SPECIAL DISCRETE DISTRIBUTIONS

- The Bernoulli process
- Binomial distribution and its mgf [Moment Generating Func^n]
- Negative Binomial dist. & its mgf
- Geometric dist. & its mgf
- Poisson dist. & its mgf.

* Bernoulli Process :-

- An experiment often consists of repeated trials, each with two possible outcomes that may be labeled <u>success</u> or <u>failure</u>.

eg:- Tossing a coin ten times & finding the prob. of no. of heads.

Head - Success

Tail - Failure

- The process is referred to as Bernoulli process. Each trial is called <u>Bernoulli trial</u>.

- <u>The Bernoulli process must process the following properties</u>:

(i) The experiment consists of repeated trials.

(ii) Each trial results in an outcome that may be classified as a <u>success</u> or a <u>failure</u>.

(iii) The probability of success, denoted by '<u>p</u>', remains $\boxed{\text{Constant}}$ from trial to trial

(iv) The repeated trials are independent of each other.

(v) The number of trials 'n' is finite.

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* Binomial Distribution :-

- Let a random experiment be performed repeatedly, each repetition being called a trial and let the occurrence of an event in a trial be called a success and its non-occurrence a failure.

- The number X of successes in 'n' bernoulli trials is called a binomial random variable. The prob. dist. this discrete rv is called the binomial dist. and its value is denoted by $(x; n, p)$.

~~eg:- no of defective items~~

~~$x = 0, 1, 2$~~

~~$P(x=2) = f(2) = b(x; n, p)$~~

~~$= b(2; 3, \frac{1}{4})$~~

Defn:- A bernoulli trial can result in a success with prob. p & a failure with prob. $q = 1-p$.

Then, the prob. dist. of the binomial random variable x, the no. of successes in 'n' independent trials, is

$$\boxed{b(x; n, p) = {}^{n}C_{x}\, p^{x}\, q^{n-x}} \quad , \quad x = 0, 1, 2, \ldots, n$$

or

$${}^{n}C_{r}\, p^{r}\, q^{n-r} \quad ; \; r = \text{No. of success cases}$$

Q:- Three items are selected at random from a manufacturing process, inspected and classified as defective or non-defective. Find the prob. dist. for no. of defectives assuming that 25% items are defective.

$\frac{25}{100} = \frac{1}{4}$

Solⁿ: Let X be no. of defectives [i.e. it is designated a success]

$S = \{NNN, NND, NDN, NDD, DNN, DND, DDN, DDD\}$

X	0	1	2	3
f(x)	$\frac{27}{64}$	$\frac{27}{64}$	$\frac{9}{64}$	$\frac{1}{64}$

$P(S) = p = \frac{1}{4}$ (success), $q = \frac{3}{4}$

~~$P(x) = P(NDD) = P(N) \cdot P(D) \cdot P(N) = \frac{3}{4} \cdot \frac{1}{4} \cdot \frac{3}{4} = \frac{9}{64}$~~

$f(0) = P(NNN) = \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} = \frac{27}{64}$, $f(2) = 3 \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{3}{4} = \frac{9}{64}$ (2 items defective)

$f(1) = 3 \cdot \frac{1}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} = \frac{27}{64}$, $f(3) = \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{64}$

or write in tabular form

$$\boxed{b(x; n, p) = b(x; 3, \tfrac{1}{4}) = {}^3C_x \left(\tfrac{1}{4}\right)^x \left(\tfrac{3}{4}\right)^{3-x} \quad ; \quad x = 0, 1, 2, 3}$$

${}^nC_x p^x q^{n-x}$

- The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what is the prob. that

(a) at least 10 survive

(b) from 3 to 8 survive

(c) exactly 5 survive?

