

Jean Baptiste Joseph Fourier



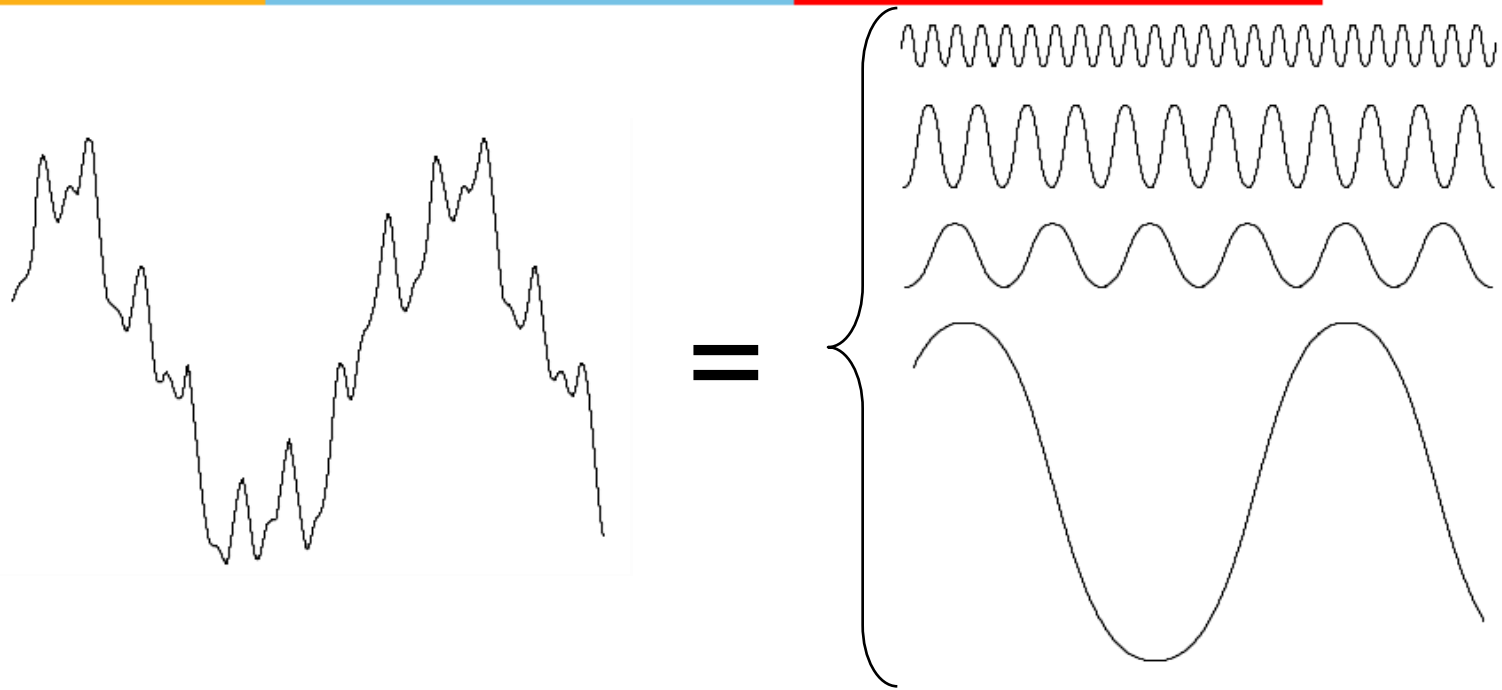
Fourier was born in Auxerre, France in 1768

- Most famous for his work “*La Théorie Analitique de la Chaleur*” published in 1822
- Translated into English in 1878: “*The Analytic Theory of Heat*”

Nobody paid much attention when the work was first published

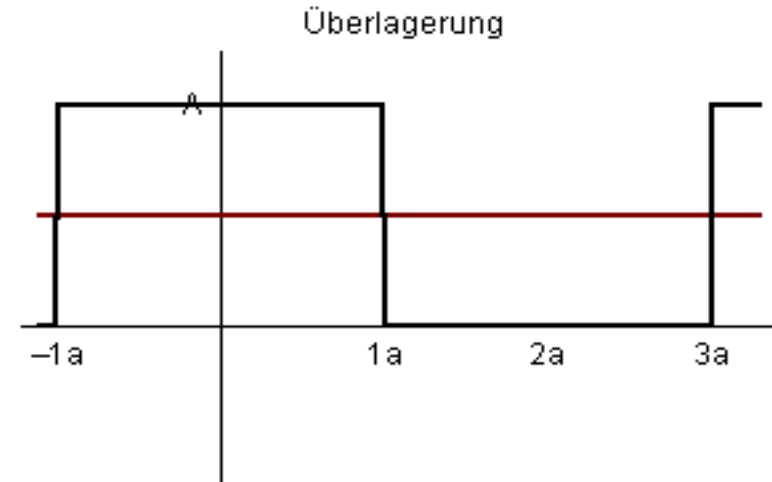
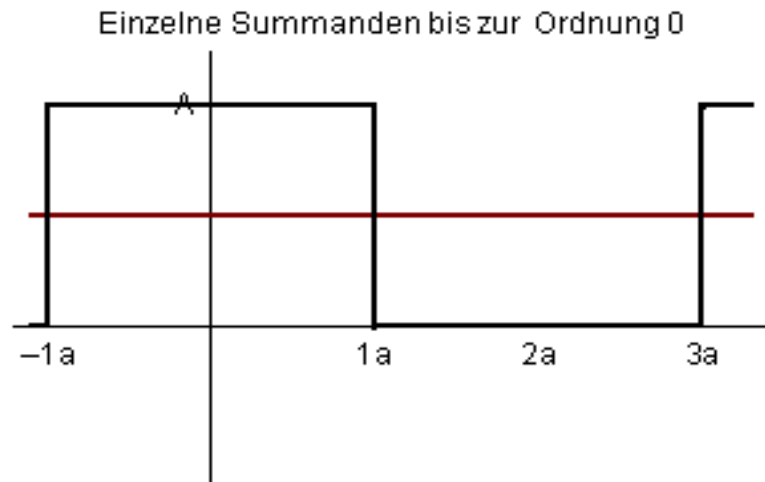
One of the most important mathematical theories in modern engineering

The Big Idea



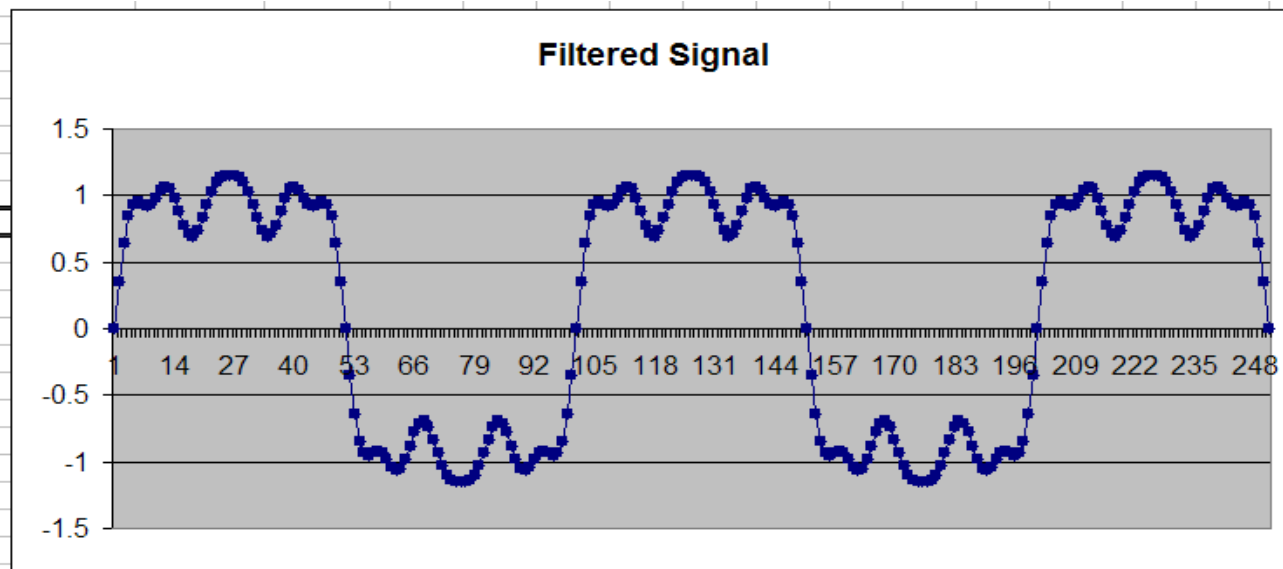
Any function that periodically repeats itself can be expressed as a sum of sines and cosines of different frequencies each multiplied by a different coefficient – a *Fourier series*

The Big Idea (cont...)

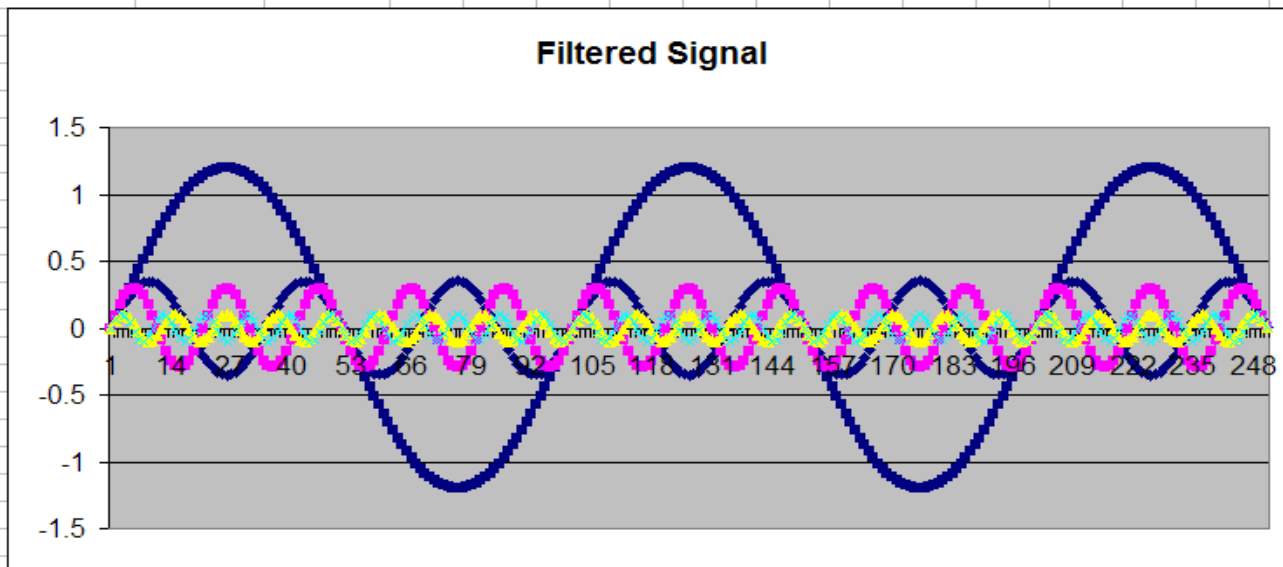


Notice how we get closer and closer to the original function as we add more and more frequencies

The Big Idea (cont...)



Frequency
domain signal
processing
example in Excel



The Discrete Fourier Transform (DFT)



The *Discrete Fourier Transform* of $f(x, y)$, for $x = 0, 1, 2 \dots M-1$ and $y = 0, 1, 2 \dots N-1$, denoted by $F(u, v)$, is given by the equation:

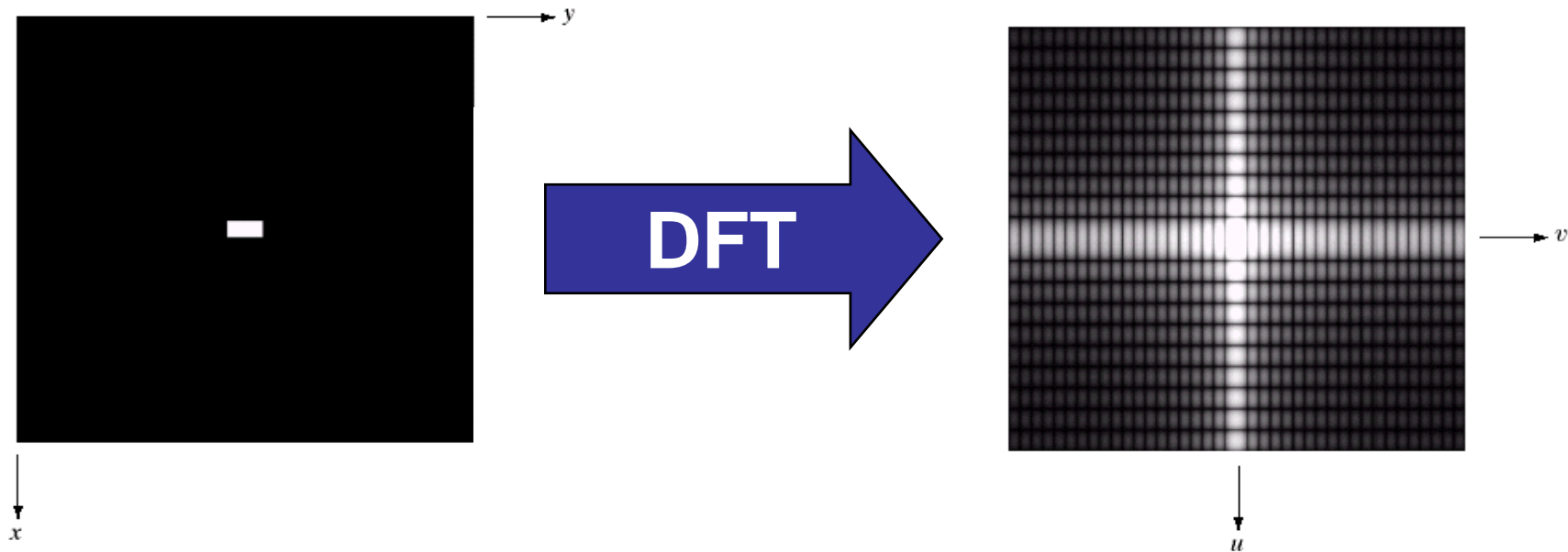
$$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(ux/M + vy/N)}$$

for $u = 0, 1, 2 \dots M-1$ and $v = 0, 1, 2 \dots N-1$.

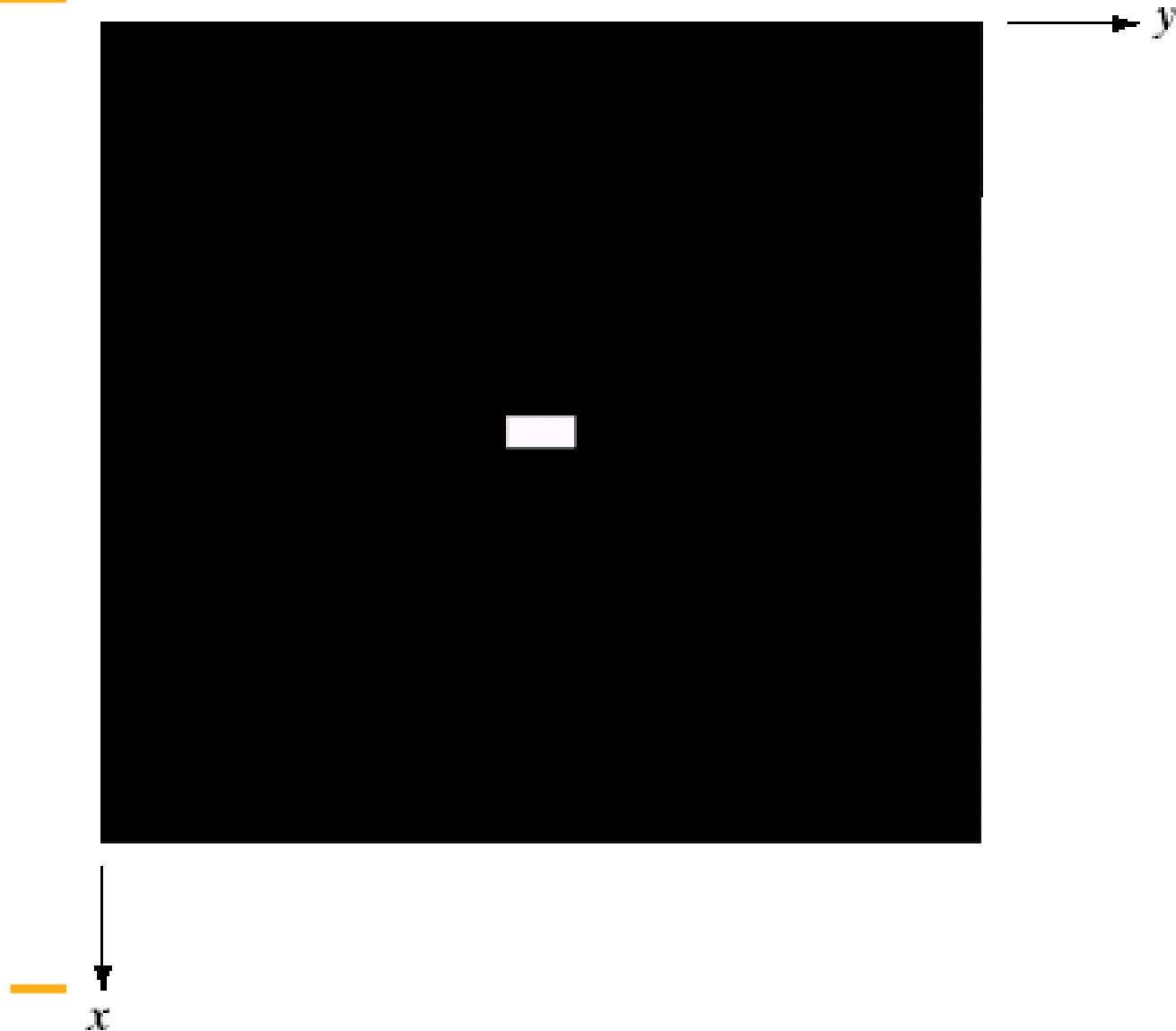
DFT & Images



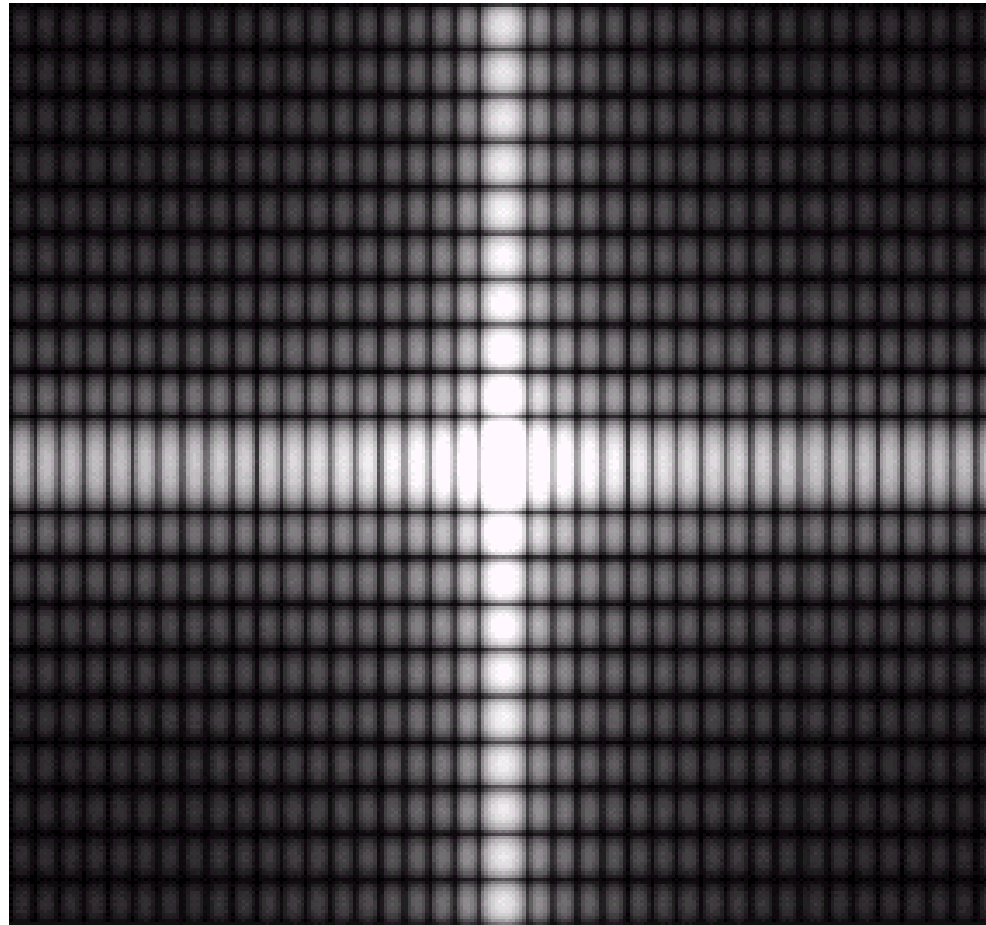
The DFT of a two dimensional image can be visualised by showing the spectrum of the images component frequencies



