Li - Diff Eq

Linear diff equation:-

$$\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \cdots + P_{n-1}\frac{dy}{dx} + P_n y$$

$$\frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + 7y = 0.$$

$y = f(x)$ Integrating factor.

$\Rightarrow x + g(y).$

$$\frac{dy}{dx} + Py = Q \quad \text{L.D.E.}$$

I.F $e^{\int P\,dx}$

$$\frac{dx}{dy} + P_1 x = Q_1$$

$\frac{dx}{dy}$ I.F $e^{\int P_1\,dy}$

general solution is denoted

$$y \times I.F = \int Q \times I.F\,dx$$

$$G.S \rightarrow x \times I.F = \int Q_1 \times I.F \cdot dy + C$$

if eq in this form:

$$\frac{dy}{dx} + Py = Q$$

then L.D.E I.F is $e^{\int P\,dy}$

then

G.S is [redacted] $\Rightarrow y \times I.F = \int Q \times I.F\,dx + C$

Follow This Steps to get answers

ex: $(x + 2y^3)\frac{dy}{dx} = y$

$$\frac{dy}{dx} = \frac{y}{x + 2y^3}$$

$$\frac{dx}{dy} = \frac{x + 2y^3}{y} = \frac{x}{y} + 2y^2.$$

$$\frac{dx}{dy} - \frac{1}{y}x = 2y^2.$$

$\because$ In this form $\frac{dx}{dy} + P_1 x = Q_1$

I.F ⇒ $e^{-\int \frac{1}{y} dy} = \frac{1}{y}$

G.S ⇒ $x \cdot \frac{1}{y} = \int 2y^2 \cdot \frac{1}{y} \cdot dy + c = y^2 + c$

⇒ $x = y\,(y^2 + c)$.

ex2: $\frac{dy}{dx} + \frac{4x}{x^2+1} \cdot y = \frac{1}{(x^2+1)^2}$

I.F ⇒ $e^{\int \frac{4x}{x^2+1} dx} = (x^2+1)^2$ (∵ Integrate).

G.S ⇒ $y \cdot (x^2+1)^2 = \int \frac{1}{x^2+1} dx + c$

+c.

⇒ $\tan^{-1} x + c$.

ex3: $(x+1)\frac{dy}{dx} - 2y = e^x (x+1)^{n+1}$.

$\frac{dy}{dx} - \frac{n}{x+1} y = e^x (x+1)^n$

I.F ⇒ $e^{-\int \frac{n}{x+1} dx}$

$= \frac{1}{(x+1)^n}$

G.S ⇒ $y \cdot \frac{1}{(x+1)^n} = \int e^x dx + c = e^x + c$.

#ProgyaNotebook

DATE ______
PAGE ______

$$\frac{dy}{dx} + Py = Q \leftarrow$$ * if 'y' has value.

$$f(y)\frac{dy}{dx} + P f(y) = Q.$$

* $z = f(y)$

1) $\frac{dz}{dx} = f'(y) \cdot \frac{dy}{dx}$.

apply 2) $\frac{dz}{dx} + Pz = Q.$

2) $\frac{d}{dx}\left[f(y) \right] = \frac{d}{dy}(f(y)) \times$

exe: $x\frac{dy}{dx} + y\log y = xye^{x}$ G.S = ?

divide by 'x' $\frac{dy}{dx} + \frac{y}{x}\log y = ye^{x}$

divide by 'y' $\frac{1}{y}\frac{dy}{dx} + \frac{1}{x}\log y = e^{x}.$

think $\log y = z$ $\frac{dz}{dx} + \frac{1}{x}z = e^{x}$

$z = \log y$

$\frac{dz}{dx} = \frac{1}{y}\frac{dy}{dx}$

$\left\{ \frac{dy}{dx} + Py = Q \right\}$ in form.