Sol: Let x be the number of people who survive

$b(x; n, p) = {}^nC_x \, p^x q^{n-x}$; $x = 0, \dots 15$ depends on qu

~~(a) $P(x \geq 10) =$~~

$p = 0.4$, $q = 0.6$, $n = 15$

(a) $P(x \geq 10) = P(x=10) + P(x=11) + P(x=12) + P(x=13) + P(x=14) + P(x=15)$

$x = 10,$

$= {}^{15}C_{10}(0.4)^{10}(0.6)^5 + {}^{15}C_{11}(0.4)^{11}(0.6)^4 + {}^{15}C_{12}(0.4)^{12}(0.6)^3$

$+ {}^{15}C_{13}(0.4)^{13}(0.6)^2 + {}^{15}C_{14}(0.4)^{14}(0.6) + {}^{15}C_{15}(0.4)^{15}$

$= 0.0338$

(b) $P(3 \leq x \leq 8) = P(x=3) + P(x=4) + P(x=5) + P(x=6) + P(x=7) + P(x=8)$

$x = 3, 4, 5, 6, 7, 8$

$= {}^{15}C_3(0.4)^3(0.6)^{12} + {}^{15}C_4(0.4)^4(0.6)^{11} + {}^{15}C_5(0.4)^5(0.6)^{10}$

$+ {}^{15}C_6(0.4)^6(0.6)^9 + {}^{15}C_7(0.4)^7(0.6)^8 + {}^{15}C_8(0.4)^8(0.6)^7$

$= 0.8779$

(c) $P(x=5) = {}^nC_x \, p^x q^{n-x}$

$= {}^{15}C_5(0.4)^5(0.6)^{10}$

$= 0.1859$

Q: The probability that a certain kind of component will survive a shock test is $\frac{3}{4}$. Find the prob. that exactly 2 of the next 4 components tested survive

Sol: $p = \frac{3}{4}$, $q = \frac{1}{4}$ $\quad n = 4$ → no. of trial $\quad x = 0, 1, 2$

$b\left(2; 4, \frac{3}{4}\right) = {}^4C_2\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^2 = 6 \cdot \frac{9}{16} \cdot \frac{1}{6} = \frac{27}{128}$

* Moment Generating Function [MGF] :-

The mgf of a random variable X (about origin) having the prob. funcⁿ $f(x)$ is given by:

$$M_X(t) = E[e^{tX}] = \begin{cases} \int e^{tx} f(x)\,dx, & \text{for continuous prob. dist.} \\ \sum e^{tx} f(x), & \text{for discrete prob. dist.} \end{cases}$$

$$M_X(t) = E(e^{tX}) = E\left[1 + tX + \frac{t^2X^2}{2!} + \cdots + \frac{t^r X^r}{r!} + \cdots\right]$$

$$= 1 + t\,E[X] + \frac{t^2}{2!}E[X^2] + \cdots + \frac{t^r}{r!}E[X^r] + \cdots$$

$$= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \cdots + \frac{t^r}{r!}\mu_r' + \cdots$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r'$$

where $\mu_r' = E[X^r] = \begin{cases} \int_{-\infty}^{\infty} x^r f(x)\,dx, & \text{for cts dist.} \\ \sum_x x^r f(x), & \text{for discrete dist.} \end{cases}$

is the r^{th} moment of X about origin.

Thus, μ_r' (about origin) = Coeff. of $\frac{t^r}{r!}$ in $M_X(t)$.

Since, $M_X(t)$ generates moments, it is known as mgf.

Or $\mu_r' = \left. \frac{d^r}{dt^r}\{M_X(t)\} \right|_{t=0}$

$\Rightarrow$ Find the Moment Generating Function of Binomial Dist. and use it to find the 'μ' & 'σ^2' :-

Let X be a binomial random variable.

$$\therefore b(x, n, p) = {}^nC_x p^x q^{n-x} \quad ; x = 0, 1, 2, \ldots n$$

$$M_X(t) = E[e^{tX}] = \sum_{x=0}^{n} e^{tx}\, {}^nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} {}^nC_x \underbrace{(pe^t)^x}_{x} \underbrace{q^{n-x}}_{a \to \text{suppose}}$$

$$= (q + pe^t)^n$$

[Binomial expansion :- $(a+x)^n = {}^nC_0 a^0 x^n + {}^nC_1 a^1 x^{n-1} + \ldots + {}^nC_n a^n x^0$

Mean :-

$\mu = E[X] = \mu_1'$, the 1st moment about origin.

