Discrete mathematics is the study of discrete structures and relationships between them. These discrete objects can be discrete sets, graphs, permutations etc.

Logic is the study of reasoning. It is concerned with whether the reasoning is correct or not.

Why is logic important?
Logic is useful in clarifying the ordinary writing.
Logic methods are used to prove the mathematical statements.
In computer science, logic is used to prove the correctness of the program. Correctness involves that program does what it is supposed to do.

Propositional Logic

A ***proposition*** is a declarative statement (that is, a sentence that declares a fact) that is either true or false but not both.
A statement that is either true or false but not both is called a ***proposition***.

More, specifically, a proposition is a declarative statement to which a true value can be assigned. A value TRUE is a true value of a proposition if the statement is correct and a value FALSE is true value of a proposition if it is false. TRUE value is denoted by T or 1 and FALSE value is denoted by F or 0.

Example 1. All the following declarative sentences are propositions.
(A) For every positive integer n there is a prime number p such that $p > n$.
(B) 1+1=2.
(C) 2+2=3.
True value of (A) is T or 1.
True value of (B) is T or 1.
True value of (C) is F or 0.

Example 2. Which of the following statements are propositions.

(A) What time is it? (This is not a proposition)
(B) Toronto is the capital of Canada. (This is a proposition with true value T)
(C) $x + 1 = 2$. (This is not a proposition)
(D) Two sides of a page contains the following statements.
Page 1. The statement on the other side of the page is correct.
Page 2. The statement on the other side of the page is incorrect.
Statement (D) is not a proposition because truth value can not be assigned to any of the two.

We now turn our attention to methods for producing new propositions from those that we already have. These methods were discussed by the English mathematician George Boole in 1854 in his book *The Laws of Thought.* Many mathematical statements are constructed by combining one or more propositions. New propositions, called **compound propositions**, are formed from existing propositions using **logical operators/logical connectives**.

Operations on propositions.

Definition 1. Let p be a proposition. The negation of p, denoted by $\neg p$ (also denoted by $\overline{p}$), is the statement

"It is not the case that p."

The proposition $\neg p$ is read "not p." The truth value of the negation of p, i.e. $\neg p$, is the opposite of the truth value of p.

Remark: $\neg p$ and p are the most common notations used in mathematics to express the negation of p, other notations that are used are $\sim p$, $-p$, p', Np, and !p

Example: Find the negation of the statement "5 is a prime number" and express the result in simple English.

Sol. The negation is "It is not the case that 5 is a prime number". This can be simply expressed as "5 is not a prime number".

Truth table gives the truth values of the propositions.

Truth table for $\neg p$			Truth table for $\neg p$	
p	$\neg p$		p	$\neg p$
0	1	OR	F	T
1	0		T	F

Definition 2: Let p and q be propositions. The ***conjunction*** of p and q, denoted by $p \wedge q$, is the proposition "p and q." The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise.

Truth table for $p \wedge q$				Truth table for $p \wedge q$		
p	q	$p \wedge q$		p	q	$p \wedge q$
0	0	0		F	F	F
0	1	0	OR	F	T	F
1	0	0		T	F	F
1	1	1		T	T	T

Definition 3: Let p and q be propositions. The *disjunction* of p and q, denoted by $p \lor q$, is the proposition "p or q." The disjunction $p \lor q$ is false when both p and q are false and is true otherwise.

Truth table for $p \lor q$				Truth table for $p \lor q$		
p	q	$p \lor q$		p	q	$p \lor q$
0	0	0		F	F	F
0	1	1	OR	F	T	T
1	0	1		T	F	T
1	1	1		T	T	T

***Definition 4*:** Let p and q be propositions. The *exclusive or* of p and q, denoted by $p \oplus q$ (or p XOR q), is the proposition that is true when exactly one of p and q is true and is false otherwise.

Example: Let p and q be the propositions that state "A student can have a salad with dinner" and "A student can have soup with dinner," respectively. What is $p \oplus q$, the exclusive or of p and q?
Sol: The exclusive or of p and q is the statement that is true when exactly one of p and q is true. That is, $p \oplus q$ is the statement "A student can have soup or salad, but not both, with dinner."

Definition 5: Let p and q be propositions. The ***conditional*** *statement* $p \to q$ is the proposition "if p, then q." The conditional statement $p \to q$ is false when p is true and q is false, and true otherwise. In the conditional statement $p \to q$, p is called the *hypothesis* (or *antecedent* or *premise*) and q is called the *conclusion* (or *consequence*). A conditional statement is also called an ***implication***.

Truth tables for ***exclusive OR*** and ***implication.***

Truth table for $p \oplus q$				Truth table for $p \to q$		
p	q	$p \oplus q$		p	q	$p \to q$
0	0	0		0	0	1
0	1	1		0	1	1
1	0	1		1	0	0
1	1	0		1	1	1

Example: Let p be the statement "Maria learns discrete mathematics" and q the statement "Maria will find a good job." Express the statement $p \to q$ as a statement in English.

Sol. "If Maria learns discrete mathematics, then she will find a good job."
"Maria will find a good job when she learns discrete mathematics."
"For Maria to get a good job, it is sufficient for her to learn discrete mathematics."
"Maria will find a good job unless she does not learn discrete mathematics."

CONVERSE, CONTRAPOSITIVE, AND INVERSE

We can form some new conditional statements starting with a conditional statement $p \to q$. In particular, there are three related conditional statements that occur so often that they have special names. The proposition $q \to p$is called the **converse** of $p \to q$. The **contrapositive** of $p \to q$ is the proposition $\neg q \to \neg p$. The proposition $\neg p \to \neg q$ is called the **inverse** of $p \to q$.

Example: Form the truth table for converse, contrapositive and inverse of $p \to q$.

p	q	$p \to q$	$q \to p$	$\neg p$	$\neg q$	$\neg q \to \neg p$	$\neg p \to \neg q$
0	0	1	1	1	1	1	1
0	1	1	0	1	0	1	0
1	0	0	1	0	1	0	1
1	1	1	1	0	0	1	1

Definition 6: Let p and q be propositions. The ***biconditional*** *statement* $p \leftrightarrow q$ is the proposition "p if and only if q." The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called *bi-implications*. There are some other common ways to express $p \leftrightarrow q$:

"p is necessary and sufficient for q"
"if p then q, and conversely"
"p iff q"
"p exactly when q"

Example: Let p be the statement "You can take the flight," and let q be the statement "You buy a ticket." Then $p \leftrightarrow q$ is the statement, "You can take the flight if and only if you buy a ticket."

Compound Propositions: We have now introduced six important logical connectives—conjunction, disjunction, exclusive or, implication, and the biconditional operator as well as negation. We can use these connectives to build up complicated compound propositions involving any number of propositional variables.

Example: Construct the truth table of the compound proposition $(p \vee \neg q) \to (p \wedge q)$.

Sol.

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \to (p \wedge q)$
0	0	1	1	0	0
0	1	0	0	0	1
1	0	1	1	0	0
1	1	0	1	1	1

Exercise: Construct the truth table for

(i) $(p \vee \neg q) \to q$ *(ii)* $(p \to q) \leftrightarrow (\neg p \vee q)$

Definition: **Tautology** is a propositional form whose truth value is true for all possible values of its propositional variable. A propositional form whose truth value is false formal possible values of its propositional variable is called a **contradiction.** A propositional form which is neither a tautology nor a contradiction is called a **contingency.**

Example: $p \vee \neg p$ is a tautology, $p \wedge \neg p$ is a contradiction and $p \vee q$ is a contingency.