$\left[\int \frac{1}{x} = \log x \right.$ $\left. e^{\log x} = x. \right]$

$\frac{1}{x} \cdot e^{\int \frac{1}{x} dx} = x$

$G.S \Rightarrow z \cdot x = \int e^{x} \cdot x\,dx + c.$

$= xe^x - \int e^x dx + C.$

$= (x-1)e^x + C.$

$x \cdot \log y = (x-1)e^x + C.$

Q: $(x^2 + y)dx = x\,dy.$ $y(1) = 2.$

at $x = 1$ then $x = 2.$

$y(2) = (2(2+1)) = 6.$

$\frac{dy}{dx}$

$\frac{dy}{dx} = x + \frac{y}{x}$

$\frac{dy}{dx} - \frac{y}{x} = x$

$\frac{dy}{dx} - \frac{y}{x} = x$

$I.F = e^{-\int \frac{1}{x} dx} = \frac{1}{x}.$

$y \cdot \frac{1}{x} = \int x \cdot \frac{1}{x} dx + C.$

$y = x(x + C.$

$2 = 1(1 + C)$

$\boxed{C = 1}$

#PragyaNotebook
DATE 26/Sep/22.
PAGE

Bernolli's equations.

$$\frac{dy}{dx}+Py=Qy^n$$

$$y^{-n}\frac{dy}{dx}+Py^{1-n}=Q \longrightarrow$$

$f(y)=y^{-n}$
$f'(y)=(1-n)y^{-n}$
$z=y^{1-n}$
$$\frac{dz}{dx}=(1-n)y^{-n}\frac{dy}{dx}$$

$$\Rightarrow \frac{dz}{dx}+P(1-n)z=P(1-n)Q.$$

$$\Rightarrow \boxed{\frac{dz}{dx}+P_1 z=Q_1} \longrightarrow \text{linear} \longrightarrow \boxed{I.F=e^{\int P_1 dx}.}$$

Q's $x\frac{dy}{dx}+y^2x=y.$

divide by x.

$$\Rightarrow \frac{dy}{dx}+y^2=\frac{y}{x}.$$

$$\Rightarrow \frac{dy}{dx}-\frac{y}{x}=-y^2.$$

divide by $-y^2$.

$$-\frac{1}{y^2}\frac{dy}{dx}+\frac{1}{xy}=1.$$

assume $\frac{1}{y}=z$

$$-\frac{1}{y^2}\frac{dy}{dx}=\frac{dz}{dx}.$$

wow

DATE
PAGE

$$\frac{dz}{dx} + \frac{1}{x} z = 1$$

$$\therefore e^{\log x} = x.$$

$$I.F \Rightarrow e^{\int \frac{1}{x} dx} = x$$

$$G.S \Rightarrow z \cdot x = \int 1 \cdot x\, dx + C$$

$$\Rightarrow \frac{1}{y} x = \frac{x^2}{2} + C.$$

(or).

$$\Rightarrow x = y\left(\frac{x^2}{2} + C\right).$$

ex: $\frac{1}{y} \frac{dy}{dx} + \frac{x}{1-x^2} = x y^{-1/2}.$ $\longrightarrow$ will come in C.A & Mid-term.

Multiply by $y^{-1/2}$.

