UNIT 6

Hypothesis Testing : Types of Error, Student t-test for single mean and difference of means, Z-test for single mean and difference of means, F-test, goodness of fit, Chi- Square Test

Tests of significance

A very important aspect of the sampling theory is the study of the tests of significance, which enable us to decide on the basis of the sample results, if

- The deviation between the observed sample statistics and the hypothetical parameter value, or
- The deviation between two independent sample statistics;

is significant or might be attributed to chance or fluctuations of sampling.

In all the testing of hypothesis,

F-test, T-test, Z-test, chi-square test,

Etc.

there is a first need of defining the hypothesis or claim.

This hypothesis is classified as

Null Hypothesis denoted by H_0

Alternative Hypothesis denoted by H_1.

Null hypothesis is the hypothesis which is tested for possible rejection under the assumption that it is true.

Any hypothesis which is complementary to the null hypothesis is called alternative hypothesis.

The alternative hypothesis is either two-tailed or one-tailed alternatives.

For example- if we want to test the null hypothesis ($H_0: \mu = \mu_0$) then the H_1 could be:

$H_1: \mu \neq \mu_0$	Two-tailed alternatives
$H_1: \mu > \mu_0$	right-tailed alternatives
$H_1: \mu < \mu_0$	left-tailed alternatives

Examples on How to write H_o and H_1

1

Example: Consider a random sample of 100 carpenters produces a sample mean $15 with standard deviation 2. You wish to show that the average hourly wage of carpenters in the state of USA is different from $14, which is the national average. Test the appropriate hypothesis at 5% level of significance.

Solution: Define the Hypothesis:

Null Hypothesis: $H_0: \mu = 14$

Alternative Hypothesis: $H_1: \mu \neq 14$ (Two-tailed)

Example: A milling process currently produces an average of 3% defectives. You are interested in showing that a simple adjustment on a machine will decrease p, *the proportion of defectives* produced in the milling process.

Solution: Define the Hypothesis:

Null Hypothesis: H_0: $p = 0.03$

Alternative Hypothesis: H_1: $p < 0.03$ (One-tailed)

Example: A group of 5 patients treated with medicines "A" weigh 42, 39, 48, 60 and 41 kgs; second group of 7 patients from the same hospital treated with medicine "B" weight 38, 42, 56, 64, 68, 69 and 62 kgs. Do you agree with the claim that the medicine "B" increases the weight significantly?

Solution: Define the Hypothesis:

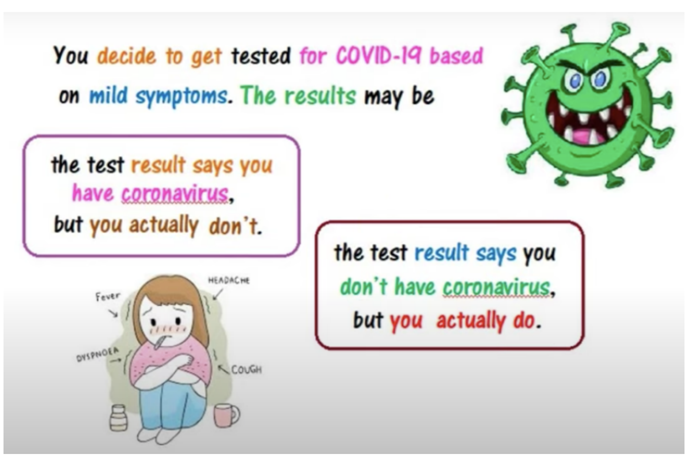
Null Hypothesis: H_0: $\mu_1 - \mu_2 = 0$

Alternative Hypothesis: H_1: $\mu_1 - \mu_2 < 0$ (One-Tailed)

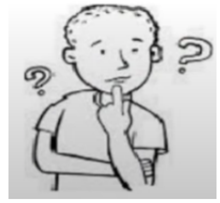
ERROR in Hypothesis

explanation with following real life situation

Examples

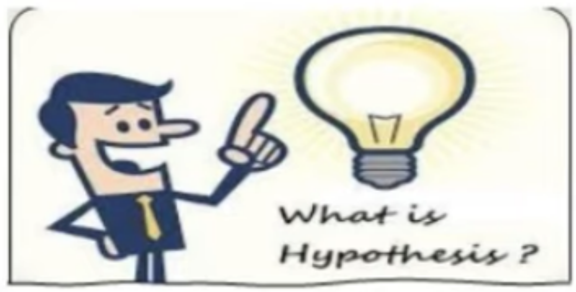


Is there any ERROR ?



Null Hypothesis: H_0

Alternative Hypothesis: H_1



To address this, we need to Study

Type I and Type II errors

Type I and Type II errors:

When conducting a hypothesis test there are two possible decisions:

- Reject the H_0 or
- Fail to reject the H_0

When we reject the H_0 there are two possibilities.

- There could really be a difference in the population, in which case we made a correct decision.
- It is possible that there is not a difference in the population (i.e., H0 is true) but our sample was different from the hypothesized value due to random sampling variation.

In that case, we made an error.

In statistics,

Type I error is a false positive conclusion,

while a

Type II error is a false negative conclusion.

Decision	Reality	
	H0 is true	H0 is false
Reject H0	Type I error	Correct decision
Fail to reject H0	Correct decision	Type II error

In other words,

The error of rejecting H_0 when H_0 is true

is called Type I error.

The error of accepting H_0 when H_0 is false

is called Type II error.

Decision	Reality	
	H0 is true	H0 is false
Reject H0	Type I error	Correct decision
Fail to reject H0	Correct decision	Type II error

Examples of Type I and Type II errors:

You decide to get tested for COVID-19 based on mild symptoms. There are two errors that could potentially occur:

- Type I error (false positive): the test result says you have coronavirus, but you actually don't.
- Type II error (false negative): the test result says you don't have coronavirus, but you actually do.

