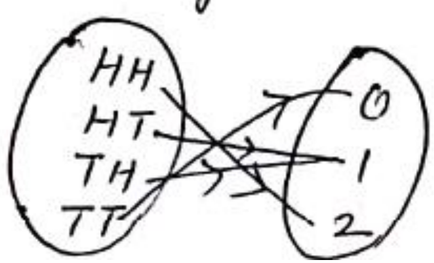
Random Variables

Let S be the sample space corresponding to a random experiment E. A random variable (RV) on a sample space defines a function that assigns a real number $X(s)$, $s \in S$. That is, a random variable defines a function $X : S \rightarrow R$ where S is the domain and range is $R = (-\infty, \infty)$

Ex: Tossing of two coins



Consider the number of heads as the random variable.

$$P(X=0) = \frac{1}{4}, \quad P(X=1) = \frac{2}{4}, \quad P(X=2) = \frac{1}{4}$$

Discrete random variable:

If a random variable X takes a finite number or countably infinite number of values, then X is called a discrete random variable. We denote the possible values taken by X as $x_1, x_2, \ldots x_n, \ldots$ which terminates in finite case. Therefore, a real valued function defined on a discrete sample space S is called discrete random variable.

Probability function of a discrete random variable

Let X be a discrete RV which takes values $x_1, x_2, \ldots$ and let $P(X=x_i) = p_i$. Then p_i is called probability function if it satisfies the following conditions

(i) $p_i \geq 0$ for all i, and

(ii) $\sum_i p_i = 1$

The collection of pairs (x_i, p_i), $i = 1, 2, 3, \ldots$

X	x_1	x_2	x_3
$P(X = x_i)$	p_1	p_2	p_3

is called the probability distribtition or the discrete probability distribution of the discrete random variable X.

The probability function is also called probability mass function or point probability function.

Distribution Function

Let X be a random variable. Then the function $F(X)$ defined by

$$F(X) = P(X \leq x) = \sum_{x_i \leq x} p(x_i)$$

is called the distribution function of X.

It has the following properties.

(i) $0 \leq F(X) \leq 1$

(ii) If $x_1 < x_2$, then $F(x_1) \leq F(x_2)$

(iii) $P(a \leq X \leq b) = F(b) - F(a)$

From the definition of distribution function

$$p(x_i) = P(X = x_i) = F(x_i) - F(x_{i-1})$$

$F(x)$ is also called the cumulative distribution function of X.

Mean and Variance of a discrete random variable.

If X is a discrete RV, then mean μ_x and variance σ_x^2 are defined as

$$\mu_x = E(X) = \sum_i x_i p_i$$

$$\sigma_x^2 = E(X^2) - (E(X))^2$$

Example 5·2. *A random variable X has the following probability distribution :*

x :	0	1	2	3	4	5	6	7
$p(x)$:	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

(i) Find k, (ii) Evaluate $P(X < 6)$, $P(X \geq 6)$, *and* $P(0 < X < 5)$, *(iii) If* $P(X \leq c) > \frac{1}{2}$, *find the minimum value of c, and (iv) Determine the distribution function of X.*

Solution. Since $\sum_{x=0}^{7} p(x) = 1$, we have

$\Rightarrow \quad k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$

$\Rightarrow \quad 10k^2 + 9k - 1 = 0$

$\Rightarrow \quad (10k - 1)(k + 1) = 0 \Rightarrow k = 1/10$

[$\because k = -1$, is rejected, since probability canot be negative.]

(*ii*) $P(X < 6) = P(X = 0) + P(X = 1) + \ldots + P(X = 5)$

$$= \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10} + \frac{1}{100} = \frac{81}{100}$$

$$P(X \geq 6) = 1 - P(X < 6) = \frac{19}{100}$$

$P(0 < X < 5) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) = 8k = 4/5$

(*iii*) $P(X \leq c) > \frac{1}{2}$. By trial, we get $c = 4$.

(*iv*)

X	$F_X(x) = P(X \leq x)$
0	0
1	$k = 1/10$
2	$3k = 3/10$
3	$5k = 5/10$
4	$8k = 4/5$
5	$8k + k^2 = 81/100$
6	$8k + 3k^2 = 83/100$
7	$9k + 10k^2 = 1$

Example 19.33 From a lot of 12 items containing 3 defective items, a sample of 4 items are drawn at random without replacement. Let a random variable X denote the number of defective items in the sample. Find the probability distribution of X.

Solution The lot contains 9 non-defective and 3 defective items. Since X denotes the number of defective items, x can take the values 0, 1, 2, 3. Four items are drawn without replacement.

