

# **CHAPTER 11**

## **LINEAR DIFFERENTIAL EQUATIONS OF SECOND AND HIGHER ORDER**

### **11.1 Introduction**

A differential equation of the form  $F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ny}{dx^n}\right) = 0$  in which the dependent variable  $y(x)$  and its derivatives viz.  $\frac{dy}{dx}$ ,  $\frac{d^2y}{dx^2}$  etc occur in first degree and are not multiplied together is called a Linear Differential Equation.

### **11.2 Linear Differential Equations (LDE) with Constant Coefficients**

A general linear differential equation of  $n^{\text{th}}$  order with constant coefficients is given by:

$$k_0 \frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + k_{n-1} \frac{dy}{dx} + k_n y = F(x)$$

where  $k$ 's are constant and  $F(x)$  is a function of  $x$  alone or constant.

$$\Rightarrow (k_0 D^n + k_1 D^{n-1} + \dots + k_{n-1} D + k_n) y = F(x)$$

Or  $f(D)y = F(x)$ , where  $D^n \equiv \frac{d^n}{dx^n}$ ,  $D^{n-1} \equiv \frac{d^{n-1}}{dx^{n-1}}$ , ...,  $D \equiv \frac{d}{dx}$  are called differential operators.

### **11.3 Solving Linear Differential Equations with Constant Coefficients**

Complete solution of equation  $f(D)y = F(x)$  is given by  $y = \text{C.F} + \text{P.I.}$

where C.F. denotes complimentary function and P.I. is particular integral.

When  $F(x) = 0$ , then solution of equation  $f(D)y = 0$  is given by  $y = \text{C.F}$

#### **11.3.1 Rules for Finding Complimentary Function (C.F.)**

Consider the equation  $f(D)y = F(x)$

$$\Rightarrow (k_0 D^n + k_1 D^{n-1} + \dots + k_{n-1} D + k_n) y = F(x)$$

Step 1: Put  $D = m$ , auxiliary equation (A.E) is given by  $f(m) = 0$

$$\Rightarrow k_0 m^n + k_1 m^{n-1} + \dots + k_{n-1} m + k_n = 0 \dots \dots \dots \textcircled{3}$$

Step 2: Solve the auxiliary equation given by  $\textcircled{3}$

- I. If the  $n$  roots of A.E. are real and distinct say  $m_1, m_2, \dots, m_n$   
 C.F. =  $c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$
- II. If two or more roots are equal i.e.  $m_1 = m_2 = \dots = m_k, k \leq n$   
 C.F. =  $(c_1 + c_2 x + c_3 x^2 + \dots + c_k x^{k-1}) e^{m_1 x} + \dots + c_n e^{m_n x}$
- III. If A.E. has a pair of imaginary roots i.e.  $m_1 = \alpha + i\beta, m_2 = \alpha - i\beta$   
 C.F. =  $e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x) + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$
- IV. If 2 pairs of imaginary roots are equal i.e.  $m_1 = m_2 = \alpha + i\beta, m_3 = m_4 = \alpha - i\beta$   
 C.F. =  $e^{\alpha x} [(c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x] + \dots + c_n e^{m_n x}$

**Example 1** Solve the differential equation:  $\frac{d^2y}{dx^2} - 8 \frac{dy}{dx} + 15y = 0$

Solution:  $\Rightarrow (D^2 - 8D + 15)y = 0$

Auxiliary equation is:  $m^2 - 8m + 15 = 0$

$$\Rightarrow (m - 3)(m - 5) = 0$$

$$\Rightarrow m = 3, 5$$

$$\text{C.F.} = c_1 e^{3x} + c_2 e^{5x}$$

Since  $F(x) = 0$ , solution is given by  $y = \text{C.F.}$

$$\Rightarrow y = c_1 e^{3x} + c_2 e^{5x}$$

**Example 2** Solve the differential equation:  $\frac{d^3y}{dx^3} - 6 \frac{d^2y}{dx^2} + 11 \frac{dy}{dx} - 6y = 0$

Solution:  $\Rightarrow (D^3 - 6D^2 + 11D - 6)y = 0$

Auxiliary equation is:  $m^3 - 6m^2 + 11m - 6 = 0 \dots \text{①}$

By hit and trial  $(m - 2)$  is a factor of ①

$\therefore$  ① May be rewritten as

$$m^3 - 2m^2 - 4m^2 + 8m + 3m - 6 = 0$$

$$\Rightarrow m^2(m - 2) - 4m(m - 2) + 3(m - 2) = 0$$

$$\Rightarrow (m^2 - 4m + 3)(m - 2) = 0$$

$$\Rightarrow (m - 3)(m - 1)(m - 2) = 0$$

$$\Rightarrow m = 1, 2, 3$$

$$\text{C.F.} = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$$

Since  $F(x) = 0$ , solution is given by  $y = \text{C.F}$

$$\Rightarrow y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$$

**Example 3** Solve  $(D^4 - 10D^3 + 35D^2 - 50D + 24)y = 0$

Solution: Auxiliary equation is:

$$m^4 - 10m^3 + 35m^2 - 50m + 24 = 0 \quad \dots\dots \textcircled{1}$$

By hit and trial  $(m - 1)$  is a factor of  $\textcircled{1}$

$\therefore \textcircled{1}$  May be rewritten as

$$m^4 - m^3 - 9m^3 + 9m^2 + 26m^2 - 26m - 24m + 24 = 0$$

$$\Rightarrow m^3(m - 1) - 9m^2(m - 1) + 26m(m - 1) - 24(m - 1) = 0$$

$$\Rightarrow (m - 1)(m^3 - 9m^2 + 26m - 24) = 0 \quad \dots\dots \textcircled{2}$$

By hit and trial  $(m - 2)$  is a factor of  $\textcircled{2}$

$\therefore \textcircled{2}$  May be rewritten as

$$(m - 1)(m^3 - 2m^2 - 7m^2 + 14m + 12m - 24) = 0$$

$$\Rightarrow (m - 1)[m^2(m - 2) - 7m(m - 2) + 12(m - 2)] = 0$$

$$\Rightarrow (m - 1)(m^2 - 7m + 12)(m - 2) = 0$$

$$\Rightarrow (m - 1)(m - 3)(m - 4)(m - 2) = 0$$

$$\Rightarrow m = 1, 2, 3, 4$$

$$\text{C.F.} = c_1 e^x + c_2 e^{2x} + c_3 e^{3x} + c_4 e^{4x}$$

Since  $F(x) = 0$ , solution is given by  $y = \text{C.F}$

$$\Rightarrow y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x} + c_4 e^{4x}$$

**Example 4** Solve the differential equation:  $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

Solution:  $\Rightarrow (D^3 + 2D^2 + D)y = 0$

Auxiliary equation is:  $m^3 + 2m^2 + m = 0$

$$\Rightarrow m(m^2 + 2m + 1) = 0$$

$$\Rightarrow m(m+1)^2 = 0$$

$$\Rightarrow m = 0, -1, -1$$

$$\text{C.F.} = c_1 + (c_2 + c_3 x)e^{-x}$$

Since  $F(x) = 0$ , solution is given by  $y = C.F$

$$\Rightarrow y = c_1 + (c_2 + c_3 x)e^{-x}$$

**Example 5** Solve the differential equation:  $\frac{d^4y}{dx^4} - 2\frac{d^2y}{dx^2} + y = 0$

$$\text{Solution: } \Rightarrow (D^4 - 2D^2 + 1)y = 0$$

Auxiliary equation is:  $m^4 - 2m^2 + 1 = 0$

$$\Rightarrow (m^2 - 1)^2 = 0$$

$$\Rightarrow (m+1)^2(m-1)^2 = 0$$

$$\Rightarrow m = -1, -1, 1, 1$$

$$\text{C.F.} = (c_1 + c_2 x)e^{-x} + (c_3 + c_4 x)e^x$$

Since  $F(x) = 0$ , solution is given by  $y = C.F$

$$\Rightarrow y = (c_1 + c_2x)e^{-x} + (c_3 + c_4x)e^x$$

**Example 6** Solve the differential equation:  $\frac{d^3y}{dx^3} - 2\frac{dy}{dx} + 4y = 0$

$$\text{Solution: } \Rightarrow (D^3 - 2D + 4)y = 0$$

Auxiliary equation is:  $m^3 - 2m + 4 = 0$  .....①

By hit and trial  $(m + 2)$  is a factor of ①

∴ ① May be rewritten as

$$m^3 + 2m^2 - 2m^2 - 4m + 2m + 4 = 0$$

$$\Rightarrow m^2(m+2) - 2m(m+2) + 2(m+2) = 0$$

$$\Rightarrow (m+2)(m^2 - 2m + 2) = 0$$

$$\Rightarrow m = -2, 1 \pm i$$

$$\text{C.F.} = c_1 e^{-2x} + e^x (c_2 \cos x + c_3 \sin x)$$

Since  $F(x) = 0$ , solution is given by  $y = C.F$

$$\Rightarrow y = c_1 e^{-2x} + e^x (c_2 \cos x + c_3 \sin x)$$

**Example 7** Solve the differential equation:  $(D^2 - 2D + 5)^2y = 0$

Solving ①, we get

$$\Rightarrow m = 1 \pm 2i, 1 \pm 2i$$

$$\text{C.F.} = e^x [(c_1 + c_2 x) \cos 2x + (c_3 + c_4 x) \sin 2x]$$

Since  $F(x) = 0$ , solution is given by  $y = C.F$

$$\Rightarrow y = e^x [(c_1 + c_2 x) \cos 2x + (c_3 + c_4 x) \sin 2x]$$

**Example 8** Solve the differential equation:  $(D^2 + 4)^3 y = 0$

Solving ①, we get

$$\Rightarrow m = \pm 2i, \pm 2i, \pm 2i$$

$$\text{C.F.} = (c_1 + c_2x + c_3x^2) \cos 2x + (c_4 + c_5x + c_6x^2) \sin 2x$$

Since  $F(x) = 0$ , solution is given by  $y = C.F$

$$\Rightarrow y = (c_1 + c_2x + c_3x^2) \cos 2x + (c_4 + c_5x + c_6x^2) \sin 2x$$

### 11.3.2 Shortcut Rules for Finding Particular Integral (P.I.)

Consider the equation  $(D)y = F(x)$ ,  $F(x) \neq 0$

$$\Rightarrow (k_0 D^n + k_1 D^{n-1} + \dots + k_{n-1} D + k_n) y = F(x)$$

Then P.I. =  $\frac{1}{f(D)} F(x)$ , Clearly P.I. = 0 if  $F(x) = 0$

### **Case I: When $F(x) = e^{ax}$**

Use the rule  $P.I = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$ ,  $f(a) \neq 0$

**In case of failure i.e. if  $f(a) = 0$**

$$\text{P.I.} = x \frac{1}{f'(D)} e^{ax} = x \frac{1}{f'(a)} e^{ax}, f'(a) \neq 0$$

If  $f'(a) = 0$ , P.I. =  $x^2 \frac{1}{f''(a)} e^{ax}$ ,  $f''(a) \neq 0$  and so on

**Example 9** Solve the differential equation:  $\frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + 10y = e^{2x}$

Solution:  $\Rightarrow (D^2 - 2D + 10)y = e^{2x}$

Auxiliary equation is:  $m^2 - 2m + 10 = 0$

$$\Rightarrow m = 1 \pm 3i$$

$$\text{C.F.} = e^x (c_1 \cos 3x + c_2 \sin 3x)$$

$$\text{P.I.} = \frac{1}{f(D)} F(x) = \frac{1}{f(D)} e^{2x} = \frac{1}{f(2)} e^{2x}, \text{ by putting } D = 2$$

$$= \frac{1}{2^2 - 2(2) + 10} e^{2x} = \frac{1}{10} e^{2x}$$

Complete solution is:  $y = \text{C.F.} + \text{P.I.}$

$$\Rightarrow y = e^x (c_1 \cos 3x + c_2 \sin 3x) + \frac{1}{10} e^{2x}$$

**Example 10** Solve the differential equation:  $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = e^x$

Solution:  $\Rightarrow (D^2 + D - 2)y = e^x$

Auxiliary equation is:  $m^2 + m - 2 = 0$

$$\Rightarrow (m + 2)(m - 1) = 0$$

$$\Rightarrow m = -2, 1$$

$$\text{C.F.} = c_1 e^{-2x} + c_2 e^x$$

$$\text{P.I.} = \frac{1}{f(D)} F(x) = \frac{1}{f(D)} e^x, \text{ putting } D = 1, f(1) = 0$$

$$\therefore \text{P.I.} = x \frac{1}{f'(D)} e^x \quad \because \text{P.I.} = x \frac{1}{f'(a)} e^{ax} \text{ if } f(a) = 0$$

$$\Rightarrow \text{P.I.} = x \frac{1}{2D+1} e^x = \frac{1}{f'(1)} e^x, f'(1) \neq 0$$

$$\Rightarrow \text{P.I.} = \frac{x e^x}{3}$$

Complete solution is:  $y = \text{C.F.} + \text{P.I}$

$$\Rightarrow y = c_1 e^{-2x} + c_2 e^x + \frac{x e^x}{3}$$

**Example 11** Solve the differential equation:  $\frac{d^2y}{dx^2} - 4y = \sinh(2x+1) + 4^x$

Solution:  $\Rightarrow (D^2 - 4)y = \sinh(2x+1) + 4^x$

Auxiliary equation is:  $m^2 - 4 = 0$

$$\Rightarrow m = \pm 2$$

$$\text{C.F.} = c_1 e^{2x} + c_2 e^{-2x}$$

$$\text{P.I.} = \frac{1}{f(D)} F(x)$$

$$= \frac{1}{f(D)} (\sinh(2x+1) + 4^x)$$

$$= \frac{1}{D^2 - 4} \left( \frac{e^{(2x+1)} - e^{-(2x+1)}}{2} \right) + \frac{1}{D^2 - 4} (e^{x \log 4})$$

$$\because \sinh x = \frac{e^x - e^{-x}}{2} \text{ and } 4^x = e^{x \log 4}$$

$$= \frac{e}{2} \frac{1}{D^2 - 4} e^{2x} - \frac{e^{-1}}{2} \frac{1}{D^2 - 4} e^{-2x} + \frac{1}{D^2 - 4} e^{x \log 4}$$

