Unit 5 MTH302

Estimation Theeory

Unit 4

Estimators

Consider a r.v X with pdf $f(x, \theta)$. It is assumed to be known except for the value of some unknown parameter(s) θ which may take any value on a set Θ. So we write pdf in the form $f(x, \theta), \theta \in \Theta$. The set Θ, which is set of all possible values of θ is called parameter space. This gives rise to family of prob distributions $\{f(x, \theta), \theta \in \Theta\}$

eg if $X \sim N(\mu, \sigma^2)$ then $\Theta = \{(\mu, \sigma^2) : -\infty < \mu < \infty, 0 < \sigma < \infty\}$

Def: Any function of the random sample $x_1, x_2, \dots, x_n$ that are being observed, say $T_n(x_1, x_2, \dots, x_n)$ is called a statistic. Clearly a statistic is a random variable. If it is used to estimate an unknown parameter θ of the distribution, it is called an estimator. A particular value of the estimator, say $T_n(x_1, x_2, \dots, x_n)$ is called an estimate of θ.

Characteristics of Estimators

The following are some of the criteria that should be satisfied by a good estimator

i) Unbiasedness
ii) Consistency
iii) Efficiency
iv) Sufficiency

Unbiasedness

Def: An estimator $T_n = T(x_1, \ldots x_n)$ is said to be an unbiased estimator of $\gamma(\theta)$ if $E(T_n) = \gamma(\theta) \quad \forall \, \theta \in \Theta$

Remark: If $E(T_n) > \theta$, T_n is said to be +vely biased and if $E(T_n) < \theta$, it is said to be negatively biased, the amount of bias $b(\theta) = E(T_n) - \gamma(\theta)$, $\theta \in \Theta$

Eg 17.1. $x_1, x_2, \ldots, x_n$ is a random sample from a normal population $N(\mu, 1)$. Show that $t = \frac{1}{n}\sum_{i=1}^{n} x_i^2$ is an unbiased estimator of $\mu^2 + 1$.

We have to prove $E(t) = \mu^2 + 1$

$$\text{i.e } E\left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right) = \mu^2 + 1$$

$$\text{i.e } \frac{1}{n} E\left(\sum_{i=1}^{n} x_i^2\right) = \mu^2 + 1.$$

Now $V(x_i) = E(x_i^2) - [E(x_i)]^2$

and $E(x_i) = \mu \quad \forall \, i = 1, 2, \ldots n$

$V(x_i) = 1$

$$E(x_i^2) = V(x_i) + [E(x_i)]^2 \quad \forall \, i = 1, 2, \ldots n$$
$$= 1 + \mu^2.$$

$$\therefore \quad E\left(\sum_{i=1}^{n} x_i^2\right) = \sum_{i=1}^{n} E(x_i^2)$$
$$= \sum_{i=1}^{n} 1 + \mu^2 = n(1 + \mu^2)$$

$$\therefore \frac{1}{n} E\left(\sum_{i=1}^{n} x_i^2\right) = 1 + \mu^2.$$

$$\therefore E(t) = 1 + \mu^2.$$

Hence, t is an unbiased estimator of $1 + \mu^2$.

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Eg 17.2 If T is an unbiased estimator of θ, show that T^2 is a biased estimator for θ^2.

Given $E(T) = \theta$

(Here T is not given explicitly so it is inconsistent data $\therefore var(T) > 0$)

$$var(T) = E(T^2) - (E(T))^2$$
$$= E(T^2) - \theta^2$$
$$\Rightarrow E(T^2) = var(T) + \theta^2$$
$$> \theta^2 \quad (\text{as } var(T) > 0)$$

$\therefore T^2$ is a biased estimator of θ^2

Eg 17.3 Show that $\frac{[\Sigma x_i(\Sigma x_i - 1)]}{n(n-1)}$ is an unbiased estimate of θ^2, for the sample $x_1, x_2, \ldots x_n$ drawn on X which takes the values 1 or 0 with respective probabilities θ and $(1-\theta)$.

Since $x_1, x_2, \ldots, x_n$ is a random sample from Bernoulli population (that results in success (with prob p) or failure (with prob $q = 1-p$)) with parameter θ

$$\therefore T = \sum_{i=1}^{n} x_i \sim B(n, \theta)$$
$$\Rightarrow E(T) = n\theta$$
$$V(T) = n\theta(1-\theta)$$

Now $E\left(\frac{\Sigma x_i(\Sigma x_i - 1)}{n(n-1)}\right)$

$$= \frac{1}{n(n-1)} E(T(T-1))$$
$$= \frac{1}{n(n-1)}\left[E(T^2) - E(T)\right]$$

Unbiasedness

$$= \frac{1}{n(n-1)} \left[V(T) + (E(T))^2 - E(T) \right]$$

$$= \frac{1}{n(n-1)} \left[n\theta(1-\theta) + n^2\theta^2 - n\theta \right]$$

$$= \frac{\theta}{n-1} \left[1 - \theta + n\theta - 1 \right]$$

$$= \frac{\theta^2(n-1)}{n-1} = \theta^2$$

$\therefore \dfrac{\Sigma x_i (\Sigma x_i - 1)}{n(n-1)}$ is an unbiased estimator of θ^2.

Eg 17.4 Let X be distributed in the Poisson form with parameter θ. Show that only unbiased estimator of $\exp\{-(k+1)\theta\}, k>0$ is $T(X) = (-k)^X$ so that $T(x) > 0$ if x is even and $T(x) < 0$ if x is odd.

$$P(x) = \frac{e^{-\theta}\theta^x}{x!}$$

To show $E(T(X)) = e^{-(k+1)\theta}$

$$E((-k)^X),\ k>0 = \sum_{x=0}^{\infty} (-k)^x \frac{e^{-\theta}\theta^x}{x!}$$

$$= e^{-\theta} \sum_{x=0}^{\infty} \frac{(-\theta k)^x}{x!}$$

$1 - \frac{\theta k}{1!} + \frac{\theta^2 k^2}{2!}$

$$= e^{-\theta} e^{-\theta k}$$

$$= e^{-(1+k)\theta}$$

$\therefore T(X) = (-k)^X$ is an unbiased estimator of $e^{-(1+k)\theta}$, $k>0$.

Consistency

Def: An estimator $T_n = T(x_1, \dots x_n)$ based on a random sample of size n, is s.t.b consistent estimator of $\gamma(\theta)$, $\theta \in \Theta$, if T_n converges to $\gamma(\theta)$ in probability i.e. if $T_n \xrightarrow{P} \gamma(\theta)$ as $n \to \infty$.

Remark

1) Sample mean $\bar{X}_n$ is always a consistent estimator of the population mean μ.

as $\bar{X}_n = \frac{\sum_{i=1}^{n} X_i}{n} \to \mu = E(X_i)$ as $n \to \infty$

Eg 17.5. Prove that in sampling from a $N(\mu, \sigma^2)$ population, the sample mean is a consistent estimator for μ.