The probabilities of type I and II errors are denoted by α and β respectively.

$$\alpha = P(\text{Rejecting } H_0 \mid H_0 \text{ is true})$$

$$\beta = P(\text{Accepting } H_0 \mid H_0 \text{ is false})$$

Following content as per textbook or reference book

12·6. Errors in Sampling. The main objective in sampling theory is to draw valid inferences about the population parameters on the basis of the sample results. In practice we decide to accept or reject the lot after examining a sample from it. As such we are liable to commit the following two types of errors :

Type I Error : *Reject H_0 when it is true.*

Type II Error : *Accept H_0 when it is wrong, i.e., accept H_0 when H_1 is true.*

If we write.

$$\left.\begin{aligned} P\{\text{Reject } H_0 \text{ when it is true}\} &= P\{\text{Reject } H_0 \mid H_0\} = \alpha \\ \text{and } P\{\text{Accept } H_0 \text{ when it is wrong}\} &= P\{\text{Accept } H_0 \mid H_1\} = \beta \end{aligned}\right\} \quad \ldots(12{\cdot}2)$$

then α and β are called the *sizes of type I error and type II error*, respectively.

In practice, type I error amounts to rejecting a lot when it is good and type II error may be regarded as accepting the lot when it is bad.

$$\left.\begin{aligned} \text{Thus} \quad & P\{\text{Reject a lot when it is good}\} = \alpha \\ \text{and} \quad & P\{\text{Accept a lot when it is bad}\} = \beta \end{aligned}\right\} \quad \ldots(12{\cdot}2a)$$

where α and β are referred to as *Producer's risk* and *Consumer's risk*, respectively.

12·7·2. Critical Values or Significant Values. The value of test statistic which separates the critical (or rejection) region and the acceptance region is called the *critical value* or *significant* value. It depends upon :

(*i*) The level of significance used, and

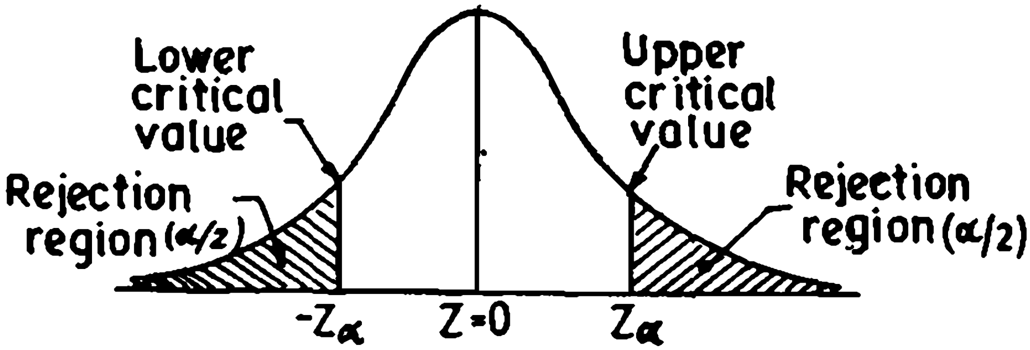
(*ii*) The alternative hypothesis, whether it is two-tailed or single-tailed.

As has been pointed out earlier, for large samples, the standardised variable corresponding to the statistic t *viz.* :

$$Z = \frac{t - E(t)}{S.E.(t)} \sim N(0, 1), \quad \ldots(*)$$

asymptotically as $n \to \infty$. The value of Z given by (*) under the null hypothesis is known as *test statistic*. The critical value of the test statistic at level of significance α for a two-tailed test is given by z_α where z_α is determined by the equation

$$P(|Z| > z_\alpha) = \alpha \quad \ldots(12{\cdot}2c)$$



Thus *the significant* or *critical value of Z for a single-tailed test (left or right) at level of significance 'α' is same as the critical value of Z for a two-tailed test at level of significance '2α'*.

Que: For the same significant or critical value 'z_α' if 'α' is the level of significance
for a two tailed test then the level of significance for a single tailed test will be equal to:

(a) 2α (b) α (c) $\frac{\alpha}{2}$ (d) *None of these.*

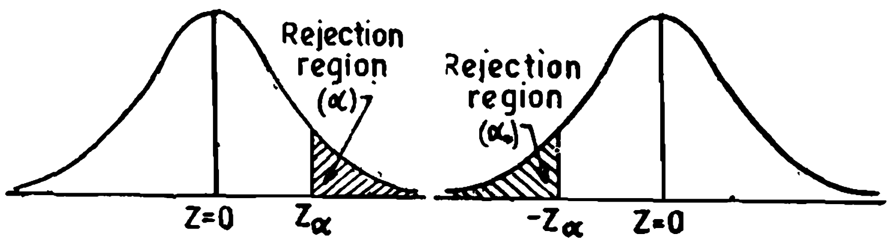
For the same significant or critical value 'z_α' the level of significance for a
single tailed test $\left(\frac{\alpha}{2}\right)$ is half the level of significance for two tailed test (α).

In case of single-tail alternative, the critical value z_α is determined so that total area to the right of it (for right-tailed test) is α and for left-tailed test the total area to the left of $-z_\alpha$ is α (See diagrams below), *i.e.*,

For Right-tail Test : $P(Z > z_\alpha) = \alpha$ ...(12·2*d*)

For Left-tail Test : $P(Z < -z_\alpha) = \alpha$ ...(12·2*e*)

RIGHT-TAILED TEST (*Level of Signifiance 'α'*) LEFT-TAILED TEST (*Level of Significance 'α'*)



CRITICAL VALUES (z_α) OF Z

Critical Values (z_α)	*Level of significance* (α) 1%	5%	10%
Two-tailed test	$\lvert Z_\alpha \rvert = 2{\cdot}58$	$\lvert Z_\alpha \rvert = 1{\cdot}96$	$\lvert Z_\alpha \rvert = 1{\cdot}645$
Right-tailed test	$Z_\alpha = 2{\cdot}33$	$Z_\alpha = 1{\cdot}645$	$Z_\alpha = 1{\cdot}28$
Left-tailed test	$Z_\alpha = -2{\cdot}33$	$Z_\alpha = -1{\cdot}645$	$Z_\alpha = -1{\cdot}28$

$$P(-3 \le Z \le 3) = 0.9973,$$
$$i.e., P(|Z| \le 3) = 0.9973$$
$$P(|Z| > 3) = 1 - P(|Z| \le 3) = 0.0027$$
$$P(-1.96 \le Z \le 1.96) = 0.95,$$
$$i.e., P(|Z| \le 1.96) = 0.95$$
$$P(|Z| > 1.96) = 1 - P(|Z| \le 1.96) = 0.05$$
$$P(|Z| \le 2.58) = 0.99 \; P(|Z| > 2.58) = 0.01$$

Procedure for Testing of Hypothesis

1. *Null Hypothesis*. Set up the Null Hypothesis H_0

2. *Alternative Hypothesis*. Set up the Alternative Hypothesis H_1.

