MOMENT GENERATING FUNCTIONS (m.g.f.)
[of Binomial, Poisson, geometric and negative binomial]

The moment generating function (m.g.f.) of a random variable X is denoted by $M_X(t)$ and is defined as

$$M_X(t) = E(e^{tX}), \quad t \in R$$

wherever this expectation exists.

u3

For discrete random variable

$$M_X(t) = \sum_x e^{tx} p(x)$$

Where $p(x)$ is the probability mass function (p.m.f.)

u4

For continuous random variable,

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

Where $f(x)$ is the probability density function (p.d.f.)

In unit 3 we have distribution for discrete data so we use formula written on left side

the rth moment about origin is denoted by μ'_r and it has following relation with expectations and mgf

For any positive integer r, we denote $\mu'_r = E(X^r)$

1st Method

$$E(X^r) = \mu'_r = \frac{d^r}{dt^r}\left(M_X(t)\right) \quad at\ t = 0$$

Mean $= E(X)$

Variance $= E(X^2) - [E(X)]^2$

2nd method:

$\mu'_r =$ Coefficient of $\frac{t^r}{r!}$ in the series expansion of $M_X(t)$

For more details,

$$M_X(t) = E(e^{tX})$$

$$= E\left[1 + tX + \frac{t^2}{2!}X^2 + \cdots \ldots \ldots \ldots\right]$$

$$= 1 + tE(X) + \frac{t^2}{2!}E(X^2) + \cdots \ldots \ldots$$

For MCQ purpose remember m.g.f. formulae for the four distributions as mentioned below:

1. m.g.f. for **binomial distribution** b(x; n, p) q=1-p

$$M_X(t) = (pe^t + q)^n$$

2. m.g.f. for **Negative binomial distribution** b^* (x; k, p)

$$M_X(t) = \left[\frac{pe^t}{1 - qe^t}\right]^k$$

3. m.g.f. for **Geometric distribution** g(x;p)

$$M_X(t) = \frac{pe^t}{1 - qe^t}$$

4. m.g.f. for **Poisson distribution** P(x; λ)

$$M_X(t) = e^{\lambda(e^t - 1)}$$

Or sometimes can also be expressed as

$$M_X(t) = e^{-\lambda(1-e^t)}$$

Example: The m.g.f. of a random variable X is given by $M_X(t) = e^{3(e^t-1)}$.

Find $P(X = 1)$.

Example: The m.g.f. of a random variable X is given by $M_X(t) = e^{3(e^t-1)}$.

Find $P(X = 1)$.

Solution: Since, the given m.g.f. can be rewritten as

$$M_X(t) = e^{-3(1-e^t)}$$

Thus, by comparing it with m.g.f. of Poisson distribution,

$$M_X(t) = e^{-\lambda(1-e^t)}$$

we get, $\lambda = 3$

Therefore, by Poisson distribution

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

Therefore, by Poisson distribution

$$\begin{aligned} P(X = 1) &= \frac{e^{-\lambda}\lambda^1}{1!} \\ &= 3e^{-3} \\ &= 0.1494 \end{aligned}$$

Example: Find the m.g.f. for the distribution where

$$f(x) = \begin{cases} \frac{2}{3}; & x = 1 \\ \frac{1}{3}; & x = 2 \\ 0\ ; & otherwise \end{cases}$$

Solution: Since X is a discrete random variable with points 1 and 2 only.

The m.g.f. is

$$\boxed{\begin{aligned} M_X(t) &= \sum e^{tx} f(x) \\ &= e^t\left(\frac{2}{3}\right) + e^{2t}\left(\frac{1}{3}\right) \\ &= \frac{2e^t + e^{2t}}{3} \end{aligned}}$$

Example: A perfect coin is tossed twice. Find the m.g.f. of the number of heads. Hence, find the mean and variance.

Solution: The p.m.f. of X, (the number of heads) is

x	0	1	2
$p(x)$	¼	½	¼

The m.g.f. is

$$M_X(t) = E(e^{tX})$$

$$= \sum e^{tx} p(x)$$

$$= e^0\left(\frac{1}{4}\right) + e^t\left(\frac{1}{2}\right) + e^{2t}\left(\frac{1}{4}\right)$$

$$= \frac{1 + 2e^t + e^{2t}}{4}$$

$$= \frac{(1 + e^t)^2}{4}$$

Mean and variance:

$$E(X) = \frac{d}{dt}M_X(t)$$
$$= \frac{1}{2}(e^t + e^{2t})$$

$$E(X^2) = \frac{d^2}{dt^2}M_X(t)$$
$$= \frac{1}{2}(e^t + 2e^{2t})$$

$$M_X(t) = \frac{(1+e^t)^2}{4}$$

At $t = 0$, we have

$$E(X) = 1;$$
$$E(X^2) = 3/2$$

Thus,

Mean = 1

$$\text{Variance} = E(X^2) - [E(X)]^2$$
$$= \frac{3}{2} - 1$$
$$= \frac{1}{2}$$

Derivations of mgf and (mean, variance):-

Example: Find the m.g.f of the Binomial distribution and hence find its mean and Variance.

