Moment Generating Function.

6·10. Moment Generating Function. The moment generating function (m.g.f.) of a random variable X (about origin) having the probability function $f(x)$ is given by

$$M_X(t) = E(e^{tX}) = \int e^{tx} f(x)\,dx,$$

(for *continuous probability distribution*)

$$= \sum_x e^{tx} f(x),$$

(for *discrete probability distribution*)

the integration or summation being extended to the entire range of x, t being the real parameter and it is being assumed that the right-hand side of (6·54) is absolutely convergent for some positive number h such that $-h < t < h$. Thus

$$M_X(t) = E(e^{tX}) = E\left[1 + tX + \frac{t^2X^2}{2!} + \ldots + \frac{t^rX^r}{r!} + \ldots\right]$$

$$= 1 + t\,E(X) + \frac{t^2}{2!}E(X^2) + \ldots + \frac{t^r}{r!}E(X^r) + \ldots$$

$$= 1 + t\,\mu_1' + \frac{t^2}{2!}\mu_2' + \ldots + \frac{t^r}{r!}\mu_r' + \ldots$$

where $\mu_r' = E(X^r) = \int x^r f(x)\,dx$, *for continuous distribution*

$$= \sum_x x^r p(x), \textit{ for discrete distribution,}$$

is the rth moment of X about origin. Thus we see that the coefficient of $\frac{t^r}{r!}$ in $M_X(t)$ gives μ_r' (above origin). Since $M_X(t)$ generates moments, it is known as moment generating function.

Differentiating (6·55) w.r.t. t and then putting $t = 0$, we get

$$\left[\frac{d^r}{dt^r}\{M_X(t)\}\right]_{t=0} = \left[\frac{\mu_r'}{r!}\cdot r! + \mu'_{r+1}\,t + \mu'_{r+2}\cdot\frac{t^2}{2!} + \ldots\right]_{t=0}$$

$$\Rightarrow \qquad \mu_r' = \left[\frac{d^r}{dt^r}\{M_X(t)\}\right]_{t=0} \qquad \ldots(6{\cdot}56)$$

In general, the moment generating function of X about the point $X = a$ is defined as

$$M_X(t) \text{ (about } X = a) = E[e^{t(X-a)}]$$

$$= E\left[1 + t(X-a) + \frac{t^2}{2!}(X-a)^2 + \ldots + \frac{t^r}{r!}(X-a)^r + \ldots\right]$$

$$= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \ldots + \frac{t^r}{r!}\mu_r' + \ldots \qquad \ldots(6{\cdot}57)$$

where $\mu_r' = E\{(X-a)^r\}$, is the rth moment about the point $X = a$.

Theorem 6·17. $M_{cX}(t) = M_X(ct)$, c being a constant.

Proof. By def.,

$$\text{L.H.S.} = M_{cX}(t) = E(e^{t.cX})$$

$$\text{R.H.S.} = M_X(ct) = E(e^{ct.X}) = \text{L.H.S.}$$

Theorem 6·18. *The moment generating function of the sum of a number of independent random variables is equal to the product of their respective moment generating functions.*

Symbolically, if $X_1, X_2, \ldots, X_n$ are independent random variables, then the moment generating function of their sum $X_1 + X_2 + \ldots + X_n$ is given by

$$M_{X_1 + X_2 + \ldots + X_n}(t) = M_{X_1}(t)\, M_{X_2}(t) \ldots M_{X_n}(t) \qquad \ldots(6{\cdot}59)$$

Proof. By definition,

$$\begin{aligned} M_{X_1 + X_2 + \ldots + X_n}(t) &= E\left[e^{t(X_1 + X_2 + \ldots + X_n)}\right] \\ &= E\left[e^{tX_1} . e^{tX_2} \ldots e^{tX_n}\right] \\ &= E(e^{tX_1})\, E(e^{tX_2}) \ldots E(e^{tX_n}) \\ &\qquad (\because X_1, X_2, \ldots, X_n \text{ are independent}) \\ &= M_{X_1}(t) . M_{X_2}(t) \ldots M_{X_n}(t) \end{aligned}$$

Hence the theorem.

Theorem 6·19. *Effect of change of origin and scale on M.G.F.* Let us transform X to the new variable U by changing both the origin and scale in X as follows :

$$U = \frac{X - a}{h}, \text{ where } a \text{ and } h \text{ are constants}$$

M.G.F. of U (about origin) is given by

$$M_U(t) = E(e^{tU}) = E\left[\exp\{t(x-a)/h\}\right].$$

$$= E\left[e^{tX/h} \cdot e^{-at/h}\right] = e^{-at/h} E(e^{tX/h})$$

$$= e^{-at/h} E(e^{Xt/h}) = e^{-at/h} M_X(t/h) \qquad \ldots(6{\cdot}60)$$

where $M_X(t)$ is the m.g.f. of X about origin.

