Lecture4

Saturday, August 28, 2021 9:50 AM

Lecture4

Topics of the lecture

- Universal quantification
- Existential quantification
- Quantifiers with restricted domains
- Precedence of Quantifiers
- Binding variables and scope of a quantifier
- Negating quantified expressions

Universal quantification ? $P(x) : x \leftarrow P$ //

The universal quantification of $P(x)$ is the statement

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Universal quantification ?

The universal quantification of $P(x)$ is the statement

"$P(x)$ for all values of x in the domain."

The notation "$\forall x P(x)$" denotes the universal quantification of $P(x)$. Here $\forall$ is called the **universal quantifier**. We read "$\forall x P(x)$" as "for all $xP(x)$" or "for every $xP(x)$". An element for which $P(x)$ is false is called a **counterexample** to $\forall x P(x)$.

$\forall x\, P(x)$

$\forall$ — universal quantifier.

$\forall x\, P(x)$

$x = k$

$P(x)$

Example

Let $P(x)$ be the statement "$x + 1 > x$". What is the truth value of the quantification $\forall x P(x)$, where the domain consists of all real numbers?

True

$P(x) : x+1 > x$

$\forall x P(x)$

$x \in \mathbb{R}$

$x+1 > x$.

Example

Let $P(x)$ be the statement "$x + 1 > x$". What is the truth value of the quantification $\forall x P(x)$, where the domain consists of all real numbers?

Since $P(x)$ is true for all real numbers x, the quantification $\forall x P(x)$ is true.

Quiz 1

Let $Q(x)$ be the statement "$x < 2$". What is the truth value of the quantification $\forall x Q(x)$, where the domain consists of all real numbers?

19 TRUE

54 FALSE

$x = 3 \in \mathbb{R}$.

$3 < 2$

Quiz 1

Let $Q(x)$ be the statement "$x < 2$". What is the truth value of the quantification $\forall x Q(x)$, where the domain consists of all real numbers?

TRUE

FALSE

FALSE. Because $Q(3)$ is false, and $x = 3$ is a counterexample for the statement $\forall x Q(x)$. Thus, $\forall x Q(x)$ is false.

Existential quantification ?

The existential quantification of $P(x)$ is the proposition

$\exists x\, P(x)$

Existential quantification ?

The existential quantification of $P(x)$ is the proposition

"There exists an element x in the domain such that $P(x)$"

Existential quantification ?

The existential quantification of $P(x)$ is the proposition

"There exists an element x in the domain such that $P(x)$"

We use the notation $\exists x P(x)$ for the existential quantification of $P(x)$. Here, $\exists$ is called the **existential quantifier.** If there is no x for which $P(x)$ is true, then $P(x)$ is false.

$\exists x P(x)$ $P(x)$

$x \in D$

$\exists x P(x)$ True//

$x = k$ $P(x)$ $\exists$

$\exists$

$x \in D$ $P(x)$

$x \notin D$ $P(x)$

for which $P(x)$ is true, then $P(x)$ is false.

$\exists x P(x)$ $P(x)$ $x \in D$ $P(x)$

$x \notin D$ $P(x)$

$\exists$ — existential quantifier.

Example

Let $P(x)$ denote the statement "$x > 4$". What is the truth value of the quantification $\exists x P(x)$, where the domain consists of all real numbers?

$P(x): x > 4$ $x \in \mathbb{R}$. $\exists x P(x)$ True

$\exists x P(x)$

Example

Let $P(x)$ denote the statement "$x > 4$". What is the truth value of the quantification $\exists x P(x)$, where the domain consists of all real numbers?

Since $P(x)$ is true for $x = 6$, the quantification $\exists x P(x)$ is true.

Quiz 2

Let $Q(x)$ be the statement "$x = x + 1$". What is the truth value of the quantification $\exists x Q(x)$, where the domain consists of all real numbers?

TRUE

FALSE

$x \in \mathbb{R}$

$x = x + 1$

$\Rightarrow 0 = 1$

$\forall x P(x)$

True if $P(x)$ is true $\forall x \in D$.

