Chapter 3

Random Variables and Probability Distributions

3.1 Discrete; continuous; continuous; discrete; discrete; continuous.

3.2 A table of sample space and assigned values of the random variable is shown next.

Sample Space	x
NNN	0
NNB	1
NBN	1
BNN	1
NBB	2
BNB	2
BBN	2
BBB	3

3.3 A table of sample space and assigned values of the random variable is shown next.

Sample Space	w
HHH	3
HHT	1
HTH	1
THH	1
HTT	−1
THT	−1
TTH	−1
TTT	−3

3.4 $S = \{HHH, THHH, HTHHH, TTHHH, TTTHHH, HTTHHH, THTHHH, HHTHHH, \dots\}$; The sample space is discrete containing as many elements as there are positive integers.

3.5 (a) $c = 1/30$ since $1 = \sum_{x=0}^{3} c(x^2+4) = 30c$.

(b) $c = 1/10$ since

$$1 = \sum_{x=0}^{2} c\binom{2}{x}\binom{3}{3-x} = c\left[\binom{2}{0}\binom{3}{3} + \binom{2}{1}\binom{3}{2} + \binom{2}{2}\binom{3}{1}\right] = 10c.$$

3.6 (a) $P(X > 200) = \int_{200}^{\infty} \frac{20000}{(x+100)^3}\, dx = -\frac{10000}{(x+100)^2}\Big|_{200}^{\infty} = \frac{1}{9}$.

(b) $P(80 < X < 200) = \int_{80}^{120} \frac{20000}{(x+100)^3}\, dx = -\frac{10000}{(x+100)^2}\Big|_{80}^{120} = \frac{1000}{9801} = 0.1020$.

3.7 (a) $P(X < 1.2) = \int_0^1 x\, dx + \int_1^{1.2}(2-x)\, dx = \frac{x^2}{2}\Big|_0^1 + \left(2x - \frac{x^2}{2}\right)\Big|_1^{1.2} = 0.68$.

(b) $P(0.5 < X < 1) = \int_{0.5}^1 x\, dx = \frac{x^2}{2}\Big|_{0.5}^1 = 0.375$.

3.8 Referring to the sample space in Exercise 3.3 and making use of the fact that $P(H) = 2/3$ and $P(T) = 1/3$, we have
$P(W = -3) = P(TTT) = (1/3)^3 = 1/27$;
$P(W = -1) = P(HTT) + P(THT) + P(TTH) = 3(2/3)(1/3)^2 = 2/9$;
$P(W = 1) = P(HHT) + P(HTH) + P(THH) = 3(2/3)^2(1/3) = 4/9$;
$P(W = 3) = P(HHH) = (2/3)^3 = 8/27$;
The probability distribution for W is then

w	-3	-1	1	3
$P(W = w)$	1/27	2/9	4/9	8/27

3.9 (a) $P(0 < X < 1) = \int_0^1 \frac{2(x+2)}{5}\, dx = \frac{(x+2)^2}{5}\Big|_0^1 = 1$.

(b) $P(1/4 < X < 1/2) = \int_{1/4}^{1/2} \frac{2(x+2)}{5}\, dx = \frac{(x+2)^2}{5}\Big|_{1/4}^{1/2} = 19/80$.

3.10 The die can land in 6 different ways each with probability 1/6. Therefore, $f(x) = \frac{1}{6}$, for $x = 1, 2, \ldots, 6$.

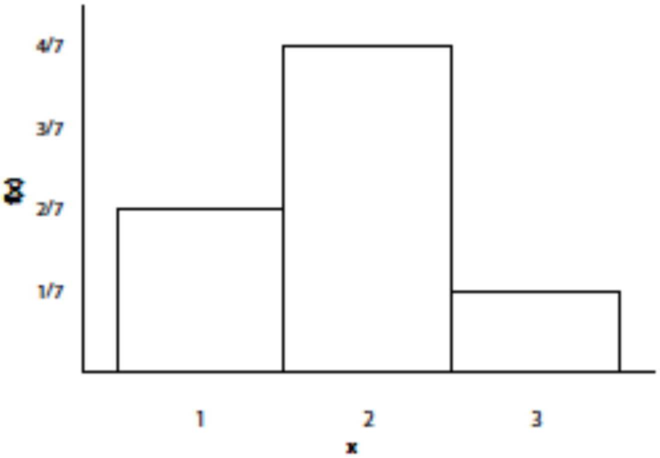
3.11 We can select x defective sets from 2, and $3 - x$ good sets from 5 in $\binom{2}{x}\binom{5}{3-x}$ ways. A random selection of 3 from 7 sets can be made in $\binom{7}{3}$ ways. Therefore,

$$f(x) = \frac{\binom{2}{x}\binom{5}{3-x}}{\binom{7}{3}}, \qquad x = 0, 1, 2.$$

In tabular form

x	0	1	2
$f(x)$	2/7	4/7	1/7

The following is a probability histogram:



3.12 (a) $P(T=5)=F(5)-F(4)=3/4-1/2=1/4$.

(b) $P(T>3)=1-F(3)=1-1/2=1/2$.

(c) $P(1.4<T<6)=F(6)-F(1.4)=3/4-1/4=1/2$.

(d) $P(T \leq 5|T \geq 2)=\frac{P(2<T<5)}{P(T\geq 2)}=\frac{3/4-1/4}{1-1/4}=\frac{2}{3}$.

3.13 The c.d.f. of X is

$$F(x)=\begin{cases}0, & \text{for } x<0,\\ 0.41, & \text{for } 0\leq x<1,\\ 0.78, & \text{for } 1\leq x<2,\\ 0.94, & \text{for } 2\leq x<3,\\ 0.99, & \text{for } 3\leq x<4,\\ 1, & \text{for } x\geq 4.\end{cases}$$

3.14 (a) $P(X<0.2)=F(0.2)=1-e^{-1.6}=0.7981$;

(b) $f(x)=F'(x)=8e^{-8x}$. Therefore, $P(X<0.2)=8\int_0^{0.2}e^{-8x}\,dx=-e^{-8x}\big|_0^{0.2}=0.7981$.

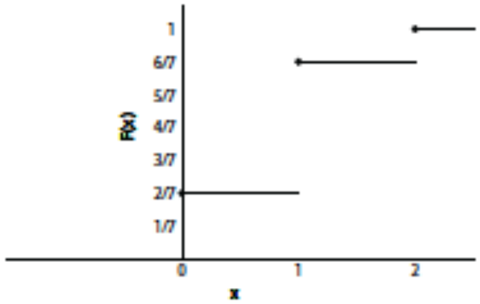
3.15 The c.d.f. of X is

$$F(x)=\begin{cases}0, & \text{for } x<0,\\ 2/7, & \text{for } 0\leq x<1,\\ 6/7, & \text{for } 1\leq x<2,\\ 1, & \text{for } x\geq 2.\end{cases}$$

(a) $P(X=1)=P(X\leq 1)-P(X\leq 0)=6/7-2/7=4/7$;

(b) $P(0<X\leq 2)=P(X\leq 2)-P(X\leq 0)=1-2/7=5/7$.

3.16 A graph of the c.d.f. is shown next.



3.17 (a) Area $= \int_1^3 (1/2)\, dx = \frac{x}{2}\Big|_1^3 = 1$.

(b) $P(2 < X < 2.5)\int_2^{2.5}(1/2)\, dx = \frac{x}{2}\Big|_2^{2.5} = \frac{1}{4}$.

(c) $P(X \le 1.6) = \int_1^{1.6}(1/2)\, dx = \frac{x}{2}\Big|_1^{1.6} = 0.3$.

3.18 (a) $P(X < 4) = \int_2^4 \frac{2(1+x)}{27}\, dx = \frac{(1+x)^2}{27}\Big|_2^4 = 16/27$.

(b) $P(3 \le X < 4) = \int_3^4 \frac{2(1+x)}{27}\, dx = \frac{(1+x)^2}{27}\Big|_3^4 = 1/3$.

3.19 For $1 \le x < 3$, $F(x) = \int_1^x (1/2)\, dt = \frac{x-1}{2}$,
Hence,

$$F(x) = \begin{cases} 0, & x < 1 \\ \frac{x-1}{2}, & 1 \le x < 3 \\ 1, & x \ge 3 \end{cases}$$

$P(2 < X < 2.5) = F(2.5) - F(2) = \frac{1.5}{2} - \frac{1}{2} = \frac{1}{4}$.

3.20 $F(x) = \frac{2}{27}\int_2^x (1+t)\, dt = \frac{2}{27}\left(t + \frac{t^2}{2}\right)\Big|_2^x = \frac{(x+4)(x-2)}{27}$,
$P(3 \le X < 4) = F(4) - F(3) = \frac{(8)(2)}{27} - \frac{(7)(1)}{27} = \frac{1}{3}$.

3.21 (a) $1 = k\int_0^1 \sqrt{x}\, dx = \frac{2k}{3}x^{3/2}\Big|_0^1 = \frac{2k}{3}$. Therefore, $k = \frac{3}{2}$.

(b) For $0 \le x < 1$, $F(x) = \frac{3}{2}\int_0^x \sqrt{t}\, dt = t^{3/2}\Big|_0^x = x^{3/2}$.
Hence,

$$F(x) = \begin{cases} 0, & x < 0 \\ x^{3/2}, & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases}$$

$P(0.3 < X < 0.6) = F(0.6) - F(0.3) = (0.6)^{3/2} - (0.3)^{3/2} = 0.3004$.

3.22 Denote by X the number of spades int he three draws. Let S and N stand for a spade and not a spade, respectively. Then
$P(X = 0) = P(NNN) = (39/52)(38/51)(37/50) = 703/1700$,
$P(X = 1) = P(SNN) + P(NSN) + P(NNS) = 3(13/52)(39/51)(38/50) = 741/1700$,
$P(X = 3) = P(SSS) = (13/52)(12/51)(11/50) = 11/850$, and
$P(X = 2) = 1 - 703/1700 - 741/1700 - 11/850 = 117/850$.
The probability mass function for X is then

x	0	1	2	3
$f(x)$	703/1700	741/1700	117/850	11/850

3.23 The c.d.f. of X is

$$F(x) = \begin{cases} 0, & \text{for } w < -3, \\ 1/27, & \text{for } -3 \le w < -1, \\ 7/27, & \text{for } -1 \le w < 1, \\ 19/27, & \text{for } 1 \le w < 3, \\ 1, & \text{for } w \ge 3, \end{cases}$$

(a) $P(W > 0) = 1 - P(W \le 0) = 1 - 7/27 = 20/27$.

(b) $P(-1 \le W < 3) = F(2) - F(-3) = 19/27 - 1/27 = 2/3$.

