Continuous Random Variables

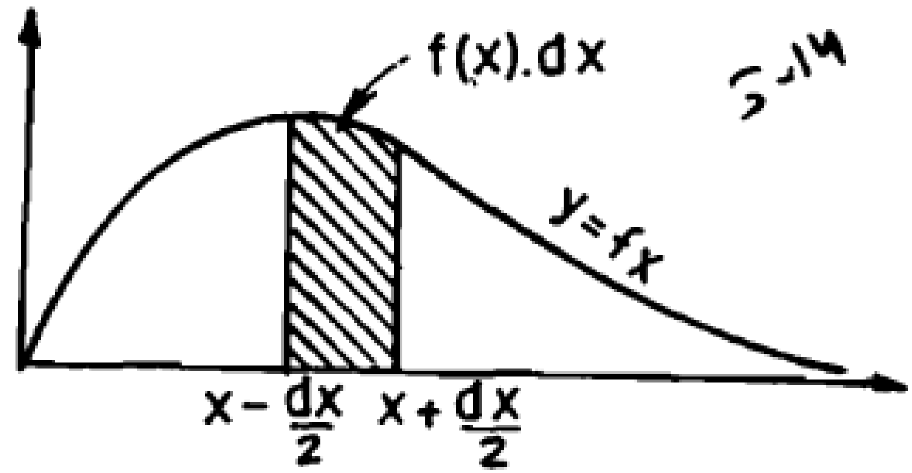
5·4. Continuous Random Variable. A random variable X is said to be continuous if it can take all possible values between certain limits. *In other words, a random variable is said to be continuous when its different values cannot be put in 1-1 correspondence with a set of positive integers.*

A continuous random variable is a random variable that (at least conceptually) can be measured to any desired degree of accuracy. Examples of continuous random variables are age, height, weight etc.

5·4·1. Probability Density Function *(Concept and Definition).* Consider the small interval $(x, x + dx)$ of length dx round the point x. Let $f(x)$ be any continuous function of x so that $f(x)\,dx$ represents the probability that X falls in the infinitesimal interval $(x, x + dx)$. Symbolically

$$P(x \le X \le x + dx) = f_X(x)\,dx \qquad \text{... (5·5)}$$

In the figure, $f(x)\,dx$ represents the area bounded by the curve $y = f(x)$, x–axis and the ordinates at the points x and $x + dx$. The function $f_X(x)$ so defined is known as *probability density function or simply density function of random variable X and is usually abbreviated as p.d.f.* The expression, $f(x)\,dx$, usually written as $dF(x)$, is known as the *probability differential* and the curve $y = f(x)$ is known as the *probability density curve* or simply *probability curve.*



Definition. p.d.f. $f_X(x)$ of the *r.v.* X is defined as :

$$f_X(x) = \lim_{\delta x \to 0} \frac{P(x \le X \le x + \delta x)}{\delta x} \qquad \text{... (5·5 a)}$$

The probability for a variate value to lie in the interval dx is $f(x)\,dx$ and hence the probability for a variate value to fall in the finite interval $[\alpha, \beta]$ is :

$$P(\alpha \le X \le \beta) = \int_{\alpha}^{\beta} f(x)\,dx \qquad \text{... (5·5 b)}$$

which represents the area between the curve $y = f(x)$, x–axis and the ordinates at $x = \alpha$ and $x = \beta$. Further since total probability is unity, we have $\int_a^b f(x)\,dx = 1$, where $[a, b]$ is the range of the random variable X. The range of the variable may be finite or infinite.

The probability density function (*p.d.f.*) of a random variable (*r.v.*) X usually denoted by $f_X(x)$ or simply by $f(x)$ has the following obvious properties

(i) $f(x) \ge 0, \; -\infty < x < \infty$ ... (5·5 *c*)

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$ ... (5·5 *d*)

(iii) The probability $P(E)$ given by

$$P(E) = \int_E f(x)\,dx \qquad \text{... (5·5 e)}$$

is well defined for any event E.

Important Remark. In case of discrete random variable, the probability at a point, *i.e.*, $P(x = c)$ is not zero for some fixed c. However, in case of continuous random variables the probability at a point is always zero, *i.e.*, $P(x = c) = 0$ for all possible values of c. This follows directly from (5·5 *b*) by taking $\alpha = \beta = c$.

This also agrees with our discussion earlier that $P(E) = 0$ does not imply that the event E is null or impossible event. This property of continuous r.v., *viz.*,

$$P(X = c) = 0, \ \forall\ c \qquad \text{... (5·5 } f\text{)}$$

leads us to the following important result :

$$P(\alpha \le X \le \beta) = P(\alpha \le X < \beta) = P(\alpha < X \le \beta) = P(\alpha < X < \beta) \qquad \text{... (5·5 } g\text{)}$$

i.e., in case of continuous r.v., it does matter whether we include the end points of the interval from α to β.