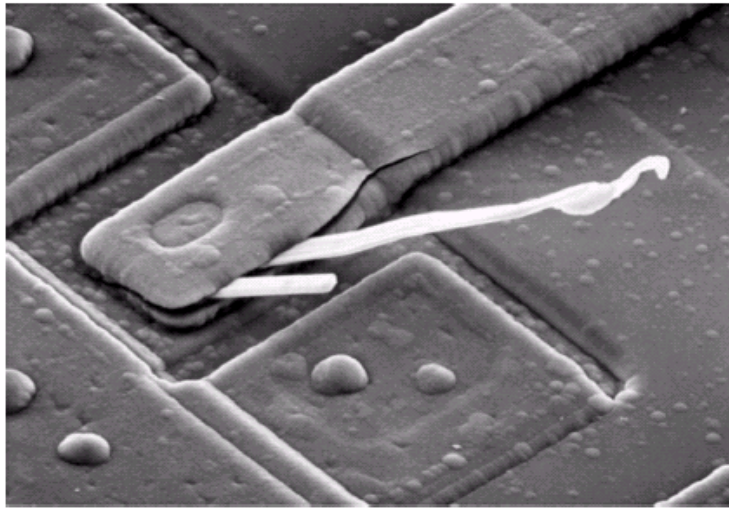
DFT & Images



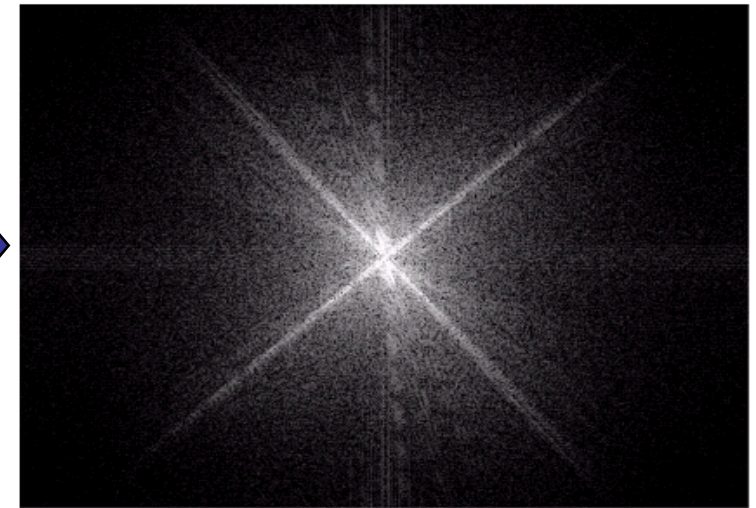
DFT & Images



DFT & Images (cont...)

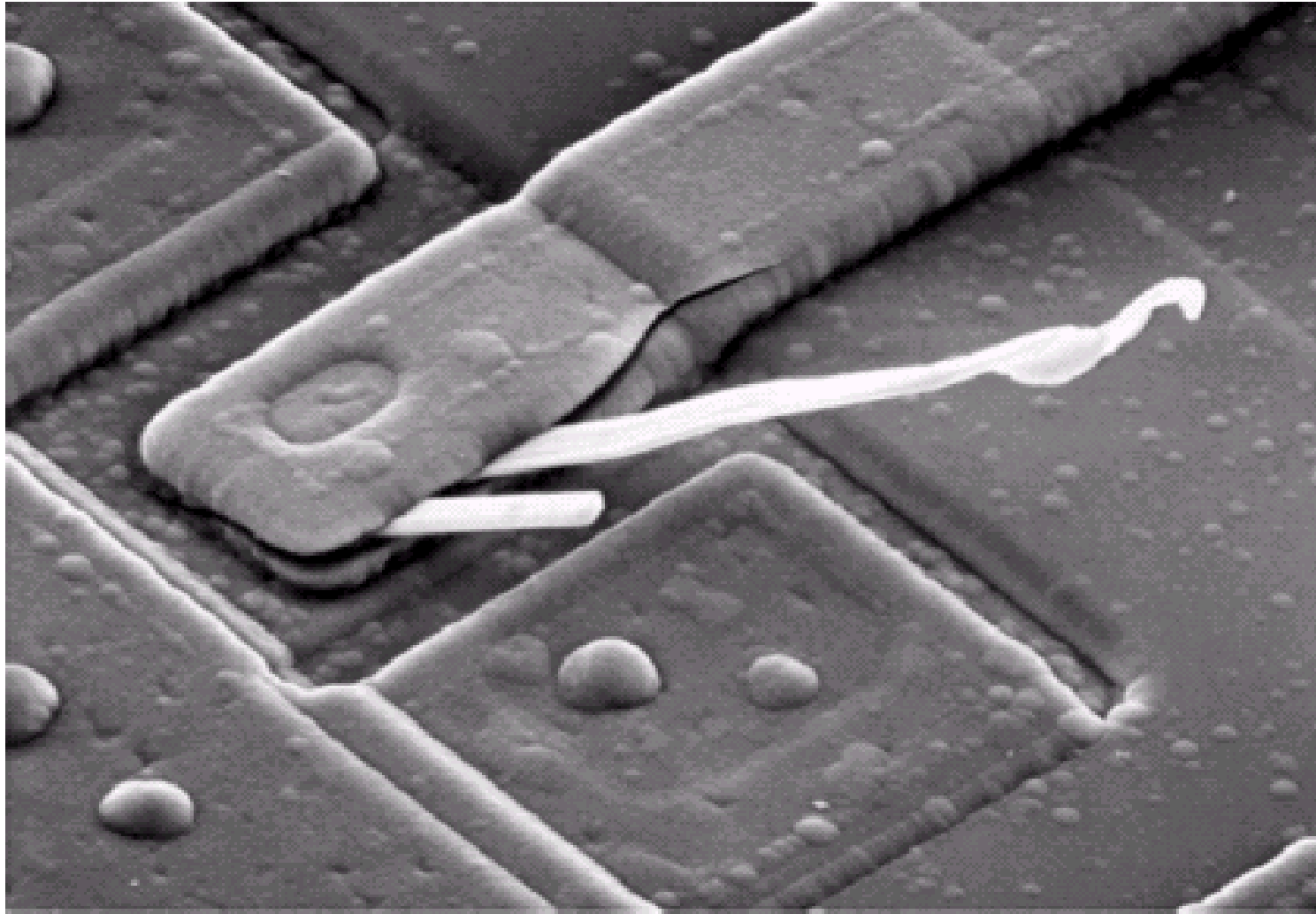


Scanning electron microscope image of an integrated circuit magnified ~2500 times

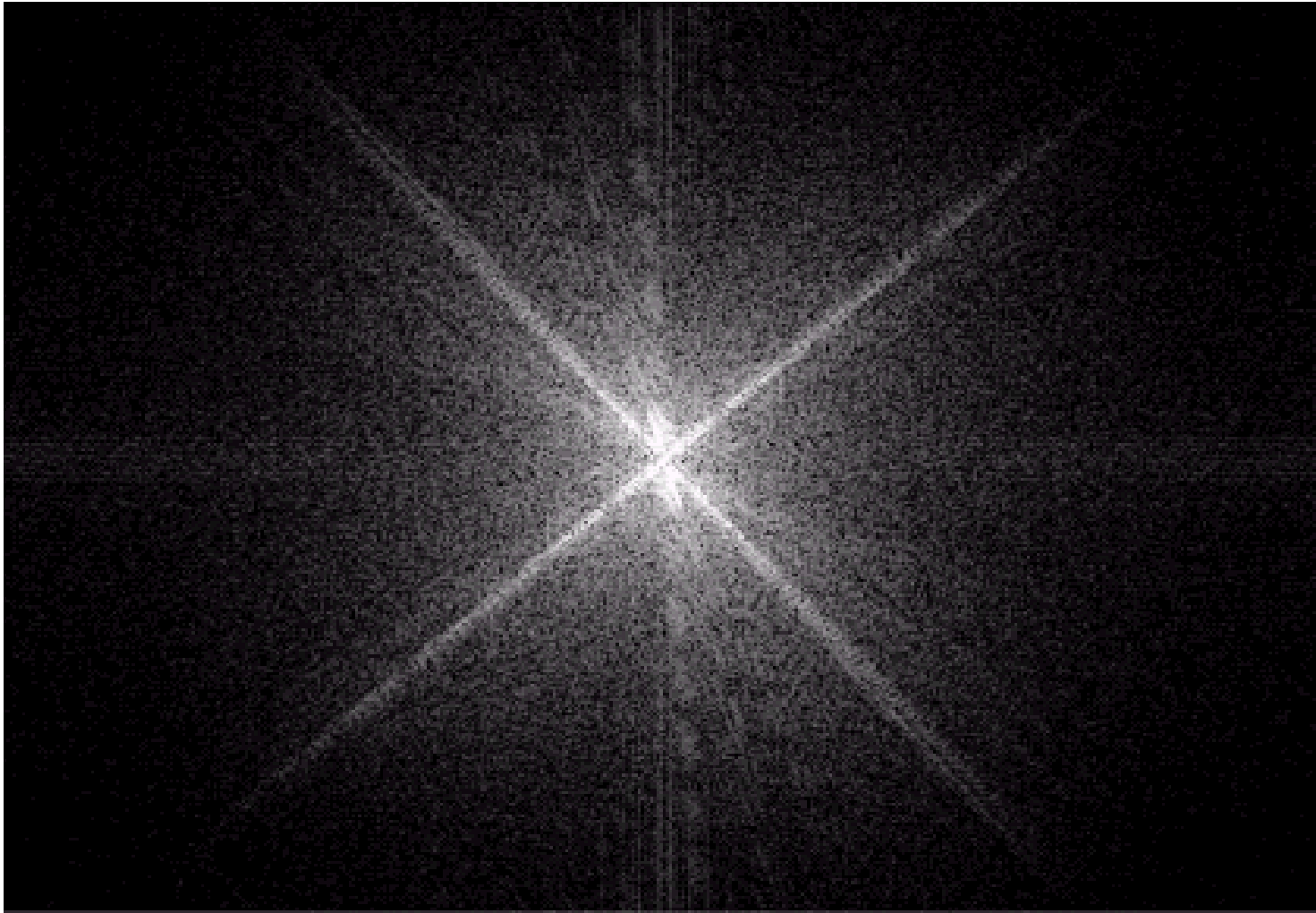


Fourier spectrum of the image

DFT & Images (cont...)



DFT & Images (cont...)



The Inverse DFT



It is really important to note that the Fourier transform is completely **reversible**

The inverse DFT is given by:

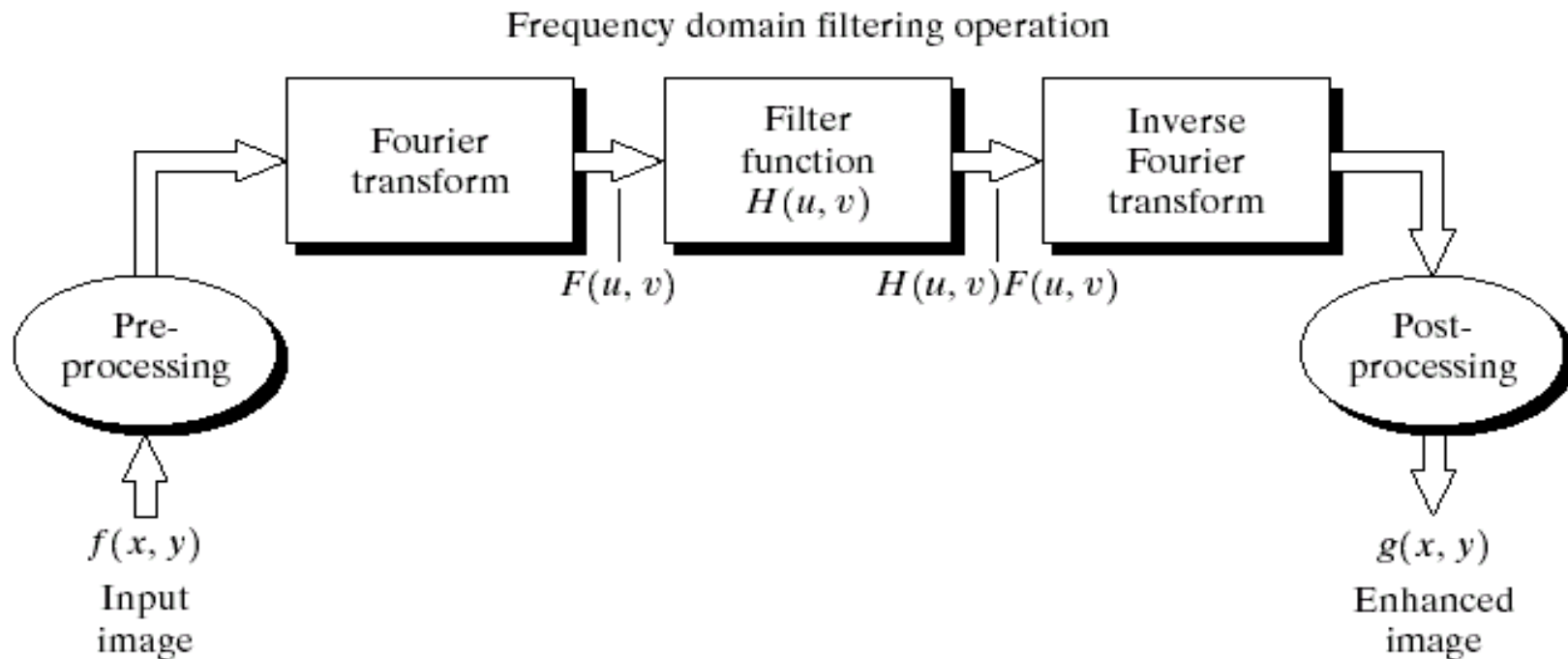
$$f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi(ux/M + vy/N)}$$

for $x = 0, 1, 2 \dots M-1$ and $y = 0, 1, 2 \dots N-1$

The DFT and Image Processing

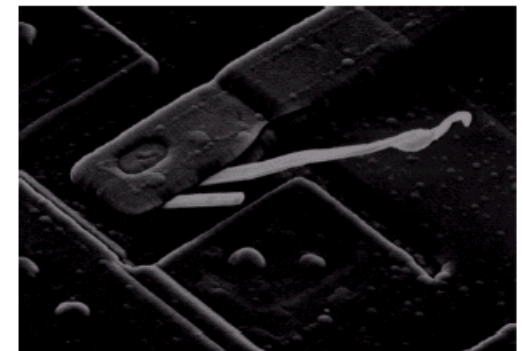
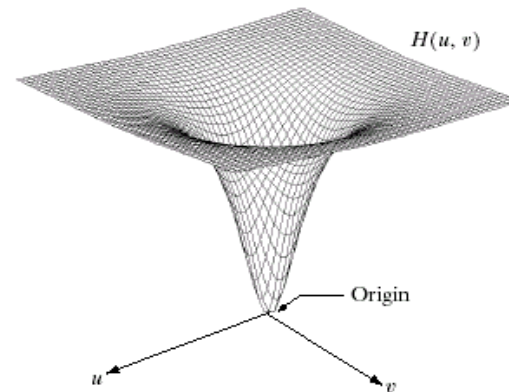
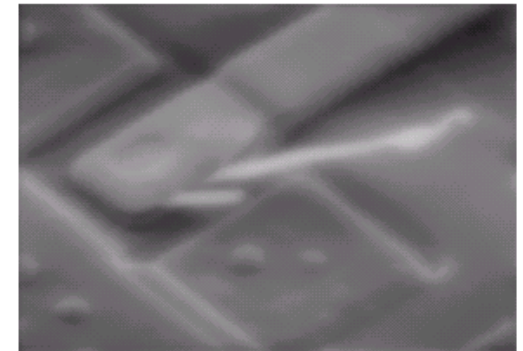
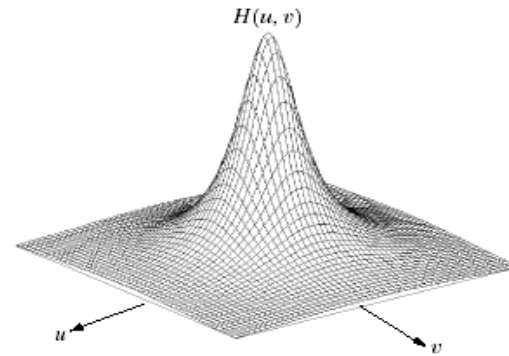
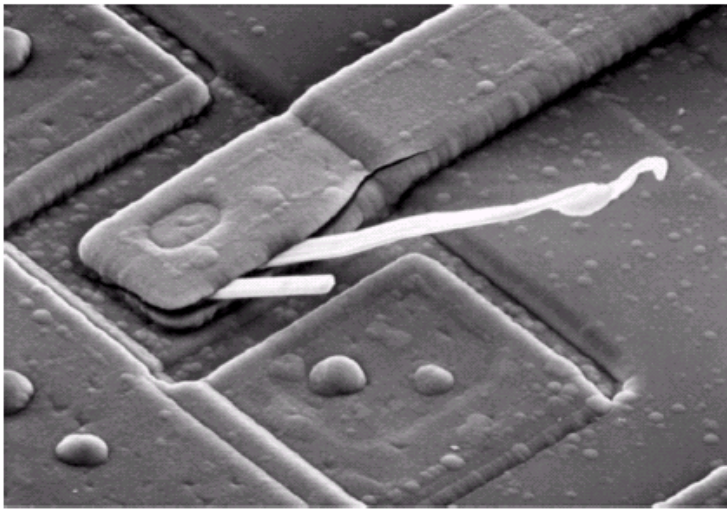
To filter an image in the frequency domain:

