Continuous Distribution Function.

If X is a continuous random

variable with the p.d.f. $f(x)$, then the function

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt\,, \quad -\infty < x < \infty\,.$$

is called the *distribution function* (d.f.) or sometimes the *cumulative distribution function* (c.d.f.) of the random variable X.

Remarks 1. $0 \le F(x) \le 1, \quad -\infty < x < \infty.$

2. From analysis (Riemann integral), we know that

$$F'(x) = \frac{d}{dx} F(x) = f(x) \ge 0 \qquad [\because f(x) \text{ is p.d.f.}]$$

$\Rightarrow$ $F(x)$ is non-decreasing function of x.

3. $$F(-\infty) = \lim_{x \to -\infty} F(x) = \lim_{x \to -\infty} \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{-\infty} f(x)\,dx = 0$$

and $$F(+\infty) = \lim_{x \to \infty} F(x) = \lim_{x \to \infty} \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx = 1$$

4. $F(x)$ is a continuous function of x on the right.
5. The discontinuities of $F(x)$ are at the most countable.
6. It may be noted that

$$P(a \le X \le b) = \int_a^b f(x)\,dx = \int_{-\infty}^{b} f(x)\,dx - \int_{-\infty}^{a} f(x)\,dx$$

$$= P(X \le b) - P(X \le a) = F(b) - F(a)$$

Similarly

$$P(a < X < b) = P(a < X \le b) = P(a \le X < b) = \int_a^b f(t)\,dt$$

7. Since $F'(x) = f(x)$, we have

$$\frac{d}{dx} F(x) = f(x) \quad \Rightarrow \quad dF(x) = f(x)\,dx$$

This is known as probability differential of X.

Example 5·15. *Verify that the following is a distribution function:*

$$F(x) = \begin{cases} 0, & x < -a \\ \dfrac{1}{2}\left(\dfrac{x}{a} + 1\right), & -a \le x \le a \\ 1, & x > a \end{cases}$$

Solution. Obviously the properties *(i)*, *(ii)*, *(iii)* and *(iv)* are satisfied. Also we observe that $F(x)$ is continuous at $x = a$ and $x = -a$, as well.

Now

$$\frac{d}{dx} F(x) = \begin{cases} \frac{1}{2a}, & -a \leq x \leq a \\ 0, & otherwise \end{cases}$$

$$= f(x), \text{ say}$$

In order that $F(x)$ is a distribution function, $f(x)$ must be a p.d.f. Thus we have to show that

$$\int_{-\infty}^{\infty} f(x)\ dx = 1$$

Now $\displaystyle\int_{-\infty}^{\infty} f(x)\ dx = \int_{-a}^{a} f(x)\ dx = \frac{1}{2a}\int_{-a}^{a} 1 \,.\, dx = 1$

Hence $F(x)$ is a d.f.

Example 5·16. *Suppose the life in hours of a certain kind of radio tube has the probability density function :*

$$f(x) = \frac{100}{x^2}, \quad \text{when } x \geq 100$$
$$= \quad 0, \quad \text{when } x < 100$$

Find the distribution function of the distribution. What is the probabilty that none of three such tubes in a given radio set will have to be replaced during the first 150 hours of operation ? What is the probability that all three of the original tubes will have been replaced during the first 150 hours ?

Solution. Probability that a tube will last for first 150 hours is given by

$$P(X \le 150) = P(0 < X < 100) + P(100 \le X \le 150)$$

$$= \int_{100}^{150} f(x)\, dx = \int_{100}^{150} \frac{100}{x^2}\, dx = \frac{1}{3}$$

Hence the probability that none of the three tubes will have to be replaced during the first 150 hours is $(1/3)^3 = 1/27$.

The probability that a tube will not last for the first 150 hours is $1 - \frac{1}{3} = \frac{2}{3}$.

Hence the probability that all three of the original tubes will have to be replaced during the first 150 hours is $(2/3)^3 = 8/27$.

Example 5·17. *Suppose that the time in minutes that a person has to wait at a certain station for a train is found to be a random phenomenon, a probability function specified by the distribution function,*

$$F(x) = \begin{cases} 0, & \text{for } x \le 0 \\ \frac{x}{2}, & \text{for } 0 \le x < 1 \\ \frac{1}{2}, & \text{for } 1 \le x < 2 \\ \frac{x}{4}, & \text{for } 2 \le x < 4 \\ 1, & \text{for } x \ge 4 \end{cases}$$

(a) Is the Distribution Function continuous ? If so, give the formula for its probability density function ?

(b) What is the probability that a person will have to wait (i) more than 3 minutes, (ii) less than 3 minutes, and (iii) between 1 and 3 minutes ?

(c) What is the conditional probability that the person will have to wait for a train for (i) more than 3 minutes, given that it is more than 1 minute, (ii) less than 3 minutes given that it is more than 1 minute ?

Solution. *(a)* Since the value of the distribution function is the same at the points $x = 0, x = 1, x = 2,$ and $x = 4$ given by the two forms of $F(x)$ for $x < 0$ and $0 \le x < 1$, $0 \le x < 1$ and $1 \le x < 2$, $1 \le x < 2$ and $2 \le x < 4$, $2 \le x < 4$ and $x \ge 4$, the distribution function is continuous.