However, this result is in general not true for discrete random variables.

Example 5·3. *The diameter of an electric cable, say X, is assumed to be a continuous random variable with p.d.f.* $f(x) = 6x(1-x),\ 0 \le x \le 1$.

(i) *Check that above is p.d.f.,*

(ii) *Determine a number b such that* $P(X < b) = P(X > b)$

Solution. Obviously, for $0 \le x \le 1$, $f(x) \ge 0$

Now $$\int_0^1 f(x)\,dx = 6\int_0^1 x(1-x)\,dx$$

$$= 6\int_0^1 (x - x^2)\,dx = 6\left|\frac{x^2}{2} - \frac{x^3}{3}\right|_0^1 = 1$$

Hence $f(x)$ is the *p.d.f.* of *r.v.* X

(ii) $$P(X < b) = P(X > b) \qquad ...(*)$$

$$\Rightarrow \qquad \int_0^b f(x)\,dx = \int_b^1 f(x)\,dx$$

$$\Rightarrow \qquad 6\int_0^b x(1-x)\,dx = 6\int_b^1 x(1-x)\,dx$$

$$\Rightarrow \qquad \left|\frac{x^2}{2} - \frac{x^3}{3}\right|_0^b = \left|\frac{x^2}{2} - \frac{x^3}{3}\right|_b^1$$

$$\Rightarrow \qquad \left(\frac{b^2}{2} - \frac{b^3}{3}\right) = \left[\left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{b^2}{2} - \frac{b^3}{3}\right)\right]$$

$$\Rightarrow \qquad 3b^2 - 2b^3 = [\,1 - 3b^2 + 2b^3\,]$$

$$\Rightarrow \qquad 4b^3 - 6b^2 + 1 = 0$$

$$(2b - 1)(2b^2 - 2b - 1) = 0$$

$$\Rightarrow \qquad 2b - 1 = 0 \quad or \quad 2b^2 - 2b - 1 = 0$$

Hence $b = 1/2$ is the only real value lying between 0 and 1 and satisfying (*).

Example 5·4. *A continuous random variable X has a p.d.f.* $f(x) = 3x^2,\ 0 \le x \le 1$. *Find a and b such that*

(i) $P\{X \le a\} = P\{X > a\}$, and

(ii) $P\{X > b\} = 0{\cdot}05$.

Solution. *(i)* Since $P(X \leq a) = P(X > a)$,
each must be equal to 1/2, because total probability is always one.

$$\therefore \quad P(X \leq a) = \frac{1}{2} \quad \Rightarrow \quad \int_0^a f(x)\,dx = \frac{1}{2}$$

$$\Rightarrow \quad 3\int_0^a x^2\,dx = \frac{1}{2} \quad \Rightarrow \quad 3\left|\frac{x^3}{3}\right|_0^a = \frac{1}{2}$$

$$\Rightarrow \quad a^3 = \frac{1}{2} \quad \Rightarrow \quad a = \left(\frac{1}{2}\right)^{\frac{1}{3}}$$

$$(ii) \quad P(X > b) = 0{\cdot}05 \quad \Rightarrow \quad \int_b^1 f(x)\,dx = 0{\cdot}05$$

$$\Rightarrow \quad 3\left|\frac{x^3}{3}\right|_b^1 = \frac{1}{20} \quad \Rightarrow \quad 1 - b^3 = \frac{1}{20}$$

$$\Rightarrow \quad b^3 = \frac{19}{20} \quad \Rightarrow \quad b = \left(\frac{19}{20}\right)^{\frac{1}{3}}.$$

Example 5·5. *Let X be a continuous random variate with p.d.f.*

$$
\begin{aligned}
f(x) &= ax, \quad 0 \le x \le 1 \\
&= a, \quad 1 \le x \le 2 \\
&= -ax + 3a, \quad 2 \le x \le 3 \\
&= 0, \text{ elsewhere}
\end{aligned}
$$

(i) Determine the constant a.

(ii) Compute $P(X \le 1{\cdot}5)$.

