Lecture5

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Lecture5

Topics of the day

lecture-5

- Introduction to proof
- (1) Direct proof
- (2) Proof by Contraposition — Indirect proof.
- Quiz

Introduction to proof

> A proof is a valid argument that establishes the truth of a mathematical statement.

A proof can use the hypothesis of the theorem, if any, axioms assumed to be true, and previously proven theorems. Using these ingredients and rules of inference, the final step of the proof establishes the truth of the statement being proved.

$p \to q$

$p \to q$

hyp

T T

Important Terminology

1. **Theorem** : A statement that can be shown to be true is called a *theorem*.

Important Terminology

Theorem : A statement that can be shown to be true is called a *theorem*.

2. **Proposition** : A less important theorem is called a *proposition*.

3. **Lemma** : A less important theorem that is helpful in the proof of other results is called a *lemma*.

Important Terminology

Theorem : A statement that can be shown to be true is called a *theorem*.

Proposition : A less important theorem is called a *proposition*.

Lemma : A less important theorem that is helpful in the proof of other results is called a *lemma*.

4 **Corollary** : A theorem that can be established directly from other theorem is called a *corollary*.

5 **Conjecture** : A statement that is being proposed to be a true statement, usually on the basis of some partial evidence, a heuristic argument, or the intuition of an expert, is called a *conjecture*.

Th^m: $\forall n \in \mathbb{N}$, $2n$ is even.

Cor:- 10 is even.

Methods of proving a theorem

In today's, lecture, we will discuss the following two methods of proof:

1 - Direct proof
2 - Proof by Contraposition — Indirect proof.

Direct proof

Direct proof : A direct proof of a conditional statement $p \to q$ is constructed when the first step is the assumption that p is true; subsequent steps are constructed using rules of inference, with the final step showing that q must also be true.

$p \to q$

p	q
T	T
F	T
F	F

$p \to q$

T → T

$p \to q$ T

Direct proof

Direct proof : A direct proof of a conditional statement $p \to q$ is constructed when the first step is the assumption that p is true; subsequent steps are constructed using rules of inference, with the final step showing that q must also be true.

A direct proof shows that the conditional statement $p \to q$ is true by showing that if p is true, then q must also be true. In direct proof, the combination p true and q false never occurs.

$p \to q$ F

T F

More about direct proof

Before we give the example of direct proof, we need to define some terminology.

Definition : The integer n is even if there exists an integer k such that $n = 2k$, and n is odd if there exists an integer k such that $n = 2k + 1$. Every integer is either even or odd, and no integer is both even and odd.

$n = 2k, \ k \in \mathbb{Z}.$

$n = 2k+1, \ k \in \mathbb{Z}.$

More about direct proof

Before we give the example of direct proof, we need to define some terminology.

Definition : The integer n is even if there exists an integer k such that $n = 2k$, and n is odd if there exists an integer k such that $n = 2k + 1$. Every integer is either even or odd, and no integer is both even and odd.

Two integers have the same *parity* when both are even or both are odd, they have opposite *parity* if one is even and the other is odd.

3, 5 same parity.

8, 14 same parity

8, 9 opp.

10, 11 opp.

Example of Direct proof

$$\underset{T}{p} \longrightarrow \underset{T}{q}.$$

Theorem : "If n is an odd integer, then n^2 is odd."

Proof :- p - n is odd, q - n^2 is odd.

Let p be True $\Rightarrow n$ is odd $\Rightarrow n = 2k+1, \ k \in \mathbb{Z}.$

Since $k \in \mathbb{Z}$.

so $2k^2 + 2k \in \mathbb{Z}$

$\Rightarrow p \in \mathbb{Z}.$

$p \to q$ T

Proof :- p - n is odd, q - n^2 is odd.

Let p be True $\Rightarrow$ n is odd $\Rightarrow$ $n = 2k+1$, $k \in \mathbb{Z}$.

$\therefore$ $n^2 = (2k+1)^2 = 4k^2+4k+1$

$= 2(2k^2+2k)+1$

$= 2p+1$, $p \in \mathbb{Z}$ $\Rightarrow$ n^2 is odd

$\Rightarrow q$ is True

$p \rightarrow q$ T

T T

Example of Direct proof

$\forall n(P(n) \rightarrow Q(n))$

Theorem : "If n is an odd integer, then n^2 is odd."