Example 6·37. *Let the random variable X assume the value 'r' with the probability law :*

$$P(X=r)=q^{r-1}p\,;\;\; r=1,2,3,\ldots$$

Find the m.g.f. of X and hence its mean and variance.

Solution. $M_X(t) = E(e^{tX})$

$$= \sum_{r=1}^{\infty} e^{tr} \quad q^{r-1} p = \frac{p}{q} \sum_{r=1}^{\infty} (qe^t)^r$$

$$= \frac{p}{q} qe^t \sum_{r=1}^{\infty} (qe^t)^{r-1} = pe^t [1 + qe^t + (qe^t)^2 + \ldots]$$

$$= \left(\frac{pe^t}{1 - qe^t}\right)$$

If dash (′) denotes the differentiation w.r.t. t, then we have

$$M_X'(t) = \frac{pe^t}{(1 - qe^t)^2}, \quad M_X''(t) = pe^t \frac{(1 + qe^t)}{(1 - qe^t)^3}$$

$\therefore$ $$\mu_1' \text{ (about origin)} = M_X'(0) = \frac{p}{(1-q)^2} = \frac{1}{p}$$

$$\mu_2' \text{ (about origin)} = M_X''(0) = \frac{p(1+q)}{(1-q)^3} = \frac{1+q}{p^2}.$$

Hence $$\text{mean} = \mu_1' \text{ (about origin)} = \frac{1}{p}$$

and $$\text{variance} = \mu_2 = \mu_2' - \mu_1'^2 = \frac{1+q}{p^2} - \frac{1}{p^2} = \frac{q}{p^2}$$

Example 6·38. *The probability density function of the random variable X follows the following probability law :*

$$p(x) = \frac{1}{2\theta} \exp\left(-\frac{|x-\theta|}{\theta}\right), \ -\infty < x < \infty$$

Find M.G.F. of X. Hence or otherwise find E (X) and V (X).

Solution. The moment generating function of X is

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} \frac{1}{2\theta} \exp\left(-\frac{|x-\theta|}{\theta}\right) e^{tx}\, dx$$

$$= \int_{-\infty}^{\theta} \frac{1}{2\theta} \exp\left(-\frac{|\theta-x|}{\theta}\right) e^{tx}\, dx$$

$$+ \int_{\theta}^{\infty} \frac{1}{2\theta} \exp\left(-\frac{|x-\theta|}{\theta}\right) e^{tx}\, dx$$

For $x \in (-\infty, \theta)$, $\quad x-\theta<0 \;\Rightarrow\; \theta-x>0$

$\therefore \qquad |x-\theta| = \theta - x \;\; \forall\, x \in (-\infty, \infty)$

Similarly, $\quad |x-\theta| = x-\theta \quad \forall\, x \in (\theta, \infty)$

$$\therefore \quad M_X(t) = \frac{e^{-1}}{2\theta} \int_{-\infty}^{\theta} \exp\left[x\left(t+\frac{1}{\theta}\right)\right] dx + \frac{e}{2\theta} \int_{\theta}^{\infty} \exp\left[-x\left(\frac{1}{\theta}-t\right)\right] dx$$

$$= \frac{e^{-1}}{2\theta} \cdot \frac{1}{\left(t+\frac{1}{\theta}\right)} \cdot \exp\left[\theta\left(t+\frac{1}{\theta}\right)\right]$$

$$+ \frac{e}{2\theta} \cdot \frac{1}{\left(\frac{1}{\theta}-t\right)} \cdot \exp\left[-\theta\left(\frac{1}{\theta}-t\right)\right]$$

$$= \frac{e^{\theta t}}{2(\theta t+1)} + \frac{e^{\theta t}}{2(1-\theta t)} = \frac{e^{\theta t}}{1-\theta^2 t^2}$$

$$= e^{\theta t}(1-\theta^2 t^2)^{-1}$$

$$= \left[1 + \theta t + \frac{\theta^2 t^2}{2!} + \ldots\right] \cdot \left[1 + \theta^2 t^2 + \theta^4 t^4 + \ldots\right]$$

$$= 1 + \theta t + \frac{3\,\theta^2 t^2}{2!} + \ldots$$

$\therefore \quad E(X) = \mu' = \text{Coefficient of } t \text{ in } M_X(t) = \theta$

$$\mu_2' = \text{Coefficient of } \frac{t^2}{2!} \text{ in } M_X(t) = 3\,\theta^2$$

