To Show

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2$$

S^2 is unbiased estimator of σ^2

If $x_1, x_2, \ldots, x_n$ be random sample with mean $E(x_i) = \mu$ and variance $Var(x_i) = \sigma^2$. Consider an estimator $S^2 = \frac{1}{n-1}\sum(x_i - \overline{x})^2$. Show S^2 is unbiased estimator of σ^2.

$$S^2 = \frac{1}{n-1}\sum(x_i - \overline{x})^2$$

$$= \frac{1}{n-1}\sum\left(x_i^2 - 2x_i\overline{x} + \overline{x}^2\right)$$

$$= \frac{1}{n-1}\left[\sum_{i=1}^{n} x_i^2 - 2\overline{x}\sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\overline{x}^2\right]$$

$$= \frac{1}{n-1}\left[\sum_{i=1}^{n} x_i^2 - 2n\overline{x}^2 + n\overline{x}^2\right]$$

$$= \frac{1}{n-1}\left[\sum_{i=1}^{n} x_i^2 - n\overline{x}^2\right]$$

Now, $$E(S^2) = \frac{1}{n-1} E\left[\sum_{i=1}^{n} x_i^2 - n\overline{x}^2\right]$$

$$= \frac{1}{n-1}\left[\sum_{i=1}^{n} E(x_i^2) - nE(\overline{x}^2)\right]$$

$$= \frac{1}{n-1}\left[\sum_{i=1}^{n} E\left(x_i^2\right) - n[Var(\overline{x}) + \{E(\overline{x})\}^2]\right]$$

$$= \frac{1}{n-1}\left[\sum_{i=1}^{n} [\sigma^2 + \mu^2] - n\left(\frac{\sigma^2}{n} + \mu^2\right)\right]$$

$$= \frac{1}{n-1}\left[n(\sigma^2 + \mu^2) - n\left(\frac{\sigma^2}{n} + \mu^2\right)\right]$$

$$= \frac{1}{n-1}[(n-1)\sigma^2]$$

$$= \sigma^2$$

i.e., S^2 is **unbiased** estimator of σ^2

NOTE

$$\overline{x} = \frac{\Sigma x_i}{n}$$

$$\Rightarrow \Sigma x_i = n\overline{x}$$

$$\overline{X} \sim N\left(\mu, \sigma^2/n\right)$$

$$E(x_i^2) - [E(x_i)]^2 = V(X_i)$$

$$\Rightarrow E(x_i^2) = \underline{\sigma^2 + \mu^2}$$

$$E(\overline{x}^2) - [E(\overline{x})]^2 = Var(\overline{x})$$

$$E(\overline{x}^2) = \frac{\sigma^3}{n} + \mu^2$$

If $x_1, x_2, \dots, x_n$ be random sample with mean $E(x_i) = \mu$ and variance $Var(x_i) = \sigma^2$. Consider an estimator $s^2 = \frac{1}{n}\sum(x_i - \overline{x})^2$. Show s^2 is biased estimator of σ^2.

$$
\begin{aligned}
s^2 &= \frac{1}{n}\sum(x_i - \overline{x})^2 \\
&= E(x_i - \overline{x})^2 \\
&= E\left(x_i^2 + \overline{x}^2 - 2\overline{x}x_i\right) \\
&= E\left(x_i^2\right) + \overline{x}^2 - 2\overline{x}E(x_i) \\
&= E\left(x_i^2\right) - \overline{x}^2 \\
&= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \overline{x}^2
\end{aligned}
$$

Now, $E(s^2) = \frac{1}{n}E\left[\sum_{i=1}^{n} x_i^2\right] - E(\overline{x}^2)$

$$
\begin{aligned}
&= \frac{1}{n}\sum_{i=1}^{n} E\left(x_i^2\right) - \left[Var(\bar{x}) + \{E(\bar{x})\}^2\right] \\
&= \frac{1}{n}\sum_{i=1}^{n} [\sigma^2 + \mu^2] - \left(\frac{\sigma^2}{n} + \mu^2\right) \\
&= \sigma^2 + \mu^2 - \frac{\sigma^2}{n} - \mu^2 \\
&= \left(1 - \frac{1}{n}\right)\sigma^2
\end{aligned}
$$

Hence, $E(s^2) \neq \sigma^2$ ✓

i.e., s^2 is biased estimator of σ^2