Lecture 7

Saturday, September 4, 2021 10:00 AM

Lecture 7

Topics of the lecture

- Proof of equivalence
- Counter examples
- Quiz

Pf by contradiction

① p is T

② $p \to q$

① $\sim p$ is T $\Rightarrow p$ is F

$\sim p$: $\sqrt{2}$ is rational contradiction

$\sqrt{2} = \frac{a}{b}$, $a, b \in \mathbb{Z}$, $b \neq 0$

gcd $(a,b) = 1$

$\begin{cases} a = 2p \\ b = 2m \end{cases}$

gcd$(a,b) = 2$

$\sim(p \to q)$ F

② $\sim(p \to q)$ is T

$\equiv p \wedge \sim q$ is T

$p \to q$

$\equiv p \,\&\, \sim q$

T T

$\equiv p$ q

T F

contradiction

Proof of equivalence

$p \leftrightarrow q$

To prove a theorem that is a biconditional statement, i.e., statement of the form $p \leftrightarrow q$, we show that both $p \to q$ and $q \to p$ are TRUE.

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

T T T

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q$	$\equiv (p \to q) \wedge (q \to p)$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

T T T T

Proof of equivalence

To prove a theorem that is a biconditional statement, i.e., statement of the form $p \leftrightarrow q$, we show that both $p \to q$ and $q \to p$ are TRUE.

The validity of this approach is based on the tautology $(p \leftrightarrow q) \leftrightarrow (p \to q) \land (q \to p)$.

T

Example of proof of equivalence

Theorem : "If n is an integer, then n is odd if and only if n^2 is odd."

Pf: $(p \leftrightarrow q)$ T $\quad (p \to q)$ T $\land$ $(q \to p)$ T

p: n is odd $\quad q$: n^2 is odd.

$p \to q$

$n = 2k+1 \Rightarrow n^2 = (2k+1)^2 = 4k^2 + 4k + 1$

$= 2(2k^2 + 2k) + 1$

$= 2p + 1, p \in \mathbb{Z}.$

$p \to q$ is T

If n^2 is odd, then n is odd.

T $q \to p \equiv \sim p \to \sim q$ T

$\sim p$: n is even $\quad \sim q$: n^2 is even

$n = 2k$

$\Rightarrow n^2 = (2k)^2 = 4k^2$

$= 2 \cdot 2k$

$= 2p, p \in \mathbb{Z}$

Example of proof of equivalence

Theorem : "If n is an integer, then n is odd if and only if n^2 is odd."

Proof : This theorem is of the form "p if and only if q", where p is "n is odd" and q is "n^2 is odd". To prove this theorem, we need to show that $p \to q$ and $q \to p$ both are true. As we are already familiar with the proofs of $p \to q$ and $q \to p$, so we conclude that $p \leftrightarrow q$ is also TRUE.

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To prove $p \to q$, direct method can be applied, but to prove $q \to p$ method of contraposition will work.

$q \to p \equiv \sim p \to \sim q$

Quiz 1

How many conditional statements we need to prove true in a proof of equivalence containing two propositions?

A. 1
(24) B. 2
C. 3
D. 4
(31)

$p \leftrightarrow q$ $\quad p \to q \wedge q \to p$

Quiz 1

How many conditional statements we need to prove true in a proof of equivalence containing two propositions?

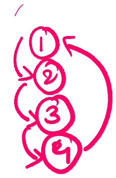
A. 1
B. 2
C. 3
D. 4

Answer : B

Proof of equivalence for more than two propositions

There may be some theorems that states that several propositions are equivalent. For example, $p_1, \; p_2, \; p_3, \; \ldots \; p_n$ are equivalent. This can be written as $p_1 \leftrightarrow p_2 \leftrightarrow \cdots \leftrightarrow p_n$, which states that

$p_1 \leftrightarrow p_2 \leftrightarrow p_3 \leftrightarrow \ldots \leftrightarrow p_n$



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all n propositions have the same truth values, and consequently, that for all i and j with $1 \leq i \leq n$ and $1 \leq j \leq n$, p_i and p_j are equivalent. One way to prove these are mutually equivalent is to use the tautology

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$$p_1 \leftrightarrow p_2 \leftrightarrow \cdots \leftrightarrow p_n \leftrightarrow \underbrace{(p_1 \to p_2)}_{T_1} \wedge \underbrace{(p_2 \to p_3)}_{T_2} \wedge \cdots \wedge \underbrace{(p_n \to p_1)}_{T_n}.$$

