Example. 4·31. *The contents of urns I, II and III are as follows:*

1 white, 2 black and 3 red balls,

2 white, 1 black and 1 red balls, and

4 white, 5 black and 3 red balls.

One urn is chosen at random and two balls drawn. They happen to be white and red. What is the probability that they come from urns I, II or III ?

Solution. Let E_1, E_2, and E_3 denote the events that the urn I, II and III is chosen, respectively, and let A be the event that the two balls taken from the selected urn are white and red. Then

$$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$$

$$P(A \mid E_1) = \frac{1 \times 3}{{}^6C_2} = \frac{1}{5}, \quad P(A \mid E_2) = \frac{2 \times 1}{{}^4C_2} = \frac{1}{3},$$

and $$P(A \mid E_3) = \frac{4 \times 3}{{}^{12}C_2} = \frac{2}{11}$$

Hence

$$P(E_2 \mid A) = \frac{P(E_2)\, P(A \mid E_2)}{\sum_{i=1}^{3} P(E_i)\, P(A \mid E_i)}$$

$$= \frac{\frac{1}{3} \times \frac{1}{3}}{\frac{1}{3} \times \frac{1}{5} + \frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{2}{11}} = \frac{55}{118}$$

Similarly

$$P(E_3 \mid A) = \frac{\frac{1}{3} \times \frac{2}{11}}{\frac{1}{3} \times \frac{1}{5} + \frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{11}} = \frac{30}{118}$$

$$\therefore \quad P(E_1 \mid A) = 1 - \frac{55}{118} - \frac{30}{118} = \frac{33}{118}$$

Sixty percent of new drivers have had driver education.

During their first year, new drivers without driver education have probability 0.08 of having an accident, but new drivers with driver education have only a 0.05 probability of an accident. What is the probability a new driver has had driver education, given that the driver has had no accident the first year?

Answer: Let A represent the new driver who has had driver education and B represent the new driver who has had an accident in his first year. Let A^c and B^c be the complement of A and B, respectively. We want to find the probability that a new driver has had driver education, given that the driver has had no accidents in the first year, that is $P(A/B^c)$.

$$\begin{aligned}
P(A/B^c) &= \frac{P(A \cap B^c)}{P(B^c)} \\
&= \frac{P(B^c/A)\,P(A)}{P(B^c/A)\,P(A) + P(B^c/A^c)\,P(A^c)} \\
&= \frac{[1 - P(B/A)]\,P(A)}{[1 - P(B/A)]\,P(A) + [1 - P(B/A^c)]\,[1 - P(A)]} \\
&= \frac{\left(\frac{60}{100}\right)\,\left(\frac{95}{100}\right)}{\left(\frac{40}{100}\right)\,\left(\frac{92}{100}\right) + \left(\frac{60}{100}\right)\,\left(\frac{95}{100}\right)} \\
&= 0.6077.
\end{aligned}$$

(1) For a certain binary communication channel the probability that a transmitted '0' is received as '0' is 0.99 and the probability that a transmitted '1' is recieved as '1' is 0.95. If the probability that a '0' is transmitted is 0.6, find the probability that

(i) a '0' is recived

(ii) a '0' was transmitted given that a '0' was recieved.

Sol: A : Event of transmitting '0'

$\overline{A}$: Event of transmitting '1'

B : Event of receiving '0'

$\overline{B}$: Event of receiving '1'.

$P(A) = 0.6$, $P(\overline{A}) = 0.4$, $P(B/A) = 0.99$

$P(\overline{B}/\overline{A}) = 0.95$, $P(B/\overline{A}) = 0.05$

(i) $P(B) = P(A)P(B/A) + P(\overline{A})P(B/\overline{A})$

$= 0.6 \times 0.99 + 0.4 \times 0.05 = 0.614$

(ii) $P(A/B) = \dfrac{P(A)P(B/A)}{P(B)} = \dfrac{0.6 \times 0.99}{0.614} = \dfrac{297}{307}$

(2) In a test, an examinee either guesses or copies or knows the answer to a multiple choice question with four choices. The probability that he makes a guess is $(1/3)$. The probability that he copies the answer is $(\frac{1}{6})$. The probability that the answer is correct, given that he copies is $(1/8)$. Find the probability that he knows the answer to the question, given that he correctly answered it.

Sol: Let A be the event that examinee gives the correct answer.

A_1: The event that examinee knows the answer

A_2: The event that examinee guesses the answer

A_3: The event that the examinee copies the answer

$$P(A_2) = \frac{1}{3}, \quad P(A_3) = \frac{1}{6}, \quad P(A_1) = 1 - \frac{1}{3} - \frac{1}{6} = \frac{1}{2}$$

$$P(A|A_1) = 1, \quad P(A|A_2) = \frac{1}{4}, \quad P(A|A_3) = \frac{1}{8}$$

$$P(A_1|A) = \frac{P(A_1)\,P(A|A_1)}{P(A_1)\,P(A|A_1) + P(A_2)\,P(A|A_2) + P(A_3)\,P(A|A_3)}$$

$$= \frac{\frac{1}{2} \times 1}{\frac{1}{2} \times 1 + \frac{1}{3} \times \frac{1}{4} + \frac{1}{6} \times \frac{1}{8}} = \frac{\frac{1}{2}}{\frac{24+4+1}{48}} = \frac{24}{29}$$

(3) A person is known to speak truth 4 times out of 5 times. He throws a die and reports that it is a six. What is the probability that it is actually a six?

Sol: A : Person speaks the truth $P(A) = 4/5$

$\bar{A}$: Person lies $P(\bar{A}) = 1/5$

B : He reports a six

$P(B/A) = \frac{1}{6}$

$P(B/\bar{A}) = 5/6$

$$P(A/B) = \frac{P(A)\,P(B/A)}{P(A)\,P(B/A) + P(\bar{A})\,P(B/\bar{A})}$$

$$= \frac{4/5 \times 1/6}{\frac{4}{5} \times \frac{1}{6} + \frac{1}{5} \times \frac{5}{6}} = \frac{4}{9}$$

(4) Box 1 contains 5000 bulbs of which 5% are defective. Box 2 contains 4000 bulbs of which 3% are defective. Two bulbs are drawn without replacement from a randomly selected box.

(i) Find the probability that both bulbs are defective

(ii) Assuming that both the bulbs are defective, find the probability that they came from box 2.

Sol: B_1: Box 1 is selected, B_2: Box 2 is selected

A: 2 defective bulbs are drawn

$$P(B_1) = P(B_2) = \frac{1}{2}, \quad P(A|B_1) = \frac{{}^{250}C_2}{{}^{5000}C_2} = \frac{249}{99980}$$

$$P(A|B_2) = \frac{{}^{120}C_2}{{}^{4000}C_2} = \frac{357}{399900}$$

(i) $$P(A) = P(B_1)P(A|B_1) + P(B_2)P(A|B_2)$$

$$= \frac{1}{2}\left(\frac{249}{99980} + \frac{357}{399900}\right) = 0.00169$$

(ii) $$P(B_2|A) = \frac{P(B_2)P(A|B_2)}{P(A)} = \frac{\frac{1}{2} \times \frac{357}{399900}}{0.00169}$$

$$= 0.264$$