1. Compute $F(u, v)$ the DFT of the image
2. Multiply $F(u, v)$ by a filter function $H(u, v)$
3. Compute the inverse DFT of the result



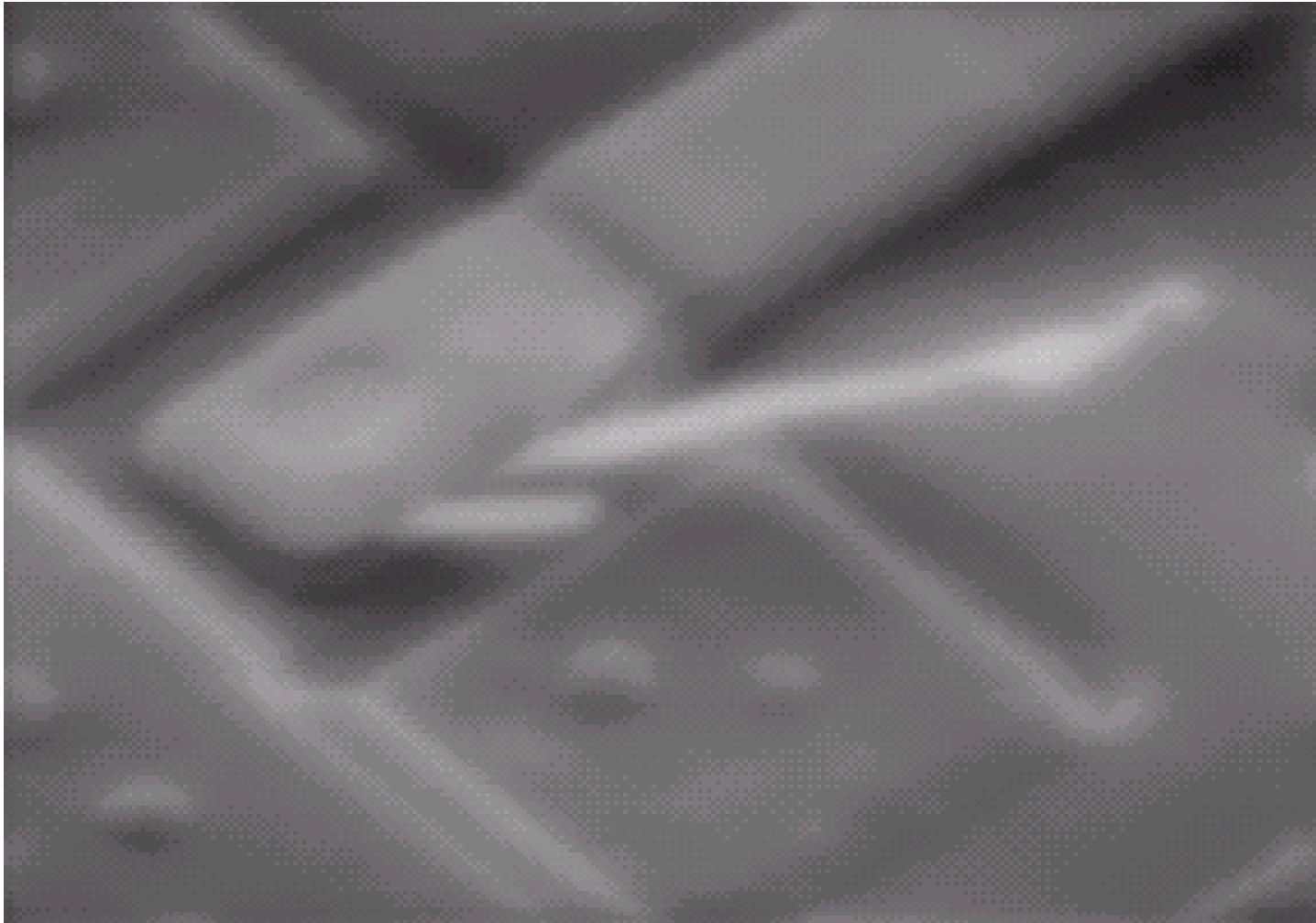
Some Basic Frequency Domain Filters

Low Pass Filter

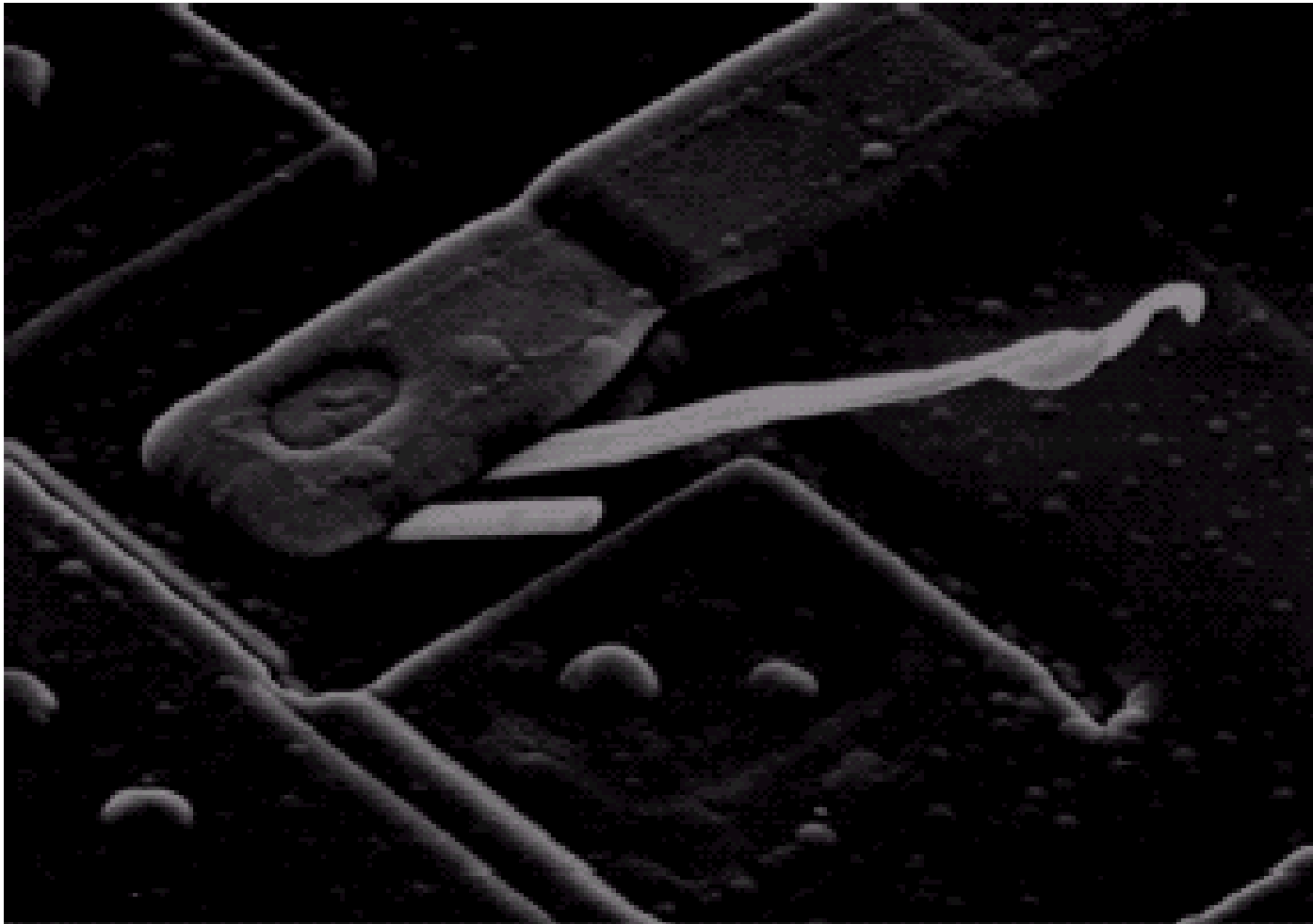


High Pass Filter

Some Basic Frequency Domain Filters



Some Basic Frequency Domain Filters



Smoothing Frequency Domain Filters



Smoothing is achieved in the frequency domain by dropping out the high frequency components

The basic model for filtering is:

$$G(u,v) = H(u,v)F(u,v)$$

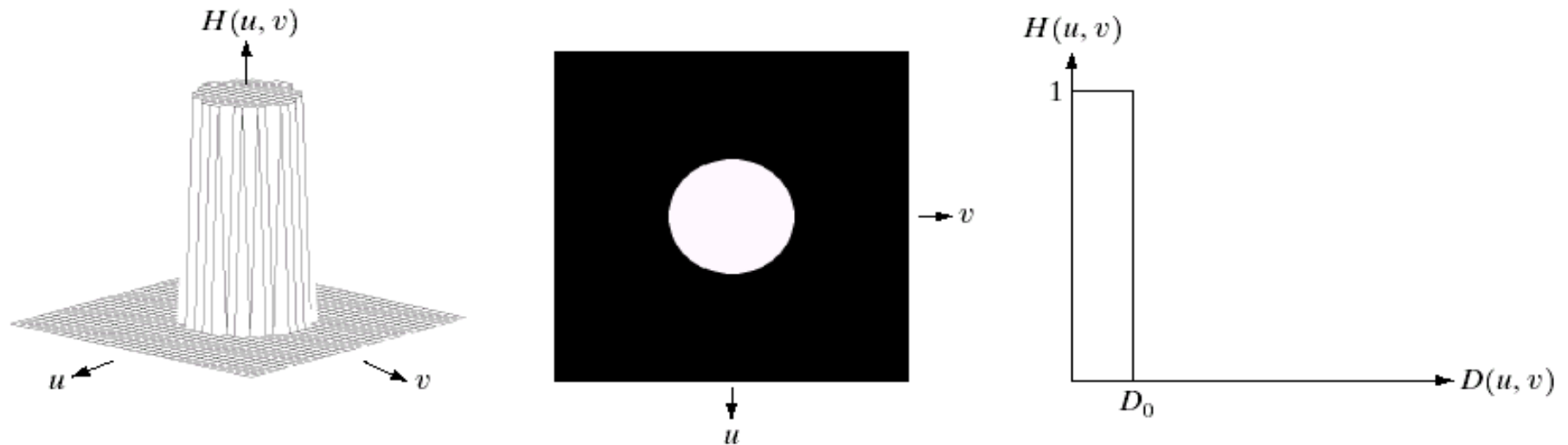
where $F(u,v)$ is the Fourier transform of the image being filtered and $H(u,v)$ is the filter transform function

Low pass filters – only pass the low frequencies, drop the high ones

Ideal Low Pass Filter



Simply cut off all high frequency components that are a specified distance D_0 from the origin of the transform



changing the distance changes the behaviour of the filter

Ideal Low Pass Filter (cont....)

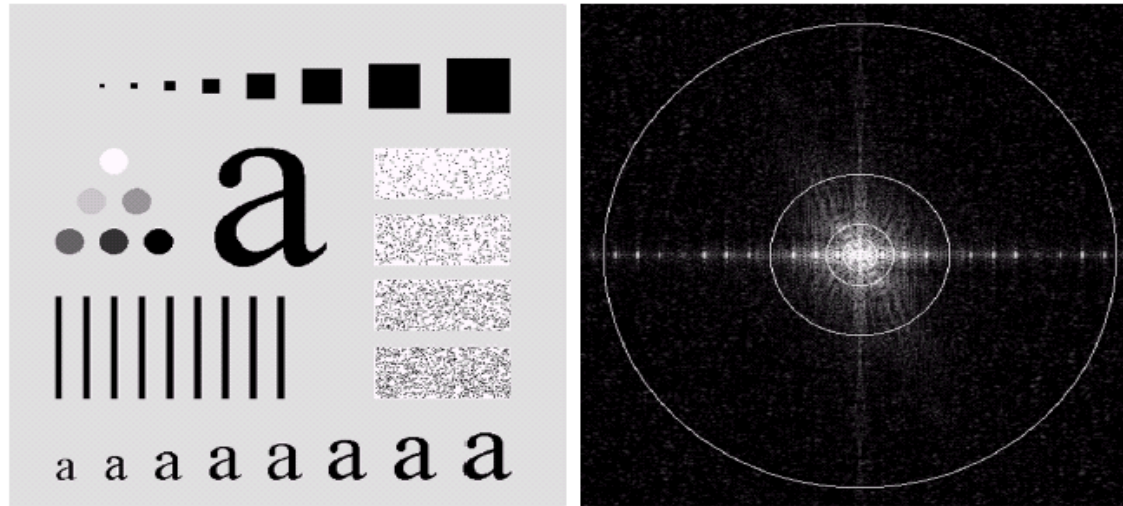
The transfer function for the ideal low pass filter can be given as:

$$H(u, v) = \begin{cases} 1 & \text{if } D(u, v) \leq D_0 \\ 0 & \text{if } D(u, v) > D_0 \end{cases}$$

where $D(u, v)$ is given as:

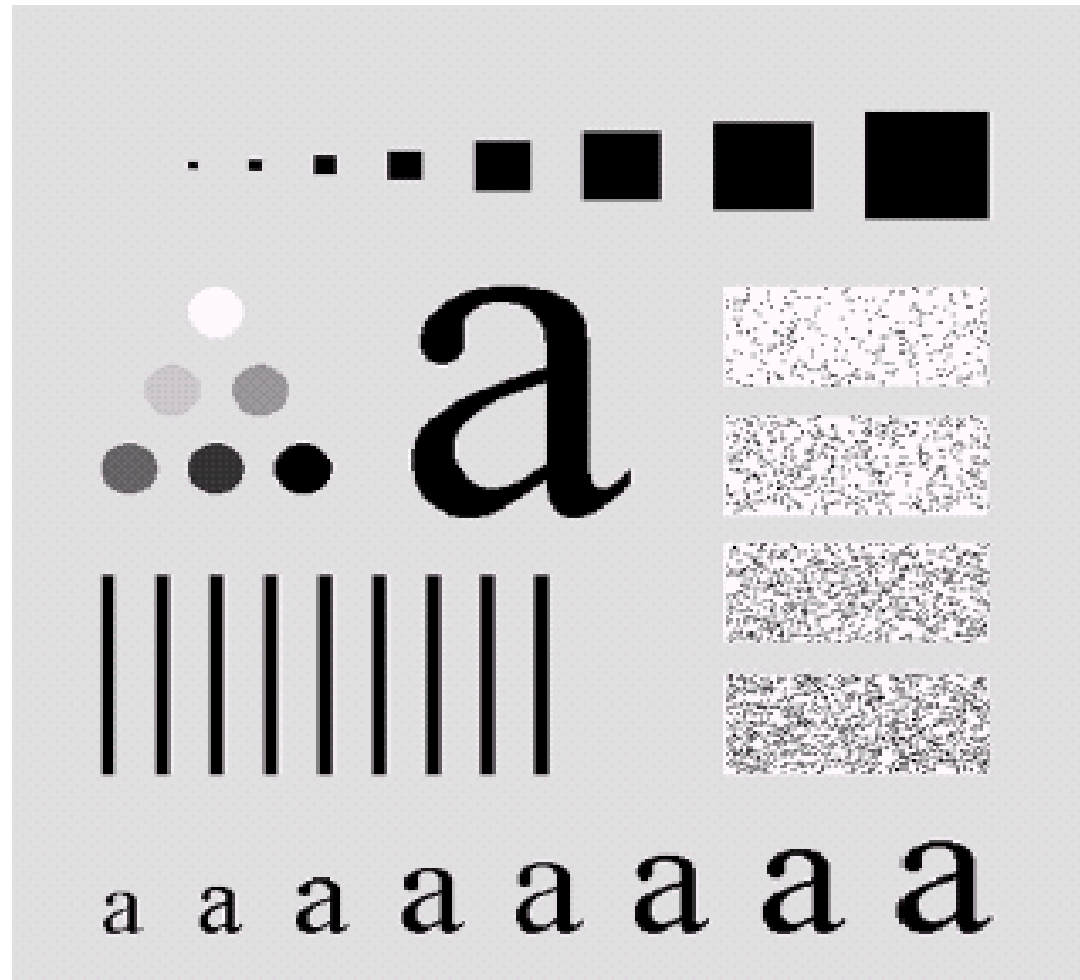
$$D(u, v) = [(u - M / 2)^2 + (v - N / 2)^2]^{1/2}$$

Ideal Low Pass Filter (cont...)