Solution: The p.m.f. for the Binomial distribution is

$$p(x) = {}^{n}C_x p^x q^{n-x}\ ; \ \ x = 0,1,2,\dots,n;\ p+q=1$$

Its m.g.f. is

$$\begin{aligned} M_X(t) &= E(e^{tX}) \\ &= \sum_{x=0}^{n} e^{tx}\, {}^{n}C_x p^x q^{n-x} \\ &= \sum_{x=0}^{n} {}^{n}C_x (e^t p)^x q^{n-x} \\ &= (e^t p + q)^n \end{aligned}$$

By using Binomial expansion of $(a+x)^n$,

provided $|x| < a$

For Mean and Variance $\quad M_X(t) = (e^t p + q)^n$

$$\begin{aligned} E(X) &= \frac{d}{dt}(e^t p + q)^n \\ &= n(e^t p + q)^{n-1} p e^t \end{aligned}$$

$$\boxed{\begin{aligned} E(X^2) &= \frac{d^2}{dt^2} M_X(t) \\ &= \frac{d}{dt}\left[np\,(e^t p + q)^{n-1} e^t\right] \\ &= np\begin{bmatrix} (n-1)(e^t p + q)^{n-2}(pe^t)e^t \\ +(e^t p + q)^{n-1} e^t \end{bmatrix} \end{aligned}}$$

At $t = 0$, we have

$$\begin{aligned} E(X) &= n(p+q)^{n-1} p \\ &= np \end{aligned}$$

$$\begin{aligned} E(X^2) &= np\begin{bmatrix} (n-1)(p+q)^{n-2} p \\ +(p+q)^{n-1} \end{bmatrix} \\ &= np[(n-1)p + 1] \end{aligned}$$

Thus,

Mean $= np$

$$\begin{aligned} \text{Variance} &= E(X^2) - [E(X)]^2 \\ &= np[(n-1)p+1] - n^2p^2 \\ &= np(1-p) \end{aligned}$$

Example: Find the m.g.f. for the geometric distribution and find its mean and variance.

Solution: The p.m.f. of the geometric distribution is

$$p(x) = q^{x-1}p \ ; \quad x = 1,2, \dots \dots$$

Thus, m.g.f. is given as

$$M_X(t) = E(e^{tX})$$
$$= \sum_{x=1}^{\infty} e^{tx} q^{x-1} p$$
$$= \frac{p}{q} \sum_{x=1}^{\infty} (qe^t)^x$$
$$= \frac{p}{\cancel{q}}\left(\frac{\cancel{q}e^t}{1-qe^t}\right)$$

Provided $qe^t < 1$

It is infinite sum of G.P.

$\sum_{n=1}^{\infty} a^n = \frac{a}{1-r}$,

provided common ratio $r < 1$.

Hence,

$$M_X(t) = \frac{pe^t}{1-q\,e^t}$$

provided $qe^t < 1$, $i.e., t < \log_e(1/q)$

Mean & Variance: $M_X(t) = \frac{pe^t}{1-q\,e^t}$

$$E(X) = \frac{d}{dr}\left[\frac{pe^t}{1-qe^t}\right]$$
$$= \frac{pe^t}{(1-qe^t)^2}$$

$$E(X^2) = \frac{d^2}{dt^2} M_X(t)$$
$$= \frac{d}{dt}\left[\frac{pe^t}{(1-qe^t)^2}\right]$$
$$= \frac{pe^t + pqe^{2t}}{(1-qe^t)^3}$$

At $t = 0$, we have

$$E(X) = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$$

$$E(X^2) = \frac{p+pq}{(1-q)^3} = \frac{p(1+q)}{p^3} = \frac{1+q}{p^2}$$

Hence,

Mean $= \frac{1}{p}$

Variance $= E(X^2) - [E(X)]^2$

$$= \frac{1+q}{p^2} - \frac{1}{p^2}$$
$$= \frac{q}{p^2}$$

Note :-In following negative binomial distribution formula symbol r th success (we used the sams as k th success in class room discusson)

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The p.m.f. of the Negative Binomial distribution with parameter r, p is

$$p(x) = \binom{x-1}{r-1} q^{x-r} p^r; x = r, r+1, r+2, \ldots$$

By definition of MGF,

$$M_X(t) = E(e^{tX})$$

$$= \sum_{x=r}^{\infty} e^{tx} \binom{x-1}{r-1} q^{x-r} p^r$$

$$= p^r e^{tr} \sum_{x=r}^{\infty} e^{t(x-r)} \binom{x-1}{r-1} q^{x-r}$$

$$= (pe^t)^r \sum_{x=r}^{\infty} \binom{x-1}{r-1} (qe^t)^{x-r}$$

$$= (pe^t)^r \sum_{k=0}^{\infty} \binom{k+r-1}{r-1} (qe^t)^k$$

Sum of a negative binomial series

$$(1-x)^{-n} = \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} x^k$$

$$= (pe^t)^r (1-qe^t)^{-r}$$

Provided $|qe^t| < 1 \Rightarrow t < \ln\left(\frac{1}{q}\right)$

$$\boxed{M_X(t) = \frac{(pe^t)^r}{(1-qe^t)^r}}$$

Is the required MGF

Mean & Variance from $M_X(t)$

$$M_X(t) = \frac{(pe^t)^r}{(1-qe^t)^r}$$

At $t = 0$,

Mean = $E(X)$

Variance = $E(X^2) - [E(X)]^2$

$$E(X) = \frac{d}{dt} M_X(t)$$

$$= \frac{(1-qe^t)^r r(pe^t)^{r-1} pe^t - (pe^t)^r r(1-qe^t)^{r-1}(-qe^t)}{(1-qe^t)^{2r}}$$