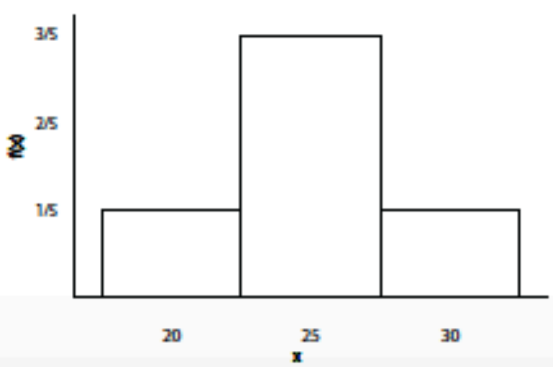
3.24 There are $\binom{10}{4}$ ways of selecting any 4 CDs from 10. We can select x jazz CDs from 5 and $4 - x$ from the remaining CDs in $\binom{5}{x}\binom{5}{4-x}$ ways. Hence

$$f(x) = \frac{\binom{5}{x}\binom{5}{4-x}}{\binom{10}{4}}, \qquad x = 0, 1, 2, 3, 4.$$

3.25 Let T be the total value of the three coins. Let D and N stand for a dime and nickel, respectively. Since we are selecting without replacement, the sample space containing elements for which $t = 20, 25$, and 30 cents corresponding to the selecting of 2 nickels and 1 dime, 1 nickel and 2 dimes, and 3 dimes. Therefore, $P(T = 20) = \frac{\binom{2}{2}\binom{4}{1}}{\binom{6}{3}} = \frac{1}{5}$,
$P(T = 25) = \frac{\binom{2}{1}\binom{4}{2}}{\binom{6}{3}} = \frac{3}{5}$,
$P(T = 30) = \frac{\binom{4}{3}}{\binom{6}{3}} = \frac{1}{5}$,
and the probability distribution in tabular form is

t	20	25	30
$P(T = t)$	1/5	3/5	1/5

As a probability histogram



3.26 Denote by X the number of green balls in the three draws. Let G and B stand for the colors of green and black, respectively.

Simple Event	x	$P(X = x)$
BBB	0	$(2/3)^3 = 8/27$
GBB	1	$(1/3)(2/3)^2 = 4/27$
BGB	1	$(1/3)(2/3)^2 = 4/27$
BBG	1	$(1/3)(2/3)^2 = 4/27$
BGG	2	$(1/3)^2(2/3) = 2/27$
GBG	2	$(1/3)^2(2/3) = 2/27$
GGB	2	$(1/3)^2(2/3) = 2/27$
GGG	3	$(1/3)^3 = 1/27$

The probability mass function for X is then

x	0	1	2	3
$P(X = x)$	8/27	4/9	2/9	1/27

3.27 (a) For $x \geq 0$, $F(x) = \int_0^x \frac{1}{2000} \exp(-t/2000)\, dt = -\exp(-t/2000)\big|_0^x$ $= 1 - \exp(-x/2000)$. So

$$F(x) = \begin{cases} 0, & x < 0, \\ 1 - \exp(-x/2000), & x \geq 0. \end{cases}$$

(b) $P(X > 1000) = 1 - F(1000) = 1 - [1 - \exp(-1000/2000)] = 0.6065$.

(c) $P(X < 2000) = F(2000) = 1 - \exp(-2000/2000) = 0.6321$.

3.28 (a) $f(x) \geq 0$ and $\int_{23.75}^{26.25} \frac{2}{5}\, dx = \frac{2}{5}t\big|_{23.75}^{26.25} = \frac{2.5}{2.5} = 1$.

(b) $P(X < 24) = \int_{23.75}^{24} \frac{2}{5}\, dx = \frac{2}{5}(24 - 23.75) = 0.1$.

(c) $P(X > 26) = \int_{26}^{26.25} \frac{2}{5}\, dx = \frac{2}{5}(26.25 - 26) = 0.1$. It is not extremely rare.

3.29 (a) $f(x) \geq 0$ and $\int_1^\infty 3x^{-4}\, dx = -3\, \frac{x^{-3}}{3}\Big|_1^\infty = 1$. So, this is a density function.

(b) For $x \geq 1$, $F(x) = \int_1^x 3t^{-4}\, dt = 1 - x^{-3}$. So,

$$F(x) = \begin{cases} 0, & x < 1, \\ 1 - x^{-3}, & x \geq 1. \end{cases}$$

(c) $P(X > 4) = 1 - F(4) = 4^{-3} = 0.0156$.

3.30 (a) $1 = k \int_{-1}^{1} (3 - x^2)\, dx = k \left(3x - \frac{x^3}{3}\right)\Big|_{-1}^{1} = \frac{16}{3}k$. So, $k = \frac{3}{16}$.

(b) For $-1 \leq x < 1$, $F(x) = \frac{3}{16} \int_{-1}^{x} (3 - t^2)\, dt = \left(3t - \frac{1}{3}t^3\right)\big|_{-1}^{x} = \frac{1}{2} + \frac{9}{16}x - \frac{x^3}{16}$.
So, $P\left(X < \frac{1}{2}\right) = \frac{1}{2} - \left(\frac{9}{16}\right)\left(\frac{1}{2}\right) - \frac{1}{16}\left(\frac{1}{2}\right)^3 = \frac{99}{128}$.

(c) $P(|X| < 0.8) = P(X < -0.8) + P(X > 0.8) = F(-0.8) + 1 - F(0.8)$
$= 1 + \left(\frac{1}{2} - \frac{9}{16}0.8 + \frac{1}{16}0.8^3\right) - \left(\frac{1}{2} + \frac{9}{16}0.8 - \frac{1}{16}0.8^3\right) = 0.164$.

3.31 (a) For $y \geq 0$, $F(y) = \frac{1}{4}\int_0^y e^{-t/4}\,dy = 1 - e^{y/4}$. So, $P(Y > 6) = e^{-6/4} = 0.2231$. This probability certainly cannot be considered as "unlikely."

(b) $P(Y \leq 1) = 1 - e^{-1/4} = 0.2212$, which is not so small either.

3.32 (a) $f(y) \geq 0$ and $\int_0^1 5(1-y)^4\,dy = -\left.(1-y)^5\right|_0^1 = 1$. So, this is a density function.

(b) $P(Y < 0.1) = -\left.(1-y)^5\right|_0^{0.1} = 1-(1-0.1)^5 = 0.4095$.

(c) $P(Y > 0.5) = (1-0.5)^5 = 0.03125$.

3.33 (a) Using integral by parts and setting $1 = k\int_0^1 y^4(1-y)^3\,dy$, we obtain $k = 280$.

(b) For $0 \leq y < 1$, $F(y) = 56y^5(1-Y)^3 + 28y^6(1-y)^2 + 8y^7(1-y) + y^8$. So, $P(Y \leq 0.5) = 0.3633$.

(c) Using the cdf in (b), $P(Y > 0.8) = 0.0563$.

3.34 (a) The event $Y = y$ means that among 5 selected, exactly y tubes meet the specification (M) and $5-y$ (M') does not. The probability for one combination of such a situation is $(0.99)^y(1-0.99)^{5-y}$ if we assume independence among the tubes. Since there are $\frac{5!}{y!(5-y)!}$ permutations of getting y Ms and $5-y$ M's, the probability of this event ($Y = y$) would be what it is specified in the problem.

(b) Three out of 5 is outside of specification means that $Y = 2$. $P(Y = 2) = 9.8 \times 10^{-6}$ which is extremely small. So, the conjecture is false.

3.35 (a) $P(X > 8) = 1 - P(X \leq 8) = \sum\limits_{x=0}^{8} e^{-6}\frac{6^x}{x!} = e^{-6}\left(\frac{6^0}{0!} + \frac{6^1}{1!} + \cdots + \frac{6^8}{8!}\right) = 0.1528$.

(b) $P(X = 2) = e^{-6}\frac{6^2}{2!} = 0.0446$.

3.36 For $0 < x < 1$, $F(x) = 2\int_0^x (1-t)\,dt = -\left.(1-t)^2\right|_0^x = 1-(1-x)^2$.

(a) $P(X \leq 1/3) = 1-(1-1/3)^2 = 5/9$.

(b) $P(X > 0.5) = (1-1/2)^2 = 1/4$.

(c) $P(X < 0.75 \mid X \geq 0.5) = \frac{P(0.5 < X < 0.75)}{P(X \geq 0.5)} = \frac{(1-0.5)^2-(1-0.75)^2}{(1-0.5)^2} = \frac{3}{4}$.

3.37 (a) $\sum\limits_{x=0}^{3}\sum\limits_{y=0}^{3} f(x,y) = c\sum\limits_{x=0}^{3}\sum\limits_{y=0}^{3} xy = 36c = 1$. Hence $c = 1/36$.

(b) $\sum\limits_{x}\sum\limits_{y} f(x,y) = c\sum\limits_{x}\sum\limits_{y} |x-y| = 15c = 1$. Hence $c = 1/15$.

3.38 The joint probability distribution of (X, Y) is

	$f(x,y)$	$x=0$	$x=1$	$x=2$	$x=3$
y	0	0	1/30	2/30	3/30
	1	1/30	2/30	3/30	4/30
	2	2/30	3/30	4/30	5/30

(a) $P(X \leq 2, Y = 1) = f(0,1) + f(1,1) + f(2,1) = 1/30 + 2/30 + 3/30 = 1/5$.

(b) $P(X > 2, Y \leq 1) = f(3,0) + f(3,1) = 3/30 + 4/30 = 7/30$.

(c) $P(X > Y) = f(1,0) + f(2,0) + f(3,0) + f(2,1) + f(3,1) + f(3,2)$
$= 1/30 + 2/30 + 3/30 + 3/30 + 4/30 + 5/30 = 3/5.$

(d) $P(X + Y = 4) = f(2,2) + f(3,1) = 4/30 + 4/30 = 4/15.$

3.39 (a) We can select x oranges from 3, y apples from 2, and $4 - x - y$ bananas from 3 in $\binom{3}{x}\binom{2}{y}\binom{3}{4-x-y}$ ways. A random selection of 4 pieces of fruit can be made in $\binom{8}{4}$ ways. Therefore,

$$f(x,y) = \frac{\binom{3}{x}\binom{2}{y}\binom{3}{4-x-y}}{\binom{8}{4}}, \qquad x = 0,1,2,3; \quad y = 0,1,2; \quad 1 \le x + y \le 4.$$

(b) $P[(X,Y) \in A] = P(X + Y \le 2) = f(1,0) + f(2,0) + f(0,1) + f(1,1) + f(0,2)$
$= 3/70 + 9/70 + 2/70 + 18/70 + 3/70 = 1/2.$

3.40 (a) $g(x) = \frac{2}{3}\int_0^1 (x + 2y)\, dy = \frac{2}{3}(x + 1)$, for $0 \le x \le 1$.

(b) $h(y) = \frac{2}{3}\int_0^1 (x + 2y)\, dx = \frac{1}{3}(1 + 4y)$, for $0 \le y \le 1$.

(c) $P(X < 1/2) = \frac{2}{3}\int_0^{1/2} (x + 1)\, dx = \frac{5}{12}$.

3.41 (a) $P(X + Y \le 1/2) = \int_0^{1/2}\int_0^{1/2-y} 24xy\, dx\, dy = 12\int_0^{1/2} \left(\frac{1}{2} - y\right)^2 y\, dy = \frac{1}{16}$.

(b) $g(x) = \int_0^{1-x} 24xy\, dy = 12x(1 - x)^2$, for $0 \le x < 1$.

(c) $f(y|x) = \frac{24xy}{12x(1-x)^2} = \frac{2y}{(1-x)^2}$, for $0 \le y \le 1 - x$.
Therefore, $P(Y < 1/8 \mid X = 3/4) = 32\int_0^{1/8} y\, dy = 1/4$.

3.42 Since $h(y) = e^{-y}\int_0^{\infty} e^{-x}\, dx = e^{-y}$, for $y > 0$, then $f(x|y) = f(x,y)/h(y) = e^{-x}$, for $x > 0$. So, $P(0 < X < 1 \mid Y = 2) = \int_0^1 e^{-x}\, dx = 0.6321$.

3.43 (a) $P(0 \le X \le 1/2,\ 1/4 \le Y \le 1/2) = \int_0^{1/2}\int_{1/4}^{1/2} 4xy\, dy\, dx = 3/8\int_0^{1/2} x\, dx = 3/64$.