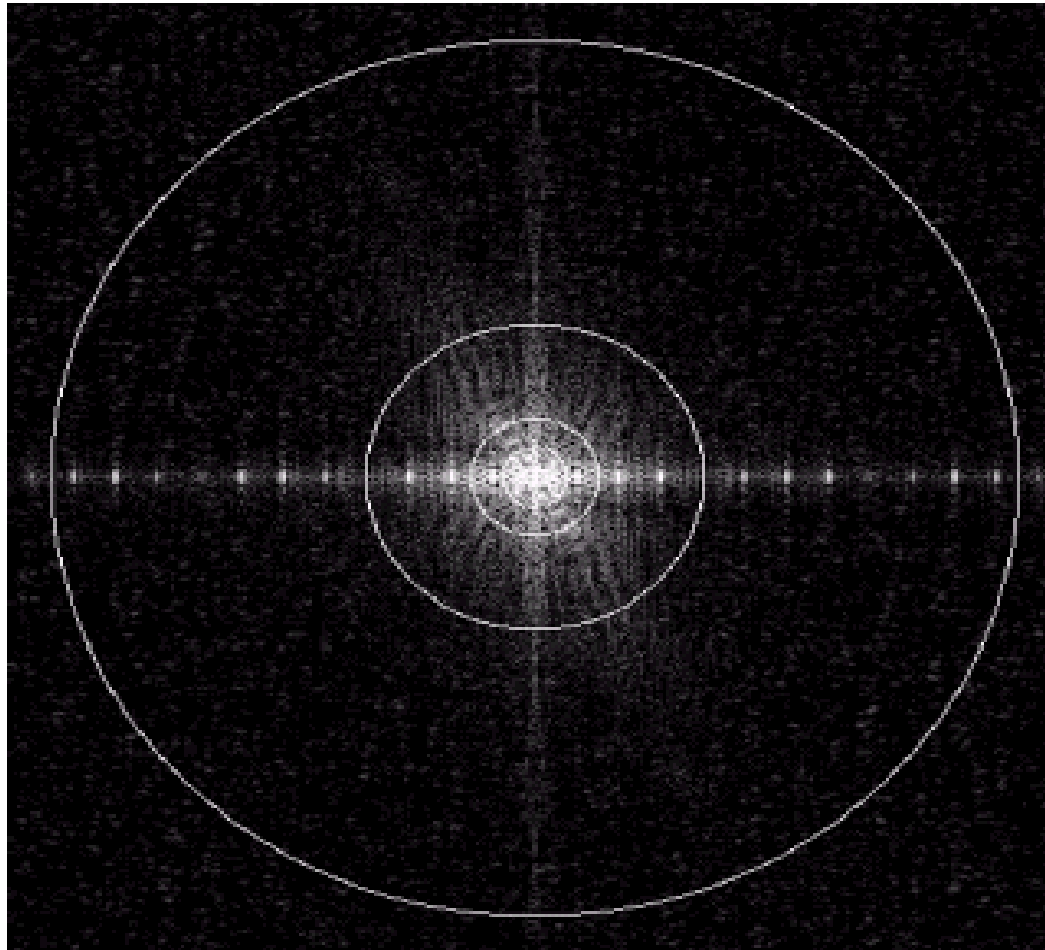


Above we show an image, it's Fourier spectrum and a series of ideal low pass filters of radius 5, 15, 30, 80 and 230 superimposed on top of it

Ideal Low Pass Filter (cont...)



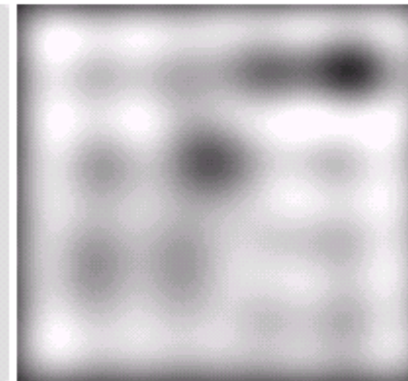
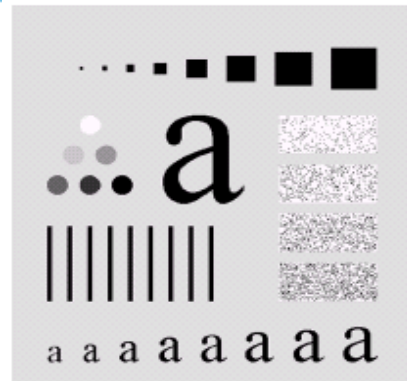
Ideal Low Pass Filter (cont...)



Ideal Low Pass Filter (cont...)

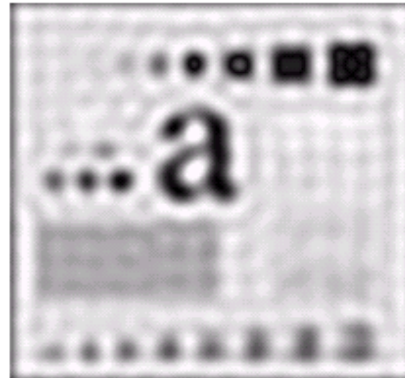


Original
image



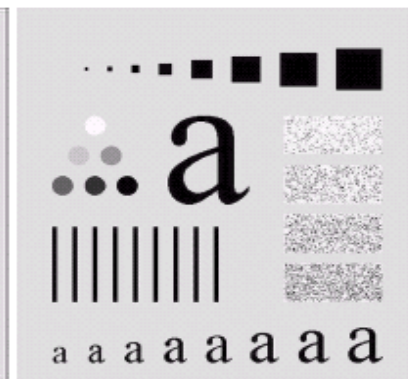
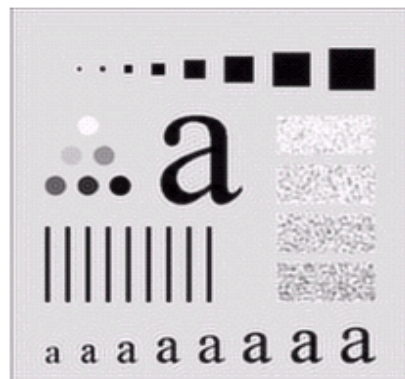
Result of filtering
with ideal low pass
filter of radius 5

Result of filtering
with ideal low pass
filter of radius 15



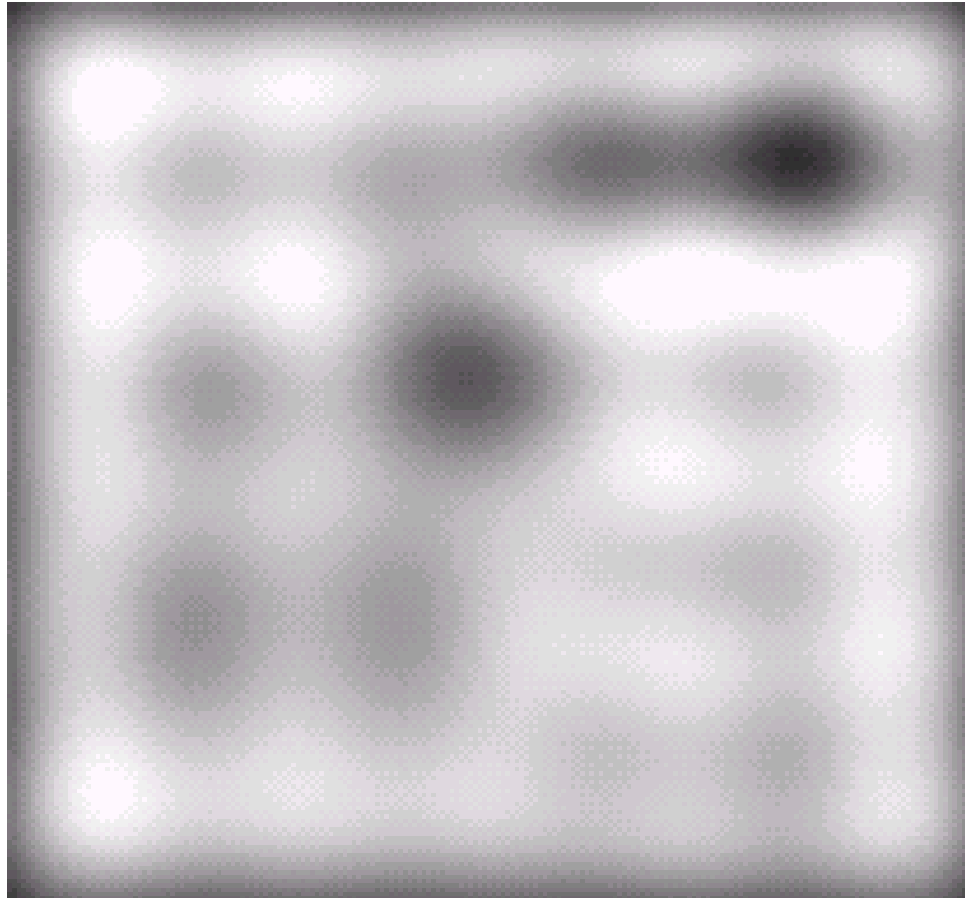
Result of filtering
with ideal low pass
filter of radius 30

Result of filtering
with ideal low pass
filter of radius 80



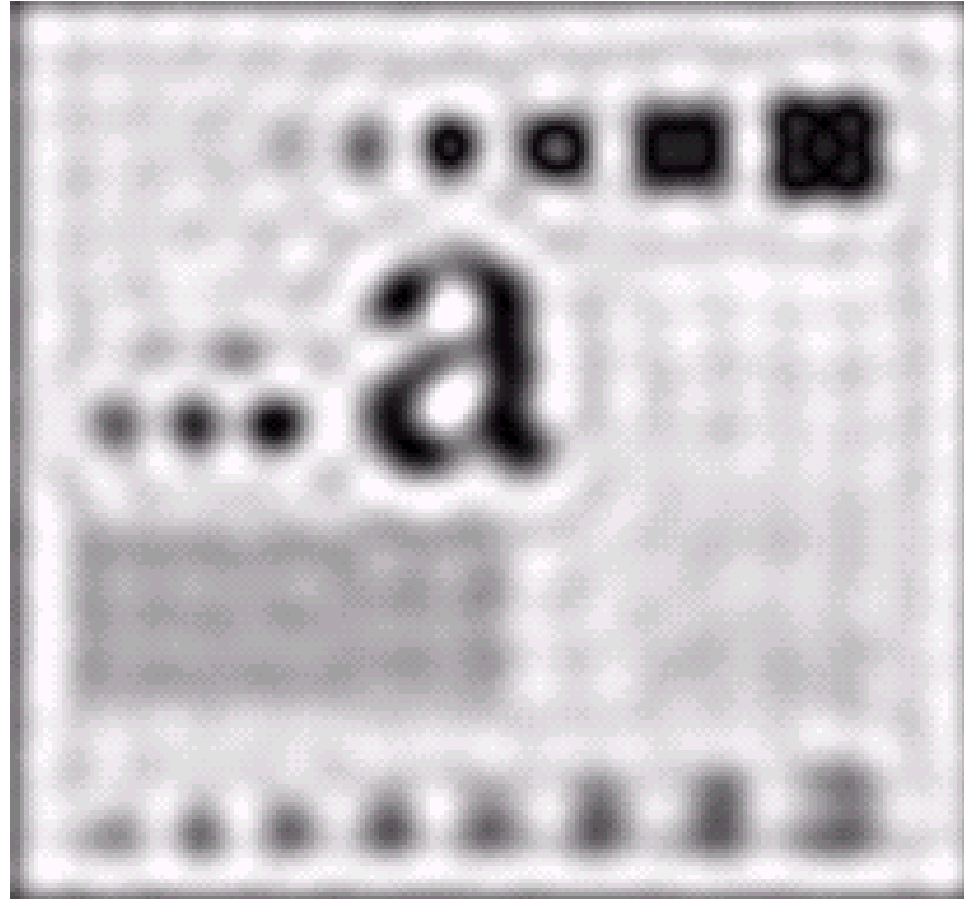
Result of filtering
with ideal low pass
filter of radius 230

Ideal Low Pass Filter (cont...)



Result of filtering
with ideal low pass
filter of radius 5

Ideal Low Pass Filter (cont...)

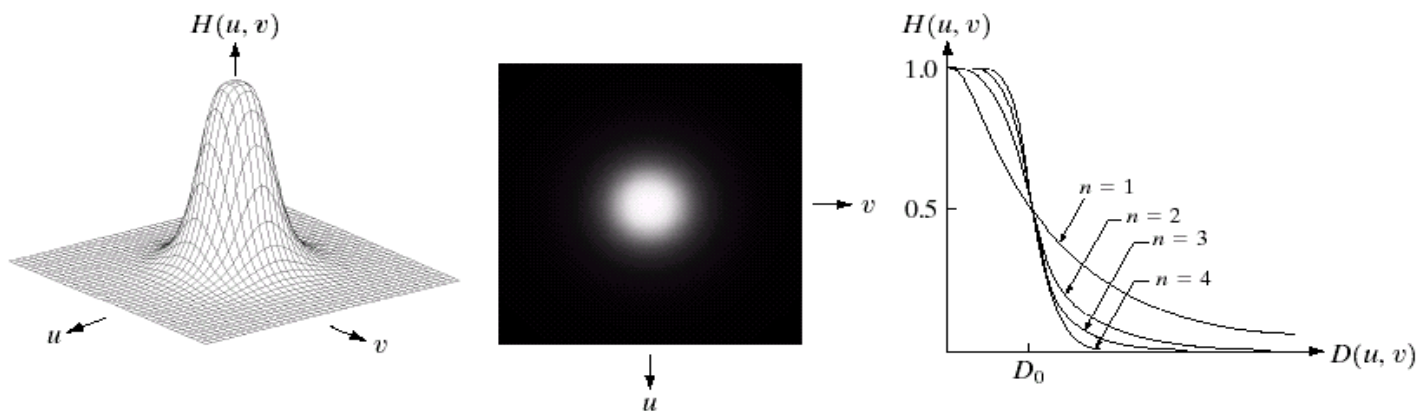


Result of filtering
with ideal low pass
filter of radius 15

Butterworth Lowpass Filters

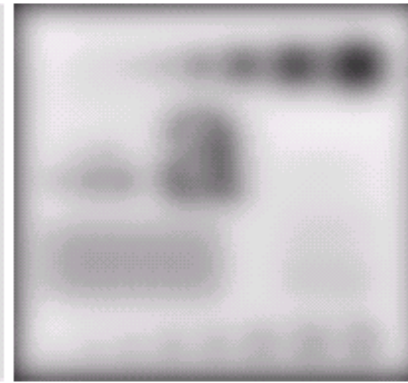
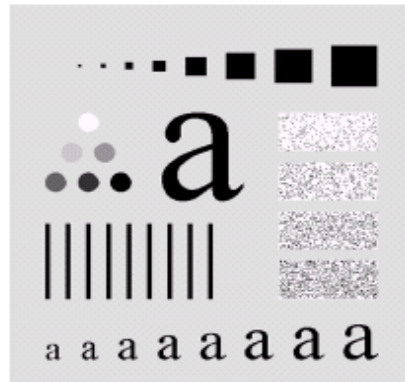
The transfer function of a Butterworth lowpass filter of order n with cutoff frequency at distance D_0 from the origin is defined as:

$$H(u, v) = \frac{1}{1 + [D(u, v) / D_0]^{2n}}$$



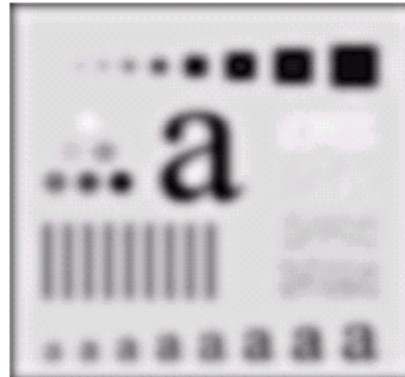
Butterworth Lowpass Filter (cont...)

Original
image



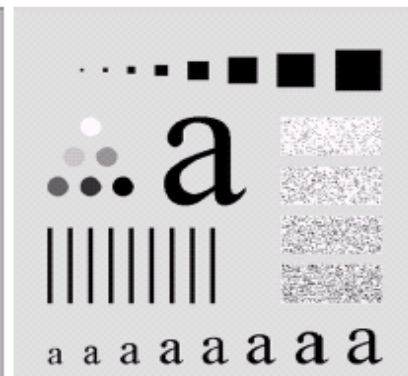
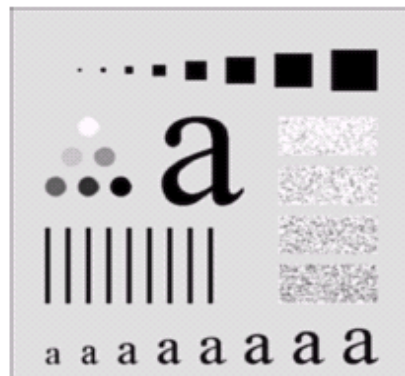
Result of filtering
with Butterworth filter
of order 2 and cutoff
radius 5

Result of filtering with
Butterworth filter of
order 2 and cutoff
radius 15



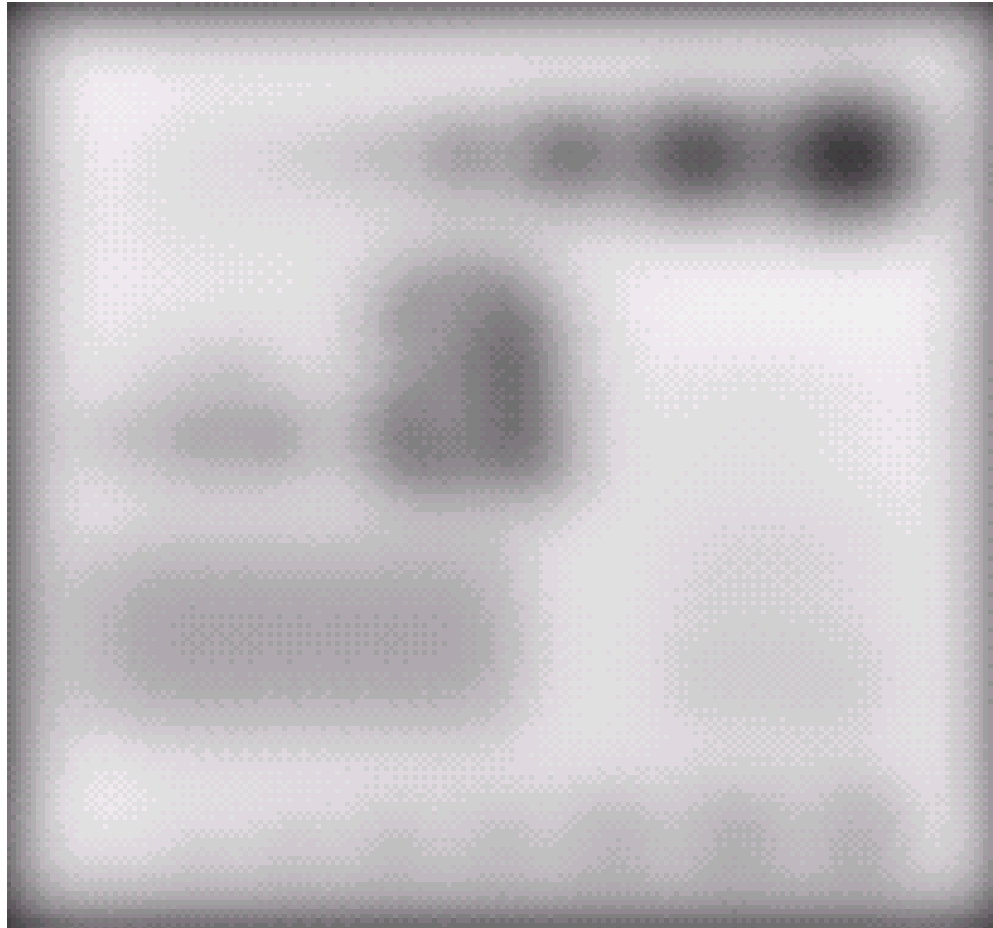
Result of filtering
with Butterworth
filter of order 2 and
cutoff radius 30

