Relation

Let A and B be two sets, then the relation R is the subset of $A \times B$.

i.e., $\boxed{R \subseteq A \times B}$

eg: $A = \{1, 2\}$, $B = \{a, b\}$

$A \times B = \{1, 2\} \times \{a, b\}$

$A \times B = \{(1, a), (1, b), (2, a), (2, b)\}$

$R_1 = \{(1, a), (2, a)\}$

$R_2 = \{(2, a), (2, b)\}$

$R_3 = \{(2, a)\}$

Counting of relations

$n(A) = 2$, $n(B) = 2$

$n(A \times B) = 2 \times 2 = 4$

$A \times B \to$ 2 — — — —

$R = \{$

Total no of Relation $= 2 \times 2 \times 2 \times 2$

$= 2^4 = 2^{2 \times \boxed{2}} = 2^{n(A) \times n(B)}$

— X —

Q Let A and B be two sets having n and m elements then find the total no of Relation from the Set A to Set B.

Sol^n:

no. of elements in Set $A = n$

no. of elements in Set $B = m$

Total no. of Relation $= 2^{n(A) \times n(B)}$

$= \boxed{2^{n \cdot m}} = 2^{\boxed{mn}}$

Total no of Relation from Set A to Set $A = 2^{n(A) \times n(A)} = 2^{n \times n} = 2^{n^2}$

— X —

— × —

① **Reflexive relation**: Let A be a non-empty set, then the relation R defined on $A \times A$ is called reflexive.

if aRa $\forall$ $a \in A$

② **Symmetric relation**: Let A be a non-empty set, then the relation R defined on $A \times A$ is called symmetric

if $(a,b) \in R$ then $(b,a) \in R$

③ **Transitive Relation**:→ Let A be a non-empty set, then the relation R defined on $A \times A$ is called transitive

if $(a,b) \in R$ and $(b,c) \in R$ $\Rightarrow$ $(a,c) \in R$

— × —

Q1 Let $A = \{1,2,3\}$,

$R = \{(1,1), (2,2), (1,2), (2,3), (1,3)\}$

Check is this relation (i) Reflexive (ii) Symmetric (iii) Transitive.

Sol^n: $A = \{1,2,3\}$

$R = \{(1,1), (2,2), (1,2), (2,3), (1,3)\}$

① Reflexive

As $(3,3) \notin R$ $\Rightarrow$ Relation is not reflexive.

② Symmetric

$(1,2) \in R$ but $(2,1) \notin R$

$\Rightarrow$ Relation is not symmetric

③ Transitive

As $(1,2) \in R$, $(2,3) \in R$ $\Rightarrow$ $(1,3) \in R$

$\Rightarrow$ Relation is transitive.

Q2 Let $\mathbb{N}$ be the set of natural nos. define a relation R

Q2 Let $\mathbb{N}$ be the Set of natural nos. define a relation R on $\mathbb{N} \times \mathbb{N}$ by $R = \{(a,b) : a \leq b\}$. Prove that this relation is reflexive, transitive but not symmetric.

Sol^n: $R = \{(a,b) : a \leq b\}$

(i) Reflexive.

As every element is less than or equal to itself

ie, $a \leq a \quad \forall\ a \in \mathbb{N}$

$\Rightarrow (a,a) \in R \quad \forall\ a \in \mathbb{N}$

This relation is reflexive.

(ii) Transitive

Let $(a,b) \in R$ and $(b,c) \in R$

$a \leq b$ and $b \leq c$

$\Rightarrow a \leq b \leq c \Rightarrow a \leq c$

$\Rightarrow (a,c) \in R$

This relation is transitive.

(iii) Symmetric

Let $(a,b) \in R$

$a \leq b$

$\not\Rightarrow b \leq a$

$(b,a) \notin R$

$\left(\begin{array}{c} 4 \leq 8 \\ 8 \nleq 4 \end{array}\right)$

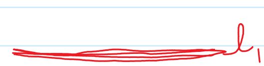
$\Rightarrow$ Relation is not symmetric

—x—

Q Let L be the Set of all lines in a plane define a relation R on $L \times L$ by $R = \{(l_1, l_2) : l_1 \perp l_2\}$. Prove that this relation is Symmetric but it is neither reflexive nor transitive.