Hence $\quad \mathrm{Var}(X) = \mu_2' - \mu_1'^{\,2} = 3\,\theta^2 - \theta^2 = 2\,\theta^2$

Moments of Binomial distribution

$$\mu_1' = E(x) = \sum_{x=0}^{n} x \, {}^nC_x \, p^x q^{n-x}$$

$${}^nC_x = \frac{\lfloor n}{\lfloor x \, \lfloor n-x} = \frac{n \lfloor (n-1)}{x \lfloor x-1 \, \lfloor (n-1)-(x-1)}$$

$$= \frac{n}{x} \, {}^{n-1}C_{x-1}$$

$${}^{n-1}C_{x-1} = \frac{\lfloor n-1}{\lfloor x-1 \, \lfloor (n-1)-(x-1)}$$

$$= \frac{(n-1) \lfloor n-2}{(x-1) \lfloor x-2 \, \lfloor (n-2)-(x-2)}$$

$$= \frac{(n-1)}{(x-1)} \, {}^{n-2}C_{x-2}$$

So $^{n}C_x = \frac{n}{x}\,^{n-1}C_{x-1} = \frac{n(n-1)}{x(x-1)}\,^{n-2}C_{x-2}$.

$$\mu_1' = \sum_{x=0}^{n} x\,\frac{n}{x}\,^{n-1}C_{x-1}\,p^x q^{n-x}$$

$$= np \sum_{x=1}^{n} {}^{n-1}C_{x-1}\,p^{x-1} q^{(n-1)-(x-1)}$$

$$= np\left({}^{n-1}C_0\,p^0 q^{n-1} + {}^{n-1}C_1\,p^1 q^{n-2} + \cdots + {}^{n-1}C_{n-1}\,p^{n-1} q^0\right)$$

$$= np(q+p)^{n-1} = np$$

$$\mu_2' = E(x^2) = \sum_{x=0}^{n} x^n \, {}^nC_x \, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \{x(x-1)+x\} \frac{n(n-1)}{x(x-1)} \, {}^{n-2}C_{x-2} \, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1) \frac{n(n-1)}{x(x-1)} \, {}^{n-2}C_{x-2} \, p^x q^{n-x}$$

$$+ \sum_{x=0}^{n} x \, {}^nC_x \, p^x q^{n-x}$$

$$= n(n-1)p^2 \sum_{x=2}^{n} {}^{n-2}C_{x-2} \, p^{x-2} q^{(n-2)-(x-2)}$$

$$+ np$$

$$= n(n-1)p^2 (q+p)^{n-2} + np$$

$$= n(n-1)p^2 + np = n(n-1)p^2 + np$$

$$\mu_3' = E(x^3) = \sum_{x=0}^{n} x^3 p(x)$$

$$= \sum_{x=0}^{n} \{x(x-1)(x-2) + 3x(x-1) + x\}\, {}^nC_x\, p^x q^{n-x}$$

continue similarly

$$\mu_3' = n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np$$

$$\mu_4' = E(x^4)$$

$$= \sum_{x=0}^{n} x^4 p(x)$$

$$x^4 = x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 7x(x-1) + x$$

$$\mu_4' = n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np$$

§ Central moments of Binomial distribution.

$$\mu_2 = \mu_2' - (\mu_1')^2 = n(n-1)p^2 + np - n^2p^2$$

$$= np - np^2 = np(1-p) = npq$$

$$\cancel{\mu_3'} = \mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

$$= npq(q-p)$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

$$= npq\{1 + 3(n-2)pq\}$$

Hence $\beta_1 = \dfrac{\mu_3^2}{\mu_2^3} = \dfrac{(1-2p)^2}{npq}$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = 3 + \frac{1-6pq}{npq}$$

$$\gamma_1 = \sqrt{\beta_1} = \frac{1-2p}{npq}, \quad \gamma_2 = \beta_2 - 3 = \frac{1-6pq}{npq}$$

Moments of the Poisson Distribution

$$\mu_1' = E(x) = \sum_{x=0}^{\infty} x\,p(x) = \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{\lfloor\!\underline{x}}$$

$$= \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{\lfloor\!\underline{x-1}} = \lambda e^{-\lambda}\left(1+\lambda+\frac{\lambda^2}{\lfloor\!\underline{2}}+\cdots\right)$$

$$= \lambda e^{-\lambda} e^{\lambda} = \lambda$$

$$\mu_2' = E(x^2) = \sum_{x=0}^{\infty} x^2 p(x)$$

$$= \sum_{x=0}^{\infty} \{x(x-1)+x\} \frac{e^{-\lambda}\lambda^x}{\lfloor\!\underline{x}}$$