Putting  $D = 2, -2$  and  $\log 4$  in the three terms respectively

$f(2) = 0$  and  $f(-2) = 0$  for first two terms

$$\therefore \text{P.I.} = \frac{e}{2} x \frac{1}{2D} e^{2x} - \frac{e^{-1}}{2} x \frac{1}{2D} e^{-2x} + \frac{1}{(\log 4)^2 - 4} e^{x \log 4}$$

$$\therefore \text{P.I.} = x \frac{1}{f'(a)} e^{ax} \text{ if } f(a) = 0$$

Now putting  $D = 2, -2$  in first two terms respectively

$$\Rightarrow \text{P.I.} = \frac{ex}{8} e^{2x} + \frac{e^{-1}x}{8} e^{-2x} + \frac{4^x}{(\log 4)^2 - 4} \quad \because e^{x \log 4} = 4^x$$

$$\Rightarrow \text{P.I.} = \frac{x}{4} \left( \frac{e^{(2x+1)} + e^{-(2x+1)}}{2} \right) + \frac{4^x}{(\log 4)^2 - 4}$$

$$\Rightarrow \text{P.I.} = \frac{x}{4} \cosh(2x+1) + \frac{4^x}{(\log 4)^2 - 4} \quad \because \cosh x = \frac{e^x + e^{-x}}{2}$$

Complete solution is:  $y = \text{C.F.} + \text{P.I}$

$$\Rightarrow y = c_1 e^{2x} + c_2 e^{-2x} + \frac{x}{4} \cosh(2x+1) + \frac{4^x}{(\log 4)^2 - 4}$$

**Case II: When  $F(x) = \sin(ax+b)$  or  $\cos(ax+b)$**

If  $F(x) = \sin(ax+b)$  or  $\cos(ax+b)$ , put  $D^2 = -a^2$ ,

$$D^3 = D^2 D = -a^2 D, D^4 = (D^2)^2 = a^4, \dots$$

This will form a linear expression in  $D$  in the denominator. Now rationalize the denominator to substitute  $D^2 = -a^2$ . Operate on the numerator term by term by taking  $D \equiv \frac{d}{dx}$

**In case of failure i.e. if  $f(-a^2) = 0$**

$$\text{P.I.} = x \frac{1}{f'(-a^2)} \sin(ax+b) \text{ or } \cos(ax+b), f'(-a^2) \neq 0$$

$$\text{If } f'(-a^2) = 0, \text{ P.I.} = x^2 \frac{1}{f''(-a^2)} \sin(ax+b) \text{ or } \cos(ax+b), f''(-a^2) \neq 0$$

**Example 12** Solve the differential equation:  $(D^2 + D - 2)y = \sin x$

Solution: Auxiliary equation is:  $m^2 + m - 2 = 0$

$$\Rightarrow (m+2)(m-1) = 0$$

$$\Rightarrow m = -2, 1$$

$$\text{C.F.} = c_1 e^{-2x} + c_2 e^x$$

$$\text{P.I.} = \frac{1}{f(D)} F(x) = \frac{1}{f(D)} \sin x = \frac{1}{D^2 + D - 2} \sin x$$

$$\text{putting } D^2 = -1^2 = -1$$

$$\text{P.I.} = \frac{1}{D-3} \sin x = \frac{D+3}{D^2-9} \sin x, \text{ Rationalizing the denominator}$$

$$= \frac{(D+3) \sin x}{-10}, \text{ Putting } D^2 = -1$$

$$\therefore \text{P.I.} = \frac{-1}{10} (D \sin x + 3 \sin x)$$

$$= \frac{-1}{10} (\cos x + 3 \sin x)$$

Complete solution is:  $y = \text{C.F.} + \text{P.I}$

$$\Rightarrow y = c_1 e^{-2x} + c_2 e^x - \frac{1}{10} (\cos x + 3 \sin x)$$

**Example 13** Solve the differential equation:  $(D^2 + 2D + 1)y = \cos^2 x$

Solution: Auxiliary equation is:  $m^2 + 2m + 1 = 0$

$$(m + 1)^2 = 0$$

$$\Rightarrow m = -1, -1$$

$$\text{C.F.} = e^{-x}(c_1 + c_2 x)$$

$$\begin{aligned}\text{P.I.} &= \frac{1}{f(D)} F(x) = \frac{1}{f(D)} \cos^2 x = \frac{1}{D^2 + 2D + 1} \left( \frac{1 + \cos 2x}{2} \right) \\ &= \frac{1}{2} \frac{1}{D^2 + 2D + 1} e^{0x} + \frac{1}{2} \frac{1}{D^2 + 2D + 1} \cos 2x\end{aligned}$$

Putting  $D = 0$  in the 1<sup>st</sup> term and  $D^2 = -2^2 = -4$  in the 2<sup>nd</sup> term

$$\begin{aligned}\text{P.I.} &= \frac{1}{2} + \frac{1}{2} \frac{1}{2D+3} \cos 2x \\ &= \frac{1}{2} + \frac{1}{2} \frac{2D+3}{4D^2-3^2} \cos 2x, \text{ Rationalizing the denominator} \\ &= \frac{1}{2} + \frac{1}{2} \frac{(2D+3) \cos 2x}{-25}, \text{ Putting } D^2 = -4 \\ \therefore \text{P.I.} &= \frac{1}{2} - \frac{1}{50} (-4 \sin 2x + 3 \cos 2x)\end{aligned}$$

Now  $y = \text{C.F.} + \text{P.I}$

$$\Rightarrow y = e^{-x}(c_1 + c_2 x) + \frac{1}{2} - \frac{1}{50} (-4 \sin 2x + 3 \cos 2x)$$

**Example 14** Solve the differential equation:  $(D^2 + 9)y = \sin 2x \cos x$

Solution: Auxiliary equation is:  $m^2 + 9 = 0$

$$\Rightarrow m = \pm 3i$$

$$\text{C.F.} = c_1 \cos 3x + c_2 \sin 3x$$

$$\text{P.I.} = \frac{1}{f(D)} F(x) = \frac{1}{f(D)} \sin 2x \cos x = \frac{1}{2} \frac{1}{D^2+9} (\sin 3x + \sin x)$$

$$= \frac{1}{2} \frac{1}{D^2+9} \sin 3x + \frac{1}{2} \frac{1}{D^2+9} \sin x$$

Putting  $D^2 = -9$  in the 1<sup>st</sup> term and  $D^2 = -1$  in the 2<sup>nd</sup> term

We see that  $f(D^2 = -9) = 0$  for the 1<sup>st</sup> term

$$\therefore \text{P.I.} = \frac{1}{2} x \frac{1}{2D} \sin 3x + \frac{1}{2} \frac{1}{8} \sin x$$

$$\because \text{P.I.} = x \frac{1}{f'(-a^2)} \sin(ax+b), f'(-a^2) \neq 0$$

$$\Rightarrow \text{P.I.} = -\frac{x}{12} \cos 3x + \frac{1}{16} \sin x$$

Complete solution is:  $y = \text{C.F.} + \text{P.I}$

$$\Rightarrow y = c_1 \cos 3x + c_2 \sin 3x - \frac{x}{12} \cos 3x + \frac{1}{16} \sin x$$

### **Case III: When $F(x) = x^n$ , $n$ is a positive integer**

$$\text{P.I.} = \frac{1}{f(D)} F(x) = \frac{1}{f(D)} x^n$$

1. Take the lowest degree term common from  $f(D)$  to get an expression of the form  $[1 \pm \phi(D)]$  in the denominator and take it to numerator to become  $[1 \pm \phi(D)]^{-1}$
2. Expand  $[1 \pm \phi(D)]^{-1}$  using binomial theorem up to  $n^{\text{th}}$  degree as  $(n+1)^{\text{th}}$  derivative of  $x^n$  is zero
3. Operate on the numerator term by term by taking  $D \equiv \frac{d}{dx}$

Following expansions will be useful to expand  $[1 \pm \phi(D)]^{-1}$  in ascending powers of  $D$

- $(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$
- $(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$
- $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots$
- $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$

**Example 15** Solve the differential equation:  $\frac{d^2y}{dx^2} - y = 5x - 2$

Solution:  $\Rightarrow (D^2 - 1)y = 5x - 2$

Auxiliary equation is:  $m^2 - 1 = 0$

$$\Rightarrow m = \pm 1$$

$$\text{C.F.} = c_1 e^x + c_2 e^{-x}$$

$$\text{P.I.} = \frac{1}{f(D)} F(x) = \frac{1}{D^2 - 1} (5x - 2)$$

$$= \frac{1}{-(1-D^2)} (5x - 2)$$

$$= -(1 - D^2)^{-1} (5x - 2)$$

$$= -[1 + D^2 + \dots] (5x - 2)$$

$$= -(5x - 2)$$

$$\therefore \text{P.I.} = -5x + 2$$

Complete solution is:  $y = \text{C.F.} + \text{P.I.}$

$$\Rightarrow y = c_1 e^x + c_2 e^{-x} - 5x + 2$$

**Example 16** Solve the differential equation:  $(D^4 + 4D^2)y = x^2 + 1$

Solution: Auxiliary equation is:  $m^4 + 4m^2 = 0$

$$\Rightarrow m^2(m^2 + 4) = 0$$

$$\Rightarrow m = 0, 0, \pm 2i$$

$$\text{C.F.} = (c_1 + c_2 x) + (c_3 \cos 2x + c_4 \sin 2x)$$

$$\text{P.I.} = \frac{1}{f(D)} F(x) = \frac{1}{D^4 + 4D^2} (x^2 + 1)$$

$$= \frac{1}{D^4 + 4D^2} (x^2 + 1)$$

$$= \frac{1}{4D^2 \left(1 + \frac{D^2}{4}\right)} (x^2 + 1)$$

$$= \frac{1}{4D^2} \left(1 + \frac{D^2}{4}\right)^{-1} (x^2 + 1)$$

$$= \frac{1}{4D^2} \left[1 - \frac{D^2}{4} + \dots\right] (x^2 + 1)$$

$$= \frac{1}{4D^2} \left(x^2 + 1 - \frac{1}{2}\right)$$

$$\begin{aligned}
&= \frac{1}{4D^2} \left( x^2 + \frac{1}{2} \right) \\
&= \frac{1}{4D} \int \left( x^2 + \frac{1}{2} \right) dx \\
&= \frac{1}{4D} \left( \frac{x^3}{3} + \frac{x}{2} \right) \\
&= \frac{1}{4} \int \left( \frac{x^3}{3} + \frac{x}{2} \right) dx \\
\therefore \text{P.I.} &= \frac{1}{4} \left( \frac{x^4}{12} + \frac{x^2}{4} \right)
\end{aligned}$$

Complete solution is:  $y = \text{C.F.} + \text{P.I}$

$$\Rightarrow y = (c_1 + c_2x) + (c_3 \cos 2x + c_4 \sin 2x) + \frac{1}{4} \left( \frac{x^4}{12} + \frac{x^2}{4} \right)$$

**Example 17** Solve the differential equation:  $(D^2 - 6D + 9)y = 1 + x + x^2$

Solution: Auxiliary equation is:  $m^2 - 6m + 9 = 0$

$$\Rightarrow (m - 3)^2 = 0$$

$$\Rightarrow m = 3, 3$$

$$\text{C.F.} = e^{3x} (c_1 + c_2x)$$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{f(D)} F(x) = \frac{1}{D^2 - 6D + 9} (1 + x + x^2) \\
&= \frac{1}{9 \left( 1 - \frac{2D}{3} + \frac{D^2}{9} \right)} (1 + x + x^2) \\
&= \frac{1}{9} \left( 1 - \left( \frac{2D}{3} - \frac{D^2}{9} \right) \right)^{-1} (1 + x + x^2) \\
&= \frac{1}{9} \left[ 1 + \left( \frac{2D}{3} - \frac{D^2}{9} \right) + \left( \frac{2D}{3} - \frac{D^2}{9} \right)^2 + \dots \right] (1 + x + x^2) \\
&= \frac{1}{9} \left[ 1 + \frac{2D}{3} - \frac{D^2}{9} + \frac{4D^2}{9} + \dots \right] (1 + x + x^2) \\
&= \frac{1}{9} \left[ 1 + \frac{2D}{3} + \frac{D^2}{3} + \dots \right] (1 + x + x^2) \\
&= \frac{1}{9} \left( 1 + x + x^2 + \frac{2}{3} + \frac{4x}{3} + \frac{2}{3} \right)
\end{aligned}$$

$$\therefore \text{P.I.} = \frac{1}{9} \left( \frac{7}{3} + \frac{7x}{3} + x^2 \right)$$

Complete solution is:  $y = \text{C.F.} + \text{P.I.}$

$$\Rightarrow y = e^{3x} (c_1 + c_2 x) + \frac{1}{9} \left( \frac{7}{3} + \frac{7x}{3} + x^2 \right)$$

**Case IV: When  $F(x) = e^{ax}g(x)$ , where  $g(x)$  is any function of  $x$**

Use the rule:  $\frac{1}{f(D)} e^{ax} g(x) = e^{ax} \left( \frac{1}{f(D+a)} g(x) \right)$