3. *Level of Significance*. Choose the appropriate level of significance (α)

α is fixed in advance.

4. *Test Statistic (or Test Criterion)*. Compute the test statistic

$$Z = \frac{t - E(t)}{S.E.(t)}$$

5. *Conclusion*. We compare z the computed value of Z in step 4 with the significant value (tabulated value) z_α, at the given level of significance, 'α'.

If $|Z| < z_\alpha$,

the null hypothesis which may therefore, be accepted.

If $|Z| > z_\alpha$,

the null hypothesis is rejected at level of significance α

Tests of Hypotheses and Tests of Significance

When we attempt to make decisions about the population on the basis of sample information, we have to make assumptions or guesses about the nature of the population involved or about the value of some parameter of the population.

Such assumptions, which may or may not be true, are called statistical hypotheses there is no significant difference between the sample statistic and the corresponding population parameter or between two sample statistics

null hypothesis H0.
a hypothesis of no difference

alternative hypothesis H1
hypothesis that is different from
(or complementary to) the null hypothesis

A procedure for deciding whether to accept or to reject a null hypothesis (and hence to reject or to accept the alternative hypothesis respectively) is called the test of hypothesis

insignificant difference

difference which is caused due to sampling fluctuations

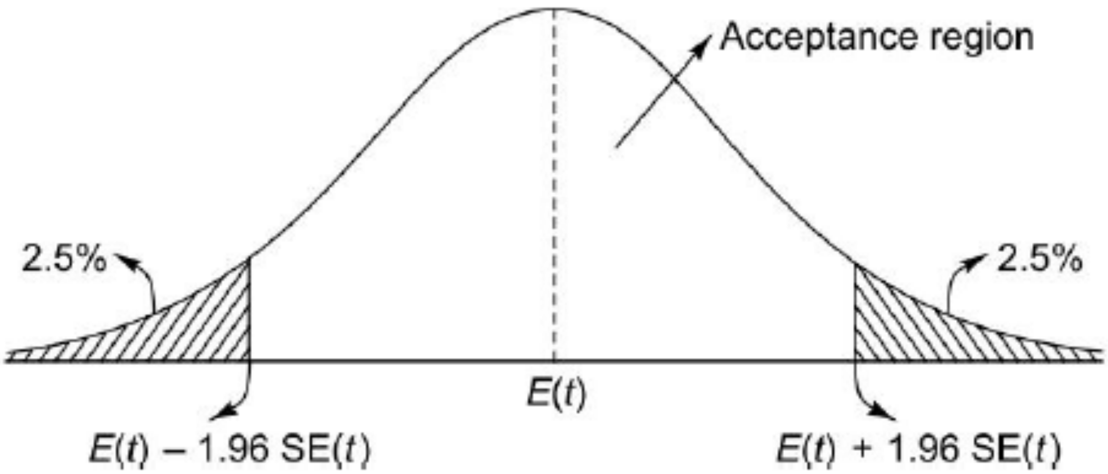
significant difference
difference that arises either because the sampling procedure is not purely random or that the sample has not been drawn from the given population

This procedure of testing whether the difference between
θ_0 and θ a is significant or not is called the test of significance

Critical Region and Level of Significance

critical region or *region of rejection.*

The region complementary to the critical region is called the *region of acceptance.*



large samples follows normal distributions

t is a statistic in large samples and it follows
a normal distribution with mean E (t),

$$Z = \frac{t - E(t)}{\mathrm{SE}(t)}$$

95% of the values of t will lie between $[E(t) \mp 1.96\ \mathrm{SE}(t)]$
or only 5% of values of t will lie outside this interval

level of significance (LOS)

the total area of the critical region expressed as $\alpha\%$ is the LOS.

From the study of normal distributions, it is known that

$$P\{E(t) - 1.96\ \mathrm{SE}(t) < t < E(t) + 1.96\ \mathrm{SE}(t)\} = 0.95$$

i.e. $$P\left\{\left|\frac{t - E(t)}{\mathrm{SE}(t)}\right| < 1.96\right\} = 0.95$$

i.e. $$P\{|Z| > 1.96\} = 0.05 \quad \text{or} \quad 5\%$$

Thus when t lies in either of the two regions constituting the critical region given above, the LOS is 5%.

Usually the LOS are taken as 5, 2 or 1%.

Errors in Testing of Hypotheses

Type I error.

The error committed in rejecting H_0, when it is really true.

Type II error.

The error committed in accepting $\mathbf{H}_0$, when it is false

The probabilities of committing Type I and II errors are denoted by α and β respectively.

The probabilities of type I and II errors are denoted by α and β respectively.

$$\alpha = P(\text{Rejecting } H_0 \mid H_0 \text{ is true})$$

$$\beta = P(\text{Accepting } H_0 \mid H_0 \text{ is false})$$

One-Tailed and Two-Tailed Tests

θ_0 is a population parameter
θ is the corresponding sample statistic
null hypothesis $H_0 : \theta = \theta_0$,
then the alternative hypothesis
can be any one of the following:

(i) $H_1 : \theta \neq \theta_0$, i.e. $\theta > \theta_0$ or $\theta < \theta_0$
(ii) $H_1 : \theta > \theta_0$
(iii) $H_1 : \theta < \theta_0$

Critical Values or Significant Values

The critical values for some standard LOSs are given in the following table both for two-tailed and one-tailed tests.