In $N(\mu, \sigma^2)$ population, the sample mean $\bar{x}$ is also normally distributed as $N(\mu, \sigma^2/n)$ i.e

$E(\bar{x}) = \mu$ and $V(\bar{x}) = \sigma^2/n$

$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$

$E(\bar{x}) = \frac{1}{n} \sum_{i=1}^{n} E(x_i)$

$= \frac{n\mu}{n} = \mu$

~~$E(\bar{x}^2) = \frac{1}{n^2} \sum E(x_i^2)$~~

$var(\bar{x}) = \frac{1}{n^2} \sum_{i=1} var(x_i)$

$= \frac{1}{n^2} \sum \sigma^2$

$= \frac{1}{n^2} \cdot n\sigma^2$

$= \frac{\sigma^2}{n}$

Thus as $n \to \infty$

$E(\bar{x}) = \mu$

$V(\bar{x}) = 0$

$\therefore \bar{x}$ is a consistent estimator for μ using the following theorem

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Theorem: Let $\{T_n\}$ be a seq of estimators such that for all $\theta \in \Theta$

i) $E_\theta(T_n) \to \gamma(\theta)$, $n \to \infty$

ii) $Var_\theta(T_n) \to 0$ as $n \to \infty$

Then T_n is a consistent estimator of $\gamma(\theta)$

This is sufficient condition for consistency.

Eg 17·6 If $X_1, X_2, \ldots, X_n$ are random observations on a Bernoulli variate X taking the value 1 with prob p and the value 0 with prob $(1-p)$ show that

$$\frac{\sum x_i}{n}\left(1 - \frac{\sum x_i}{n}\right) \text{ is a consistent estimator of } p(1-p)$$

Since X_i $i=1,\ldots n$ are are independent identically distributed (iid) Bernoulli variates with parameter p

$E(x_i) = 1 \cdot p + 0 \cdot (1-p) = p$

$E(x_i^2) = 1^2 p + 0^2 (1-p) = p$

$\therefore T = \sum_{i=1}^{n} x_i \sim B(n, p)$

$\Rightarrow E(T) = np$, $V(T) = np(1-p)$

$V(x_i) = p - p^2 = p(1-p)$

Now $\bar{X} = \dfrac{\sum_{i=1}^{n} x_i}{n}$, $\bar{X} = \dfrac{T}{n}$

$$E(\bar{X}) = E\left(\frac{T}{n}\right) = \frac{E(T)}{n} = \frac{np}{n} = p$$

$$V(\bar{X}) = V\left(\frac{T}{n}\right) = \frac{1}{n^2} V(T) = \frac{np(1-p)}{n^2}$$

$$= \frac{p(1-p)}{n}$$

$$\to 0 \text{ as } n \to \infty$$

$\because E(\bar{X}) = p$ & $V(\bar{X}) \to 0$ as $n \to \infty$

$\therefore \bar{X}$ is a consistent estimator of p

Now $\dfrac{\sum x_i}{n} - \left(\dfrac{\sum x_i}{n}\right)^2$

$\Rightarrow \bar{X} - (\bar{X})^2$

Here $\bar{X} - (\bar{X})^2$ is a (poly f^n) cts function of $\bar{X}$ so $\bar{X} - (\bar{X})^2 = \bar{X}(1-\bar{X})$ is consistent

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estimator of $p(1-p)$
using invariance property of consistent estimators i.e if T_n is a consistent estimator of $\gamma(\theta)$ and $\Psi(\gamma(\theta))$ is a cts f^n of $\gamma(\theta)$ then $\Psi(T_n)$ is a consistent estimator of $\Psi(\gamma(\theta))$

Efficient estimators

Now, there may exist two estimators which are both unbiased and consistent estimators of unknown parameter for eg: sample mean $\bar{x}$ and median are both unbiased and consistent estimators of μ when σ^2 is known for $N(\mu, \sigma^2)$ population. In this case, we look for a criterion which is based on variances of sampling distribution of estimators usually known as efficiency.

If, of the two consistent estimators T_1, T_2 of a certain parameter θ, we have $V(T_1) < V(T_2)$ $\forall n$ then T_1 is more efficient than T_2 for all sample sizes

Most efficient estimator is the one with least variance.

Efficient 'Estimators.

If, of the two consistent estimators T_1, T_2 of a certain parameter θ, we have

$$V(T_1) < V(T_2), \quad \text{for all } n \qquad \ldots(15{\cdot}11)$$

then T_1 is more efficient than T_2 for all samples sizes.

Efficiency *(Def.) If T_1 is the most efficient estimator with variance V_1 and T_2 is any other estimator with variance V_2, then the efficiency E of T_2 is defined as :*

$$E = \frac{V_1}{V_2} \qquad \ldots(15{\cdot}12)$$

Obviously, E cannot exceed unity.

Theorem : lot {T_} l..

<u>Eg 17.7</u> A random sample $x_1, x_2, \ldots, x_5$ of size 5 is drawn from a normal population with unknown mean μ. Consider the following estimators to estimate μ:

i) $t_1 = \dfrac{x_1 + x_2 + x_3 + x_4 + x_5}{5}$

ii) $t_2 = \dfrac{x_1 + x_2}{2} + x_3$

iii) $t_3 = \dfrac{2x_1 + x_2 + \lambda x_3}{3}$ where λ is such that t_3 is an unbiased estimator of μ. Find λ. Are t_1 and t_2 unbiased? State giving reasons, the estimator which is best among t_1, t_2 and t_3.

Here, we have $E(x_i) = \mu$
$V(x_i) = \sigma^2 \quad i = 1, \ldots 5$

Since t_3 is unbiased estimator of μ
$E(t_3) = \mu$.

$\therefore E\left(\dfrac{2x_1 + x_2 + \lambda x_3}{3}\right) = \mu$

$\Rightarrow \frac{2}{3}E(x_1) + \frac{1}{3}E(x_2) + \frac{\lambda}{3}E(x_3) = \mu$

$\Rightarrow \frac{2}{3}\mu + \frac{1}{3}\mu + \frac{\lambda}{3}\mu = \mu$

$\Rightarrow \mu + \frac{\lambda}{3}\mu = \mu$

$\Rightarrow \frac{\lambda}{3}\mu = 0$.

$\Rightarrow \lambda = 0 \qquad (\because \mu \neq 0)$ generally.

Now $E(t_1) = \frac{1}{5}\sum_{i=1}^{5} E(x_i) = \frac{1}{5} \times 5\mu = \mu$.

$\therefore t_1$ is unbiased estimator of μ.

$E(t_2) = \frac{1}{2}[E(x_1) + E(x_2)] + E(x_3)$
$= \frac{1}{2} \times 2\mu + \mu = 2\mu \neq \mu$

$\therefore t_2$ is biased estimator of μ.

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Now $V(t_1) = V\left(\frac{X_1 + X_2 + X_3 + X_4 + X_5}{5}\right)$

$= \frac{1}{25}[V(X_1) + V(X_2) + V(X_3) + V(X_4) + V(X_5)]$

$= \frac{1}{25} \times 5\sigma^2 = \frac{\sigma^2}{5}$

$V(t_2) = V\left(\frac{X_1 + X_2}{2} + X_3\right)$

$= \frac{1}{4}(V(X_1) + V(X_2)) + V(X_3)$

$= \frac{2\sigma^2}{4} + \sigma^2 = \frac{3}{2}\sigma^2$

$V(t_3) = V\left(\frac{2X_1 + X_2}{3}\right)$

$= \frac{1}{9}(4V(X_1) + V(X_2))$

$= \frac{1}{9}(4\sigma^2 + \sigma^2) = \frac{5}{9}\sigma^2$

Since $V(t_1)$ is least $\therefore$ t_1 is best estimator of μ.

