Problems on Probability

The odds that person X speaks the truth are 3 :2 and the odds that person Y speaks the truth are 5:3. In what percentage of cases are they likely to contradict each other on an-identical point.

Solution. Let us define the events:

$A : X$ speaks the truth, $\quad B : Y$ speaks the truth

Then $\overline{A}$ and $\overline{B}$ represent the complementary events that X and Y tell a lie respectively. We are given:

$$P(A) = \frac{3}{3+2} = \frac{3}{5} \quad \Rightarrow \quad P(\overline{A}) = 1 - \frac{3}{5} = \frac{2}{5}$$

and $$P(B) = \frac{5}{5+3} = \frac{5}{8} \quad \Rightarrow \quad P(\overline{B}) = 1 - \frac{5}{8} = \frac{3}{8}$$

The event E that X and Y contradict each other on an identical point can happen in the following mutually exclusive ways:

(i) X speaks the truth and Y tells a lie, *i.e.*, the event $A \cap \overline{B}$ happens,

(ii) X tells a lie and Y speaks the truth, *i.e.*, the event $\overline{A} \cap B$ happens.

Hence by addition theorem of probability the required probability is given by:

$$P(E) = P(i) + P(ii) = P(A \cap \overline{B}) + P(\overline{A} \cap B)$$

$$= P(A) \cdot P(\overline{B}) + P(\overline{A}) \cdot P(B),$$

[Since A and B are independent]

$$= \frac{3}{5} \times \frac{3}{8} + \frac{2}{5} \times \frac{5}{8} = \frac{19}{40} = 0{\cdot}475$$

Hence A and B are likely to contradict each other on an identical point in 47·5% of the cases.

(b) One shot is fired from each of the three guns. E_1, E_2, E_3 *denote the events that the target is hit by the first, second and third gun respectively. If* $P(E_1) = 0{\cdot}5$, $P(E_2) = 0{\cdot}6$ *and* $P(E_3) = 0{\cdot}8$ *and* E_1, E_2, E_3 *are independent events, find the probability that (a) exactly one hit is registered, (b) at least two hits are registered.*

(a) Exactly one hit can be registered in the following mutually exclusive ways: *(i)* $E_1 \cap \overline{E_2} \cap \overline{E_3}$ happens, *(ii)* $\overline{E_1} \cap E_2 \cap \overline{E_3}$ happens, *(iii)* $\overline{E_1} \cap \overline{E_2} \cap E_3$ happens.

Hence by addition probability theorem, the required probability 'p' is given by :

$$p = P(E_1 \cap \overline{E_2} \cap \overline{E_3}) + P(\overline{E_1} \cap E_2 \cap \overline{E_3}) + P(\overline{E_1} \cap \overline{E_2} \cap E_3)$$

$$= P(E_1)\, P(\overline{E_2})\, P(\overline{E_3}) + P(\overline{E_1})\, P(E_2)\, P(\overline{E_3}) + P(\overline{E_1})\, P(\overline{E_2})\, P(E_3)$$

(Since E_1, E_2 and E_3 are independent)

$$= 0{\cdot}5 \times 0{\cdot}4 \times 0{\cdot}2 + 0{\cdot}5 \times 0{\cdot}6 \times 0{\cdot}2 + 0{\cdot}5 \times 0{\cdot}4 \times 0{\cdot}8 = 0{\cdot}26.$$

(b) At least two hits can be registered in the following mutually exclusive ways: *(i)* $E_1 \cap E_2 \cap \overline{E_3}$ happens *(ii)* $E_1 \cap \overline{E_2} \cap E_3$ happens, *(iii)* $\overline{E_1} \cap E_2 \cap E_3$ happens. *(iv)* $E_1 \cap E_2 \cap E_3$ happens.

Required probability

$$= P(E_1 \cap E_2 \cap \overline{E_3}) + P(E_1 \cap \overline{E_2} \cap E_3) + P(\overline{E_1} \cap E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3)$$

$$= 0{\cdot}5 \times 0{\cdot}6 \times 0{\cdot}2 + 0{\cdot}5 \times 0{\cdot}4 \times 0{\cdot}8 + 0{\cdot}5 \times 0{\cdot}6 \times 0{\cdot}8 + 0{\cdot}5 \times 0{\cdot}6 \times 0{\cdot}8$$

$$= 0{\cdot}06 + 0{\cdot}16 + 0{\cdot}24 + 0{\cdot}24 = 0{\cdot}70$$

If A and B are two independent events such that $P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{5}$, then

