where do we use recurrence relation in real life.



# Lecture-4

## Recurrence Relation

# Applications of Recurrence Relations

In this section we will show that such relations can be used to study and to solve counting problems. For example, suppose that the number of bacteria in a colony doubles every hour. If a colony begins with five bacteria, how many will be present in $n$ hours? ~~To solve this problem,~~

Sol$^n$ Let there be $a_n$ no. of bacteria at the end of $n$ hours. Also, It is given that no of bacteria doubles every hour.

CODERINDEED

$$a_n = 2a_{n-1} \quad ; \quad a_0 = 5 \longrightarrow (1)$$

$n-(n-1)$

$\cancel{n}-\cancel{n}+1$

Change $n$ to $n-1$ in (1)

$$a_{n-1} = 2a_{n-2} \longrightarrow (2)$$

$$\vdots$$

$$a_2 = 2a_1 \longrightarrow (n-1)$$

$$a_1 = 2a_0 \longrightarrow (n)$$

$\therefore$ from (n),

$$a_1 = 2 \cdot 5$$

from (n-1)

$$a_2 = 2a_1 = 2(2 \cdot 5) = 2^2 \cdot 5$$

$$\vdots$$

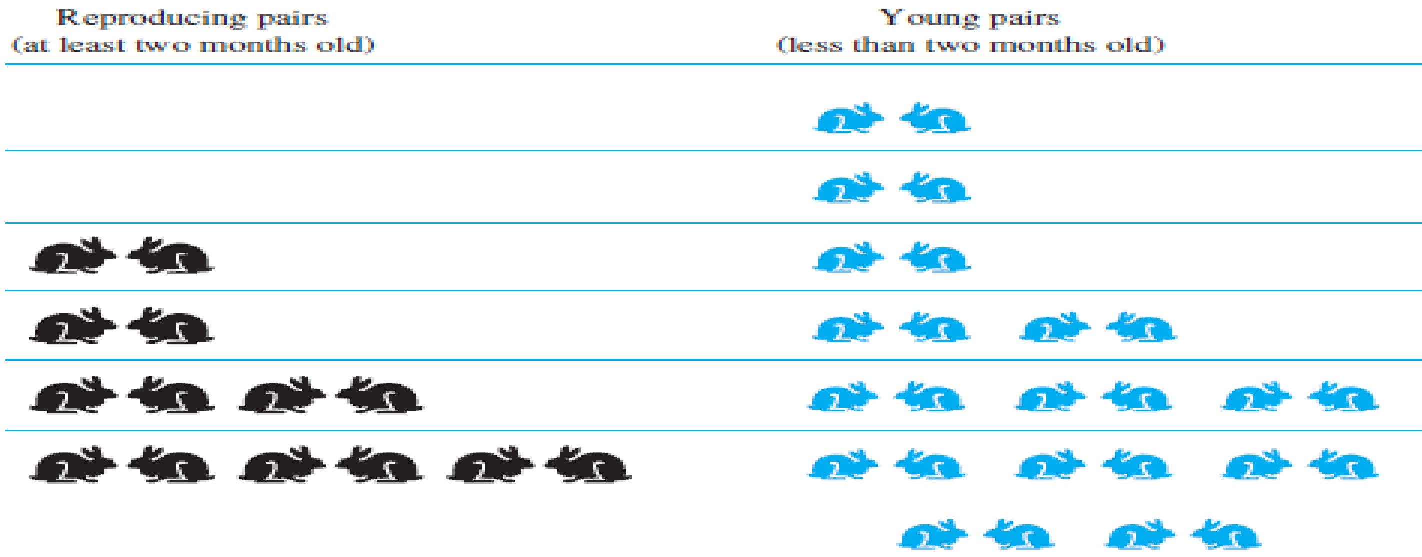
$$\boxed{a_n = 2^n \cdot 5}$$

This is called explicit sol^n of (

## 8.1.2 Modeling With Recurrence Relations

**EXAMPLE 1** **Rabbits and the Fibonacci Numbers** Consider this problem, which was originally posed by Leonardo Pisano, also known as Fibonacci, in the thirteenth century in his book *Liber abaci*. A young pair of rabbits (one of each sex) is placed on an island. A pair of rabbits does not breed until they are 2 months old. After they are 2 months old, each pair of rabbits produces another pair each month, as shown in Figure 1. Find a recurrence relation for the number of pairs of rabbits on the island after $n$ months, assuming that no rabbits ever die.