Now, $E[X^r] = \frac{d^r}{dt^r}\left[M_X(t)\right]_{t=0}$

$$\therefore E[X] = \frac{d}{dt}\left[M_X(t)\right]_{t=0} = \frac{d}{dt}\left[(q+pe^t)^n\right]_{t=0}$$

$$= \left| n(q+pe^t)^{n-1} \cdot pe^t \right|_{t=0}$$

$$= np(q+p)^{n-1} \qquad [\because p+q=1]$$

$$\boxed{E[X] = np = \text{Mean}}$$

$$\mu = E(X) = np$$

Variance :-

$Var(x) = E[x^2] - \{E(x)\}^2 \quad -(1)$

$$E(x^2) = \left| \frac{d^2}{dt^2} (M_x(t)) \right|_{t=0}$$

$$= \left| \frac{d}{dt} \left(npe^t (q + pe^t)^{n-1} \right) \right|_{t=0}$$

$$= \left| np \left\{ e^t (n-1) (q + pe^t)^{n-2} \cdot pe^t + e^t (q + pe^t)^{n-1} \right\} \right|_{t=0}$$

$$\Rightarrow = \left[np \left\{ (n-1)(q+p)^{n-2} p + (q+p)^{n-1} \right\} \right]. \qquad [\because q + p = 1]$$

$$= np \{ (n-1) p + 1 \}$$

$$= np [np - p + 1]$$

$$E(x^2) = n^2p^2 - np^2 + np$$

eq^r(1)

$$Var(x) = \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$= np - np^2$$

$$= np(1-p) \qquad \left[\because q + p = 1 \Rightarrow q = 1 - p\right]$$

$$\boxed{\sigma^2 = npq}$$

* Negative Binomial and Geometric Distribution

⇒ Negative Binomial experiments :-

Consider an exp. where the properties are the same as those listed for a binomial exp., with the exception that the trials will be replaced until a fixed number of successes occur.

Therefore, instead of the probability of 'x' successes in 'n' trials, where n is fixed, we are now interested in the prob. that the k^{th} success occurs on the x^{th} trial.

Experiments of this kind are called negative binomial experiments.

⇒ Negative Binomial Random Variable

The number X of trials required to produce 'k' successes in a negative binomial exp. is called a negative binomial random variable and its prob. dist. is called the negative binomial dist.

⇒ Negative Binomial dist.

If repeated independent trials can result in a success with prob. 'p' & a failure with prob. $q = 1-p$, then the prob. dist. of the rv X, the no. of the trial on which the k^{th} success occurs is

$$b^*(x; k, p) = \binom{x-1}{k-1} p^k q^{x-k} \quad ; x = k, k+1, \ldots$$

Q:- Find the prob that a person flipping a coin gets
(a) the third head on the seventh flip.
(b) the first head on " fourth flip.

Sol: (a) $b^*(7, 3, \frac{1}{2}) = {}^6C_2 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^4$
$= \frac{15}{128} = 0.1172$

Mean $\boxed{\mu = \frac{k}{p}}$

variance, $= \boxed{\frac{kq}{p^2}}$

(b) $b^*(4, 1, \frac{1}{2}) = {}^3C_0 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^3$
$= \frac{1}{16} = 0.0625$

Q:- In an NBA championship series, the team that wins 4 games out of seven is the winner. Suppose that the teams A and B face each other in the championship games and that team A has prob. 0.55 of winning a game over team B.
(a) What is the prob that team A will win the series in 6 games?
(b) what is the prob. that team A will win the series?
(c) If teams A & B were facing each other in a regional play off series, which is decided by winning 3 out of five games. What is the prob. that team A would win the series?

Ans (a) $b^*(6; 4, 0.55) = {}^5C_3 (0.55)^4 (0.45)^2$
$= 0.1853$

(b) $P(\text{A will win the series}) = P(X \geq 4)$
$= b^*(4; 4, 0.55) + b^*(5; 4, 0.55) + b^*(6; 4, 0.55) + b^*(7; 4, 0.55)$
$= {}^3C_3 (0.55)^4 (0.45)^0 + {}^4C_3 (0.55)^4 (0.45) + {}^5C_3 (0.55)^4 (0.45)^2 + {}^6C_3 (0.55)^4 (0.45)^3$
$= 0.6083$

(c) P(team A wins the play off) = $P(X \geq 3)$

$= b^*(3;3,0.55) + b^*(4;3,0.55) + b^*(5;3;0.55)$

$= {}^2C_2 (0.55)^3 (0.45)^0 + \text{\_\_\_\_\_\_}$

$= 0.5931$

* If we consider the special case of the negative binomial dist. where $\underline{K=1}$, we have a prob. dist. for the no. of trials required for a single success.

For eg. Tossing a coin until head occurs. we might be interested in the prob. that the first head occurs on the fourth toss.