$$= \sum_{x=0}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^x}{x(x-1)\lfloor\!\underline{x-2}} + \sum_{x=0}^{\infty} x\frac{e^{-\lambda}\lambda^x}{\lfloor\!\underline{x}}$$

$$= \lambda^2 e^{-\lambda} \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{\lfloor\!\underline{x-2}} + \lambda$$

$$= \lambda^2 e^{-\lambda} e^{\lambda} + \lambda = \lambda^2+\lambda$$

$$\mu_3' = E(x^3) = \sum_{x=0}^{\infty} x^3 p(x)$$

Follow as in Binomial

$$\mu_3' = \lambda^3 + 3\lambda^2 + \lambda$$

$$\mu_4' = E(x^4) = \sum_{x=0}^{\infty} x^4 p(x)$$

$$\mu_4' = \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda$$

$$\mu_2 = \mu_2' - (\mu_1')^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

$$= \lambda^3 + 3\lambda^2 + \lambda - 3\lambda(\lambda^2 + \lambda) + 2\lambda^3 = \lambda$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

$$= 3\lambda^2 + \lambda$$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{1}{\lambda}, \quad \gamma_1 = \sqrt{\beta_1} = \frac{1}{\sqrt{\lambda}}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = 3 + \frac{1}{\lambda}, \quad \gamma_2 = \beta_2 - 3 = \frac{1}{\lambda}$$

Moment Generating Function of Binomial distribution

$$M_X(t) = E(e^{tX}) = \sum_{x=0}^{n} e^{tx}\, {}^nC_x\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} {}^nC_x\, (pe^t)^x q^{n-x}$$

$$= {}^nC_0\, (pe^t)^0 q^n + {}^nC_1\, (pe^t)^1 q^{n-1} + \cdots + {}^nC_n (pe^t)^n q^0$$

$$M_X(t) = (q + pe^t)^n$$

$$\mu_r' = \left| \frac{d^r}{dt^r} M_X(t) \right|_{\text{at } t=0}$$

$$\mu_1' = \left| \frac{d}{dt} M_X(t) \right|_{\text{at } t=0}$$

$$\frac{d}{dt} (q + pe^t)^n = n (q + pe^t)^{n-1} pe^t$$

at $t=0$, $\mu_1' = np\, e^0 (q + pe^0)^{n-1}$

$$= np \cdot 1\, (q+p)^{n-1} = np.$$

$$\mu_2' = \frac{d^2}{dt^2} M_X(t) = \frac{d}{dt} \left\{ np\, e^t (q + pe^t)^{n-1} \right\}$$

$$= np \left\{ e^t (n-1)(q + pe^t)^{n-2} pe^t + (q + pe^t)^{n-1} e^t \right\}$$

at $t=0$, $\mu_2' = np\{(n-1)p + 1\} = n(n-1)p^2 + np$

Mean $= np$, Var $= npq$

Moment Generating function of the Poisson Distribution

$$M_X(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{\lfloor x}$$

$$= \sum_{x=0}^{\infty} e^{-\lambda} \frac{(\lambda e^t)^x}{\lfloor x}$$

$$= e^{-\lambda} \left\{ 1 + \lambda e^t + \frac{(\lambda e^t)^2}{\lfloor 2} + \cdots \right)$$

$$= e^{-\lambda} e^{\lambda e^t}$$

$$M_X(t) = e^{\lambda(e^t - 1)}$$

$$\mu_r' = \left\{ \frac{d^r}{dt^r} M_X(t) \right\}_{\text{at } t=0}$$

$$\mu_1' = \left| \frac{d}{dt} M_X(t) \right|_{\text{at } t=0}$$

$$\frac{d}{dt} e^{\lambda(e^t - 1)} = e^{\lambda(e^t - 1)} e^t \cdot \lambda$$

$$\text{at } t=0, \ \mu_1' = \lambda e^0 e^{\lambda(e^0 - 1)}$$

$$= \lambda \cdot 1 \, e^{\lambda(1-1)} = \lambda \cdot e^0 = \lambda$$

$$\frac{d^2}{dt^2} M_X(t) = \frac{d}{dt} \lambda e^t e^{\lambda(e^t-1)}$$

$$= \lambda \left[e^t e^{\lambda(e^t-1)} \cdot \lambda e^t + e^{\lambda(e^t-1)} \cdot e^t \right]$$

at $t=0$, $\mu_2' = \lambda\left[\lambda \cdot e^0 e^0 e^{\lambda(e^0-1)} + e^0 e^{\lambda(e^0-1)}\right]$

$$= \lambda[\lambda+1) = \lambda^2+\lambda$$

Mean $= \lambda$, Var $= \lambda^2+\lambda-\lambda^2 = \lambda$