Sol^n $R = \{(l_1, l_2) : l_1 \perp l_2\}$

(i) Reflexive



No line can be $\perp$ to itself

ie, $l_1 \not\perp l_1$

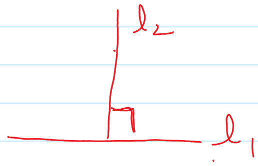
No line can be $\perp$ to itself

ie, $l_1 \not\perp l_1$

$\Rightarrow (l_1, l_1) \notin R \quad \forall\, l_1 \in L$

$\Rightarrow$ This relation is not reflexive.

② Symmetric



Let $(l_1, l_2) \in R$

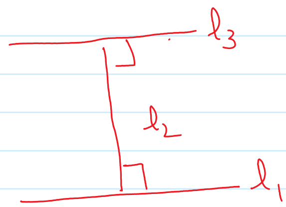
$\Rightarrow l_1 \perp l_2$

$\Rightarrow l_2 \perp l_1$

$\Rightarrow (l_2, l_1) \in R$

$\Rightarrow$ This relation is Symmetric.

③ Transitive



Let $(l_1, l_2) \in R$ and $(l_2, l_3) \in R$

$l_1 \perp l_2$ and $l_2 \perp l_3$

$\not\Rightarrow l_1 \perp l_3$

$\Rightarrow (l_1, l_3) \notin R$

$\Rightarrow$ This relation is not transitive.

—— X ——

• Equivalence relation : Let A be a non-empty set. Define a relation R on $A \times A$. Then this relation is called an equivalence relation if it is

① Reflexive
② Symmetric
③ Transitive

$a|b = \frac{b}{a}$ ✓

Q. Let $\mathbb{N}$ be the set of natural no.s. define a relation R on $N \times N$ as $R = \{(a,b) : a|b\}$. Check that relation is not an equivalence relation.

Sol^n $R = \{(a,b) : a|b\}$

$\not\Rightarrow b|a$

$\Rightarrow (b, a) \notin R$

$4|8$

$8 \nmid 4$

Sol^n $R = \{(a,b) : a | b\}$

① Reflexive

Every element divides itself

i.e., $a | a \quad \forall\ a \in \mathbb{N}$

$\Rightarrow (a,a) \in R$

$\Rightarrow$ Relation is reflexive.

② Symmetric

Let $(a,b) \in R$

$\Rightarrow a | b$

$\not\Rightarrow b | a$ $\qquad 4|8$ $\qquad 8 \nmid 4$

$\Rightarrow (b,a) \notin R$

$\Rightarrow$ Relation is not Symmetric.

$\therefore$ we can say that this relation is not an equivalence relation.

—×—

Q① Let $\mathbb{Z}$ be the Set of Integers define a Relation R on $\mathbb{Z}$ by $R = \{(a,b) : 2 \text{ divides } a-b\}$. Prove that this relation is an equivalence relation.

Sol^n $R = \{(a,b) : 2 \text{ divides } a-b\}$ $\qquad \frac{0}{2}$

① Reflexive.

We know that $2 | 0$ ✓

$2 | a-a \qquad \forall\ a \in \mathbb{Z}$

2 divides $a-a \qquad \forall\ a \in \mathbb{Z}$

$(a,a) \in R$

$\Rightarrow$ Relation is reflexive.

$\Rightarrow$ Relation is Sy

② Symmetric

Let $(a,b) \in R$

$\Rightarrow$ $2 \mid (a-b)$, Then $\exists$ integer k s.t

$\frac{a-b}{2} = k$ $\quad (k \in \mathbb{Z})$

$a-b = 2k$

Multiply both sides by -1

$-(a-b) = 2(-k)$

$(b-a) = 2(-k)$

$2 \mid (b-a)$

$(b,a) \in R$

$\Rightarrow$ Relation is Symmetric

$2|4,\ 2|6 \Rightarrow 2|(4+6)$

③ Transitive.