<u>Eg 17.8</u> X_1, X_2 and X_3 is a random sample of size 3 from a population with mean value μ and variance σ^2. T_1, T_2, T_3 are the estimators used to estimate μ where

$T_1 = X_1 + X_2 - X_3$

$T_2 = 2X_1 + 3X_3 - 4X_2$

$T_3 = \frac{1}{3}(\lambda X_1 + X_2 + X_3)$

i) Are T_1 & T_2 unbiased estimators?

$E(T_1) = E(X_1 + X_2 - X_3)$

$= E(X_1) + E(X_2) - E(X_3) = \mu + \mu - \mu = \mu$

T_1 is unbiased estimator of μ.

$E(T_2) = 2E(X_1) + 3E(X_3) - 4E(X_2)$

$= 2\mu + 3\mu - 4\mu = \mu$.

$\Rightarrow T_2$ is unbiased estimator of μ.

ii) Find λ s.t T_3 is unbiased estimator for μ.

$E(T_3) = \mu \Rightarrow \frac{1}{3}\left[\lambda E(X_1) + E(X_2) + E(X_3)\right] = \mu$

$\Rightarrow \lambda\mu + \mu + \mu = 3\mu$

$\Rightarrow \lambda\mu = \mu$

$\Rightarrow \lambda = 1$

iii) With this value of λ, is T_3 a consistent estimator ?

Here $T_3 = \frac{1}{3}(X_1 + X_2 + X_3)$

$= \overline{X}$ sample mean.

sample mean i.e T_3 is consistent estimator of population mean μ

iv) Which is the best estimator ?

$V(T_1) = V(X_1) + V(X_2) + V(X_3)$

$= 3\sigma^2$

$V(T_2) = 4V(X_1) + 9V(X_3) + 16V(X_2)$

$= 4\sigma^2 + 9\sigma^2 + 16\sigma^2$

$= 29\sigma^2$.

$V(T_3) = \frac{1}{9}\left(V(X_1) + V(X_2) + V(X_3)\right)$

$= \frac{1}{9} \times 3\sigma^2 = \frac{\sigma^2}{3}$.

i.e. T_3 is best estimator of μ as var(T_3) is minimum.

Min variance unbiased estimator (MVUE)

If a statistic $T = T(x_1, .. x_n)$ based on sample of size n is such that

i) T is unbiased for $\gamma(\theta)$ $\forall \theta \in \Theta$

ii) It has smallest variance among all unbiased estimator of $\gamma(\theta)$ then T is called the min variance unbiased estimator of $\gamma(\theta)$

Minimum Variance Unbiased (M.V.U.) Estimators.

If a statistic $T = T(x_1, x_2, \ldots, x_n)$, based on sample of size n is such that :

(i) T is unbiased for $\gamma(\theta)$, for all $\theta \in \Theta$ and

(ii) It has the smallest variance among the class of all unbiased estimators of $\gamma(\theta)$,

then T is called the minimum variance unbiased estimator (MVUE) of $\gamma(\theta)$.

More precisely

T is MVUE of $\gamma(\theta)$ if

$$E_\theta(T) = \gamma(\theta) \quad \text{for all } \theta \in \Theta$$

$$\mathrm{Var}_\theta(T) \leq \mathrm{Var}_\theta(T') \quad \text{for all } \theta \in \Theta$$

where T' is any other unbiased estimator of $\gamma(\theta)$.

Sufficient estimator

Likelihood function or Joint density function.

Let $x_1, x_2, \ldots x_n$ be a random sample of size n from a population with density function $f(x, \theta)$ then the likelihood function of the sample values $x_1, \ldots x_n$ usually denoted by $L = L(\theta)$ is their joint density function given by

$$L = f(x_1, \theta)\, f(x_2, \theta) \ldots f(x_n, \theta)$$
$$= \prod_{i=1}^{n} f(x_i, \theta)$$

$$L = \prod_{i=1}^{n} f(x_i, \theta)$$

Condition for sufficient estimator

Factorization theorem :

If $T = t(x_1, x_2, \ldots, x_n)$ is an estimator of a parameter θ based on sample $x_1, x_2, \ldots, x_n$ of size n from the population with density $f(x, \theta)$ such that the conditional distribution of $x_1, x_2, \ldots, x_n$ given T, is independent of θ , then T is sufficient estimator for θ.

Factorization Theorem (Neymann)

It provides the necessary and sufficient condition for a distribution to be sufficient statistic.

Factorization Theorem (Neymann)

$T = t(x)$ is sufficient for θ if and only if the joint density function L of the sample values can be expressed as

$$L = g(T, \theta)\, h(x)$$

Where

$g(T, \theta)$ depends of θ and x only through value of $T = t(x)$

$h(x)$ is independent of θ

2- step approach to find the sufficient estimator

1. Define the likelihood function $L = \prod_{i=1}^{n} f(x_i, \theta)$

   Simplify it

2. Check whether it can be written in the form of $g(T, \theta)h(x)$

   where

   $g(T, \theta)$ depends of θ and x only through value of $T = t(x)$

   $h(x)$ is independent of θ

~~answers / options on rough sheet~~ ... to solve questions, otherwise it will be considered as UMC.
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Eg 17.15 Let $X_1, X_2, \ldots, X_n$ be a random sample from a population with pdf $f(x;\theta) = \theta x^{\theta-1}$, $0<x<1$, $\theta>0$

Show that $t_1 = \prod_{i=1}^{n} X_i$ is sufficient for θ.

OR

Example: Let $x_1, x_2, \ldots, x_n$ be a random sample from a population with pdf

$$f(x,\theta) = \theta x^{\theta-1}; 0 < x < 1\,; \theta > 0$$

Find the sufficient estimator for θ.

Solution: Define the likelihood function

$$L = \prod_{i=1}^{n} f(x_i, \theta)$$

$$= \prod_{i=1}^{n} \theta x_i^{\theta-1}$$

$$= \theta^n \left(\prod_{i=1}^{n} x_i\right)^{\theta-1}$$

$$= \theta^n \left(\prod_{i=1}^{n} x_i\right)^{\theta} \cdot \frac{1}{\prod_{i=1}^{n} x_i}$$

Which is of the form $g(T,\theta)h(x)$

Hence, by Factorization theorem,

we get

$\prod_{i=1}^{n} x_i$ is the sufficient estimator for θ.

Example: Let $x_1, x_2, \ldots, x_n$ be a random sample from exponential distribution

$$f(x,\theta) = \frac{1}{\theta} \cdot e^{-\frac{x}{\theta}}; 0 < x, \theta < \infty.$$

Find the sufficient estimator for θ.

Solution: Define the likelihood function

$$L = \prod_{i=1}^{n} f(x_i, \theta)$$

$$= \prod_{i=1}^{n} \frac{1}{\theta} \cdot e^{-\frac{x_i}{\theta}}$$

$$= \left(\frac{1}{\theta}\right)^n e^{-\frac{\sum x_i}{\theta}}$$

Which is of the form $g(T, \theta)h(x)$
Where $h(x) = 1$

Hence, by Factorization theorem, we get $\sum x_i$ is the sufficient estimator for θ.

Example: Let $x_1, x_2, \ldots, x_n$ be a random sample from Cauchy distribution

$$f(x,\theta) = \frac{1}{\pi} \cdot \frac{1}{1 + (x - \theta)^2} ; -\infty < x, \theta < \infty$$

Examine if there exists a sufficient statistic for θ.