$p_1 \to p_2 \quad p_2 \to p_3 \quad p_3 \to p_1$

T

$p_1 \leftrightarrow p_2 \leftrightarrow p_3$

$p_1 \to p_2$

$p_2 \to p_1$

$p_2 \to p_3$

$p_3 \to p_2$

$p_1 \to p_3$

$p_3 \to p_1$

$p_1 \leftrightarrow p_2 \leftrightarrow \cdots \leftrightarrow p_n$

$\equiv (p_1 \to p_2) \wedge (p_2 \to p_3) \wedge \cdots \wedge (p_n \to p_1)$

1 p_1 　 $p_1 \to p_2$

2 p_2 　 $p_2 \to p_3$

3 p_3 　 $p_3 \to p_1$

$p_1 \to p_2 \wedge p_2 \to p_3 \to p_3 \to p_1$

$p_1 \to p_3 \quad p_2 \to p_1 \quad p_3 \to p_2$

Proof of equivalence cont . . .

Proof of equivalence cont . . .

Thus, it is shown that if the n conditional statements $p_1 \to p_2,\ p_2 \to p_3, \cdots, p_n \to p_1$ can be shown to be true, then the propositions $p_1,\ p_2,\ p_3, \cdots, p_n$ are all equivalent.

$p_1 \Leftrightarrow p_2 \Leftrightarrow \cdots \Leftrightarrow p_n$

T

Example

Show that the following statements about the integer n are equivalent:

- p_1 : n is even.
- p_2 : $n - 1$ is odd.
- p_3 : n^2 is even.

$p_3 \to p_1 \equiv \sim p_1 \to \sim p_3$ T

n^2 even $\Rightarrow n$ is even T

n is odd $\Rightarrow n^2$ is odd.

$p_1 \Leftrightarrow p_2 \Leftrightarrow p_3$ T

$\equiv (p_1 \to p_2) \wedge (p_2 \to p_3) \to (p_3 \to p_1) \equiv T \equiv p_1 \Leftrightarrow p_2 \Leftrightarrow p_3$ T Bored

T T T

p_1 is True $\Rightarrow n = 2k \Rightarrow n-1 = 2k-1,\ k \in \mathbb{Z} \Rightarrow n-1$ is odd $\Rightarrow p_2$ is T

$(p_1 \to p_2)$ is T

p_2 is True $\Rightarrow n-1$ is odd $\Rightarrow n-1 = 2k+1$

$\Rightarrow n = 2k+2 \Rightarrow n = 2p \Rightarrow n^2 = 4p^2 = 2\cdot 2p^2 = 2m$

p_3 T

Example

Show that the following statements about the integer n are equivalent:

- p_1 : n is even.
- p_2 : $n - 1$ is odd.
- p_3 : n^2 is even.

Proof : We will show that these three statements are equivalent by showing that the conditional statements $p_1 \to p_2,\ p_2 \to p_3$ and $p_3 \to p_1$ are true.

Proof continues . . .

To prove $p_1 \to p_2$, we use direct method. Let n be even. Then, $n = 2k$, for some integer k. Consequently, $n-1 = 2k-1 = 2(k-1)+1 = 2m+1$, where $m = k-1$, another integer. This implies that $n-1$ is odd.Thus, it proves $p_1 \to p_2$.

Proof continues . . .

To prove $p_1 \to p_2$, we use direct method. Let n be even. Then, $n = 2k$, for some integer k. Consequently, $n-1 = 2k-1 = 2(k-1)+1 = 2m+1$, where $m = k-1$, another integer. This implies that $n-1$ is odd.Thus, it proves $p_1 \to p_2$.

We again use direct method to prove $p_2 \to p_3$. Suppose that $n-1$ is odd. Then, $n-1 = 2k+1$ for some integer k. Hence, $n = 2k+2$, so that $n^2 = (2k+2)^2 = 4k^2 + 8k + 4 = 2(2k^2 + 4k + 2) = 2m$, where $m = 2k^2 + 4k + 2$, another integer. Thus, n^2 is even and this proves $p_2 \to p_3$.

Proof continues . . .

Finally, to prove $p_3 \to p_1$, we use a method of contraposition, since direct method will not work here. That is, we assume n is odd and show that n^2 is also odd, which is easy to prove.