Let $(a,b) \in R$ and $(b,c) \in R$

$2 \mid (a-b)$ and $2 \mid (b-c)$

$2 \mid (a-\cancel{b})+(\cancel{b}-c)$

$2 \mid (a-c)$

$\Rightarrow$ $(a,c) \in R$

$\Rightarrow$ This relation is Transitive.

$\therefore$ This relation is an equivalence relation.

—×—

Q Let $\mathbb{R}$ be the set of real nos. Define a Relation $R = \{(a,b) : |a| = |b|\}$

Prove that this relation is an equivalence relation.

Solⁿ $R = \{(a,b) : |a| = |b|\}$

① Reflexive Relation.

As we know that

$|a| = |a| \quad \forall\, a \in \mathbb{R}$

$\Rightarrow$ $(a,a) \in R \quad \forall\, a \in \mathbb{R}$

This relation is reflexive

② Symmetric

Let $(a,b) \in R$

$|a| = |b|$

or $|b| = |a|$

$(b,a) \in R$

This relation is symmetric

③ Transitive.

Let $(a,b) \in R$ and $(b,c) \in R$

$|a| = |b|$ and $|b| = |c|$

$|a| = |b| = |c|$

$\Rightarrow$ $|a| = |c|$

$\Rightarrow$ $(a,c) \in R$

$\therefore$ This Relation is Transitive

Hence, we can say that this relation is an equivalence relation.

—×—

$R = \{(a,b) : a \leq b\}$

This relation is not Symmetric

Let $(a,b) \in R$

$\Rightarrow$ $a \leq b$

$\not\Rightarrow$ $b \leq a$

$(b,a) \notin R$

$4 \leq 8$

$8 \leq 4$

$\Rightarrow \quad b \leq a$

$(b, a) \notin R$

$\therefore$ This relation is not an equivalence relation.

① Let R be the relation over the set $N \times N$ and is defined by $(a,b) \, R \, (c,d) \Leftrightarrow ad = bc$ ~~$a+d = b+c$~~. Prove that this relation is an equivalence relation.

Sol^n ① Reflexive

As we know that

$a+b = b+a$

$(a,b) \, R \, (a,b) \quad \forall \, (a,b) \in N \times N$

$\Rightarrow$ Relation is reflexive

② Symmetric

Let $(a,b) \, R \, (c,d)$

$a+d = b+c$

$d+a = c+b$

$c+b = d+a$

$(c,d) \, R \, (a,b)$

$\Rightarrow$ This relation is symmetric

Reflexive:

As we know that

$a+b = b+a$

$(a,b) \, R \, (a,b) \quad \forall \, (a,b) \in N \times N$

Transitive

Let $(a,b) \, R \, (c,d)$ and $(c,d) \, R \, (e,f)$

$a+d = b+c \longrightarrow ①$ $\qquad c+f = d+e \longrightarrow ②$

Adding ① & ②, we get

$a+\cancel{d}+\cancel{c}+f = b+\cancel{c}+\cancel{d}+e$

$a+f = b+e$

$(a,b) \, R \, (e,f)$

$\Rightarrow$ Relation is transitive

$\therefore$ Relation is an equivalence relation

— x —

Anti-Symmetric relation: Let A be a non-empty set, then the relation R is called anti-symmetric if

$(a,b) \in R$ and $(b,a) \in R$ then $\boxed{a = b}$

eg: $A = \{1, 2, 3\}$

$R = \{ (1,1), (2,2),$

① Reflexive (This relation is not reflexive)

(1) Reflexive (This relation is not reflexive)
(2) Anti-symmetric (yes)

How to express a relation in matrix form.