Solution: Define the likelihood function

$$L = \prod_{i=1}^{n} f(x_i, \theta)$$

$$= \left(\frac{1}{\pi}\right)^n \prod_{i=1}^{n} \frac{1}{1 + (x_i - \theta)^2}$$

which cannot be written

in the form $g(T, \theta)h(x)$

Hence, by Factorization theorem,
there is no single statistic,
which is sufficient estimator for θ.

Example: Let $x_1, x_2, \dots, x_n$ be a random sample from $N(\mu, \sigma^2)$ population. Find sufficient estimators for μ and σ^2.

Solution: Define the likelihood function

$$L = \prod_{i=1}^{n} f(x_i, \theta)$$

$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2\right\}$$

$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \exp\left\{-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n} x_i^2 - 2\mu\sum_{i=1}^{n} x_i + n\mu^2\right)\right\}$$

Which is of the form $g(T, \theta)h(x)$

Where $h(x) = 1$

Here, $T = (t_1(x), t_2(x))$

$= (\sum x, \sum x^2)$

Thus,

$\sum x$ is the sufficient estimator for μ

And $\sum x^2$ is sufficient estimator for σ^2

Example: If $x_1, x_2, \dots, x_n$ are independent and follows Bernoulli distribution with parameter p. Find the sufficient estimator for p.

Solution. The p.m.f of the Bernoulli distribution with parameter p is

$$P(X = x) = p^x(1-p)^{1-x}\ ; \quad x = 0,1$$

The joint density function is

$$L = P(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n)$$
$$= P(X_1 = x_1)P(X_2 = x_2) \dots P(X_n = x_n)$$
$$= p^{x_1}(1-p)^{1-x_1}\, p^{x_2}(1-p)^{1-x_2} \dots .\, p^{x_n}(1-p)^{1-x_n}$$
$$= p^{x_1+x_2+\dots+x_n}(1-p)^{n-x_1-x_2-\dots-x_n}$$
$$= p^{\Sigma x_i}(1-p)^{n-\Sigma x_i}$$

Which is of form $g(T,p)h(x)$

Where $h(x) = 1$

Hence,

by Factorization theorem

$T = \sum x_i$ is the sufficient estimator for p.

Example: If $x_1, x_2, \dots, x_n$ are independent and follows Poisson distribution with parameter λ. Find the sufficient estimator for λ.

Solution. The p.m.f of the passion distribution with parameter λ is

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}\ ; \quad x = 0,1,2, \dots$$

The joint density function is

$$L = P(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n)$$
$$= P(X_1 = x_1)P(X_2 = x_2) \dots P(X_n = x_n)$$
$$= \frac{e^{-\lambda}\lambda^{x_1}}{x_1!} \cdot \frac{e^{-\lambda}\lambda^{x_2}}{x_2!} \dots \dots \frac{e^{-\lambda}\lambda^{x_n}}{x_n!}$$
$$= \frac{e^{-n\lambda}\lambda^{x_1+x_2+\dots+x_n}}{x_1!\,x_2! \dots x_n!}$$

Which is of form $g(T,\lambda)h(x)$

Where $g(T,\lambda) = e^{-n\lambda}\lambda^{x_1+x_2+\cdots+x_n}$

And $\lambda(x) = \frac{1}{x_1!x_2!\ldots x_n!}$

Hence, by Factorization theor

$T = \sum x_i$ is the sufficient

estimator for λ.

Example: If $x_1, x_2, \ldots, x_n$ are independent and distributed as $\Gamma(\alpha,\beta)$ where α,β are unknown parameter of a Gamma distribution then show that $(\prod x_i, \sum x_i)$ is a two-dimensional sufficient statistic for (α,β).

Solution: The pdf of the Gamma distribution is

$$f(x,\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} \quad ; -\infty < x < \infty$$

The likelihood function is

$$L = \prod_{i=1}^{n} f(x_i,\alpha,\beta)$$

$$= \left(\frac{1}{\Gamma(\alpha)\beta^\alpha}\right)^n \left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} e^{-\frac{\sum x}{\beta}}$$

Which is of form $g(T,\theta)h(x)$

Where $\lambda(x) = 1$

Hence, by Factorization theorem,

$T = (\prod x_i, \sum x_i)$ is the

sufficient estimator for (α, β).

Properties of Sufficient Estimator:

- A sufficient estimator is never unique in any sense.
- If $T_1 = t(x_1, x_2, \ldots, x_n)$ is a sufficient statistic and $g(x)$ is a one-to-one function, then $T_2 = g(T_1)$ is also a sufficient statistic.
- Sufficient estimator need not be an unbiased.
- Sufficient estimator always a consistent estimator.
- If a unique MLE of θ exists, it must be a function of sufficient estimator.

Maximum Likelihood Estimator (MLE)

For finding MLE for the population parameter θ, we have to maximize the likelihood function

$$L(x, \theta) = f(x_1, x_2, \ldots, x_n, \theta) = \prod_{i=1}^{n} f(x_i, \theta).$$

$\therefore$ we have

$$\frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{\partial (\log L)}{\partial \theta} = 0.$$

$$\frac{\partial^2 L}{\partial \theta^2} < 0 \Rightarrow \frac{\partial^2 (\log L)}{\partial \theta^2} < 0$$

Remark: MLE's are always consistent estimator but need not be unbiased.

Definition

Let $x_1, x_2, \ldots, x_n$ be observations from n independent and identically distributed random variables drawn from a Probability Distribution that depends on some parameters θ.

The goal of MLE is to maximize the likelihood function:

$$L = f(x_1, x_2, \ldots, x_n|\theta) = \prod_{i=1}^{n} f(x_i|\theta)$$

For maximization, we have

$$\boxed{\frac{dL}{d\theta} = 0 \; ; \quad \frac{d^2L}{d\theta^2} < 0}$$

Since logarithm is a non-decreasing function, so for maximizing L, it is equivalently correct to maximize $\log L$, i.e.,

$$\frac{1}{L}\frac{dL}{d\theta} = 0 \Rightarrow \frac{d \log L}{d\theta} = 0$$

$$L = \prod_{i=1}^{n} f(x_i|\theta)$$

In other words, the log-likelihood function is easier to work with:

$$\log L = \sum_{i=1}^{n} \log f(x_i|\theta)$$

Remark: MLE's are always consistent estimator but need not be unbiased

Question: Obtain MLE for p from a random sample of size n from a population with pdf $f(x, p) = p^{x}(1-p)^{1-x}$ $-\infty < x < \infty$ $\cancel{x = 0, 1, \dots, n}$

or

Example: For a random sample $x_1, x_2, \dots, x_n$. Assume that x_i's are independent Bernoulli random variables of the students picking a course of Statistics with unknown parameter p, find the maximum likelihood estimator of p, the proportion of students who select Statistics subject.

Solution: Define a Bernoulli random variable as

$x_i = 1$; if a randomly selected student selects a Statistics subject

$x_i = 0$; if a randomly selected student does not select a Statistics subject

The p.m.f. of Bernoulli random variable x_i is

$$f(x_i) = p^{x_i}(1-p)^{1-x_i}$$

The likelihood function L is defined as

$$L = \prod_{i=1}^{n} f(x_i|\theta)$$

$$= \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i}$$

$$= p^{\sum x_i}(1-p)^{\sum(1-x_i)}$$

$$\Rightarrow \log L = \sum x_i \log p + \sum(1-x_i)\log(1-p)$$

$$= \log p \sum x_i + \left(n - \sum x_i\right)\log(1-p)$$

To maximize the L, we have

$$\frac{d}{dp}\log L = 0 \Rightarrow \frac{1}{p}\sum x_i + \left(n - \sum x_i\right)\left(-\frac{1}{1-p}\right) = 0$$

$$\Rightarrow \frac{\sum x_i}{p} = \frac{n - \sum x_i}{1-p}$$

$$\Rightarrow \sum x_i - p\sum x_i = np - p\sum x_i$$

$$\Rightarrow p = \frac{\sum x_i}{n}$$

Further, $\frac{d^2}{dp^2}\log L < 0$

Thus,

an estimator of p is $\frac{\sum x_i}{n}$

Example: For a random sample $x_1, x_2, \dots, x_n$. Assume that x_i's are independent Binomial random variables with unknown parameter p, find the maximum likelihood estimator of p.