Nature of test	*LOS* 1% (0.01)	2% (0.02)	5% (0.05)	10% (0.1)
Two-tailed	$\lvert z_\alpha \rvert = 2.58$	$\lvert z_\alpha \rvert = 2.33$	$\lvert z_\alpha \rvert = 1.96$	$\lvert z_\alpha \rvert = 1.645$
Right-tailed	$z_\alpha = 2.33$	$z_\alpha = 2.055$	$z_\alpha = 1.645$	$z_\alpha = 1.28$
Left-tailed	$z_\alpha = -2.33$	$z_\alpha = -2.055$	$z_\alpha = -1.645$	$z_\alpha = -1.28$

Procedure for Testing of Hypothesis

1. Null hypothesis H_0 is defined.
2. Alternative hypothesis H_1 is also defined after a careful study of the problem and also the nature of the test (whether one-tailed or two-tailed) is decided.
3. LOS α is fixed or taken from the problem if specified and z_α is noted.
4. The test-statistic $z = \dfrac{t - E(t)}{\mathrm{SE}(t)}$ is computed.
5. Comparison is made between $|z|$ and z_α. If $|z| < z_\alpha$, H_0 is accepted or H_1 is rejected, i.e. it is concluded that the difference between t and $E(t)$ is not significant at α% LOS.

On the other hand, if $|z| > z_\alpha$, H_0 is rejected or H_1 is accepted, i.e. it is concluded that the difference between t and $E(t)$ is significant at α% LOS.

Interval Estimation of Population Parameters

The interval within which the parameter is expected to lie is called the confidence interval for that parameter

The end points of the confidence interval are called confidence limits or fiducial limits.

$$P\{|z| \leq 1.96\} = 0.95$$

i.e. $$P\left\{\left|\frac{t - E(t)}{\text{SE}(t)}\right| \leq 1.96\right\} = 0.95$$

i.e. $$P\{t - 1.96\text{SE}(t) \leq E(t) \leq t + 1.96\text{SE}(t)\} = 0.95$$

This means that we can assert, with 95% confidence, that the parameter $E(t)$ will lie between $t - 1.96$ SE(t) and $t + 1.96$SE(t). Thus $\{t - 1.96\text{SE}(t), t + 1.96\ \text{SE}(t)$ are the 95% confidence limits for $E(t)$.

Similarly $\{t - 2.58\ \text{SE}(t), t + 2.58\ \text{SE}(t)\}$ is the 99% confidence interval for $E(t)$.

Tests of Significance for Large Samples

size of the sample exceeds 30.

Test [redacted]

Test of significance of the difference between sample mean and population mean.

the test statistic z is given by

$$z = \frac{\overline{X} - \mu}{\sigma / \sqrt{n}}.$$

As usual, if $|z| \le z_\alpha$, the difference between the sample mean $\overline{X}$ and the population mean μ is not significant at $\alpha\%$ LOS.

Note

1. *If σ is not known, the sample SD s can be used in its place, as s is nearly equal to σ when n is large.*

2. *95% confidence limits for μ are given by* $\dfrac{|\mu - \bar{X}|}{\sigma/\sqrt{n}} \le 1.96$, *i.e.*

   $\left(\bar{X} - 1.96\dfrac{\sigma}{\sqrt{n}}, \bar{X} + 1.96\dfrac{\sigma}{\sqrt{n}}\right)$, *if σ is known. If σ is not known, then*

   the 95% confidence interval is $\left(\bar{X} - \dfrac{1.96s}{\sqrt{n}}, \bar{X} + \dfrac{1.96s}{\sqrt{n}}\right)$.

Example 12·15. *A sample of 900 members has a mean 3·4 cms,. and s.d. 2·61 cms. Is the sample from a large population of mean 3·25 cms. and s.d. 2·61 cms. ?*

If the population is normal and its mean is unknown, find the 95% and 98% fiducial limits of true mean.

Solution. *Null hypothesis,* (H_0) : The sample has been drawn from the population with mean $\mu = 3{\cdot}25$ cms., and S.D. $\sigma = 2{\cdot}61$ cms.

Alternative Hypothesis, $H_1 : \mu \neq 3{\cdot}25$ (Two-tailed).

Test Statistic. Under H_0, the test statistic is :

$$Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1), \text{ (since } n \text{ is large)}$$

Here, we are given

$$\bar{x} = 3{\cdot}4 \text{ cms., } n = 900 \text{ cms., } \mu = 3{\cdot}25 \text{ cms. and } \sigma = 2{\cdot}61 \text{ cms.}$$

$$Z = \frac{3{\cdot}40 - 3{\cdot}25}{2{\cdot}61/\sqrt{900}} = \frac{0{\cdot}15 \times 30}{2{\cdot}61} = 1{\cdot}73$$

Since $|Z| < 1{\cdot}96$, we conclude that the data don't provide us any evidence against the null hypothesis (H_0) which may, therefore, be accepted at 5% level of significance.

95% *fiducial limits* **for the population mean μ are :**

$$\bar{x} \pm 1{\cdot}96\, \sigma/\sqrt{n} \Rightarrow 3{\cdot}40 \pm 1{\cdot}96 \times 2{\cdot}61/\sqrt{900}$$

$$\Rightarrow \quad 3{\cdot}40 \pm 0{\cdot}1705, \text{ i.e., } 3{\cdot}5705 \text{ and } 3{\cdot}2295$$

98% *fiducial limits* **for μ are given by :**

$$\bar{x} \pm 2{\cdot}33 \frac{\sigma}{\sqrt{n}}, \text{ i.e., } 3{\cdot}40 \pm 2{\cdot}33 \times \frac{2{\cdot}61}{30}$$

$$\Rightarrow \quad 3{\cdot}40 \pm 0{\cdot}2027 \text{ i.e., } 3{\cdot}6027 \text{ and } 3{\cdot}1973$$

Example 9 *A sample of 100 students is taken from a large population. The mean height of the students in this sample is 160 cm. Can it be reasonably regarded that, in the population, the mean height is 165 cm, and the SD is 10 cm?*

Solution Here $\bar{x} = 160$, $n = 100$, $\mu = 165$ and $\sigma = 10$.

H_0: $\mu_0 = \mu$ (i.e. the difference between μ_0 and μ is not significant).

H_1: $\mu_0 \neq \mu$.

Two-tailed test is to be used.