**Example 18** Solve the differential equation:  $(D^2 + 2)y = x^2 e^{3x}$

Solution: Auxiliary equation is:  $m^2 + 2 = 0$

$$\Rightarrow m^2 = -2$$

$$\Rightarrow m = \pm \sqrt{2}i$$

$$\text{C.F.} = (c_1 \cos(\sqrt{2}x) + c_2 \sin(\sqrt{2}x))$$

$$\begin{aligned} \text{P.I.} &= \frac{1}{f(D)} F(x) = \frac{1}{D^2+2} x^2 e^{3x} \\ &= e^{3x} \frac{1}{(D+3)^2+2} x^2 \\ &= e^{3x} \frac{1}{D^2+6D+11} x^2 \\ &= \frac{e^{3x}}{11} \frac{1}{\left(1+\frac{6D}{11}+\frac{D^2}{11}\right)} x^2 \\ &= \frac{e^{3x}}{11} \left(1 + \left(\frac{6D}{11} + \frac{D^2}{11}\right)\right)^{-1} x^2 \\ &= \frac{e^{3x}}{11} \left[1 - \left(\frac{6D}{11} + \frac{D^2}{11}\right) + \left(\frac{6D}{11} + \frac{D^2}{11}\right)^2 + \dots\right] x^2 \\ &= \frac{e^{3x}}{11} \left[1 - \frac{6D}{11} - \frac{D^2}{11} + \frac{36D^2}{121} + \dots\right] x^2 \\ &= \frac{e^{3x}}{11} \left[1 - \frac{6D}{11} + \frac{25D^2}{121} + \dots\right] x^2 \\ &= \frac{e^{3x}}{11} \left(x^2 - \frac{12x}{11} + \frac{50}{121}\right) \end{aligned}$$

$$\therefore P.I = \frac{e^{3x}}{11} \left( x^2 - \frac{12x}{11} + \frac{50}{121} \right)$$

Complete solution is:  $y = \text{C.F.} + \text{P.I}$

$$\Rightarrow y = (c_1 \cos(\sqrt{2}x) + c_2 \sin(\sqrt{2}x)) + \frac{e^{3x}}{11} \left( x^2 - \frac{12x}{11} + \frac{50}{121} \right)$$

**Example 19** Solve the differential equation:  $(D^3 + 1)y = e^{2x} \sin x$

Solution: Auxiliary equation is:  $m^3 + 1 = 0$

$$\Rightarrow m^3 = -1$$

$$\Rightarrow m = -1, \frac{1 \pm \sqrt{3}i}{2}$$

$$\text{C.F.} = c_1 e^{-x} + e^{\frac{x}{2}} \left( c_2 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_3 \sin\left(\frac{\sqrt{3}}{2}x\right) \right)$$

$$\text{P.I.} = \frac{1}{f(D)} F(x) = \frac{1}{D^3+1} e^{2x} \sin x$$

$$= e^{2x} \frac{1}{(D+2)^3+1} \sin x$$

$$= e^{2x} \frac{1}{D^3+6D^2+12D+9} \sin x$$

$$= e^{2x} \frac{1}{-D-6+12D+9} \sin x, \text{ Putting } D^2 = -1$$

$$= e^{2x} \frac{1}{11D+3} \sin x$$

$$= e^{2x} \frac{11D-3}{121D^2-9} \sin x, \text{ Rationalizing the denominator}$$

$$= -\frac{e^{2x}}{130} (11D-3) \sin x, \text{ Putting } D^2 = -1$$

$$\therefore \text{P.I.} = -\frac{e^{2x}}{130} (11 \cos x - 3 \sin x)$$

Complete solution is:  $y = \text{C.F.} + \text{P.I}$

$$\Rightarrow y = c_1 e^{-x} + e^{\frac{x}{2}} \left( c_2 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_3 \sin\left(\frac{\sqrt{3}}{2}x\right) \right)$$

$$-\frac{e^{2x}}{130} (11 \cos x - 3 \sin x)$$

**Example 20** Solve the differential equation:  $\frac{d^2y}{dx^2} - 4y = x \sinh x$

Solution:  $\Rightarrow (D^2 - 4)y = x \sinh x$

Auxiliary equation is:  $m^2 - 4 = 0$

$$\Rightarrow m = \pm 2$$

$$\text{C.F.} = c_1 e^{2x} + c_2 e^{-2x}$$

$$\text{P.I.} = \frac{1}{f(D)} F(x)$$

$$= \frac{1}{f(D)} (x \sinh x)$$

$$= \frac{1}{D^2 - 4} \left( x \frac{e^x - e^{-x}}{2} \right) \quad \because \sinh x = \frac{e^x - e^{-x}}{2}$$

$$= \frac{1}{D^2 - 4} \left( x \frac{e^x}{2} - x \frac{e^{-x}}{2} \right)$$

$$= \frac{e^x}{2} \frac{1}{(D+1)^2 - 4} x - \frac{e^{-x}}{2} \frac{1}{(D-1)^2 - 4} x$$

$$= \frac{e^x}{2} \frac{1}{(D^2 + 2D - 3)} x - \frac{e^{-x}}{2} \frac{1}{D^2 - 2D - 3} x$$

$$= \frac{e^x}{2} \frac{1}{-3 \left( 1 - \frac{D^2 - 2D}{3} \right)} x - \frac{e^{-x}}{2} \frac{1}{-3 \left( 1 - \frac{D^2 + 2D}{3} \right)} x$$

$$= -\frac{e^x}{6} \left[ 1 - \left( \frac{D^2}{3} + \frac{2D}{3} \right) \right]^{-1} x + \frac{e^{-x}}{6} \left[ 1 - \left( \frac{D^2}{3} - \frac{2D}{3} \right) \right]^{-1} x$$

$$= -\frac{e^x}{6} \left( 1 + \frac{2D}{3} \right) x + \frac{e^{-x}}{6} \left( 1 - \frac{2D}{3} \right) x$$

$$= -\frac{e^x}{6} \left( x + \frac{2}{3} \right) + \frac{e^{-x}}{6} \left( x - \frac{2}{3} \right)$$

$$= -\frac{x}{3} \left( \frac{e^x - e^{-x}}{2} \right) - \frac{2}{9} \left( \frac{e^x + e^{-x}}{2} \right)$$

$$\therefore \text{P.I.} = -\frac{x}{3} \sinh x - \frac{2}{9} \cosh x$$

Complete solution is:  $y = \text{C.F.} + \text{P.I.}$

$$\Rightarrow y = c_1 e^{2x} + c_2 e^{-2x} - \frac{x}{3} \sinh x - \frac{2}{9} \cosh x$$

**Example 21** Solve the differential equation:  $(D^2 + 1)y = x^2 \sin 2x$

Solution: Auxiliary equation is:  $m^2 + 1 = 0$

$$\Rightarrow m^2 = -1$$

$$\Rightarrow m = \pm i$$

$$C.F. = c_1 \cos x + c_2 \sin x$$

$$P.I. = \frac{1}{f(D)} F(x) = \frac{1}{D^2+1} x^2 \sin 2x$$

$$= \text{Imaginary part of } \frac{1}{D^2+1} x^2 e^{i2x}$$

$$\begin{aligned} \text{Now } \frac{1}{D^2+1} x^2 e^{i2x} &= e^{i2x} \frac{1}{(D+2i)^2+1} x^2 \\ &= e^{i2x} \frac{1}{D^2+4i^2+4iD+1} x^2 \\ &= e^{i2x} \frac{1}{D^2+4iD-3} x^2 \end{aligned}$$

$$\begin{aligned} &= \frac{-e^{i2x}}{3} \left[ 1 - \left( \frac{D^2}{3} + \frac{4iD}{3} \right) \right]^{-1} x^2 \\ &= \frac{-e^{i2x}}{3} \left[ 1 + \left( \frac{D^2}{3} + \frac{4iD}{3} \right) + \left( \frac{D^2}{3} + \frac{4iD}{3} \right)^2 \right] x^2 \\ &= \frac{-e^{i2x}}{3} \left[ 1 + \frac{D^2}{3} + \frac{4iD}{3} + \frac{16i^2D^2}{9} \right] x^2 \\ &= \frac{-e^{i2x}}{3} \left[ 1 - \frac{13D^2}{9} + \frac{4iD}{3} \right] x^2 \\ &= \frac{-e^{i2x}}{3} \left[ x^2 - \frac{26}{9} + i \frac{8x}{3} \right] \\ &= -\frac{1}{3} (\cos 2x + i \sin 2x) \left[ x^2 - \frac{26}{9} + i \frac{8x}{3} \right] \end{aligned}$$

$$\begin{aligned} \therefore P.I. &= \text{Imaginary part of } \frac{1}{D^2+1} x^2 e^{i2x} = -\frac{1}{3} \left( \frac{8x}{3} \cos 2x + \left( x^2 - \frac{26}{9} \right) \sin 2x \right) \\ &= -\frac{8x}{9} \cos 2x + \frac{1}{27} (26 - 9x^2) \sin 2x \end{aligned}$$

Complete solution is:  $y = \text{C.F.} + \text{P.I}$

$$\Rightarrow y = c_1 \cos x + c_2 \sin x - \frac{8x}{9} \cos 2x + \frac{1}{27} (26 - 9x^2) \sin 2x$$

**Example 22** Solve the differential equation:  $(D^2 - 4D + 4)y = x^2 e^{2x} \sin 2x$

Solution: Auxiliary equation is:  $m^2 - 4m + 4 = 0$

$$\Rightarrow (m - 2)^2$$

$$\Rightarrow m = 2, 2$$

$$\text{C.F.} = (c_1 + c_2 x) e^{2x}$$

$$\begin{aligned}\text{P.I.} &= \frac{1}{f(D)} F(x) = \frac{1}{D^2 - 4D + 4} x^2 e^{2x} \sin 2x \\ &= e^{2x} \frac{1}{(D+2)^2 - 4(D+2) + 4} x^2 \sin 2x \\ &= e^{2x} \frac{1}{D^2} x^2 \sin 2x \\ &= e^{2x} \frac{1}{D} \int x^2 \sin 2x \, dx \\ &= e^{2x} \frac{1}{D} \left[ (x^2) \left( \frac{-\cos 2x}{2} \right) - (2x) \left( \frac{-\sin 2x}{4} \right) + (2) \left( \frac{\cos 2x}{8} \right) \right] \\ &= e^{2x} \frac{1}{D} \left[ -\frac{1}{2} x^2 \cos 2x + \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x \right] \\ &= e^{2x} \left[ -\frac{1}{2} \int x^2 \cos 2x \, dx + \frac{1}{2} \int x \sin 2x \, dx + \frac{1}{4} \int \cos 2x \, dx \right] \\ &= e^{2x} \left[ -\frac{1}{2} \left[ (x^2) \left( \frac{\sin 2x}{2} \right) - (2x) \left( \frac{-\cos 2x}{4} \right) + (2) \left( \frac{-\sin 2x}{8} \right) \right] + \right. \\ &\quad \left. 12x^2 \cos 2x - 1 - \sin 2x + 4 + 14 \sin 2x \right] \\ \therefore \text{P.I.} &= e^{2x} \left[ \frac{-x^2}{4} \sin 2x - \frac{x}{2} \cos 2x + \frac{3}{8} \sin 2x \right]\end{aligned}$$

Complete solution is:  $y = \text{C.F.} + \text{P.I}$

$$\Rightarrow y = (c_1 + c_2 x) e^{2x} + e^{2x} \left[ \frac{-x^2}{4} \sin 2x - \frac{x}{2} \cos 2x + \frac{3}{8} \sin 2x \right]$$

**Case V: When  $F(x) = x g(x)$ , where  $g(x)$  is any function of  $x$**

Use the rule:  $\frac{1}{f(D)} (x g(x)) = x \frac{1}{f(D)} g(x) + \left( \frac{d}{dD} \frac{1}{f(D)} \right) g(x)$

**Example 23** Solve the differential equation:  $(D^2 + 9)y = x \cos x$

Solution: Auxiliary equation is:  $m^2 + 9 = 0$

$$\Rightarrow m^2 = -9$$

$$\Rightarrow m = \pm 3i$$

$$\text{C.F.} = (c_1 \cos 3x + c_2 \sin 3x)$$

$$\begin{aligned}\text{P.I.} &= \frac{1}{f(D)} F(x) = \frac{1}{D^2+9} x \cos x \\&= x \frac{1}{D^2+9} \cos x + \frac{-2D}{(D^2+9)^2} \cos x \\&= x \frac{1}{-1+9} \cos x + \frac{-2D}{(-1+9)^2} \cos x, \quad \text{Putting } D^2 = -1 \\&= \frac{x \cos x}{8} - \frac{2D \cos x}{64} \\&= \frac{x \cos x}{8} - \frac{2D \cos x}{64} \\&\therefore \text{P.I.} = \frac{x \cos x}{8} + \frac{\sin x}{32}\end{aligned}$$

Complete solution is:  $y = \text{C.F.} + \text{P.I}$

$$\Rightarrow y = c_1 \cos 3x + c_2 \sin 3x + \frac{x \cos x}{8} + \frac{\sin x}{32}$$