$$y^{-1/2} \frac{dy}{dx} + \frac{x}{1-x^2} y^{1/2} = x.$$

assume. then $y^{1/2} = z$

$$\frac{1}{2} y^{1/2} \frac{dy}{dx} = \frac{dz}{dx}.$$

$$\Rightarrow \frac{dz}{dx} + \frac{x}{2(1-x^2)} z = \frac{x}{2}.$$

$$I.F = e^{\int \frac{x}{2(1-x^2)} dx}.$$

$$= e^{-1/4 \int \frac{-2x}{1-x^2} dx}.$$

$$= e^{\log (1-x^2)^{-1/4}} = \frac{1}{(1-x^2)^{1/4}}.$$

$$G.S \Rightarrow z \cdot \frac{1}{(1-x^2)^{1/4}}.$$

$$= \int \frac{x}{2} \cdot \frac{1}{(1-x^2)^{1/4}} \, dx + C$$

assume
then $1-x^2 = t$.
$-2x\,dx = dt$

$$= \int \frac{x}{2} (1-x^2)^{-1/4} \, dx.$$

$$= \frac{1}{2} \int \left(-\frac{1}{2}\right) \cdot t^{-1/4} \, dt.$$

$$= \frac{-1}{\cancel{4}} t^{3/4} \times \frac{\cancel{4}}{3}$$

$$\text{G.S} \implies -\frac{1}{3}(1-x^2)^{3/4} + C \ \text{Ans.}$$

$$\Rightarrow 5y = (1-x^2)^{1/4} \left[-\frac{1}{3}(1-x^2)^{3/4} + C \right]$$

Ex 3 $\quad (x^3y^3 + xy) \frac{dy}{dx} = 1$

$$\frac{dx}{dx} = x^3y^3 + xy.$$

$$\frac{dx}{dy} - xy = x^3y^3.$$

divide by x^3.

assume

$\frac{1}{x^2} = z$

then $-2 \frac{1}{x^3} \frac{dx}{dy} = \frac{dz}{dy}$

$$\frac{1}{x^3} \frac{dx}{dy} - \frac{y}{x^2} = y^3.$$

$$\frac{1}{x^3} \frac{dx}{dy} - y\left(\frac{1}{x^3}\right) = y^3$$

$$(-2)\frac{1}{x^3}\frac{dx}{dy} + (-2)\,y\left(\frac{1}{x^2}\right) = y^3(-2).$$

$$\Rightarrow \frac{dz}{dy} + 2yz = -2y^3.$$

$$I.F = e^{\int 2y\,dy} = e^{y^2}$$

$$G.S \Rightarrow z\cdot e^{y^2} = \int -2y^3\cdot e^{y^2}\cdot dy + c.$$

assume $y^2 = t$

$2y\,dy = dt$

$dy = dt$

$$= \int -2t\,e^{t}\,dt + c$$

$$= -(t-1)\,e^{t} + c$$

$$\Rightarrow \boxed{e^{y^2} = x^2(1-y^2)\cdot e^{y^2} + c}$$

Linear differential equation of 2^{nd} order:-

$$\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2 y = 0. \Longrightarrow$$ 2nd homogeneous linear D.E with constant eq.

$$\frac{d}{dx} = D$$

$$\frac{dy}{dx} = Dy. \qquad \frac{d^2y}{dx^2} = D^2y \qquad \frac{d^ny}{dx^n} = D^ny$$

$$D^2y + P_1Dy + P_2y = 0$$

$$(D^2 + P_1D + P_2)y = 0.$$

$$y = e^{mx}$$

$$\frac{dy}{dx} = me^{mx}$$

$$\frac{d^2y}{dx^2} = m^2e^{mx}$$

$$m^2e^{mx} + P_1me^{mx} + P_2e^{mx} = 0$$

$$(m^2 + P_1m + P_2)\,e^{mx} = 0$$

$$\Rightarrow m^2 + P_1m + P_2 = 0$$ Auxiliary eq.

C.F = complentary function.

(i) $m_1 \neq m_2$ — for real & destined

$C.F = C_1 e^{m_1 x} + C_2 e^{m_2 x}$

(ii) $m_1 = m_2 = m_3$

$C.F = (C_1 + C_2 x) e^{mx}$ — for real & equal

(iii) $m_1 = \alpha + i\beta$ — for complex conjugate.

$m_2 = \alpha - i\beta$

$C.F = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$.

G.S $\Rightarrow y = C.F$. — if homogeneous.

ex:- $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$.

$m^2 - 5m + 6 = 0$

$m = 2,3$

$y = C_1 e^{2x} + C_2 e^{3x}$

ex2:- $y'' + 4y = 0$ — Show $\sin 2x$, $\cos 2x$ are solutions & linearly independent.