Links




| Month | Reproducing pairs | Young pairs | Total pairs |
|---|---|---|---|
| 1 | 0 | 1 | 1 |
| 2 | 0 | 1 | 1 |
| 3 | 1 | 1 | 2 |
| 4 | 1 | 2 | 3 |
| 5 | 2 | 3 | 5 |
| 6 | 3 | 5 | 8 |

**FIGURE 1** **Rabbits on an island.**

**Definition 1**

A *linear homogeneous recurrence relation of degree k with constant coefficients* is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

**EXAMPLE 1** The recurrence relation $P_n = (1.11)P_{n-1}$ is a linear homogeneous recurrence relation of degree one. The recurrence relation $f_n = f_{n-1} + f_{n-2}$ is a linear homogeneous recurrence relation of

**EXAMPLE 2** The recurrence relation $a_n = a_{n-1} + a_{n-2}^2$ is not linear. The recurrence relation $H_n = 2H_{n-1} + 1$ is not homogeneous. The recurrence relation $B_n = nB_{n-1}$ does not have constant coefficients. ◀

# Trignometric function Values

$$\frac{0}{4}, \quad \frac{1}{4}, \quad \frac{\cancel{2}}{\cancel{4}}, \quad \frac{3}{4}, \quad \frac{\cancel{4}}{\cancel{4}}$$

$$\sqrt{0}, \quad \sqrt{\frac{1}{4}}, \quad \sqrt{\frac{1}{2}}, \quad \sqrt{\frac{3}{4}}, \quad \sqrt{1}$$

$$0, \quad \frac{1}{2}, \quad \frac{1}{\sqrt{2}}, \quad \frac{\sqrt{3}}{2}, \quad 1$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

| $\theta$ | $0$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ |
|---|---|---|---|---|---|
| $\sin\theta$ | $0$ | $\frac{1}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{\sqrt{3}}{2}$ | $1$ |
| $\cos\theta$ | $1$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{2}$ | $0$ |
| $\tan\theta$ | $0$ | $\frac{1}{\sqrt{3}}$ | $1$ | $\sqrt{3}$ | $\infty$ |

$\sin\theta$

$\operatorname{cosec}\theta$

$2^{nd}$ Quadrant

$\pi - \theta$

School

$1^{st}$ Quadrant

All

$\theta$

$-\pi \le \theta < \pi$

$\tan\theta$

$\cot\theta$

$3^{rd}$ Quadrant

$-\pi + \theta$

to

$4^{th}$ Quadrant

$-\theta$

College

$\cos\theta$

$\sec\theta$

$\sin\theta$ $\cos\theta$ $\tan\theta$

$\operatorname{cosec}\theta$ $\sec\theta$ $\cot\theta$

$$x+iy = r(\cos\theta + i\sin\theta) \longrightarrow (*)$$

Compare real and Img parts.