Result of filtering with
Butterworth filter of
order 2 and cutoff
radius 80



Result of filtering
with Butterworth
filter of order 2 and
cutoff radius 230

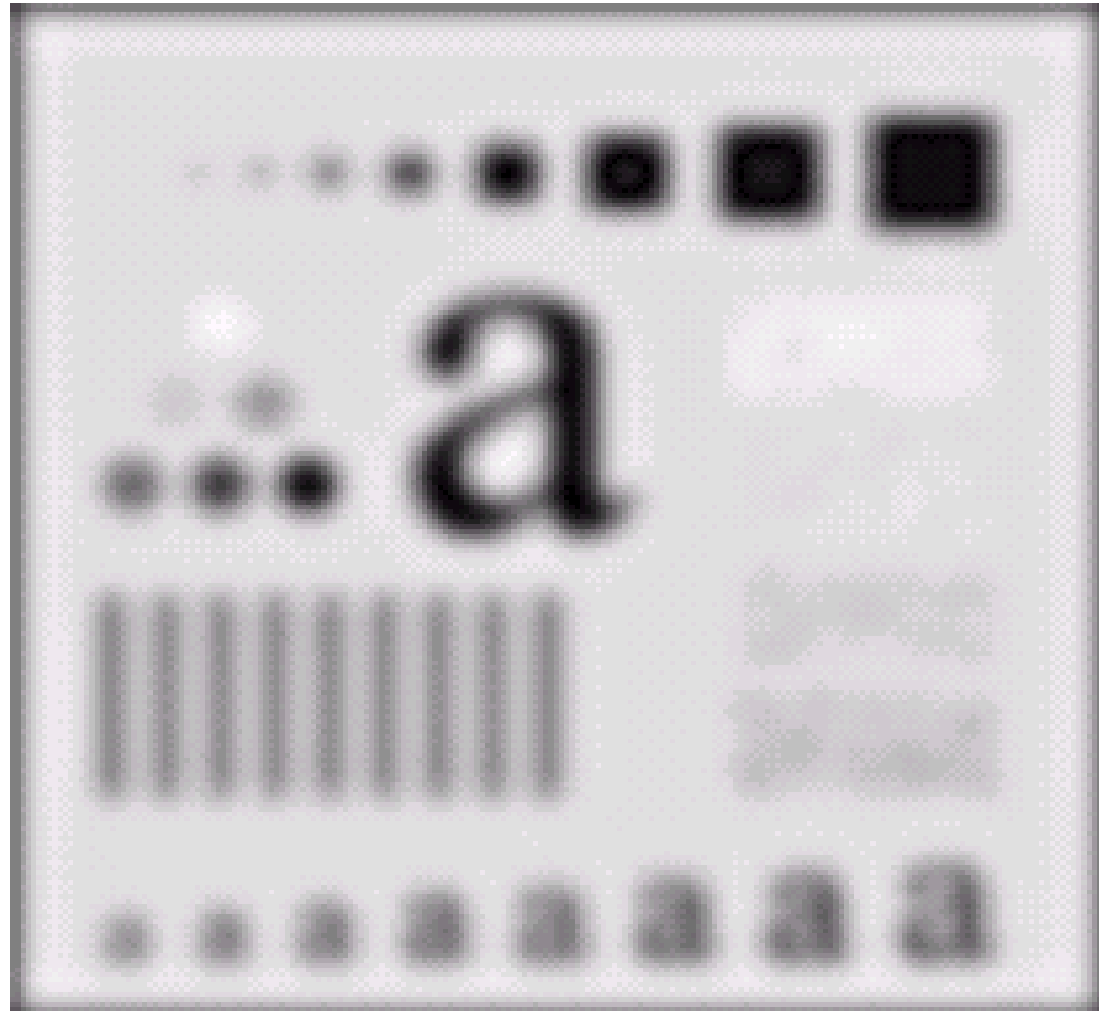
Butterworth Lowpass Filter (cont...)



Result of filtering
with Butterworth filter
of order 2 and cutoff
radius 5

Butterworth Lowpass Filter (cont...)

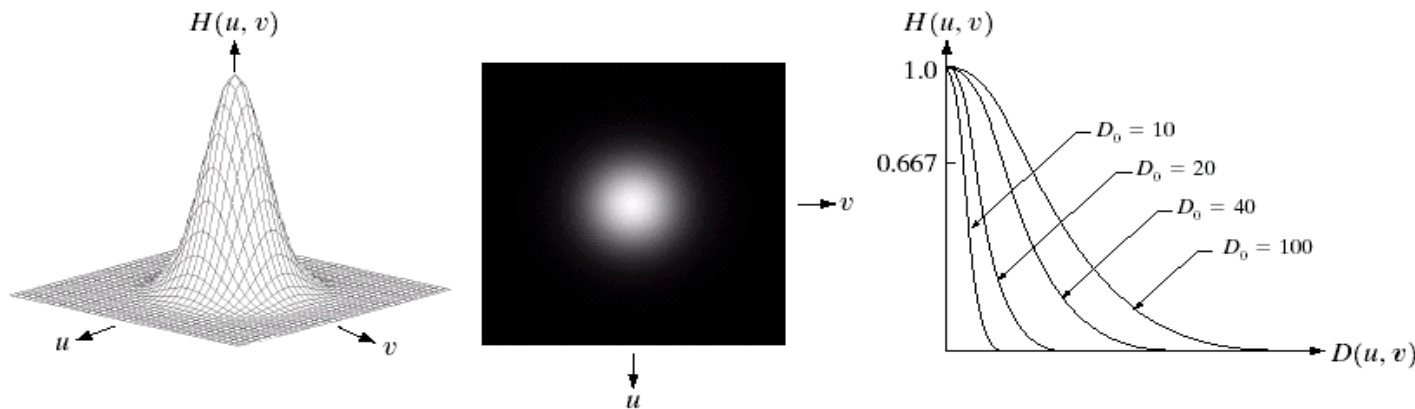
Result of filtering with
Butterworth filter of
order 2 and cutoff
radius 15



Gaussian Lowpass Filters

The transfer function of a Gaussian lowpass filter is defined as:

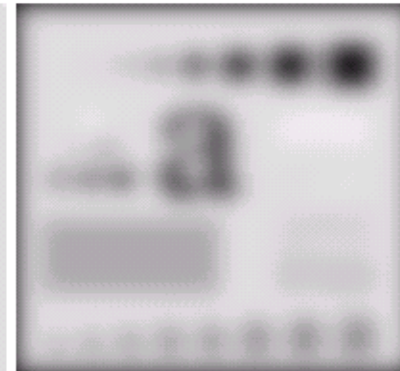
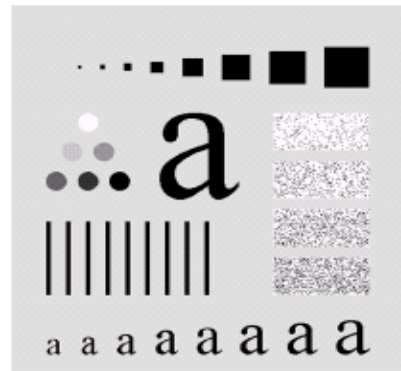
$$H(u, v) = e^{-D^2(u, v) / 2D_0^2}$$



Gaussian Lowpass Filters (cont...)

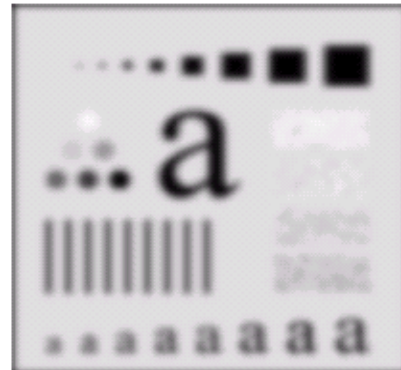


Original
image



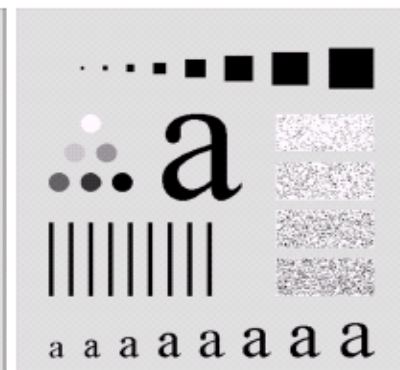
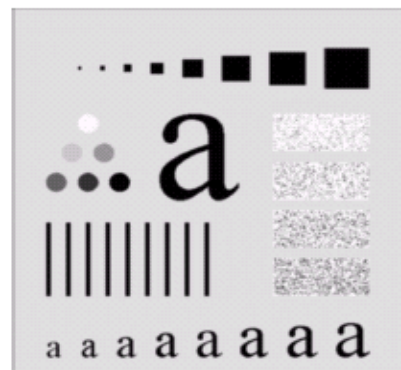
Result of filtering
with Gaussian filter
with cutoff radius 5

Result of filtering
with Gaussian
filter with cutoff
radius 15



Result of filtering
with Gaussian filter
with cutoff radius 30

Result of filtering
with Gaussian
filter with cutoff
radius 85

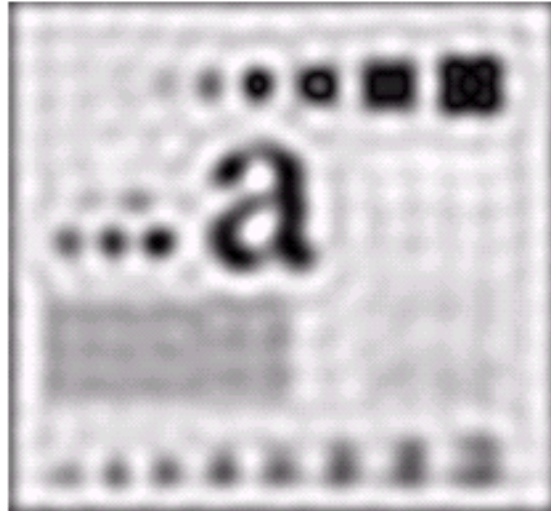


Result of filtering
with Gaussian filter
with cutoff radius
230

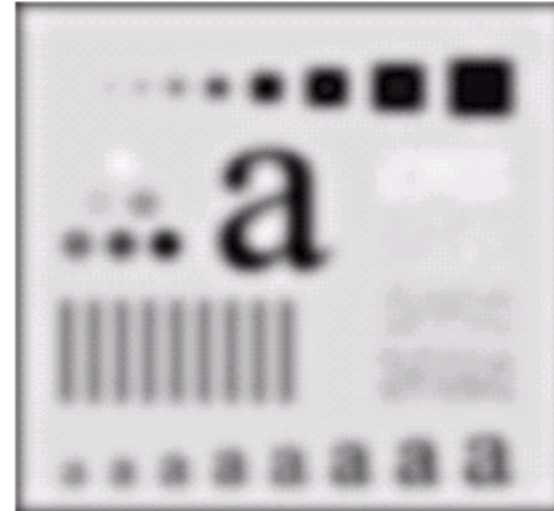
Lowpass Filters Compared



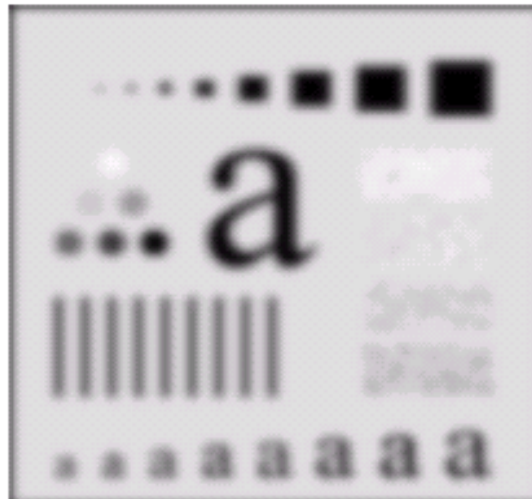
Result of filtering
with ideal low pass
filter of radius 15



Result of filtering
with Butterworth
filter of order 2
and cutoff radius
15

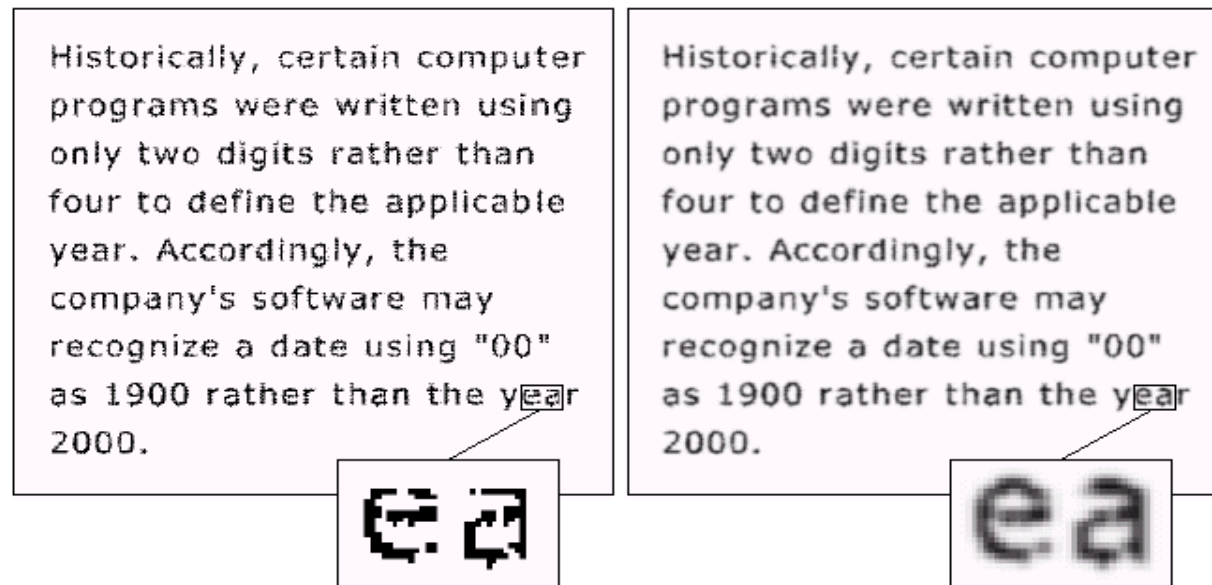


Result of filtering
with Gaussian
filter with cutoff
radius 15



Lowpass Filtering Examples

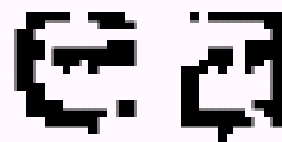
A low pass Gaussian filter is used to connect broken text



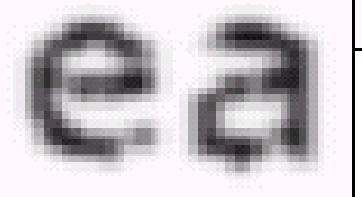
Lowpass Filtering Examples



Historically, certain computer programs were written using only two digits rather than four to define the applicable year. Accordingly, the company's software may recognize a date using "00" as 1900 rather than the year 2000.



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Lowpass Filtering Examples (cont...)

Different lowpass Gaussian filters used to remove blemishes in a photograph

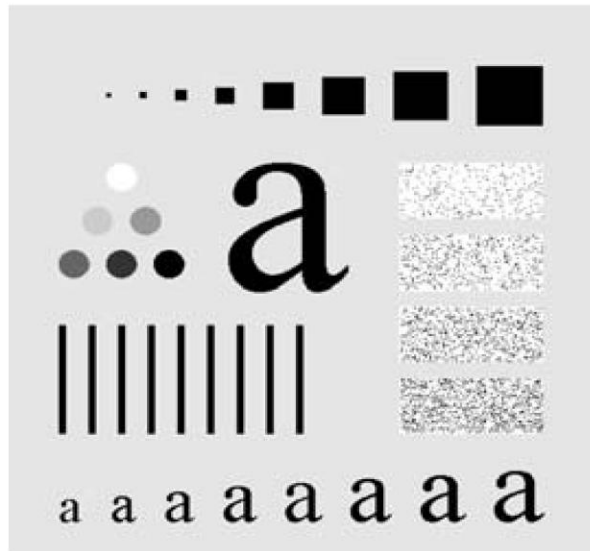


Lowpass Filtering Examples (cont..)

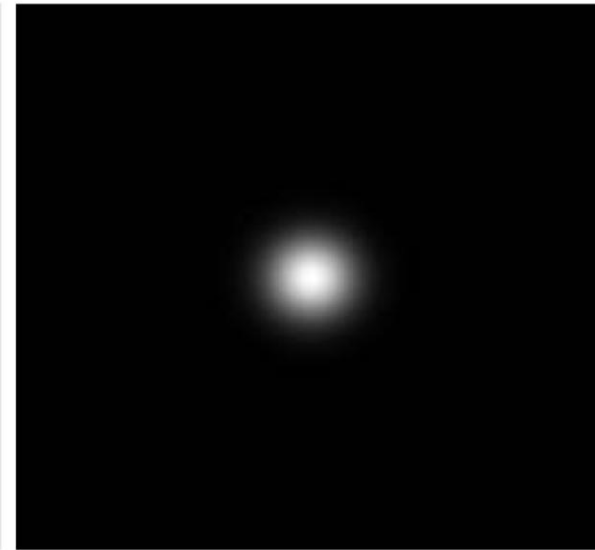


Lowpass Filtering Examples (cont...)