The negative binomial reduces to

$b^*(x;1,p) = pq^{x-1}$; $x = 1,2,3, \ldots$

* Geometric Distribution :-

If repeated independent trials can result in a success with prob. 'p' and a failure with prob. 'q' = 1-p, then, the prob. dist. of the r.v X, the no. of the trial on which the first success occurs, is

$$\boxed{g(x;p) = pq^{x-1},} \quad x = 1,2,3 \ldots$$

Mean

$$\boxed{\mu = \frac{1}{p}}$$

Variance

$$\boxed{\text{Var.} = \frac{q}{p^2}} \text{ or } \frac{1-p}{p^2}$$

Q:- For a certain manufacturing process, it is known that, on the average, 1 in every 100 items is defective. What is the prob. that the fifth item inspected is the 1st defective item found?

$p = \frac{1}{100} = 0.01$

$q = 0.99$

Sol: $g(\overset{\text{succ}}{5}; \overset{\frac{1}{100}}{0.01}) = (0.01)(0.99)^4$

$= 0.0096$

Q: At a busy time, a telephone exchange is very near capacity, so callers have difficulty placing their calls. It may be of interest to know the no. of attempts necessary in order to make a connection. Suppose that we let $p = 0.05$ be the prob. of a connection during a busy time.
We are interested in knowing the prob. that 5 attempts are necessary for a successful call.

Sol: $g(5; 0.05) = (0.05)(0.95)^4$

$= 0.0407$

Q: In the above example, find the expected no. of calls necessary to make a connection?

Sol: $\mu = \frac{1}{p}$

$= \frac{1}{0.05}$

$= 20$

* MGF of Negative Binomial Dist.

$$M_X(t) = E[e^{tX}] = \sum_{x=k}^{\infty} e^{tx} \binom{x-1}{k-1} q^{x-k} p^k$$

$$= p^k e^{tk} \sum_{x=k}^{\infty} e^{tx} \cdot e^{tk-tk} \binom{x-1}{k-1} q^{x-k} p^k$$

$$= p^k e^{tk} \sum_{x=k}^{\infty} e^{t(x-k)} \binom{x-1}{k-1} q^{x-k}$$

$$= p^k e^{tk} \sum_{x=k}^{\infty} \binom{x-1}{k-1} (qe^t)^{x-k}$$

$$= p^k e^{tk} \sum_{r=0}^{\infty} \binom{k+r-1}{k-1} (qe^t)^{x} \qquad \text{Let } r = x-k \Rightarrow x = k+r$$

$$= (pe^t)^k \left[1 + k(qe^t) + \frac{k(k+1)}{2!}(qe^t)^2 + \frac{k(k+1)(k+2)}{3!}(qe^t)^3 + \cdots\right]$$

$$= (pe^t)^k \left[1-qe^t\right]^{-k}$$

$$\left[\because (1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!} + \cdots\right] \text{ provided } |x|<1$$

$$\therefore M_X(t) = \frac{(pe^t)^k}{(1-qe^t)^k}, \quad \text{provided } |qe^t| < 1$$

Mean

$$E[X] = \frac{d}{dt} M_X(t)$$

$$= \frac{d}{dt}\left[\frac{(pe^t)^k}{(1-qe^t)^k}\right]$$

$$= \frac{(1-qe^t)^k \cdot k(pe^t)^{k-1} pe^t - (pe^t)^k \cdot k(1-qe^t)^{k-1} \cdot (-qe^t)}{(1-qe^t)^{2k}}$$

$$= \frac{K(1-qe^t)^{k-1}(pe^t)^k\left[1-qe^t+qe^t\right]}{(1-qe^t)^{2k}}$$

$$= \frac{K(1-qe^t)^{k-1}(pe^t)^k}{(1-qe^t)^{2k}}$$

At $t=0$,

$$\mu = \frac{K(1-q)^{k-1}p^k}{(1-q)^{2k}} = \frac{Kp^{k-1}p^k}{p^{2k}} = \frac{Kp^{2k-1}}{p^{2k}}$$

$$\boxed{\mu = \frac{k}{p}}$$

__Variance__

$$E[X^2] = \frac{d^2}{dt^2} M_X(t) \Big|_{t=0}$$

$$= \frac{d}{dt}\left[\frac{k(1-qe^t)^{k-1}(pe^t)^k}{(1-qe^t)^{2k}}\right]_{t=0}$$