False if $P(x)$ is false for atleast one $x \in D$.

$\exists x P(x)$

True if $P(x)$ is true for atleast one $x \in D$.

False if $P(x)$ is false $\forall x \in D$.

Quiz 2

Let $Q(x)$ be the statement "$x = x + 1$". What is the truth value of the quantification $\exists x Q(x)$, where the domain consists of all real numbers?

TRUE

FALSE

FALSE. Because $Q(3)$ is false for every real number x, the existential quantification of $Q(x)$, i.e., $\exists x Q(x)$ is false.

Quantifiers with restricted domains

Quantifiers are meaningless if the variables they bind do not have a **domain**.

$\forall x \in D$ //

$\exists x \in D$ //

$\forall x (\in D)$

$\exists x (\in D)$

=

many

few

Quantifiers with restricted domains

Quantifiers are meaningless if the variables they bind do not have a **domain**.

The restriction of a universal quantification is the same as the universal quantification of a conditional statement. For example, $\forall x < 0 (x^2 > 0)$ is another way of expressing $\forall x (x < 0 \rightarrow x^2 > 0)$.

$\rightarrow \forall x < 0 \ (x^2 > 0)$

$\|$

$\forall x (x < 0 \rightarrow x^2 > 0)$

Quantifiers with restricted domains

Quantifiers are meaningless if the variables they bind do not have a **domain**.

The restriction of a universal quantification is the same as the universal quantification of a conditional statement. For example, $\forall x < 0(x^2 > 0)$ is another way of expressing $\forall x(x < 0 \to x^2 > 0)$.

The restriction of an existential quantification is the same as the existential quantification of a conjunction. For example, $\exists z > 0(z^2 = 2)$ is another way of expressing $\exists z(z > 0 \wedge z^2 = 2)$.

$\exists z > 0 (z^2 = 2)$ $\exists z(\underbrace{z > 0 \wedge z^2 = 2}})$

$\forall x<0(x^2>0)$

$\forall x(x<0 \to x^2>0)$

$\exists z>0(z^2=2)$

$\exists z(z>0 \wedge z^2=2)$.

Precedence of Quantifiers

The quantifiers $\forall$ and $\exists$ have higher precedence than all logical operators from propositional calculus.

$\vee$, $\wedge$ $\Rightarrow$

$\sim$ $\Leftrightarrow$ $\forall$ $\wedge$ $\sim$

$\forall$

BODMAS

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Precedence of Quantifiers

The quantifiers $\forall$ and $\exists$ have higher precedence than all logical operators from propositional calculus.

For example, $\forall x P(x) \vee Q(x)$ is the disjunction of $\forall x P(x)$ and $Q(x)$. Making it more clear, it means $(\forall x P(x)) \vee Q(x)$ rather than $\forall x(P(x) \vee Q(x))$.

$$(\forall x P(x)) \vee Q(x)$$
$$\neq \quad \forall x (P(x) \vee Q(x))$$

Binding variables

When a quantifier is used on a variable x, the occurrence of the variable is bound. An occurrence of a variable that is not bound by a quantifier or set equal to a particular value is said to be free.

$$\forall x \, P(x)$$
$$\exists x \, P(x)$$

Binding variables

When a quantifier is used on a variable x, the occurrence of the variable is bound. An occurrence of a variable that is not bound by a quantifier or set equal to a particular value is said to be free.

All the variables that occur in a propositional function must be bound or set equal to a particular value to turn it into a proposition. And, this can be done by using a combination of universal quantifiers, existential quantifiers and value assignments.

$P(x)$

$\forall x P(x)$

$\in D$

Binding variables

When a quantifier is used on a variable x, the occurrence of the variable is bound. An occurrence of a variable that is not bound by a quantifier or set equal to a particular value is said to be free.

All the variables that occur in a propositional function must be bound or set equal to a particular value to turn it into a proposition. And, this can be done by using a combination of universal quantifiers, existential quantifiers and value assignments.

For example, in the statement $\exists x(x + y = 1)$, variable x is bound and variable y is free.