Proof : This theorem states that : $\forall nP((n) \rightarrow Q(n))$, where $P(n)$ is "n is an odd integer" and $Q(n)$ is "n^2 is odd". By the definition of odd integer, let $n = 2k + 1$, where k is some integer. So, we get $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 = 2m + 1$, where $m = 2k^2 + 2k$, an integer. Thus, by the definition of an odd integer, we can conclude that n^2 is an odd integer.

Quiz 1

$\forall n\ (P(n) \to Q(n))$

$p \longrightarrow q$

Let the statement be "If n is not an odd integer then square of n is not odd." Then, if $P(n)$ is "n is an not an odd integer" and $Q(n)$ is "(square of n) is not odd." For direct proof we should prove

$\forall n(\sim P(n) \to \sim Q(n))$

A. $\forall nP((n) \to Q(n))$
B. $\exists nP((n) \to Q(n))$
C. $\forall n \sim (P(n) \to Q(n))$ $\longleftrightarrow$ $\forall n\ (P(n) \to Q(n))$
D. $\forall nP((n) \to \sim (Q(n)))$

Quiz 1

Let the statement be "If n is not an odd integer then square of n is not odd." Then, if $P(n)$ is "n is an not an odd integer" and $Q(n)$ is "(square of n) is not odd." For direct proof we should prove

A. $\forall nP((n) \to Q(n))$
B. $\exists nP((n) \to Q(n))$
C. $\forall n \sim (P(n) \to Q(n))$
D. $\forall nP((n) \to \sim (Q(n)))$

Answer : A. From the definition of direct proof.

Quiz 2

Which of the following can only be used in disproving the statements?

A. Direct proof

B. Contrapositive proofs

C. Counter Example

D. Mathematical Induction

Quiz 2

Which of the following can only be used in disproving the statements?

A. Direct proof

B. Contrapositive proofs

C. Counter Example

D. Mathematical Induction

Answer : C

Proof by Contraposition Indirect method

$\boxed{p \to q} \equiv \sim q \to \sim p$

This is an extremely useful type of indirect proof. This method make use of the fact that the conditional statement $p \to q$ is equivalent to its contrapositive, $\sim q \to \sim p$.

Converse:- $q \to p$.

Inverse:- $\sim p \to \sim q$

Contrapositive:- $\sim q \to \sim p$

p	q	$p \to q$	$q \to p$	$\sim p$	$\sim q$	$\sim p \to \sim q$	$\sim q \to \sim p$
T	T	T	T	F	F	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Proof by Contraposition

This is an extremely useful type of indirect proof. This method make use of the fact that the conditional statement $p \to q$ is equivalent to its contrapositive, $\sim q \to \sim p$.

That is, the conditional statement $p \to q$ can be proved by showing that its contrapositive $\sim q \to \sim p$ is true.

Example of Proof by Contraposition

p q

Theorem : If n is an integer and $3n+2$ is odd, then n is odd.

3n+2 is odd

↳ n is odd.

T

Proof :- $p \to q \equiv \sim q \to \sim p$. T

$\underbrace{n \text{ is even} \to 3n+2 \text{ is even}}$

Let n be even $\Rightarrow n = 2k,\ k \in \mathbb{Z}$.

$\Rightarrow 3n+2 = 3 \times 2k + 2$

$= 6k+2 = 2(3k+1) = 2p,\ p \in \mathbb{Z}$.

$\Rightarrow 3n+2$ is even.

Example of Proof by Contraposition

Theorem : If n is an integer and $3n+2$ is odd, then n is odd.

Proof: The first step is assuming that the conclusion of the given conditional statement is false, i.e., n is even. Then, by the definition of even integers, we have $n = 2k$, for any integer k. After this, $3n+2 = 3 \times 2k + 2 = 6k + 2 = 2(3k+1) = 2m$, where $m = 3k+1$, an integer. This tells us that $3n+2$ is also even.

Example of Proof by Contraposition

Theorem : If n is an integer and $3n+2$ is odd, then n is odd.

