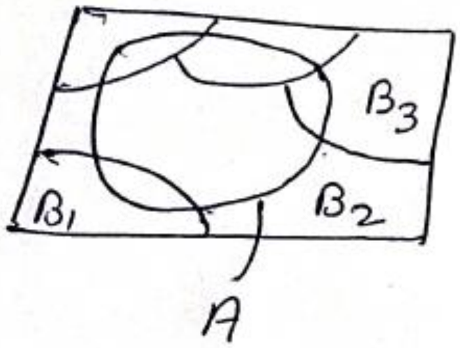
Theorem of Total probability and Baye's theorem

Theo: Let $B_1, B_2, \dots, B_n$ be a set of <u>exhaustive</u> and mutually exclusive events of the sample space S with $P(B_k) \neq 0$, $k = 1, 2, \dots, n$ and let A be any event of S. Then

$$P(A) = \sum_{i=1}^{n} P(B_i \cap A) = \sum_{i=1}^{n} P(B_i) P(A|B_i)$$

Proof:



Let $B_1, B_2, \ldots B_n$ are exhaustive and mutually exclusive events of S, we have

$$S = \bigcup_{i=1}^{n} B_i$$

Since $B_i \cap B_j = \phi$, $i \neq j$. Therefore

$A \cap B_1$, $A \cap B_2$, ---, $A \cap B_n$ are also mutually exclusive and

$$A = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n)$$

Hence $P(A) = P(A \cap B_1) + P(A \cap B_2) + \cdots + P(A \cap B_n)$

So, $$P(A) = \sum_{i=1}^{n} P(B_i \cap A) = \sum_{i=1}^{n} P(B_i) P(A|B_i)$$

Theo: (Baye's theorem): Let $B_1, B_2, \ldots, B_n$ be a set of of exhaustive and mutually exclusive events of the sample space S with $P(B_k) \neq 0$, $k = 1, 2, \ldots, n$ and A be any event of S with $P(A) \neq 0$. Then

$$P(B_k/A) = \frac{P(B_k)\,P(A/B_k)}{\sum_{i=1}^{n} P(B_k)\,P(A/B_k)}, \quad k = 1, 2, \ldots, n$$

Proof: $$P(B_k/A) = \frac{P(B_k \cap A)}{P(A)} \quad \text{— (1)}$$

From theorem of total probability

$$P(A) = \sum_{k=1}^{n} P(B_k)\, P(A|B_k) \quad \text{— (2)}$$

also $$P(B_k \cap A) = P(B_k)\, P(A|B_k) \quad \text{— (3)}$$

Thus $$P(B_k/A) = \frac{P(B_k)\, P(A|B_k)}{\sum_{k=1}^{n} P(B_k)\, P(A|B_k)}$$

Example 19.28 Two players A and B participate in a game of throwing two dice. The first player who gets a sum of 7 is awarded the prize. If A starts the game, find the probabilities of their winning.

Solution A sum of 7 is obtained if the numbers appearing on the dice are (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1).

Hence,
$$P(\text{sum is } 7) = \frac{6}{36} = \frac{1}{6} = p.$$

$$P(\text{not getting sum as } 7) = 1 - \frac{1}{6} = \frac{5}{6} = q.$$

Player A wins, if he gets a sum of 7 in the first or third or fifth, ..., throw. Player B wins, if he gets a sum of 7 in the second or fourth or sixth, ..., throw.

$$P(\text{sum of 7 in first throw}) = p = \frac{1}{6}$$

$$P(\text{sum of 7 in third throw}) = q^2 p = \left(\frac{5}{6}\right)^2 \left(\frac{1}{6}\right).$$

$$P(\text{sum of 7 in fifth throw}) = q^4 p = \left(\frac{5}{6}\right)^4 \left(\frac{1}{6}\right), \ldots .$$

$$P(A \text{ wins}) = p + q^2 p + q^4 p + \ldots$$

$$= p(1 + q^2 + q^4 + \ldots) = \frac{p}{1 - q^2} = \frac{1}{6}\left(\frac{36}{11}\right) = \frac{6}{11}.$$

$$P(B \text{ wins}) = 1 - P(A \text{ wins}) = \frac{5}{11}.$$

Example 19.29 A factory has four independent units A, B, C and D which produce 40%, 30%, 20%, and 10% of identical items, respectively. The percentages of defective items produced by these units are 2%, 1%, 0.5% and 0.25% respectively. If an item is selected at random, find the probability that the item is defective.