$\forall y\ (x^2 + y + \frac{2}{3} = 0)$

x ~~bound~~ free

bound

$\exists x$

$\exists x$

$\forall y$

$\exists y$

Scope of a quantifier

The part of a logical expression to which a quantifier is applied is called the scope of the quantifier. And, a variable is free if it is outside the scope of all quantifiers in the formula that specify this variable.

$\underline{\exists x(x-2=0)} \wedge y=3.$

$x-2=0$

free

Scope of a quantifier

The part of a logical expression to which a quantifier is applied is called the scope of the quantifier. And, a variable is free if it is outside the scope of all quantifiers in the formula that specify this variable.

For example, in the statement $\exists x(P(x) \wedge Q(x)) \vee \forall x R(x)$, all variables are bound. The scope of the first quantifier, $\exists x$, is the expression $P(x) \wedge Q(x)$ and the scope of the second quantifier, $\forall x$, is the expression $R(x)$.

$\exists x(P(x) \wedge Q(x)) \vee \forall x R(x)$

$P(x) \wedge Q(x)$

$R(x)$

Negating quantified expressions

$\sim\forall = \exists$
$\sim\exists = \forall$

The negation of a universal quantification $\forall x P(x)$ is same as the existential quantification $\exists x \sim P(x)$. That is, $\sim \forall x P(x) \equiv \exists x \sim P(x)$.

The negation of an existential quantification $\exists x Q(x)$ is same as the universal quantification $\forall x \sim Q(x)$. That is, $\sim \exists x Q(x) \equiv \forall x \sim Q(x)$.

$\sim \forall x P(x) \equiv \exists x \sim P(x)$

$\sim(\forall x \in \mathbb{Z}, x^2 > 0)$

$\equiv \exists x \in \mathbb{Z}, x^2 \leq 0$

$\sim(\exists x \in \mathbb{Z}, x^2 > 0)$

$\equiv \forall x \in \mathbb{Z}, x^2 \leq 0.$

Negating quantified expressions

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The negation of an existential quantification $\exists x Q(x)$ is same as the universal quantification $\forall x \sim Q(x)$. That is, $\sim \exists x Q(x) \equiv \forall x \sim Q(x)$.

Quiz 3

The negation of the statement $\forall x(x^2 > x)$ is

A. $\forall x(x^2 < x)$
B. $\forall x(x^2 \leq x)$
C. $\exists x(x^2 < x)$
D. $\exists x(x^2 \leq x)$

$\sim \forall x(x^2 > x)$

$\equiv \exists x(x^2 \leq x)$

$x > 0$

$x \leq 0$

$> = <$

Quiz 3

The negation of the statement $\forall x(x^2 > x)$ is

A. $\forall x(x^2 < x)$
B. $\forall x(x^2 \leq x)$
C. $\exists x(x^2 < x)$
D. $\exists x(x^2 \leq x)$

Answer: D

Quiz 4

The negation of the statement $\exists x(x^2 = 2)$ is

A. $\forall x(x^2 \neq 2)$

B. $\forall x(x^2 < 2)$ $x^2 > 2$

C. $\forall x(x^2 > 2)$ $x^2 < 2$

D. $\exists x(x^2 \leq 2)$

Quiz 4

The negation of the statement $\exists x(x^2 = 2)$ is

A. $\forall x(x^2 \neq 2)$

B. $\forall x(x^2 < 2)$

C. $\forall x(x^2 > 2)$

D. $\exists x(x^2 \leq 2)$

Answer: A

Quiz 5

What is the truth value of the statement $\forall x(x^2 + 2 > 1)$ is the domain of x consists of all real numbers?

TRUE

FALSE

$x^2+2>1$

$\Rightarrow x^2>1-2$

$\Rightarrow x^2>-1$

$x^2 \geq 0$

$x \in \mathbb{R}$

Quiz 5

What is the truth value of the statement $\forall x(x^2 + 2 > 1)$ is the domain of x consists of all real numbers?

TRUE

FALSE

TRUE.