<u>Logical identities</u>

1. $p \leftrightarrow (p \vee p)$ and $p \leftrightarrow (p \wedge p)$ Idempotence Laws
2. $(p \vee q) \leftrightarrow (q \vee p)$ and $(p \wedge q) \leftrightarrow (q \wedge p)$ Commutative Laws
3. $[(p \vee q) \vee r] \leftrightarrow [p \vee (q \vee r)]$ and $[(p \wedge q) \wedge r] \leftrightarrow [p \wedge (q \wedge r)]$ Associative Laws
4. $\neg(p \vee q) \leftrightarrow (\neg q \wedge \neg q)$ and $\neg(p \wedge q) \leftrightarrow (\neg q \vee \neg q)$ De'Morgans Laws
5. $[(p \vee q) \wedge r] \leftrightarrow [(p \wedge r) \vee (q \wedge r)]$ and $[(p \wedge q) \vee r] \leftrightarrow [(p \vee r) \wedge (q \vee r)]$ Distributive Laws
6. $(p \vee 0) \leftrightarrow p$ and $(p \wedge 1) \leftrightarrow p$ Identity Laws
7. $(p \vee 1) \leftrightarrow 1$ and $(p \wedge 0) \leftrightarrow 0$ Domination Laws
8. $(p \to q) \leftrightarrow (\neg p \vee q)$ [Verify this using truth table]
9. $\neg(\neg p) \leftrightarrow p$ Double Negation
10. $p \vee (p \wedge q) \leftrightarrow p$ and $p \wedge (p \vee q) \leftrightarrow p$ Absorption Laws
11. $p \vee \neg p \leftrightarrow 1$ and $p \wedge \neg p \leftrightarrow 0$ Negation Laws

Exercise: Show that the following are tautologies.

1. $[(p \to q) \wedge (q \to p)] \leftrightarrow (p \leftrightarrow q)$
2. $[(p \wedge q) \to r] \leftrightarrow [p \to (q \to r)]$

Exercise: Show that the implication operator is not associative for the propositional variables p, q and r.

Sol.

p	q	r	$p \to q$	$q \to r$	$(p \to q) \to r$	$p \to (q \to r)$
0	0	0				
0	0	1				
0	0	1				
0	1	1				
1	0	0				
1	0	1				
1	0	1				
1	1	1				

As the last two columns in the truth table are not identical, therefore, implication operator is not associative.

Example: Show that $\neg(p \to q)$ and $p \wedge \neg q$ are logically equivalent.

Sol. We could use a truth table to show that these compound propositions are equivalent but here we will use the logical identities to prove it.

$\neg(p \to q) \leftrightarrow \neg(\neg p \vee q)$ by the conditional-disjunction equivalence

$\leftrightarrow \neg(\neg p) \wedge \neg q$ by the second De Morgan law

$\leftrightarrow p \wedge \neg q$ by the double negation law

Exercise: Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences.

Sol:

$$
\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{by the second De Morgan law} \\
&\equiv \neg p \wedge [\neg(\neg p) \vee \neg q] && \text{by the first De Morgan law} \\
&\equiv \neg p \wedge (p \vee \neg q) && \text{by the double negation law} \\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{by the second distributive law} \\
&\equiv \mathbf{F} \vee (\neg p \wedge \neg q) && \text{because } \neg p \wedge p \equiv \mathbf{F} \\
&\equiv (\neg p \wedge \neg q) \vee \mathbf{F} && \text{by the commutative law for disjunction} \\
&\equiv \neg p \wedge \neg q && \text{by the identity law for } \mathbf{F}
\end{aligned}
$$

Consequently $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

Predicates:

The statement "x is greater than 3" has two parts. The first part, the variable x, is the subject of the statement. The second part—the **predicate**, "is greater than 3"—refers to a property that the subject of the statement can have. We can denote the statement "x is greater than 3" by $P(x)$, where P denotes the predicate "is greater than 3" and x is the variable. The statement $P(x)$ is also said to be the value of the **propositional function** P at x. Once a value has been assigned to the variable x, the statement $P(x)$ becomes a proposition and has a truth value.

Example: Let $P(x)$ denote the statement "$x > 3$." What are the truth values of $P(4)$ and $P(2)$?

Solution: We obtain the statement $P(4)$ by setting $x = 4$ in the statement "$x > 3$." Hence, $P(4)$, which is the statement "$4 > 3$," is true. However, $P(2)$, which is the statement "$2 > 3$," is false.

We can also have statements that involve more than one variable. For instance, consider the statement "$x = y + 3$." We can denote this statement by $Q(x, y)$, where x and y are variables and Q is the predicate. When values are assigned to the variables x and y, the statement $Q(x, y)$ has a truth value.

Example: Let $Q(x, y)$ denote the statement "$x = y + 3$." What are the truth values of the propositions $Q(1, 2)$ and $Q(3, 0)$?

Solution: To obtain $Q(1, 2)$, set $x = 1$ and $y = 2$ in the statement $Q(x, y)$. Hence, $Q(1, 2)$ is the statement "$1 = 2 + 3$," which is false. The statement $Q(3, 0)$ is the proposition "$3 = 0 + 3$," which is true.

Exercise: $R(x, y, z)$ denote the statement "$x + y = z$." What are the truth values of the propositions $R(1, 2, 3)$ and $R(0, 0, 1)$

In general, a statement involving n variable say, $x_1, x_2, \ldots, x_n$ is denoted by $P(x_1, x_2, \ldots, x_n)$ is called the propositional function of $n-$variables or is also known as $n-$place predicate or an $n-$array prediction.

Quantifiers

When each variable in a propositional function is assigned a value then it becomes a proposition and has a certain truth value. The values assigned to the variables of a propositional function are called quantifiers. The part of logic that deals with predicates and quantifiers is called the predicate calculus.

Types of Quantification

The universal quantification: The universal quantification of $P(x)$ is the statement "$P(x)$ for all values of x in the domain." The notation $\forall x P(x)$ denotes the universal quantification of $P(x)$. Here $\forall$ is called the universal quantifier. We read $\forall xP(x)$ as "for all $xP(x)$" or "for every $xP(x)$." An element for which $P(x)$ is false is called a counterexample to $\forall xP(x)$.

Example: Let $P(x)$ be the statement "$x + 1 > x$." What is the truth value of the quantification $\forall xP(x)$, where the domain consists of all real numbers?

Sol. Because $P(x)$ is true for all real numbers x, the quantification $\forall xP(x)$ is true.

The Existential Quantifier: The *existential quantification* of $P(x)$ is the proposition "there exists an element x in the domain such that $P(x)$."

We use the notation $\exists xP(x)$ for the existential quantification of $P(x)$. Here $\exists$ is called the *existential quantifier*.

Example: Let $P(x)$ denote the statement "$x > 3$." What is the truth value of the quantification $\exists xP(x)$, where the domain consists of all real numbers?

Sol. Because "$x > 3$" is sometimes true—for instance, when $x = 4$—the existential quantification of $P(x)$, which is $\exists xP(x)$, is true.

Example: Let $Q(x)$ denote the statement "$x = x + 1$." What is the truth value of the quantification $\exists xQ(x)$, where the domain consists of all real numbers?

Sol: Because $Q(x)$ is false for every real number x, the existential quantification of $Q(x)$, which is $\exists xQ(x)$, is false.

The Uniqueness Quantifier: (Uniqueness quantifier, denoted by $\exists!$ or $\exists_1$) The notation $\exists!xP(x)$ [or $\exists_1 xP(x)$] states "There exists a unique x such that $P(x)$ is true."

Example: $\exists!x(x - 1 = 0)$, where the domain is the set of real numbers, states that there is a unique real number x such that $x-1 = 0$. This is a true statement, as $x = 1$ is the unique real number such that $x - 1 = 0$. Thus, the truth value of uniqueness quantifier is true.

Negation of Quantifiers:

Negation of universal quantifier of $P(x)$ is the existential quantifier of the negation of $P(x)$, that is, $\neg\forall x P(x) \leftrightarrow \exists x \neg P(x)$ and Negation of universal quantifier of $P(x)$ is the existential quantifier of the negation of $P(x)$, that is, $\neg\exists x P(x) \leftrightarrow \forall x \neg P(x)$

Example: What are the negations of the statements $\forall x(x^2 > x)$ and $\exists x(x^2 = 2)$?