[illegible] of total probability, we obtain n Let the defective item be denoted by M. By the theorem of total probability, we obtain

$$P(\text{defective item}) = P(A)\,P(M/A) + P(B)\,P(M/B) + P(C)\,P(M/C) + P(D)\,P(M/D)$$

$$= 0.4(0.02) + 0.3(0.01) + 0.2(0.005) + 0.1(0.0025) = 0.01225.$$

Example 4·30. *In 1989 there were three candidates for the position of principal – Mr. Chatterji, Mr. Ayangar and Dr. Singh – whose chances of getting the appointment are in the proportion 4:2:3 respectively. The probability that Mr. Chatterji if selected would introduce co-education in the college is 0·3. The probabilities of Mr. Ayangar and Dr. Singh doing the same are respectively 0·5 and 0·8. What is the probability that there was co-education in the college in 1990?*

Solution. Let the events and probabilities be defined as follows:

A : Introduction of co-education

E_1 : Mr. Chatterji is selected as principal

E_2 : Mr. Ayangar is selected as principal

E_3 : Dr. Singh is selected as principal.

Then

$$P(E_1) = \frac{4}{9}, \quad P(E_2) = \frac{2}{9} \quad \text{and} \quad P(E_3) = \frac{3}{9}$$

$$P(A \mid E_1) = \frac{3}{10}, \quad P(A \mid E_2) = \frac{5}{10} \quad \text{and} \quad P(A \mid E_3) = \frac{8}{10}$$

$$\begin{aligned} \therefore \quad P(A) &= P[(A \cap E_1) \cup (A \cap E_2) \cup (A \cap E_3)] \\ &= P(A \cap E_1) + P(A \cap E_2) + P(A \cap E_3) \\ &= P(E_1)\,P(A \mid E_1) + P(E_2)\,P(A \mid E_2) + P(E_3)\,P(A \mid E_3) \\ &= \frac{4}{9} \cdot \frac{3}{10} + \frac{2}{9} \cdot \frac{5}{10} + \frac{3}{9} \cdot \frac{8}{10} = \frac{23}{45} \end{aligned}$$

Example 4·33. *In a bolt factory machines A, B and C manufacture respectively 25%, 35% and 40% of the total. Of their output 5, 4, 2 per cent are defective bolts. A bolt is drawn at random from the product and is found to be defective. What are the probabilities that it was manufactured by machines A, B and C?*

Solution. Let E_1, E_2 and E_3 denote the events that a bolt selected at random is manufactured by the machines A, B and C respectively and let E denote the event of its being defective. Then we have

$$P(E_1) = 0{\cdot}25,\ P(E_2) = 0{\cdot}35,\ P(E_3) = 0{\cdot}40$$

The probability of drawing a defective bolt manufactured by machine A is $P(E \mid E_1) = 0{\cdot}05$.

Similarly, we have

$$P(E \mid E_2) = 0{\cdot}04, \text{ and } P(E \mid E_3) = 0{\cdot}02$$

Hence the probability that a defective bolt selected at random is manufactured by machine A is given by

$$P(E_1 \mid E) = \frac{P(E_1)\,P(E \mid E_1)}{\sum_{i=1}^{3} P(E_i)\,P(E \mid E_i)}$$

$$= \frac{0{\cdot}25 \times 0{\cdot}05}{0{\cdot}25 \times 0{\cdot}05 + 0{\cdot}35 \times 0{\cdot}04 + 0{\cdot}40 \times 0{\cdot}02} = \frac{125}{345} = \frac{25}{69}$$

Similarly

$$P(E_2 \mid E) = \frac{0{\cdot}35 \times 0{\cdot}04}{0{\cdot}25 \times 0{\cdot}05 + 0{\cdot}35 \times 0{\cdot}04 + 0{\cdot}40 \times 0{\cdot}02} = \frac{140}{345} = \frac{28}{69}$$

and

$$P(E_3 \mid E) = 1 - [P(E_1 \mid E) + P(E_2 \mid E)] = 1 - \frac{25}{69} - \frac{28}{69} = \frac{16}{69}$$

This example illustrates one of the chief applications of Bayes Theorem.

Example 72. A letter is known to have come from either TATANAGAR or CALCUTTA. On the envelope, just two consecutive letters, TA, are visible. The probability that the letter has come from CALCUTTA is 4/11.

Solution Let E_1 denote the event that the letter from TATANAGAR and E_2 the event that the letter came from CALCUTTA. Let A denote the event that the two consecutive alphabets visible on the envelope are TA. We have $P(E_1) = 1/2$, $P(E_2) = 1/2$, $P(A \mid E_1) = 2/8 = 1/4$ and $P(A \mid E_2) = 1/7$. Therefore, by Bayes' theorem we have

$$P(E_2 \mid A) = \frac{P(E_2)\,P(A \mid E_2)}{P(E_1)\,P(A \mid E_1) + P(E_2)\,P(A \mid E_2)} = \frac{4}{11}$$