$$= k\left\{\frac{(1-qe^t)^{2k}\frac{d}{dt}\left[(1-qe^t)^{k-1}(pe^t)^k\right] \mp (1-qe^t)^{k-1}(pe^t)^k \frac{d}{dt}(1-qe^t)^{2k}}{(1-qe^t)^{4k}}\right\}_{t=0}$$

$$= k\left\{\frac{(1-qe^t)^{2k}\left[(1-qe^t)^{k-1}\frac{d}{dt}(pe^t)^k + (pe^t)^k\frac{d}{dt}(1-qe^t)^{k-1}\right] \mp (1-qe^t)^{k-1}(pe^t)^k \cdot 2k(1-qe^t)(-qe^t)}{(1-qe^t)^{4k}}\right\}$$

$$= k\left\{\frac{(1-qe^t)^{2k}\left[(1-qe^t)^{k-1}\cdot k(pe^t)^{k-1}(pe^t) + (pe^t)^k(k-1)(1-qe^t)^{k-2}(-qe^t)\right] \mp (1-qe^t)^{k-1}(pe^t)^k 2k(1-qe^t)^{2k-1}(-qe^t)}{(1-qe^t)^{4k}}\right\}_{t=0}$$

$$= k\left\{\frac{(1-q)^{2k}\left[(1-q)^{k-1}k(p)^{k-1}p + (p)^k(k-1)(1-q)^{k-2}(-q)\right] \mp (1-q)^{k-1}(p)^k 2k(1-q)^{2k-1}(-q)}{(1-q)^{4k}}\right\}$$

$$= k\left\{\frac{(1-q)^{2k+k-1}k\,p^{k-1+1} + p^k(k-1)(1-q)^{2k+k-2}(-q) \mp p^k 2k(-q)(1-q)^{k-1+2k-1}}{(1-q)^{4k}}\right\}$$

$$= k\left\{\frac{p^{3k-1+k}k + p^{k+3k-2}(k-1)(-q) + 2k(-q)p^{k+3k-2}}{(1-q)^{4k}}\right\}$$

$$= k\left\{\frac{kp^{4k-1} - q(k-1)p^{4k-2} + 2kq\,p^{4k-2}}{p^{4k}}\right\}$$

$$= k\left[\frac{kp^{4k-1} - qkp^{4k-2} + qp^{4k-2} + 2kqp^{4k-2}}{p^{4k}}\right]$$

$$= k\left[\frac{kp^{4k-1} + qp^{4k-2} + q^k p^{4k-2}}{p^{4k}}\right]$$

$$= k \cdot \frac{p^{4k}}{p^{4k}}\left[\frac{k}{p} + \frac{q}{p^2} + \frac{qk}{p^2}\right]$$

$$= k\left[\frac{k}{p} + \frac{q}{p^2} + \frac{qk}{p^2}\right]$$

$$= k\left[\frac{kp+q+qk}{p^2}\right] = k\left[\frac{q+k(p+q)}{p^2}\right]$$

$$= k\left[\frac{q+k}{p^2}\right]$$

$$\text{variance} = \frac{k}{p^2}(k+q) - \frac{k^2}{p^2}$$

$$= \cancel{\frac{k^2}{p^2}} + \frac{kq}{p^2} - \cancel{\frac{k^2}{p^2}}$$

$$\boxed{\text{Var.} = \frac{kq}{p^2}}$$

* MGF of Geometric distributions :-

$$M_x(t) = E[e^{tx}] = \sum_{x=1}^{\infty} e^{tx} p q^{x-1}$$

$$= \frac{p}{q}\sum_{x=1}^{\infty} e^{tx} q^x$$

$$= \frac{p}{q}\sum_{x=1}^{\infty} [qe^t]^x$$

$$= \frac{p}{q}\frac{qe^t}{1-qe^t}, \text{ provided } qe^t<1$$

$$\boxed{M_x(t) = \frac{pe^t}{1-qe^t}} \text{ provided } qe^t<1$$

Infinite GP

$$\sum_{n=1}^{\infty} a^n = \frac{a}{1-r},$$

provided common ratio $r<1$

Mean

$$\mu_1' = E[X] = \left| \frac{d}{dt} M_X(t) \right|_{t=0}$$

$$= \left| \frac{d}{dt} \left(\frac{pe^t}{1-qe^t} \right) \right|_{t=0}$$

$$= \left. \frac{(1-qe^t)pe^t - pe^t(-qe^t)}{(1-qe^t)^2} \right|_{t=0}$$

$$= \frac{pe^t[1-\cancel{qe^t}+\cancel{qe^t}]}{(1-qe^t)^2}$$

$$= \left. \frac{pe^t}{(1-qe^t)^2} \right|_{t=0}$$

$$E(X) = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p} \Rightarrow \boxed{\mu = \frac{1}{p}}$$