Proved

Proof: The first step is assuming that the conclusion of the given conditional statement is false, i.e., n is even. Then, by the definition of even integers, we have $n = 2k$, for any integer k. After this, $3n+2 = 3 \times 2k + 2 = 6k + 2 = 2(3k+1) = 2m$, where $m = 3k+1$, an integer. This tells us that $3n+2$ is also even.

Thus, if n is even, then $3n+2$ is also even. Hence, by the method of Contraposition, we can conclude that if $3n+2$ is odd, then n is odd. This proves the given theorem.

Quiz 3

$\sim P(n) \rightarrow Q(n)$

$\forall n(\sim Q(n) \rightarrow \sim P(n))$

Let the statement be "If n is not an odd integer then sum of n with some not odd number will not be odd.". Then, if $P(n)$ is "n is an not an odd integer" and $Q(n)$ is "sum of n with some not odd number will not be odd." A proof by contraposition will be

A. $\forall n(P(n) \rightarrow Q(n))$

B. $\exists n P((n) \rightarrow Q(n))$

C. $\forall n \sim (P(n) \rightarrow Q(n))$

D. $\forall n(\sim (Q((n)) \rightarrow \sim (P(n)))$

Quiz 3

Let the statement be "If n is not an odd integer then sum of n with some not odd number will not be odd.". Then, if $P(n)$ is "n is an not an odd integer" and $Q(n)$ is "sum of n with some not odd number will not be odd." A proof by contraposition will be

A. $\forall n(P(n) \rightarrow Q(n))$
B. $\exists nP((n) \rightarrow Q(n))$
C. $\forall n \sim (P(n) \rightarrow Q(n))$
D. $\forall n(\sim (Q((n)) \rightarrow \sim (P(n)))$

Answer : D. From the definition of proof by Contraposition.

Quiz 4

A theorem used to prove other theorems is known as

A. Lemma
B. Corollary
C. Conjecture
D. None of the above.

Quiz 4

A theorem used to prove other theorems is known as

A. Lemma

B. Corollary

C. Conjecture

D. None of the above.

Answer : A. From the definition of a Lemma.

Quiz 5

$p \longrightarrow q$

Consider the statement, "If *n* is divisible by 30 then *n* is divisible by 2 and by 3 and by 5." Which of the following statements is equivalent to this statement?

&

$q \equiv a \wedge b \wedge c$

$\sim q \equiv \sim a \vee \sim b \vee \sim c$

or

$p \to q \equiv \sim q \longrightarrow \sim p$

$\not\equiv q \to p$

$\not\equiv \sim p \to \sim q.$

~~A.~~ If *n* is not divisible by 30 then *n* is divisible by 2 or divisible by 3 or divisible by 5.

$\sim p \to \sim q$ ~~B.~~ If *n* is not divisible by 30 then *n* is not divisible by 2 or not divisible by 3 or not divisible by 5.

$q \to p$ ~~C.~~ If *n* is divisible by 2 and divisible by 3 and divisible by 5 then *n* is divisible by 30.

$\sim q \to \sim p$ D. If *n* is not divisible by 2 or not divisible by 3 or not divisible by 5 then *n* is not divisible by 30.

Quiz 5

Thankyou all . . .

Consider the statement, "If *n* is divisible by 30 then *n* is divisible by 2 and by 3 and by 5." Which of the following statements is equivalent to this statement?

A. If *n* is not divisible by 30 then *n* is divisible by 2 or divisible by 3 or divisible by 5.

B. If *n* is not divisible by 30 then *n* is not divisible by 2 or not divisible by

Quiz 5

Thankyou all...

Consider the statement, “If n is divisible by 30 then n is divisible by 2 and by 3 and by 5.” Which of the following statements is equivalent to this statement?

A. If n is not divisible by 30 then n is divisible by 2 or divisible by 3 or divisible by 5.
B. If n is not divisible by 30 then n is not divisible by 2 or not divisible by 3 or not divisible by 5.
C. If n is divisible by 2 and divisible by 3 and divisible by 5 then n is divisible by 30.
D. If n is not divisible by 2 or not divisible by 3 or not divisible by 5 then n is not divisible by 30.

Answer : D. $p \to q \equiv \sim q \to \sim p$.