**Example 24** Solve the differential equation:

$$(D^2 - 1)y = x \sin x + (1 + x^2)e^x$$

Solution: Auxiliary equation is:  $m^2 - 1 = 0$

$$\Rightarrow m = \pm 1$$

$$\text{C.F.} = c_1 e^x + c_2 e^{-x}$$

$$\begin{aligned}\text{P.I.} &= \frac{1}{f(D)} F(x) = \frac{1}{D^2-1} (x \sin x + (1 + x^2)e^x) \\&= \frac{1}{D^2-1} x \sin x + \frac{1}{D^2-1} (1 + x^2)e^x\end{aligned}$$

$$\text{Now } \frac{1}{D^2-1} x \sin x = x \frac{1}{D^2-1} \sin x + \frac{-2D}{(D^2-1)^2} \sin x$$

$$= x \frac{1}{(-1-1)} \sin x + \frac{-2D}{(-1-1)^2} \sin x, \quad \text{Putting } D^2 = -1$$

$$= -\frac{1}{2}(x \sin x + \cos x)$$

$$\text{Also } \frac{1}{D^2-1} (1+x^2) e^x = e^x \frac{1}{(D+1)^2-1} (1+x^2)$$

$$= e^x \frac{1}{D^2+2D} (1+x^2)$$

$$= e^x \frac{1}{2D(1+\frac{D}{2})} (1+x^2)$$

$$= e^x \frac{1}{2D} \left(1 + \frac{D}{2}\right)^{-1} (1+x^2)$$

$$= e^x \frac{1}{2D} \left[1 - \frac{D}{2} + \frac{D^2}{4}\right] (1+x^2)$$

$$= e^x \frac{1}{2D} \left[1 + x^2 - x + \frac{1}{2}\right]$$

$$= e^x \frac{1}{2D} \left[x^2 - x + \frac{3}{2}\right]$$

$$= \frac{e^x}{2} \left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{3x}{2}\right]$$

$$\therefore \text{P.I.} = -\frac{1}{2}(x \sin x + \cos x) + \frac{e^x}{2} \left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{3x}{2}\right]$$

Complete solution is:  $y = \text{C.F.} + \text{P.I}$

$$\Rightarrow y = c_1 e^x + c_2 e^x - \frac{1}{2}(x \sin x + \cos x) + \frac{e^x}{2} \left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{3x}{2}\right]$$

**Case VI: When  $F(x)$  is any general function of  $x$  not covered in shortcut methods I to V above**

Resolve  $f(D)$  into partial fractions and use the rule:

$$\frac{1}{D+a} F(x) = e^{-ax} \int e^{ax} F(x) dx$$

**Example 25** Solve the differential equation:  $(D^2 + 3D + 2)y = e^x$

Solution: Auxiliary equation is:  $m^2 + 3m + 2 = 0$

$$\Rightarrow (m+1)(m+2) = 0$$

$$\Rightarrow m = -1, -2$$

$$\text{C.F.} = c_1 e^{-x} + c_2 e^{-2x}$$

$$\begin{aligned}\text{P.I.} &= \frac{1}{f(D)} F(x) = \frac{1}{D^2+3D+2} e^{ex} \\&= \frac{1}{(D+1)(D+2)} e^{ex} \\&= \left(\frac{1}{(D+1)} - \frac{1}{(D+2)}\right) e^{ex} \\&= e^{-x} \int e^x e^{ex} dx - e^{-2x} \int e^{2x} e^{ex} dx \\&= e^{-x} \int De^{ex} dx - e^{-2x} \int e^x De^{ex} dx \\&= e^{-x} e^{ex} - e^{-2x} [e^x e^{ex} - \int e^x e^{ex} dx], \text{ Integrating 2nd term by parts} \\&= e^{-x} e^{ex} - e^{-2x} [e^x e^{ex} - \int De^{ex} dx] \\&= e^{-x} e^{ex} - e^{-2x} [e^x e^{ex} - e^{ex}] \\&\therefore \text{P.I.} = e^{-2x} e^{ex}\end{aligned}$$

Complete solution is:  $y = \text{C.F.} + \text{P.I}$

$$\Rightarrow y = c_1 e^{-x} + c_2 e^{-2x} + e^{-2x} e^{ex}$$

**Example 26** Solve the differential equation:  $(D^2 + 4)y = \tan 2x$

Solution: Auxiliary equation is:  $m^2 + 4 = 0$

$$\Rightarrow m = \pm 2i$$

$$\text{C.F.} = c_1 \cos 2x + c_2 \sin 2x$$

$$\begin{aligned}\text{P.I.} &= \frac{1}{f(D)} F(x) = \frac{1}{D^2+4} \tan 2x \\&= \frac{1}{(D-2i)(D+2i)} \tan 2x \\&= \frac{1}{4i} \left( \frac{1}{(D-2i)} - \frac{1}{(D+2i)} \right) \tan 2x \\&\text{P.I.} = \frac{1}{4i} \left( \frac{1}{D-2i} \tan 2x \right) - \frac{1}{4i} \left( \frac{1}{D+2i} \tan 2x \right) \dots\dots \textcircled{1}\end{aligned}$$

$$\begin{aligned}
\text{Now } \frac{1}{D-2i} \tan 2x &= e^{2ix} \int e^{-2ix} \tan 2x \, dx \\
&= e^{2ix} \int (\cos 2x - i \sin 2x) \tan 2x \, dx \\
&= e^{2ix} \int (\sin 2x - i \frac{\sin^2 2x}{\cos 2x}) \, dx \\
&= e^{2ix} \int (\sin 2x - i \sec 2x + i \cos 2x) \, dx \\
&= e^{2ix} \left( -\frac{1}{2} \cos 2x - \frac{i}{2} \log |\sec 2x + \tan 2x| + \frac{i}{2} \sin 2x \right) \\
\therefore \frac{1}{D-2i} \tan 2x &= e^{2ix} \left( -\frac{1}{2} e^{-2ix} - \frac{i}{2} \log |\sec 2x + \tan 2x| \right) \dots \textcircled{2}
\end{aligned}$$

Replacing  $i$  by  $-i$

$$\frac{1}{D+2i} \tan 2x = e^{-2ix} \left( -\frac{1}{2} e^{2ix} + \frac{i}{2} \log |\sec 2x + \tan 2x| \right) \dots \textcircled{3}$$

Using \textcircled{2} and \textcircled{3} in \textcircled{1}

$$\begin{aligned}
\text{P.I.} &= \frac{1}{4i} \left[ e^{2ix} \left( -\frac{1}{2} e^{-2ix} - \frac{i}{2} \log |\sec 2x + \tan 2x| \right) \right] \\
&\quad - \frac{1}{4i} \left[ e^{-2ix} \left( -\frac{1}{2} e^{2ix} + \frac{i}{2} \log |\sec 2x + \tan 2x| \right) \right] \\
&= \frac{1}{4i} \left[ -\frac{1}{2} - \frac{i}{2} e^{2ix} \log |\sec 2x + \tan 2x| + \frac{1}{2} - \frac{i}{2} e^{-2ix} \log |\sec 2x + \tan 2x| \right] \\
&= \frac{1}{4i} \left[ -i \frac{e^{2ix} + e^{-2ix}}{2} \log |\sec 2x + \tan 2x| \right] \\
\therefore \text{P.I.} &= -\frac{1}{4} [\cos 2x \log |\sec 2x + \tan 2x|]
\end{aligned}$$

Complete solution is:  $y = \text{C.F.} + \text{P.I.}$

$$\Rightarrow y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4} [\cos 2x \log |\sec 2x + \tan 2x|]$$

### Exercise 11A

**Solve the following differential equations:**

$$1. \quad (D^3 + D^2 - 5D + 3)y = 0 \quad \text{Ans. } y = (c_1 x + c_2)e^x + c_3 e^{-3x}$$

$$2. \frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{3x} \quad \text{Ans. } y = c_1 e^{2x} + c_2 e^{3x} + e^{3x}(x - 1)$$

$$3. \frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = e^x \cosh 2x$$

$$\text{Ans. } y = c_1 e^{-3x} + c_2 e^{2x} + \frac{1}{12} e^{3x} - \frac{1}{12} e^{-x}$$

$$4. (D - 1)^2(D^2 + 1)^2y = e^x$$

$$\text{Ans. } y = (c_1 x + c_2)e^x + (c_3 x + c_4) \cos x + (c_5 x + c_6) \sin x + \frac{x^2}{8}e^x$$

$$5. (D^2 - 6D + 9)y = x^2 + 2e^{2x}$$

$$\text{Ans. } y = (c_1 x + c_2)e^{3x} + \frac{1}{9} \left( x^2 + \frac{4x}{8} + \frac{2}{3} \right) + 2e^{2x}$$

$$6. (D^2 + D - 2)y = x + \sin x$$

$$\text{Ans. } y = c_1 e^{-2x} + c_2 e^x - \frac{1}{4}(2x + 1) - \frac{1}{10}(\cos x + 3 \sin x)$$

$$7. (D^2 + D)y = (1 + e^x)^{-1}$$

$$\text{Ans. } y = c_1 + c_2 e^{-x} + x - (1 + e^{-x}) \log(1 + e^x)$$

$$8. (D^2 + 5D + 6)y = e^{-2x} \sec^2 x (1 + 2 \tan x)$$

$$\text{Ans. } y = c_1 e^{-2x} + c_2 e^{-3x} + e^{-2x}(\tan x - 1)$$

$$9. \frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 12y = (x - 1)e^{2x}$$

$$\text{Ans. } y = c_1 e^{2x} + c_2 e^{-6x} + \frac{e^{2x}}{8} \left( \frac{x^2}{2} - \frac{9x}{8} \right)$$

$$10. \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4x + e^{3x}, \text{ given } y = 1, \frac{dy}{dx} = -1 \text{ when } x = 0$$

$$\text{Ans. } y = -\frac{1}{2}e^x - 2e^{2x} + 2x + 3 + \frac{e^{3x}}{2}$$

## 11.4 Differential Equations Reducible to Linear Form with Constant Coefficients

Some special type of homogenous and non homogeneous linear differential equations with variable coefficients after suitable substitutions can be reduced to linear differential equations with constant coefficients.

### 11.4.1 Cauchy's Linear Differential Equation

The differential equation of the form:

$$k_0 x^n \frac{d^n y}{dx^n} + k_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + k_{n-1} x \frac{dy}{dx} + k_n y = F(x)$$

is called Cauchy's linear equation and it can be reduced to linear differential equations with constant coefficients by following substitutions:

$$x = e^t \Rightarrow \log x = t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{dy}{dt} \frac{1}{x}$$

$$\Rightarrow x \frac{dy}{dx} = \frac{dy}{dt} = Dy, \text{ where } D \equiv \frac{d}{dt}$$

Similarly  $x^2 \frac{d^2y}{dx^2} = D(D - 1)y$ ,  $x^3 \frac{d^3y}{dx^3} = D(D - 1)(D - 2)y$  and so on.

**Example 27** Solve the differential equation:

$$x^3 \frac{d^3y}{dx^3} + 3x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + 8y = 13 \cos(\log x), x > 0 \quad \dots \dots \dots \textcircled{1}$$

Solution: This is a Cauchy's linear equation with variable coefficients.

$$\text{Putting } x = e^t \therefore \log x = t$$

$$\Rightarrow x \frac{dy}{dx} = Dy, x^2 \frac{d^2y}{dx^2} = D(D - 1)y \text{ and } x^3 \frac{d^3y}{dx^3} = D(D - 1)(D - 2)y$$

$\therefore \textcircled{1}$  May be rewritten as

$$(D(D - 1)(D - 2) + 3D(D - 1) + D + 8)y = 13 \cos t$$

$$\Rightarrow (D^3 + 8)y = 13 \cos t, D \equiv \frac{d}{dt}$$

Auxiliary equation is:  $m^3 + 8 = 0$

$$\Rightarrow (m + 2)(m^2 - 2m + 2) = 0$$

$$\Rightarrow m = -2, 1 \pm \sqrt{3}i$$

$$\text{C.F.} = c_1 e^{-2t} + e^t (c_2 \cos \sqrt{3}t + c_3 \sin \sqrt{3}t)$$

$$= \frac{c_1}{x^2} + x(c_2 \cos(\sqrt{3} \log x) + c_3 \sin(\sqrt{3} \log x))$$

$$\text{P.I.} = \frac{1}{f(D)} F(x) = 13 \frac{1}{D^3 + 8} \cos t$$

$$= 13 \frac{1}{-D+8} \cos t, \text{ Putting } D^2 = -1$$

$$= 13 \frac{(8+D)}{64-D^2} \cos t = 13 \frac{(8+D)}{65} \cos t \quad \text{Putting } D^2 = -1$$

$$\begin{aligned}
\therefore \text{P.I.} &= \frac{1}{5} (8 \cos t + D \cos t) \\
&= \frac{1}{5} (8 \cos t - \sin t) \\
&= \frac{1}{5} (8 \cos(\log x) - \sin(\log x))
\end{aligned}$$

Complete solution is:  $y = \text{C.F.} + \text{P.I}$

$$\begin{aligned}
\Rightarrow y &= \frac{c_1}{x^2} + x(c_2 \cos(\sqrt{3} \log x) + c_3 \sin(\sqrt{3} \log x)) + \\
&\quad \frac{1}{5} (8 \cos(\log x) - \sin(\log x))
\end{aligned}$$

**Example 28** Solve the differential equation:

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - y = \frac{x^3}{1+x^2} \quad \dots\dots\dots \textcircled{1}$$

Solution: This is a Cauchy's linear equation with variable coefficients.