$$x = r\cos\theta \longrightarrow (1)$$

$$y = r\sin\theta \longrightarrow (2)$$

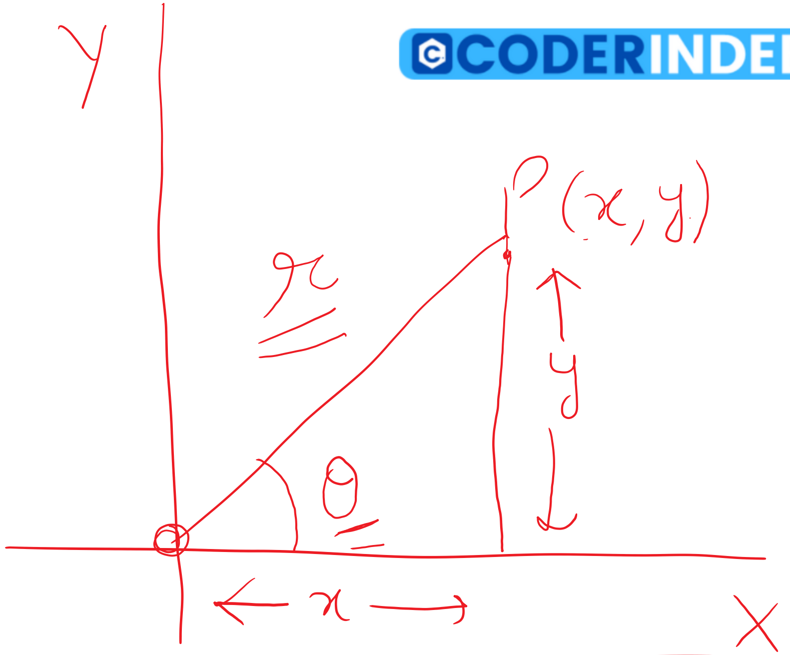
Squaring and adding (1) & (2);

$$x^2+y^2 = (r\cos\theta)^2+(r\sin\theta)^2$$

$$x^2+y^2 = r^2[\cos^2\theta+\sin^2\theta]$$

$$r^2 = x^2+y^2$$

$$r = \sqrt{x^2+y^2} \checkmark$$




(2) ÷ (1) ⇒

$$\frac{\cancel{r}\sin\theta}{\cancel{r}\cos\theta} = \frac{y}{x}$$

$$\tan\theta = \frac{y}{x}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

# De-moiver's Theorem

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta}$$

Solve the recurrence relation.

$$a_n + a_{n-1} + a_{n-2} = 0$$

Sol$^n$ :- The given recurrence relation is.

$$a_n + a_{n-1} + a_{n-2} = 0 \longrightarrow (1)$$

Its order $= n-(n-2) = \cancel{n} - \cancel{n} + 2 = \underline{\underline{2}}$

$\therefore$ Its characteristic Eqn is

$$\alpha^2 + \alpha + 1 = 0$$

$$\alpha^2+\alpha+1=0$$

$$a=1,\quad b=1,\quad c=1$$

$$\alpha=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-1\pm\sqrt{(1)^2-4(1)(1)}}{2(1)}$$

$$=\frac{-1\pm\sqrt{1-4}}{2}$$

$$=\frac{-1\pm\sqrt{-3}}{2}=\frac{-1}{2}\pm\frac{i\sqrt{3}}{2}$$

$$=-\frac{1}{2}+\frac{i\sqrt{3}}{2},\quad -\frac{1}{2}-\frac{i\sqrt{3}}{2}$$

$$\therefore a_n=c_1\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^n+c_2\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^n$$

2

Consider

$$-\frac{1}{2}+i\frac{\sqrt{3}}{2} = r(\cos\theta + i\sin\theta)$$

Equate real and Imz parts.

$$r\cos\theta = -\frac{1}{2} \longrightarrow 3$$

$$r\sin\theta = \frac{\sqrt{3}}{2} \longrightarrow 4$$

Squaring and adding 3 & 4

$$r^2 = \left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$

$$r^2 = \frac{1}{4} + \frac{3}{4} = \frac{\cancel{4}}{\cancel{4}} = 1$$

$$r^2 = 1 \Rightarrow \boxed{r = 1}$$

$\therefore$ from 3 & 4, we get,

$$\cos\theta = \frac{-1}{2}, \quad \sin\theta = \frac{\sqrt{3}}{2}$$

As $\sin\theta$ is +ve and $\cos\theta$ is negitive, $\theta$ lies in 2nd quadra

$$\theta = \pi - \frac{\pi}{3} = \frac{3\pi - \pi}{3} = \frac{2\pi}{3}$$

CODERINDEED

$$-\frac{1}{2}+i\frac{\sqrt{3}}{2} = r(\cos\theta + i\sin\theta)$$

$$-\frac{1}{2}+i\frac{\sqrt{3}}{2} = \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