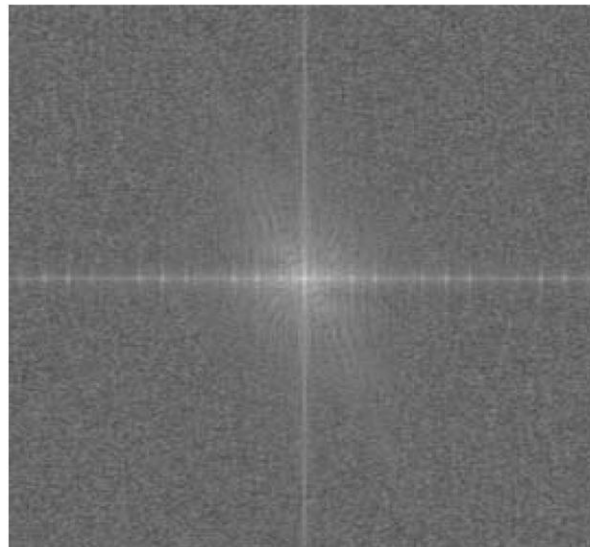
Original image



Gaussian lowpass filter



Spectrum of original image



Processed image



Sharpening in the Frequency Domain

Edges and fine detail in images are associated with high frequency components

High pass filters – only pass the high frequencies, drop the low ones

High pass frequencies are precisely the reverse of low pass filters, so:

$$H_{hp}(u, v) = 1 - H_{lp}(u, v)$$

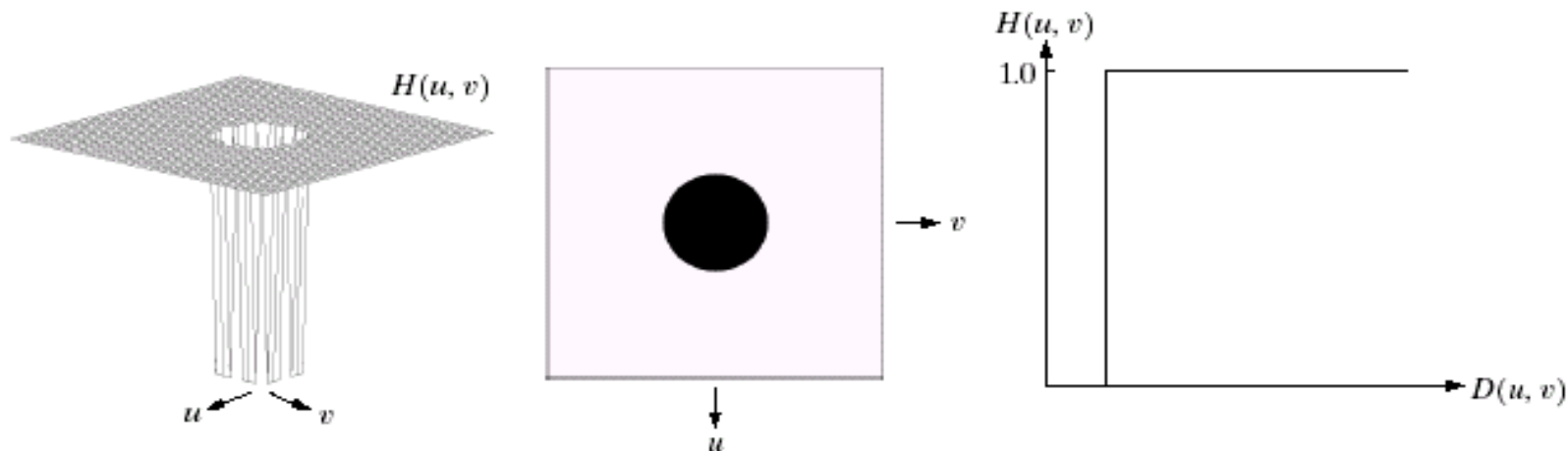
Ideal High Pass Filters



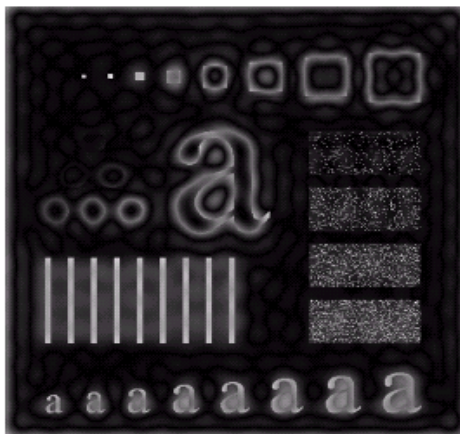
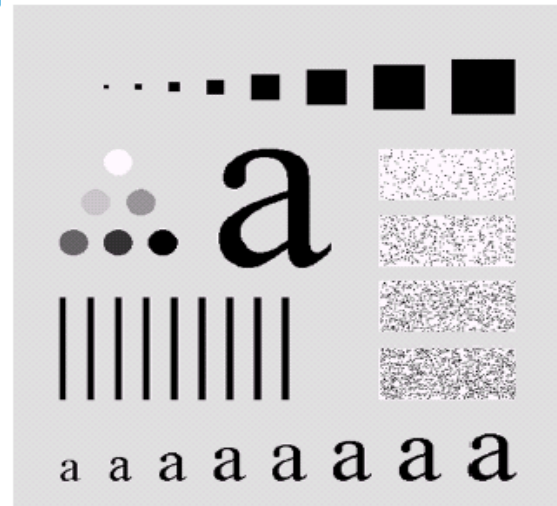
The ideal high pass filter is given as:

$$H(u, v) = \begin{cases} 0 & \text{if } D(u, v) \leq D_0 \\ 1 & \text{if } D(u, v) > D_0 \end{cases}$$

where D_0 is the cut off distance as before



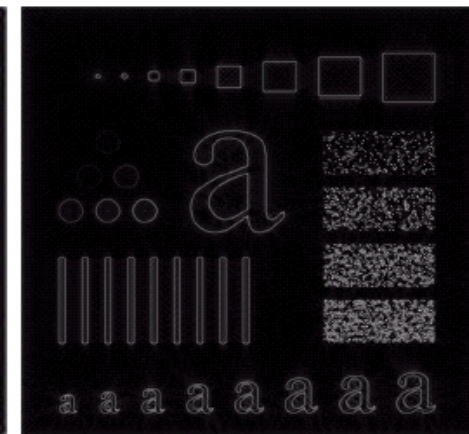
Ideal High Pass Filters (cont...)



Results of ideal
high pass filtering
with $D_0 = 15$



Results of ideal
high pass filtering
with $D_0 = 30$



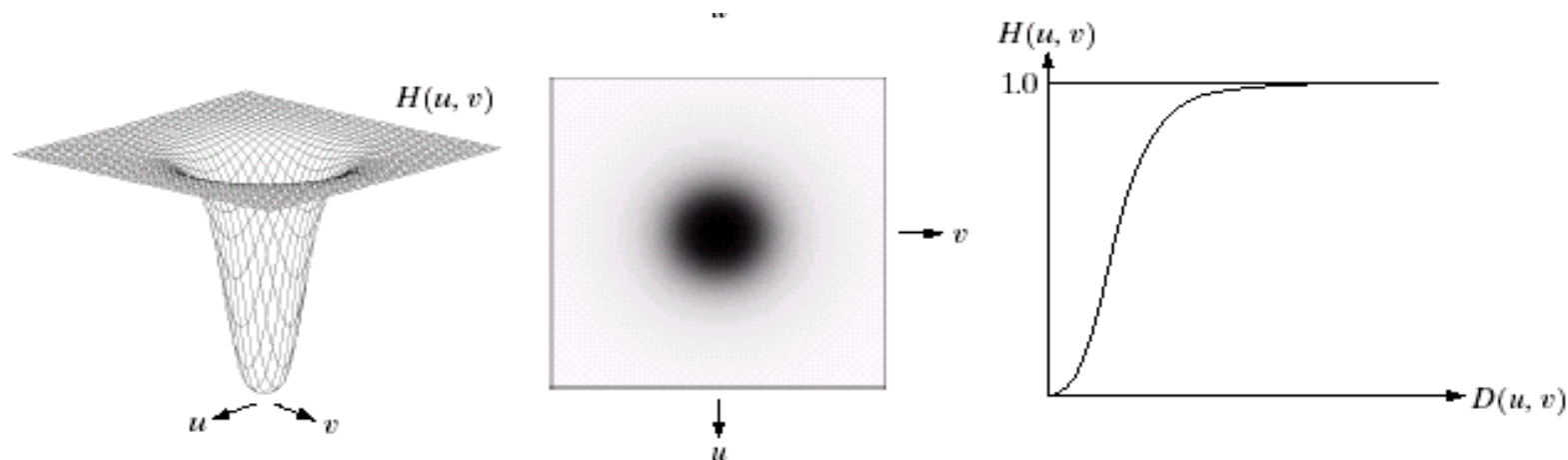
Results of ideal
high pass filtering
with $D_0 = 80$

Butterworth High Pass Filters

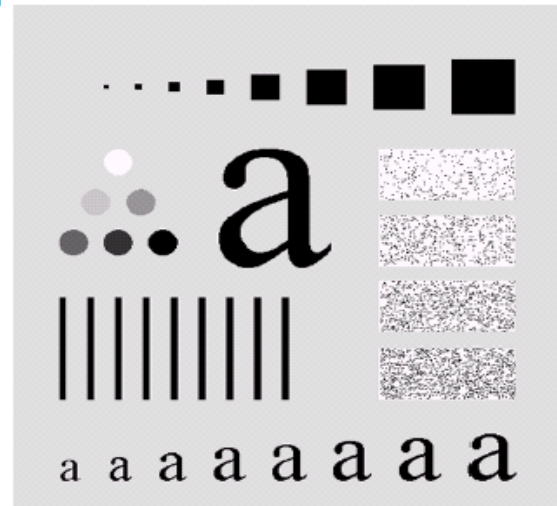
The Butterworth high pass filter is given as:

$$H(u, v) = \frac{1}{1 + [D_0 / D(u, v)]^{2n}}$$

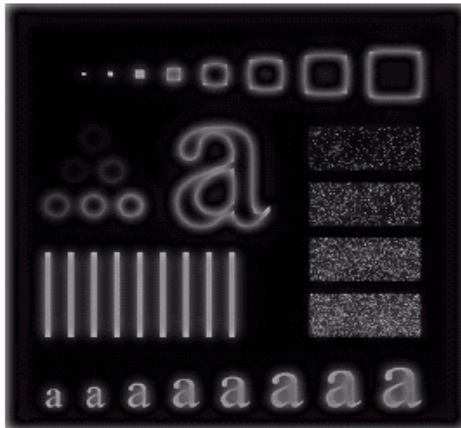
where n is the order and D_0 is the cut off distance as before



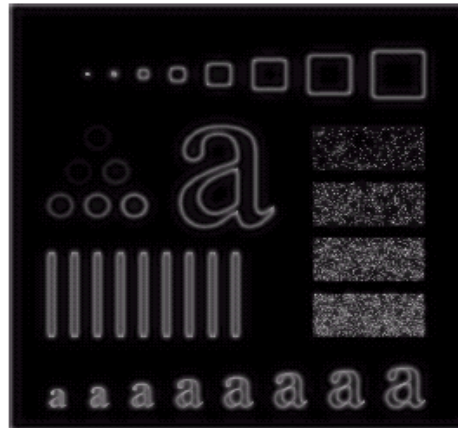
Butterworth High Pass Filters (cont...)



Results of
Butterworth
high pass
filtering of
order 2 with
 $D_0 = 15$



Results of Butterworth high pass
filtering of order 2 with $D_0 = 30$



Results of
Butterworth
high pass
filtering of
order 2 with
 $D_0 = 80$

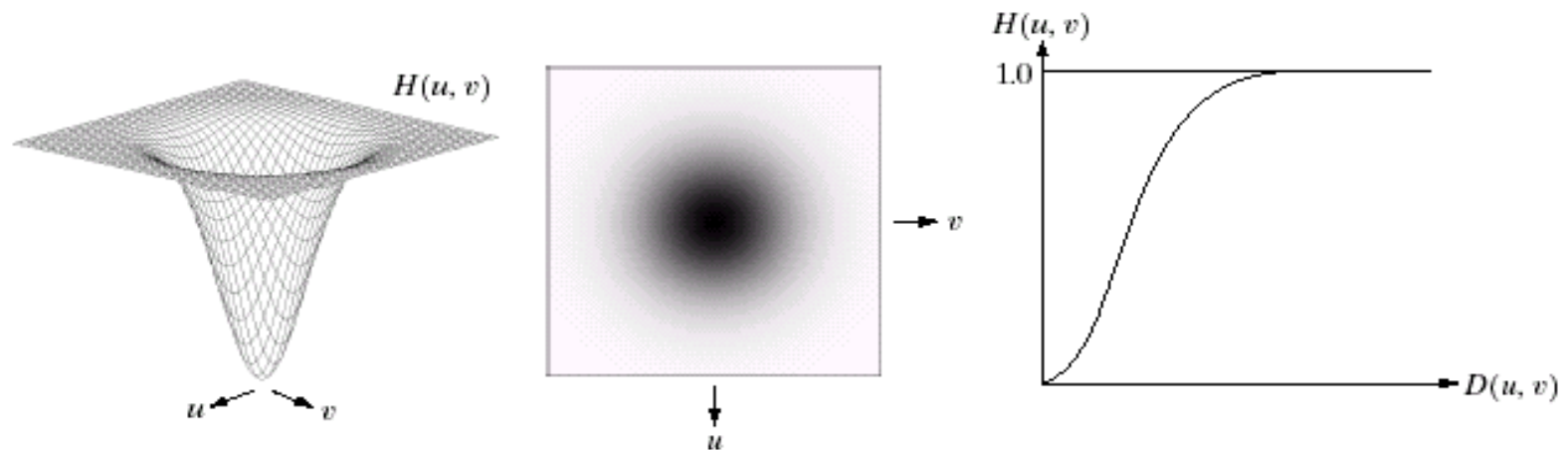


Gaussian High Pass Filters

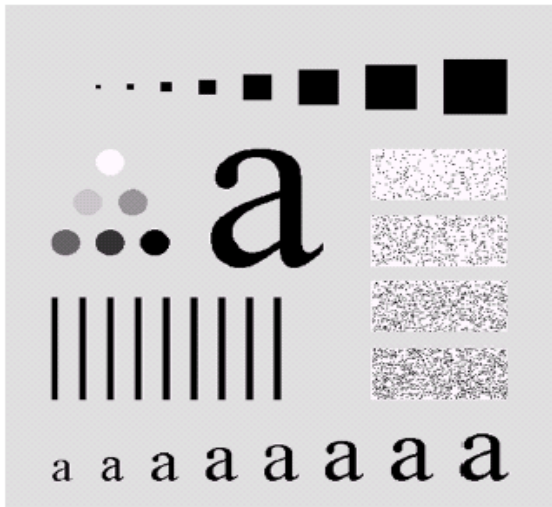
The Gaussian high pass filter is given as:

$$H(u, v) = 1 - e^{-D^2(u, v) / 2D_0^2}$$

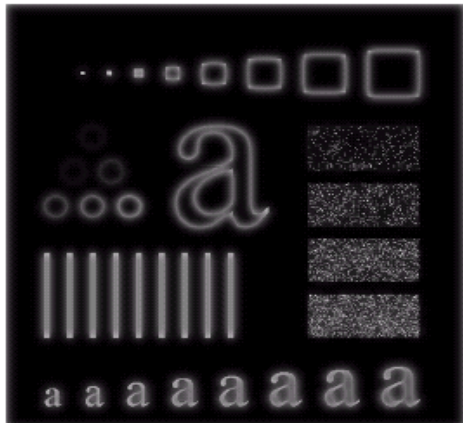
where D_0 is the cut off distance as before



Gaussian High Pass Filters (cont...)



Results of
Gaussian
high pass
filtering with
 $D_0 = 15$



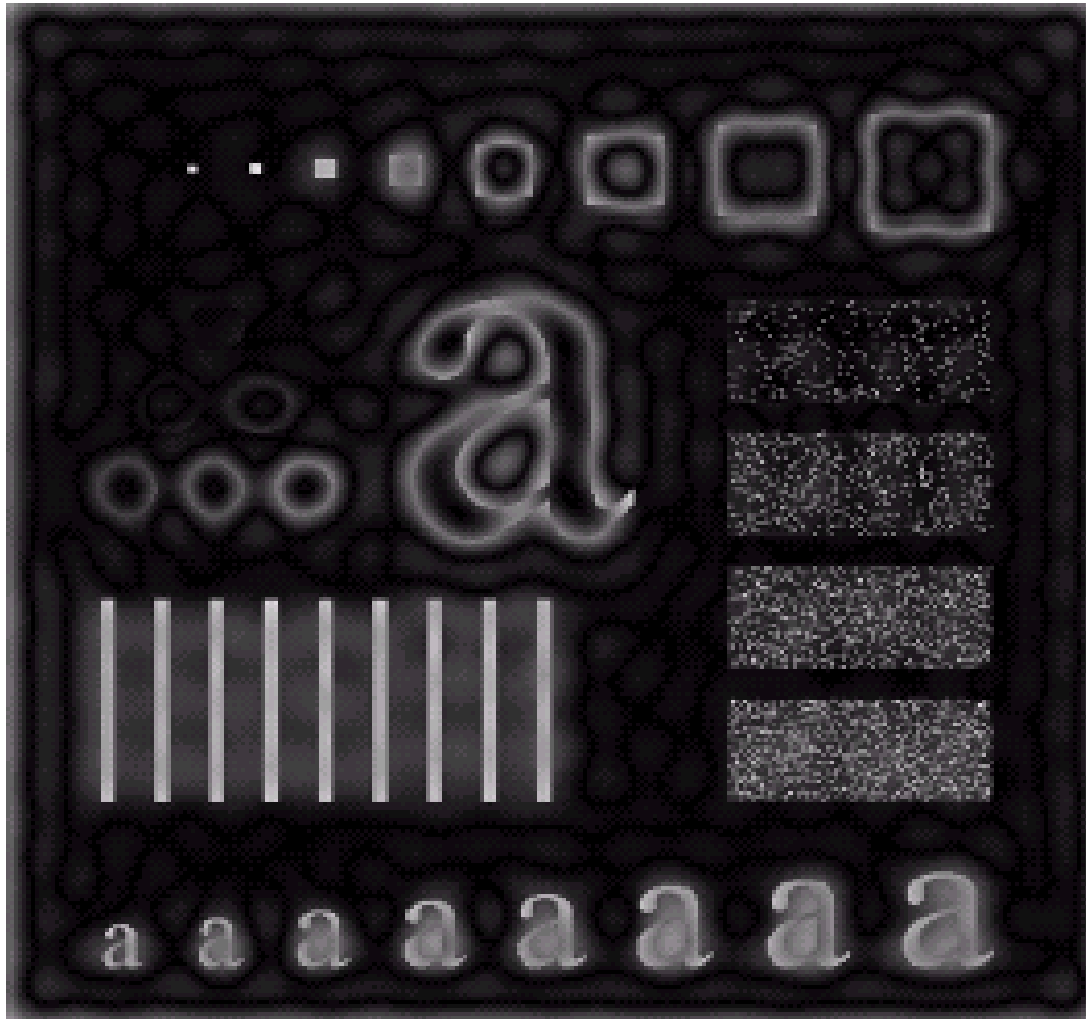
Results of Gaussian high pass
filtering with $D_0 = 30$