Solution: The negation of $\forall x(x^2 > x)$ is the statement $\neg\forall x(x^2 > x)$, which is equivalent to $\exists x\neg(x^2 > x)$. This can be rewritten as $\exists x(x^2 \leq x)$. The negation of $\exists x(x^2 = 2)$ is the statement $\neg\exists x(x^2 = 2)$, which is equivalent to $\forall x\neg(x^2 = 2)$. This can be rewritten as $\forall x(x^2 \neq 2)$. The truth values of these statements depend on the domain.

Exercise: Show that $\neg\forall x(P(x) \rightarrow Q(x))$ and $\exists x(P(x) \wedge \neg Q(x))$ are logically equivalent.

Quantifiers with restricted domains.

What do the statements $\forall x<0(x^2>0), \forall y\neq 0(y^3\neq 0)$, and $\exists z>0(z^2=2)$mean, where the domain in each case consists of the real numbers?

***Sol*:** The statement $\forall x < 0\ (x^2 > 0)$ states that for every real number x with $x < 0$, $x^2 > 0$. That is, it states "The square of a negative real number is positive." This statement is the same as $\forall x(x<0\rightarrow x^2>0)$.

The statement $\forall y \neq 0\ (y^3 \neq 0)$ states that for every real number y with $y \neq 0$, we have $y^3 \neq 0$. That is, it states "The cube of every nonzero real number is nonzero." This statement is equivalent to $\forall y(y \neq 0 \rightarrow y^3 \neq 0)$.

Finally, the statement $\exists z > 0\ (z^2 = 2)$ states that there exists a real number z with $z > 0$ such that $z^2 = 2$. That is, it states "There is a positive square root of 2." This statement is equivalent to$\exists z(z>0\wedge z^2=2)$.

Note: (1) The restriction of a universal quantification is the same as the universal quantification of a conditional statement. For instance, $\forall x < 0\ (x^2 > 0)$ is another way of expressing $\forall x(x < 0 \rightarrow x^2 > 0)$.

(2) The restriction of an existential quantification is the same as the existential quantification of a conjunction. For instance, $\exists\, z > 0\ (z^2 = 2)$ is another way of expressing $\exists z(z > 0 \wedge z^2 = 2)$.

Precedence of Quantifiers

The quantifiers $\forall$ (for all) and $\exists$ (there exists) have higher precedence than all logical operators from propositional calculus. For example, $\forall xP(x) \vee Q(x)$ is the disjunction of $\forall xP(x)$ and $Q(x)$. In other words, it means $(\forall xP(x)) \vee Q(x)$ rather than $\forall x(P(x) \vee Q(x))$.

Introduction to Proofs

A proof is a valid argument that establishes the truth of a mathematical statement.

Some Terminology

A **theorem** is a statement that can be shown to be true. In mathematical writing, the term theorem is usually reserved for a statement that is considered at least somewhat important. Less important theorems sometimes are called **propositions**. A less important theorem that is **helpful** in the proof of other results is called a **lemma.** Complicated proofs are usually easier to understand when they are proved using a series of lemmas, where each lemma is proved individually. A **corollary** is a theorem that can be established directly from a theorem that has been proved. A **conjecture** is a statement that is being proposed to be a true statement, usually on the basis of some partial evidence, a heuristic argument, or the intuition of an expert. When a proof of a conjecture is found, the conjecture becomes a theorem. Many times conjectures are shown to be false, so they are not theorems.

Direct Proofs

A **direct proof** of a conditional statement $p \rightarrow q$ is constructed when the first step is the assumption that p is true; subsequent steps are constructed using rules of logic, with the final step showing that q must also be true. A direct proof shows that a conditional statement $p \rightarrow q$ is true by showing that if p is true, then q must also be true, so that the combination p true and q false never occurs. In a direct proof, we assume that p is true and use axioms, definitions, and previously proven theorems, together with rules of logic, to show that q must also be true.

Exercise.

Give a direct proof of the theorem "If n is an odd integer, then n^2 is odd."

Solution: Note that this theorem states $\forall n P((n) \rightarrow Q(n))$, where $P(n)$ is "n is an odd integer" and $Q(n)$ is "n^2 is odd." As we have said, we will follow the usual convention in mathematical proofs by showing that $P(n)$ implies $Q(n)$, and not explicitly using universal instantiation. To begin a direct proof of this theorem, we assume that the hypothesis of this conditional statement is true, namely, we assume that n is odd. By the definition of an odd integer, it follows that $n = 2k + 1$, where k is some integer. We want to show that n^2 is also odd. We can square both sides of the equation $n = 2k + 1$ to obtain a new equation that expresses n^2. When we do this, we find that $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. By the definition of an odd integer, we can conclude that n^2 is an odd integer (it is one more than twice an integer). Consequently, we have proved that if n is an odd integer, then n^2 is an odd integer. ◀

Exercise: Give a direct proof that if m and n are both perfect squares, then nm is also a perfect square. (An integer a is a **perfect square** if there is an integer b such that $a = b^2$.)

Proof by contraposition

An extremely useful type of indirect proof is known as **proof by contraposition**. Proofs by contraposition make use of the fact that the conditional statement $p \rightarrow q$ is equivalent to its contrapositive, $\neg q \rightarrow \neg p$. This means that the conditional statement $p \rightarrow q$ can be proved by showing that its contrapositive, $\neg q \rightarrow \neg p$, is true. In a proof by contraposition of $p \rightarrow q$, we take $\neg q$ as a premise, and using axioms, definitions, and previously proven theorems, together with rules of inference, we show that $\neg p$ must follow.

VACUOUS AND TRIVIAL PROOFS

We can quickly prove that a conditional statement $p \rightarrow q$ is true when we know that p is false, because $p \rightarrow q$ must be true when p is false. Consequently, if we can show that p is false, then we have a proof, called a **vacuous proof**, of the conditional statement $p \rightarrow q$.

Prove that if n is an integer and $3n + 2$ is odd, then n is odd.

Solution: We first attempt a direct proof. To construct a direct proof, we first assume that $3n + 2$ is an odd integer. From the definition of an odd integer, we know that $3n + 2 = 2k + 1$ for some integer k. Can we use this fact to show that n is odd? We see that $3n + 1 = 2k$, but there does not seem to be any direct way to conclude that n is odd. Because our attempt at a direct proof failed, we next try a proof by contraposition.

The first step in a proof by contraposition is to assume that the conclusion of the conditional statement "If $3n + 2$ is odd, then n is odd" is false; namely, assume that n is even. Then, by the definition of an even integer, $n = 2k$ for some integer k. Substituting $2k$ for n, we find that $3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$. This tells us that $3n + 2$ is even (because it is a multiple of 2), and therefore not odd. This is the negation of the premise of the theorem. Because the negation of the conclusion of the conditional statement implies that the hypothesis is false, the original conditional statement is true. Our proof by contraposition succeeded; we have proved the theorem "If $3n + 2$ is odd, then n is odd." ◀

Give a direct proof that if m and n are both perfect squares, then nm is also a perfect square. (An integer a is a **perfect square** if there is an integer b such that $a = b^2$.)

Example: Show that the proposition $P(0)$ is true, where $P(n)$ is "If $n > 1$, then $n^2 > n$" and the domain consists of all integers.

Solution: Note that $P(0)$ is "If $0 > 1$, then $0^2 > 0$." We can show $P(0)$ using a vacuous proof. Indeed, the hypothesis $0 > 1$ is false. This tells us that $P(0)$ is automatically true.

Exercise: Prove that if n is an integer with $10 \leq n \leq 15$ which is a perfect square, then n is also a perfect cube.

Exercise: Prove that the sum of two rational numbers is rational.

Exercise: Prove that if n is an integer and n^2 is odd, then n is odd.

We can also quickly prove a conditional statement $p \rightarrow q$ if we know that the conclusion q is true. By showing that q is true, it follows that $p \rightarrow q$ must also be true. A proof of $p \rightarrow q$ that uses the fact that q is true is called a **trivial proof**.