Let LOS be 1%. Therefore, $z_\alpha = 2.58$.

$$z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} = \frac{160 - 165}{10 / \sqrt{100}} = -5$$

$\therefore \qquad |z| > z_\alpha.$

Therefore, the difference between μ_0 and μ is significant at 1% level, i.e., H_0 rejected.

That is, it is not statistically correct to assume that $\mu = 165$.

Example 10 *The mean breaking strength of the cables supplied by a manufacturer is 1800 with a SD of 100. By a new technique in the manufacturing process, it is claimed that the breaking strength of the cable has increased. To test this claim, a sample of 50 cables is tested and it is found that the mean breaking strength is 1850. Can we support the claim at 1% LOS?*

Solution $\bar{x} = 1850$, $n = 50$, $\mu = 1800$ and $\sigma = 100$.

H_0: $\mu_0 = \mu$.

H_1: $\mu_0 > \mu$.

One-tailed (right-tailed) test is to be used.

Let LOS = 1%. Therefore, $z_\alpha = 2.33$.

$$z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} = \frac{1850 - 1800}{100 / \sqrt{50}} = 3.54$$

$\therefore \qquad |z| > z_\alpha$

Therefore the difference between $\bar{x}$ and μ is significant at 1% level, i.e., H_0 is rejected and H_1 is accepted.

That is, based on the sample data, we may support the claim of increase in breaking strength.

Example 11 *The mean value of a random sample of 60 items was found to be 145, with a SD of 40. Find the 95% confidence limits for the population mean. What size of the sample is required to estimate the population mean within 5 of its actual value with 95% or more confidence, using the sample mean?*

Solution Since the population SD σ too is not given, we can approximate it by the sample SDs. Therefore 95% confidence limits for μ are given by

$$\frac{|\mu - \bar{x}|}{s/\sqrt{n}} \le 1.96,$$

i.e. $$\bar{x} - 1.96\frac{s}{\sqrt{n}} \le \mu \le \bar{x} + 1.96\frac{s}{\sqrt{n}}.$$

i.e. $$145 - \frac{1.96 \times 40}{\sqrt{60}} \le \mu \le 145 + \frac{1.96 \times 40}{\sqrt{60}}$$

i.e. $$134.9 \le \mu \le 155.1$$

We have to find the value of n such that

$$P\{\bar{x} - 5 \le \mu \le \bar{x} + 5\} \ge 0.95$$

i.e. $$P\{-5 \le \mu - \bar{x} \le 5\} \ge 0.95$$

i.e. $$P\{|\mu - \bar{x}| \le 5\} \ge 0.95 \quad \text{or}$$

$$P\{|\bar{x} - \mu| \le 5\} \ge 0.95$$

$$\therefore \; P\left\{\frac{|\bar{x} - \mu|}{\sigma/\sqrt{n}} \le \frac{5}{\sigma/\sqrt{n}}\right\} \ge 0{\cdot}95$$

i.e. $$P\left\{|z| \le \frac{5\sqrt{n}}{\sigma}\right\} \ge 0{\cdot}95,$$

where z is the standard normal variate. (1)

We know that $P\{|z| \leq 1.96\} = 0{\cdot}95$. Therefore, the least value of $n = n_L$ that will satisfy (1) is given by

$$\frac{5\sqrt{n_L}}{\sigma} = 1.96$$

i.e. $$\sqrt{n_L} = \frac{1.96\,s}{5} \quad (\because \quad \sigma \simeq s)$$

i.e. $$n_L = \left(\frac{1.96 \times 40}{5}\right)^2$$

i.e. $$n_L = 245.86$$

Therefore, least size of the sample is 246.

LARGE SAMPLE STATISTICAL TEST for $\mu_1 - \mu_2$

1) Define the Hypothesis

Null Hypothesis: $H_0: \mu_1 - \mu_2 = D$ where D is some specified difference that you wish to test.

Alternative Hypothesis:

One Tailed Test	Two-Tailed test
$H_1: (\mu_1 - \mu_2) > D$ Or $H_1: (\mu_1 - \mu_2) < D$	$H_1: \mu_1 - \mu_2 \neq D$

2) Test statistic:

$$z \approx \frac{(\overline{x_1} - \overline{x_2}) - D}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

OR p-value

One Tailed Test	Two-Tailed test
p-value $= P(Z > z)$ Or $P(Z < -z)$	p-value = $P(Z > z) + P(Z < -z)$

OR

Test 2

Test of significance of the difference between the means of two samples.

The test statistic z is given by

$$z = \frac{\overline{X}_1 - \overline{X}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

If $|z| \le z_\alpha$, the difference between $(\bar{X}_1 - \bar{X}_2)$ and 0 or the difference between $\bar{X}_1$ and $\bar{X}_2$ is not significant at α% LOS.

Note 1. *If the samples are drawn from the same population, i.e. if $\sigma_1 = \sigma_2 = \sigma$, then*

$$z = \frac{\bar{X}_1 - \bar{X}_2}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \qquad (2)$$

2. *If σ_1 and σ_2 are not known and $\sigma_1 \neq \sigma_2$, σ_1 and σ_2 can be approximated by the sample SDs s_1 and s_2. Hence, in such a situation [from (1)],*

$$z = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} \qquad (3)$$

3. *If σ_1 and σ_2 are equal and not known, then $\sigma_1 = \sigma_2 = \sigma$ is approximated by $\sigma^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2}$. Hence, in such a situation, [from (2)],*

$$z = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\left(\frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2}\right)\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}},$$

i.e.

$$z = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{s_1^2}{n_2} + \frac{s_2^2}{n_1}}} \qquad (4)$$

4. *The difference in the denominators of the values of z given in (3) and (4) may be noted.*

Example 12·23. *The means of two single large samples of 1000 and 2000 members are 67·5 inches and 68·0 inches respectively. Can the samples be regarded as drawn from the same population of standard deviation 2·5 inches? (Test at 5% level of significance).*

Solution. We are given :

$$n_1 = 1000,\ n_2 = 2000;\ \bar{x}_1 = 67{\cdot}5 \text{ inches},\ \bar{x}_2 = 68{\cdot}0 \text{ inches}.$$

Null hypothesis, $H_0 : \mu_1 = \mu_2$ and $\sigma = 2{\cdot}5$ inches, *i.e.*, the samples have been drawn from the same population of standard deviation 2·5 inches.

