STOCHASTIC VOLATILITY MODELS IN FINANCE

MA498 Project 1

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- The three major types of financial instruments are derivatives, equities, and debt. The term "derivative" refers to a product whose value is determined by the value of an underlying asset.
- Financial institutions are responsible for carefully managing huge amounts of assets received from various sources.
- Thus, in the financial industry, quick and accurate pricing models are highly required. The market's most basic requirement is to be provided with the fair price of the derivatives every second; otherwise, hedging activities cannot be executed.

Volatility Smile

- The Black-Scholes Model predicts that the implied volatility curve is flat when plotted against varying strike prices.
- However in reality, plotting the market volatility of options with the same expiration date and asset but different strike prices produces a smile.
- Volatility smiles started occurring in options pricing after the 1987 stock market crash.
- The options that are farther in the money or out of the money tend to have higher implied volatility.
- Options with the lowest implied volatility have strike prices at the money (Asset price is close to the strike price).

Volatility Smile

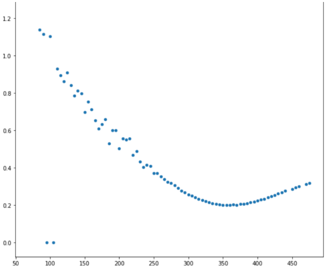

Figure: Market Implied Volatility plotted against the strike price

Therefore, the possibility for varying volatility needed to be factored into options pricing.

The Heston Model

Heston model was constructed by factoring in the randomness of volatility into the Classical Black-Scholes Model. Thus, there is an extra Brownian motion W_{t2}.

Under some Probability measure μ_p, it can be described as following:

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_{t1} \tag{1}$$

$$dv_t = k(\theta - v_t)dt + \sqrt{v_t} dW_{t2} \tag{2}$$

$$\mathbb{E}_p[dW_{t1} dW_{t2}] = \rho dt \tag{3}$$

μ : Drift of the process for the stock

θ : Mean reversion level for the variance

k : Mean reversion rate for variance

σ : Volatility of the variance

ρ : the correlation between the Brownian motions W_{t1} and W_{t2}

Model Under Risk Neutral Measure

The discounted price of the stock has to be a martingale under the risk neutral probability measure.

Hence the model turns out to be

$$dS_t = rS_t dt + \sqrt{v_t} S_t dW_{t1}^{RN} \tag{4}$$

$$dv_t = k^*(\theta^* - v_t)dt + \sigma \sqrt{v_t} dW_{t2}^{RN} \tag{5}$$

$$dW_{t1}^{RN} = dW_{t1} + \frac{\mu - r}{\sqrt{v_t}} dt, dW_{t2}^{RN} = dW_{t2} + \frac{\lambda \sqrt{v_t}}{\sigma} dt \tag{6}$$

$$E^{RN}[dW_{t1}^{RN} dW_{t2}^{RN}] = \rho dt \tag{7}$$

$$k^* = k + \lambda \quad and \quad \theta^* = \frac{k\theta}{k + \lambda} \tag{8}$$

Here λ is the risk premium.

Contd.

Equating the value of $C(S, v, t)$, price of call option, to the discounted value of the expectation of the final payoff under the risk neutral measure

$$\begin{aligned} C(S, v, t) &= e^{-r(T-t)} E^{RN}[(S_T - K)^+] \\ &= e^{-r(T-t)} E^{RN}[(S_T - K)\mathbf{1}_{S_T > K}] \\ &= e^{-r(T-t)} E^{RN}[S_T \mathbf{1}_{S_T > K}] - K e^{-r(T-t)} E^{RN}[\mathbf{1}_{S_T > K}] \end{aligned} \tag{9}$$

$$E^{RN}[\mathbf{1}_{S_T > K}] = \mathbf{P}^{RN}(S_T > K) = \mathbf{P}^{RN}(lnS_T > lnK)$$

$$e^{-r(T-t)} E^{RN}[S_T \mathbf{1}_{S_T > K}] = S_t E^{RN}\left(\frac{\frac{S_T}{S_t}}{e^{r(T-t)}} \mathbf{1}_{S_T > K}\right) = S_t \mathbf{P}^Q[lnS_T > lnK] \tag{10}$$

μ^Q is a measure where $\frac{d\mu^{RN}}{d\mu^Q} = \frac{e^{r(T-t)}}{\frac{S_T}{S_t}}$

Pricing under The Heston Model

We get:

$$C(S, v, t) = S_t P_1 - K e^{-r(T-t)} P_2 \tag{11}$$

where

$P_1 = P^Q(lnS_T > lnK)$ is the probability that the call option expires in-the-money under the measure μ^Q($where \quad \frac{d\mu^{RN}}{d\mu^Q} = \frac{e^{r(T-t)}}{\frac{S_T}{S_t}}$)

and $P_2 = P^{RN}(lnS_T > lnK)$ is the probability that the call option expires in-the-money under the risk neutral measure μ^{RN}.