Solution. *(i)* Constant 'a' is determined from the consideration that total probability is unity, *i.e.*,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{0}^{1} ax\,dx + \int_{1}^{2} a.dx + \int_{2}^{3} (-ax + 3a)\,dx = 1$$

$$\Rightarrow a\left|\frac{x^2}{2}\right|_{0}^{1} + a\left|x\right|_{1}^{2} + a\left|-\frac{x^2}{2} + 3x\right|_{2}^{3} = 1$$

$$\Rightarrow \frac{a}{2} + a + a\left[\left(-\frac{9}{2} + 9\right) - (-2 + 6)\right] = 1$$

$$\Rightarrow \frac{a}{2} + a + \frac{a}{2} = 1 \quad \Rightarrow \quad 2a = 1 \quad \Rightarrow \quad a = \frac{1}{2}$$

(ii) $P(X \le 1{\cdot}5) = \int_{-\infty}^{1\cdot5} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{1\cdot5} f(x)\,dx$

$$= a\int_{0}^{1} x\,dx + \int_{1}^{1\cdot5} a.dx$$

$$= a\left|\frac{x^2}{2}\right|_{0}^{1} + a\left|x\right|_{1}^{1\cdot5} = \frac{a}{2} + 0{\cdot}5\,a$$

$$= a = \frac{1}{2} \qquad \left[\because a = \tfrac{1}{2},\ \textit{Part (i)}\right]$$

Example 5·12. *The time one has to wait for a bus at a downtown bus stop is observed to be random phenomenon (X) with the following probability density function :*

$$
\begin{aligned}
f_X(x) &= 0, && \text{for } x < 0 \\
&= \tfrac{1}{9}(x+1), && \text{for } 0 \le x < 1 \\
&= \tfrac{4}{9}\left(x - \tfrac{1}{2}\right), && \text{for } 1 \le x < \tfrac{3}{2} \\
&= \tfrac{4}{9}\left(\tfrac{5}{2} - x\right), && \text{for } \tfrac{3}{2} \le x < 2 \\
&= \tfrac{1}{9}(4 - x), && \text{for } 2 \le x < 3 \\
&= \tfrac{1}{9}, && \text{for } 3 \le x < 6 \\
&= 0, && \text{for } 6 \le x,
\end{aligned}
$$

Let the events A and B be defined as follows :

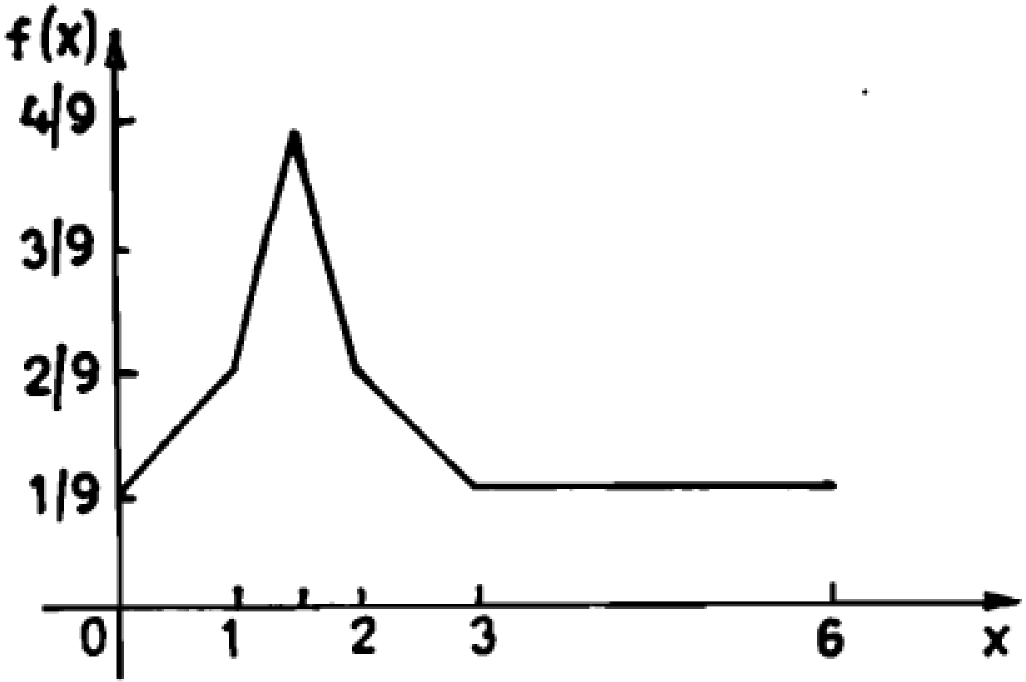
A : One waits between 0 to 2 minutes inclusive:

B : One waits between 0 to 3 minutes inclusive.

(i) Draw the graph of probability density function.

(ii) Show that (a) $P(B \mid A) = \dfrac{2}{3}$, *(b)* $P(\bar{A} \cap \bar{B}) = \dfrac{1}{3}$

Solution. *(i)* The graph of p.d.f. is given below.