Proofs by Contradiction

Exercise: Show that the statement "Every positive integer is the sum of the squares of two integers" is false.

Suppose we want to prove that a statement p is true. Furthermore, suppose that we can find a contradiction q such that $\neg p \rightarrow q$ is true. Because q is false, but $\neg p \rightarrow q$ is true, we can conclude that $\neg p$ is false, which means that p is true. *How can we find a contradiction q that might help us prove that p is true in this way?*

Because the statement $r \wedge \neg r$ is a contradiction whenever r is a proposition, we can prove that p is true if we can show that $\neg p \to (r \wedge \neg r)$ is true for some proposition r. Proofs of this type are called **proofs by contradiction**.

Example: Prove that $\sqrt{2}$ is irrational by giving a proof by contradiction.

Exercise: Prove the theorem "If n is an integer, then n is odd if and only if n^2 is odd."

Exercise: Show that the statement "Every positive integer is the sum of the squares of two integers" is false.

Mistakes in Proof

What is wrong with this "proof" that 1 = 2?
"Proof": We use these steps, where a and b are two equal positive integers.

Step	**Reason**
1. $a = b$	*Given*
2. $a^2 = ab$	Multiply both sides of (1) by a
3. $a^2 - b^2 = ab - b^2$	Subtract b^2 from both sides of (2)
4. $(a-b)(a+b) = b(a-b)$	Factor both sides of (3)
5. $a+b=b$	Divide both sides of (4) by $a - b$
6. $2b = b$	Replace a by b in (5) because *$a=b$ and simplify*
7. $2 = 1$	Divide both sides of (6) by b

Example: What is wrong with this "proof"?
"Theorem": If n^2 is positive, then n is positive.

"Proof ": Suppose that n^2 is positive. Because the conditional statement "If n is positive, then n^2 is positive" is true, we can conclude that n is positive.

Solution: Let $P(n)$ be "n is positive" and $Q(n)$ be "n^2 is positive." Then our hypothesis is $Q(n)$. The statement "If n is positive, then n^2 is positive" is the statement $\forall n(P(n) \to Q(n))$. From the hypothesis $Q(n)$ and the statement $\forall n(P(n) \to Q(n))$ we cannot conclude $P(n)$, because we are not using a valid rule of inference. Instead, this is an example of the fallacy of affirming the conclusion. A counterexample is supplied by $n = -1$ for which $n^2 = 1$ is positive, but n is negative.

Exercise: What is wrong with this "proof"?

"Theorem": If n is not positive, then n^2 is not positive. (This is the contrapositive of the "theorem" in Example 17.)

"Proof ": Suppose that n is not positive. Because the conditional statement "If n is positive, then n^2 is positive" is true, we can conclude that n^2 is not positive.

UNIT 2

Sequence: An expression of the form $a_1, a_2, a_3, \ldots$, where a_i is a real number for each i is called a real sequence. This is denoted by $\langle a_n \rangle$ or $\{a_n\}$.

Example: $1,4,7,10,\ldots$, then $\{a_n\} = \{2n-1\},\ n \in \mathbb{N}$.

$1, -1,\ 1, -1,\ 1, -1, \ldots$, then $\{a_n\} = \{(-1)^{n-1}\},\ n \in \mathbb{N}$

Recurrence Relation: A reaction that determines the subsequent terms of a sequence by those that precede them is called a recurrence relation.

Example: $a_n = a_{n-1} + 1,\ a_0 = 1$

$a_n = a_{n-1} - a_{n-2},\ a_0 = 2,\ a_1 = 5$, etc are recurrence relations.

Rabbits and the Fibonacci Numbers Consider this problem, which was originally posed by Leonardo Pisano, also known as Fibonacci, in the thirteenth century in his book *Liber abaci*. A young pair of rabbits (one of each sex) is placed on an island. A pair of rabbits does not breed until they are 2 months old. After they are 2 months old, each pair of rabbits produces another pair each month. Find a recurrence relation for the number of pairs of rabbits on the island after n months, assuming that no rabbits ever die.

Let M_rF_r denote a pair of rabbits whose age is r months. Then the problem can be modeled as follows.

Month	Pairs of Rabbits	Number of pairs
1	M_1F_1	1
2	M_2F_2	1
3	$M_3F_3,\ M_1F_1$(Newborn pair)	2
4	M_4F_4, M_1F_1, M_2F_2	3
5	$M_5F_5, M_1F_1, M_2F_2, M_3F_3, M_1F_1$	5

Denote by f_n the number of pairs of rabbits after n months. At the end of the first month, the number of pairs of rabbits on the island is $f_1 = 1$. Because this pair does not breed during the second month, $f_2 = 1$ also. To find the number of pairs after n months, add the number on the island the previous month, f_{n-1}, and the number of newborn pairs, which equals f_{n-2}, because each newborn pair comes from a pair at least 2 months old.

Consequently, the sequence $\{f_n\}$ satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2}$

for $n \geq 3$ together with the initial conditions $f_1 = 1$ and $f_2 = 1$.

Fibonacci sequence: A sequence $\{f_n\}$, where $f_n = f_{n-1} + f_{n-2}$ with $f_1 = 1$ and $f_2 = 1$ is called a Fibonacci sequence.

The Tower of Hanoi Puzzle A popular puzzle of the late nineteenth century invented by the French mathematician Edouard Lucas, called the Tower of Hanoi, consists of three pegs mounted on a board together with disks of different sizes. Initially these disks are placed on the first peg in order of size, with the largest on the bottom (see Figure 2). The rules of the puzzle allow disks to be moved one at a time from one peg to another as long as a disk is never placed on top of a smaller disk. The goal of the puzzle is to have all the disks on the second peg in order of size, with the largest on the bottom.

Let H_n denote the number of moves needed to solve the Tower of Hanoi puzzle with n disks. Set up a recurrence relation for the sequence $\{H_n\}$.

Solution: Begin with n disks on peg 1. We can transfer the top $n - 1$ disks, following the rules of the puzzle, to peg 3 using H_{n-1} moves (see Figure 3 for an illustration of the pegs and disks at this point). We keep the largest disk fixed during these moves. Then, we use one move to transfer the largest disk to the second peg. Finally, we transfer the $n - 1$ disks on peg 3 to peg 2 using H_{n-1} moves, placing them on top of the largest disk, which always stays fixed on the bottom of peg 2. This shows that we can solve the Tower of Hanoi puzzle for n disks using $2H_{n-1} + 1$ moves.