To solve for P_1 and P_2, we will first derive the PDE's for P_1 and P_2. To find the PDEs, we need to construct a hedging portfolio.

Consider a portfolio M, which includes one option $V = V(S, v, t), \delta_1$ units of the stock, and δ_2 units of another option $U(S, v, t)$. Then,
Solving above equation using Ito's formula, we get

$$\frac{\frac{\partial V}{\partial t} + \frac{1}{2} vS^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma vS \frac{\partial^2 V}{\partial S \partial v} + \frac{1}{2} v\sigma^2 \frac{\partial^2 V}{\partial v^2} + rS\frac{\partial V}{\partial S} - rV}{\frac{\partial V}{\partial v}} = f(S, v, t) \quad (12)$$

We claim this function is,

$$f(S, v, t) = k(\theta - v) + \lambda(S, v, t) \quad (13)$$

Let us consider a portfolio H, which has one share of call option $V(S, v, t)$ and shorts Δ shares of S.
$E^{RN}(dH - rHdt) = 0$ implies the equation (12)

Pricing under The Heston Model

Using the same setup, it is demonstrated that the value of any asset $C(S, v, t)$ must satisfy the following PDE:

$$\frac{vS^2}{2}\frac{\partial^2 C}{\partial S^2}+\rho\sigma vS\frac{\partial^2 C}{\partial S \partial v}+\frac{v\sigma^2}{2}\frac{\partial^2 C}{\partial v^2}+rS\frac{\partial C}{\partial S}+[k(\theta-v)+(S,v,t)]\frac{\partial C}{\partial v} \\ -rC+\frac{\partial C}{\partial t}=0 \tag{14}$$

Setting $x = ln(S)$ and replacing C by equation (10), we get

$$\frac{1}{2}v\frac{\partial^2 P_j}{\partial x^2}+\rho\sigma v\frac{\partial^2 P_j}{\partial x \partial v}+\frac{1}{2}v\sigma^2\frac{\partial^2 P_j}{\partial v^2}+(r+u_j v)\frac{\partial P_j}{\partial x}+(a-b_j v)\frac{\partial P_j}{\partial v}+\frac{\partial P_j}{\partial t}=0 \tag{15}$$

for j = 1,2 where $u_1 = \frac{1}{2}, u_2 = -\frac{1}{2}, a = k\theta, b_1 = k + \lambda - \rho\sigma, b_2 = k + \lambda$
Instead of solving P_1 and P_2 directly, we consider the characteristic function.

Theorem

Inversion Theorem

$$Pr(x > k) = \frac{1}{2} + \frac{1}{\pi}\int_{-\infty}^{\infty} Re(\frac{e^{-iuk} f(u)}{iu})\, du \qquad (16)$$

where f(u) is the characteristic function of the random variable x.

$$P_j = Pr^j(lnS_T > lnk) = \frac{1}{2} + \frac{1}{\pi}\int_{0}^{\infty} Re(\frac{e^{-i\phi lnk} f_j(\phi, x, v, T)}{i\phi})\, d\phi \qquad (17)$$

Here $f_j(\phi, x, v, T)$ is the characteristic function of $X_T = lnS_T$.
Note that $f_j(\phi) = E(e^{i\phi lnS_T}) = E(e^{i\phi lnS_T}|F_t)$.

Using Multidimensional Feynman-Kac theorem ,the PDE for the characteristic functions f_1,f_2 is

$$\frac{1}{2}v\frac{\partial^2 f_j}{\partial x^2} + \rho\sigma v\frac{\partial^2 f_j}{\partial x \partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 f_j}{\partial v^2} + (r + u_j v)\frac{\partial f_j}{\partial x} + (a - b_j v)\frac{\partial f_j}{\partial v} - \frac{\partial f_j}{\partial \tau} = 0 \quad (18)$$

subject to

$$f_j(\phi, x, v, T) = e^{i\phi lnS_T} \quad (19)$$

Let us assume that the characteristic function has the following form

$$f_j(\phi, x, v, T) = exp(C_j(\tau, \phi) + D_j(\tau, \phi)v + i\phi x) \quad (20)$$