Putting  $x = e^t \therefore \log x = t$

$$\Rightarrow x \frac{dy}{dx} = Dy, x^2 \frac{d^2y}{dx^2} = D(D-1)y$$

$\therefore \textcircled{1}$  May be rewritten as

$$(D(D-1) + D - 1)y = \frac{e^{3t}}{1+e^{2t}}$$

$$\Rightarrow (D^2 - 1)y = \frac{e^{3t}}{1+e^{2t}}, \quad D \equiv \frac{d}{dt}$$

Auxiliary equation is:  $m^2 - 1 = 0$

$$\Rightarrow m = \pm 1$$

$$\text{C.F.} = c_1 e^{-t} + c_2 e^t$$

$$= \frac{c_1}{x} + c_2 x$$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{f(D)} F(x) = \frac{1}{D^2-1} \frac{e^{3t}}{1+e^{2t}} \\
&= \frac{1}{(D-1)(D+1)} \frac{e^{3t}}{1+e^{2t}} = \frac{1}{2} \left( \frac{1}{(D-1)} - \frac{1}{(D+1)} \right) \frac{e^{3t}}{1+e^{2t}}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \left( \frac{1}{(D-1)} \frac{e^{3t}}{1+e^{2t}} - \frac{1}{(D+1)} \frac{e^{3t}}{1+e^{2t}} \right) \\
&= \frac{1}{2} \left( e^t \int e^{-t} \frac{e^{3t}}{1+e^{2t}} dt - e^{-t} \int e^t \frac{e^{3t}}{1+e^{2t}} dt \right) \because \frac{1}{D+a} F(x) = e^{-ax} \int e^{ax} F(x) dx \\
&= \frac{1}{2} \left( e^t \int \frac{e^{2t}}{1+e^{2t}} dt - e^{-t} \int \frac{e^{4t}}{1+e^{2t}} dt \right)
\end{aligned}$$

Put  $e^{2t} = u \Rightarrow 2e^{2t}dt = du$

$$\begin{aligned}
\therefore P.I &= \frac{1}{4} \left( e^t \int \frac{1}{1+u} du - e^{-t} \int \frac{u}{1+u} du \right) \\
&= \frac{1}{4} \left( e^t \log(1+u) - e^{-t} \int \frac{1+u-1}{1+u} du \right) \\
&= \frac{1}{4} \left( e^t \log(1+u) - e^{-t} \int \left(1 - \frac{1}{1+u}\right) du \right) \\
&= \frac{1}{4} (e^t \log(1+u) - e^{-t} (u - \log(1+u))) \\
&= \frac{1}{4} (e^t \log(1+e^{2t}) - e^{-t} (e^{2t} - \log(1+e^{2t}))) \\
&= \frac{1}{4} \left( x \log(1+x^2) - \frac{1}{x} (x^2 - \log(1+x^2)) \right) \\
&= \frac{1}{4} \left( x + \frac{1}{x} \right) \log(1+x^2) - \frac{x}{4}
\end{aligned}$$

Complete solution is:  $y = C.F. + P.I$

$$\begin{aligned}
\Rightarrow y &= \frac{c_1}{x} + c_2 x + \frac{1}{4} \left( x + \frac{1}{x} \right) \log(1+x^2) - \frac{x}{4} \\
\Rightarrow y &= \frac{c_1}{x} + c_3 x + \frac{1}{4} \left( x + \frac{1}{x} \right) \log(1+x^2), c_3 = c_2 - \frac{1}{4}
\end{aligned}$$

**Example 29** Solve the differential equation:

$$x^2 D^2 - 2xD - 4y = x^2 + 2 \log x, \quad x > 0 \quad \dots \dots \textcircled{1}$$

Solution: This is a Cauchy's linear equation with variable coefficients.

Putting  $x = e^t \therefore \log x = t$

$$\Rightarrow xD = \delta y, x^2 D^2 = \delta(\delta - 1)y, \delta \equiv \frac{d}{dt}$$

$\therefore \textcircled{1}$  May be rewritten as

$$(\delta(\delta - 1) - 2\delta - 4)y = e^{2t} + 2t$$

$$\Rightarrow (\delta^2 - 3\delta - 4)y = e^{2t} + 2t$$

Auxiliary equation is:  $m^2 - 3m - 4 = 0$

$$\Rightarrow (m + 1)(m - 4) = 0$$

$$\Rightarrow m = -1, 4$$

$$\text{C.F.} = c_1 e^{-t} + c_2 e^{4t}$$

$$= \frac{c_1}{x} + \frac{c_2}{x^4}$$

$$\text{P.I.} = \frac{1}{f(\delta)} F(x) = \frac{1}{\delta^2 - 3\delta - 4} (e^{2t} + 2t)$$

$$= \frac{1}{\delta^2 - 3\delta - 4} e^{2t} + \frac{1}{\delta^2 - 3\delta - 4} 2t$$

$$= \frac{1}{-6} e^{2t} + 2 \frac{1}{-4 \left(1 - \frac{\delta^2}{4} + \frac{3\delta}{4}\right)} t \quad \text{Putting } \delta = 2 \text{ in the 1st term}$$

$$= \frac{-e^{2t}}{6} - \frac{1}{2} \left(1 - \left(\frac{\delta^2}{4} - \frac{3\delta}{4}\right)\right)^{-1} t$$

$$= \frac{-e^{2t}}{6} - \frac{1}{2} \left[1 + \frac{\delta^2}{4} - \frac{3\delta}{4} + \dots\right] t$$

$$= \frac{-e^{2t}}{6} - \frac{1}{2} \left[t - \frac{3}{4}\right]$$

$$\therefore \text{P.I.} = \frac{-x^2}{6} - \frac{1}{2} \left[\log x - \frac{3}{4}\right]$$

Complete solution is:  $y = \text{C.F.} + \text{P.I}$

$$\Rightarrow y = \frac{c_1}{x} + \frac{c_2}{x^4} - \frac{x^2}{6} - \frac{1}{2} \left[\log x - \frac{3}{4}\right]$$

### 11.4.2 Legendre's Linear Differential Equation

The differential equation of the form:  $k_0(ax + b)^n \frac{d^n y}{dx^n} +$

$$k_1 (ax + b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + k_{n-1} (ax + b) \frac{dy}{dx} + k_n y = F(x)$$

is called Legendre's linear equation and it can be reduced to linear differential equations with constant coefficients by following substitutions:

$$(ax + b) = e^t \Rightarrow t = \log(ax + b)$$

$$\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{dy}{dt} \frac{a}{ax+b}$$

$$\Rightarrow (ax + b) \frac{dy}{dx} = a \frac{dy}{dt} = aDy, \text{ where } D \equiv \frac{d}{dt}$$

$$\text{Similarly } (ax + b)^2 \frac{d^2y}{dx^2} = a^2 D(D - 1)y$$

$$(ax + b)^3 \frac{d^3y}{dx^3} = a^3 D(D - 1)(D - 2)y \text{ and so on.}$$

**Example 30** Solve the differential equation:

$$(3x + 2)^2 \frac{d^2y}{dx^2} + 3(3x + 2) \frac{dy}{dx} - 36y = 3x^2 + 4x + 1 \quad \dots\dots\dots (1)$$

Solution: This is a Legendre's linear equation with variable coefficients.

$$\text{Putting } (3x + 2) = e^t \therefore t = \log(3x + 2)$$

$$\Rightarrow (3x + 2) \frac{dy}{dx} = 3Dy, (3x + 2)^2 \frac{d^2y}{dx^2} = 3^2 D(D - 1)y$$

$$\text{Also } 3x^2 + 4x + 1 = \frac{1}{3}(9x^2 + 12x + 3)$$

$$= \frac{1}{3}((3x)^2 + 2.3.2x + 4 - 4 + 3)$$

$$= \frac{1}{3}((3x + 2)^2 - 1)$$

$$= \frac{1}{3}(e^{2t} - 1)$$

$\therefore (1)$  May be rewritten as

$$(9D(D - 1) + 9D - 36)y = \frac{1}{3}(e^{2t} - 1)$$

$$\Rightarrow 9(D^2 - 4)y = \frac{1}{3}(e^{2t} - 1)$$

Auxiliary equation is:  $9(m^2 - 4) = 0$

$$\Rightarrow m = \pm 2$$

$$\text{C.F.} = c_1 e^{-2t} + c_2 e^{2t}$$

$$= \frac{c_1}{(3x+2)^2} + c_2(3x + 2)^2$$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{f(D)} F(x) = \frac{1}{9(D^2-4)} \frac{1}{3} (e^{2t} - 1) \\
&= \frac{1}{27} \left( \frac{1}{(D^2-4)} e^{2t} - \frac{1}{(D^2-4)} e^{0t} \right) \\
&= \frac{1}{27} \left( \frac{t}{2.2} e^{2t} - \frac{1}{(0-4)} e^{0t} \right), \text{ Putting } D = 2 \text{ in 1st term, it is a} \\
&\text{case of failure } \therefore \frac{1}{(D^2-4)} e^{2t} = t \frac{1}{f'(2)} e^{2x}, \text{ also } D = 0 \text{ in the 2nd term.}
\end{aligned}$$

$$\begin{aligned}
\therefore \text{P.I.} &= \frac{1}{27} \left( \frac{t}{4} e^{2t} + \frac{1}{4} \right) \\
&= \frac{1}{27} \left( \frac{\log(3x+2)}{4} (3x+2)^2 + \frac{1}{4} \right) \\
&= \frac{1}{108} [(3x+2)^2 \log(3x+2) + 1]
\end{aligned}$$

Complete solution is:  $y = \text{C.F.} + \text{P.I}$

$$\Rightarrow y = \frac{c_1}{(3x+2)^2} + c_2(3x+2)^2 + \frac{1}{108} [(3x+2)^2 \log(3x+2) + 1]$$

**Example 31** Solve the differential equation:

$$(x+1)^2 \frac{d^2y}{dx^2} + (x+1) \frac{dy}{dx} + y = 2 \sin(\log(x+1)), \quad x > -1 \dots\dots \textcircled{1}$$

Solution: This is a Legendre's linear equation with variable coefficients.

$$\text{Putting } (x+1) = e^t \quad \therefore t = \log(x+1)$$

$$\Rightarrow (x+1) \frac{dy}{dx} = Dy, (x+1)^2 \frac{d^2y}{dx^2} = D^2(D-1)y$$

$\therefore \textcircled{1}$  May be rewritten as

$$(D(D-1) + D + 1)y = 2 \sin t$$

$$\Rightarrow (D^2 + 1)y = 2 \sin t$$

Auxiliary equation is:  $(m^2 + 1) = 0$

$$\Rightarrow m = \pm i$$

$$\text{C.F.} = c_1 \cos t + c_2 \sin t$$

$$= c_1 \cos(\log(x+1)) + c_2 \sin(\log(x+1))$$

$$= \frac{c_1}{(3x+2)^2} + c_2(3x+2)^2$$

$$\text{P.I.} = \frac{1}{f(D)} F(x) = \frac{1}{D^2+1} 2 \sin t$$

$=2t \frac{1}{2D} \sin t$ , Putting  $D^2 = -1$ , case of failure

$$\therefore \frac{1}{(D^2+1)} \sin t = t \frac{1}{f'(D)} \sin t$$

$$= t \int \sin t \, dt = -t \cos t$$

$$\therefore P.I = -\log(x+1) \cos(\log(x+1))$$

Complete solution is:  $y = \text{C.F.} + \text{P.I}$

$$y = c_1 \cos(\log(x+1)) + c_2 \sin(\log(x+1)) - \log(x+1) \cos(\log(x+1))$$

## 11.5 Method of Variation of Parameters for Finding Particular Integral

Method of Variation of Parameters enables us to find the solution of 2<sup>nd</sup> and higher order differential equations with constant coefficients as well as variable coefficients.

## Working rule

Consider a 2<sup>nd</sup> order linear differential equation:

- Find complimentary function given as: C.F. =  $c_1 y_1 + c_2 y_2$ ,  
where  $y_1$  and  $y_2$  are two linearly independent solutions of ①
  - Calculate  $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ ,  $W$  is called Wronskian of  $y_1$  and  $y_2$
  - Compute  $u_1 = - \int \frac{y_2 F(x)}{W} dx$ ,  $u_2 = \int \frac{y_1 F(x)}{W} dx$
  - Find P.I. =  $u_1 y_1 + u_2 y_2$
  - Complete solution is given by:  $y = \text{C.F.} + \text{P.I.}$

Note: Method is commonly used to solve 2<sup>nd</sup> order differential but it can be extended to solve differential equations of higher orders.