(b) $P(X < Y) = \int_0^1\int_0^y 4xy\, dx\, dy = 2\int_0^1 y^3\, dy = 1/2$.

3.44 (a) $1 = k\int_{30}^{50}\int_{30}^{50} (x^2 + y^2)\, dx\, dy = k(50 - 30)\left(\int_{30}^{50} x^2\, dx + \int_{30}^{50} y^2\, dy\right) = \frac{392k}{3} \cdot 10^4$. So, $k = \frac{3}{392} \cdot 10^{-4}$.

(b) $P(30 \le X \le 40,\ 40 \le Y \le 50) = \frac{3}{392} \cdot 10^{-4}\int_{30}^{40}\int_{40}^{50} (x^2 + y^2)\, dy\, dx$
$= \frac{3}{392} \cdot 10^{-3}(\int_{30}^{40} x^2\, dx + \int_{40}^{50} y^2\, dy) = \frac{3}{392} \cdot 10^{-3}\left(\frac{40^3-30^3}{3} + \frac{50^3-40^3}{3}\right) = \frac{49}{196}$.

(c) $P(30 \le X \le 40,\ 30 \le Y \le 40) = \frac{3}{392} \cdot 10^{-4}\int_{30}^{40}\int_{30}^{40} (x^2 + y^2)\, dx\, dy$
$= 2\frac{3}{392} \cdot 10^{-4}(40 - 30)\int_{30}^{40} x^2\, dx = \frac{3}{196} \cdot 10^{-3}\frac{40^3-30^3}{3} = \frac{37}{196}$.

3.45 $P(X + Y > 1/2) = 1 - P(X + Y < 1/2) = 1 - \int_0^{1/4}\int_x^{1/2-x} \frac{1}{y}\, dy\, dx$
$= 1 - \int_0^{1/4}\left[\ln\left(\frac{1}{2} - x\right) - \ln x\right]\, dx = 1 + \left[\left(\frac{1}{2} - x\right)\ln\left(\frac{1}{2} - x\right) - x\ln x\right]\Big|_0^{1/4}$
$= 1 + \frac{1}{4}\ln\left(\frac{1}{4}\right) = 0.6534.$

3.46 (a) From the column totals of Exercise 3.38, we have

x	0	1	2	3
$g(x)$	1/10	1/5	3/10	2/5

(b) From the row totals of Exercise 3.38, we have

y	0	1	2
$h(y)$	1/5	1/3	7/15

3.47 (a) $g(x) = 2\int_x^1 dy = 2(1-x)$ for $0 < x < 1$;
$h(y) = 2\int_0^y dx = 2y$, for $0 < y < 1$.
Since $f(x, y) \neq g(x)h(y)$, X and Y are not independent.

(b) $f(x|y) = f(x, y)/h(y) = 1/y$, for $0 < x < y$.
Therefore, $P(1/4 < X < 1/2 \mid Y = 3/4) = \frac{4}{3}\int_{1/4}^{1/2} dx = \frac{1}{3}$.

3.48 (a) $g(2) = \sum_{y=0}^{2} f(2, y) = f(2, 0) + f(2, 1) + f(2, 2) = 9/70 + 18/70 + 3/70 = 3/7$. So, $f(y|2) = f(2, y)/g(2) = (7/3)f(2, y)$.
$f(0|2) = (7/3)f(2, 0) = (7/3)(9/70) = 3/10$, $f(1|2) = 3/5$ and $f(2|2) = 1/10$. In tabular form,

y	0	1	2
$f(y\|2)$	3/10	3/5	1/10

(b) $P(Y = 0 \mid X = 2) = f(0|2) = 3/10$.

3.49 (a)

x	1	2	3
$g(x)$	0.10	0.35	0.55

(b)

y	1	2	3
$h(y)$	0.20	0.50	0.30

(c) $P(Y = 3 \mid X = 2) = \frac{0.1}{0.05+0.10+0.20} = 0.2857$.

3.50

	$f(x,y)$	x = 2	4	$h(y)$
y	1	0.10	0.15	0.25
	3	0.20	0.30	0.50
	5	0.10	0.15	0.25
	$g(x)$	0.40	0.60	

(a)

x	2	4
$g(x)$	0.40	0.60

(b)

y	1	3	5
$h(y)$	0.25	0.50	0.25

3.51 (a) If (x, y) represents the selection of x kings and y jacks in 3 draws, we must have $x = 0, 1, 2, 3$; $y = 0, 1, 2, 3$; and $0 \leq x + y \leq 3$. Therefore, $(1, 2)$ represents the selection of 1 king and 2 jacks which will occur with probability

$$f(1, 2) = \frac{\binom{4}{1}\binom{4}{2}}{\binom{12}{3}} = \frac{6}{55}.$$

Proceeding in a similar fashion for the other possibilities, we arrive at the following joint probability distribution:

	$f(x,y)$	x = 0	1	2	3
y	0	1/55	6/55	6/55	1/55
	1	6/55	16/55	6/55	
	2	6/55	6/55		
	3	1/55			

(b) $P[(X,Y) \in A] = P(X+Y \geq 2) = 1 - P(X+Y < 2) = 1 - 1/55 - 6/55 - 6/55 = 42/55$.

3.52 (a) $P(H) = 0.4$, $P(T) = 0.6$, and $S = \{HH, HT, TH, TT\}$. Let (W, Z) represent a typical outcome of the experiment. The particular outcome $(1,0)$ indicating a total of 1 head and no heads on the first toss corresponds to the event TH. Therefore, $f(1,0) = P(W=1, Z=0) = P(TH) = P(T)P(H) = (0.6)(0.4) = 0.24$. Similar calculations for the outcomes $(0,0)$, $(1,1)$, and $(2,1)$ lead to the following joint probability distribution:

	$f(w,z)$	w = 0	1	2
z	0	0.36	0.24	
	1		0.24	0.16

(b) Summing the columns, the marginal distribution of W is

w	0	1	2
$g(w)$	0.36	0.48	0.16

(c) Summing the rows, the marginal distribution of Z is

z	0	1
$h(z)$	0.60	0.40

(d) $P(W \geq 1) = f(1,0) + f(1,1) + f(2,1) = 0.24 + 0.24 + 0.16 = 0.64$.

3.53 $g(x) = \frac{1}{8}\int_2^4 (6 - x - y)\, dy = \frac{3-x}{4}$, for $0 < x < 2$.
So, $f(y|x) = \frac{f(x,y)}{g(x)} = \frac{6-x-y}{2(3-x)}$, for $2 < y < 4$,
and $P(1 < Y < 3 \mid X = 1) = \frac{1}{4}\int_2^3 (5 - y)\, dy = \frac{5}{8}$.

3.54 Since $f(1,1) \neq g(1)h(1)$, the variables are not independent.

3.55 X and Y are independent since $f(x,y) = g(x)h(y)$ for all (x,y).

3.56 (a) $h(y) = 6\int_0^{1-y} x\, dx = 3(1-y)^2$, for $0 < y < 1$. Since $f(x|y) = \frac{f(x,y)}{h(y)} = \frac{2x}{(1-y)^2}$, for $0 < x < 1 - y$, involves the variable y, X and Y are not independent.

(b) $P(X > 0.3 \mid Y = 0.5) = 8\int_{0.3}^{0.5} x\, dx = 0.64$.

3.57 (a) $1 = k\int_0^2\int_0^1\int_0^1 xy^2z\, dx\, dy\, dz = 2k\int_0^1\int_0^1 y^2z\, dy\, dz = \frac{2k}{3}\int_0^1 z\, dz = \frac{k}{3}$. So, $k = 3$.

(b) $P\left(X < \frac{1}{4}, Y > \frac{1}{2}, 1 < Z < 2\right) = 3\int_1^2\int_{1/2}^1\int_0^{1/4} xy^2z\, dx\, dy\, dz = \frac{9}{2}\int_0^{1/4}\int_{1/2}^1 y^2z\, dy\, dz$
$= \frac{21}{16}\int_1^2 z\, dz = \frac{21}{512}$.

3.58 $g(x) = 4\int_0^1 xy\ dy = 2x$, for $0 < x < 1$; $h(y) = 4\int_0^1 xy\ dx = 2y$, for $0 < y < 1$. Since $f(x,y) = g(x)h(y)$ for all (x,y), X and Y are independent.

3.59 $g(x) = k\int_{30}^{50}(x^2+y^2)\ dy = k\left(x^2y + \frac{y^3}{3}\right)\Big|_{30}^{50} = k\left(20x^2 + \frac{98{,}000}{3}\right)$, and $h(y) = k\left(20y^2 + \frac{98{,}000}{3}\right)$.
Since $f(x,y) \neq g(x)h(y)$, X and Y are not independent.

3.60 (a) $g(y,z) = \frac{4}{9}\int_0^1 xyz^2\ dx = \frac{2}{9}yz^2$, for $0 < y < 1$ and $0 < z < 3$.

(b) $h(y) = \frac{2}{9}\int_0^3 yz^2\ dz = 2y$, for $0 < y < 1$.

(c) $P\left(\frac{1}{4} < X < \frac{1}{2}, Y > \frac{1}{3}, Z < 2\right) = \frac{4}{9}\int_1^2\int_{1/3}^1\int_{1/4}^{1/2} xyz^2\ dx\ dy\ dz = \frac{7}{162}$.

(d) Since $f(x|y,z) = \frac{f(x,y,z)}{g(y,z)} = 2x$, for $0 < x < 1$, $P\left(0 < X < \frac{1}{2} \mid Y = \frac{1}{4}, Z = 2\right) = 2\int_0^{1/2} x\ dx = \frac{1}{4}$.

3.61 $g(x) = 24\int_0^{1-x} xy\ dy = 12x(1-x)^2$, for $0 < x < 1$.

(a) $P(X \geq 0.5) = 12\int_{0.5}^1 x(1-x)^2\ dx = \int_{0.5}^1(12x - 24x^2 + 12x^3)\ dx = \frac{5}{16} = 0.3125$.

(b) $h(y) = 24\int_0^{1-y} xy\ dx = 12y(1-y)^2$, for $0 < y < 1$.

(c) $f(x|y) = \frac{f(x,y)}{h(y)} = \frac{24xy}{12y(1-y)^2} = \frac{2x}{(1-y)^2}$, for $0 < x < 1-y$.
So, $P\left(X < \frac{1}{8} \mid Y = \frac{3}{4}\right) = \int_0^{1/8} \frac{2x}{1/16}\ dx = 32\int_0^{1/8} x = 0.25$.

3.62 (a)

x	1	3	5	7
$f(x)$	0.4	0.2	0.2	0.2

(b) $P(4 < X \leq 7) = P(X \leq 7) - P(X \leq 4) = F(7) - F(4) = 1 - 0.6 = 0.4$.

3.63 (a) $g(x) = \int_0^\infty ye^{-y(1+x)}\ dy = -\frac{1}{1+x}ye^{-y(1+x)}\Big|_0^\infty + \frac{1}{1+x}\int_0^\infty e^{-y(1+x)}\ dy$
$= -\frac{1}{(1+x)^2}\ e^{-y(1+x)}\Big|_0^\infty$
$= \frac{1}{(1+x)^2}$, for $x > 0$.
$h(y) = ye^{-y}\int_0^\infty e^{-yx}\ dx = -e^{-y}\ e^{-yx}\Big|_0^\infty = e^{-y}$, for $y > 0$.

(b) $P(X \geq 2, Y \geq 2) = \int_2^\infty\int_2^\infty ye^{-y(1+x)}\ dx\ dy = -\int_2^\infty e^{-y}\ e^{-yx}\Big|_2^\infty dy = \int_2^\infty e^{-3y}dy$
$= -\frac{1}{3}e^{-3y}\Big|_2^\infty = \frac{1}{3e^6}$.

