Lecture 6

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Lecture 6

Topics of the lecture — lecture - 6

- Vacuous and Trivial Proofs
- Proof by Contradiction
- Proof Strategy
- Quiz

Vacuous Proof

In a conditional statement $p \to q$, we know that, if hypothesis p is false, then whatever may be the truth value of conclusion q, the truth value of $p \to q$ is always true.

$p \to q$ T

F T,F

p	q	$p \to q$
T	T	T
T	F	F
F	T ✗	T
F	F ✗	T

Vacuous Proof

In a conditional statement $p \to q$, we know that, if hypothesis p is false, then whatever may be the truth value of conclusion q, the truth value of $p \to q$ is always true.

This fact can be used to proof any theorem which is called **vacuous proof**.

Example of Vacuous Proof

Ex- Show that the proposition $P(0)$ is true, where $P(n)$ is "If $n > 1$, then $n^2 > n$" and the domain consists of all integers.

$P(n)$: If $\underbrace{n>1}_{p}$, then $\underbrace{n^2>n}_{q}$.

$\therefore P(0)$: If $\underbrace{0>1}_{p}$, then $\underbrace{0^2>0}_{q}$.

$p \longrightarrow q$ ×

$p : 0>1$ False

$P(0)$ — true

Proved//

Example of Vacuous Proof

Show that the proposition $P(0)$ is true, where $P(n)$ is "If $n > 1$, then $n^2 > n$" and the domain consists of all integers.

According to the given statement, $P(0)$ is "If $0 > 1$, then $0^2 > 0$". Here, the hypothesis $0 > 1$ is FALSE. So, according to the concept of vacuous proof, we can directly say that $P(0)$ is TRUE. Also, we don't need to look at the truth value of the conclusion $0^2 > 0$ (Though, it is FALSE).

Example of Vacuous Proof

Show that the proposition $P(0)$ is true, where $P(n)$ is "If $n > 1$, then $n^2 > n$" and the domain consists of all integers.

According to the given statement, $P(0)$ is "If $0 > 1$, then $0^2 > 0$". Here, the hypothesis $0 > 1$ is FALSE. So, according to the concept of vacuous proof, we can directly say that $P(0)$ is TRUE. Also, we don't need to look at the truth value of the conclusion $0^2 > 0$ (Though, it is FALSE).

Remark: The fact that the conclusion of the given conditional statement, $0^2 > 0$, FALSE, is irrelevant to the truth value of the conditional statement, because a conditional statement with a false hypothesis is guaranteed to be true.

Trivial Proof

In a conditional statement $p \to q$, if we know that conclusion q is true, then whatever may be the truth value of hypothesis p, the truth value of the conditional statement is always true.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

True

Trivial Proof

In a conditional statement $p \to q$, if we know that conclusion q is true, then whatever may be the truth value of hypothesis p, the truth value of the conditional statement is always true.

A proof of $p \to q$ that uses the fact that q is true is called a **trivial proof**.

Example of Trivial Proof

$n = 0, \pm 1, \pm 2, \ldots$

Ex: Let $P(n)$ be "If a and b are positive integers with $a \geq b$, then $a^n \geq b^n$", where the domain consists of all non-negative integers. Show that $P(0)$ is true.

$a, b = 1, 2, 3, \ldots$

$P(0)$ is true//

Trivial proof//

$P(n)$: If $\underbrace{a \geq b}_{p}$, then $\underbrace{a^n \geq b^n}_{q}$.

$P(0)$: If $\underbrace{a \geq b}_{\times p}$, then $\underbrace{a^0 \geq b^0}_{q \; T}$. $1 \geq 1$ True

Example of Trivial Proof

Let $P(n)$ be "If a and b are positive integers with $a \geq b$, then $a^n \geq b^n$", where the domain consists of all non-negative integers. Show that $P(0)$ is true.

The proposition $P(0)$ is "If $a \geq b$, then $a^0 \geq b^0$". Since $a^0 = b^0 = 1$, so the conclusion of the given conditional statement is TRUE. Hence, by the concept of trivial proof, the conditional statement $P(0)$ is true.

Find the truth value of $P(2)$.

$P(2)$: If $a \geq b$, then $a^2 \geq b^2$.

Direct proof//

$a \geq b$
$\Rightarrow a^2 \geq b^2$

$a, b \in \mathbb{Z}^+$

Ex: If $\underbrace{a \leq b}_{p}$, then $\underbrace{a^3 \geq b^3}_{q \; F}$. F

Example of Trivial Proof

Let $P(n)$ be "If a and b are positive integers with $a \geq b$, then $a^n \geq b^n$", where the domain consists of all non-negative integers. Show that $P(0)$ is true.

The proposition $P(0)$ is "If $a \geq b$, then $a^0 \geq b^0$". Since $a^0 = b^0 = 1$, so the conclusion of the given conditional statement is TRUE. Hence, by the concept of trivial proof, the conditional statement $P(0)$ is true.

