Solution to Question No. 1 part B:

B.1.1 Formation of ODE and its solution:

The Lake has initially $5 \times 10^6\ m^3$ of water with $25 \times 10^6\ kg$ of dissolved pollutants. Sewage water containing $40\frac{kg}{m^3}$ of pollutants enters the lake at $20800\frac{m^3}{hr}$, also an industry discharges waste containing $50\frac{kg}{m^3}$ of pollutants at a rate of $200\frac{m^3}{hr}$. The water leaves the lake at a rate of $21000\frac{m^3}{hr}$.

Let's assume a differential element $dQ(t)$ that is an infinitesimally small amount of pollutant at a time t.

The rate of change of amount of pollutants in the river with respect to t will be equal to the difference between the rate of amount of pollutants coming IN and the rate of amount of pollutants going OUT.

$$dQ(t) = IN.dt - OUT.dt$$

$$IN = \left(40\frac{kg}{m^3} \times 20800\frac{m^3}{hr}\right) + \left(50\frac{kg}{m^3} \times 200\frac{m^3}{hr}\right)$$

$$IN = 842000\frac{kg}{hr}$$

$$OUT = \frac{Q(t)}{5 \times 10^6}\frac{kg}{m^3} \times 21000\frac{m^3}{hr}$$

$$OUT = \frac{Q(t)}{5 \times 10^6} \times 21000\frac{kg}{hr}$$

$$dQ(t) = 842000dt - \frac{Q(t) \times 21000}{5 \times 10^6}dt$$

$$\frac{dQ(t)}{dt} = 842000 - \frac{Q(t) \times 21}{5 \times 10^3} \quad -(1)$$

This is a First Order Linear Ordinary Differential Equation of the form,

$$y' + ay = g(x)$$

Whole General solution is,

$$ye^{ax} = \int e^{ax}g(x)dx + C$$

Hence the General Solution of (1) is,

$$Q(t) \times e^{\frac{21t}{5\times10^3}} = \int 842000 \times e^{\frac{21t}{5\times10^3}}\,dt + C$$

$$Q(t) = \frac{4210}{21} \times 10^6 + C \times e^{-\frac{21t}{5\times10^3}}$$

This is an Initial Value Problem where initially the amount of pollutants in the lake is $25 \times 10^6\ kg$,

So, $Q(0) = 25 \times 10^6$

$$25 \times 10^6 = \frac{4210}{21} \times 10^6 + C$$

$$C = -\frac{3685}{21} \times 10^6$$

Hence the final Differential Equation becomes,

$$Q(t) = \frac{4210}{21} \times 10^6 - \frac{3685}{21} \times 10^6 \times e^{-\frac{21t}{5\times10^3}}$$

After 5 days or 120 hours, the amount of pollutants in the lake is,

$$Q(120) = \frac{4210}{21} \times 10^6 - \frac{3685}{21} \times 10^6 \times e^{-\frac{21\times120}{5\times10^3}}$$

$$Q(120) = 94469377.34172088\ kg$$

And the Volume of the lake can also be modelled similarly.

$$IN = 20800\frac{m^3}{hr} + 200\frac{m^3}{hr} = 21000\frac{m^3}{hr}$$

$$OUT = 21000\frac{m^3}{hr}$$

Since $IN - OUT = 0$, the amount of water present in the lake will be constant for the 5 days.

After 5 days, the pollutants are no longer coming in. and fresh water comes in at $15000\frac{m^3}{hr}$.

The Volume at a time will be $5 \times 10^6 m^3 - (21000 - 15000)\frac{m^3}{hr} \times (t - 120)hr$

Or $5 \times 10^6 - 6000 \times (t - 120)$

$$IN = 0\frac{kg}{m^3} \times 15000\frac{m^3}{hr} = 0\frac{kg}{hr}$$

$$OUT = \frac{Q(t)}{5 \times 10^6 - 6000 \times (t - 120)}\frac{kg}{m^3} \times 21000\frac{m^3}{hr} = \frac{Q(t) \times 21}{5 \times 10^3 - 6(t - 120)}\frac{kg}{hr}$$

$$\frac{dQ(t)}{dt} = -\frac{Q(t) \times 21}{5720 - 6t}$$

Which is a First Order Linear Differential Equation that can be solved by Variable separable.

$$\int \frac{dQ(t)}{Q(t)} = -\int \frac{21}{5720 - 6t} dt$$

$$\ln Q(t) = \frac{21}{6}\ln(5720 - 6t) + \ln C$$

$$Q(t) = C \times (5720 - 6t)^{\frac{21}{6}}$$

Using the Initial Value, the amount of Pollutants at $t = 120$ is $94469377.34172088\ kg$.

$$\frac{94469377.34172088}{(5720 - 6 \times 120)^{\frac{21}{6}}} = C$$

$$C = 1.068798997 \times 10^{-5}$$

Hence,

$$Q(t) = 1.068798997 \times 10^{-5} \times (5720 - 6t)^{\frac{21}{6}}$$

Now the final solution becomes

$$Q(t) = \begin{cases} \frac{4210}{21} \times 10^6 - \frac{3685}{21} \times 10^6 \times e^{-\frac{21t}{5\times 10^3}}; 0 \leq t \leq 120 \\ 1.068798997 \times 10^{-5} \times (5720 - 6t)^{\frac{21}{6}}; t \geq 120 \end{cases}$$

B.1.2 Time required for the lake to become pollutant free:

The lake will be pollutant free when $Q(t) = 0$, it cannot be zero during the first 5 days or 120 hours since the lake is being polluted at that stage continuously, hence using the second equation,