$A = \{1, 2, 3\}$

$\rightarrow R = \{(1,1), (1,2), (2,3), (2,2)\}$

$(1,2) \in R$, $(2,3) \in R$

$(1,3) \notin R$

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}_{3\times3}$$

a_{12}, a_{13}, a_{21}, a_{23}, a_{31}, a_{32}

	1	2	3
1	(1,1)	(1,2)	(1,3)
2	(2,1)	(2,2)	(2,3)
3	(3,1)	(3,2)	(3,3)

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}_{4\times4}$$

find the Relation

$A = \{1, 2, 3, 4\}$

$R = \{(1,1), (1,4), (2,2), (2,4), (3,1), (3,2), (4,1), (4,4)\}$

Def^n of Partial order relation (P.O.R)

Let A be a non-empty set, then the Relation R on A is ~~////~~ called Partial order relation if it is

(1) Reflexive

~~(2) Symmetric~~

~~(2) Symmetric~~

(2) Anti-symmetric

(3) Transitive.

Q) Let N be the set of natural no.s Define a relation R on N by $R = \{(a,b) : a | b\}$. Show that this relation is a partial order relation.

Soln (1) Reflexive.

As every natural no. divides itself.

i.e. $a | a \quad \forall\, a \in \mathbb{N}$

$(a,a) \in R \quad \forall\, a \in \mathbb{N}$

$\Rightarrow$ This relation is reflexive relation.

Anti-symmetric

Let $(a,b) \in R$ and $(b,a) \in R$

$a | b$ and $b | a$

$\left(\begin{matrix} 4 | 8 \\ 8 \nmid 4 \end{matrix} \right)$

This is possible only when $\boxed{a = b}$

$\therefore$ Relation is anti-symmetric relation.

Transitive.

Let $(a,b) \in R$ and $(b,c) \in R$

$a | b$ and $b | c$

$2|4,\ 4|8$

$2|8$

$\Rightarrow a | c$

$\Rightarrow (a,c) \in R$

This relation is transitive.

$\therefore$ we can say that this relation is a partial order relation.

— x —

Q2 Let S be the non-empty set and P(S) be the power set of S.

Then define a relation R on P(S) by

$R = \{(A,B) : A \subseteq B\}$. Prove that this relation is a partial order relation.

$R = \{ (A, B) : A \subseteq B \}$. Prove that this relation is a partial order relation.

$S = \{1, 2\}$

$P(S) = \{ \phi, \{1\}, \{2\}, \{1,2\} \}$

Solⁿ: Reflexive:

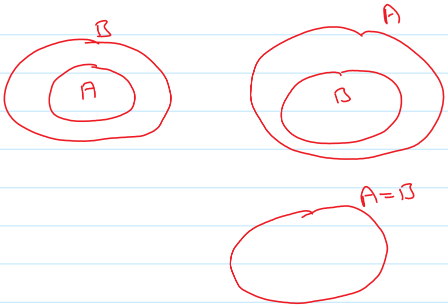
Every set is subset of itself.

i.e., $A \subseteq A \quad \forall A \in P(S)$

$(A, A) \in R \quad \forall A \in P(S)$

$\Rightarrow$ Relation is reflexive.

Anti-Symmetric:



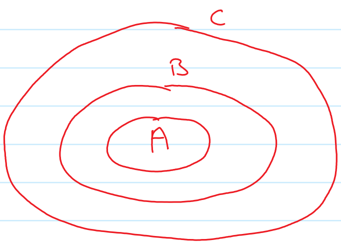
Let $(A, B) \in R$ and $(B, A) \in R$

$\Rightarrow A \subseteq B$ and $B \subseteq A$

This is possible when $A = B$.

$\Rightarrow$ This relation is Anti-Symmetric

Transitive



Let $(A, B) \in R$ and $(B, C) \in R$

$\Rightarrow A \subseteq B$ and $B \subseteq C$

$\Rightarrow A \subseteq C$

$\Rightarrow (A, C) \in R$

$\Rightarrow$ This relation is transitive

Hence, Relation is a Partial order relation.