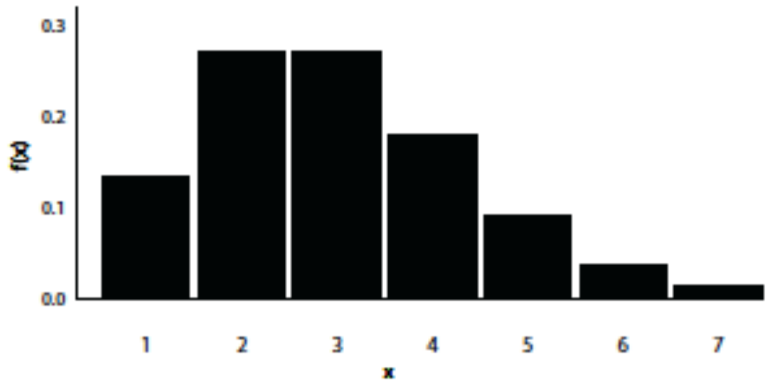
3.64 (a) $P\left(X \leq \frac{1}{2}, Y \leq \frac{1}{2}\right) = \frac{3}{2}\int_0^{1/2}\int_0^{1/2}(x^2+y^2)\ dydx = \frac{3}{2}\int_0^{1/2}\left(x^2y + \frac{y^3}{3}\right)\Big|_0^{1/2}\ dx$
$= \frac{3}{4}\int_0^{1/2}\left(x^2 + \frac{1}{12}\right)\ dx = \frac{1}{16}$.

(b) $P\left(X \geq \frac{3}{4}\right) = \frac{3}{2}\int_{3/4}^1\left(x^2 + \frac{1}{3}\right)\ dx = \frac{53}{128}$.

3.65 (a)

x	0	1	2	3	4	5	6
$f(x)$	0.1353	0.2707	0.2707	0.1804	0.0902	0.0361	0.0120

(b) A histogram is shown next.



(c)

x	0	1	2	3	4	5	6
$F(x)$	0.1353	0.4060	0.6767	0.8571	0.9473	0.9834	0.9954

3.66 (a) $g(x) = \int_0^1 (x+y)\,dy = x + \frac{1}{2}$, for $0 < x < 1$, and $h(y) = y + \frac{1}{2}$ for $0 < y < 1$.

(b) $P(X > 0.5, Y > 0.5) = \int_{0.5}^1 \int_{0.5}^1 (x+y)\,dx\,dy = \int_{0.5}^1 \left(\frac{x^2}{2} + xy\right)\Big|_{0.5}^1\,dy$
$= \int_{0.5}^1 \left[\left(\frac{1}{2} + y\right) - \left(\frac{1}{8} + \frac{y}{2}\right)\right]\,dy = \frac{3}{8}$.

3.67 $f(x) = \binom{5}{x}(0.1)^x(1-0.1)^{5-x}$, for $x = 0, 1, 2, 3, 4, 5$.

3.68 (a) $g(x) = \int_1^2 \left(\frac{3x-y}{9}\right)\,dy = \frac{3xy - y^2/2}{9}\Big|_1^2 = \frac{x}{3} - \frac{1}{6}$, for $1 < x < 3$, and
$h(y) = \int_1^3 \left(\frac{3x-y}{9}\right)\,dx = \frac{4}{3} - \frac{2}{9}y$, for $1 < y < 2$.

(b) No, since $g(x)h(y) \neq f(x, y)$.

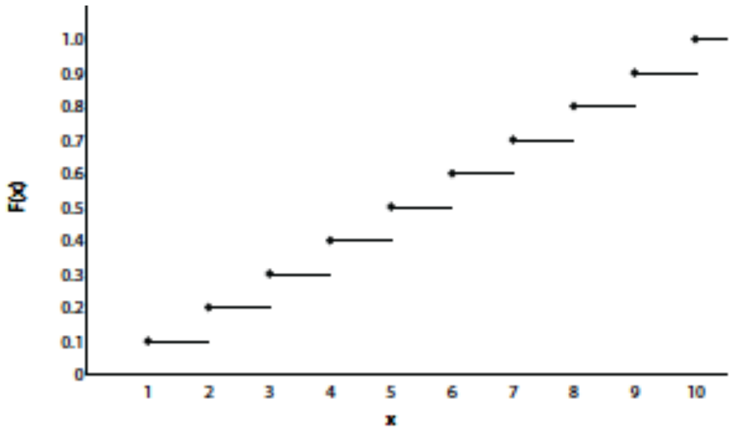
(c) $P(X > 2) = \int_2^3 \left(\frac{x}{3} - \frac{1}{6}\right)\,dx = \left(\frac{x^2}{6} - \frac{x}{6}\right)\Big|_2^3 = \frac{2}{3}$.

3.69 (a) $f(x) = \frac{d}{dx}F(x) = \frac{1}{50}e^{-x/50}$, for $x > 0$.

(b) $P(X > 70) = 1 - P(X \leq 70) = 1 - F(70) = 1 - (1 - e^{-70/50}) = 0.2466$.

3.70 (a) $f(x) = \frac{1}{10}$, for $x = 1, 2, \ldots, 10$.

(b) A c.d.f. plot is shown next.



3.71 $P(X \geq 3) = \frac{1}{2}\int_3^\infty e^{-y/2} = e^{-3/2} = 0.2231$.

3.72 (a) $f(x) \geq 0$ and $\int_0^{10} \frac{1}{10}\, dx = 1$. This is a continuous uniform distribution.

(b) $P(X \leq 7) = \frac{1}{10}\int_0^7\, dx = 0.7$.

3.73 (a) $f(y) \geq 0$ and $\int_0^1 f(y)\, dy = 10\int_0^1 (1-y)^9\, dy = -\frac{10}{10}(1-y)^{10}\Big|_0^1 = 1$.

(b) $P(Y > 0.6) = \int_{0.6}^1 f(y)\, dy = -(1-y)^{10}\Big|_{0.6}^1 = (1-0.6)^{10} = 0.0001$.

3.74 (a) $P(Z > 20) = \frac{1}{10}\int_{20}^\infty e^{-z/10}\, dz = -e^{-z/10}\Big|_{20}^\infty = e^{-20/10} = 0.1353$.

(b) $P(Z \leq 10) = -e^{-z/10}\Big|_0^{10} = 1 - e^{-10/10} = 0.6321$.

3.75 (a) $g(x_1) = \int_{x_1}^1 2\, dx_2 = 2(1-x_1)$, for $0 < x_1 < 1$.

(b) $h(x_2) = \int_0^{x_2} 2\, dx_1 = 2x_2$, for $0 < x_2 < 1$.

(c) $P(X_1 < 0.2, X_2 > 0,5) = \int_{0.5}^1\int_0^{0.2} 2\, dx_1\, dx_2 = 2(1-0.5)(0.2-0) = 0.2$.

(d) $f_{X_1|X_2}(x_1|x_2) = \frac{f(x_1,x_2)}{h(x_2)} = \frac{2}{2x_2} = \frac{1}{x_2}$, for $0 < x_1 < x_2$.

3.76 (a) $f_{X_1}(x_1) = \int_0^{x_1} 6x_2\, dx_2 = 3x_1^2$, for $0 < x_1 < 1$. Apparently, $f_{X_1}(x_1) \geq 0$ and $\int_0^1 f_{X_1}(x_1)\, dx_1 = \int_0^1 3x_1^2\, dx_1 = 1$. So, $f_{X_1}(x_1)$ is a density function.

(b) $f_{X_2|X_1}(x_2|x_1) = \frac{f(x_1,x_2)}{f_{X_1}(x_1)} = \frac{6x_2}{3x_1^2} = 2\frac{x_2}{x_1^2}$, for $0 < x_2 < x_1$.
So, $P(X_2 < 0.5 \mid X_1 = 0.7) = \frac{2}{0.7^2}\int_0^{0.5} x_2\, dx_2 = \frac{25}{49}$.

3.77 (a) $g(x) = \frac{9}{(16)4^y}\sum_{x=0}^\infty \frac{1}{4^x} = \frac{9}{(16)4^y}\frac{1}{1-1/4} = \frac{3}{4}\cdot\frac{1}{4^x}$, for $x = 0,1,2,\ldots$; similarly, $h(y) = \frac{3}{4}\cdot\frac{1}{4^y}$, for $y = 0,1,2,\ldots$. Since $f(x,y) = g(x)h(y)$, X and Y are independent.

(b) $P(X+Y < 4) = f(0,0)+f(0,1)+f(0,2)+f(0,3)+f(1,0)+f(1,1)+f(1,2)+f(2,0)+f(2,1)+f(3,0) = \frac{9}{16}\left(1+\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\frac{1}{4^2}+\frac{1}{4^3}+\frac{1}{4^3}\right) = \frac{9}{16}\left(1+\frac{2}{4}+\frac{3}{4^2}+\frac{4}{4^3}\right) = \frac{63}{64}$.

3.78 $P(\text{the system works}) = P(\text{all components work}) = (0.95)(0.99)(0.92) = 0.86526$.

3.79 $P(\text{the system does not fail}) = P(\text{at least one of the components works})$
$= 1 - P(\text{all components fail}) = 1 - (1-0.95)(1-0.94)(1-0.90)(1-0.97) = 0.999991.$

3.80 Denote by X the number of components (out of 5) work.
Then, $P(\text{the system is operational}) = P(X \geq 3) = P(X = 3) + P(X = 4) + P(X = 5) =$
$\binom{5}{3}(0.92)^3(1-0.92)^2 + \binom{5}{4}(0.92)^4(1-0.92) + \binom{5}{5}(0.92)^5 = 0.9955.$

Chapter 4

Mathematical Expectation

4.1 $\mu = E(X) = (0)(0.41) + (1)(0.37) + (2)(0.16) + (3)(0.05) + (4)(0.01) = 0.88.$

4.2 $E(X) = \sum_{x=0}^{3} x\, f(x) = (0)(27/64) + (1)(27/64) + (2)(9/64) + (3)(1/64) = 3/4.$

4.3 $\mu = E(X) = (20)(1/5) + (25)(3/5) + (30)(1/5) = 25$ cents.

4.4 Assigning wrights of $3w$ and w for a head and tail, respectively. We obtain $P(H) = 3/4$ and $P(T) = 1/4$. The sample space for the experiment is $S = \{HH, HT, TH, TT\}$. Now if X represents the number of tails that occur in two tosses of the coin, we have

$$P(X = 0) = P(HH) = (3/4)(3/4) = 9/16,$$
$$P(X = 1) = P(HT) + P(TH) = (2)(3/4)(1/4) = 3/8,$$
$$P(X = 2) = P(TT) = (1/4)(1/4) = 1/16.$$

The probability distribution for X is then

x	0	1	2
$f(x)$	9/16	3/8	1/16

from which we get $\mu = E(X) = (0)(9/16) + (1)(3/8) + (2)(1/16) = 1/2.$

4.5 Let c = amount to play the game and Y = amount won.

y	$5-c$	$3-c$	$-c$
$f(y)$	2/13	2/13	9/13

$E(Y) = (5-c)(2/13) + (3-c)(2/13) + (-c)(9/13) = 0$. So, $13c = 16$ which implies $c = \$1.23$.