Solution: For binomial distribution, we have

$$p(x_i) = \binom{n}{x_i} p^{x_i}(1-p)^{n-x_i} \quad ; \quad x_i = 0,1,2,\dots,n \, ; p \in [0,1]$$

The likelihood function L is defined as

$$L = \prod_{i=1}^{n} p(x_i|\theta) = \prod_{i=1}^{n} \binom{n}{x_i} p^{x_i}(1-p)^{n-x_i}$$

The likelihood function L is defined as

$$\begin{aligned} L &= \prod_{i=1}^{n} p(x_i|\theta) \\ &= \prod_{i=1}^{n} \binom{n}{x_i} p^{x_i}(1-p)^{n-x_i} \end{aligned}$$

$$\Rightarrow \log L = \sum_{i=1}^{n} \left[\log \binom{n}{x_i} + \log p^{x_i} + \log(1-p)^{n-x_i} \right]$$

$$\Rightarrow \log L = \sum_{i=1}^{n} \left[\log \binom{n}{x_i} + x_i \log p + (n - x_i) \log(1 - p)\right]$$

$$\Rightarrow \log L = \sum_{i=1}^{n} \log \binom{n}{x_i} + \log p \sum_{i=1}^{n} x_i + \log(1 - p) \sum_{i=1}^{n} (n - x_i)$$

To maximize L, we have

$$\frac{d}{dp}\log L = 0 \Rightarrow \frac{1}{p}\sum x_i - \frac{1}{1-p}\sum(n - x_i) = 0$$

$$\Rightarrow \frac{1}{p}\sum x_i - \frac{n^2}{1-p} + \frac{1}{1-p}\sum x_i = 0$$

$$\Rightarrow \frac{1}{p(1-p)}\sum x_i = \frac{n^2}{1-p}$$

$$\Rightarrow \frac{1}{p}\sum x_i = n^2$$

$$\Rightarrow p = \frac{\sum x_i}{n^2}$$

$$\frac{d^2}{dp^2}\log L = -\frac{1}{p^2}\sum x_i + \frac{1}{(1-p)^2}\sum(n - x_i) < 0$$

Hence, the MLE of p is $\frac{\sum x_i}{n^2}$

Eg 17.32 Prove that the MLE of the parameter α of a population having density function $\frac{2}{\alpha^2}(\alpha - x)$, $0 < x < \alpha$, for a sample of unit size is $2x$, x being the sample value. Show also that the estimate is biased.

Sample of unit size $\therefore n = 1$, the likelihood function is

$$L(x,\alpha) = \frac{2}{\alpha^2}(\alpha - x), \quad 0 < x < \alpha$$

$$0 = \frac{d}{d\alpha}\log L = \frac{d}{d\alpha}\left(\log 2 + \log(\alpha - x) - \log \alpha^2\right)$$

$$= \frac{d}{d\alpha}\left(\log 2 + \log(\alpha - x) - 2\log\alpha\right)$$

$$0 = \frac{1}{\alpha - x} - \frac{2}{\alpha}.$$

$$\frac{2}{\alpha} = \frac{1}{\alpha - x}$$

$$2\alpha - 2x = \alpha$$

$$\alpha = 2x$$

Hence MLE of α is $\hat{\alpha} = 2x$.

$$E(\hat{\alpha}) = E(2x) = \int_0^\alpha 2x\, f(x,\alpha)\, dx$$

$$= \int_0^\alpha 2x \times \frac{2}{\alpha^2}(\alpha - x)\, dx$$

$$= \frac{4}{\alpha^2}\int_0^\alpha (\alpha x - x^2)\,dx$$

$$= \frac{4}{\alpha^2}\left(\frac{\alpha x^2}{2} - \frac{x^3}{3}\right)\Big|_0^\alpha$$

$$= \frac{4}{\alpha^2}\left(\frac{\alpha^3}{2} - \frac{\alpha^3}{3}\right)$$

$$= 4\left(\frac{\alpha}{2} - \frac{\alpha}{3}\right) = \frac{4\alpha}{6} = \frac{2\alpha}{3} \neq \alpha$$

$\therefore$ $\hat{\alpha} = 2x$ is not an unbiased estimate of α

Eg 17.33 a) Find the MLE for the parameter of a Poisson distribution on the basis of a sample of size n. Also find its variance
b) show that the sample mean $\bar{x}$, is sufficient for estimating the parameter λ of the Poisson distribution.

$$P(X=x) = f(x,\lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!}, \quad x=0,1,2,\ldots$$

Likelihood function of random sample $x_1, x_2, \ldots, x_n$ of n observations from this population is

$$L = \prod_{i=1}^{n} f(x_i,\lambda)$$

$$= \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}$$

$$= \frac{e^{-n\lambda}\lambda^{\sum_{i=1}^{n} x_i}}{x_1!\, x_2! \ldots x_n!}$$

$$\log L = -n\lambda + \sum_{i=1}^{n} x_i \log\lambda - \sum_{i=1}^{n} \log(x_i!)$$

$$\frac{\partial \log L}{\partial \lambda} = 0 \qquad\qquad \frac{\partial^2 \log L}{\partial \lambda^2} = -\frac{1}{\lambda^2}\sum_{i=1}^{n} x_i$$

$$\Rightarrow -n + \sum_{i=1}^{n} x_i \times \frac{1}{\lambda} = 0 \qquad\qquad = -\frac{n\bar{x}}{\lambda^2}$$

$$\Rightarrow n = \frac{1}{\lambda}\sum_{i=1}^{n} x_i$$

$$\Rightarrow \lambda = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}$$

MLE for λ is sample mean $\bar{x}$. $\hat{\lambda} = \bar{x}$

Variance of the estimate is given by

$$V(\hat{\lambda}) = \frac{1}{E\left(-\frac{\partial^2}{\partial\lambda^2}\log L\right)}$$

$$= \frac{1}{E(n\bar{x}/\lambda^2)} = \frac{\lambda^2}{n\, E(\bar{x})} = \frac{\lambda^2}{n\lambda}$$

$$= \frac{\lambda}{n}$$

$$\text{var}(\hat{\lambda}) = \frac{\lambda}{n}.$$

Sheet No.__________ Registration No.__________

Note: i) This sheet must be submitted to the invigilator along with the question paper on completion of examination.
ii) Exchange of rough sheet will be considered as UMC.
iii) Use only for rough work which is required to solve questions, otherwise it will be considered as UMC.
iv) Writing of answers / options on rough sheet will be considered as UMC.