(ii) *(a)* $P(A) = \int_0^2 f(x)\,dx = \int_0^1 \frac{1}{9}(x+1)\,dx + \int_1^{3/2} \frac{4}{9}\left(x - \frac{1}{2}\right)dx$

$$+ \int_{3/2}^{2} \frac{4}{9}\left(\frac{5}{2} - x\right)dx$$

$= \frac{1}{2}$ (on simplification)

$$P(A \cap B) = P(1 \le X \le 2) = \int_1^2 f(x)\,dx$$

$$= \int_1^{3/2} \frac{4}{9}\left(x - \frac{1}{2}\right)dx + \int_{3/2}^2 \frac{4}{9}\left(\frac{5}{2} - x\right)dx$$

$$= \frac{4}{9}\left[\frac{x^2}{2} - \frac{x}{2}\right]_1^{3/2} + \frac{4}{9}\left[\frac{5}{2}x - \frac{x^2}{2}\right]_{3/2}^2 = \frac{1}{3}$$

(on simplification)

$$\therefore \; P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{1/3}{1/2} = \frac{2}{3}$$

(b) $\bar{A} \cap \bar{B}$ means that waiting time is more than 3 minutes.

$$\therefore \; P(\bar{A} \cap \bar{B}) = P(X > 3) = \int_3^\infty f(x)\,dx = \int_3^6 f(x)\,dx + \int_6^\infty f(x)\,dx$$

$$= \int_3^6 \frac{1}{9}\,dx = \frac{1}{9}\left| x \right|_3^6 = \frac{1}{3}$$

Example 5·13. *The amount of bread (in hundreds of pounds) X that a certain bakery is able to sell in a day is found to be a numerical valued random phenomenon, with a probability function specified by the probability density function $f(x)$, given by*

$$f(x) = A \cdot x, \quad \text{for } 0 \le x < 5$$
$$= A(10 - x), \quad \text{for } 5 \le x < 10$$
$$= 0, \quad \textit{otherwise}$$

(a) Find the value of A such that $f(x)$ is a probability density function.

(b) What is the probability that the number of pounds of bread that will be sold tomorrow is

(i) more than 500 pounds,

(ii) less than 500 pounds,

(iii) between 250 and 750 pounds?

(c) Denoting by A, B, C the events that the pounds of bread sold are as in b (i), b (ii) and b (iii) respectively, find $P(A \mid B)$, $P(A \mid C)$. Are (i) A and B independent events? (ii) Are A and C independent events?

Solution. *(a)* In order that $f(x)$ should be a probability density function

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

i.e.,
$$\int_0^5 A\, x\, dx + \int_5^{10} A\,(10 - x)\, dx = 1$$

$$\Rightarrow \qquad A = \frac{1}{25} \qquad \text{(On simplification)}$$

(b) (i) The probability that the number of pounds of bread that will be sold tomorrow is more than 500 pounds, *i.e.,*

$$P(5 \le X \le 10) = \int_5^{10} \frac{1}{25}(10 - x)\, dx = \frac{1}{25}\left| 10x - \frac{x^2}{2} \right|_5^{10}$$

$$= \frac{1}{25}\left(\frac{25}{2}\right) = \frac{1}{2} = 0{\cdot}5$$

(ii) The probability that the number of pounds of bread that will be sold tomorrow is less than 500 pounds, *i.e.,*

$$P(0 \le X \le 5) = \int_0^5 \frac{1}{25} \cdot x\, dx = \frac{1}{25}\left| \frac{x^2}{2} \right|_0^5 = \frac{1}{2} = 0{\cdot}5$$

(iii) The required probability is given by

$$P(2{\cdot}5 \le X \le 7{\cdot}5) = \int_{2\cdot5}^{5} \frac{1}{25}\, x\, dx + \int_{5}^{7\cdot5} \frac{1}{25}\,(10-x)\, dx = \frac{3}{4}$$

(c) The events A, B and C are given by

$$A : 5 < X \le 10\,; \quad B : 0 \le X < 5\,; \quad C : 2{\cdot}5 < X < 7{\cdot}5$$

Then from parts *b (i)*, *(ii)* and *(iii)*, we have

$$P(A) = 0{\cdot}5\,, \quad P(B) = 0{\cdot}5\,, \quad P(C) = \frac{3}{4}$$

The events $A \cap B$ and $A \cap C$ are given by

$$A \cap B = \phi \quad \text{and} \quad A \cap C \;:\; 5 < X < 7{\cdot}5$$

$$\therefore \qquad P(A \cap B) = P(\phi) = 0$$

and $$P(A \cap C) = \int_{5}^{7\cdot5} f(x)\, dx = \frac{1}{25}\int_{5}^{7\cdot5} (10-x)\, dx$$

$$= \frac{1}{25} \times \frac{75}{8} = \frac{3}{8}$$

$$P(A)\,.\,P(C) = \frac{1}{2} \times \frac{3}{4} = \frac{3}{8} = P(A \cap C)$$

$\Rightarrow$ A and C are independent.