Probability density function $= f(x) = \dfrac{d}{dx} F(x)$

$$
\begin{aligned}
\therefore \qquad f(x) &= 0, \quad \text{for } x < 0 \\
&= \frac{1}{2}, \quad \text{for } 0 \le x < 1. \\
&= 0, \quad \text{for } 1 \le x < 2 \\
&= \frac{1}{4}, \quad \text{for } 2 \le x < 4 \\
&= 0, \quad \text{for } x \ge 4
\end{aligned}
$$

(b) Let the random variable X represent the waiting time in minutes.

Then

(i) Required probability $= P(X > 3) = 1 - P(X \le 3) = 1 - F(3)$

$$= 1 - \frac{1}{4}.3 = \frac{1}{4}$$

(ii) Required probability $= P(X < 3) = P(X \le 3) - P(X = 3)$

$$= F(3) = \frac{3}{4}$$

(Since, the probability that a continuous variable takes a fixed value is zero) *(iii)* Required Probability $= P(1 < X < 3) = P(1 < X \leq 3)$

$$= F(3) - F(1) = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$$

(c) Let A denote the event that a person has to wait for more than 3 minutes and B the event that he has to wait for more than 1 minute. Then

$$P(A) = P(X > 3) = \frac{1}{4} \qquad [\text{c.f. } (b), (i)]$$

$$P(B) = P(X > 1) = 1 - P(X \leq 1) = 1 - F(1) = 1 - \frac{1}{2} = \frac{1}{2}$$

$$P(A \cap B) = P(X > 3 \cap X > 1) = P(X > 3) = \frac{1}{4}$$

(i) Required probability is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{1/2} = \frac{1}{2}$$

(ii) Required probability $= P(\bar{A} \mid B) = \dfrac{P(\bar{A} \cap B)}{P(B)}$

Now $P(\bar{A} \cap B) = P(X \leq 3 \cap X > 1) = P(1 < X \leq 3) = F(3) - F(1) = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$

$$\therefore \quad P(\bar{A} \mid B) = \frac{1/4}{1/2} = \frac{1}{2}$$

Example 5·18. *A petrol pump is supplied with petrol once a day. If its daily volume X of sales in thousands of litres is distributed by*

$$f(x) = 5(1-x)^4, \quad 0 \leq x \leq 1,$$

what must be the capacity of its tank in order that the probability that its supply will be exhausted in a given day shall be 0·01 ?

Solution. Let the capacity of the tank (in '000 of litres) be 'a' such that

$$P(X \geq a) = 0{\cdot}01 \quad \Rightarrow \quad \int_a^1 f(x)\,dx = 0{\cdot}01$$

$$\Rightarrow \quad \int_a^1 5(1-x)^4\,dx = 0{\cdot}01 \quad \text{or} \quad \left[5 \cdot \frac{(1-x)^5}{(-5)}\right]_a^1 = 0{\cdot}01$$

$$\Rightarrow \quad (1-a)^5 = 1/100 \quad \text{or} \quad 1-a = (1/100)^{1/5}$$

$$\therefore \quad a = 1-(1/100)^{1/5} = 1-0{\cdot}3981 = 0{\cdot}6019$$

Hence the capacity of the tank $= 0{\cdot}6019 \times 1000$ *litres* $= 601{\cdot}9$ *litres*.

9. A bombing plane carrying three bombs flies directly above a railroad track. If a bomb falls within 40 feet of track, the track will be sufficiently damaged to disrupt the traffic. Within a certain bomb site the points of impact of a bomb have the probability density function :

$$
\begin{aligned}
f(x) &= (100 + x)/10{,}000, \quad \textit{when } -100 \le x \le 0 \\
&= (100 - x)/10{,}000, \quad \textit{when } 0 \le x \le 100 \\
&= 0, \quad \textit{elsewhere}
\end{aligned}
$$

where x represents the vertical deviation (in feet) from the aiming point, which is the track in this case. Find the distribution function. If all the bombs are used, what is the probability that track will be damaged ?

Hint. Probability that track will be damaged by the bomb is given by

$$P(|X| < 40) = P(-40 < X < 40)$$

$$= \int_{-40}^{0} f(x)\,dx + \int_{0}^{40} f(x)\,dx$$

$$= \int_{-40}^{0} \frac{100+x}{10,000}\,dx + \int_{0}^{40} \frac{100-x}{10,000}\,dx = \frac{16}{25}$$

$\therefore$ Probability that a bomb will not damage the track $= 1 - \frac{16}{25} = \frac{9}{25}$

Probability that none of the three bombs damages the track

$= \left(\frac{9}{25}\right)^3 = 0{\cdot}046656$

Required probability that the track will be damaged $= 1 - 0{\cdot}046656 = 0{\cdot}953344$.

10. The length of time (in minutes) that a certain lady speaks on the telephone is found to be random phenomenon, with a probability function specified by the probability density function $f(x)$ as

$$f(x) = A e^{-x/5}, \quad \text{for } x \geq 0$$
$$= 0, \quad \textit{otherwise}$$

(a) Find the value of A that makes $f(x)$ a p.d.f.

Ans. $A = 1/5$

(b) What is the probability that the number of minutes that she will talk over the phone is

(i) More than 10 minutes, *(ii)* less than 5 minutes, and *(iii)* between 5 and 10 minutes?

Ans. *(i)* $\frac{1}{e^2}$, *(ii)* $\frac{e-1}{e}$, *(iii)* $\frac{e-1}{e^2}$.