(b) For the Poisson distribution with parameter λ, we have using the following remark

$$\frac{\partial}{\partial \lambda} \log L = -n + \frac{1}{\lambda}\sum_{i=1}^{n} x_i$$

$$= -n + \frac{n\bar{x}}{\lambda}$$

$$= n\left(\frac{\bar{x}}{\lambda} - 1\right)$$

$$= g(\bar{x}, \lambda), \text{ a fn of } \bar{x} \text{ \& } \lambda \text{ only}$$

$\therefore$ $\bar{x}$ is sufficient estimator of λ.

Remark: If $\frac{\partial}{\partial \theta} \log L$ can be expressed in the form $\psi(t, \theta)$, as a function of statistic and parameter alone, then the statistic is regarded as a sufficient estimator of the parameter. If $\frac{\partial}{\partial \theta} \log L$ can't be expressed in $\psi(t, \theta)$, no sufficient estimator exists in that case.

Eg 17.39 Obtain MLE of θ in $f(x, \theta) = (1+\theta)x^{\theta}$, $0 < x < 1$, based on an independent sample of size n.

$$L(x, \theta) = \prod_{i=1}^{n} f(x_i, \theta)$$

$$= (1+\theta)^n \left(\prod_{i=1}^{n} x_i\right)^{\theta}$$

$$\log L = n \log(1+\theta) + \theta \log\left(\prod_{i=1}^{n} x_i\right)$$

$$= n \log(1+\theta) + \theta \sum_{i=1}^{n} \log x_i$$

$$\frac{\partial}{\partial \theta} \log L = \frac{n}{1+\theta} + \sum_{i=1}^{n} \log x_i = 0$$

$$\Rightarrow n + \sum_{i=1}^{n} \log x_i + \theta \sum_{i=1}^{n} \log x_i = 0$$

$$\Rightarrow \theta = \frac{-n}{\sum_{i=1}^{n} \log x_i} - 1$$

$$\Rightarrow \theta = \frac{-n}{\log\left(\prod_{i=1}^{n} x_i\right)} - 1$$

$$\therefore \frac{-n}{\log\left(\prod_{i=1}^{n} x_i\right)} - 1 \text{ is MLE of } \theta.$$

Sampling Distribution of Means

A random sample of n observations is taken from a normal population with mean μ and variance σ^2. Each observation X_i, $i = 1, 2, \dots, n$, of the random sample will then have the same normal distribution as the population being sampled.

$$\bar{X} = \frac{1}{n}(X_1 + X_2 + \cdots + X_n)$$

Mean $\bar{X}$ also follows normal distribute with
Mean

$$\mu_{\bar{X}} = \frac{1}{n}(\underbrace{\mu + \mu + \cdots + \mu}_{n \text{ terms}}) = \mu$$

And Variance

$$\sigma^2_{\bar{X}} = \frac{1}{n^2}(\underbrace{\sigma^2 + \sigma^2 + \cdots + \sigma^2}_{n \text{ terms}}) = \frac{\sigma^2}{n}.$$

Central Limit Theorem (C L T)

If $X_1, X_2, \cdots, X_n, \cdots$, be a sequence of independent identically distributed RVs with $E(X_i) = \mu$ and $\mathrm{Var}(X_i) = \sigma^2$, $i = 1, 2, \cdots$, and

If $\bar{X} = \frac{1}{n}(X_1 + X_2 + \cdots + X_n)$,

then $E(\bar{X}) = \mu$ and $\mathrm{Var}(\bar{X}) = \frac{1}{n^2}(n\sigma^2) = \frac{\sigma^2}{n}$

$\bar{X}$ follows $N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$ as $n \to \infty$

$\bar{X}$ follows a normal distribution with mean μ and variance $\frac{\sigma^2}{n}$ as n tends to infinity.

the standard normal distribution $n(z; 0, 1)$. $\quad Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$

Central Limit Theorem (C L T)

If $X_1, X_2, \cdots, X_n, \cdots$, be a sequence of independent identically distributed RVs with $E(X_i) = \mu$ and $\mathrm{Var}(X_i) = \sigma^2$, $i = 1, 2, \cdots$, and

$$\text{if } S_n = X_1 + X_2 + \cdots + X_n \qquad S_n = \sum X_i$$

then S_n follows $N(n\mu, \sigma\sqrt{n})$ as $n \to \infty$

S_n follows a normal distribution with mean $n\mu$ and variance $n\sigma^2$ as n tends to infinity.

the standard normal distribution $n(z;0,1)$. $$Z = \frac{S_n - n\mu}{\sqrt{n}\sigma}$$

Central Limit Theorem

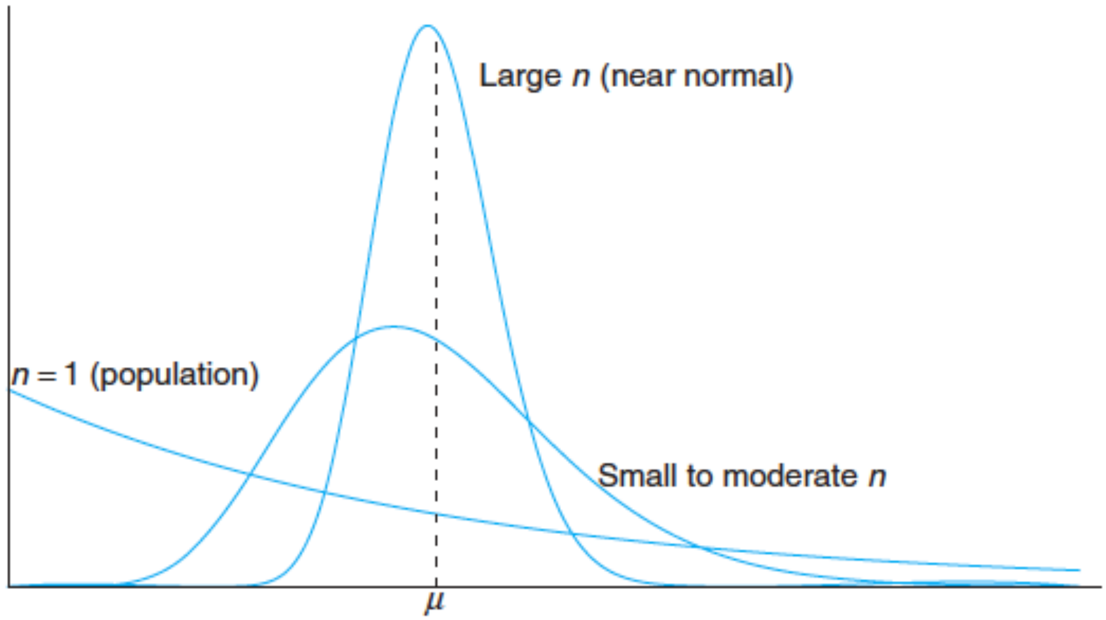
Central Limit Theorem: If $\bar{X}$ is the mean of a random sample of size n taken from a population with mean μ and finite variance σ^2, then the limiting form of the distribution of

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}},$$

as $n \to \infty$, is the standard normal distribution $n(z;0,1)$.

Note:-

The normal approximation for $\bar{X}$ will generally be good if $n \geq 30$, provided the population distribution is not terribly skewed
If $n < 30$, the approximation is
good only if the population is not too different from a normal distribution

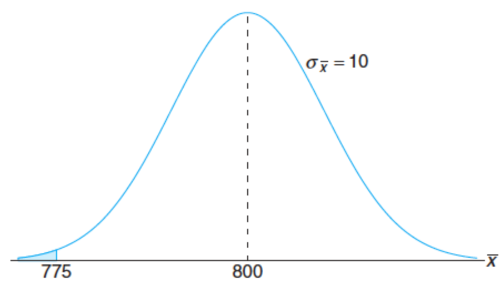


Example

An electrical firm manufactures light bulbs that have a length of life that is approximately normally distributed, with mean equal to 800 hours and a standard deviation of 40 hours. Find the probability that a random sample of 16 bulbs will have an average life of less than 775 hours.