Here, the hypothesis, which is the statement $a \geq b$, was not actually needed in the proof.

Proof by the method of Contradiction

$p \to q$ T

In this method, we proof a proposition p to be true by assuming that its negation $\sim p$ is true and the arriving at a contradictory statement or contradiction. Also, conditional statement $p \to q$ is shown to be true by showing unsatisfiability of its negation.

$\sim(p \to q)$ T F $\Rightarrow p \to q$ T

$\Rightarrow p \to q$ F

$\Rightarrow p \,\&\, q$

T F

contradiction

$\sim(p \to q)$ T

$\equiv p \wedge \sim q$

p T

$\sim p$ T $\Rightarrow p$ F

$\sim p$ F

$\Rightarrow p$ T

valid argument

contradiction

T

p T
q T
$\sim q$ T

p T
q F

$p \to q$
T F

Proof by the method of Contradiction

In this method, we proof a proposition p to be true by assuming that its negation $\sim p$ is true and the arriving at a contradictory statement or contradiction. Also, conditional statement $p \to q$ is shown to be true by showing unsatisfiability of its negation.

Next question is, what is the negation of $p \to q$? Its $p \wedge \sim q$. This will be true only if p and $\sim q$ are both true.

p T }

p T
q F

Proof by the method of Contradiction

In this method, we proof a proposition p to be true by assuming that its negation $\sim p$ is true and the arriving at a contradictory statement or contradiction. Also, conditional statement $p \to q$ is shown to be true by showing unsatisfiability of its negation.

Next question is, what is the negation of $p \to q$? Its $p \wedge \sim q$. This will be true only if p and $\sim q$ are both true.

Thus, showing p and $\sim q$ both true, will lead to the negation of $p \to q$ that leads to contradiction.

Example of Proof by Contradiction

Ex:- Prove that $\sqrt{2}$ is irrational by giving a proof by contradiction.

p : $\sqrt{2}$ is irrational.

Let $\sim p$ be True.

i.e., $\sqrt{2}$ is rational

$\Rightarrow \sqrt{2} = \frac{a}{b}$, $a, b \in \mathbb{Z}$, $b \neq 0$ & a, b donot have any common factor.

Example of Proof by Contradiction

Prove that $\sqrt{2}$ is irrational by giving a proof by contradiction.

Let p be the proposition "$\sqrt{2}$ is irrational". To start a proof by contradiction, we suppose that $\sim p$ is true. Now, $\sim p$ is the statement "$\sqrt{2}$ is rational". After this, we will show that assuming $\sim p$ is true leads to a contradiction.

Cont . . .

$\sqrt{2}$ is irrational.

If $\sqrt{2}$ is rational, then there exists integers a and b with $\sqrt{2} = \frac{a}{b}$, where $b \neq 0$ and a and b have no common factors. Now,

$$(\sqrt{2})^2 = \left(\frac{a}{b}\right)^2$$

$$\Rightarrow 2 = \frac{a^2}{b^2}$$

$$\Rightarrow 2b^2 = a^2$$

$(\sqrt{2})^2 = \left(\frac{a}{b}\right)^2$

$\Rightarrow 2 = \frac{a^2}{b^2}$

$\Rightarrow a^2 = 2b^2$

$\Rightarrow a^2$ is even

$\Rightarrow a$ is even

$\Rightarrow a = 2c,\ c \in \mathbb{Z}.$

$2b^2 = (2c)^2$

$\Rightarrow 2b^2 = 4c^2$

$\Rightarrow b^2 = 2c^2$

$\Rightarrow b^2$ is even

$\Rightarrow b$ is even

$\Rightarrow b = 2m,\ m \in \mathbb{Z}.$

$\gcd(a,b) = 2$

$\gcd(a,b) = 1$

$1 = 2$?

No

Cont . . .

If $\sqrt{2}$ is rational, then there exists integers a and b with $\sqrt{2} = \frac{a}{b}$, where $b \neq 0$ and a and b have no common factors. Now,

$$\begin{aligned}(\sqrt{2})^2 &= \left(\frac{a}{b}\right)^2 \\ \Rightarrow 2 &= \frac{a^2}{b^2} \\ \Rightarrow 2b^2 &= a^2\end{aligned}$$

So, a^2 is even. Also, if a^2 is even, then a must also be even. Let $a = 2c$, where c is some integer. After this, we get, $2b^2 = 4c^2 \quad \Rightarrow \quad b^2 = 2c^2$, which implies that b^2 is even, i.e., b is also even.

Cont . . .

Now, since a and b are both even, so 2 is the common factor of both a and b. Thus, our assumption that a and b do not have any common factors is violated, we arrive at a contradiction.

Cont . . .

Now, since a and b are both even, so 2 is the common factor of both a and b. Thus, our assumption that a and b do not have any common factors is violated, we arrive at a contradiction.