Alternative Hypothesis, $H_1 : \mu_1 \neq \mu_2$ (Two tailed.)

Test Statistic. Under H_0, the test statistic is (since samples are large)

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\sigma^2 \left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \sim N(0, 1)$$

Now $$Z = \frac{67{\cdot}5 - 68{\cdot}0}{2{\cdot}5 \times \sqrt{\frac{1}{1000} + \frac{1}{2000}}} = \frac{-0{\cdot}5}{2{\cdot}5 \times 0{\cdot}0387} = -5{\cdot}1$$

Conclusion. Since $|Z| > 3$, the value is highly significant and we reject the null hypothesis and conclude that samples are certainly not from the same population with standard deviation 2·5.

Example 12·25. *The average hourly wage of a sample of 150 workers in a plant 'A' was Rs. 2·56 with a standard deviation of Rs. 1·08. The average wage of a sample of 200 workers in plant 'B' was Rs. 2·87 with a standard deviation of Rs. 1·28. Can an applicant safely assume that the hourly wages paid by plant 'B' are higher than those paid by plant 'A'?*

Solution. Let X_1 and X_2 denote the hourly wages (in *Rs.*) of workers in plant A and plant B respectively. Then we are given :

$$n_1 = 150,\ \bar{x}_1 = 2{\cdot}56,\ s_1 = 1{\cdot}08 = \hat{\sigma}_1$$

$$n_2 = 200,\ \bar{x}_2 = 2{\cdot}87,\ s_2 = 1{\cdot}28 = \hat{\sigma}_2$$

Null hypothesis, $H_0 : \mu_1 = \mu_2$, *i.e.*, there is no significant difference between the mean level of wages of workers in plant A and plant B.

Alternative hypothesis, $H_1 : \mu_2 > \mu_1$ *i.e.*, $\mu_1 < \mu_2$ (Left-tailed test)

Test Statistic. Under H_0, the test statistic (for large samples) is :

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\left(\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)}} = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)}} \sim N(0, 1)$$

$$\therefore \qquad Z = \frac{2{\cdot}56 - 2{\cdot}87}{\sqrt{\left\{\frac{(1{\cdot}08)^2}{150} + \frac{(1{\cdot}28)^2}{200}\right\}}} = \frac{-0{\cdot}31}{\sqrt{0{\cdot}016}} = \frac{-0{\cdot}31}{0{\cdot}126} = -2{\cdot}46.$$

Critical region. For a one-tailed test, the critical value of Z at 5% level of significance is $1{\cdot}645$. The critical region for left-tailed test thus consists of all values of $Z \leq -1{\cdot}645$.

Conclusion. Since calculated value of Z $(-2{\cdot}46)$ is less than critical value $(-1{\cdot}645)$, it is significant at 5% level of significance. Hence the null hypothesis is rejected at 5% level of significance and we conclude that the average hourly wages paid by plant '*B*' are certainly higher than those paid by plant '*A*'.

Example 13 *In a random sample of size 500, the mean is found to be 20. In another independent sample of size 400, the mean is 15. Could the samples have been drawn from the same population with SD 4?*

Solution Here $\bar{x}_1 = 20$, $n_1 = 500$, $\bar{x}_2 = 15$, $n_2 = 400$; $\sigma = 4$

H_0: $\mu_1 = \mu_2$, i.e. the samples have been drawn from the same population.

H_1: $\mu_1 \neq \mu_2$.

Two-tailed test is to be used.

Let LOS be 1%. Therefore, $z_\alpha = 2.58$.

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

$$= \frac{20 - 15}{4\sqrt{\frac{1}{500} + \frac{1}{400}}} = 18.6$$

$\therefore \qquad |z| > z_\alpha$

Therefore, the difference between $\bar{x}_1$ and $\bar{x}_2$ is significant at 1% level, i.e., H_0 is rejected

That is, the samples could not have been drawn from the same population.

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Example 14 *A simple sample of heights of 6400 English men has a mean of 170 cm and a SD of 6.4 cm, while a simple sample of heights of 1600 Americans has a mean of 172 cm and a SD of 6.3 cm. Do the data indicate that Americans are, on the average, taller than the Englishmen?*

Solution Here $n_1 = 6400$, $\bar{x}_1 = 170$ and

$s_1 = 6.4$; $n_2 = 1600$, $\bar{x}_2 = 172$ and $s_2 = 6.3$.

H_0: $\mu_1 = \mu_2$ or $\cancel{\bar{x}_1 = \bar{x}_2}$,

i.e. the samples have been drawn from two different populations with the same mean.

H_1: $\cancel{x_1 < x_2}$ or $\mu_1 < \mu_2$.

Left-tailed test is to be used.

Let LOS be 1%. Therefore, $z_\alpha = -2.33$.

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

[$\because \sigma_1 \simeq s_1$ and $\sigma_2 \simeq s_2$. Refer to Note 2 under Test 4]

$$= \frac{170 - 172}{\sqrt{\frac{(6.4)^2}{6400} + \frac{(6.3)^2}{1600}}} = -11.32$$

$\therefore \qquad |z| > |z_\alpha|$

Therefore, the difference between $\bar{x}_1$ and $\bar{x}_2$ (or μ_1 and μ_2) is significant at 1% level, i.e., H_0 is rejected and H_1 is accepted.

That is, the Americans are, on the average, taller than the Englishmen.

Example 15 *Test the significance of the difference between the means of the samples, drawn from two normal populations with the same SD using the following data:*

	Stize	*Mean*	*SD*
Sample 1	100	61	4
Sample 2	200	63	6

Solution H_0: $\cancel{\bar{x}_1 = \bar{x}_2}$ ~~or~~ $\mu_1 = \mu_2$.

H_1: $\cancel{\bar{x}_1 \neq \bar{x}_2}$ ~~or~~ $\mu_1 \neq \mu_2$.

Two-tailed test is to be used.