Soln

The sampling distribution of $\bar{X}$ will be approximately normal



with $\mu_{\bar{X}} = 800$ and
$\sigma_{\bar{X}} = 40/\sqrt{16} = 10.$
Corresponding to $\bar{x} = 775$, we find that

$$z = \frac{775 - 800}{10} = -2.5,$$

and therefore

$$P(\bar{X} < 775) = P(Z < -2.5) = 0.0062.$$

Example 1 *The lifetime of a certain brand of an electric bulb may be considered a RV with mean 1200 h and standard deviation 250 h. Find the probability, using central limit theorem, that the average lifetime of 60 bulbs exceeds 1250 h.*

Solution Let X_i represent the lifetime of the bulb.
$E(X_i) = 1200$ and $\text{Var}(X_i) = 250^2$

Let $\bar{X}$ denote the mean lifetime of 60 bulbs.
By corollory of Lindeberg-Levy form of CLT

$$\bar{X} \text{ follows } N\left(1200, \frac{250}{\sqrt{60}}\right)$$

$$P(\overline{X} > 1250) = P\left(\frac{\overline{X} - 1200}{\frac{250}{\sqrt{60}}} > \frac{1250 - 1200}{\frac{250}{\sqrt{60}}}\right)$$

$$= P\left(z > \frac{\sqrt{60}}{5}\right)$$

$$= P(z > 1.55),$$

where z is the standard normal variable

$$= 0.0606$$

(from the table of areas under normal curve)

Example 3 *If $X_1, X_2, \cdots X_n$ are Poisson variates with parameter $\lambda = 2$, use the central limit theorem to estimate $P(120 \leq S_n \leq 160)$, where $S_n = X_1 + X_2 + \cdots X_n$ and $n = 75$.* (MU — Apr. 96)

Solution $E(X_i) = \lambda = 2$ and $\mathrm{Var}(X_i) = \lambda = 2$

By CLT, S_n follows $N(n\mu, \sigma\sqrt{n})$

i.e., S_n follows $N(150, \sqrt{150})$

$$P\{120 \leq S_n \leq 160\} = P\left\{\frac{-30}{\sqrt{150}} \leq \frac{S_n - 150}{\sqrt{150}} \leq \frac{10}{\sqrt{150}}\right\}$$

$$= P\{-2.45 \leq z \leq 0.85\}$$

where z is the standard normal variable.

$$= 0.4927 + 0.2939, \text{ (from the normal tables)}$$

$$= 0.7866$$

Example 2 *A distribution with unknown mean μ has variance equal to 1.5. Use central limit theorem to find how large a sample should be taken from the distribution in order that the probability will be at least 0.95 that the sample mean will be within 0.5 of the population mean.* (MKU – Apr. 97)

Solution Let n be the size of the sample, a typical member of which is X_i.
Given: $E(X_i) = \mu$ and $\text{Var}(X_i) = 1.5$.

Let $\overline{X}$ denote the sample mean.
By corollary under CLT,

$\overline{X}$ follows $N\left(\mu, \dfrac{\sqrt{1.5}}{\sqrt{n}}\right)$

We have to find n such that

$$P\{\mu - 0.5 < \overline{X} < \mu + 0.5\} \geq 0.95$$

i.e., $$P\{-0.5 < \overline{X} - \mu < 0.5\} \geq 0.95$$

i.e., $$P\{|\overline{X} - \mu| < 0.5\} \geq 0.95$$

i.e., $$P\left\{\frac{|\overline{X} - \mu|}{\sqrt{\frac{1.5}{n}}} < \frac{0.5}{\sqrt{\frac{1.5}{n}}}\right\} \geq 0.95$$

i.e., $$P\{|z| < 0.4082\sqrt{n}\} \geq 0.95$$

where z is the standard normal variable.
The least value of n is obtained from

$$P\{|z| < 0.4082\sqrt{n}\} = 0.95$$

From the table of areas under normal curve

$$P\{|z| < 1.96\} = 0.95$$

Therefore, least n is given by $0.4082\sqrt{n} = 1.96$, i.e., least $n = 24$.
Therefore, the size of the sample must be at least 24.

NEXT

Sampling Distribution of the Difference between Two Means

mean

$$\mu_{\bar{X}_1 - \bar{X}_2} = \mu_{\bar{X}_1} - \mu_{\bar{X}_2} = \mu_1 - \mu_2$$

variance

$$\sigma^2_{\bar{X}_1-\bar{X}_2} = \sigma^2_{\bar{X}_1} + \sigma^2_{\bar{X}_2} = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}.$$

The Central Limit Theorem can be easily extended to the two-sample, two-population case.

If independent samples of size n_1 and n_2 are drawn at random from two populations, discrete or continuous, with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2, respectively, then the sampling distribution of the differences of means, $\bar{X}_1 - \bar{X}_2$, is approximately normally distributed with mean and variance given by

$$\mu_{\bar{X}_1-\bar{X}_2} = \mu_1 - \mu_2 \text{ and } \sigma^2_{\bar{X}_1-\bar{X}_2} = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}.$$

Hence,

$$Z = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{(\sigma_1^2/n_1) + (\sigma_2^2/n_2)}}$$

is approximately a standard normal variable.

Example

The television picture tubes of manufacturer A have a mean lifetime of 6.5 years and a standard deviation of 0.9 year, while those of manufacturer B have a mean lifetime of 6.0 years and a standard deviation of 0.8 year. What is the probability that a random sample of 36 tubes from manufacturer A will have a mean lifetime that is at least 1 year more than the mean lifetime of a sample of 49 tubes from manufacturer B?

Solution:-

Population 1	Population 2
$\mu_1 = 6.5$	$\mu_2 = 6.0$
$\sigma_1 = 0.9$	$\sigma_2 = 0.8$
$n_1 = 36$	$n_2 = 49$

the sampling distribution of $\bar{X}_1 - \bar{X}_2$
will be approximately normal and will have a mean and standard deviation

$$\mu_{\bar{X}_1-\bar{X}_2} = 6.5 - 6.0 = 0.5 \quad \text{and} \quad \sigma_{\bar{X}_1-\bar{X}_2} = \sqrt{\frac{0.81}{36} + \frac{0.64}{49}} = 0.189.$$

$\bar{x}_1 - \bar{x}_2 = 1.0$

$$z = \frac{1.0 - 0.5}{0.189} = 2.65,$$

and hence

$$P(\bar{X}_1 - \bar{X}_2 \geq 1.0) = P(Z > 2.65) = 1 - P(Z < 2.65)$$
$$= 1 - 0.9960 = 0.0040.$$

Solve the following questions

8.26 The amount of time that a drive-through bank teller spends on a customer is a random variable with a mean $\mu = 3.2$ minutes and a standard deviation $\sigma = 1.6$ minutes. If a random sample of 64 customers is observed, find the probability that their mean time at the teller's window is

(a) at most 2.7 minutes;

(b) more than 3.5 minutes;