(a) $P(A \cup B) = 3/5$ (b) $P(A|B) = 1/2$

(c) $P(A|A \cup B) = 5/6$ (d) $P(A \cap B | A' \cup B') = 0$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{2} + \frac{1}{5} - \frac{1}{2} \times \frac{1}{5} = 3/5$$

$$P(A/B) = 1/2$$

$$P(A \mid A \cup B) = \frac{P(A \cap (A \cup B))}{P(A \cup B)} = \frac{P(A)}{P(A \cup B)} = \frac{\frac{1}{2}}{3/5} = 5/6$$

$$P(A \cap B \mid A' \cup B') = P(A \cap B \mid (A \cap B)') = 0$$

For two events A and B, if $P(A) = P(A/B) = 1/4$ and $P(B/A) = 1/2$, then

(a) A and B are independent

(b) A and B are mutually exclusive

(c) $P(A'/B) = 3/4$ (d) $P(B'/A') = 1/2$

$$P(A) = P(A/B) \Rightarrow P(A) = \frac{P(A \cap B)}{P(B)} \Rightarrow P(A \cap B) = P(A)P(B)$$

A and B are independent

$$P(A \cap B) = P(A)\,P(B/A)$$

$$= \frac{1}{4} \times \frac{1}{2} = \frac{1}{8} \neq 0$$

$$P(A'/B) = P(A') = 1 - P(A) = 1 - \frac{1}{4} = \frac{3}{4}$$

$$P(B'/A') = P(B') = 1 - P(B) = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\left[\because P(B) = P(B/A) = \frac{1}{2}\right]$$

If A and B are two independent events such that $P(A' \cap B) = 2/15$ and $P(A \cap B') = 1/6$, then find $P(B)$

$P(A' \cap B) = \frac{2}{15} \Rightarrow P(A')\,P(B) = \frac{2}{15}$

$\Rightarrow (1-a)\,b = \frac{2}{15}$ $[\,P(A) = a,\ P(B) = b\,]$ —— (I)

$P(A \cap B') = \frac{1}{6} \Rightarrow P(A)\,P(B') = \frac{1}{6}$

$\Rightarrow a(1-b) = \frac{1}{6}$ —— (II)

From (I) and (II)

$$\begin{array}{r} b - ab = 2/15 \\ \underline{\overset{-}{}a \overset{-}{+} ab = \overset{-}{}1/6} \end{array}$$

$\Rightarrow b - a = \frac{2}{15} - \frac{1}{6} = \frac{4-5}{30} = -\frac{1}{30}$

$\Rightarrow a = b + \frac{1}{30}$

$\left(b + \frac{1}{30}\right)(1-b) = \frac{1}{6} \Rightarrow (30b+1)(1-b) = 5$

$\Rightarrow 30b + 1 - 30b^2 - b = 5 \Rightarrow 30b^2 - 29b + 4 = 0$

$\Rightarrow 30b^2 - 24b - 5b + 4 = 0 \Rightarrow 6b(5b-4) - 1(5b-4) = 0$

$\Rightarrow (5b-4)(6b-1) = 0 \Rightarrow b = 4/5,\ 1/6$

$P(B) = \frac{1}{6},\ 4/5$

A student appears for tests I, II, and III. The student is successful if he passes either in tests I and II or test I and test III. The probabilities of the student passing in tests I, II, III are p, q and $\frac{1}{2}$ respectively. If the probability that the student is successful is $1/2$ then

(a) $p = 1,\ q = 0$ (b) $p = 2/3,\ q = 1/2$

(c) $p = 3/5,\ q = 2/3$ (d) There are infinitely many values of p and q

P (the student is successful)

$$= P(A \cap B \cap \overline{C}) + P(A \cap \overline{B} \cap C) + P(A \cap B \cap C)$$

$$= P(A)\,P(B)\,P(\overline{C}) + P(A)\,P(\overline{B})\,P(C) + P(A)\,P(B)\,P(C)$$

$$= pq(1-1/2) + p(1-q)\frac{1}{2} + pq \cdot \frac{1}{2}$$

$$= \frac{1}{2}\left[pq + p(1-q) + pq\right] = \frac{1}{2}\left[p + pq\right]$$

$$\Rightarrow \frac{1}{2} = \frac{1}{2}(1+q)p \Rightarrow p(1+q) = 1$$

There are 10 pairs of shoes in a cupboard, from which 4 shoes are picked at random. The probability that there is at least one pair is ___

Req. prob $= 1 - \dfrac{{}^{10}C_4 \times 2^4}{{}^{20}C_4}$