$\frac{d^2y}{dx^2} + 4y = 0$ $\therefore$ assume $\frac{d}{dx} = D$

$D^2 y + 4y = 0$ — A.E is $m^2 + 4 = 0$

$(D^2 + 4) y = 0$

$m = \sqrt{-4}$

$m = \pm 2i$ — $\alpha = 0$ & $\beta = 2$

$C.F = e^{0x}(C_1 \cos 2x + C_2 \sin 2x)$

#PragyaNotebook

DATE ______

PAGE ______

c_1, c_2 are arbitrary constants.

$$G.S = c_1 \cos 2x + c_2 \sin 2x.$$

$$W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$$= 2(\sin^2 2x + \cos^2 2x)$$

$$= 2 \neq 0$$

$(D^n + P_1 D^{n-1} + \cdots + P_n) y = 0$

if it is n^{th} order linear diff eqn then

$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$.

then any $y = y_1$ or y_2 or y_n are solutions.

$\because \sin^2 x + \cos^2 x = 0$.

Wronskian – is a tool to check functions as linearly independent or not.

(W) ↳ is determinant.

if $y_1, y_2, \cdots y_n$ functions.

$$c_1 y_1 + c_2 y_2 + \cdots + c_n y_n = 0$$

$$\Rightarrow c_1 = c_2 = \cdots c_n = 0.$$

then $y_1, y_2, y_3, \cdots y_n$ are linearly independent.

if any of $c_i \neq 0$ then $y_1, y_2, \cdots y_n$ are linearly dependent.

if n functions

$$W = \begin{vmatrix} y_1, & y_2 & \cdots & y_n \\ y_1', & y_2' & \cdots & y_n' \\ y_1'', & y_2'' & \cdots & y_n'' \\ \vdots & & & \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

Q: Obj: What is used to check?
1) independent 2) dependent (3) both.
(4) Non.

Case(1): if $W \neq 0$, then
$y_1, y_2, \dots y_n$ functions are independent linearly.

Case(2): if $y_1, y_2, \dots y_n$ are linearly dependent
then $W = 0$.

in both cases converse are 0.

* offcourse if $W = 0$
both will be independent.

ex: $y_1 = x^2$, $y_2 = x|x|$

How? Check c_1, c_2 by linear combination
let $c_1 y_1 + c_2 y_2 = 0$ then.

for $x > 0$: as it is for $x < 0$:

$c_1 x^2 + c_2 x^2 = 0$ $c_1 x^2 - c_2 x^2 = 0$.

$$\left.\begin{aligned} c_1 + c_2 = 0 \\ c_1 - c_2 = 0 \end{aligned}\right\}$$
$\Rightarrow c_1 = c_2 = 0$.

for $x > 0$: for $x < 0$:

$$W = \begin{vmatrix} x^2 & x^2 \\ 2x & 2x \end{vmatrix} = 0 \qquad W = \begin{vmatrix} x^2 & x^2 \\ -2x & -2x \end{vmatrix} = 0$$

$\therefore y_1 \,\&\, y_2$ are linear independent.

DATE
PAGE

ex 3: $\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 0$ given: $x(0) = 0$ $\frac{dx}{dt}(0) = 15$.

final solution?

$AE = m^2 + 5m + 6 = 0$.

$(m+2)(m+3) = 0 \Rightarrow m = -2, -3$.

"t" now?

$C.F \Rightarrow C_1 e^{-2t} + C_2 e^{-3t}$

In complentary function there is no "y".

$G.S \Rightarrow \cancel{y} = C_1 e^{-2t} + C_2 e^{-3t}$.

at $t = 0 : x = 0$

$\Rightarrow 0 = C_1 + C_2$.