Let LOS be 5%. Therefore, $z_\alpha = 1.96$

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2}{n_2} + \dfrac{s_2^2}{n_1}}}$$

(refer to Note 3 under Test 4; the populations have the same SD)

$$= \frac{61 - 63}{\sqrt{\dfrac{4^2}{200} + \dfrac{6^2}{100}}} = -3.02$$

$\therefore \quad |z| > z_\alpha$

Therefore, the difference between $\bar{x}_1$ and $\bar{x}_2$ (or μ_1 and μ_2) is significant at 5% level, i.e., H_0 is rejected and H_1 is accepted.

Example 16 *The average marks scored by 32 boys is 72 with a SD of 8, while that for 36 girls is 70 with a SD of 6. Test at 1% LOS whether the boys perform better than girls.*

Solution H_0: $\cancel{\bar{x}_1 = \bar{x}_2}$ (or $\mu_1 = \mu_2$).

H_1: $\mu_1 > \mu_2$.

Right-tailed test is to be used.

Let LOS be 1%. Therefore, $z_\alpha = 2.33$.

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}}$$

(The two populations are assumed to have SD s $\sigma_1 \simeq s_1$ and $\sigma_2 \simeq s_2$)

$$= \frac{72 - 70}{\sqrt{\dfrac{8^2}{32} + \dfrac{6^2}{36}}} = 1.15$$

$\therefore \quad |z| < z_\alpha$

Therefore, the difference between $\bar{x}_1$ and $\bar{x}_2$ (μ_1 and μ_2) is not significant at 1% level, i.e. H_0 is accepted and H_1 is rejected.

That is, statistically we cannot concluede that boys perform better than girls.

Tests for small samples (n<30)

Test 1

t-Test for Single Mean.

(i) The sample has been drawn from the population with mean μ or (ii) there is no significant difference between the sample mean $\overline{x}$ and the population mean μ,

the statistic $$t = \frac{\overline{x} - \mu_0}{S/\sqrt{n}} \qquad \text{...(14·6)}$$

where $$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \text{ and } S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2, \qquad \text{...[14·6}(a)]$$

follows Student's t-distribution with $(n-1)$ d.f.

We now compare the calculated value of t with the tabulated value at certain level of significance. If calculated $|t| >$ tabulated t, null hypothesis is rejected and if calculated $|t| <$ tabulated t, H_0 may be accepted at the level of significance adopted

2. We know, the sample variance

$$s^2 = \frac{1}{n}\sum_i (x_i - \overline{x})^2$$

$$ns^2 = (n-1)S^2$$

$$\Rightarrow \qquad \frac{S^2}{n} = \frac{s^2}{n-1}$$

Hence for numerical problems, the test statistic (14·6) on using [14·6(e)] becomes

$$t = \frac{\overline{x} - \mu_0}{\sqrt{S^2/n}} = \frac{\overline{x} - \mu_0}{\sqrt{s^2/(n-1)}} \sim t_{n-1} \qquad \text{...[14·6}(f)]$$

3. Assumptions for Student's t-test. The following assumptions are made in the Student's t-test :

(*i*) The parent population from which the sample is drawn is normal.

(*ii*) The sample observations are independent, *i.e.*, the sample is random.

(*iii*) The population standard deviation σ is unknown.

Sometimes $t = \dfrac{\bar{x} - \mu}{s / \sqrt{n-1}}$ is also taken as $t = \dfrac{\bar{x} - \mu}{S / \sqrt{n}}$,

where $S^2 = \dfrac{1}{n-1} \sum_{r=1}^{n} \left(x_r - \bar{x}\right)^2$ and is called student's t. We shall use only

$t = \dfrac{\bar{x} - \mu}{s / \sqrt{n-1}}$, where s is the sample SD.

We get the value of $t_\nu(\alpha)$ for the LOS α and $\nu = n - 1$ from the t-table.

If the calculated value of t satisfies $|t| < t_\nu(\alpha)$, the null hypothesis H_0 is accepted at LOS α otherwise H_0 is rejected at LOS α.

Note *95% confidence interval of μ is given by*

$$\left| \frac{\bar{x} - \mu}{s / \sqrt{n-1}} \right| \le t_{0.05}, \text{ since } P\left\{ \left| \frac{\bar{x} - \mu}{s / \sqrt{n-1}} \right| \le t_{0.05} \right\} = 0.95$$

i.e. by $\bar{x} - t_{0.05} \dfrac{s}{\sqrt{n-1}} \le \mu \le \bar{x} + t_{0.05} \times \dfrac{s}{\sqrt{n-1}}$, *where $t_{0.05}$ is the 5% critical value of t for ν (= $n - 1$) degrees of freedom for a two-tailed test.*

Example 14·2. *A machinist is making engine parts with axle diameters of 0·700 inch. A random sample of 10 parts shows a mean diameter of 0·742 inch with a standard deviation of 0·040 inch. Compute the statistic you would use to test whether the work is meeting the specifications. Also state how you would proceed further.*

Solution. Here we are given :

$\mu = 0{\cdot}700$ inches, $\bar{x} = 0{\cdot}742$ inches, $s = 0{\cdot}040$ inches and $n = 10$

Null Hypothesis, $H_0 : \mu = 0{\cdot}700$, *i.e.. the product is conforming to specifications.*

Alternative Hypothesis, $H_1 : \mu \neq 0{\cdot}700$

Test Statistic. Under H_0, the test statistic is :

$$t = \frac{\bar{x} - \mu}{\sqrt{S^2/n}} = \frac{\bar{x} - \mu}{\sqrt{s^2/(n-1)}} \sim t_{(n-1)}$$

Now
$$t = \frac{\sqrt{9}(0{\cdot}742 - 0{\cdot}700)}{0{\cdot}040} = 3{\cdot}15$$

Example 14·3. *The mean weekly sales of soap bars in departmental stores was 146·3 bars per store. After an advertising campaign the mean weekly sales in 22 stores for a typical week increased to 153·7 and showed a standard deviation of 17·2. Was the advertising campaign successful ?*

Solution. We are given : $n = 22$, $\bar{x} = 153{\cdot}7$, $s = 17{\cdot}2$.

Null Hypothesis. The advertising campaign is not successful, *i.e.,*

$H_0 : \mu = 146{\cdot}3$

Alternative Hypothesis. $H_1 : \mu > 146{\cdot}3$.(Right-tail).