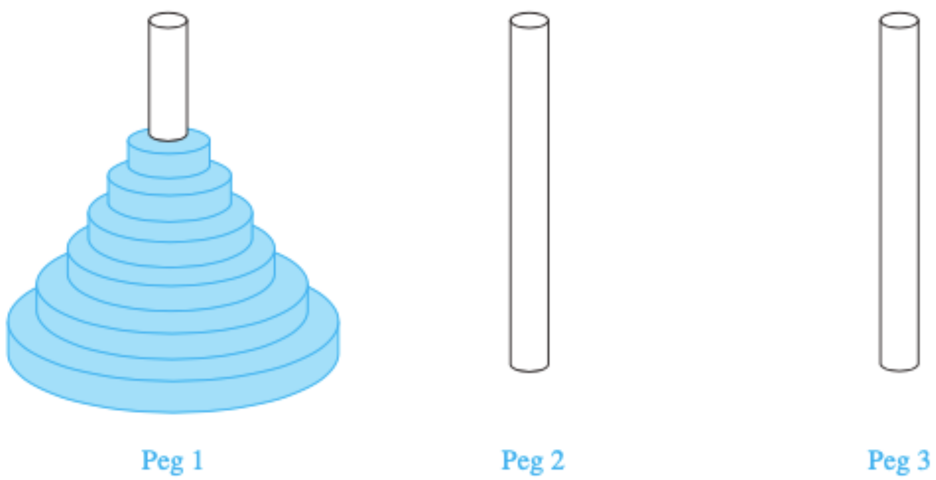


FIGURE 2 **The initial position in the Tower of Hanoi.**

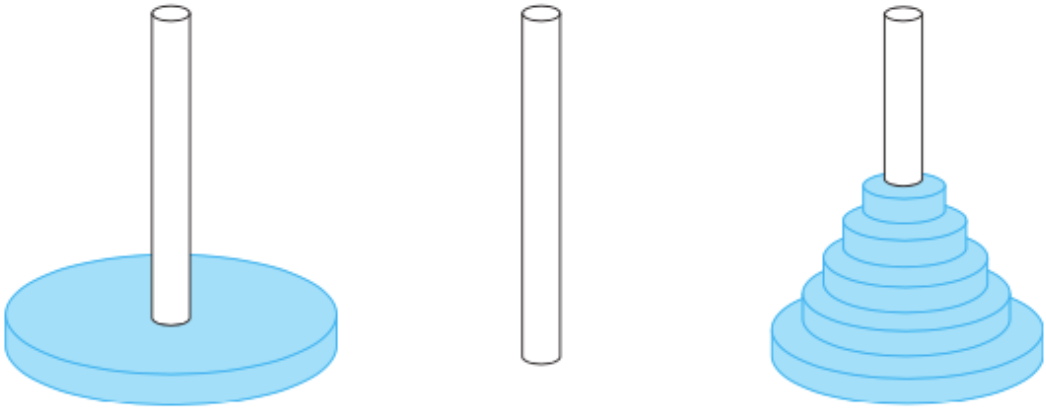


FIGURE 3 **An intermediate position in the Tower of Hanoi.**

We can use an iterative approach to solve this recurrence relation. Note that

$$\begin{aligned} H_n &= 2H_{n-1} + 1 \\ &= 2(2H_{n-2} + 1) + 1 = 2^2 H_{n-2} + 2 + 1 \\ &= 2^2(2H_{n-3} + 1) + 2 + 1 = 2^3 H_{n-3} + 2^2 + 2 + 1 \\ &\vdots \\ &= 2^{n-1}H_1 + 2^{n-2} + 2^{n-3} + \cdots + 2 + 1 = 2^{n-1} + 2^{n-2} + \cdots + 2 + 1 \\ &= 2^n - 1. \end{aligned}$$

THINK!

How much time does it require to transfer 64 disks from one peg to another, according to the rules of the puzzle assuming that it takes one second to move a disk?

From the explicit formula, the number of steps it require equals

$$2^{64} - 1 = 18{,}446{,}744{,}073{,}709{,}551{,}615$$

Thus, making one move per second, it will take them more than 500 billion years to complete the transfer, so the world will end when the puzzle finishes.

METHODS OF SOLVING RECURRENCE RELATIONS

Definition: A recurrence relation of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$, where $c_1, c_2, \ldots, c_k$ are constants and $c_k \neq 0$ is called a Linear Homogeneous Recurrence Relations of Degree k with Constant Coefficients.

Example: $f_n = f_{n-1} + f_{n-2}$ is linear homogeneous recurrence relation of degree 1.
The recurrence relation $a_n = a_{n-1} + a^2_{n-2}$ is not linear.
The recurrence relation $H_n = 2H_{n-1} + 1$ is not homogeneous.
The recurrence relation $B_n = nB_{n-1}$ does not have constant coefficients.

Basic Rule for solution: For linear homogenous recurrence relations with constant coefficients. We use two key ideas to find all their solutions. First, these recurrence relations have solutions of the form $a_n = r^n$, where r is a constant. To see this, observe that $a_n = r^n$, is a solution of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k}$ if and only if

$$r^n = c_1r^{n-1} + c_2r^{n-2} + \cdots + c_kr^{n-k}.$$

When both sides of this equation are divided by r^{n-k} (when $r \neq 0$) and the right-hand side is subtracted from the left, we obtain the equation

$$r^k - c_1r^{k-1} - c_2r^{k-2} - \cdots - c_k = 0.$$

And the second key observation is that a linear combination of two solutions of a linear homogeneous recurrence relation is also a solution.

Definition: The equation $r^k - c_1r^{k-1} - c_2r^{k-2} - \cdots - c_k = 0$ is called the c**haracteristic equation** of the recurrence relation. The solutions of this equation are called the **characteristic roots** of the recurrence relation.

Theorem: If the roots of the characteristic equation of the recurrence relation

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k},$$

are all distinct, say, $r_1, r_2, \ldots, r_k$, then the solution of the recurrence relation is given by

$$a_n = \alpha_1r_1^n + \alpha_2r_2^n + \cdots + \alpha_kr_k^n,$$

where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are arbitrary constants.

Example: Find the solution of $a_n = 4a_{n-1} - a_{n-2} - 6a_{n-3}$.

Sol. The recurrence relation can be written as $a_n - 4a_{n-1} + a_{n-2} + 6a_{n-3} = 0$. Then the characteristic equation is $r^3 - 4r^2 + 4r + 6 = 0$ and the roots are $r = -1, 2, 3$. Thus, the general solution is $a_n = \alpha_1(-1)^n + \alpha_2.2^n + \alpha_3.3^n$.

Example: Solve the recurrence relation $a_n = 2a_{n-1} - 3a_{n-2}$, where $a_0 = -1$, $a_1 = 3$.

Sol. The recurrence relation can be written as $a_n - 2a_{n-1} + 3a_{n-2} = 0$. Then the characteristic equation is $r^2 - 2r + 3 = 0 = (r+1)(r-3)$, so that $r = -1, 3$. Thus, the general solution is $a_n = \alpha_1(-1)^n + \alpha_2.3^n$.

Using the initial conditions, $a_0 = -1$, $a_1 = 3$, we obtain;

$a_0 = \alpha_1(-1)^0 + \alpha_2.3^0$, which gives, $\alpha_1 + \alpha_2 = -1$ and

$a_1 = \alpha_1(-1)^1 + \alpha_2.3^1$, which gives, $-\alpha_1 + 3\alpha_2 = 3$. These equations give, $\alpha_1 = -3/2$, $\alpha_2 = 1/2$.

Thus, the solution is $a_n = \dfrac{-3}{2} \cdot (-1)^n + \dfrac{1}{2} \cdot 3^n$

Theorem: If r is the characteristic root of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} \ ,$$

with multiplicity j, then the solution of the recurrence relation corresponding to r is

$$\left[\alpha_1 + \alpha_2 \cdot n + \alpha_3 \cdot n^2 + \cdots + \alpha_j \cdot n^{j-1}\right] r^j \ ,$$

where $\alpha_1, \alpha_2, \ldots, \alpha_j$ are arbitrary constants.

Example: Find the solution of $a_n = 8a_{n-1} - 24a_{n-2} + 32a_{n-3} - 8a_{n-4}$.

Sol. The recurrence relation can be written as $a_n - 8a_{n-1} + 24a_{n-2} - 32a_{n-3} + 8a_{n-4} = 0$. Then the characteristic equation is $r^4 - 8r^3 + 24r^2 - 32r + 8 = 0$ and the roots are $r = 2,2,2,2$. Thus, the general solution is $\left[\alpha_1 + \alpha_2 \cdot n + \alpha_3 \cdot n^2 + \alpha_4 \cdot n^3\right] r^4$.

The general case:

THEOREM 4 Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$, respectively, so that $m_i \geq 1$ for $i = 1, 2, \ldots, t$ and $m_1 + m_2 + \cdots + m_t = k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$\begin{aligned} a_n = {} & (\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n \\ & + (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n \\ & + \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n \end{aligned}$$

for $n = 0, 1, 2, \ldots$, where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i - 1$.

Example: Find the solution the recurrence relation whose roots are $-1, -1, -1, 2, 3, 5, 5, 5, 5$.

