t, v_{1,t}, v_{2,t}) &= E[e^{i\phi x_T + i\phi_1 v_{1,T} + i\phi_2 v_{2,T}} | F_t] \\ &= e^{[A(\tau,\phi,\phi_1,\phi_2) + B_0(\tau,\phi,\phi_1,\phi_2)x_t + B_1(\tau,\phi,\phi_1,\phi_2)v_{1,t} + B_2(\tau,\phi,\phi_1,\phi_2)v_{2,t}]} \end{aligned} \tag{3.11}$$

The formation for the call price under the double Heston Model is presented as following:

$$C(S, v, t) = S_t P_1 - Ke^{-r(T-t)} P_2 \tag{3.12}$$

$$P_1 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re(\frac{e^{-i\phi lnK} f(\phi - i, x, v_1, v_2)}{i\phi Se^{r\tau}})\, d\phi \tag{3.13}$$

$$P_2 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re(\frac{e^{-i\phi lnK} f(\phi - i, x, v_1, v_2)}{i\phi})\, d\phi \tag{3.14}$$

$$f(\phi, x_t, v_1, v_2) = e^{(A(\tau,\phi) + i\phi x_t + B_1(\tau,\phi)v_1 + B_2(\tau,\phi)v_2)} \tag{3.15}$$

The implied volatility from the double Heston model is substantially closer to the market implied volatility than the implied volatility from the Heston model, regardless of whether the maturity is short or long.

Chapter 4

Calibration

In the previous chapters, we introduced the closed-form solution of the Heston and Double Heston model.
However, we have not estimated meaningful parameters for the models i.e., calibrated the models.
Trading firms need software to help them make decisions as quickly as possible. Calibrating these models takes the most time. A straight forward way of doing this is the Monte Carlo simulation. However,it is slow as it has to generate thousands of possible paths of the stock price and then take the average.
Two other common ways are the non-gradient and the gradient based optimization algorithms.

4.1 Objective function

The market call price is denoted by $C_M(\tau, K)$ to represent the market price and C(τ, K, Θ) to represent the model price. The parameters are the elements determining the price.

Θ is the collection of parameters $(\kappa, \theta, \sigma, v_0, \rho)$ in the Heston model.

In the Double heston model, Θ is the collection of $(\kappa_1, \kappa_2, \theta_1, \theta_2, \sigma_1, \sigma_2, v_{01}, v_{02}\rho_1, \rho_2)$

$IV_M(\tau, K)$, $IV(\tau, K, \theta)$ represents the Market implied volatility and Model implied volatility.

For a set of n maturities τ_i and a set of n strikes K_j the two objective functions are:

$F_1(\Theta) = \frac{1}{mn}\sum_{\tau_i}^{n}\sum_{K_j}^{m}(C_M(\tau_i, K_j) - C(\tau_i, K_j, \Theta))^2$

$F_2(\Theta) = \frac{1}{mn}\sum_{\tau_i}^{n}\sum_{K_j}^{m}(IV_M(\tau_i, K_j) - IV(\tau_i, K_j, \Theta))^2$

The objective of the calibration is to find the parameter estimates that somehow minimize the objective function $F_1(\Theta)$ or $F_2(\Theta)$ under constraints like the feller's condition. $F_2(\Theta)$ is more significant since the implied volatility plays a very important role in derivative trading.

But it is more expensive to calculate since the root-finding algorithm must be employed to find the implied volatility.

4.2 Conclusion

We have emphasised on Heston Model and Double Heston Model in this report. Further study on implementing the model calibration and sensitivity analysis will be done in the later part of the Project.

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