$\frac{dx}{dt} = -2C_1 e^{-2t} - 3C_2 e^{-3t}$

$15 = -2C_1 - 3C_2$

$\cancel{15 = C_1}$:

$C_1 = 15$

$C_2 = -15$

then $x = 15(e^{-2t} - e^{-3t})$

2nd OL But Non-Homo

there will be extra solution that is complementary function

for homo & Non-homo

#PragyaNotebook

DATE

PAGE

C.F. is always there

for non-Homo ⇒ we have there PI → Particular Integral

G.S.

2nd order linear but non-homogeneous:-

$$\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2 y = Q$$ then non-homogeneous

$$\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2 y = 0$$

C.F → Complementary function.

P.I → particular integral. free from arbitrary constant.

$y = C.F + P.I.$ If n^{th} order, then n AC

PI - free from AC
① 1 AC
② 2 AC
③ 3 AC

→ operator

$$(D^2 + P_1 D + P_2)y = Q.$$ find roots.

$\to f(D)$.

For non-homogeneous - then $P.I = \frac{1}{f(D)} Q$,

⇒ integration, an

if (1) $Q = e^{ax}$ then

then

$\frac{1}{f(D)} Q = \frac{1}{f(a)} e^{ax}$ $f(a) \neq 0$

this is

for 1st order - multiply by x
for 2nd order - " x^2

ex: $(D^2 + 5D + 4)y = e^{2x}$.

for this.

$$\frac{1}{D^2 + 5D + 4} e^{2x}$$

$$= \frac{1}{4 + 10 + 4} e^{2x}$$

$$P.I. = \frac{1}{18} e^{2x}.$$

$G.S \Rightarrow y = C_1 e^{-x} + C_2 e^{-4x} + \frac{1}{18} e^{2x}$

#PragyaNotebook

DATE

PAGE

~~if f(a) ≠ 0~~

if f(a) = 0 then

$(D^2+5D+4)y = e^{-x}$

$\frac{1}{(D^2+5D+4)} e^{-x}$

if $D=-1$.

$\frac{1}{1-5+4} e^{-x}$

$= \frac{1}{0} e^{-x}$ if it is "0" then

find derivative of f(D).

$= \frac{1}{D^2+5D+4} e^{-x} = \frac{x}{2D+5} e^{-x}$

$= \frac{x}{-2+5} e^{-x}$

$= \frac{x}{3} e^{-x}$.

formula is

$\frac{1}{f(D)} e^{ax}$, if $f(a) \neq 0$

to this

$= \frac{1}{f(a)} e^{ax}$, if $f(a) = 0$ and $f'(a) \neq 0$

$= \frac{x}{f'(a)} e^{ax}$, if $f'(a) = 0$ and $f''(a) \neq 0$

$= \frac{x}{f''(a)}$, if it will continue till it is non zero

at end we get 1 or constant

Q $\frac{1}{F(D)}$ $\sin ax / \cos ax$ if $a = \sin/\cos$.

then. we make $F(D) = F(D^2)$.

$= (D^2+5D+4)y = \sin 2x$.

$= \frac{1}{D^2+5D+4} \sin 2x$.

$\Rightarrow$ in place of D^2 we put $-a^2$ $\boxed{D^2 \Rightarrow -a^2}$

$= \frac{1}{-a^2+5D+4} \sin 2x$

$= \frac{1}{-4+5D+4} \sin 2x$

$= \frac{1}{5D} \sin 2x$

$D = -a^2$

case $= \frac{1}{5(-4)}$

$= \frac{1}{1}$

$= 1$

$= \frac{1}{D^2+5D+4} \sin 3x$

$= \frac{1}{-3^2+5D+4} \sin 3x$.

$= \frac{1}{-5+5D} \sin 3x$.

$= \frac{1}{5(D-1)} \sin 3x$. $\boxed{\text{multiply by } D+1}$

$= \frac{D+1}{5(D-1)(D+1)} \sin 3x$

derivative

$= \frac{1}{5} \frac{D+1}{D^2-1} \sin 3x$.