Solving for f_j, we get,

Closed-form solution of The Heston Model

$$C(S, v, t) = S_t P_1 - K e^{-r(T-t)} P_2 \tag{21}$$

$$P_j = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} Re\left(\frac{e^{-i\phi lnk} f_j(\phi, x, v, T)}{i\phi}\right) d\phi \tag{22}$$

$$f_j(\phi, x_t, v_t) = exp(C_j(\tau, \phi) + D_j(\tau, \phi) v_t + i\phi x_t) \tag{23}$$

$$C_j(\tau, \phi) = ri\phi\tau + k\frac{\theta}{\sigma^2}[(b_j - \rho\sigma i\phi + d_j)\tau - 2ln(\left(\frac{1 - m_j e^{-d_j\tau}}{1 - m_j}\right)))] \tag{24}$$

$$D_j(\tau, \phi) = \frac{b_j - \rho\sigma i\phi + d_j}{\sigma^2}\left(\frac{1 - e^{-d_j\tau}}{1 - m_j e^{-d_j\tau}}\right) \tag{25}$$

where

$$d_j = \sqrt{(\rho\sigma i\phi - b_j)^2 - \sigma^2(2u_j i\phi - \phi^2)}, \quad g_j = \frac{b_j - \rho\sigma i\phi + d_j}{b_j - \rho\sigma i\phi - d_j} \tag{26}$$

$$u_1 = \tfrac{1}{2}, u_2 = -\tfrac{1}{2}, a = k\theta, b_1 = k + \lambda - \rho\sigma, b_2 = k + \lambda$$

The Double Heston Model

The Heston model fails to fit the implied volatility smile in some cases.We introduce the double Heston model, an extension of the original Heston model.
Under some probability measure μ_p, the Double Heston model can be described as following:

$$dS = rSdt + \sqrt{v_1}SdW_1 + \sqrt{v_2}SdW_2 \tag{27}$$

$$dv_1 = \kappa_1(\theta_1 - v_1)dt + \sigma_1\sqrt{v_1}dW_3 \tag{28}$$

$$dv_2 = \kappa_2(\theta_2 - v_2)dt + \sigma_2\sqrt{v_2}dW_4 \tag{29}$$

Here, the wiener processes W_1, W_3 has zero covariance with W_2 and W_4 and vice versa.

$$E[dW_1 dW_3] = \rho_1 dt \tag{30}$$

$$E[dW_2 dW_4] = \rho_2 dt \tag{31}$$

$$E[dW_1 dW_2] = E[dW_3 dW_4] = E[dW_1 dW_4] = E[dW_2 dW_3] = 0 \tag{32}$$

Pricing under the Double Heston Model

The formation for the call price under the double Heston Model is presented as following:

$$C(S, v, t) = S_t P_1 - Ke^{-r(T-t)} P_2 \tag{33}$$

$$P_1 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re\left(\frac{e^{-i\phi lnK} f(\phi - i, x, v_1, v_2)}{i\phi S e^{r\tau}}\right) d\phi \tag{34}$$

$$P_2 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re\left(\frac{e^{-i\phi lnK} f(\phi, x, v_1, v_2)}{i\phi}\right) d\phi \tag{35}$$

$$f(\phi, x_t, v_1, v_2) = e^{(A(\tau,\phi) + i\phi x_t + B_1(\tau,\phi) v_1 + B_2(\tau,\phi) v_2)} \tag{36}$$

Contd.

where

$$B_j(\tau,\phi) = \frac{\kappa_j - \rho_j\sigma_j\phi i - d_j}{\sigma_j^2}\left[\frac{1-e^{-d_j\tau}}{1-c_je^{-d_j\tau}}\right] \tag{37}$$

$$A(\tau,\phi) = r\phi i\tau + \sum_{j=1}^{2}\frac{\kappa_j\theta_j}{\sigma_j^2}\left[(\kappa_j - \rho_j\sigma_j\phi i - d_j)\tau - 2ln(\frac{1-c_je^{-d_j\tau}}{1-c_j})\right] \tag{38}$$

$$g_j = \frac{\kappa_j - \rho_j\sigma_j\phi i + d_j}{\kappa_j - \rho_j\sigma_j\phi i - d_j} \quad , \quad c_j = \frac{1}{g_j} \tag{39}$$

$$d_j = \sqrt{(\kappa_j - \rho_j\sigma_j\phi i)^2 + \sigma_j^2\phi(\phi + i)} \tag{40}$$

The implied volatility from the Double Heston model comes out to be a much closer fit to the market implied volatility than the Heston model in the case of both short and long maturities.

Calibration

Trading firms need software to help them make decisions as quickly as possible. Thereby, we need estimates of the parameters so that the models are equipped to make the best fit possible.
The objective of the calibration is to find the parameters that minimize the objective functions.
For a set of n maturities τ_i and a set of n strikes K_j the objective functions are:

$$F_1(\Theta) = \frac{1}{mn}\sum_{\tau_i}^{n}\sum_{K_j}^{m}(C_M(\tau_i, K_j) - C(\tau_i, K_j, \Theta))^2$$

$$F_2(\Theta) = \frac{1}{mn}\sum_{\tau_i}^{n}\sum_{K_j}^{m}(IV_M(\tau_i, K_j) - IV(\tau_i, K_j, \Theta))^2$$

Θ is the collection of parameters in the Heston and Double Heston models.
$IV_M(\tau, K)$, $IV(\tau, K, \Theta)$ represents the Market implied volatility and Model implied volatility.

THANK YOU!