4.6 $\mu = E(X) = (\$7)(1/12) + (\$9)(1/12) + (\$11)(1/4) + (\$13)(1/4) + (\$15)(1/6) + (\$17)(1/6)$
$= \$12.67.$

4.7 Expected gain $= E(X) = (4000)(0.3) + (-1000)(0.7) = \$500.$

4.8 Let X = profit. Then

$$\mu = E(X) = (250)(0.22) + (150)(0.36) + (0)(0.28) + (-150)(0.14) = \$88.$$

4.9 For the insurance of \$200,000 pilot, the distribution of the claim the insurance company would have is as follows:

Claim Amount	\$200,000	\$100,000	\$50,000	0
$f(x)$	0.002	0.01	0.1	0.888

So, the expected claim would be

$$(\$200,000)(0.002) + (\$100,000)(0.01) + (\$50,000)(0.1) + (\$0)(0.888) = \$6,400.$$

Hence the insurance company should charge a premium of $\$6,400 + \$500 = \$6,900$.

4.10 $\mu_X = \sum xg(x) = (1)(0.17) + (2)(0.5) + (3)(0.33) = 2.16$,
$\mu_Y = \sum yh(y) = (1)(0.23) + (2)(0.5) + (3)(0.27) = 2.04$.

4.11 $E(X) = \frac{4}{\pi}\int_0^1 \frac{x}{1+x^2}\,dx = \frac{\ln 4}{\pi}$.

4.12 $E(X) = \int_0^1 2x(1-x)\,dx = 1/3$. So, $(1/3)(\$5,000) = \$1,667.67$.

4.13 $E(X) = \int_0^1 x^2\,dx + \int_1^2 x(2-x)\,dx = 1$. Therefore, the average number of hours per year is $(1)(100) = 100$ hours.

4.14 $E(X) = \int_0^1 \frac{2x(x+2)}{5}\,dx = \frac{8}{15}$.

4.15 $E(X) = \frac{1}{\pi a^2}\int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} x\,dx\,dy = \frac{1}{\pi a^2}\left[\left(\frac{a^2-y^2}{2}\right) - \left(\frac{a^2-y^2}{2}\right)\right]\,dy = 0$.

4.16 $P(X_1 + X_2 \geq 1) = 1 - P(X_1 = 0, X_2 = 0)$
$= 1 - \frac{\binom{980}{2}\binom{20}{0}}{\binom{1000}{2}} = 1 - 0.9604 = 0.040$.

4.17 The probability density function is,

x	-3	6	9
$f(x)$	1/6	1/2	1/3
$g(x)$	25	169	361

$\mu_{g(X)} = E[(2X+1)^2] = (25)(1/6) + (169)(1/2) + (361)(1/3) = 209$.

4.18 $E(X^2) = (0)(27/64) + (1)(27/64) + (4)(9/64) + (9)(1/64) = 9/8$.

4.19 Let $Y = 1200X - 50X^2$ be the amount spent.

x	0	1	2	3
$f(x)$	1/10	3/10	2/5	1/5
$y = g(x)$	0	1150	2200	3150

$\mu_Y = E(1200X - 50X^2) = (0)(1/10) + (1150)(3/10) + (2200)(2/5) + (3150)(1/5)$
$= \$1,855.$

4.20 $E[g(X)] = E(e^{2X/3}) = \int_0^\infty e^{2x/3}e^{-x}\,dx = \int_0^\infty e^{-x/3}\,dx = 3.$

4.21 $E(X^2) = \int_0^1 2x^2(1-x)\,dx = \frac{1}{6}$. Therefore, the average profit per new automobile is $(1/6)(\$5000.00) = \$833.33$.

4.22 $E(Y) = E(X+4) = \int_0^\infty 32(x+4)\frac{1}{(x+4)^3}\,dx = 8$ days.

4.23 (a) $E[g(X,Y)] = E(XY^2) = \sum_x \sum_y xy^2 f(x,y)$
$= (2)(1)^2(0.10) + (2)(3)^2(0.20) + (2)(5)^2(0.10) + (4)(1)^2(0.15) + (4)(3)^2(0.30)$
$+ (4)(5)^2(0.15) = 35.2.$

(b) $\mu_X = E(X) = (2)(0.40) + (4)(0.60) = 3.20,$
$\mu_Y = E(Y) = (1)(0.25) + (3)(0.50) + (5)(0.25) = 3.00.$

4.24 (a) $E(X^2Y - 2XY) = \sum_{x=0}^{3}\sum_{y=0}^{2}(x^2y - 2xy)f(x,y) = (1-2)(18/70) + (4-4)(18/70) + \cdots +$
$(8-8)(3/70) = -3/7.$

(b)

x	0	1	2	3
$g(x)$	5/70	30/70	30/70	5/70

y	0	1	2
$h(y)$	15/70	40/70	15/70

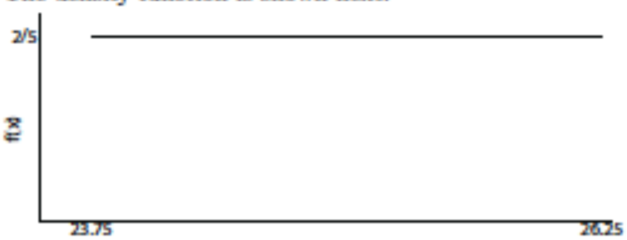
$\mu_X = E(X) = (0)(5/70) + (1)(30/70) + (2)(30/70) + (3)(5/70) = 3/2,$
$\mu_Y = E(Y) = (0)(15/70) + (1)(40/70) + (2)(15/70) = 1.$
Hence $\mu_X - \mu_Y = 3/2 - 1 = 1/2.$

4.25 $\mu_{X+Y} = E(X+Y) = \sum_{x=0}^{3}\sum_{y=0}^{3}(x+y)f(x,y) = (0+0)(1/55) + (1+0)(6/55) + \cdots + (0+3)(1/55) =$
$2.$

4.26 $E(Z) = E(\sqrt{X^2+Y^2}) = \int_0^1\int_0^1 4xy\sqrt{x^2+y^2}\,dx\,dy = \frac{4}{3}\int_0^1 [y(1+y^2)^{3/2} - y^4]\,dy$
$= 8(2^{3/2} - 1)/15 = 0.9752.$

4.27 $E(X) = \frac{1}{2000}\int_0^\infty x\exp(-x/2000)\,dx = 2000\int_0^\infty y\exp(-y)\,dy = 2000.$

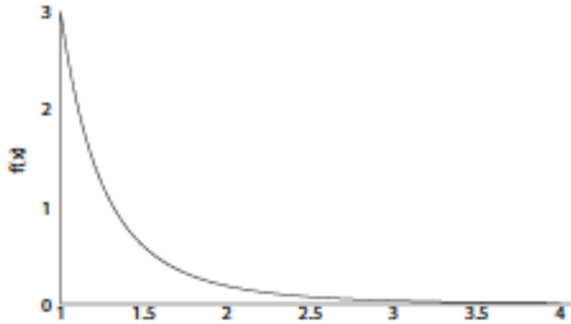
4.28 (a) The density function is shown next.



(b) $E(X) = \frac{2}{5}\int_{23.75}^{26.25} x\,dx = \frac{1}{5}(26.25^2 - 23.75^2) = 25.$

(c) The mean is exactly in the middle of the interval. This should not be surprised due to the symmetry of the density at 25.

4.29 (a) The density function is shown next



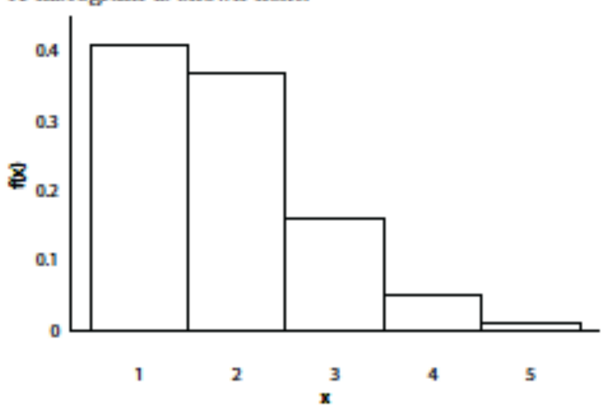
(b) $\mu = E(X) = \int_1^\infty 3x^{-3}\, dx = \frac{3}{2}$.

4.30 $E(Y) = \frac{1}{4}\int_0^\infty ye^{-y/4}\, dy = 4$.

4.31 (a) $\mu = E(Y) = 5\int_0^1 y(1-y)^4\, dy = -\int_0^1 y\, d(1-y)^5 = \int_0^\infty (1-y)^5\, dy = \frac{1}{6}$.

(b) $P(Y > 1/6) = \int_{1/6}^1 5(1-y)^4\, dy = -\left.(1-y)^5\right|_{1/6}^1 = (1-1/6)^5 = 0.4019$.

4.32 (a) A histogram is shown next.



(b) $\mu = (0)(0.41) + (1)(0.37) + (2)(0.16) + (3)(0.05) + (4)(0.01) = 0.88$.

(c) $E(X^2) = (0)^2(0.41) + (1)^2(0.37) + (2)^2(0.16) + (3)^2(0.05) + (4)^2(0.01) = 1.62$.

4.33 $\mu = \$500$. So, $\sigma^2 = E[(X-\mu)^2] = \sum_x (x-\mu)^2 f(x) = (-1500)^2(0.7) + (3500)^2(0.3) = \$5,250,000$.

4.34 $\mu = (-2)(0.3) + (3)(0.2) + (5)(0.5) = 2.5$ and
$E(X^2) = (-2)^2(0.3) + (3)^2(0.2) + (5)^2(0.5) = 15.5$.
So, $\sigma^2 = E(X^2) - \mu^2 = 9.25$ and $\sigma = 3.041$.

4.35 $\mu = (2)(0.01) + (3)(0.25) + (4)(0.4) + (5)(0.3) + (6)(0.04) = 4.11$,
$E(X^2) = (2)^2(0.01) + (3)^2(0.25) + (4)^2(0.4) + (5)^2(0.3) + (6)^2(0.04) = 17.63$.
So, $\sigma^2 = 17.63 - 4.11^2 = 0.74$.

4.36 $\mu = (0)(0.4) + (1)(0.3) + (2)(0.2) + (3)(0.1) = 1.0$,
and $E(X^2) = (0)^2(0.4) + (1)^2(0.3) + (2)^2(0.2) + (3)^2(0.1) = 2.0$.
So, $\sigma^2 = 2.0 - 1.0^2 = 1.0$.

4.37 It is know $\mu = 1/3$.
So, $E(X^2) = \int_0^1 2x^2(1-x)\,dx = 1/6$ and $\sigma^2 = 1/6 - (1/3)^2 = 1/18$. So, in the actual profit, the variance is $\frac{1}{18}(5000)^2$.

4.38 It is known $\mu = 8/15$.
Since $E(X^2) = \int_0^1 \frac{2}{5}x^2(x+2)\,dx = \frac{11}{30}$, then $\sigma^2 = 11/30 - (8/15)^2 = 37/450$.

4.39 It is known $\mu = 1$.
Since $E(X^2) = \int_0^1 x^2 \cdot x\,dx + \int_1^2 x^2(2-x)\,dx = 7/6$, then $\sigma^2 = 7/6 - (1)^2 = 1/6$.

4.40 $\mu_{g(X)} = E[g(X)] = \int_0^1 (3x^2+4)\left(\frac{2x+4}{5}\right)\,dx = \frac{1}{5}\int_0^1 (6x^3 + 12x^2 + 8x + 16)\,dx = 5.1$.
So, $\sigma^2 = E[g(X) - \mu]^2 = \int_0^1 (3x^2 + 4 - 5.1)^2\left(\frac{2x+4}{5}\right)\,dx$
$= \int_0^1 (9x^4 - 6.6x^2 + 1.21)\left(\frac{2x+4}{5}\right)\,dx = 0.83$.