Proof continues . . .

Finally, to prove $p_3 \to p_1$, we use a method of contraposition, since direct method will not work here. That is, we assume n is odd and show that n^2 is also odd, which is easy to prove.

With this final step, we complete the proof of the given theorem.

Proved!!

Quiz 2

How many conditional statements we need to prove true in a proof of equivalence containing n propositions?

A. 2
B. $n - 2$
C. $n - 1$
D. n

$p_1 \Leftrightarrow p_2 \Leftrightarrow p_3 \Leftrightarrow p_4$

$p_1 \to p_2$ T
$p_2 \to p_3$ T
$p_3 \to p_4$ T
$p_4 \to p_1$ T

Quiz 2

How many conditional statements we need to prove true in a proof of equivalence containing n propositions?

A. 2
B. $n - 2$
C. $n - 1$
D. n

Answer : D.

Quiz 3

The proof of a biconditional statement $p \leftrightarrow q$ is based on which of the following tautology?

A. $(p \leftrightarrow q) \overset{\equiv}{\leftrightarrow} (p \to q) \wedge (q \to p)$
B. $(p \leftrightarrow q) \leftrightarrow (p \to q) \vee (q \to p)$
C. $(p \leftrightarrow q) \leftrightarrow (p \to q) \to (q \to p)$
D. $(p \leftrightarrow q) \leftrightarrow \ (q \to p)$

Quiz 3

The proof of a biconditional statement $p \leftrightarrow q$ is based on which of the following tautology?

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B. $(p \leftrightarrow q) \leftrightarrow (p \to q) \vee (q \to p)$
C. $(p \leftrightarrow q) \leftrightarrow (p \to q) \to (q \to p)$
D. $(p \leftrightarrow q) \leftrightarrow \ (q \to p)$

Answer : A

Counterexamples

We know that, to show that a statement of the form $\forall x P(x)$ is FALSE, we need only to find a **counterexample**, that is, an example x for which $P(x)$ is false.

$x = k$
$p(k)$ is false//

Example

Show that the statement "Every positive integer is the sum of the squares of two integers" is FALSE.

$\rightarrow \forall x\, P(x)$

$x \neq a^2 + b^2$

$x \in \mathbb{Z}^+,\ a, b \in \mathbb{Z}.$

$1 = 1^2 + 0^2$

$2 = 1^2 + 1^2$

$3 \neq a^2 + b^2,\ a, b \in \mathbb{Z}$

$x = 3$

$\forall x\, P(x)$ False//

Example

Show that the statement "Every positive integer is the sum of the squares of two integers" is FALSE.

Solution : To show that this statement is false, we need to look for a counterexample, which is a particular positive integer that is not the sum of the squares of two integers.

Example

Show that the statement "Every positive integer is the sum of the squares of two integers" is FALSE.

Solution : To show that this statement is false, we need to look for a counterexample, which is a particular positive integer that is not the sum of the squares of two integers.

If we start with the first positive integer 1, then we see that $1 = 0^2 + 1^2 = 0^2 + (-1)^2$. So, 1 is not a counterexample. Now, for 2, we can write $2 = 1^2 + 1^2 = (-1)^2 + 1^2 = 1^2 + (-1)^2 = (-1)^2 + (-1)^2$, so 2 is also not a counterexample. Again, we notice that no such representation exists for 3.

$3 \neq a^2 + b^2$

Example cont . . .

Thus, 3 is a counterexample for the positive integer that is not written as the sum of the squares of two integers. That is, the given statement is FALSE.

Quiz 4

Which of the following is a counterexample to the statement $\forall x(x^3 \geq 0)$, where x is in the set of integers.?

False

$x^3 \geq 0$

$-1 \geq 0$

A. $x = 0$

B. $x = 1$

C. $x = -1$

D. $x = 2$

Quiz 4

Which of the following is a counterexample to the statement $\forall x(x^3 \geq 0)$, where x is in the set of integers.?

A. $x = 0$

B. $x = 1$

C. $x = -1$

D. $x = 2$

Answer : C

Quiz 5

Which of the following is equivalent to the statement that $3x + 2$ is even, where x is any integer?

A. $x + 6$ is odd

B. $x - 5$ is even

C. $x^2 + 5$ is even

D. $x^2 - 3$ is odd

3x+2 even

x even

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C. $x^2 + 5$ is even

D. $x^2 - 3$ is odd

Answer : D