Sol. The solution is

$(\alpha_1 + \alpha_2 \cdot n + \alpha_2 \cdot n^2)(-1)^n + \alpha_4 \cdot 2^n + \alpha_5 \cdot 3^n + (\alpha_6 + \alpha_7 \cdot n + \alpha_8 \cdot n^2 + \alpha_9 \cdot n^3) \cdot 5^n$

Definition: A recurrence relation of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + g(n)$, where $c_1, c_2, \ldots, c_k$ are constants, $c_k \neq 0$ and $g(n)$ is a non zero function of n is called a *Linear* Non *Homogeneous Recurrence Relations of Degree k with Constant Coefficients.*

The recurrence relation $a_n = 3a_{n-1} + 2n$ is an example of a linear non-homogeneous recurrence relation with constant coefficients, that is, a recurrence relation of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + g(n)$, where $c_1, c_2, \ldots, c_k$ are real numbers and $g(n)$ is a function not identically zero depending only on n. The recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ is called the associated homogeneous recurrence relation.

Shift Operator: Let $\{a_n\}$ be a sequence and $S = \{a_n : n \in \mathbb{N}\}$ be the set of terms of $\{a_n\}$. A function $E : S \to S$ defined as $Ea_n = a_{n+1}$ is called a shift operator.

An obvious property of shift property is $E^k a_n = a_{n+k}$, $k \geq 1$.

***Difference Operator*:** A function Δ defined as $\Delta a_n = a_{n+1} - a_n$ is called the difference operator.

Relation between shift operator and difference operator:

$$\begin{aligned} \Delta a_n &= a_{n+1} - a_n \\ &= Ea_n - a_n \\ &= (E-1)a_n, \text{ that is, } \Delta = E - 1. \end{aligned}$$

Consider $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + g(n)$, $c_k \neq 0$, which gives

$$a_n - c_1 a_{n-1} - c_2 a_{n-2} - \cdots - c_k a_{n-k} = g(n)$$

Replacing n by $n + k$, we obtain, $a_{n+k} - c_1 a_{n+(k-1)} - c_2 a_{n+(k-2)} - \cdots - c_k a_{n+(k-k)} = g(n+k)$,

Using $E^k a_n = a_{n+k}$, $k \geq 1$, gives, $E^k a_n - E^{k-1} a_n - \cdots - E^{k-k} a_n = g(n+k)$, which gives

$$\left(E^k - E^{k-1} - \cdots - E^{k-k}\right) a_n = g(n+k)$$

i.e. $F(E)a_n = g(n+k)$, where $F(E)$ is the characteristic polynomial of the associated homogeneous part of $a_n - c_1 a_{n-1} - c_2 a_{n-2} - \cdots - c_k a_{n-k} = g(n)$.

Theorem: The solution of the non homogeneous linear recurrence relation with constant coefficients $a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k} + g(n)$, is given by $a_n = a^{(h)} + a^{(p)}$, where $\{a^{(h)}\}$ is a solution of the associated homogeneous recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k}$, and $a_n^{(p)}$ is the particular solution (P. S.) which is defined as

$$a_n^{(p)} = \frac{1}{F(E)} \cdot g(n+k).$$

Rules for finding particular solution P. S.

Case I: When $F(E) = a^n$ for some constant a, and $F(a) \neq 0$, then

$$a_n^{(p)} = \frac{1}{F(E)} \cdot g(n+k)$$

$$\text{i.e., } a_n^{(p)} = \frac{1}{F(a)} \cdot a^{n+k}.$$

Case II: When $F(E) = a^n$ for some constant a, and $F(a) = 0$, then

$$a_n^{(p)} = \frac{1}{F(E)} \cdot g(n+k)$$

$$\text{i.e., } a_n^{(p)} = \frac{1}{(E-a)^j G(E)} \cdot a^{n+k}, \text{ where } G(a) \neq 0$$

$$\text{i.e., } a_n^{(p)} = \binom{n}{k} \frac{1}{G(a)} \cdot a^{n+k-j}$$

Special Case: When $g(n) = \cos n\alpha$ or $\sin n\alpha$.

We have $\cos n\alpha = \dfrac{e^{in\alpha} + e^{-in\alpha}}{2} = \dfrac{(e^{in\alpha})^n + (e^{-i\alpha})^n}{2} = \dfrac{1}{2}[a^n + b^n]$, where $a = e^{i\alpha}$, $b = e^{-i\alpha}$.
Similarly, $\sin n\alpha = \dfrac{e^{in\alpha} - e^{-in\alpha}}{2i} = \dfrac{(e^{in\alpha})^n - (e^{-i\alpha})^n}{2i} = \dfrac{1}{2i}[a^n - b^n]$, Thus, we can use the above Case I and Case II for finding such particular solutions.

Example: Solve **(i)** $a_n = 2a_{n-1} - 3a_{n-2} + 5^n$ **(ii)** $a_n - 8a_{n-2} + 16a_{n-4} = 4^n - 5^n$

Sol: (i) By replacing n by $n+2$, *the* given recurrence relation can be written as $F(E)a_n = 5^{n+2}$, where $F(E) = E^2 - 2E + 3$.

Thus, the characteristic equation of the homogeneous part is $F(E) = E^2 - 2E + 3 = 0$, which gives, $E = -1, 3$. Thus, $a_n^{(h)} = \alpha \cdot (-1)^n + \beta \cdot 3^n$.

Now, the particular solution is

$$a_n^{(p)} = \frac{1}{F(E)} \cdot 5^{n+2} = \frac{1}{E^2 - 2E + 3} \cdot 5^{n+2} = \frac{1}{5^2 - 2 \times 5 + 3} \cdot 5^{n+2} = \frac{1}{18} \cdot 5^{n+2} = \frac{25}{18} \cdot 5^n.$$

Example: Solve (i) $a_n = 2a_{n-1} - 3a_{n-2} + 3^n + (-1)^n$

(ii) $a_n - 8a_{n-2} + 16a_{n-4} = 2^n + (-2)^n + 5^n$

Sol: (i) By replacing n by $n+1$, the given recurrence relation can be written as $F(E)a_n = 3^{n+1} + (-1)^{n+1}$, where $F(E) = E^2 - 2E + 3$.

Thus, the characteristic equation of the homogeneous part is $F(E) = E^2 - 2E + 3 = 0$, which gives, $E = -1,\ 3$. Thus, $a_n^{(h)} = \alpha \cdot (-1)^n + \beta \cdot 3^n$.

Now, the particular solution is

$$\begin{aligned}
a_n^{(p)} = \frac{1}{F(E)} \cdot 5^{n+2} &= \frac{1}{E^2 - 2E + 3} \cdot [3^{n+2} + (-1)^{n+2}] \\
&= \frac{1}{(E+1)(E-3)} \cdot 3^{n+2} + \frac{1}{(E+1)(E-3)} \cdot (-1)^{n+2} \\
&= \frac{1}{E+1} \cdot \frac{1}{E-3} \cdot 3^{n+2} + \frac{1}{E+1} \cdot \frac{1}{E-3} \cdot (-1)^{n+2} \\
&= \frac{1}{3+1} \cdot \binom{n}{1} \cdot 3^{n+2-1} + \binom{n}{1} \cdot \frac{1}{-1-3} \cdot (-1)^{n+2-1} \\
&= \frac{1}{4} \cdot \frac{n(n-1)}{2} \cdot 3^{n+1} + \frac{n(n-1)}{2} \cdot \frac{1}{-4} \cdot (-1)^{n+1} \\
&= \frac{n(n-1)}{8} \left[3^{n+1} - (-1)^{n+1}\right]
\end{aligned}$$

Therefore, $a_n = \alpha \cdot (-1)^n + \beta \cdot 3^n + \dfrac{n(n-1)}{8} \left[3^{n+1} - (-1)^{n+1}\right]$.