4.41 It is known $\mu_{g(X)} = E[(2X+1)^2] = 209$. Hence
$\sigma^2_{g(X)} = \sum\limits_x [(2X+1)^2 - 209]^2 g(x)$
$= (25-209)^2(1/6) + (169-209)^2(1/2) + (361-209)^2(1/3) = 14,144$.
So, $\sigma_{g(X)} = \sqrt{14,144} = 118.9$.

4.42 It is known $\mu_{g(X)} = E(X^2) = 1/6$. Hence
$\sigma^2_{g(X)} = \int_0^1 2\left(x^2 - \frac{1}{6}\right)^2 (1-x)\,dx = 7/180$.

4.43 $\mu_Y = E(3X-2) = \frac{1}{4}\int_0^\infty (3x-2)e^{-x/4}\,dx = 10$. So
$\sigma^2_Y = E\{[(3X-2) - 10]^2\} = \frac{9}{4}\int_0^\infty (x-4)^2 e^{-x/4}\,dx = 144$.

4.44 $E(XY) = \sum\limits_x\sum\limits_y xyf(x,y) = (1)(1)(18/70) + (2)(1)(18/70)$
$+ (3)(1)(2/70) + (1)(2)(9/70) + (2)(2)(3/70) = 9/7$;
$\mu_X = \sum\limits_x\sum\limits_y xf(x,y) = (0)f(0,1) + (0)f(0,2) + (1)f(1,0) + \cdots + (3)f(3,1) = 3/2$,
and $\mu_Y = 1$.
So, $\sigma_{XY} = E(XY) - \mu_X\mu_Y = 9/7 - (3/2)(1) = -3/14$.

4.45 $\mu_X = \sum\limits_x xg(x) = 2.45$, $\mu_Y = \sum\limits_y yh(y) = 3.20$, and
$E(XY) = \sum\limits_x\sum\limits_x xyf(x,y) = (1)(0.05) + (2)(0.05) + (3)(0.10) + (2)(0.05)$
$+ (4)(0.10) + (6)(0.35) + (3)(0) + (6)(0.20) + (9)(0.10) = 7.85$.
So, $\sigma_{XY} = 7.85 - (2.45)(3.20) = 0.01$.

4.46 From previous exercise, $k = \left(\frac{3}{392}\right)10^{-4}$, and $g(x) = k\left(20x^2 + \frac{98000}{3}\right)$, with
$\mu_X = E(X) = \int_{30}^{50} xg(x)\,dx = k\int_{30}^{50}\left(20x^3 + \frac{98000}{3}x\right)\,dx = 40.8163$.
Similarly, $\mu_Y = 40.8163$. On the other hand,
$E(XY) = k\int_{30}^{50}\int_{30}^{50} xy(x^2+y^2)\,dy\,dx = 1665.3061$.
Hence, $\sigma_{XY} = E(XY) - \mu_X\mu_Y = 1665.3061 - (40.8163)^2 = -0.6642$.

4.47 $g(x) = \frac{2}{3}\int_0^1 (x+2y)\ dy = \frac{2}{3}(x+1)$, for $0 < x < 1$, so $\mu_X = \frac{2}{3}\int_0^1 x(x+1)\ dx = \frac{5}{9}$;
$h(y) = \frac{2}{3}\int_0^1 (x+2y)\ dx = \frac{2}{3}\left(\frac{1}{2}+2y\right)$, so $\mu_Y = \frac{2}{3}\int_0^1 y\left(\frac{1}{2}+2y\right)\ dy = \frac{11}{18}$; and
$E(XY) = \frac{2}{3}\int_0^1\int_0^1 xy(x+2y)\ dy\ dx = \frac{1}{3}$.
So, $\sigma_{XY} = E(XY) - \mu_X\mu_Y = \frac{1}{3} - \left(\frac{5}{9}\right)\left(\frac{11}{18}\right) = -0.0062$.

4.48 Since $\sigma_{XY} = Cov(a+bX, X) = b\sigma_X^2$ and $\sigma_Y^2 = b^2\sigma_X^2$, then
$\rho = \frac{\sigma_{XY}}{\sigma_X}\sigma_Y = \frac{b\sigma_X^2}{\sqrt{\sigma_X^2 b^2\sigma_X^2}} = \frac{b}{|b|} =$ sign of b.
Hence $\rho = 1$ if $b > 0$ and $\rho = -1$ if $b < 0$.

4.49 $E(X) = (0)(0.41) + (1)(0.37) + (2)(0.16) + (3)(0.05) + (4)(0.01) = 0.88$
and $E(X^2) = (0)^2(0.41) + (1)^2(0.37) + (2)^2(0.16) + (3)^2(0.05) + (4)^2(0.01) = 1.62$.
So, $Var(X) = 1.62 - 0.88^2 = 0.8456$ and $\sigma = \sqrt{0.8456} = 0.9196$.

4.50 $E(X) = 2\int_0^1 x(1-x)\ dx = 2\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = \frac{1}{3}$ and
$E(X^2) = 2\int_0^1 x^2(1-x)\ dx = 2\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 = \frac{1}{6}$. Hence,
$Var(X) = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{18}$, and $\sigma = \sqrt{1/18} = 0.2357$.

4.51 The joint and marginal probability mass functions are given in the following table.

		y			
		0	1	2	$f_X(x)$
	0	0	$\frac{1}{35}$	$\frac{3}{70}$	$\frac{1}{14}$
x	1	$\frac{3}{70}$	$\frac{9}{35}$	$\frac{9}{70}$	$\frac{3}{7}$
	2	$\frac{9}{70}$	$\frac{9}{35}$	$\frac{3}{70}$	$\frac{3}{7}$
	3	$\frac{3}{70}$	$\frac{1}{35}$	0	$\frac{1}{14}$
	$f_X(x)$	$\frac{3}{14}$	$\frac{4}{7}$	$\frac{3}{14}$	

Hence, $E(X) = \frac{3}{2}$, $E(Y) = 1$, $E(XY) = \frac{9}{7}$, $Var(X) = \frac{15}{28}$, and $Var(Y) = \frac{3}{7}$. Finally, $\rho_{XY} = -\frac{1}{\sqrt{5}}$.

4.52 Since $f_X(x) = 2(1-x)$, for $0 < x < 1$, and $f_Y(y) = 2y$, for $0 < y < 1$, we obtain $E(X) = \frac{1}{3}$, $E(Y) = \frac{2}{3}$, $Var(X) = Var(Y) = \frac{1}{18}$, and $E(XY) = \frac{1}{4}$. Hence, $\rho_{XY} = \frac{1}{2}$.

4.53 Previously we found $\mu = 4.11$ and $\sigma^2 = 0.74$, Therefore,
$\mu_{g(X)} = E(3X-2) = 3\mu - 2 = (3)(4.11) - 2 = 10.33$ and $\sigma_{g(X)} = 9\sigma^2 = 6.66$.

4.54 Previously we found $\mu = 1$ and $\sigma^2 = 1$. Therefore,
$\mu_{g(X)} = E(5X+3) = 5\mu + 3 = (5)(1) + 3 = 8$ and $\sigma_{g(X)} = 25\sigma^2 = 25$.

4.55 Let $X =$ number of cartons sold and $Y =$ profit.
We can write $Y = 1.65X + (0.90)(5-X) - 6 = 0.75X - 1.50$. Now
$E(X) = (0)(1/15) + (1)(2/15) + (2)(2/15) + (3)(3/15) + (4)(4/15) + (5)(3/15) = 46/15$,
and $E(Y) = (0.75)E(X) - 1.50 = (0.75)(46/15) - 1.50 = \$0.80$.

4.56 $\mu_X = E(X) = \frac{1}{4}\int_0^\infty xe^{-x/4}\ dx = 4$.
Therefore, $\mu_Y = E(3X-2) = 3E(X) - 2 = (3)(4) - 2 = 10$.
Since $E(X^2) = \frac{1}{4}\int_0^\infty x^2e^{-x/4}\ dx = 32$, therefore, $\sigma_X^2 = E(X^2) - \mu_X^2 = 32 - 16 = 16$.
Hence $\sigma_Y^2 = 9\sigma_X^2 = (9)(16) = 144$.

4.57 $E(X) = (-3)(1/6) + (6)(1/2) + (9)(1/3) = 11/2$,
$E(X^2) = (-3)^2(1/6) + (6)^2(1/2) + (9)^2(1/3) = 93/2$. So,
$E[(2X+1)^2] = 4E(X^2) + 4E(X) + 1 = (4)(93/2) + (4)(11/2) + 1 = 209$.

4.58 Since $E(X) = \int_0^1 x^2\,dx + \int_1^2 x(2-x)\,dx = 1$, and
$E(X^2) = \int_0^1 x^3 2\,dx + \int_1^2 x^2(2-x)\,dx = 7/6$, then

$$E(Y) = 60E(X^2) + 39E(X) = (60)(7/6) + (39)(1) = 109 \text{ kilowatt hours.}$$

4.59 The equations $E[(X-1)^2] = 10$ and $E[(X-2)^2] = 6$ may be written in the form:

$$E(X^2) - 2E(X) = 9, \qquad E(X^2) - 4E(X) = 2.$$

Solving these two equations simultaneously we obtain

$$E(X) = 7/2, \quad \text{and} \quad E(X^2) = 16.$$

Hence $\mu = 7/2$ and $\sigma^2 = 16 - (7/2)^2 = 15/4$.

4.60 $E(X) = (2)(0.40) + (4)(0.60) = 3.20$, and
$E(Y) = (1)(0.25) + (3)(0.50) + (5)(0.25) = 3$. So,

(a) $E(2X - 3Y) = 2E(X) - 3E(Y) = (2)(3.20) - (3)(3.00) = -2.60$.

(b) $E(XY) = E(X)E(Y) = (3.20)(3.00) = 9.60$.

4.61 $E(2XY^2 - X^2Y) = 2E(XY^2) - E(X^2Y)$. Now,
$E(XY^2) = \sum_{x=0}^{2}\sum_{y=0}^{2} xy^2 f(x,y) = (1)(1)^2(3/14) = 3/14$, and
$E(X^2Y) = \sum_{x=0}^{2}\sum_{y=0}^{2} x^2 y f(x,y) = (1)^2(1)(3/14) = 3/14$.
Therefore, $E(2XY^2 - X^2Y) = (2)(3/14) - (3/14) = 3/14$.

4.62 $\sigma_Z^2 = \sigma^2_{-2X+4Y-3} = 4\sigma_X^2 + 16\sigma_Y^2 = (4)(5) + (16)(3) = 68$.

4.63 $\sigma_Z^2 = \sigma^2_{-2X+4Y-3} = 4\sigma_X^2 + 16\sigma_Y^2 - 16\sigma_{XY} = (4)(5) + (16)(3) - (16)(1) = 52$.

4.64 $E(Z) = E(XY) = E(X)E(Y) = \int_0^1 \int_2^\infty 16xy(y/x^3)\,dx\,dy = 8/3$.

4.65 It is easy to see that the expectations of X and Y are both 3.5. So,

(a) $E(X+Y) = E(X) + E(Y) = 3.5 + 3.5 = 7.0$.

(b) $E(X-Y) = E(X) - E(Y) = 0$.

(c) $E(XY) = E(X)E(Y) = (3.5)(3.5) = 12.25$.

4.66 $\mu_X = \mu_Y = 3.5$. $\sigma_X^2 = \sigma_Y^2 = [(1)^2 + (2)^2 + \cdots + (6)^2](1/6) - (3.5)^2 = \frac{35}{12}$.