Variance

$$E[X^2] = \left| \frac{d^2}{dt^2} \left(\frac{pe^t}{1-qe^t} \right) \right|_{t=0}$$

$$= \left| \frac{d}{dt} \left[\frac{pe^t}{(1-qe^t)^2} \right] \right|_{t=0}$$

$$= \left. \frac{(1-qe^t)^2 \cdot pe^t - pe^t \cdot 2(1-qe^t)(-qe^t)}{(1-qe^t)^4} \right|_{t=0}$$

$$E[X^2] = \frac{(1-q)^2 p + 2p(1-q)q}{(1-q)^4}$$

$$= \frac{p^3 + 2p^2 q}{p^4} = \frac{1}{p} + \frac{2q}{p^2}$$

$$\text{Var.} = \frac{1}{p} + \frac{2q}{p^2} - \frac{1}{p^2} = \frac{p+2q-1}{p^2} = \frac{1+q-1}{p^2} \Rightarrow \boxed{\text{Var.} = \frac{q}{p^2}}$$

* Poisson Process and Poisson Distribution

Some experiments results in counting the number of particular events occur in given time interval or in a specified region, known as Poisson experiment.

The time interval may be of any length, such as a minute, a day, a week, a month or even a year.

eg:- (1) No. of telephone calls received per hour by an office.

(2) How many vehicles pass through a traffic signal in a day.

⇒ Poisson Random variable and Poisson distribution:-

The number X of outcomes occuring during a Poisson exp. is called a Poisson r.v. and its prob. dist. is called the Poisson ~~distribution~~ distribution.

⇒ Poisson distribution:-

The prob. dist. of the Poisson random variable X, representing the number of outcomes occuring in a given time interval or specified region denoted by 't', is

$$\boxed{P(x;\lambda t) = \frac{e^{-\lambda t}(\lambda t)^x}{x!}} \quad ; \; x = 0, 1, 2, \ldots$$

where λ is the avg. no. of outcomes per unit time, distance, area or volume and $e = 2.71828\ldots$

Q: During a laboratory exp., the avg. no. of radioactive particles passing through a counter in 1 milisecond is 4. What is the prob. that 6 particles enter the counter in a given milisecond?

Sol: $\lambda t = 4 \times 1 = 4$, $x = 6$

$p(x; \lambda t) = p(6; 4)$

$= \frac{e^{-4}(4)^6}{6!} = \frac{(0.0183)(4096)}{720}$

$= 0.1041$

$\boxed{\text{Mean, } \mu = \lambda t \quad \text{var, } \sigma^2 = \lambda t}$

* **Approximation of Binomial dist. by a Poisson dist.:**

Poisson dist. is a limiting case of the binomial dist. under the following cond^n s:

(i) n, the no. of trials is indefinitely large i.e. $n \to \infty$

(ii) p, the constant prob. of success for each trial is indefinitely small i.e. $p \to 0$

(iii) $np = \mu$, is finite.

Th^m:- Let X be a binomial r.v, with prob. dist. $b(x; n, p)$. When $n \to \infty$, $p \to 0$ and $np \to \mu$, remains constant,

$b(x; n, p) \xrightarrow{n \to \infty} p(x; \mu)$

$\boxed{p(x; \mu) = \frac{e^{-\mu}\mu^x}{x!}}$, $x = 0, 1, 2, \ldots$

Q: In a certain industrial facility, accidents occur infrequently. It is known that the prob. of an accident on any given day is 0.005 and accidents are independent of each other.

(a) What is the prob. that in any given period of 400 days there will be an accident on one day?

(b) What is the prob. that there are atmost three days with accidents?