— X —

V. Important topic

Counting of reflexive relations

Let A be a set having n elements.

Then the no. of elements in $A \times A = n \times n$

$= n^2$

$$\begin{array}{c|ccccc} & 1 & 2 & 3 & \cdots & n \\ \hline 1 & (1,1) & (1,2) & (1,3) & \cdots & (1,n) \\ 2 & (2,1) & (2,2) & (2,3) & \cdots & (2,n) \\ 3 & \vdots & & & & \\ \vdots & \vdots & & & & \\ n & (n,1) & (n,2) & (n,3) & \cdots & (n,n) \end{array} \quad n^2$$

$\boxed{n^2 - n}$ ✓

Total no of reflexive relations $= \underbrace{2 \times 2 \times 2 \times \cdots \times 2}_{n^2 - n}$

$$\boxed{= 2^{n^2 - n}}$$

$= 2^{n^2-n}$ $\boxed{n^2 - n}$

$R = \{ (1,1), (2,2), \_\_\_, (n,n),$

Q. If a Set A has 7 elements. Then how many reflexive relations are there?

(a) 2^{40} (b) 2^{42} ✓ (c) 2^{44} (d) 2^{46}

Solⁿ: Total no. of reflexive relations $= \boxed{2^{n^2-n}}$

$$= 2^{7^2-7} = 2^{49-7} = 2^{42}$$

— X —

Counting of Total no. of Symmetric relations

Let A be a non-empty set having n elements.

then total no. of elements in $A \times A = n \times n = n^2$

$$\begin{array}{c|cccccc} & 1 & 2 & 3 & \cdots & (n-1) & n \\ \hline 1 & (1,1) & (1,2) & (1,3) & & (1,n-1) & (1,n) \\ 2 & (2,1) & (2,2) & (2,3) & \cdots & (2,n-1) & (2,n) \\ 3 & & & & & & \\ \vdots & \vdots & & & & & \\ n-1 & & & & & & \\ n & (n,1) & (n,2) & \cdots & \cdots & & (n,n) \end{array}$$

Total no. of Symmetric relations $= \overset{1}{2} \times \overset{2}{2} \times \overset{3}{2} \times \cdots \overset{n}{2}$

$$= 2^{1+2+\cdots+n}$$

$R = \{ \_ (1,n), (n,1),$

$$= 2^{\Sigma n} = 2^{n(n+1)/2}$$ ✓

$\Sigma n = 1+2+\cdots+n = \frac{n(n+1)}{2}$

Q. If A is a Set having 5 elements, Then how many Symmetric relations are there.

Solⁿ: Total no of Symmetric relations $= 2^{\Sigma n}$

$$= 2^{n(n+1)/2}$$

$5 (\overset{3}{\cancel{6}})$

$= 2^{\ldots/2}$

$= 2^{5(\cancel{6}^{3})/\cancel{2}} = 2^{15}$

—X—

A → 8

Composition of Relations.

R, S

SoS SoR RoS RoR

① Let R be a relation from {1, 2, 3, 4} to {1, 2, 3}

with $R = \{(1,1), (1,4), (2,3), (3,1), (3,4)\}$

and S is the relation from {1, 2, 3, 4} to {0, 1, 2}

$S = \{(1,0), (2,0), (3,1), (3,2), (4,1)\}$

find the composition SoR.

Ans: $R = \{(1,1), (1,4), (2,3), (3,1), (3,4)\}$

$S = \{(1,0), (2,0), (3,1), (3,2), (4,1)\}$

$SoR = \{(1,0), (1,1), (2,1), (2,2), (3,0), (3,1)\}$

$(1,\cancel{1})\ (\cancel{1},0) \rightarrow (1,0)$

$(1,\cancel{4}) \rightarrow (\cancel{4},1) \rightarrow (1,1)$

$(2,\cancel{3}) \rightarrow (\cancel{3},1) \rightarrow (2,1)$

$(2,3) \rightarrow (3,2) \rightarrow (2,2)$

$(3,\cancel{1}) \rightarrow (\cancel{1},0) \rightarrow (3,0)$

$(3,\cancel{4}) \rightarrow (\cancel{4},1) \rightarrow (3,1)$

—X—

Partial order relation.