(a) $\sigma_{2X-Y} = 4\sigma_X^2 + \sigma_Y^2 = \frac{175}{12}$;

(b) $\sigma_{X+3Y-5} = \sigma_X^2 + 9\sigma_Y^2 = \frac{175}{6}$.

4.67 $E[g(X,Y)] = E(X/Y^3 + X^2Y) = E(X/Y^3) + E(X^2Y)$.
$E(X/Y^3) = \int_1^2 \int_0^1 \frac{2x(x+2y)}{7y^3}\, dx\, dy = \frac{2}{7}\int_1^2 \left(\frac{1}{3y^3} + \frac{1}{y^2}\right)\, dy = \frac{15}{84}$;
$E(X^2Y) = \int_1^2 \int_0^1 \frac{2x^2y(x+2y)}{7}\, dx\, dy = \frac{2}{7}\int_1^2 y\left(\frac{1}{4} + \frac{2y}{3}\right)\, dy = \frac{139}{252}$.
Hence, $E[g(X,Y)] = \frac{15}{84} + \frac{139}{252} = \frac{46}{63}$.

4.68 $P = I^2R$ with $R = 50$, $\mu_I = E(I) = 15$ and $\sigma_I^2 = Var(I) = 0.03$.
$E(P) = E(I^2R) = 50E(I^2) = 50[Var(I) + \mu_I^2] = 50(0.03 + 15^2) = 11251.5$. If we use the approximation formula, with $g(I) = I^2$, $g'(I) = 2I$ and $g''(I) = 2$, we obtain,

$$E(P) \approx 50\left[g(\mu_I) + 2\frac{\sigma_I^2}{2}\right] = 50(15^2 + 0.03) = 11251.5.$$

Since $Var[g(I)] \approx \left[\frac{\partial g(i)}{\partial i}\right]^2_{i=\mu_I} \sigma_I^2$, we obtain

$$Var(P) = 50^2 Var(I^2) = 50^2(2\mu_I)^2\sigma_I^2 = 50^2(30)^2(0.03) = 67500.$$

4.69 For $0 < a < 1$, since $g(a) = \sum_{x=0}^{\infty} a^x = \frac{1}{1-a}$, $g'(a) = \sum_{x=1}^{\infty} xa^{x-1} = \frac{1}{(1-a)^2}$ and
$g''(a) = \sum_{x=2}^{\infty} x(x-1)a^{x-2} = \frac{2}{(1-a)^3}$.

(a) $E(X) = (3/4)\sum_{x=1}^{\infty} x(1/4)^x = (3/4)(1/4)\sum_{x=1}^{\infty} x(1/4)^{x-1} = (3/16)[1/(1-1/4)^2]$
$= 1/3$, and $E(Y) = E(X) = 1/3$.
$E(X^2) - E(X) = E[X(X-1)] = (3/4)\sum_{x=2}^{\infty} x(x-1)(1/4)^x$
$= (3/4)(1/4)^2 \sum_{x=2}^{\infty} x(x-1)(1/4)^{x-2} = (3/4^3)[2/(1-1/4)^3] = 2/9$.
So, $Var(X) = E(X^2) - [E(X)]^2 = [E(X^2) - E(X)] + E(X) - [E(X)]^2$
$2/9 + 1/3 - (1/3)^2 = 4/9$, and $Var(Y) = 4/9$.

(b) $E(Z) = E(X) + E(Y) = (1/3) + (1/3) = 2/3$, and
$Var(Z) = Var(X+Y) = Var(X) + Var(Y) = (4/9) + (4/9) = 8/9$, since X and Y are independent (from Exercise 3.79).

4.70 (a) $g(x) = \frac{3}{2}\int_0^1 (x^2 + y^2)\, dy = \frac{1}{2}(3x^2 + 1)$ for $0 < x < 1$ and
$h(y) = \frac{1}{2}(3y^2 + 1)$ for $0 < y < 1$.
Since $f(x,y) \neq g(x)h(y)$, X and Y are not independent.

(b) $E(X+Y) = E(X) + E(Y) = 2E(X) = \int_0^1 x(3x^2+1)\, dx = 3/4 + 1/2 = 5/4$.
$E(XY) = \frac{3}{2}\int_0^1 \int_0^1 xy(x^2+y^2)\, dx\, dy = \frac{3}{2}\int_0^1 y\left(\frac{1}{4} + \frac{y^2}{2}\right)\, dy$
$= \frac{3}{2}\left[\left(\frac{1}{4}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\right] = \frac{3}{8}$.

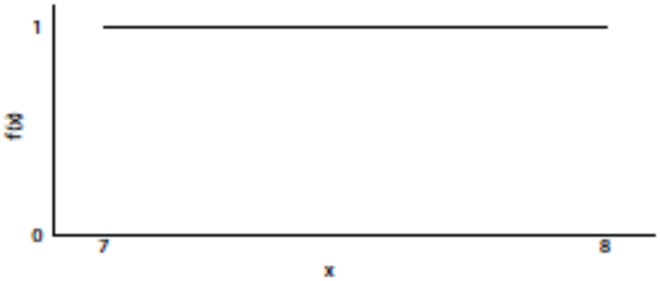
(c) $Var(X) = E(X^2) - [E(X)]^2 = \frac{1}{2}\int_0^1 x^2(3x^2+1)\, dx - \left(\frac{5}{8}\right)^2 = \frac{7}{15} - \frac{25}{64} = \frac{73}{960}$, and $Var(Y) = \frac{73}{960}$. Also, $Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{3}{8} - \left(\frac{5}{8}\right)^2 = -\frac{1}{64}$.

(d) $Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y) = 2\frac{73}{960} - 2\frac{1}{64} = \frac{29}{240}$.

4.71 (a) $E(Y) = \int_0^{\infty} ye^{-y/4}\, dy = 4$.

(b) $E(Y^2) = \int_0^\infty y^2 e^{-y/4}\, dy = 32$ and $Var(Y) = 32 - 4^2 = 16$.

4.72 (a) The density function is shown next.



(b) $E(Y) = \int_7^8 y\, dy = \frac{1}{2}[8^2 - 7^2] = \frac{15}{2} = 7.5$,
$E(Y^2) = \int_7^8 y^2\, dy = \frac{1}{3}[8^3 - 7^3] = \frac{169}{3}$, and $Var(Y) = \frac{169}{3} - \left(\frac{15}{2}\right)^2 = \frac{1}{12}$.

4.73 Using the exact formula, we have

$$E(e^Y) = \int_7^8 e^y\, dy = e^y|_7^8 = 1884.32.$$

Using the approximation, since $g(y) = e^y$, so $g''(y) = e^y$. Hence, using the approximation formula,

$$E(e^Y) \approx e^{\mu_Y} + e^{\mu_Y}\frac{\sigma_Y^2}{2} = \left(1 + \frac{1}{24}\right) e^{7.5} = 1883.38.$$

The approximation is very close to the true value.

4.74 Using the exact formula, $E(Z^2) = \int_7^8 e^{2y}\, dy = \frac{1}{2}\, e^{2y}|_7^8 = 3841753.12$. Hence,

$$Var(Z) = E(Z^2) - [E(Z)]^2 = 291091.3.$$

Using the approximation formula, we have

$$Var(e^Y) = (e^{\mu_Y})^2 Var(Y) = \frac{e^{(2)(7.5)}}{12} = 272418.11.$$

The approximation is not so close to each other. One reason is that the first order approximation may not always be good enough.

4.75 $\mu = 900$ hours and $\sigma = 50$ hours. Solving $\mu - k\sigma = 700$ we obtain $k = 4$.
So, using Chebyshev's theorem with $P(\mu - 4\sigma < X < \mu + 4\sigma) \geq 1 - 1/4^2 = 0.9375$, we obtain $P(700 < X < 1100) \geq 0.9375$. Therefore, $P(X \leq 700) \leq 0.03125$.

4.76 Using $\mu = 60$ and $\sigma = 6$ and Chebyshev's theorem

$$P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - \frac{1}{k^2},$$

since from $\mu + k\sigma = 84$ we obtain $k = 4$.
So, $P(X < 84) \geq P(36 < X < 84) \geq 1 - \frac{1}{4^2} = 0.9375$. Therefore,

$$P(X \geq 84) \leq 1 - 0.9375 = 0.0625.$$

Since $1000(0.0625) = 62.5$, we claim that at most 63 applicants would have a score as 84 or higher. Since there will be 70 positions, the applicant will have the job.

4.77 (a) $P(|X - 10| \geq 3) = 1 - P(|X - 10| < 3)$
$= 1 - P[10 - (3/2)(2) < X < 10 + (3/2)(2)] \leq 1 - \left[1 - \frac{1}{(3/2)^2}\right] = \frac{4}{9}$.

(b) $P(|X - 10| < 3) = 1 - P(|X - 10| \geq 3) \geq 1 - \frac{4}{9} = \frac{5}{9}$.

(c) $P(5 < X < 15) = P[10 - (5/2)(2) < X < 10 + (5/2)(2)] \geq 1 - \frac{1}{(5/2)^2} = \frac{21}{25}$.

(d) $P(|X - 10| \geq c) \leq 0.04$ implies that $P(|X - 10| < c) \geq 1 - 0.04 = 0.96$.
Solving $0.96 = 1 - \frac{1}{k^2}$ we obtain $k = 5$. So, $c = k\sigma = (5)(2) = 10$.

4.78 $\mu = E(X) = 6\int_0^1 x^2(1-x)\,dx = 0.5$, $E(X^2) = 6\int_0^1 x^3(1-x)\,dx = 0.3$, which imply $\sigma^2 = 0.3 - (0.5)^2 = 0.05$ and $\sigma = 0.2236$. Hence,

$$\begin{aligned}P(\mu - 2\sigma < X < \mu + 2\sigma) &= P(0.5 - 0.4472 < X < 0.5 + 0.4472)\\ &= P(0.0528 < X < 0.9472) = 6\int_{0.0528}^{0.9472} x(1-x)\,dx = 0.9839,\end{aligned}$$

compared to a probability of at least 0.75 given by Chebyshev's theorem.

4.79 Define $I_1 = \{x_i |\ |x_i - \mu| < k\sigma\}$ and $I_2 = \{x_i |\ |x_i - \mu| \geq k\sigma\}$. Then

$$\begin{aligned}\sigma^2 = E[(X-\mu)^2] &= \sum_x (x-\mu)^2 f(x) = \sum_{x_i \in I_1} (x_i - \mu)^2 f(x_i) + \sum_{x_i \in I_2} (x_i - \mu)^2 f(x_i)\\ &\geq \sum_{x_i \in I_2} (x_i - \mu)^2 f(x_i) \geq k^2\sigma^2 \sum_{x_i \in I_2} f(x_i) = k^2\sigma^2 P(|X - \mu| \geq k\sigma),\end{aligned}$$

which implies

$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}.$$

Hence, $P(|X - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}$.

4.80 $E(XY) = \int_0^1 \int_0^1 xy(x+y)\,dx\,dy = \frac{1}{3}$, $E(X) = \int_0^1\int_0^1 x(x+y)\,dx\,dy = \frac{7}{12}$ and $E(Y) = \frac{7}{12}$.
Therefore, $\sigma_{XY} = E(XY) - \mu_X\mu_Y = \frac{1}{3} - \left(\frac{7}{12}\right)^2 = -\frac{1}{144}$.

4.81 $E(Y - X) = \int_0^1 \int_0^y 2(y - x)\,dx\,dy = \int_0^1 y^2\,dy = \frac{1}{3}$. Therefore, the average amount of kerosene left in the tank at the end of each day is $(1/3)(1000) = 333$ liters.