**Example 32** Solve the differential equation:  $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$

using method of variation of parameters.

$$\text{Solution: } \Rightarrow (D^2 + 1)y = \operatorname{cosec} x$$

$$\text{Auxiliary equation is: } (m^2 + 1) = 0$$

$$\Rightarrow m = \pm i$$

$$\text{C.F.} = c_1 \cos x + c_2 \sin x = c_1 y_1 + c_2 y_2$$

$$\therefore y_1 = \cos x \text{ and } y_2 = \sin x$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$u_1 = - \int \frac{y_2 F(x)}{W} dx = - \int \sin x \operatorname{cosec} x dx = - \int 1 dx = -x$$

$$u_2 = \int \frac{y_1 F(x)}{W} dx = \int \cos x \operatorname{cosec} x dx = \int \cot x dx = \log|\sin x|$$

$$\therefore \text{P.I.} = u_1 y_1 + u_2 y_2$$

$$= -x \cos x + \sin x \log|\sin x|$$

$$\text{Complete solution is: } y = \text{C.F.} + \text{P.I}$$

$$\Rightarrow y = c_1 \cos x + c_2 \sin x - x \cos x + \sin x \log|\sin x|$$

**Example 33** Solve the differential equation:  $(D^2 - 2D + 1)y = e^x$

using method of variation of parameters.

$$\text{Solution: Auxiliary equation is: } (m^2 - 2m + 1) = 0$$

$$\Rightarrow m = 1, 1$$

$$\text{C.F.} = (c_1 + c_2 x)e^x = c_1 e^x + c_2 x e^x = c_1 y_1 + c_2 y_2$$

$$\therefore y_1 = e^x \text{ and } y_2 = x e^x$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & x e^x \\ e^x & x e^x + e^x \end{vmatrix} = e^{2x}$$

$$u_1 = - \int \frac{y_2 F(x)}{W} dx = - \int \frac{x e^x e^x}{e^{2x}} dx = - \int x dx = -\frac{x^2}{2}$$

$$u_2 = \int \frac{y_1 F(x)}{W} dx = \int \frac{e^x e^x}{e^{2x}} dx = \int 1 dx = x$$

$$\therefore \text{P.I} = u_1 y_1 + u_2 y_2$$

$$= -\frac{x^2}{2} e^x + x^2 e^x = \frac{x^2}{2} e^x$$

Complete solution is:  $y = \text{C.F.} + \text{P.I}$

$$\Rightarrow y = (c_1 + c_2 x) e^x + \frac{x^2}{2} e^x$$

**Example 34** Solve the differential equation:  $\frac{d^2y}{dx^2} + 4y = x \sin 2x$

using method of variation of parameters.

$$\text{Solution: } \Rightarrow (D^2 + 4)y = x \sin 2x$$

$$\text{Auxiliary equation is: } (m^2 + 4) = 0$$

$$\Rightarrow m = \pm 2i$$

$$\text{C.F.} = c_1 \cos 2x + c_2 \sin 2x = c_1 y_1 + c_2 y_2$$

$$\therefore y_1 = \cos 2x \text{ and } y_2 = \sin 2x$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

$$u_1 = - \int \frac{y_2 F(x)}{W} dx = -\frac{1}{2} \int x \sin^2 2x dx = -\frac{1}{4} \int x(1 - \cos 4x) dx$$

$$= -\frac{1}{4} \left[ \frac{x^2}{2} - \left[ (x) \left( \frac{\sin 4x}{4} \right) - (1) \left( -\frac{\cos 4x}{16} \right) \right] \right]$$

$$= \left[ -\frac{x^2}{8} + \frac{x \sin 4x}{16} + \frac{\cos 4x}{64} \right]$$

$$u_2 = \int \frac{y_1 F(x)}{W} dx = \frac{1}{2} \int x \sin 2x \cos 2x dx = \frac{1}{4} \int x \sin 4x dx$$

$$= \frac{1}{4} \left[ (x) \left( -\frac{\cos 4x}{4} \right) - (1) \left( -\frac{\sin 4x}{16} \right) \right]$$

$$= \left[ -\frac{x \cos 4x}{16} + \frac{\sin 4x}{64} \right]$$

$$\therefore \text{P.I} = u_1 y_1 + u_2 y_2$$

$$= \cos 2x \left[ -\frac{x^2}{8} + \frac{x \sin 4x}{16} + \frac{\cos 4x}{64} \right] + \sin 2x \left[ -\frac{x \cos 4x}{16} + \frac{\sin 4x}{64} \right]$$

$$\begin{aligned}
&= \frac{x}{16} (\sin 4x \cos 2x - \cos 4x \sin 2x) + \frac{1}{64} (\cos 4x \cos 2x + \sin 4x \sin 2x) \\
&\quad - \frac{x^2}{8} \cos 2x = \frac{x}{16} \sin 2x + \frac{1}{64} \cos 2x - \frac{x^2}{8} \cos 2x
\end{aligned}$$

Complete solution is:  $y = \text{C.F.} + \text{P.I}$

$$\Rightarrow y = c_1 \cos 2x + c_2 \sin 2x + \frac{x}{16} \sin 2x + \frac{1}{64} \cos 2x - \frac{x^2}{8} \cos 2x$$

**Example 35** Solve the differential equation:  $(D^2 - D - 2)y = e^{(e^x+3x)}$

using method of variation of parameters.

Solution: Auxiliary equation is:  $(m^2 - m - 2) = 0$

$$\Rightarrow m = -1, 2$$

$$\text{C.F.} = c_1 e^{-x} + c_2 e^{2x} = c_1 y_1 + c_2 y_2$$

$$\therefore y_1 = e^{-x} \text{ and } y_2 = e^{2x}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{2x} \\ -e^{-x} & 2e^{2x} \end{vmatrix} = 3e^x$$

$$u_1 = - \int \frac{y_2 F(x)}{W} dx = - \int \frac{e^{2x} e^{(e^x+3x)}}{3e^x} dx = - \int \frac{e^{2x} e^{e^x} e^{3x}}{3e^x} dx$$

$$= - \frac{1}{3} \int e^{4x} e^{e^x} dx, \text{ Putting } e^x = t \Rightarrow e^x dx = dt$$

$$u_1 = - \frac{1}{3} \int t^3 e^t dt = - \frac{1}{3} [(t^3)(e^t) - (3t^2)(e^t) + (6t)(e^t) - (6)(e^t)]$$

$$\Rightarrow u_1 = - \frac{e^{e^x}}{3} [e^{3x} - 3e^{2x} + 6e^x - 6]$$

$$u_2 = \int \frac{y_1 F(x)}{W} dx = \int \frac{e^{-x} e^{(e^x+3x)}}{3e^x} dx = \int \frac{e^{-x} e^{e^x} e^{3x}}{3e^x} dx \frac{1}{3} \int e^x e^{e^x} dx = \frac{e^{e^x}}{3}$$

$$\therefore \text{P.I.} = u_1 y_1 + u_2 y_2$$

$$= - \frac{e^{e^x} e^{-x}}{3} [e^{3x} - 3e^{2x} + 6e^x - 6] + \frac{e^{e^x} e^{2x}}{3}$$

$$= \frac{e^{e^x}}{3} [3e^x - 6 + 6e^{-x}]$$

Complete solution is:  $y = \text{C.F.} + \text{P.I}$

$$\Rightarrow y = c_1 e^{-x} + c_2 e^{2x} + \frac{e^{e^x}}{3} [3e^x - 6 + 6e^{-x}]$$

**Example 36** Given that  $y_1 = x$  and  $y_2 = \frac{1}{x}$  are two linearly independent solutions of the differential equation:  $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - y = x, x \neq 0$

Find the particular integral and general solution using method of variation of parameters.

Solution: Rewriting the equation as:  $\frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} - \frac{1}{x^2} y = \frac{1}{x}$

Given that  $y_1 = x$  and  $y_2 = \frac{1}{x}$

$$\therefore \text{C.F.} = c_1 y_1 + c_2 y_2 = c_1 x + \frac{c_2}{x}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix} = -\frac{2}{x}$$

$$u_1 = - \int \frac{y_2 F(x)}{W} dx = \int \frac{1}{x} \cdot \frac{1}{x} \cdot \frac{x}{2} dx = \frac{1}{2} \int \frac{1}{x} dx = \frac{1}{2} \log x$$

$$u_2 = \int \frac{y_1 F(x)}{W} dx = - \int x \cdot \frac{1}{x} \cdot \frac{x}{2} dx = -\frac{x^2}{4}$$

$$\therefore \text{P.I.} = u_1 y_1 + u_2 y_2$$

$$= \frac{x}{2} \log x - \frac{x}{4}$$

Complete solution is:  $y = \text{C.F.} + \text{P.I.}$

$$\Rightarrow y = c_1 x + \frac{c_2}{x} + \frac{x}{2} \log x - \frac{x}{4}$$

**Example 37** Solve the differential equation:  $x^2 \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + 6y = x^2 \log x$

using method of variation of parameters.

Solution: This is a Cauchy's linear equation with variable coefficients.

Putting  $x = e^t \quad \therefore \log x = t$

$$\Rightarrow x \frac{dy}{dx} = Dy, x^2 \frac{d^2y}{dx^2} = D(D-1)y$$

∴ Given differential equation may be rewritten as

$$(D(D - 1) - 4D + 6)y = te^{2t}$$

$$\Rightarrow (D^2 - 5D + 6)y = te^{2t}$$

Auxiliary equation is:  $(m - 2)(m - 3) = 0$

$$\Rightarrow m = 2, 3$$

$$\text{C.F.} = c_1 e^{2t} + c_2 e^{3t} = c_1 y_1 + c_2 y_2$$

$$\therefore y_1 = e^{2t} \text{ and } y_2 = e^{3t}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{vmatrix} = e^{5t}$$

$$u_1 = - \int \frac{y_2 F(t)}{W} dt = - \int \frac{e^{3t} te^{2t}}{e^{5t}} dt = - \int t dt = - \frac{t^2}{2}$$

$$\begin{aligned} u_2 &= \int \frac{y_1 F(t)}{W} dt = \int \frac{e^{2t} te^{2t}}{e^{5t}} dt = \int t e^{-t} dt = [(t)(-e^{-t}) - (1)(e^{-t})] \\ &= -te^{-t} - e^{-t} \end{aligned}$$

$$\therefore \text{P.I.} = u_1 y_1 + u_2 y_2$$

$$= -\frac{t^2}{2} e^{2t} - (te^{-t} + e^{-t}) e^{3t}$$

$$= -\frac{t^2}{2} e^{2t} - te^{2t} - e^{2t} = -e^{2t} \left( \frac{t^2}{2} + t + 1 \right)$$

Complete solution is:  $y = \text{C.F.} + \text{P.I.}$

$$\Rightarrow y = c_1 e^{2t} + c_2 e^{3t} - e^{2t} \left( \frac{t^2}{2} + t + 1 \right)$$

$$\text{or } y = c_1 x^2 + c_2 x^3 - x^2 \left( \frac{(\log x)^2}{2} + \log x + 1 \right)$$

$$\Rightarrow y = c_3 x^2 + c_2 x^3 - \frac{x^2}{2} (\log x)^2 - x^2 \log x, c_3 = c_1 - 1$$

## 11.6 Solving Simultaneous Linear Differential Equations

Linear differential equations having two or more dependent variables with single independent variable are called simultaneous differential equations and can be of two types:

**Type 1:**  $f_1(D)x + f_2(D)y = F(t)$ ,  $g_1(D)x + g_2(D)y = G(t)$ ,  $D \equiv \frac{d}{dt}$

Consider a system of ordinary differential equations in two dependent variables  $x$  and  $y$  and an independent variable  $t$ :

$$f_1(D)x + f_2(D)y = F(t), \quad g_1(D)x + g_2(D)y = G(t), \quad D \equiv \frac{d}{dt}$$

Given system can be solved as follows:

1. Eliminate  $y$  from the given system of equations resulting a differential equation exclusively in  $x$ .
  2. Solve the differential equation in  $x$  by usual methods to obtain  $x$  as a function of  $t$ .
  3. Substitute value of  $x$  and its derivatives in one of the simultaneous equations to get an equation in  $y$ .
  4. Solve for  $y$  by usual methods to obtain its value as a function of  $t$ .

**Example 38** Solve the system of equations:  $\frac{dx}{dt} + y = e^t$ ,  $\frac{dy}{dt} - x = e^{-t}$

Solution: Rewriting given system of differential equations as:

$$Dx + y = e^t \dots \text{.....(1)}$$

$$Dy - x = e^{-t} \dots \textcircled{2}, \quad D \equiv \frac{d}{dt}$$

Multiplying ① by  $D$

Subtracting ② from ③, we get

which is a linear differential equation in  $x$  with constant coefficients.

To solve ④ for  $x$ , Auxiliary equation is  $m^2 + 1 = 0$

$$\Rightarrow m = \pm i$$

$$\text{C.F.} = c_1 \cos t + c_2 \sin t$$

$$\text{P.I.} = \frac{1}{f(D)} F(t) = \frac{1}{D^2+1} (e^t - e^{-t}) = \frac{1}{D^2+1} e^t - \frac{1}{D^2+1} e^{-t}$$

$= \frac{1}{2}e^t - \frac{1}{2}e^{-t}$ , Putting  $D = 1$  and  $D = -1$  in 1<sup>st</sup> and 2<sup>nd</sup> terms respectively

Using ⑤ in ①  $\Rightarrow D \left[ c_1 \cos t + c_2 \sin t + \frac{1}{2}e^t - \frac{1}{2}e^{-t} \right] + y = e^t$

$$\Rightarrow \left[ -c_1 \sin t + c_2 \cos t + \frac{1}{2}e^t + \frac{1}{2}e^{-t} \right] + y = e^t$$

⑤ and ⑥ give the required solution.

**Example 39** Solve the system of equations:  $t \frac{dx}{dt} + y = 0$ ,  $\frac{dy}{dt} + x = 0$

given that  $x(1) = 1$ ,  $y(-1) = 0$

**Solution:** Given system of equations is:

Multiplying ① by  $t \frac{d}{dt}$

$$t \frac{d}{dt} \left( t \frac{dx}{dt} + y \right) = 0$$

$$\Rightarrow t^2 \frac{d^2x}{dt^2} + t \frac{dx}{dt} + t \frac{dy}{dt} = 0 \quad \dots \dots \dots \text{③}$$

Subtracting ② from ③, we get

which is Cauchy's linear differential equation in  $x$  with variable coefficients.

.Putting  $t = e^k \quad \therefore \log t = k$

$$\Rightarrow t \frac{dx}{dt} = Dx, t^2 \frac{d^2x}{dt^2} = D(D-1)x, D \equiv \frac{d}{dk}$$

$\therefore$  ④ may be rewritten as

$$(D(D - 1) + D - 1)x = 0 \quad \dots\dots \textcircled{5}$$

$$\Rightarrow (D^2 - 1)x = 0$$

To solve ⑤ for  $x$ , Auxiliary equation is  $m^2 - 1 = 0$

$$\Rightarrow m = \pm 1$$

$$\text{C.F.} = c_1 e^k + c_2 e^{-k} = c_1 t + \frac{c_2}{t}$$

$$\text{Using ⑥ in ①} \Rightarrow t \frac{d}{dt} \left( c_1 t + \frac{c_2}{t} \right) + y = 0$$

$$\Rightarrow c_1 t - \frac{c_2}{t} + y = 0$$

Also given that at  $t = 1, x = 1$  and at  $t = -1, y = 0$

Using in ⑥and ⑦  $c_1 + c_2 = 1$ ,  $c_1 - c_2 = 0 \Rightarrow c_1 = c_2 = \frac{1}{2}$

Using  $c_1 = c_2 = \frac{1}{2}$  in ⑥ and ⑦, we get

$$x = \frac{1}{2} \left( t + \frac{1}{t} \right), \quad y = \frac{1}{2} \left( \frac{1}{t} - t \right)$$

**Example 40** Solve the system of equations:

$$\frac{d^2x}{dt^2} + y = \sin t, \quad \frac{d^2y}{dt^2} + x = \cos t$$

Solution: Rewriting given system of differential equations as:

$$D^2y + x = \cos t \quad \dots \dots \textcircled{2}, \quad D \equiv \frac{d}{dt}$$

Multiplying ① by  $D^2$

$$D^2(D^2x + y) = D^2\sin t$$

Subtracting ② from ③, we get

which is a linear differential equation in  $x$  with constant coefficients.