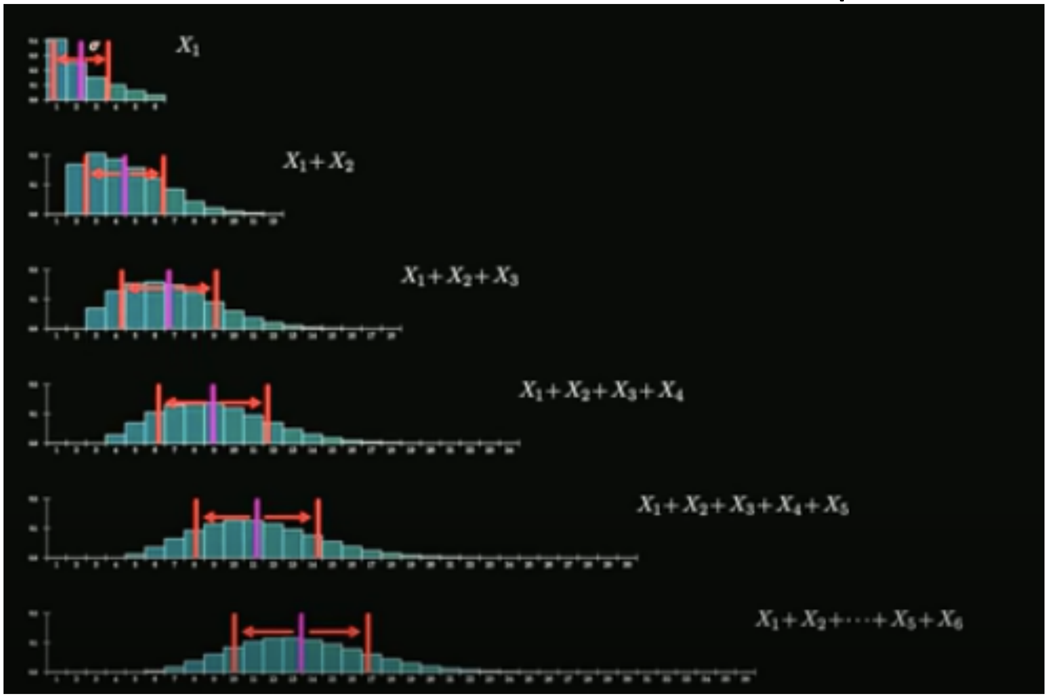
(c) at least 3.2 minutes but less than 3.4 minutes.

8.28 A random sample of size 25 is taken from a normal population having a mean of 80 and a standard deviation of 5. A second random sample of size 36 is taken from a different normal population having a mean of 75 and a standard deviation of 3. Find the probability that the sample mean computed from the 25 measurements will exceed the sample mean computed from the 36 measurements by at least 3.4 but less than 5.9. Assume the difference of the means to be measured to the nearest tenth.

Textbook questions on CLT from 8.17 to 8.34

Extra questions on Central Limit Theorem

Central limit Theorem can also be used for sum of random samples



If $X_i(i = 1,2,\ldots,n)$ be independently distributed random variables such that

$$E(X_i) = \mu_i \text{ and } Var(X_i) = \sigma_i^2$$

then as $n \to \infty$, the distribution of the sum of these random variables, namely

$$S_n = X_1 + X_2 + \cdots + X_n$$

tends to the normal distributed with mean μ and variance σ^2, where

$$\mu = \sum_{i=1}^{n} \mu_i \quad ; \quad \sigma^2 = \sum_{i=1}^{n} \sigma_i^2$$

Here $S_n = \sum X_i$ also follow normal distribution with

$$Z = \frac{S_n - n\mu}{\sqrt{n}\sigma}$$

Example: A random sample of size 100 is taken from a population whose mean is 60 and variance is 400. Using central limit theorem, with what probability can we assert that the mean of the sample will not differ from $\mu = 60$ by more than 4?

Solution: Given that $n = 100,\ \mu_i = 60;\ \sigma_i^2 = 400$

The sample mean is $\bar{X} = \frac{X_1+X_2+\cdots+X_{100}}{100}$

$$E(\bar{X}) = \frac{1}{100}\sum_{i=1}^{100} E(X_i)$$

$$= \frac{1}{100}[100 \times 60]$$

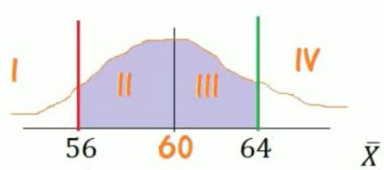
$$= 60$$

$$Var(\bar{X}) = \frac{1}{(100)^2}\sum_{i=1}^{100} Var(X_i)$$

$$= \frac{1}{10000}[100 \times 400]$$

$$= 4$$

$$Z = \frac{\bar{X}-60}{2}$$



Required probability

$$= P(|\bar{X} - 60| \le 4)$$

$$= P(-4 \le \bar{X} - 60 \le 4)$$

$$= P(56 \le \bar{X} \le 64)$$

$$= P(-2 \le Z \le 2)$$

$$= 0.4772 + 0.4772$$

$$= 0.9544$$

Given that $P(Z < -2) = 0.0228$ from Normal Table

Example: If $X_1, X_2, \dots, X_n$ are Poisson variables with parameter 2, use the central limit theorem to estimate $P(120 \le S_n \le 160)$ where $S_n = X_1 + X_2 + \dots + X_n, n = 75$

Solution: Given that $\lambda = 2$ (a Poisson distribution parameter)

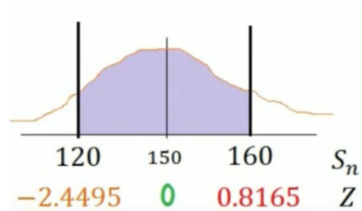
$$\therefore E(X_i) = 2; \quad Var(X_i) = 2$$

$$E(S_n) = \sum_{i=1}^{75} E(X_i) = 75 \times 2 = 150$$

$$Var(S_n) = \sum_{i=1}^{75} Var(X_i) = 75 \times 2 = 150$$

Therefore

$$Z = \frac{S_n - 150}{\sqrt{150}}$$



Required probability

$$= P(120 \le S_n \le 160)$$

$$= P(-2.4495 \le Z \le 0.8165)$$

$$= 0.4929 + 0.2939$$

$$= 0.7868$$

Given that $P(Z < 0.8165) = 0.7939$;
$P(Z < -2.45) = 0.0071$
from Normal Table

Using CLT Solve the following examples

1

Example: Let $\{X_i\}$ be independent and identically distributed random variables with mean 3 and variance ½. Use Central limit theorem to Estimate $P(340 \le S_n \le 370)$, where $S_n = X_1 + X_2 + \dots + X_n$ and $n = 120$

2.

Example: A random sample of size 100 is taken from a population whose mean is 80 and variance is 400. Using central limit theorem, with what probability can we assert that the mean of the sample will not differ from $\mu = 80$ by more than 6?

3.

Example: The lifetime of a certain type of electric bulb may be considered as an exponential random variable with mean 50 hours. Using central limit theorem, find the approximate probability that 100 of these electric bulbs will provide a total of more than 6000 hours of burning time.

4.

Example: 20 dice are thrown. Find the approximate probability that the sum obtained is between 65 and 75 using central limit theorem.

5.

Example: A distribution with unknown mean has variance 1.5. Use central limit theorem to find how large a sample should be taken from the distribution in order that the probability will be at least 0.95 that the sample mean will be within 0.5 of the population mean.

NOTE:-

FOR MORE QUESTIONS FOR PRACTICE FOLLOW TEXBOOK AND REFERENCE BOOK