① Reflexive ② Anti-Symmetric ③ Transitive.

(Eg.1) $R = \{(a,b) : a \mid b\}$ → This is a partial order relation.

$R = \{(A,B) : A \subseteq B\}$ → This relation is also a partial order relation.

(Hasse diagram of the Partial order relation)

(Hasse diagram of the Partial order relation)

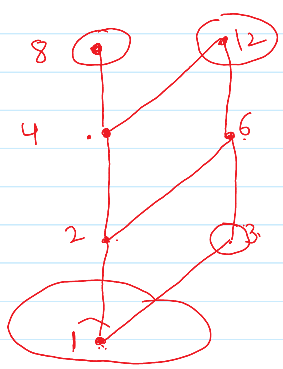
Hasse diagram is the pictorial / Graphical representation of Partial order relation.

—x—

Q1) Draw the Hasse diagram representing the partial ordering. $\{(a,b) \mid a \text{ divides } b\}$ on $A = \{1, 2, 3, 4, 6, 8, 12\}$

Sol^n $A = \{1, 2, 3, 4, 6, 8, 12\}$

$R = \{(1,1), (1,2), (1,3), (1,4), (1,6), (1,8), (1,12), (2,2), (2,4), (2,6), (2,8), (2,12), (3,3), (3,6), (3,12), (4,4), (4,8), (4,12), (6,6), (6,12), (8,8), (12,12)\}$



Hasse diagram

Maximal elements = 8, 12

Minimal element = 1

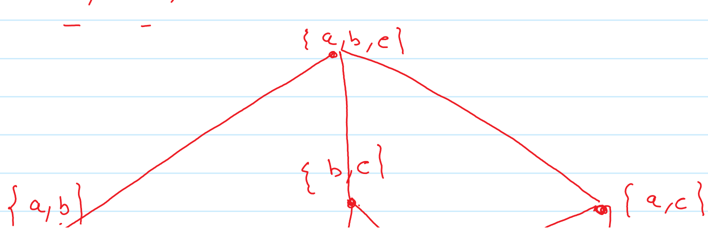
Q Draw the Hasse diagram of the Partial ordering $\{(A,B) : A \subseteq B\}$ on the $P(S)$ where $S = \{a, b, c\}$

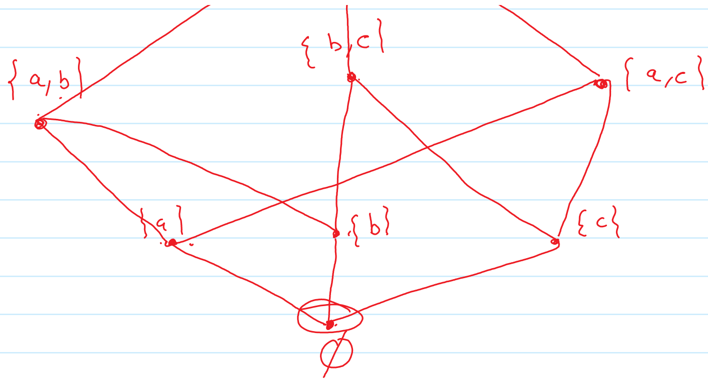
Sol^n $S = \{a, b, c\}$

$\phi \subseteq \{a\}$

$P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}$

$\{a\} \subseteq \{a,b\}$



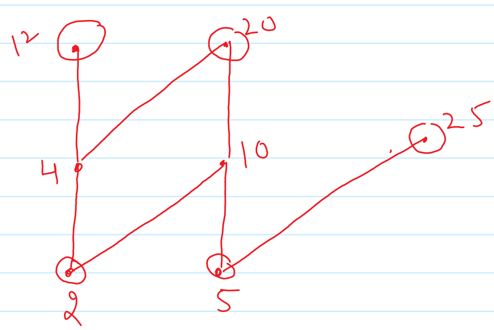


Maximal element = {a, b, c}, Minimal element = ϕ.