4.82 (a) $E(X) = \int_0^\infty \frac{x}{5}e^{-x/5}\,dx = 5$.

(b) $E(X^2) = \int_0^\infty \frac{x^2}{5}e^{-x/5}\,dx = 50$, so $Var(X) = 50 - 5^2 = 25$, and $\sigma = 5$.

(c) $E[(X+5)^2] = E\{[(X-5)+10]^2\} = E[(X-5)^2] + 10^2 + 20E(X-5)$
$= Var(X) + 100 = 125$.

4.83 $E(XY) = 24\int_0^1 \int_0^{1-y} x^2y^2\, dx\, dy = 8\int_0^1 y^2(1-y)^3\, dy = \frac{2}{15}$,
$\mu_X = 24\int_0^1 \int_0^{1-y} x^2y\, dx\, dy = \frac{2}{5}$ and $\mu_Y = 24\int_0^1 \int_0^{1-y} xy^2\, dx\, dy = \frac{2}{5}$. Therefore, $\sigma_{XY} = E(XY) - \mu_X\mu_Y = \frac{2}{15} - \left(\frac{2}{5}\right)^2 = -\frac{2}{75}$.

4.84 $E(X+Y) = \int_0^1 \int_0^{1-y} 24(x+y)xy\, dx\, dy = \frac{4}{5}$.

4.85 (a) $E(X) = \int_0^\infty \frac{x}{900} e^{-x/900}\, dx = 900$ hours.

(b) $E(X^2) = \int_0^\infty \frac{x^2}{900} e^{-x/900}\, dx = 1620000$ hours^2.

(c) $Var(X) = E(X^2) - [E(X)]^2 = 810000$ hours^2 and $\sigma = 900$ hours.

4.86 It is known $g(x) = \frac{2}{3}(x+1)$, for $0 < x < 1$, and $h(y) = \frac{1}{3}(1+4y)$, for $0 < y < 1$.

(a) $\mu_X = \int_0^1 \frac{2}{3}x(x+1)\, dx = \frac{5}{9}$ and $\mu_Y = \int_0^1 \frac{1}{3}y(1+4y)\, dy = \frac{11}{18}$.

(b) $E[(X+Y)/2] = \frac{1}{2}[E(X) + E(Y)] = \frac{7}{12}$.

4.87 $Cov(aX, bY) = E[(aX - a\mu_X)(bY - b\mu_Y)] = abE[(X-\mu_X)(Y-\mu_Y)] = abCov(X,Y)$.

4.88 It is known $\mu = 900$ and $\sigma = 900$. For $k = 2$,

$$P(\mu - 2\sigma < X < \mu + 2\sigma) = P(-900 < X < 2700) \geq 0.75$$

using Chebyshev's theorem. On the other hand,

$$P(\mu - 2\sigma < X < \mu + 2\sigma) = P(-900 < X < 2700) = 1 - e^{-3} = 0.9502.$$

For $k = 3$, Chebyshev's theorem yields

$$P(\mu - 3\sigma < X < \mu + 3\sigma) = P(-1800 < X < 3600) \geq 0.8889,$$

while $P(-1800 < X < 3600) = 1 - e^{-4} = 0.9817$.

4.89 $g(x) = \int_0^1 \frac{16y}{x^3}\, dy = \frac{8}{x^3}$, for $x > 2$, with $\mu_X = \int_2^\infty \frac{8}{x^2}\, dx = -\left.\frac{8}{x}\right|_2^\infty = 4$,
$h(y) = \int_2^\infty \frac{16y}{x^3}\, dx = -\left.\frac{8y}{x^2}\right|_2^\infty = 2y$, for $0 < y < 1$, with $\mu_Y = \int_0^1 2y^2 = \frac{2}{3}$, and
$E(XY) = \int_2^\infty \int_0^1 \frac{16y^2}{x^2}\, dy\, dx = \frac{16}{3}\int_2^\infty \frac{1}{x^2}\, dx = \frac{8}{3}$. Hence,
$\sigma_{XY} = E(XY) - \mu_X\mu_Y = \frac{8}{3} - (4)\left(\frac{2}{3}\right) = 0$.

4.90 Since $\sigma_{XY} = 1$, $\sigma_X^2 = 5$ and $\sigma_Y^2 = 3$, we have $\rho = \frac{\sigma_{XY}}{\sigma_X\sigma_Y} = \frac{1}{\sqrt{(5)(3)}} = 0.2582$.

4.91 (a) From Exercise 4.37, we have $\sigma^2 = 1/18$, so $\sigma = 0.2357$.

(b) Also, $\mu_X = 1/3$ from Exercise 4.12. So,

$$\begin{aligned} P(\mu - 2\sigma < X < \mu + 2\sigma) &= P[1/3 - (2)(0.2357) < X < 1/3 + (2)(0.2357)] \\ &= P(0 < X < 0.8047) = \int_0^{0.8047} 2(1-x)\, dx = 0.9619. \end{aligned}$$

Using Chebyshev's theorem, the probability of this event should be larger than 0.75, which is true.

(c) $P(\text{profit} > \$500) = P(X > 0.1) = \int_{0.1}^1 2(1-x) = 0.81$.

4.92 Since $g(0)h(0) = (0.17)(0.23) \neq 0.10 = f(0,0)$, X and Y are not independent.

4.93 $E(X) = (-5000)(0.2) + (10000)(0.5) + (30000)(0.3) = \$13,000$.

4.94 (a) $f(x) = \binom{3}{x}(0.15)^x(0.85)^{3-x}$, for $x = 0, 1, 2, 3$.

x	0	1	2	3
$f(x)$	0.614125	0.325125	0.057375	0.003375

(b) $E(X) = 0.45$.

(c) $E(X^2) = 0.585$, so $Var(X) = 0.585 - 0.45^2 = 0.3825$.

(d) $P(X \leq 2) = 1 - P(X = 3) = 1 - 0.003375 = 0.996625$.

(e) 0.003375.

(f) Yes.

4.95 (a) $E(X) = (-\$15\text{k})(0.05) + (\$15\text{k})(0.15) + (\$25\text{k})(0.30) + (\$40\text{k})(0.15) + (\$50\text{k})(0.10) + (\$100\text{k})(0.05) + (\$150\text{k})(0.03) + (\$200\text{k})(0.02) = \$33.5\text{k}$.

(b) $E(X^2) = 2,697,500,000$ dollars^2. So, $\sigma = \sqrt{E(X^2) - [E(X)]^2} = \$39.689\text{k}$.

4.96 (a) $E(X) = \frac{3}{4 \times 50^3} \int_{-50}^{50} x(50^2 - x^2)\, dx = 0$.

(b) $E(X^2) = \frac{3}{4 \times 50^3} \int_{-50}^{50} x^2(50^2 - x^2)\, dx = 500$.

(c) $\sigma = \sqrt{E(X^2) - [E(X)]^2} = \sqrt{500 - 0} = 22.36$.

4.97 (a) The marginal density of X is

x_1	0	1	2	3	4
$f_{X_1}(x_1)$	0.13	0.21	0.31	0.23	0.12

(b) The marginal density of Y is

x_2	0	1	2	3	4
$f_{X_2}(x_2)$	0.10	0.30	0.39	0.15	0.06

(c) Given $X_2 = 3$, the conditional density function of X_1 is $f(x_1, 3)/0.15$. So

x_1	0	1	2	3	4
$f_{X_2}(x_2)$	$\frac{7}{15}$	$\frac{1}{5}$	$\frac{1}{15}$	$\frac{1}{5}$	$\frac{1}{15}$

(d) $E(X_1) = (0)(0.13) + (1)(0.21) + (2)(0.31) + (3)(0.23) + (4)(0.12) = 2$.

(e) $E(X_2) = (0)(0.10) + (1)(0.30) + (2)(0.39) + (3)(0.15) + (4)(0.06) = 1.77$.

(f) $E(X_1|X_2 = 3) = (0)\left(\frac{7}{15}\right) + (1)\left(\frac{1}{5}\right) + (2)\left(\frac{1}{15}\right) + (3)\left(\frac{1}{5}\right) + (4)\left(\frac{1}{15}\right) = \frac{18}{15} = \frac{6}{5} = 1.2$.

(g) $E(X_1^2) = (0)^2(0.13) + (1)^2(0.21) + (2)^2(0.31) + (3)^2(0.23) + (4)^2(0.12) = 5.44$.
So, $\sigma_{X_1} = \sqrt{E(X_1^2) - [E(X_1)]^2} = \sqrt{5.44 - 2^2} = \sqrt{1.44} = 1.2$.

4.98 (a) The marginal densities of X and Y are, respectively,

x	0	1	2
$g(x)$	0.2	0.32	0.48

y	0	1	2
$h(y)$	0.26	0.35	0.39

The conditional density of X given $Y = 2$ is

x	0	1	2
$f_{X\|Y=2}(x\|2)$	$\frac{4}{39}$	$\frac{5}{39}$	$\frac{30}{39}$

(b) $E(X) = (0)(0.2) + (1)(0.32) + (2)(0.48) = 1.28$,
$E(X^2) = (0)^2(0.2) + (1)^2(0.32) + (2)^2(0.48) = 2.24$, and
$Var(X) = 2.24 - 1.28^2 = 0.6016$.

(c) $E(X|Y = 2) = (1)\frac{5}{39} + (2)\frac{30}{39} = \frac{65}{39}$ and $E(X^2|Y = 2) = (1)^2\frac{5}{39} + (2)^2\frac{30}{39} = \frac{125}{39}$. So, $Var(X) = \frac{125}{39} - \left(\frac{65}{39}\right)^2 = \frac{650}{1521} = \frac{50}{117}$.

4.99 The profit is $8X + 3Y - 10$ for each trip. So, we need to calculate the average of this quantity. The marginal densities of X and Y are, respectively,

x	0	1	2
$g(x)$	0.34	0.32	0.34

y	0	1	2	3	4	5
$h(y)$	0.05	0.18	0.15	0.27	0.19	0.16

So, $E(8X + 3Y - 10) = (8)[(1)(0.32) + (2)(0.34)] + (3)[(1)(0.18) + (2)(0.15) + (3)(0.27) + (4)(0.19) + (5)(0.16)] - 10 = \$6.55$.

4.100 Using the approximation formula, $Var(Y) \approx \sum_{i=1}^{k} \left[\frac{\partial h(x_1, x_2, \dots, x_k)}{\partial x_i}\right]^2\Big|_{x_i=\mu_i,\ 1\le i\le k} \sigma_i^2$, we have

$$Var(\hat{Y}) \approx \sum_{i=0}^{2} \left(\frac{\partial e^{b_0+b_1k_1+b_2k_2}}{\partial b_i}\right)^2\Bigg|_{b_i=\beta_i,\ 0\le i\le 2} \sigma_{b_i}^2 = e^{2(\beta_0+k_1\beta_1+k_2\beta_2)}(\sigma_0^2 + k_1^2\sigma_1^2 + k_2^2\sigma_2^2).$$

4.101 (a) $E(Y) = 10\int_0^1 y(1-y)^9\,dy = -\,y(1-y)^{10}\big|_0^1 + \int_0^1 (1-y)^{10}\,dy = \frac{1}{11}$.

(b) $E(1 - Y) = 1 - E(Y) = \frac{10}{11}$.

(c) $Var(Z) = Var(1 - Y) = Var(Y) = E(Y^2) - [E(Y)]^2 = \frac{10}{11^2\times 12} = 0.006887$.