Again $P(A)\,.\,P(B) = \dfrac{1}{4} \ne P(A \cap B)$

$\Rightarrow$ A and B are not independent.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = 0$$

$$P(A \mid C) = \frac{P(A \cap C)}{P(C)} = \frac{3/8}{3/4} = \frac{1}{2}$$

Example 5·14. *The mileage C in thousands of miles which car owners get with a certain kind of tyre is a random variable having probability density function*

$$f(x) = \frac{1}{20}\, e^{-x/20}\,, \text{ for } x > 0$$
$$= 0\,, \text{ for } x \le 0$$

Find the probabilities that one of these tyres will last

(i) at most 10,000 miles,

(ii) anywhere from 16,000 to 24,000 miles.

(iii) at least 30,000 miles.

Solution. Let *r.v.* X denote the mileage (in '000 miles) with a certain kind of tyre. Then required probability is given by:

(i) $$P(X \le 10) = \int_0^{10} f(x)\, dx = \frac{1}{20} \int_0^{10} e^{-x/20}\, dx$$

$$= \frac{1}{20} \left| \frac{e^{-x/20}}{-1/20} \right|_0^{10} = 1 - e^{-1/2}$$

$$= 1 - 0{\cdot}6065 = 0{\cdot}3935$$

(ii) $$P(16 \le X \le 24) = \frac{1}{20} \int_{16}^{24} \exp\left(-\frac{x}{20}\right) dx = \left| -e^{-x/20} \right|_{16}^{24}$$

$$= e^{-16/20} - e^{-24/20} = e^{-4/5} - e^{-6/5}$$

$$= 0{\cdot}4493 - 0{\cdot}3012 = 0{\cdot}1481$$

(iii) $$P(X \ge 30) = \int_{30}^{\infty} f(x)\, dx = \frac{1}{20} \left| \frac{e^{-x/20}}{-1/20} \right|_{30}^{\infty}$$

$$= e^{-1.5} = 0{\cdot}2231$$

1. *(a)* A continuous random variable X follows the probability law

$$f(x) = A x^2, \; 0 \leq x \leq 1$$

Determine A and find the probability that *(i)* X lies between 0·2 and 0·5, *(ii)* X is less than 0·3, *(iii)* $1/4 < X < 1/2$ and *(iv)* $X > 3/4$ given $X > 1/2$.

Ans. $A = 0{\cdot}3$, *(i)* $0{\cdot}117$, *(ii)* $0{\cdot}027$, *(iii)* $15/256$ and *(iv)* $27/56$.

(*b*) If a random variable X has the density function

$$f(x) = \begin{cases} 1/4, & -2 < x < 2 \\ 0, & elsewhere. \end{cases}$$

Obtain (*i*) $P(X < 1)$, (*ii*) $P(|X| > 1)$ (*iii*) $P(2X + 3 > 5)$

Ans. *(i)* 3/4, *(ii)* 1/2 *(iii)* 1/4.

6. A continuous distribution of a variable X in the range $(-3, 3)$ is defined by

$$f(x) = \frac{1}{16}(3+x)^2, \quad -3 \le x \le -1$$
$$= \frac{1}{16}(6-2x^2), \quad -1 \le x \le 1$$
$$= \frac{1}{16}(3-x)^2, \quad 1 \le x \le 3$$

(i) Verify that the area under the curve is unity.

(ii) Find the mean and variance of the above distribution.

19. A random variable X has the p.d.f. :

$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \textit{otherwise} \end{cases}$$

Find (i) $P\left(X < \frac{1}{2}\right)$, (ii) $P\left(\frac{1}{4} < X < \frac{1}{2}\right)$, (iii) $P\left(X > \frac{3}{4} \,\middle|\, X > \frac{1}{2}\right)$, and (iv) $P\left(X < \frac{3}{4} \,\middle|\, X > \frac{1}{2}\right)$.

Ans. *(i)* 1/4 , *(ii)* 3/16 , *(iii)* $\dfrac{P(X > 3/4)}{P(X > 1/2)} = \dfrac{7/16}{3/4} = \dfrac{7}{12}$; *(iv)* $\dfrac{P(1/2 < X < 3/4)}{P(X > 1/2)}$