To solve ④ for  $x$ , Auxiliary equation is  $m^4 - 1 = 0$

$$\Rightarrow (m^2 - 1)(m^2 + 1) = 0$$

$$\Rightarrow m = \pm 1, \pm i$$

$$\text{C.F.} = c_1 e^t + c_2 e^{-t} + (c_3 \cos t + c_4 \sin t)$$

$$\text{P.I.} = \frac{1}{f(D)} F(t) = \frac{1}{D^4 - 1} (-\sin t - \cos t) = -\frac{1}{D^4 - 1} \sin t - \frac{1}{D^4 - 1} \cos t$$

Putting  $D^2 = -1$  i.e.  $D^4 = 1$  in 1<sup>st</sup> and 2<sup>nd</sup> terms, it is a case of failure

$$\begin{aligned}\therefore \text{P.I.} &= -t \frac{1}{4D^3} \sin t - t \frac{1}{4D^3} \cos t \\ &= \frac{t}{4D} \sin t + \frac{t}{4D} \cos t \quad \text{putting } D^2 = -1 \\ &= -\frac{t}{4} \cos t + \frac{t}{4} \sin t\end{aligned}$$

$$\therefore x = (c_1 e^t + c_2 e^{-t}) + (c_3 \cos t + c_4 \sin t) + \frac{t}{4} (\sin t - \cos t) \dots \textcircled{5}$$

Using \textcircled{5} in \textcircled{1}

$$\begin{aligned}\Rightarrow D^2 \left[ c_1 e^t + c_2 e^{-t} + (c_3 \cos t + c_4 \sin t) + \frac{t}{4} (\sin t - \cos t) \right] + y &= \sin t \\ \Rightarrow D \left[ c_1 e^t - c_2 e^{-t} - c_3 \sin t + c_4 \cos t + \frac{t}{4} (\cos t + \sin t) + \frac{1}{4} (\sin t - \cos t) \right] \\ &+ y = \sin t \\ \Rightarrow \left[ c_1 e^t + c_2 e^{-t} - c_3 \cos t - c_4 \sin t + \frac{t}{4} (-\sin t + \cos t) + \frac{1}{4} (\cos t + \sin t) \right. \\ &\quad \left. + \frac{1}{4} (\cos t + \sin t) \right] + y = \sin t \\ \Rightarrow y &= -(c_1 e^t + c_2 e^{-t}) + (c_3 \cos t + c_4 \sin t) + \left( \frac{t}{4} + \frac{1}{2} \right) (\sin t - \cos t) \dots \textcircled{6}\end{aligned}$$

\textcircled{5} and \textcircled{6} give the required solution.

**Type II: Symmetric simultaneous equations of the form**  $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Simultaneous differential equations in the form  $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$  can be solved by the method of grouping or the method of multipliers or both to get two independent solutions:  $u = c_1, v = c_2$ ; where  $c_1$  and  $c_2$  are arbitrary constants.

**Method of grouping:** In this method, we consider a pair of fractions at a time which can be solved for an independent solution.

**Method of multipliers:** In this method, we multiply each fraction by suitable multipliers (not necessarily constants) such that denominator becomes zero.

If  $a, b, c$  are multipliers, then  $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{adx+b dy+c dz}{aP+bQ+cR}$

**Example 41** Solve the set of simultaneous equations:

$$\frac{dx}{(z^2 - 2yz - y^2)} = \frac{dy}{(xy + zx)} = \frac{dz}{(xy - zx)}$$

Taking  $x, y, z$  as multipliers, each fraction equals

$$\frac{xdx + ydy + zdz}{(xz^2 - 2xyz - xy^2 + xy^2 + xyz + xyz - xz^2)} = \frac{xdx + ydy + zdz}{0}$$

$$\Rightarrow xdx + ydy + zdz = 0$$

Integrating, we get  $\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c'_1$

1<sup>st</sup> independent solution is:  $u = x^2 + y^2 + z^2 = c_1 \dots \dots \dots \textcircled{1}$

Now for 2<sup>nd</sup> independent solution, taking last two members of the set of

$$\text{equations: } \frac{dy}{x(y+z)} = \frac{dz}{x(y-z)}$$

$$\Rightarrow (y - z)dy = (y + z)dz$$

$$\Rightarrow ydy - (zdy + ydz) - zdz = 0$$

$$\Rightarrow ydy - d(yz) - zdz = 0$$

Integrating, we get

$$\frac{y^2}{2} - yz - \frac{z^2}{2} = c'_2$$

$$\Rightarrow v = y^2 - 2yz - z^2 = c_2 \dots \dots \dots \textcircled{2}$$

$\textcircled{1}$  and  $\textcircled{2}$  give the required solution.

### Exercise 11B

**Q1. Solve the following differential equations:**

i.  $x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 4y = (1+x)^2$

Ans.  $\langle y = (c_1 + c_2 \log x)x^2 + \frac{1}{4} + 2x + \frac{x^2}{2}(\log x)^2 \rangle$

ii.  $x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + 2y = e^x$

Ans.  $\langle y = \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{e^x}{x^2} \rangle$

iii.  $(2x+3)^2 \frac{d^2y}{dx^2} - (2x+3) \frac{dy}{dx} - 12y = 6x$

Ans.  $\langle y = c_1(2x+3)^{\frac{3+\sqrt{57}}{4}} + c_2(2x+3)^{\frac{3-\sqrt{57}}{4}} - \frac{3}{14}(2x+3) + \frac{3}{4} \rangle$

iv.  $(x+1)^2 \frac{d^2y}{dx^2} + (x+1) \frac{dy}{dx} + y = 4 \cos(\log(x+1))$

Ans.  $\langle y = c_1 \cos(\log(x+1)) + c_2 \sin(\log(x+1)) + 2 \log(x+1) \sin \log x + 1 \rangle$

**Q2. Solve the following differential equations using method of variation of parameters**

i.  $\frac{d^2y}{dx^2} + y = x \sin x$

Ans.  $\langle y = c_1 \cos x + c_2 \sin x + \frac{1}{8} \cos x + \frac{x}{4} \sin x - \frac{x^2}{4} \cos x \rangle$

ii.  $(D^2 - 1)y = e^{-2x} \sin e^{-x}$

Ans.  $\langle y = c_1 e^x + c_2 e^{-x} - \sin e^{-x} - e^x \cos e^{-x} \rangle$

iii.  $(D^2 - 2D)y = e^x \sin x$

Ans.  $\langle y = c_1 + c_2 e^{2x} - \frac{1}{2} e^x \sin x \rangle$

iv.  $\frac{d^2y}{dx^2} - 2 \frac{dy}{dx} = e^x \log x$

Ans.  $\langle y = c_1 + c_2 e^{2x} + \frac{x^2}{4} e^x (2 \log x - 3) \rangle$

**Q2. Solve the following set of simultaneous differential equations**

i.  $\frac{dx}{dt} - 7x + y = 0, \quad \frac{dy}{dt} - 2x - 5y = 0$

Ans.  $\langle x = e^{6t}(c_1 \cos t + c_2 \sin t), y = e^{6t}(c_1 - c_2) \cos t + (c_1 + c_2) \sin t \rangle$

ii.  $(D+1)x + (2D+1)y = e^t, \quad (D-1)x + (D+1)y = 1$

$$\text{Ans: } \langle x = c_1 e^t + c_2 e^{-2t} + 2e^{-t}, y = 3c_1 e^t + 2c_2 e^{-2t} + 3e^{-t} \rangle$$

$$\text{iii. } \frac{dx}{(z^2 - 2yz - y^2)} = \frac{dy}{(xy + zx)} = \frac{dz}{(xy - zx)}$$

$$\text{Ans: } \langle xy - z = c_1, x^2 - y^2 + z^2 = c_2 \rangle$$

## 11.7 Previous Years Solved Questions

Q1. Solve  $(D^2 + D + 1)^2(D - 2)y = 0$

**⟨Q1(h), GGSIPU, December 2012⟩**

Solution: Auxiliary equation is:  $(m^2 + m + 1)^2(m - 2)y = 0$ .....①

Solving ①, we get

$$\Rightarrow m = 2, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i, -\frac{1}{2} \mp \frac{\sqrt{3}}{2}i$$

$$\text{C.F.} = c_1 e^{2x} + e^{\frac{-x}{2}} [(c_2 + c_3 x) \cos \frac{\sqrt{3}}{2} x + (c_4 + c_5 x) \sin \frac{\sqrt{3}}{2} x]$$

Since  $F(x) = 0$ , solution is given by  $y = C.F$

$$\Rightarrow y = c_1 e^{2x} + e^{\frac{-x}{2}} [(c_2 + c_3 x) \cos \frac{\sqrt{3}}{2} x + (c_4 + c_5 x) \sin \frac{\sqrt{3}}{2} x]$$

Q2. Solve  $(D^2 - 1)y = \cosh x \cos x$

**⟨Q8(b), GGSIPU, December 2012⟩**

Solution: Auxiliary equation is:  $m^2 - 1 = 0$

$$\Rightarrow m = \pm 1$$

$$\text{C.F.} = c_1 e^x + c_2 e^{-x}$$

$$\text{P.I.} = \frac{1}{f(D)} F(x)$$

$$= \frac{1}{D^2 - 1} \left( \frac{e^x + e^{-x}}{2} \cos x \right) \quad \because \cosh x = \frac{e^x + e^{-x}}{2}$$

$$= \frac{1}{D^2 - 1} \left( \frac{e^x}{2} \cos x + \frac{e^{-x}}{2} \cos x \right)$$

$$= \frac{e^x}{2} \frac{1}{(D+1)^2 - 1} \cos x + \frac{e^{-x}}{2} \frac{1}{(D-1)^2 - 1} \cos x$$

$$\begin{aligned}
&= \frac{e^x}{2} \frac{1}{(D^2+2D)} \cos x + \frac{e^{-x}}{2} \frac{1}{D^2-2D} \cos x \\
&= \frac{e^x}{2} \frac{1}{2D-1} \cos x + \frac{e^{-x}}{2} \frac{1}{-2D-1} \cos x \quad \text{Putting } D^2 = -1 \\
&= \frac{e^x}{2} \frac{2D+1}{4D^2-1} \cos x - \frac{e^{-x}}{2} \frac{2D-1}{4D^2-1} \cos x \\
&= -\frac{e^x}{10} (2D+1) \cos x + \frac{e^{-x}}{10} (2D-1) \cos x \quad \text{Putting } D^2 = -1 \\
&= -\frac{e^x}{10} (-2 \sin x + \cos x) + \frac{e^{-x}}{10} (-2 \sin x - \cos x)
\end{aligned}$$

$$\therefore \text{P.I.} = \frac{e^x}{10} (2 \sin x - \cos x) - \frac{e^{-x}}{10} (2 \sin x + \cos x)$$

Complete solution is:  $y = \text{C.F.} + \text{P.I}$

$$\Rightarrow y = c_1 e^x + c_2 e^{-x} + \frac{e^x}{10} (2 \sin x - \cos x) - \frac{e^{-x}}{10} (2 \sin x + \cos x)$$

**Q3.** Solve  $\frac{d^2y}{dx^2} + 4y = 4 \tan 2x$  by the method of variation of parameters.

**(Q9(a), GGSIPU, December 2012)**

Solution:  $\Rightarrow (D^2 + 4)y = 4 \tan 2x$

Auxiliary equation is:  $(m^2 + 4) = 0$

$$\Rightarrow m = \pm 2i$$

$$\text{C.F.} = c_1 \cos 2x + c_2 \sin 2x = c_1 y_1 + c_2 y_2$$

$$\therefore y_1 = \cos 2x \text{ and } y_2 = \sin 2x$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2 \sin 2x & 2 \cos 2x \end{vmatrix} = 2$$

$$u_1 = - \int \frac{y_2 F(x)}{W} dx = - \frac{4}{2} \int \sin 2x \tan 2x dx = -2 \int \frac{\sin^2 2x}{\cos 2x} dx$$

$$-2 \int \frac{1 - \cos^2 2x}{\cos 2x} dx = -2 \int (\sec 2x - \cos 2x) dx$$

$$= -2 \left[ \frac{1}{2} \log |\sec 2x + \tan 2x| - \frac{1}{2} \sin 2x \right]$$

$$= [\sin 2x - \log|\sec 2x + \tan 2x|]$$

$$\begin{aligned} u_2 &= \int \frac{y_1 F(x)}{w} dx = \frac{1}{2} \int 4 \tan 2x \cos 2x \, dx = 2 \int \sin 2x \, dx \\ &= -\cos 2x \end{aligned}$$

$$\therefore P.I = u_1 y_1 + u_2 y_2$$

$$\begin{aligned} &= \cos 2x [\sin 2x - \log|\sec 2x + \tan 2x|] - \sin 2x \cos 2x \\ &= -\cos 2x \log|\sec 2x + \tan 2x| \end{aligned}$$

Complete solution is:  $y = C.F. + P.I$

$$\Rightarrow y = c_1 \cos 2x + c_2 \sin 2x - \cos 2x \log|\sec 2x + \tan 2x|$$

**Q4.** Solve the system of equations:  $\frac{dx}{dt} + x = y + e^t$ ,  $\frac{dy}{dt} + y = x + e^t$

**(Q9(b), GGSIPU, December 2012)**

Solution: Rewriting given system of differential equations as:

$$(D + 1)x - y = e^t \dots\dots \textcircled{1}$$

$$(D + 1)y - x = e^t \dots\dots \textcircled{2}, D \equiv \frac{d}{dt}$$

Multiplying  $\textcircled{1}$  by  $(D + 1)$

$$\Rightarrow (D + 1)^2 x - (D + 1)y = (D + 1)e^t$$

$$(D^2 + 2D + 1)x - (D + 1)y = 2e^t \dots\dots \textcircled{3}$$

Adding  $\textcircled{2}$  and  $\textcircled{3}$ , we get

$$(D^2 + 2D)x = 3e^t \dots\dots \textcircled{4}$$

which is a linear differential equation in  $x$  with constant coefficients.