Example: Solve (i) $a_n = 6a_{n-1} + \cos(n-6)$

(ii) $a_n = 2a_{n-1} - 3a_{n-2} + \sin(n-2) + \cos(n-1)$

Sol: (i) The characteristic equation of the homogeneous part is $F(E) = E - 6 = 0$, which gives, $E = 6$. Thus, $a_n^{(h)} = \alpha \cdot 6^n$.

Now, $a_n^{(p)} = \dfrac{1}{F(E)} \cdot g(n+k) = \dfrac{1}{E-6} \cdot \cos n = \dfrac{1}{E-6}\dfrac{1}{2}[a^n + b^n]$, where $a = e^{i\alpha}$, $b = e^{-i\alpha}$,

thus, $a_n^{(p)} = \dfrac{1}{E-6}\dfrac{1}{2}[a^n + b^n] = \dfrac{1}{2}\left[\dfrac{1}{e^i - 6}e^{in} + \dfrac{1}{e^i - 6}e^{-in}\right]$.

Factorial Polynomial: A polynomial denoted by $n^{(m)}$ and defined as

$$n^{(m)} = \frac{n!}{(n-m)!} = n(n-1)(n-2)\cdots\big(n-(m-1)\big)$$

is called a factorial polynomial.

Example: (i) $n^{(0)} = \dfrac{n!}{(n-0)!} = 1$

(ii) $n^{(1)} = \dfrac{n!}{(n-1)!} = n$

(iii) $n^{(2)} = \dfrac{n!}{(n-2)!} = n(n-1) = n^2 - n$

(iv) $n^{(3)} = \dfrac{n!}{(n-3)!} = n(n-1)(n-2) = (n^2-n)(n-2) = n^3 - 3n^2 + 2n$

Example: Write the polynomial n^2, n^3 and $1 + n + n^2 + n^3$ in terms of factorial polynomial.

Sol. We have (i) We have $n^{(2)} = \dfrac{n!}{(n-2)!} = n(n-1) = n^2 - n$, this gives,

$$n^2 = n^{(2)} + n$$

$$= n^{(2)} + n^{(1)}, \text{ because } n^{(1)} = n$$

(ii) We have $n^{(3)} = n^3 - 3n^2 + 2n$, this gives,

$$n^3 = n^{(3)} + 3n^2 - 2n$$

$$= n^{(3)} + 3(n^{(2)} + n^{(1)}) - 2n^{(1)}$$

$$= n^{(3)} + 3n^{(2)} + n^{(1)}$$

Formulae: $\Delta n^{(m)} = m \cdot n^{(m-1)}$ and $\dfrac{1}{\Delta} n^{(m)} = \dfrac{n^{(m+1)}}{m+1}$

Case III. When $g(n)$ is a polynomial function, we replace E by $1 + \Delta$, so that

$a_n^{(p)} = \dfrac{1}{F(E)} \cdot g(n+k) = \dfrac{1}{F(1+\Delta)} \cdot g(n+k)$, and then expand $\dfrac{1}{F(1+\Delta)}$ in increasing powers of Δ.

Example: Solve the following recurrence relations.

(i) $3a_n = 2a_{n-1} + (n-1)^3$ (ii) $a_n = -2a_{n-1} - a_{n-2} + 3 \cdot n^3$

(iii) $a_n = 2a_{n-1} - a_{n-2} + 3n^2 + n$ (iv) $a_n = a_{n-1} + 56a_{n-2} + 3(n^{(2)} + n^2)$

Sol. (i) The given recurrence relation can be put in the from $F(E)a_n = n^3$**,** where $F(E) = 3E - 2$ after replacing n by $n+1$. The solution is $a_n = a_n^{(h)} + a_n^{(p)}$.

Now, as $E = 2/3$ is the characteristic root, thus, $a_n^{(h)} = \alpha(2/3)^n$.

Now, $a_n^{(h)} = \dfrac{1}{F(E)} \cdot n^3 = \dfrac{1}{3E-2} \cdot n^3 = \dfrac{1}{3(1+\Delta)-2} \cdot n^3 = \dfrac{1}{1+3\Delta} \cdot n^3$

$$= (1+3\Delta)^{-1} \cdot \left(n^{(3)} + 3n^{(2)} + n^{(1)}\right)$$

$$= \left(1 - 3\Delta + 9\Delta^2 - 27\Delta^3 + \cdots\right) \cdot \left(n^{(3)} + 3n^{(2)} + n^{(1)}\right)$$

$$= 1 \cdot \left(n^{(3)} + 3n^{(2)} + n^{(1)}\right) - 3\Delta \cdot \left(n^{(3)} + 3n^{(2)} + n^{(1)}\right) + 9\Delta^2 \cdot \left(n^{(3)} + 3n^{(2)} + n^{(1)}\right)$$
$$-27\Delta^3 \cdot \left(n^{(3)} + 3n^{(2)} + n^{(1)}\right)$$

$$= \left(n^{(3)} + 3n^{(2)} + n^{(1)}\right) - 3 \cdot \left(3n^{(2)} + 3.2n^{(1)} + 1\right) + 9 \cdot \left(6n^{(1)} + 0\right) - 27 \cdot (3+0)$$

$$= n^{(3)} - 6n^{(2)} + 37n^{(1)} - 78$$

$$= (n^3 - 3n^2 + 2n) - 6(n^2 - n) + 37n - 78$$

$$= n^3 - 9n^2 + 45n - 78$$

Therefore, $a_n = a_n^{(h)} + a_n^{(p)} = \alpha(2/3)^n + n^3 - 9n^2 + 45n - 78$ is the solution.

Generating functions: Let $\{a_n\}$ be a sequence. A generating function of $\{a_n\}$ is the power series

$$G(x) = \sum_{i=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

Example: Write the generating functions of the following sequences.

(i) 1, 1, 1,… (ii) 1, – 1, 1, – 1,…
(iii) 1, 0, 1, 0, 1, 0,… (iv) 1, 2, 3, 4,…

Sol. (i) The generating function is $G(x) = \sum_{i=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots = 1 + x + x^2 + \cdots$
$= \dfrac{1}{1-x}$

(iv) The generating function is

$$G(x) = \sum_{i=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots = 1 + 2x + 3x^2 + 4x^3 + \cdots = \frac{1}{(1-x)^2}$$

Binomial Expansions:

(i) $$\frac{1}{1-ax} = (1-ax)^{-1} = 1 + ax + (ax)^2 + \cdots = \sum_{n=0}^{\infty} (ax)^n$$

(ii) $$\frac{1}{1+ax} = (1+ax)^{-1} = 1 - (ax) + (ax)^2 - (ax)^3 + \cdots = \sum_{n=0}^{\infty} (-ax)^n$$

(iii) $$\frac{1}{(1-x)^2} = (1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n=0}^{\infty} (n+1)(ax)^n$$

(iv) $$\frac{1}{(1+x)^2} = (1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots = \sum_{n=0}^{\infty} (n+1)(-ax)^n$$

(v) $$\frac{1}{(1-ax)^2} = (1-ax)^{-2} = 1 + 2(ax) + 3(ax)^2 + 4(ax)^3 + \cdots = \sum_{n=0}^{\infty} (n+1)(ax)^n$$

(vi) $$\frac{1}{(1+ax)^2} = (1+ax)^{-2} = 1 - 2(ax) + 3(ax)^2 - 4(ax)^3 + \cdots = \sum_{n=0}^{\infty} (n+1)(-ax)^n$$

Example: Solve the recurrence relation $a_n = 2a_{n-1} + 3^n$, $a_0 = 2$ using generating functions.