$3\cos 3x +$

$\therefore$ put 3 in D^2

$= \frac{1}{5}\left(\frac{1}{D^2-1}\right)$

$= \frac{1}{5}\left[\frac{3}{-9-1}\cos 3x + \frac{1}{-a^2-1}\right.$

$= \frac{1}{50}\left[3\cos 3x + \sin 3x\right]$

#PragyaNotebook
DATE
PAGE

If where no "0"

$\Rightarrow \frac{1}{D^2+4}\cos 3x \Rightarrow \frac{1}{-9+4}\cos 3x \Rightarrow -\frac{1}{5}\cos 3x$

we put $D^2 = -a^2$.

But if $\frac{1}{D^2+4}\cos 2x$ this will be "0"

two steps of methods can be followed

∴ as 1st order "x" by x.

① $= \frac{x}{2D}\cos 2x$

$= \frac{x}{2}\cdot\frac{1}{D}\cos 2x$

$= \frac{x}{2}\int \cos 2x \cdot dx$

$= \frac{x}{2}\cdot\frac{\sin 2x}{2} \Rightarrow$ P.I.

② if u don't want to integrate then.

$= \frac{x}{2D}\cos 2x$

$= \frac{x}{2}\cdot\frac{D}{D^2}\cos 2x$

$= \frac{x}{2}\cdot\frac{-2\sin 2x}{D^2}$

$= \frac{x}{2}(-2)\left(\frac{1}{-4}\right)\sin 2x$

$= \frac{x}{4}\sin 2x$,

sin 3x

sin 2x

1^{st} order examples

Ex: $(x+1)\frac{dy}{dx} - y = e^{3x}(x+1)^2$

divide by $(x+1)$.

$\frac{dy}{dx} = \frac{y}{x+1} + e^{3x}(x+1)$

$\frac{dy}{dx} - \frac{y}{x+1} = e^{3x}(x+1) \longrightarrow$

$\Rightarrow e^{-\int \frac{1}{x+1}dx}$

$= e^{\log(x+1)^{-1}}$

$= \frac{1}{x+1}$

$y \cdot \left(\frac{1}{x+1}\right) = \int e^{3x}(x+1)^2 \frac{1}{x+1} dx + c$

$= \int e^{3x}(x+1)\,dx + C \longrightarrow$

$\therefore uv \Rightarrow (x+1)\frac{e^{3x}}{3} - \frac{1}{9}e^{3x}$

$= \frac{e^{3x}}{9}(3x+3-1) + C$

$\Rightarrow \frac{9y}{x+1} = e^{3x}(3x+2) - C$ //

Ex: $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1 \longrightarrow$ for MCQ type questions must only 2 marks.

$\frac{dy}{dx} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}$

$\frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \longrightarrow$

$e^{\int \frac{1}{\sqrt{x}}dx} = e^{2\sqrt{x}}$

$y \cdot e^{2\sqrt{x}} = \int e^{2\sqrt{x}} \cdot \frac{e^{-2\sqrt{x}}}{\sqrt{x}} dx + c.$

#PragyaNotebook

DATE ______

PAGE ______

$y e^{2\sqrt{x}} = 2\sqrt{x} + C.$

$y \cdot e^{2\sqrt{x}} - 2\sqrt{x} = C$

Ex: $3x(1-x^2) y^2 \frac{dy}{dx} + (2x^2-1) y^3 = ax^3.$

$y^3 = z.$

$3y^2 \frac{dy}{dx} = \frac{dz}{dx}.$

$x(1-x^2) \frac{dz}{dx} + (2x^2-1) z = ax^3$

divide by $x(1-x^2)$

$\frac{dz}{dx} + \frac{2x^2-1}{x(1-x^2)} z = \frac{ax^2}{1-x^2}$

$\text{I.F} \Rightarrow e^{\int \frac{2x^2-1}{x(1-x^2)} dx} = \frac{1}{x\sqrt{x^2-1}}.$