To solve  $\textcircled{4}$  for  $x$ , Auxiliary equation is  $m^2 + 2m = 0$

$$\Rightarrow m = 0, -2$$

$$C.F. = c_1 + c_2 e^{-2t}$$

$$P.I. = \frac{1}{f(D)} F(t) = 3 \frac{1}{D^2 + 2D} e^t$$

$$= e^t \quad \text{Putting } D = 1$$

$$\therefore x = c_1 + c_2 e^{-2t} + e^t \dots\dots \textcircled{5}$$

$$\text{Using ⑤ in ①} \Rightarrow D[c_1 + c_2 e^{-2t} + e^t] + c_1 + c_2 e^{-2t} + e^t - y = e^t$$

$$\Rightarrow -2c_2e^{-2t} + e^t + c_1 + c_2e^{-2t} - y = 0$$

⑤ and ⑥ give the required solution.

**Q5.** Solve by method of variation of parameters  $y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$

$\langle Q8(a), GGSIPU, December 2013 \rangle, \langle Q3(b), GGSIPU, 2^{nd} term 2014 \rangle$

Solution: Auxiliary equation is:  $m^2 - 6m + 9 = 0$

$$(m - 3)^2 = 0$$

$$\Rightarrow m = 3,3$$

$$\text{C.F.} = (c_1 + c_2 x)e^{3x} = c_1 e^{3x} + c_2 x e^{3x} = c_1 y_1 + c_2 y_2$$

$$\therefore y_1 = e^{3x} \text{ and } y_2 = xe^{3x}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & 3xe^{3x} + e^{3x} \end{vmatrix} = e^{6x}$$

$$u_1 = - \int \frac{y_2 F(x)}{W} dx = - \int \frac{x e^{3x} e^{3x}}{x^2 e^{6x}} dx = - \int \frac{1}{x} dx = -\log x$$

$$u_2 = \int \frac{y_1 F(x)}{W} dx = \int \frac{e^{3x} e^{3x}}{x^2 e^{6x}} dx = \int \frac{1}{x^2} dx = -\frac{1}{x}$$

$$\therefore \text{P.I} = u_1 y_1 + u_2 y_2$$

$$= -e^{3x} \log x - e^{3x} = -e^{3x}(1 + \log x)$$

Complete solution is:  $y = \text{C.F.} + \text{P.I}$

$$\Rightarrow y = (c_1 + c_2 x)e^{3x} - e^{3x}(1 + \log x)$$

**Q6.** Solve the differential equation:  $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{2x} + \sin 2x$

*⟨Q8(b), GGSIPU, December 2013⟩*

$$\text{Solution: } \Rightarrow (D^3 + 2D^2 + D)y = e^{2x} + \sin 2x$$

Auxiliary equation is:  $m^3 + 2m^2 + m = 0$

$$\Rightarrow m(m^2 + 2m + 1) = 0$$

$$\Rightarrow m(m + 1)^2 = 0$$

$$\Rightarrow m = 0, -1, -1$$

$$\text{C.F.} = c_1 + e^{-x}(c_2 + c_3x)$$

$$\text{P.I.} = \frac{1}{f(D)} F(x) = \frac{1}{D^3 + 2D^2 + D} (e^{2x} + \sin 2x)$$

$$= \frac{1}{D^3 + 2D^2 + D} e^{2x} + \frac{1}{D^3 + 2D^2 + D} \sin 2x$$

$$= \frac{1}{18} e^{2x} + \frac{1}{-4D-8+D} \sin 2x, \text{ putting } D = 2 \text{ in 1}^{\text{st}} \text{ term, } D^2 = -4 \text{ in the 2}^{\text{nd}} \text{ term}$$

$$= \frac{1}{18} e^{2x} - \frac{3D-8}{(3D+8)(3D-8)} \sin 2x = \frac{1}{18} e^{2x} - \frac{3D-8}{(9D^2-64)} \sin 2x$$

$$= \frac{1}{18} e^{2x} + \frac{1}{100} (3D - 8) \sin 2x$$

$$= \frac{1}{18} e^{2x} + \frac{1}{100} (6\cos 2x - 8\sin 2x)$$

Complete solution is:  $y = \text{C.F.} + \text{P.I}$

$$\Rightarrow y = c_1 + e^{-x}(c_2 + c_3x) + \frac{1}{18} e^{2x} + \frac{1}{100} (6\cos 2x - 8\sin 2x)$$

**Q7.** Solve  $(D^2 - 2D + 1)y = xe^x \cos x$

*(Q8(a), GGSIPU, December 2014)*

Solution: Auxiliary equation is:  $m^2 - 2m + 1 = 0$

$$\Rightarrow (m - 1)^2$$

$$\Rightarrow m = 1, 1$$

$$\text{C.F.} = (c_1 + c_2x)e^x$$

$$\text{P.I.} = \frac{1}{f(D)} F(x) = \frac{1}{D^2 - 2D + 1} xe^x \cos x$$

$$= e^x \frac{1}{(D+1)^2 - 2(D+1) + 1} x \cos x$$

$$= e^x \frac{1}{D^2} x \cos x$$

$$\begin{aligned}
&= e^x \frac{1}{D} \int x \cos x \, dx \\
&= e^x \frac{1}{D} [(x)(\sin x) - (1)(-\cos x)] \\
&= e^x \frac{1}{D} [x \sin x + \cos x] \\
&= e^x [\int x \sin x \, dx + \int \cos x \, dx] \\
&= e^x [(x)(-\cos x) - (1)(-\sin x)] + \sin x \\
&\therefore \text{P.I.} = e^x [-x \cos x + 2 \sin x]
\end{aligned}$$

Complete solution is:  $y = \text{C.F.} + \text{P.I}$

$$\Rightarrow y = (c_1 + c_2 x) e^x + e^x [-x \cos x + 2 \sin x]$$

**Q8.** Solve by M.O.V.P.  $\frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + y = e^x \log x$

**(Q8(b), GGSIPU, December 2014)**

Solution: Given differential equation may be rewritten as

$$(D^2 - 2D + 1)y = e^x \log x$$

: Auxiliary equation is:  $m^2 - 2m + 1 = 0$

$$\Rightarrow (m - 1)^2$$

$$\Rightarrow m = 1, 1$$

$$\begin{aligned}
\text{C.F.} &= (c_1 + c_2 x) e^x = c_1 y_1 + c_2 y_2 \\
\therefore y_1 &= e^x \text{ and } y_2 = x e^x
\end{aligned}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & x e^x \\ e^x & x e^x + e^x \end{vmatrix} = e^{2x}$$

$$u_1 = - \int \frac{y_2 F(x)}{W} dx = - \int \frac{x e^x e^x \log x}{e^{2x}} dx = - \int x \log x \, dx$$

$$\int x \log x \, dx = I = \left[ (x)(x \log x - x) - (1) \left( I - \frac{x^2}{2} \right) \right]$$

$$\therefore \int \log x \, dx = x \log x - x$$

$$\Rightarrow 2I = x^2 \log x - x^2 + \frac{x^2}{2}$$

$$\Rightarrow I = \int x \log x \, dx = \frac{x^2}{2} \log x - \frac{x^2}{4}$$

$$\therefore u_1 = \frac{x^2}{4} - \frac{x^2}{2} \log x$$

$$u_2 = \int \frac{y_1 F(x)}{W} dx = \int \frac{e^x e^x \log x}{e^{2x}} dx = \int \log x \, dx = x \log x - x$$

$$\begin{aligned}\therefore P.I. &= \left( \frac{x^2}{4} - \frac{x^2}{2} \log x \right) e^x + (x \log x - x) x e^x \\ &= e^x \left( \frac{x^2}{4} - \frac{x^2}{2} \log x + x^2 \log x - x^2 \right) \\ &= \frac{x^2}{2} e^x \left( \log x - \frac{3}{2} \right)\end{aligned}$$

Complete solution is:  $y = C.F. + P.I$

$$\Rightarrow y = (c_1 + c_2 x) e^x + \frac{x^2}{2} e^x \left( \log x - \frac{3}{2} \right)$$

$$\textbf{Q9. Solve } (D - 1)^2(D + 1)^2 = \sin^2 \frac{x}{2} + e^x + x$$

**(Q1(a), GGSIPU, December 2015)**

Solution: Auxiliary equation is:  $(m - 1)^2(m + 1)^2 = 0$

$$\Rightarrow m = 1, 1, -1, -1$$

$$C.F. = (c_1 + c_2 x) e^x + (c_3 + c_4 x) e^{-x}$$

$$\begin{aligned}P.I. &= \frac{1}{f(D)} F(x) = \frac{1}{((D-1)(D+1))^2} (\sin^2 \frac{x}{2} + e^x + x) \\ &= \frac{1}{2 D^4 - 2 D^2 + 1} (1 - \cos x) + \frac{1}{D^4 - 2 D^2 + 1} e^x + \frac{1}{D^4 - 2 D^2 + 1} x \\ &= \frac{1}{2 D^4 - 2 D^2 + 1} e^{0x} - \frac{1}{2 D^4 - 2 D^2 + 1} \cos x + \frac{1}{D^4 - 2 D^2 + 1} e^x + \frac{1}{D^4 - 2 D^2 + 1} x\end{aligned}$$

$$\text{Now } \frac{1}{2 D^4 - 2 D^2 + 1} e^{0x} = \frac{1}{2}, \text{ putting } D = 0$$

$$\text{Also } \frac{1}{2 D^4 - 2 D^2 + 1} \cos x = \frac{1}{8} \cos x \text{ putting } D^2 = -1$$

$$\text{Again } \frac{1}{D^4 - 2 D^2 + 1} e^x = x \frac{1}{4 D^3 - 4 D} e^x \text{ as } f(1) = 0, \text{ a case of failure 2 times}$$

$$= x^2 \frac{1}{12D^2-4} e^x = \frac{x^2}{8} e^x, \text{ putting } D = 1$$

$$\text{And } \frac{1}{D^4-2D^2+1} x = \frac{1}{1+(D^4-2D^2)} x = [1 + (D^4 - 2D^2)]^{-1} x = x$$

$$\therefore \text{P.I.} = \frac{1}{2} - \frac{1}{8} \cos x + \frac{x^2}{8} e^x + x$$

Complete solution is:  $y = \text{C.F.} + \text{P.I}$

$$\Rightarrow y = (c_1 + c_2 x) e^x + (c_3 + c_4 x) e^{-x} - \frac{1}{8} \cos x + \frac{x^2}{8} e^x + x + \frac{1}{2}$$

**Q.10** Solve  $x^2 \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + 6y = x^4 \sin x$

(Q3(b), GGSIPU, December 2015)

Solution: This is a Cauchy's linear equation with variable coefficients.

$$\text{Putting } x = e^t \quad \therefore \log x = t$$

$$\Rightarrow x \frac{dy}{dx} = Dy, x^2 \frac{d^2y}{dx^2} = D(D-1)y$$

$\therefore$  Equation may be rewritten as

$$(D(D-1) - 4D + 6)y = e^{4t} \sin e^t$$

$$\Rightarrow (D^2 - 5D + 6)y = e^{4t} \sin e^t, D \equiv \frac{d}{dt}$$

Auxiliary equation is:  $m^2 - 5m + 6 = 0$

$$\Rightarrow (m-2)(m-3) = 0$$

$$\Rightarrow m = 2, 3$$

$$\text{C.F.} = c_1 e^{2t} + c_2 e^{3t}$$

$$= c_1 x^2 + c_2 x^3$$

$$\text{P.I.} = \frac{1}{f(D)} F(x) = \frac{1}{D^2-5D+6} e^{4t} \sin e^t$$

$$= e^{4t} \frac{1}{(D+4)^2-5(D+4)+6} \sin e^t$$

$$= e^{4t} \frac{1}{D^2+3D+2} \sin e^t = e^{4t} \frac{1}{(D+1)(D+2)} \sin e^t$$

$$\begin{aligned}
&= e^{4t} \left[ \frac{1}{(D+1)} - \frac{1}{(D+2)} \right] \sin e^t = e^{4t} \left[ \frac{1}{(D+1)} \sin e^t - \frac{1}{(D+2)} \sin e^t \right] \\
&= e^{4t} [e^{-t} \int e^t \sin e^t dt - e^{-2t} \int e^{2t} \sin e^t dt] \\
&\quad \because \frac{1}{(D+a)} F(t) = e^{-at} \int e^{at} F(t) dt
\end{aligned}$$

Solving the two integrals by putting  $e^t = u$ ,  $\therefore e^t dt = du$

$$\therefore P.I = -e^{2t} \sin e^t = -x^2 \sin x$$

Complete solution is:  $y = C.F. + P.I$

$$\Rightarrow y = c_1 x^2 + c_2 x^3 - x^2 \sin x$$