Raising power n on both sides

$$\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^n = \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^n = \cos\frac{2n\pi}{3} + i\sin\frac{2n\pi}{3}$$

$$\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)^n = \cos\frac{2n\pi}{3} - i\sin\frac{2n\pi}{3}$$

$$a_n = c_1\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^n + c_2\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)^n \longrightarrow (2)$$

$$= c_1\left[\cos\frac{2n\pi}{3} + i\sin\frac{2n\pi}{3}\right] + c_2\left(\cos\frac{2n\pi}{3} - i\sin\frac{2n\pi}{3}\right)$$

$$= (c_1+c_2)\cos\frac{2n\pi}{3} + (ic_1 - ic_2)\sin\frac{2n\pi}{3}$$

$$= \boxed{A\cos\frac{2n\pi}{3} + B\sin\frac{2n\pi}{3}}$$

If the roots of characteristic eqn are $\underline{1,1,1,1}$, 2, 2, 3, 3, 3. Then write the general Solⁿ.

<u>Solⁿ</u> For the roots 1, 1, 1, 1 the Solⁿ is $(c_1+c_2n+c_3n^2+c_4n^3)(1)^n$

For the roots 2, 2 the Solⁿ is $(c_5+c_6n)(2)^n$

For the roots 3, 3, 3 the Solⁿ is $(c_7+c_8n+c_9n^2)(3)^n$

∴ The general Solⁿ is:

$$(c_1+c_2n+c_3n^2+c_4n^3)(1)^n+(c_5+c_6n)(2)^n+(c_7+c_8n+c_9n^2)\,($$

—x—

If the roots of characteristic eqn are

$\underline{\underline{1,1,1}}$, $\underline{\underline{2,2}}$, $\underline{1 \pm i\sqrt{3}}$, then write the general Sol$^n$.

For the roots $1 \pm i\sqrt{3}$ the Sol$^n$ is

$$c_6 (1+i\sqrt{3})^n + c_7 (1-i\sqrt{3})^n$$

**EXAMPLE 3** What is the solution of the recurrence relation

Extra Examples

$$a_n = a_{n-1} + 2a_{n-2}$$

with $a_0 = 2$ and $a_1 = 7$?

**EXAMPLE 4** Find an explicit formula for the Fibonacci numbers.

$1 + i\sqrt{3} = r(\cos\theta + i\sin\theta)$

Comparing real and Img parts.

$r\cos\theta = 1 \longrightarrow (1)$

$r\sin\theta = \sqrt{3} \longrightarrow (2)$

Squaring and adding (1) & (2).

$r^2 = (1)^2 + (\sqrt{3})^2$

$r^2 = 1 + 3$

$r^2 = 4$

$\boxed{r = 2}$

(2) ÷ (1) ⇒

$$\frac{\cancel{r}\sin\theta}{\cancel{r}\cos\theta} = \frac{\sqrt{3}}{1}$$

$\tan\theta = \sqrt{3}$

$\theta = \frac{\pi}{3}$

$$1+i\sqrt{3} = 2\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right]$$

$$(1+i\sqrt{3})^n = 2^n\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right]^n = 2^n\left[\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right]$$

$$(1-i\sqrt{3})^n = 2^n\left[\cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3}\right]$$

$$C_6(1+i\sqrt{3})^n + C_7(1-i\sqrt{3})^n$$

$$= C_6\left[2^n\right]\left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right) + C_7\, 2^n\left[\cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3}\right]$$

$$= 2^n\left\{(C_6+C_7)\cos\frac{n\pi}{3} + (iC_6 - iC_7)\sin\frac{n\pi}{3}\right\} = 2^n\left[A\cos\frac{n\pi}{3} + B\sin\right.$$