— x —

Q Which elements of the poset ({2, 4, 5, 10, 12, 20, 25}, |) are maximal and minimal elements.

Solⁿ:

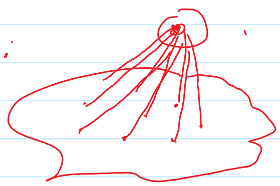


Maximal elements = 12, 20, 25

Minimal elements = 2, 5

— x —

Greatest element: If every element of the set is in relation with element a, then element is called greatest element.

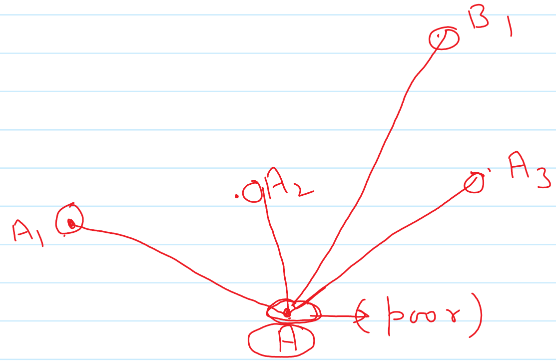


$A = \{1, 2, 3, 4, 5\}$ $\boxed{a}$

~~least elemen~~

least element: An element a which is in relation with all other elements of the set the the element is called least element.

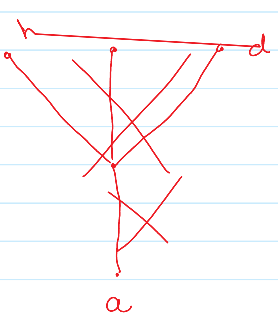
with all other elements of the set the the element is called least element.

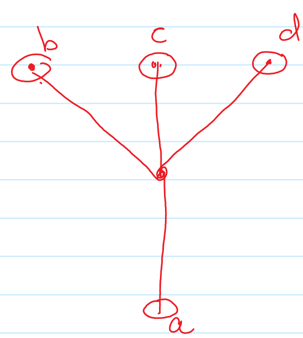


A is called the least element

✓ Greatest element is always found at the top of the Hasse diagram.

✓ least element is always found at the bottom of the Hasse diagram.





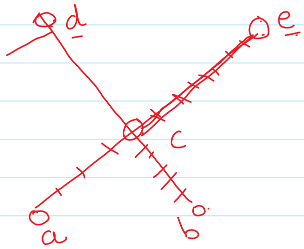
Maximal elements = b, c, d.

Minimal element = a

Greatest element = doesn't Exist

least element = a

— X —



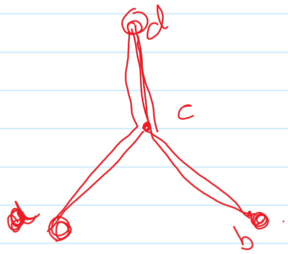
Maximal elements = d, e.

Minimal elements = a, b

Greatest element = doesn't Exist

least elements = doesn't Exist

— x —



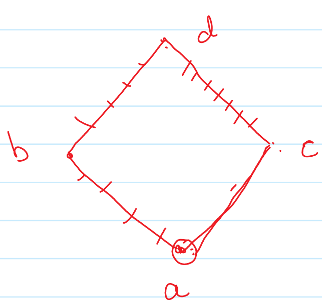
Maximal element = d

Minimal elements = a, b.