$$= \frac{r(1-qe^t)^{r-1}(pe^t)^r[(1-qe^t)+qe^t]}{(1-qe^t)^{2r}}$$

$$= \frac{r(1-qe^t)^{r-1}(pe^t)^r}{(1-qe^t)^{2r}}$$

At $t = 0$, we have

$$E(X) = \frac{r(1-q)^{r-1}p^r}{(1-q)^{2r}}$$

$$= \frac{rp^{r-1}p^r}{p^{2r}}$$

$$= \frac{r}{p}$$

4.

Example: Find the m.g.f of the Poisson distribution. Does $M_X(t)$ exist for all values of t?

Solution: The p.m.f. of the Poisson distribution with parameter λ is

$$p(x) = \frac{e^{-\lambda}\lambda^x}{x!}; x = 0,1,2, \dots$$

Thus, m.g.f. of it is

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx}\frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda}\sum_{x=0}^{\infty}\frac{(e^t\lambda)^x}{x!}$$

$$= e^{-\lambda} e^{e^t\lambda}$$

Thus,

$$M_X(t) = e^{-\lambda(1-e^t)}$$

which exist for all values of t.

Because, $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$ and is convergent for all x.

Find Mean and Variance: $M_X(t) = e^{-\lambda(1-e^t)}$

$$E(X) = \frac{d}{dt}\left[e^{-\lambda(1-e^t)}\right]$$

$$= e^{-\lambda(1-e^t)}(\lambda e^t)$$

$$E(X^2) = \frac{d^2}{dt^2}\left[e^{-\lambda(1-e^t)}\right]$$

$$= \frac{d}{dt}\left[e^{-\lambda(1-e^t)}(\lambda e^t)\right]$$

$$= e^{-\lambda(1-e^t)}(\lambda e^t) + e^{-\lambda(1-e^t)}(\lambda e^t)^2$$

At $t = 0$, $E(X) = \lambda$

$E(X^2) = \lambda + \lambda^2$

Therefore,

Mean $= E(X)$

$= \lambda$

Variance $= E(X^2) - [E(X)]^2$

$= \lambda + \lambda^2 - \lambda^2$

$= \lambda$

Mean & Variance from $M_X(t)$

$$M_X(t) = \frac{(pe^t)^r}{(1-qe^t)^r}$$

$$E(X^2) = \frac{d^2}{dt^2} M_X(t)$$

$$= rp^r \left[\frac{(1-qe^t)^{2r}\left[re^{rt}(1-qe^t)^{r-1} + e^{rt}(r-1)(-qe^t)(1-qe^t)^{r-2}\right] - e^{rt}(1-qe^t)^{r-1}(2r)(1-qe^t)^{2r-1}(-qe^t)}{(1-qe^t)^{4r}} \right]$$

At $t = 0$, we have

$$E(X^2) = rp^r \left[\frac{(1-q)^{2r}\left[r(1-q)^{r-1} + (r-1)(-q)(1-q)^{r-2}\right] - (1-q)^{r-1}(2r)(1-q)^{2r-1}(-q)}{(1-q)^{4r}} \right]$$

$$= rp^r \left[\frac{p^{2r}(rp^{r-1} - q(r-1)p^{r-2}) + 2rqp^{r-1}p^{2r-1}}{p^{4r}} \right]$$

At $t = 0$, we have

$$E(X^2) = rp^r \left[\frac{(1-q)^{2r}\left[r(1-q)^{r-1} + (r-1)(-q)(1-q)^{r-2}\right] - (1-q)^{r-1}(2r)(1-q)^{2r-1}(-q)}{(1-q)^{4r}} \right]$$

$$= rp^r \left[\frac{p^{2r}(rp^{r-1} - q(r-1)p^{r-2}) + 2rqp^{r-1}p^{2r-1}}{p^{4r}} \right]$$

$$= \frac{rp^r}{p^{2r}}\left[rp^{r-1} - rqp^{r-2} + qp^{r-2} + 2rqp^{r-2}\right]$$

$$= r\left[\frac{r}{p} + \frac{rq}{p^2} + \frac{q}{p^2}\right]$$

$$= \frac{r}{p^2}[rp + rq + q]$$

$$= \frac{r}{p^2}[r + q]$$

Thus,

$$\text{Variance} = E(X^2) - \left(E(X)\right)^2$$

$$= \frac{r}{p^2}[r+q] - \frac{r^2}{p^2}$$

$$= \frac{r}{p^2}[r + q - r]$$

$$= \frac{rq}{p^2}$$