$\int \frac{2x^2-1}{x(1-x^2)}$

$= \int \left(\frac{x}{1-x^2} - \frac{1}{x} \right) dx$

$= -\frac{1}{2} \int \frac{2x}{x^2-1} dx - \int \frac{1}{x} dx$

$\frac{2x^2-1}{x} = \frac{Ax}{(1-x^2)} + \frac{B}{x}$

$= \frac{Ax^2 + B - Bx^2}{x(1-x^2)}$

$\Rightarrow A - B = 2 \quad B = -1.$

$A = 1$

$= -\frac{1}{2} \log (x^2-1) - \log x$

$= -\log x\sqrt{x^2-1}$

$$z \cdot \frac{1}{x\sqrt{x^2-1}} = \int \frac{ax^2}{1-x^2} \cdot \frac{1}{x\sqrt{x^2-1}}\,dx + c$$

$$= -a\int x(x^2-1)^{-3/2}\,dx + c \qquad x^2-1 = t$$

$$2x\,dx = dt$$

$$= -\frac{a}{2}\int t^{-3/2}\,dt + c$$

$$z = +\frac{a}{2} \cdot 2\frac{1}{\sqrt{t}} + c.$$

$$z \Rightarrow y^3 = x\sqrt{x^2-1}\left[a\frac{1}{\sqrt{x^2-1}} + c\right]$$

$$\Rightarrow y^3 = ax + cx\sqrt{x^2-1}$$

Ex: $y(\log y)\,dx + (x - \log y)\,dy = 0$.

$y(\log y)\,dx = (\log y - x)\,dy$.

$\frac{dx}{dy} = \frac{\log y - x}{y\log y} = \frac{1}{y} - \frac{x}{y\log y}$.

$\frac{dx}{dy} + \frac{x}{y\log y} = \frac{1}{y}$

$I.F \Rightarrow e^{\int \frac{1}{y\log y} dy} \longrightarrow$ $\therefore \int \frac{1}{y} = \log y$ then $\log y = \frac{1}{y}$.

$= e^{\log(\log y)} = \log y$.

$G.S \Rightarrow x \cdot \log y = \int \frac{1}{y} \log y \, dy + C.$

$= \int z\,dz + c = \frac{z^2}{2} + c$ $\qquad \log y = z.$

$\frac{1}{y} dy = dz.$

$= \frac{(\log y)^2}{2} + c = -\log x \sqrt{x^2 - 1}$

$\Rightarrow 2x\log y = (\log y)^2 + c$

#Pragya

DATE

PAGE

2nd order Non-Homo.

Ex $\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2y = Q$

(i) $Q = e^{ax}$

(ii) $Q = \sin ax \,|\, \cos ax$

(iii) $Q = e^{ax}.V$ "V" is any function of x.

$V = f_1(x)$.

$P.I = \frac{1}{f(D)} = \frac{1}{F(D)} e^{ax}.V$

$= e^{ax} \frac{1}{F(D+a)}.V$

(iv) $\frac{1}{F(D)} P(x)$. only polynomials. (x^{n-2}...etc)

$= [F(D)]^{-1} P(x)$.

$= [1 \pm F_1(D)]^{-1} P(x)$.

ex: $\frac{1}{D^2+7}(x^2+x+1)$

$= \frac{1}{7}\left(1+\frac{D^2}{7}\right)^{-1}(x^2+x+1)$ then $(1+x)^{-1}$ Formula

$= \frac{1}{7}\left(1-\frac{D^2}{7}+\frac{D^4}{49}\ldots\right)(x^2+x+1)$

$= \left[\frac{1}{7}(x^2+x+1) - \frac{1}{49}(2)\right]$ from 3rd & 4th derivative are "0"

#PragyaNotebook

DATE

PAGE

Ex: $(D^2 - 3D + 2)y = e^{3x}$

$m^2 - 3m + 2$

$m = +1, +2.$

$AE = m^2 - 3m + 2 = 0.$

destined roots.

then

$C.F \Rightarrow C_1 e^{x} + C_2 e^{2x}.$ $\rightarrow$ GS $\Rightarrow$ $y = C_1 e^{x} + C_2 e^{2x} + \frac{e^{3x}}{2}$.