For $\quad x = 0, \quad p(x) = \dfrac{9C_4}{12C_4} = \dfrac{14}{55}.$

For $\quad x = 1, \quad p(x) = \dfrac{(9C_3)(3C_1)}{12C_4} = \dfrac{28}{55}.$

For $\quad x = 2, \quad p(x) = \dfrac{(9C_2)(3C_2)}{12C_4} = \dfrac{12}{55}.$

For $\quad x = 3, \quad p(x) = \dfrac{(9C_1)(3C_3)}{12C_4} = \dfrac{1}{55}.$

We have the following probability distribution.

x	0	1	2	3
$p(x)$	$\frac{14}{55}$	$\frac{28}{55}$	$\frac{12}{55}$	$\frac{1}{55}$

Example 19.34 A random variable X has the following probability distribution

x	0	1	2	3	4
$p(x)$	c	$2c$	$2c$	c^2	$5c^2$

Find the value of c. Evaluate $P(X < 3)$, $P(0 < X < 4)$. Determine the distribution function of X. Find the mean and variance of X.

Solution Since $\sum_{x=0}^{4} p(x) = 1$, we get

$$c + 2c + 2c + c^2 + 5c^2 = 1, \quad \text{or} \quad 6c^2 + 5c - 1 = 0,$$

or $(6c - 1)(c + 1) = 0$, or $c = 1/6, -1$.
Since $p(x) \geq 0$, the possible value is $c = 1/6$. Now,

$$P(X < 3) = P(X = 0) + P(x = 1) + P(x = 2)$$

$$= c + 2c + 2c = 5c = 5/6.$$

$$P(0 < X < 4) = P(X = 1) + P(X = 2) + P(X = 3)$$

$$= 2c + 2c + c^2 = \frac{25}{36}.$$

We have the following results for the probability distribution and distribution function.

x	0	1	2	3	4
$p(x)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{2}{6}$	$\frac{1}{36}$	$\frac{5}{36}$
$F(x)$	$\frac{1}{6}$	$\frac{3}{6}$	$\frac{5}{6}$	$\frac{31}{36}$	1

$$\text{mean} = \mu_X = \Sigma x_i p_i = 0 + \frac{2}{6} + \frac{4}{6} + \frac{3}{36} + \frac{20}{36} = \frac{59}{36}.$$

Variance can be obtained by either of the formulas in (19.25ii). We have

$$\text{Variance} = \sigma_X^2 = E(x^2) - [E(x)]^2$$

$$= \left[0\left(\frac{1}{6}\right) + 1\left(\frac{2}{6}\right) + 4\left(\frac{2}{6}\right) + 9\left(\frac{1}{36}\right) + 16\left(\frac{5}{36}\right)\right] - \left(\frac{59}{36}\right)^2 = 1.4529.$$

1. (*a*) A student is to match three historical events (Mahatma Gandhi's Birthday, India's freedom, and First World War) with three years (1947, 1914, 1896). If he guesses with no knowledge of the correct answers, what is the probability distribution of the number of answers he gets correctly ?

(*b*) From a lot of 10 items containing 3 defectives, a sample of 4 items is drawn at random. Let the random variable X denote the number of defective items in the sample. Answer the following when the sample is drawn without replacement.

(*i*) Find the probability distribution of X,

(*ii*) Find $P(X \leq 1)$, $P(X < 1)$ and $P(0 < X < 2)$

(a)

x	0	1	2	3
$p(x)$	$\frac{1}{3}$	$\frac{1}{2}$	0	$\frac{1}{6}$

(b) (i)

x	0	1	2	3
$p(x)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{30}$

(ii) 2/3, 5/6, 1/2

4. *(a)* A random variable X has the following probability function :

Values of X, x :	-2	-1	0	1	2	3
$p(x)$:	$0{\cdot}1$	k	$0{\cdot}2$	$2k$	$0{\cdot}3$	k

(i) Find the value of k, and calculate mean and variance.

(ii) Construct the c.d.f. $F(X)$

Ans. *(i)* 0·1, 0·8 and 2·16, *(ii)* $F(X)$ = 0·1, 0·2, 0·4, 0·6, 0·9, 1·0

7. If

$$p(x) = \frac{x}{15};\ x = 1, 2, 3, 4, 5$$

$$= 0, \text{ elsewhere}$$

Find *(i)* $P\{X = 1 \text{ or } 2\}$, and *(ii)* $P\left\{\frac{1}{2} < X < \frac{5}{2} \mid X > 1\right\}$

$$(i)\ P\{X = 1 \text{ or } 2\} = P(X = 1) + P(X = 2) = \frac{1}{15} + \frac{2}{15} = \frac{1}{5}$$

$$(ii)\ P\left\{\frac{1}{2} < X < \frac{5}{2} \,\middle|\, X > 1\right\} = \frac{P\left\{\left(\frac{1}{2} < X < \frac{5}{2}\right) \cap X > 1\right\}}{P(X > 1)}$$

$$= \frac{P\{(X = 1 \text{ or } 2) \cap X > 1\}}{P(X > 1)} = \frac{P(X = 2)}{1 - P(X = 1)} = \frac{2/15}{1 - (1/15)} = \frac{1}{7}$$