Greatest element = d

least element = doesn't exist

— x —



Maximal element = d ✓

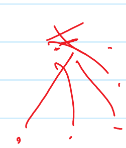
Minimal element = a ✓

Greatest element = d ✓

least element = a ✓

— x —

Q. Is there a greatest element and a least element in the poset $(Z^+, |)$.



Soln. For any integer m, 1 divides m, $m \in Z^+$

Sol^n: For any integer m, 1 divides m; $m \in Z^+$

$\Rightarrow$ 1 is the least element

There is no greatest element because there is no integer that is divisible by all the integer

— X —

$A = \{\ldots\}$
$S = \{\ldots\}$
$S \subseteq A$

Upper bound:

If u is an element of S s.t $a \leq u \;\; \forall a \in A$

Then u is called an upper bound of A.

There may be many upper bounds. but the least element of the upper bound is called the least upper bound.

$\boxed{2}, 4, 6, 8 \longrightarrow (u.b)$

$2 \rightarrow$ l.u.b (least upper bound)

$A = \{\ldots\}$
$S = \{\ \}$

lower bound:

If l is an element of S, s.t $l \leq a \;\; \forall a \in A$

then l is called the lower bound of the set A.

There may be many bounds.

$l_1, l_2, l_3, \ldots, \boxed{l_n}$

Then the ~~maximum~~ greatest lower bound among the lower bounds is called the greatest lower bound

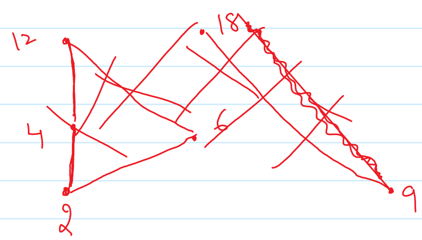
Q) For the poset

$(\{2, 4, 6, 9, 12, 18, 27, 36, 48, 60, 72\}, |)$

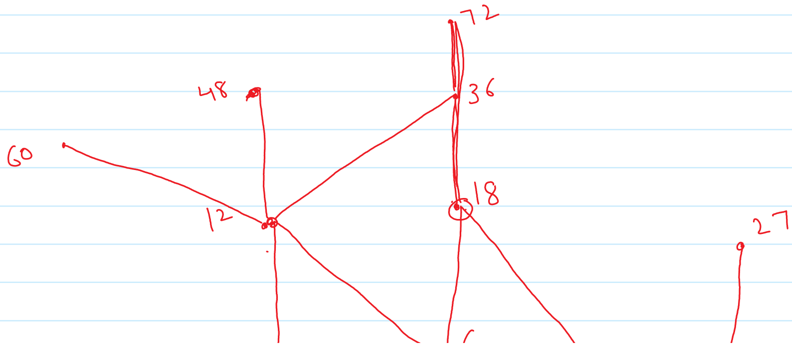
(i) upper bound of $\{2, 9\}$

(ii) The greatest lower bound of $\{60, 72\}$

Sol^n



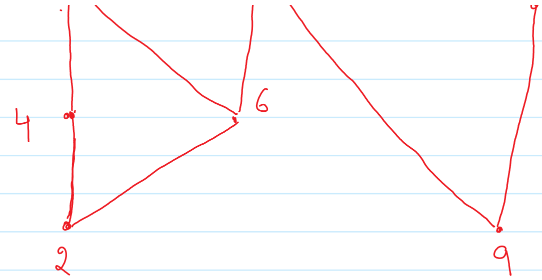
$(\{2, 4, 6, 9, 12, 18, 27, 36, 48, 60, 72\}, |)$



(i) upper bound of $\{2, 9\}$

18, 36, 72

(ii) Greatest lower bound $\{60, 72\}$



{60, 72}

2, 4, 12, 6

2, 4, 6, $\boxed{12}$

Greatest lower bound = 12

(a) Upper bound of {2, 9} → 18, 36, 72

(b) lower bound of {60, 72} → 2, 4, 6, 12,

Greatest lower bound = 12

—x—

Q Find the lower bound of {1, 4}

lower bound is 1

Find the upper bound of

{4, 12}

$\boxed{12}$, 36, 48, 60, 72

What is the least upper bound = 12

—x—