Results of
Gaussian
high pass
filtering with
 $D_0 = 80$

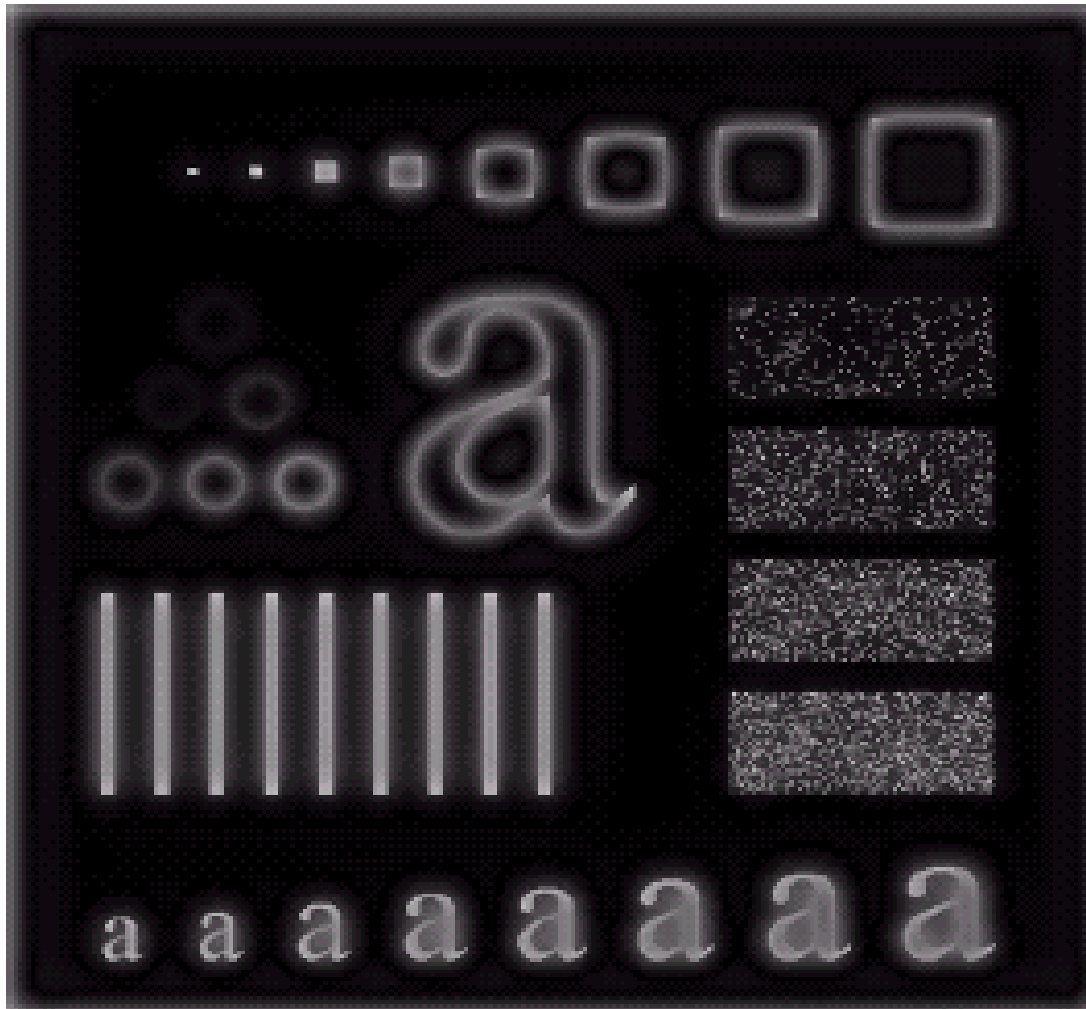


Highpass Filter Comparison



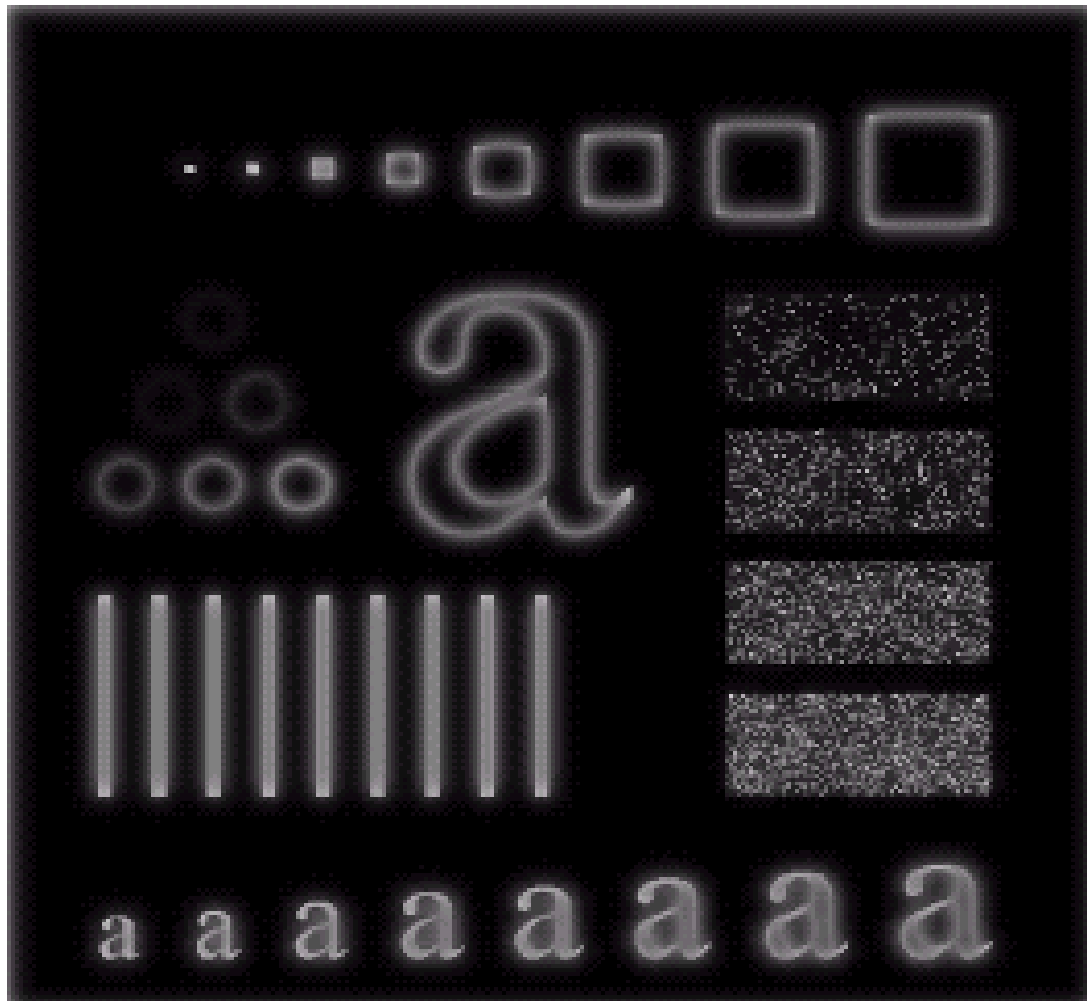
Results of ideal
high pass filtering
with $D_0 = 15$

Highpass Filter Comparison



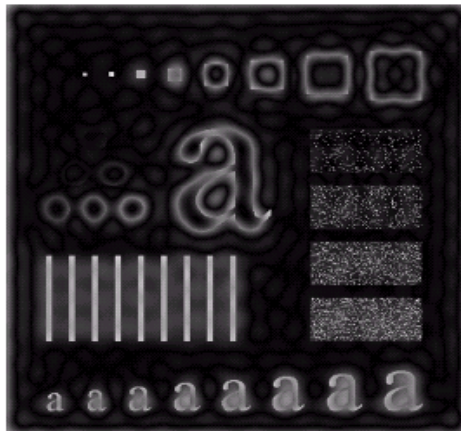
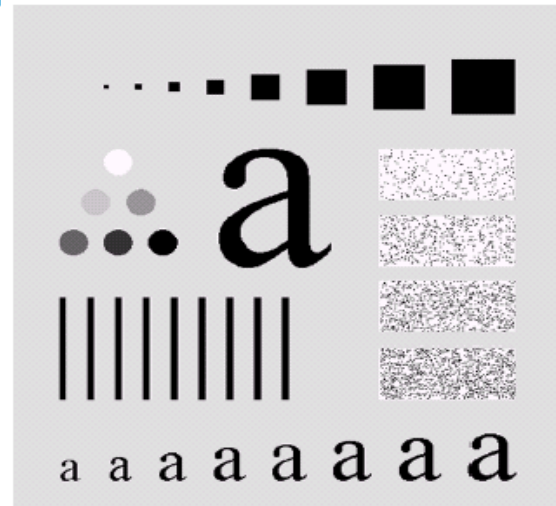
Results of Butterworth
high pass filtering of order
2 with $D_0 = 15$

Highpass Filter Comparison

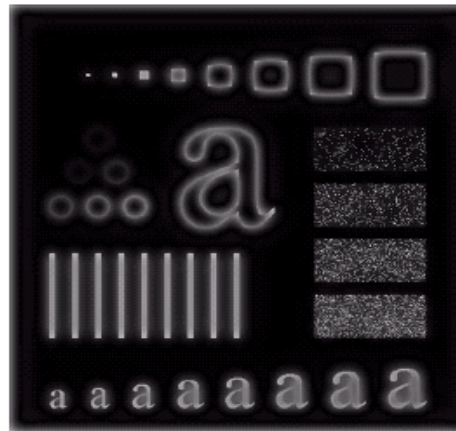


Results of Gaussian
high pass filtering with
 $D_0 = 15$

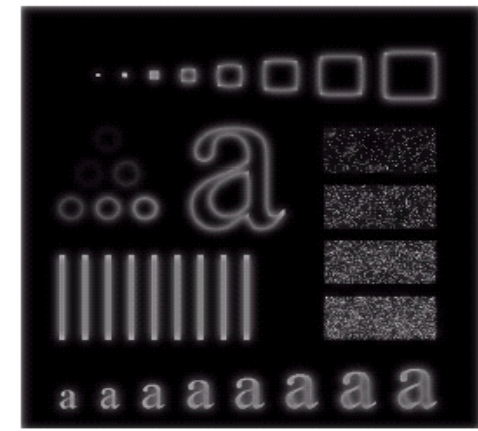
Highpass Filter Comparison



Results of ideal
high pass filtering
with $D_0 = 15$

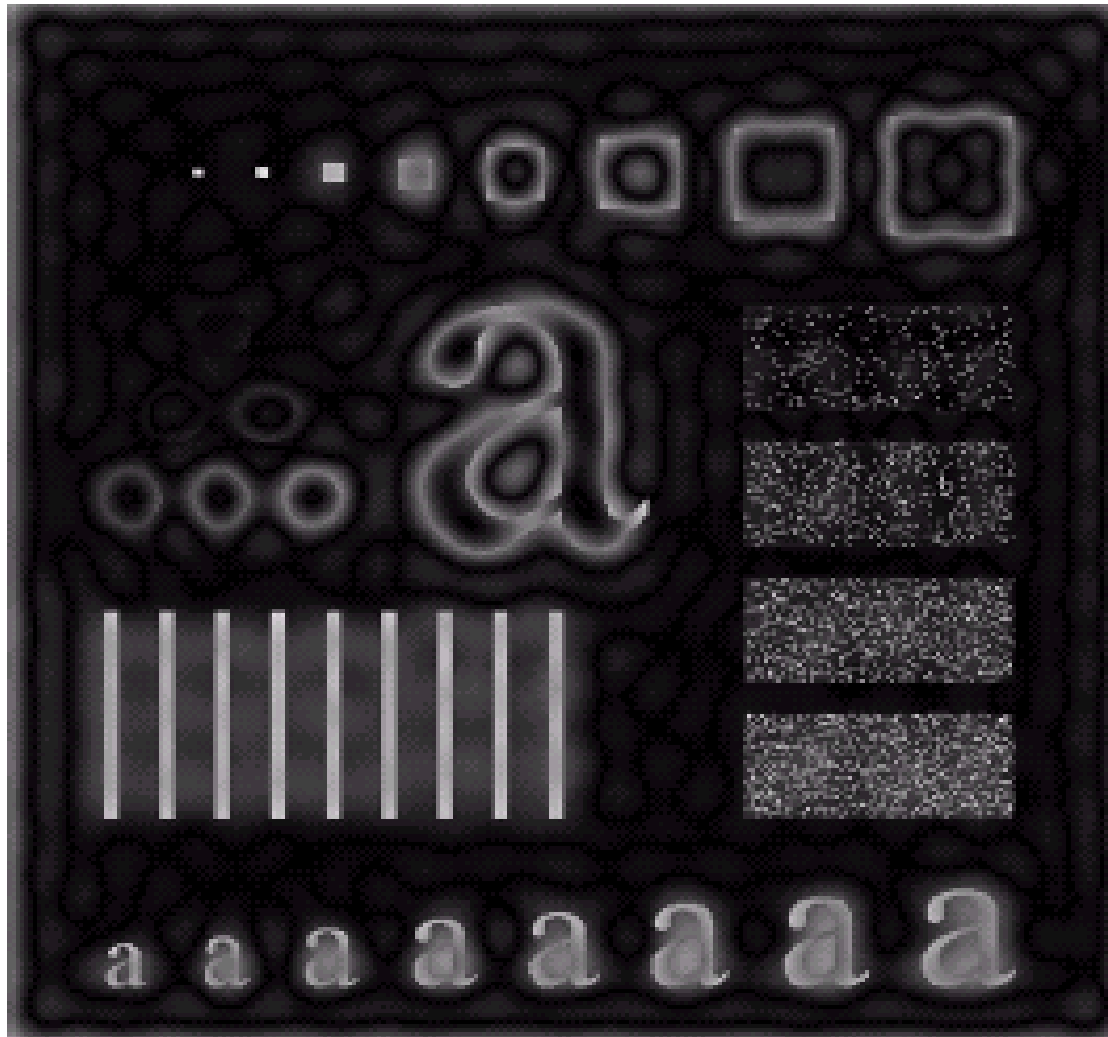


Results of Butterworth
high pass filtering of order
2 with $D_0 = 15$



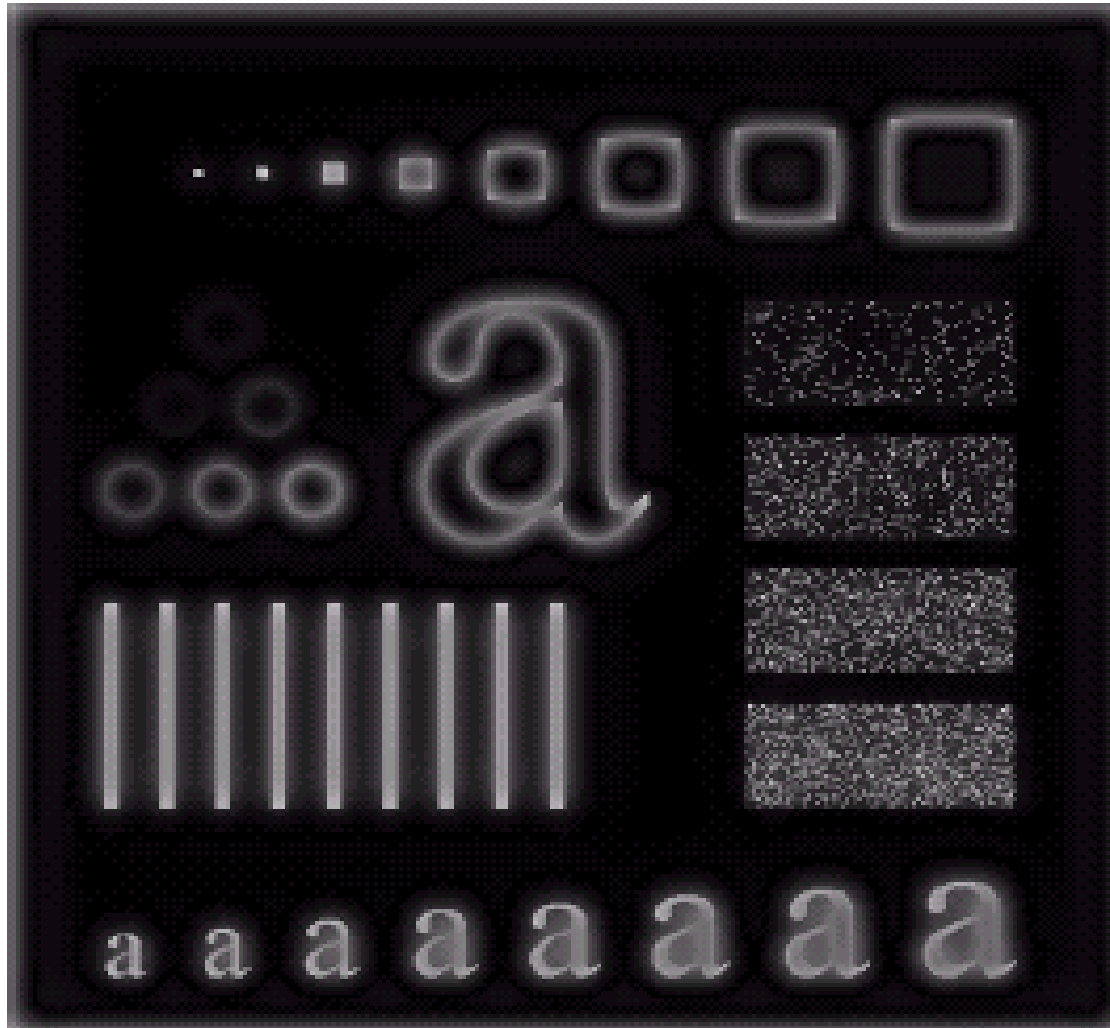
Results of Gaussian
high pass filtering with
 $D_0 = 15$

