UNIT 2 : Recurrence Relations

- Introduction to Recurrence Relation
- Modelling with recurrence relation
- Quiz

Introduction to Recurrence relation

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For example, let the number of bacteria in a colony doubles every hour. If the colony begins with five bacteria, how many bacteria will be there after n hours?

Example cont . . .

To solve the previous "bacteria" problem, let a_n be the number of bacteria at the end of n hours. Because the number of bacteria doubles every hour, the relationship $a_n = 2a_{n-1}$ holds whenever n is a positive integer.

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This recurrence relation $a_n = 2a_{n-1}$, together with initial condition $a_0 = 5$, uniquely determines a_n for all nonnegative integers n.

Also, we can find a formula for a_n using an iterative approach. Here, the formula for a_n is $a_n = 5.2^n$, for all nonnegative integers n.

Application of recurrence relation

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Now, we will try to study a variety of counting problems that can be modelled using recurrence relations.

Modelling with Recurrence Relation

We can use recurrence relations to model a wide variety of problems, such as finding compound interest, counting rabbits on an island, determining the number of moves in the tower of Hanoi puzzle, and counting bit strings with certain properties.

Example 1 - Rabbits and Fibonacci numbers

Let us consider the problem that was originally posed by Leonardo Pisano, also known as Fibonacci, in the thirteenth century in his book *Liber abaci*. A young pair of rabbits is placed on an island. A pair of rabbits does not breed until they are 2 months old. After they are 2 months old, each pair of rabbits produces another pair each month.

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To find a recurrence relation for the number of pairs on the island after n months, assuming that no rabbits ever die.

Example 1 cont . . .

Solution : Denote by f_n the number of pairs of rabbits after n months. We will show that f_n, $n = 1, 2, 3, ...$ are the terms of the Fibonacci sequence.

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The rabbit population can be modelled using a recurrence relation. At the end of first month, the number of pairs of rabbits on the island is $f_1 = 1$. Because this pair does not breed during the second month, $f_2 = 1$ also.

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The rabbit population can be modelled using a recurrence relation. At the end of first month, the number of pairs of rabbits on the island is $f_1 = 1$. Because this pair does not breed during the second month, $f_2 = 1$ also.

To find the number of pairs after n months, add the number on the island the previous month, f_{n-1}, and the number of newborn pairs, which equals f_{n-2}, because each newborn pair comes from a pair at least 2 months old.

Example 1 cont . . .

Consequently, the sequence f_n satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2}$, for $n \geq 3$ together with the initial conditions $f_1 = 1$ and $f_2 = 1$. Because this recurrence relation and initial conditions uniquely determine the sequence, so the number of pais of rabbits on the island after n months is given by the nth Fibonacci number.

Quiz 1

Consider the recurrence relation $a_1 = 4,\ a_n = 5n + a_{n-1}$. The value of a_{64} is

A. 10399

B. 23760

C. 75100

D. 53700

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Answer : A

Quiz 2

Consider the recurrence relation $a_1 = 3,\ a_2 = 5$ and $a_n = a_{n-1} + 2a_{n-2}$. The value of a_7 is

A. 43

B. 85

C. 171

D. 265

Quiz 2

Consider the recurrence relation $a_1 = 3,\ a_2 = 5$ and $a_n = a_{n-1} + 2a_{n-2}$. The value of a_7 is

A. 43

B. 85

C. 171

D. 265

Answer : C

Example 2 - Hanoi Tower

A popular puzzle of the late nineteenth century invented by the French mathematician E. Lucas, called the Tower of Hanoi, consists of three pegs(rods) on a board together with disks of different sizes. Initially, these disks are placed on the first peg in order of size, with the largest on the bottom.

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The rules of the puzzle allow disks to be moved one at a time from one peg to another as long as a disk is never placed on the top of a smaller disk.

The goal of the puzzle is to have all the disks on the second peg in order of size, with the largest on the bottom.

Hanoi Tower - Recurrence relation

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Solution: Begin with n disks on peg 1. We can transfer the top $n-1$ disks, following the rules of the puzzle, to peg 3 using H_{n-1} moves. We kept the largest disk fixed during these moves. Then, we use one move to transfer the largest disk to the second peg. Finally, we transfer the $n-1$ disks from peg 3 to peg 2 using H_{n-1} moves, placing them on the top of the largest disk, which always stays fixed on the bottom of peg 2. This shows that we can solve the Tower of Hanoi puzzle for n disks using $2H_{n-1}+1$ moves.

Solution of Hanoi Tower

We can use an iterative approach to solve this recurrence relation as follows:

$$
\begin{aligned}
H_n &= 2H_{n-1} + 1 \\
&= 2(2H_{n-2} + 1) + 1 \\
&= 2^2(2H_{n-3} + 1) + 2 + 1 \\
&= 2^3 H_{n-3} + 2^2 + 2 + 1 \\
&= 2^{n-1} H_1 + 2^{n-2} + 2^{n-3} + \cdots + 2 + 1 \\
&= 2^{n-1} + 2^{n-2} + \cdots + 2 + 1 \\
&= 2^n - 1
\end{aligned}
$$