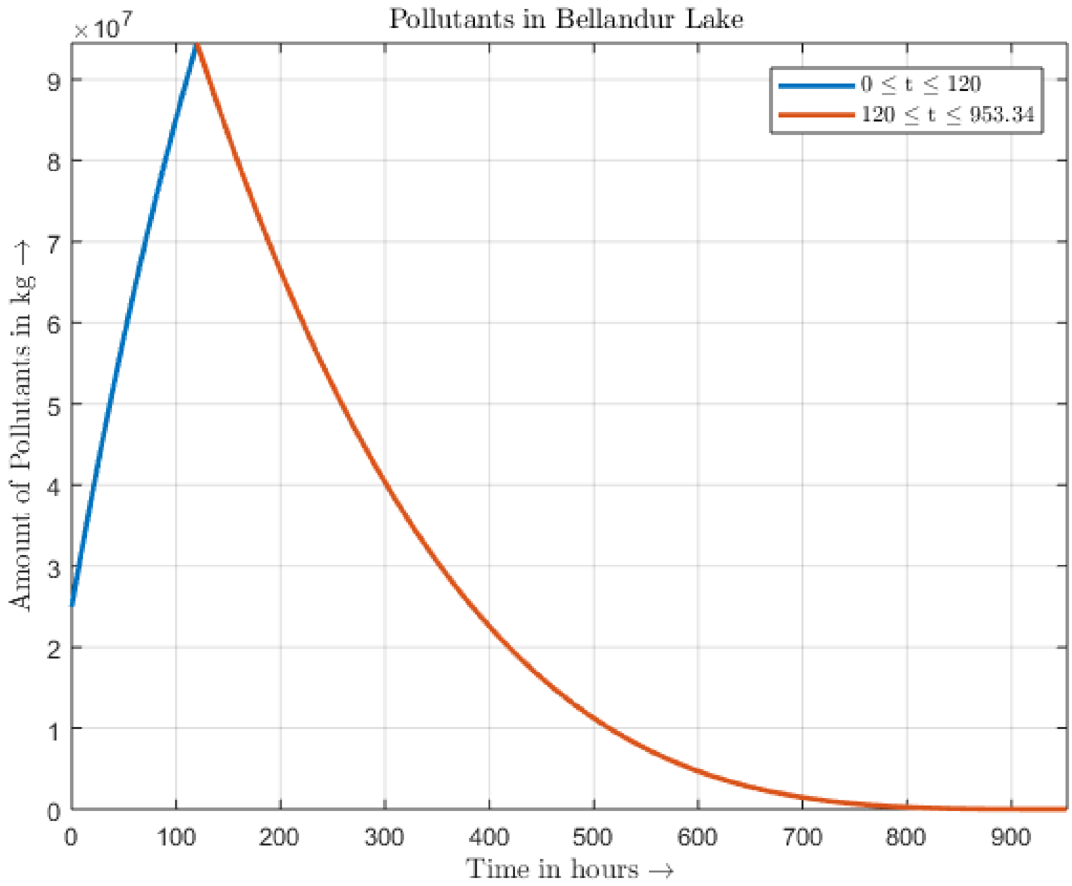
$$1.068798997 \times 10^{-5} \times (5720 - 6t)^{\frac{21}{6}} = 0$$

$$6t = 5720$$

$$t = \frac{5720}{6} = 953.333\ hrs$$

Hence the lake will becomes pollutant free after $953.33\ hours$ or $39.72\ days$, on the $40th$ day it will be completely pollutant free.

B.1.3 Plot of $Q(t)$ versus time:

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Q1 = @(x) (4210/21)*(10^6) - (3685/21)*(10^6)*exp(-21*x/(5*(10^3)));
Q2 = @(x) 1.068798997*10^(-5)*(5720-6*x)^(21/6)
fplot(Q1, [0, 120], 'LineWidth', 2);
hold on;
fplot(Q2, [120,953.34], 'LineWidth', 2);
grid on;
xlabel('Time in hours $\rightarrow$', 'Interpreter', 'latex');
ylabel('Amount of Pollutants in kg $\rightarrow$', 'Interpreter', 'latex');
title('Pollutants in Bellandur Lake $ $', 'Interpreter', 'latex');
%legend(func2str(Q1), func2str(Q2));
legend({'0 $\leq$ t $\leq$ 120','120 $\leq$ t $\leq$ 953.34'},'Interpreter',
'latex');
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B.1.4 Time required for pollutants to become thrice and one tenth of the initial quantity:

The time required for pollutants to become thrice or $75 \times 10^6 kg$,

$$75 \times 10^6 = \frac{4210}{21} \times 10^6 - \frac{3685}{21} \times 10^6 \times e^{-\frac{21t}{5\times10^3}}$$

$$\frac{4210 - 75 \times 21}{3685} = e^{-\frac{21t}{5\times10^3}}$$

$$\frac{3685}{2635} = e^{\frac{21t}{5\times10^3}}$$

$$t = \frac{5 \times 10^3}{21} \ln \frac{3685}{2635}$$

$$t = 79.854129 \ \approx 79.85 \ hours$$

The time required for pollutants to become one tenth or $2.5 \times 10^6 \ kg$,

$$2.5 \times 10^6 = 1.068798997 \times 10^{-5} \times (5720 - 6t)^{\frac{21}{6}}$$

$$5720 - 6t = \left(\frac{2.5 \times 10^{11}}{1.068798997}\right)^{\frac{6}{21}}$$

$$t = \frac{5720}{6} - \frac{1}{6}\left(\frac{2.5 \times 10^{11}}{1.068798997}\right)^{\frac{6}{21}}$$

$$t = 658.1124 \ hours$$

Hence after $658.1124 \ hours$ the pollutants will becomes $\frac{1}{10} th$ of the initial value.