Sol: Let X be a binomial rv with $n = 400$ & $p = 0.005$

Thus, $np = 400 \times 0.05$
$= 2 = \mu$

Using poisson process

(a) $P(X=1) = \dfrac{e^{-\mu}\mu^{x}}{x!} = \dfrac{e^{-2}(2)^1}{1!} = (0.1353)(2) = 0.2706$

(b) $P(X \le 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$= \dfrac{e^{-2}(2)^0}{0!} + \dfrac{e^{-2}(2)^1}{1!} + \dfrac{e^{-2}(2)^2}{2!} + \dfrac{e^{-2}(3)^2}{3!}$

$= 0.8569$

Q:- In a manufacturing process, where glass products are made, defects or bubbles occur, occasionaly rendering the piece undesirable for marketing. It is known that, on average, 1 in every 1000 of these items produced has ~~an~~ one or more bubbles. What is the prob. that a rv sample of 8000 will yield fewer than 7 items posessing bubbles?

Sol: It is a binomial exp. with
$n = 8000$ & $p = 0.001$

Since p is very close to 0 & n is quite large,
we will use Poisson dist.

$$\mu \overset{np}{=} 8000 \times 0.001 = 8$$

Let x represent the no. of bubbles.

$$P(x<7) = P(x=0) + \cdots + P(x=6)$$
$$= e^{-8}\left[\frac{8^0}{0!} + \frac{8^1}{1!} + \cdots \frac{8^6}{6!}\right]$$
$$= 0.3134$$

Q:- A manufacturer who produces medicine bottles, finds that 0.1% of the bottles are defective. The bottles are packed in boxes containing 500 bottles. A drug manufactures buys 100 boxes from the producer of bottles. Using poisson dist., find how many boxes will contain
(i) no defective (ii) at least two defectives.

Sol: $n = 500$, $p = 0.001$, $np = 0.5$

X: No. of defective bottles in a box of 500.

$$P(X = x) = \frac{e^{-0.5}(0.5)^x}{x!} \; ; \; x = 0, 1, 2 \ldots$$

The no. of boxes containing x defective bottles in a consignment of 100 boxes is

$$100 \times P(X = x) = 100 \times \frac{e^{-0.5}(0.5)^x}{x!} \; ; \; x = 0, 1, 2, \ldots$$

(i) No. of boxes containing no defective bottles is

$$100 \times P(X=0) = 100 \times \frac{e^{-0.5} \times (0.5)^0}{0!}$$

$$= 100 \times 0.6065$$

$$= 6065 \simeq 61$$

(ii) No. of boxes containing atleast two defective bottles is

$$100 \times P(X \geq 2) = 100\,[1 - P(X<2)]$$

$$= 100\,[1 - P(X=0) - P(X=1)]$$

$$= 100\left[1 - 0.60651 \times \left(1 - \frac{0.6065(0.5)}{1!}\right)\right]$$

$$= 9.02$$

$$\simeq 9$$

* MGF of Poisson dist.

$$M_X(t) = E[e^{tX}] = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^t\lambda)^x}{x!} \qquad \left[e^x = 1 + x + \frac{x^2}{2!} + \cdots\right]$$

$$= e^{-\lambda}\, e^{e^t\lambda}$$

$$M_X(t) = e^{-\lambda(1-e^t)}$$

Mean

$$E(X) = \left[\frac{d}{dt}[M_X(t)]\right]_{t=0}$$

$$= e^{-\lambda(1-e^t)}\,(-\lambda)\,(-e^t)$$

$$= (\lambda e^t)\, e^{-\lambda(1-e^t)}$$

$$\cancel{\lambda e^{t}}$$

$$= \lambda e^{t-\lambda+\lambda e^{t}} \Big|_{t=0}$$

$$= \lambda e^{-\lambda+\lambda}$$

$$= \lambda$$

$$\boxed{E(X) = \lambda}$$

<u>Variance</u>

$$E(X^2) = \frac{d^2}{dt^2}\left[M_X(t)\right]$$

$$= \frac{d}{dt}\left[(\lambda e^{t})\, e^{-\lambda(1-e^{t})}\right]$$

$$= \lambda\left[e^{t} e^{-\lambda(1-e^{t})} + e^{t} e^{-\lambda(1-e^{t})}(\lambda e^{t})\right] \Big|_{t=0}$$

$$= \lambda\left[e^{-\lambda(0)} + e^{0} e^{-\lambda(1-1)} \lambda\right]$$

$$= \lambda(1+\lambda)$$

$$= \lambda+\lambda^2$$

$$\text{Var.} = \lambda + \cancel{\lambda^2} - \cancel{\lambda^2}$$

$$\boxed{\text{Var.} = \lambda}$$