Highpass Filter Comparison



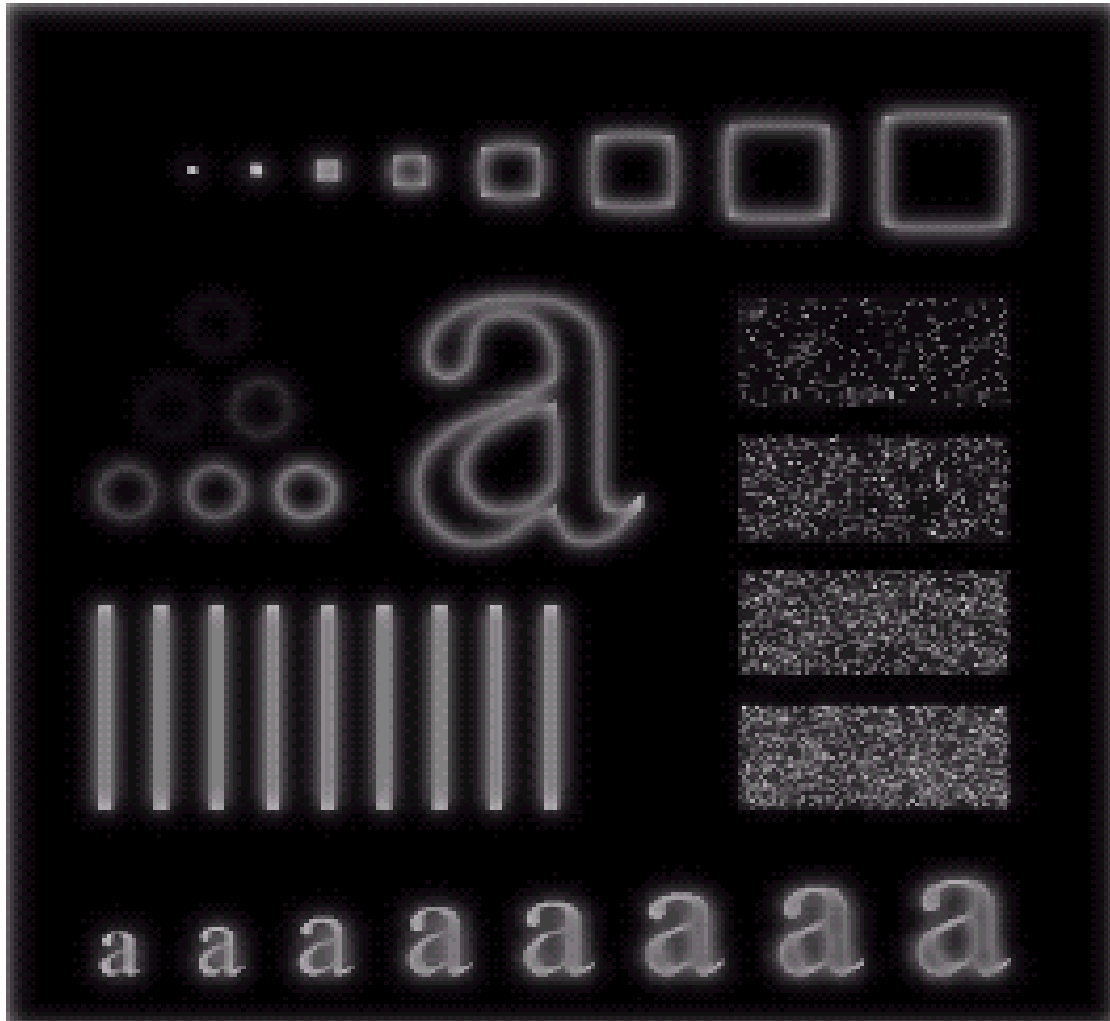
Results of ideal
high pass filtering
with $D_0 = 15$

Highpass Filter Comparison



Results of Butterworth
high pass filtering of order
2 with $D_0 = 15$

Highpass Filter Comparison



Results of Gaussian
high pass filtering with
 $D_0 = 15$

Highpass Filtering Example



Original image



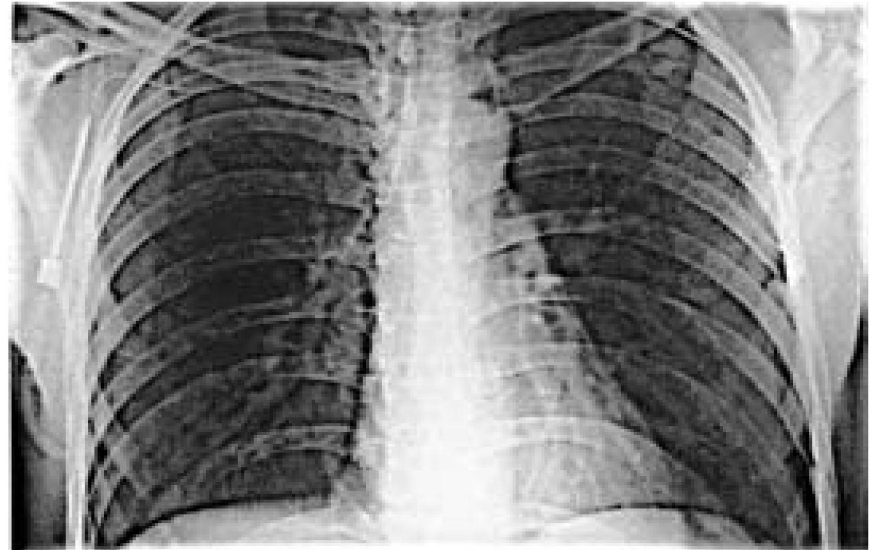
Highpass filtering result



High frequency
emphasis result

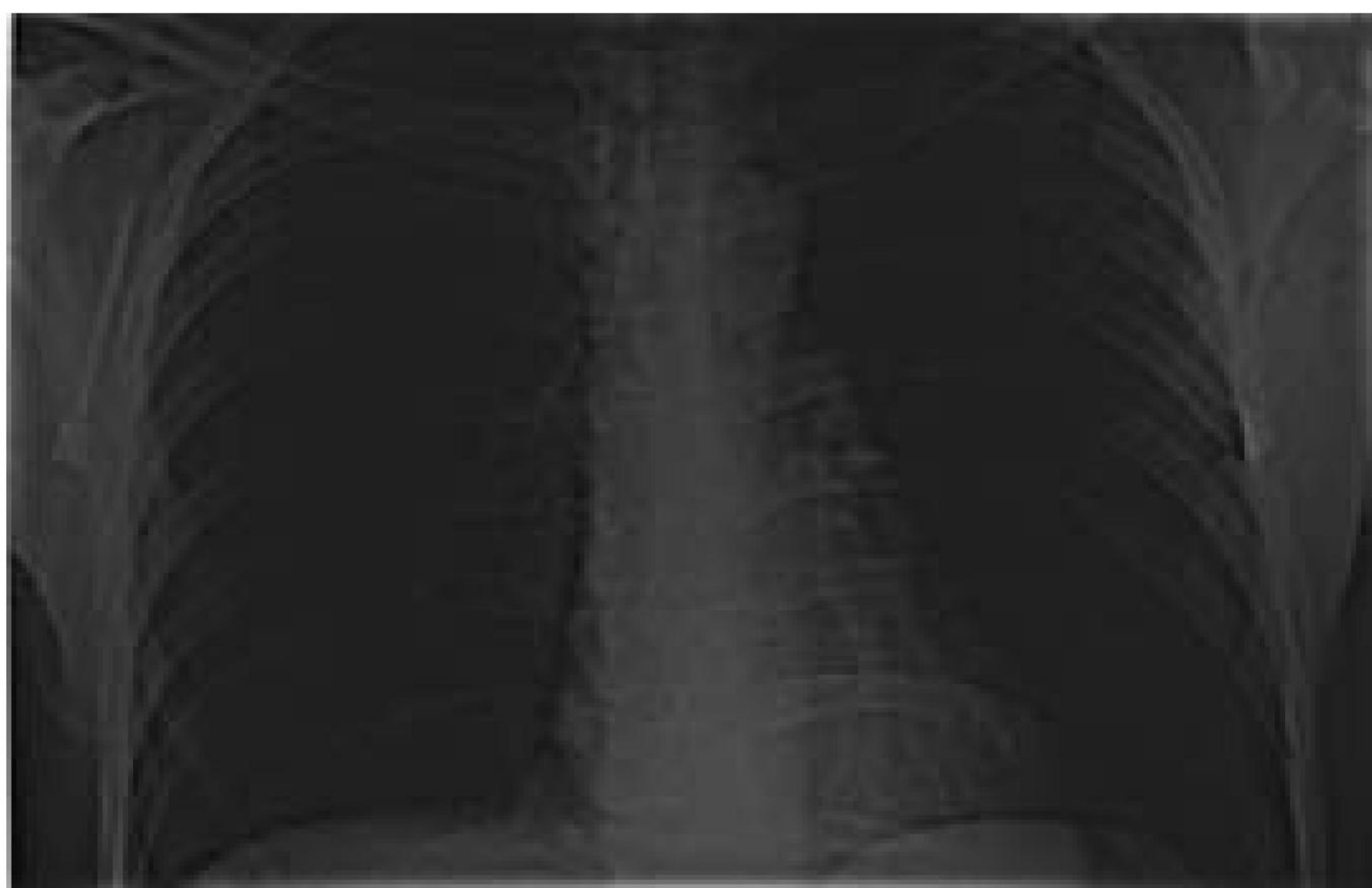


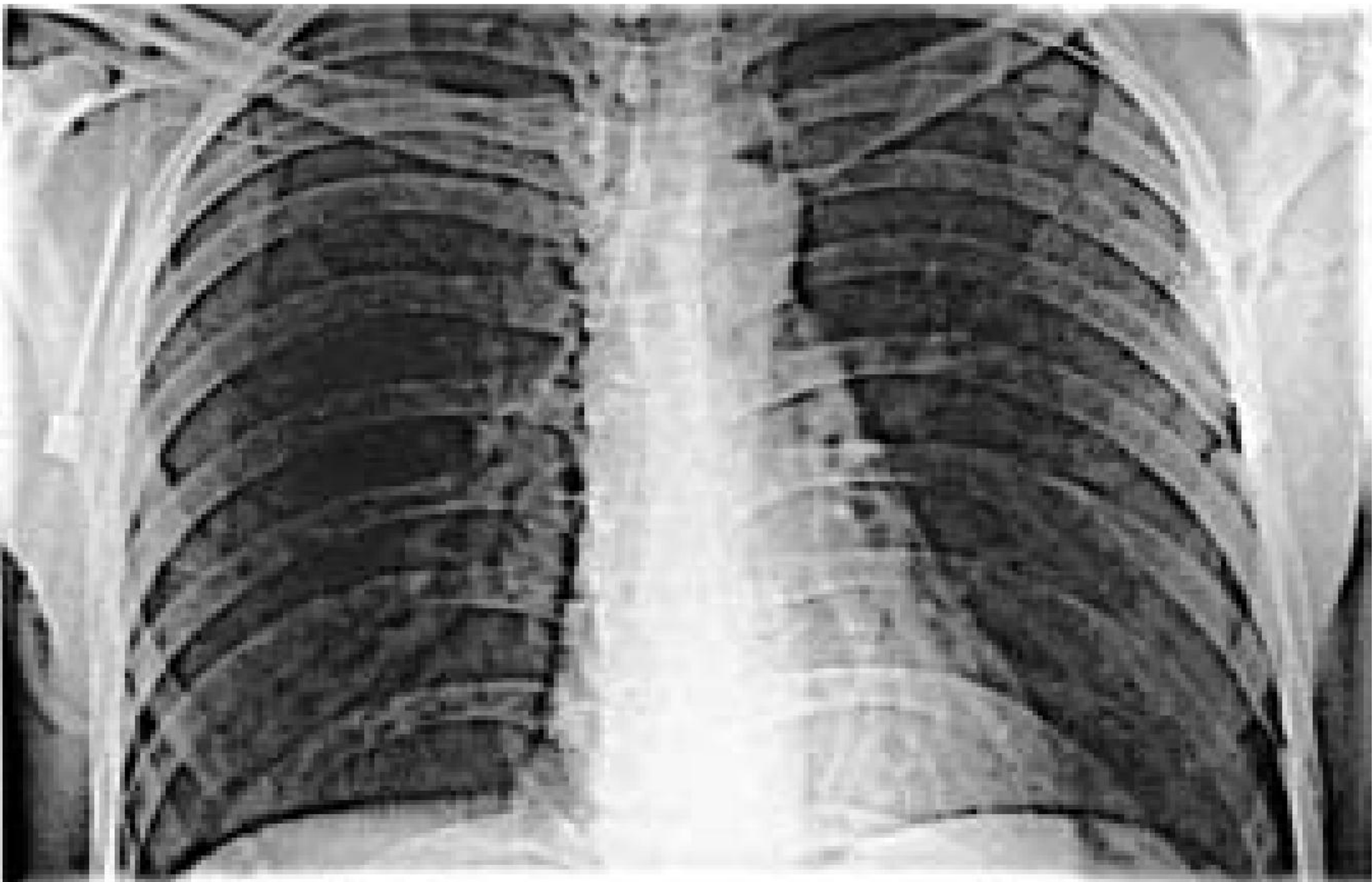
After histogram
equalisation





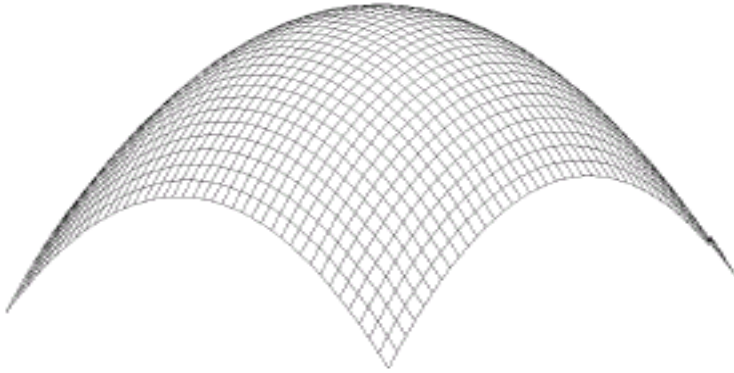






Laplacian In The Frequency Domain

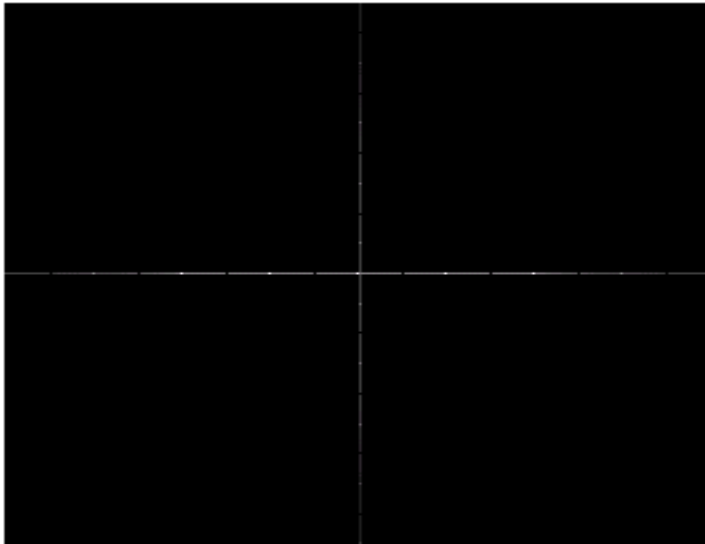
Laplacian in the frequency domain



2-D image of Laplacian in the frequency domain

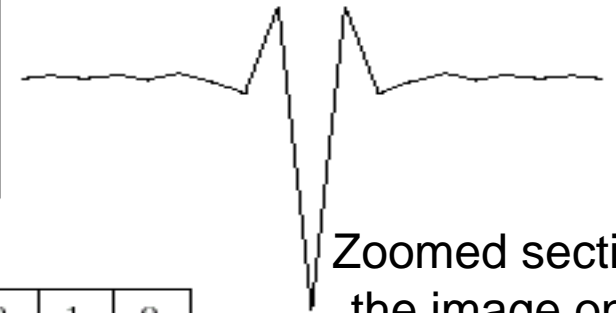


Inverse DFT of Laplacian in the frequency domain



0	1	0
1	-4	1
0	1	0

Zoomed section of the image on the left compared to spatial filter



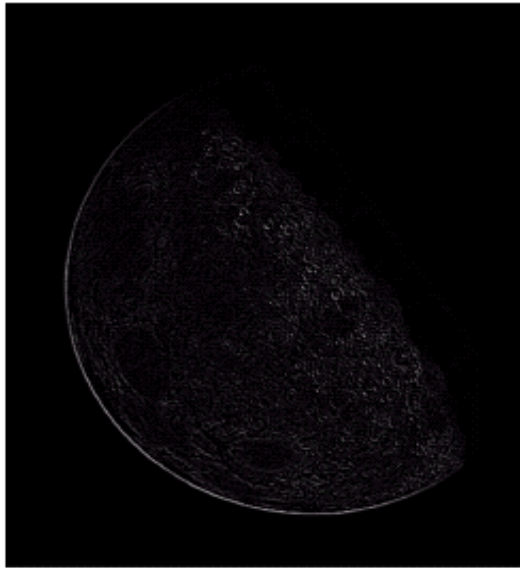
Frequency Domain Laplacian Example



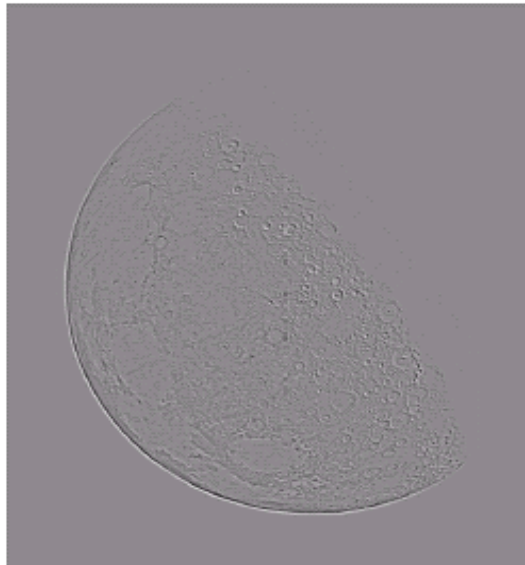
Original
image



Laplacian
filtered
image



Laplacian
image scaled



Enhanced
image



Fast Fourier Transform



The reason that Fourier based techniques have become so popular is the development of the *Fast Fourier Transform (FFT)* algorithm

Allows the Fourier transform to be carried out in a reasonable amount of time

Reduces the amount of time required to perform a Fourier transform by a factor of 100 – 600 times!

Frequency Domain Filtering & Spatial Domain Filtering



Similar jobs can be done in the spatial and frequency domains

Filtering in the spatial domain can be easier to understand

Filtering in the frequency domain can be much faster – especially for large images

Summary



In this lecture we examined image enhancement in the frequency domain

- The Fourier series & the Fourier transform
- Image Processing in the frequency domain
 - Image smoothing
 - Image sharpening
- Fast Fourier Transform

Next time we will begin to examine image restoration using the spatial and frequency based techniques we have been looking at

Interesting Application Of Frequency Domain Filtering



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Interesting Application Of Frequency Domain Filtering