$PI = \frac{1}{D^2 - 3D + 2} e^{3x}$

$= \frac{1}{(D-1)(D-2)} e^{3x}$

Replace "D" by "3"

$= \frac{1}{(3-1)(3-2)} e^{3x}$

$= \frac{1}{2(1)} e^{3x}$

$= \frac{1}{2} e^{3x}$

DATE
PAGE

Ex: $(2D+1)^2 y = 4e^{-\frac{x}{2}}$

$(4D^2+1+4D)y = 4e^{-\frac{x}{2}}$.

~~$(4m^2+4m+1)y = 4e^{-x}=0$~~

$4m^2+4m+1=0$. $m=-\frac{1}{2},-\frac{1}{2}$. real & equal roots.

$CF \Rightarrow (C_1+C_2x)e^{-\frac{1}{2}x}$ $\because y = e^{mx}$.

$P.I \Rightarrow \frac{1}{(2D+1)^2} 4e^{-\frac{x}{2}}$

$\frac{1}{\left(2\left(-\frac{1}{2}\right)+1\right)^2} = \frac{1}{0}$ ✗

then

$\Rightarrow 4 \frac{1}{2(2D+1)2} e^{-\frac{x}{2}}$

$=0$ again take derivative

$\Rightarrow \frac{x^2}{2} e^{-\frac{x}{2}}$

$G.S \Rightarrow y = (C_1+C_2x)e^{-\frac{1}{2}x} + \frac{x^2}{2}e^{-x/2}$

$\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2 y = 0$. **Proof**

$y = e^{mx}$

$\frac{dy}{dx} = me^{mx}$

$\frac{d^2y}{dx^2} = m^2e^{mx}$

$m^2e^{mx} + P_1 me^{mx} + P_2 e^{mx} = 0$

$\Rightarrow e^{mx}(m^2+P_1m+P_2) = 0$

for non trivial solution $m^2+P_1m+P_2$ will be '0'

or

e^{mx} will be the trivial solution,

$C_1 \quad C_2 \quad (C_2 \cos\ldots + C_3 \sin\ldots)$

Ex: $(D^3-1)y = (e^x+1)^2$.

A.E $\Rightarrow m^3-1=0$.

$(m-1)(m^2+m+1)=0$.

$m = 1, \dfrac{-1 \pm \sqrt{3}i}{2}$. $\Longrightarrow$ real part in this is $-\frac{1}{2}$, imaginary is $\frac{\sqrt{3}}{2}$.

C.F $\Rightarrow c_1 e^{1 \cdot x} + e^{-\frac{1}{2}x}\left(c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right)$

for $(e^x+1)^2$ (Non-homogeneous).

P.I $\Rightarrow \dfrac{1}{D^3-1}(e^x+1)^2 = \dfrac{1}{D^3-1}(e^{2x}+2e^x+1)$

$$\cancel{= \frac{1}{(2)^3-1}(e^x+1)^2 = \frac{1}{8-1}(e^x+1)^2 = \frac{1}{7}(e^x+1)^2.}$$

$$= \frac{1}{D^3-1}e^{2x} + \frac{1}{D^3-1}2e^x + \frac{1}{D^3-1}1 \cdot e^{0x}$$

$$\cancel{= \frac{1}{8-1}e^{2x} + 2\frac{x}{3D^2} + \frac{x^2}{D}}$$

~~G.S $\Rightarrow y = (c_1 + c_2$~~

$$\cancel{= \frac{1}{7}e^{2x} + \frac{2x}{7}e^x}$$

$$= \frac{1}{7}e^{2x} + 2\frac{x}{3D^2}e^x + \frac{1}{D^3-1}e^{0x}.$$

$$= \left[\frac{1}{7}e^{2x} + \frac{2x}{3e}x - 1\right]$$

$\Rightarrow y = C.F + P.I.$