This proves that our assumption that $\sqrt{2}$ is rational is not correct. Hence, by the method of contradiction, $\sqrt{2}$ is irrational.

$\therefore p \to q$ is True //

Proved //

If n is even, then n^2 is even.

$p \to q$.

Sol^n: Let $p \to q$ be F

$\Rightarrow p$ is T and q is F

$\Rightarrow n$ is even & n^2 is odd

$\Rightarrow n = 2k$

$n^2 = 2p+1$

$n^2 = 4k^2 \neq 2p+1$ // $\forall k, p \in \mathbb{Z}$.

Vacuous p false

Trivial $\to$ q True //

Proof Strategy

Two important approaches for proving theorems of the form $\forall(P(x) \to Q(x))$; direct proof and proof by contraposition. Following is the strategy to prove a given theorem.

- First, we should evaluate whether the **direct method of proof** looks promising. If yes, go with it.
- Second, if it seems that direct method doesn't work, then we should try method of **proof by Contraposition**.
- In the third and last stage, if above two methods are not helping us anyway, we can proceed towards **method of proof by Contradiction**.

Example of Proof Strategy

proved

Theorem : If n is an integer and $3n+2$ is odd, then n is odd.

Pf: T $p \to q \equiv \sim q \to \sim p$. T

3n+2 is odd   n odd   n even   3n+2 even

$n = 2k$

$\Rightarrow 3n+2 = 3 \cdot 2k + 2$

$= 6k+2$

$= 2(3k+1)$

$= 2p, \; p \in \mathbb{Z}$

Example of Proof Strategy

Theorem : If n is an integer and $3n+2$ is odd, then n is odd.

Proof: If we try to use direct method of proof, it will not be promising to us, because assuming $3n+2$ is odd and showing n is odd is not that simple. So, next we will try for method by **Contraposition**.

Cont . . .

The first step is assuming that the conclusion of the given conditional statement is false, i.e., n is even. Then, by the definition of even integers, we have $n = 2k$, for any integer k. After this, $3n + 2 = 3 \times 2k + 2 = 6k + 2 = 2(3k + 1) = 2m$, where $m = 3k + 1$, an integer. This tells us that $3n + 2$ is also even.

Cont . . .

The first step is assuming that the conclusion of the given conditional statement is false, i.e., n is even. Then, by the definition of even integers, we have $n = 2k$, for any integer k. After this, $3n + 2 = 3 \times 2k + 2 = 6k + 2 = 2(3k + 1) = 2m$, where $m = 3k + 1$, an integer. This tells us that $3n + 2$ is also even.

Thus, if n is even, then $3n + 2$ is also even. Hence, by the method of Contraposition, we can conclude that if $3n + 2$ is odd, then n is odd. This proves the given theorem most effectively by method of contraposition.

$\sqrt{5}$ is irrational.

$p \to q$

$\sim q \to \sim p$

Quiz 1

When to proof $p \to q$ true, we proof p false, that type of proof is known as

A. Direct proof $\overset{T}{p} \to \overset{T}{q}$

B. Proof by Contraposition $\sim q \to \sim p$

C. Vacuous proof

D. Trivial proof

vacuous proof

$p \to q$

F ?

T

Quiz 1

When to proof $p \rightarrow q$ true, we proof p false, that type of proof is known as

A. Direct proof
B. Proof by Contraposition
C. Vacuous proof
D. Trivial proof

Answer : C. From the definition of vacuous proof.

Quiz 2

A proof that $p \rightarrow q$ is true based on the fact that q is true is known as

Trivial

$p \rightarrow q$: True

T, F

A. Direct proof
B. Proof by Contraposition
C. Vacuous proof
D. Trivial proof

Quiz 2

A proof that $p \rightarrow q$ is true based on the fact that q is true is known as

A. Direct proof

B. Proof by Contraposition

C. Vacuous proof

D. Trivial proof

Answer : D. From the definition of trivial proof.

Quiz 3

In proving $\sqrt{5}$ as irrational, we begin with assumption $\sqrt{5}$ is rational in which type of proof?

A. Direct proof

B. Proof by Contradiction

C. Vacuous proof

D. Trivial proof

Quiz 3

In proving $\sqrt{5}$ as irrational, we begin with assumption $\sqrt{5}$ is rational in which type of proof?

A. Direct proof
B. Proof by Contradiction
C. Vacuous proof
D. Trivial proof

Answer: B. From the definition of proof by contradiction.

(Q) while proving $p \to q$ to be True, we assume p to be T & q to be F in which proof?

✓(a) Contradiction — 8
× (b) Contraposition — 21
× (c) Trivial — 3
× (d) Direct — 34

$p \to q$ T

$\sim(p \to q)$ T

$\equiv p \wedge \sim q$ T

$\equiv p$ T, $\sim q$ T $\left.\begin{matrix} p & T \\ q & F \end{matrix}\right\}$