Sol. The generating function of the recurrence relation is

$$\begin{aligned}
G(x) &= \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \\
&= 2 + (2a_0 + 3)x + (2a_1 + 3^2)x^2 + (2a_3 + 3^3)x^3 + \cdots \\
&= 2 + 2(a_0 x + a_1 x^2 + a_3 x^3 + \cdots) + 3x + 3^2x^2 + 3^3x^3 + \cdots \\
&= 2 + 2xG(x) + \frac{3x}{1-3x}
\end{aligned}$$

Therefore, $$G(x) = \frac{2}{1-2x} + \frac{3x}{(1-2x)(1-3x)} = \frac{2}{1-2x} + \frac{-3}{1-2x} + \frac{3}{1-3x}$$

$$= -(1-2x)^{-1} + 3(1-3x)^{-1} = -\sum_{n=0}^{\infty} (2x)^n + 3\sum_{n=0}^{\infty} (3x)^n$$

Thus, $\sum_{n=0}^{\infty} a_n x^n = (-2^n + 3^{n+1}) \sum_{n=0}^{\infty} x^n$, which gives, $a_n = 3^{n+1} - 2^n$.

Exercise: Solve the following recurrence relations using generating functions.

(i) $2a_n = 4a_{n-1} + 3^n$, $a_0 = 1$ (ii) $H_n = 2H_{n-1} + 1$, $H_0 = 1$

(iii) $2a_n = 4a_{n-1} + n$, $a_0 = 1$ ***(iv)*** $2a_n = 4a_{n-1} + n(-1)^n$, $a_0 = 1$

UNIT 3

Counting Principles.

THE PRODUCT RULE: If there are n_1 ways to do a task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 n_2$ ways to do the two tasks simultaneously.

Exercise: A new company with just two employees, Sanchez and Patel, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?

Solution: The procedure of assigning offices to these two employees consists of assigning an office to Sanchez, which can be done in 12 ways, then assigning an office to Patel different from the office assigned to Sanchez, which can be done in 11 ways. By the product rule, there are $12 \cdot 11 = 132$ ways to assign offices to these two employees.

Generalized Product Rule: If each task T_i, $i = 1,2,\ldots,n$, can be done in n_i ways, regardless of how the previous tasks were done, then there are $n_1 \cdot n_2 \cdot \cdots \cdot n_m$ ways to carry out the tasks simultaneously.

Exercise: How many different bit strings of length seven are there?

Solution: Each of the seven bits can be chosen in two ways, because each bit is either 0 or 1. Therefore, the product rule shows there are a total of $2^7 = 128$ different bit strings of length seven.

Example: (**Counting Functions**) How many functions are there from a set with m elements to a set with n elements?

Solution: A function corresponds to a choice of one of the n elements in the codomain for each of the m elements in the domain. Hence, by the product rule there are $n \cdot n \cdot \cdots \cdot n = n^m$ functions from a set with m elements to one with n elements. For example, there are $5^3 = 125$ different functions from a set with 3 elements to a set with 5 elements.

Definition: **One-to-One Function.** A function f is one-to-one if every different in the domain is mapped with different elements of the codomain.

Example: (**Counting One-to-One Functions**) How many one-to-one functions are there from a set with m elements to one with n elements?

Solution: First note that when $m > n$ there are no one-to-one functions from a set with m elements to a set with n elements.

Now let $m \leq n$. Then there are $n(n-1)(n-2)\cdots(n-(m-1)) = \dfrac{n!}{(n-m)!}$ one-to-one functions from a set with m elements to one with n elements.

For example, there are $5 \cdot 4 \cdot 3 = 60$ one-to-one functions from a set with three elements to a set with five elements.

Example: (**Counting the number of Onto functions**): First note that when $m < n$ there are no onto functions from a set with m elements to a set with n elements.

Now let $m \geq n$. Then there are

$$\sum_{n=0}^{n-1} (-1)^{n-1} \binom{n}{k} (n-k)^m$$

$$= n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m + \cdots + (-1)^{n-1}\binom{n}{n-1}(1)^m$$

one-to-one functions from a set with m elements to one with n elements.

Example: Number of onto functions from a set with m elements to a set of two elements is equal to $n^m - 2$

Example: Number of onto functions from a set with 5 elements to a set of 3 elements is equal to $3^5 - \binom{3}{1}(3-1)^5 + \binom{3}{2}(3-2)^5 = 243 - 3 \cdot 2^5 + 3 \cdot 1^5 = 150$

Example: Number of onto functions from a set with 6 elements to a set of 4 elements is equal to $4^6 - \binom{4}{1}(4-1)^6 + \binom{4}{2}(4-2)^6 - \binom{4}{3}(4-3)^6$

$$= 2^{12} - 4 \cdot 3^6 + 6 \cdot 2^6 + 4 \cdot 1^6 = 4096 - 2916 + 96 - 4 = 1472$$

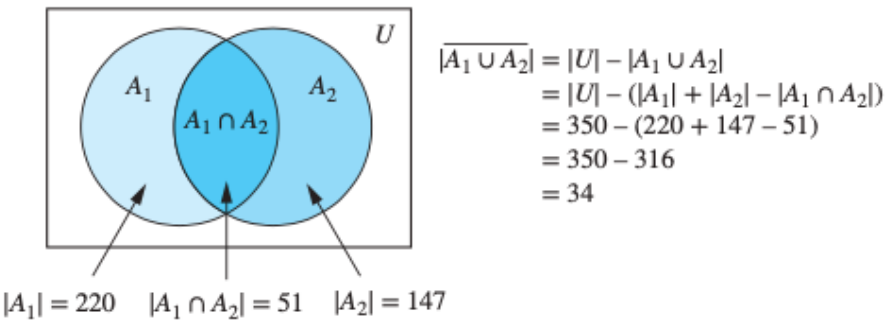
THE SUM RULE If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.

Example: A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?

Solution: The student can choose a project by selecting a project from the first list, the second list, or the third list. Because no project is on more than one list, by the sum rule there are 23 + 15 + 19 = 57 ways to choose a project.

The Inclusion-Exclusion Principle: Suppose A and B are two finite sets. Then

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$



Rule for three sets: $n(A \cup B \cup C) = n(A) + n(B + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

Example. How many integers not exceeding 2023 are divisible by 2 or 3 or 5.

Sol. Let A, B, C denote the sets of integers, not exceeding 2023, that are divisible by 2, 3 and 5 respectively.

Then we have to find $n(A \cup B \cup C)$.

We have $n(A \cup B \cup C) = n(A) + n(B + n(C) - n(A \cap B) - n(B \cap C)$

$$-n(A \cap C) + n(A \cap B \cap C)$$

Now, $n(A) = \left[\dfrac{2023}{2}\right] = 1011$, $n(B) = \left[\dfrac{2023}{3}\right] = 674$, $n(C) = \left[\dfrac{2023}{5}\right] = 404$,

$n(A \cap B) = \left[\dfrac{2023}{6}\right] = 337, n(B \cap C) = \left[\dfrac{2023}{15}\right] = 134, n(A \cap C) = \left[\dfrac{2023}{10}\right] = 202,$

$n(A \cap B \cap C) = \left[\dfrac{2023}{2}\right] = 67.$

Thus, $n(A \cup B \cup C) = 1011 + 674 + 404 - 337 - 134 - 202 + 67 = 1483$

THE PIGEONHOLE PRINCIPLE If k is a positive integer and $k + 1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Examples:

(i) Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.

(ii) In any group of 27 English words, there must be at least two that begin with the same letter, because there are 26 letters in the English alphabet.

THE GENERALIZED PIGEONHOLE PRINCIPLE If N objects are placed into k boxes, then there is at least one box containing at least $\left\lceil \frac{N}{k} \right\rceil$ objects.

Example.

(i) Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month.

(ii) Among every group of 22 people, there must be $\left\lceil \frac{22}{7} \right\rceil = 4$ whose birthday falls on the same day of a week.

(iii) What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F?

Sol. The minimum number of students needed to ensure that at least six students receive the same grade is the smallest integer N such that $\lceil N/5 \rceil = 6$. The smallest such integer is $N = 5 \cdot 5 + 1 = 26$. If you have only 25 students, it is possible for there to be five who have received each grade so that no six students have received the same grade. Thus, 26 is the minimum number of students needed to ensure that at least six students will receive the same grade.

Relation: A relation on a set A is a subset of $A \times A$.

Example: Let