Test Statistic. Under the null hypothesis, the test statistic is :

$$t = \frac{\bar{x} - \mu}{\sqrt{s^2/(n-1)}} \sim t_{22-1} = t_{21}$$

Now
$$t = \frac{153{\cdot}7 - 146{\cdot}3}{\sqrt{(17{\cdot}2)^2/21}} = \frac{7{\cdot}4 \times \sqrt{21}}{17{\cdot}2} = 9{\cdot}03$$

Conclusion. Tabulated value of t for 21 *d.f.* at 5% level of significance for *single-tailed test* is 1·72. Since calculated value is much greater than the tabulated value, it is highly significant. Hence we reject the null hypothesis and conclude that the advertising campaign was definitely successful in promoting sales.

Example 14·4. *A random sample of 10 boys had the following I.Q.'s : 70, 120, 110, 101, 88, 83, 95, 98, 107, 100. Do these data support the assumption of a population mean I.Q. of 100 ? Find a reasonable range in which most of the mean I.Q. values of samples of 10 boys lie.*

Solution. *Null hypothesis,* H_0 *: The data are consistent with the assumption of a mean I.Q. of 100 in the population, i.e.,* $\mu = 100$.

Alternative hypothesis, $H_1 : \mu \neq 100$.

Test Statistic. Under H_0, the test statistic is :

$$t = \frac{(\bar{x} - \mu)}{\sqrt{S^2/n}} \sim t_{(n-1)},$$

where $\bar{x}$ and S^2 are to be computed from the sample values of I.Q.'s.

CALCULATIONS FOR SAMPLE MEAN AND S.D.

X	$(X - \bar{x})$	$(X - \bar{x})^2$
70	−27·2	739·84
120	22·8	519·84
110	12·8	163·84
101	3·8	14·44
88	−9·2	84·64
83	−14·2	201·64
95	−2·2	4·84
98	0·8	0·64
107	9·8	96·04
100	2·8	7·84
Total 972		1833·60

Hence $n = 10, \bar{x} = \frac{972}{10} = 97{\cdot}2$ and $S^2 = \frac{1833{\cdot}60}{9} = 203{\cdot}73$

$$|t| = \frac{|97{\cdot}2 - 100|}{\sqrt{203{\cdot}73/10}} = \frac{2{\cdot}8}{\sqrt{20{\cdot}37}} = \frac{2{\cdot}8}{4{\cdot}514} = 0{\cdot}62$$

Tabulated $t_{0{\cdot}05}$ for $(10 - 1)$ *i.e.,* 9 d.f. for two-tailed test is 2·262.

Conclusion. Since calculated t is less than tabulated $t_{0{\cdot}05}$ for 9 d.f., H_0 may be accepted at 5% level of significance and we may conclude that the data are consistent with the assumption of mean I.Q. of 100 in the population.

The 95% confidence limits within which the mean I.Q. values of samples of 10 boys will lie are given by

$$\bar{x} \pm t_{0{\cdot}05}\, S/\sqrt{n} = 97{\cdot}2 \pm 2{\cdot}262 \times 4{\cdot}514$$

$= 97{\cdot}2 \pm 10{\cdot}21 = 107{\cdot}41$ and $86{\cdot}99$

Hence the required 95% confidence interval is [86·99, 107·41].

Test 2

t-Test for Difference of Means.

14·2·10. t-Test for Difference of Means. Suppose we want to test if two independent samples x_i $(i = 1, 2, \ldots, n_1)$ and y_j, $(j = 1, 2, \ldots, n_2)$ of sizes n_1 and n_2 have been drawn from two normal populations with means μ_X and μ_Y respectively.

Under the null hypothesis (H_0) *that the samples have been drawn from the normal populations with means μ_X and μ_Y and under the assumption that the population variance are equal, i.e.,* $\sigma_X^2 = \sigma_Y^2 = \sigma^2$ (say), the statistic

$$t = \frac{(\bar{x} - \bar{y}) - (\mu_X - \mu_Y)}{S\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} \qquad \ldots(14{\cdot}7)$$

where $\bar{x} = \dfrac{1}{n_1}\sum_{i=1}^{n_1} x_i, \qquad \bar{y} = \dfrac{1}{n_2}\sum_{j=1}^{n_2} y_j$

and

$$S^2 = \frac{1}{n_1 + n_2 - 2}\left[\sum_i (x_i - \bar{x})^2 + \sum_j (y_j - \bar{y})^2\right] \qquad \ldots[14{\cdot}7(a)]$$

is an unbiased estimate of the common population variance σ^2, follows Student's t-distribution with $(n_1 + n_2 - 2)$ d.f.

Under the null hypothesis H_0 that (*a*) *samples have been drawn from t… populations with the same means,i.e.,* $\mu_X = \mu_Y$ *or (b) the sample means* $\bar{x}$ a… $\bar{y}$ *do not differ significantly*, [From (14·7)] the statistic :

$$t = \frac{\bar{x} - \bar{y}}{S\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \quad [\because \mu_X = \mu_Y, \text{ under } H_0]$$

where symbols are defined in (14·7*a*),follows Student's *t*-distribution with $(n_1 + n_2 - 2)$ d.f.

Note 1. *If f $n_1 = n_2 = n$ and if the samples are independent i.e., the observations in the two samples are not at all related, then the test statistic is given by*

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2 + s_2^2}{n-1}}} \quad \text{with} \quad \nu = 2n - 2 \qquad (2)$$

2. *If $n_1 = n_2 = n$ and if the pairs of values of x_1 and x_2 are associated in some way (or correlated), the formula (2) for t in Note (1) should not be used. In this case, we shall assume that $H_0 : \bar{d}\ (= \bar{x} - \bar{y}) = 0$ and test the significance of the difference between $\bar{d}$ and 0, using the test statistic $t = \frac{\bar{d}}{s/\sqrt{n-1}}$ with $\nu = n - 1$, where $d_i = x_i - y_i$ $(i = 1, 2, \ldots, n)$, $\bar{d} = \bar{x} - \bar{y}$; and $s = SD\ of\ d = \frac{1}{n}\sum_{i=1}^{n}\left(d_i - \bar{d}\right)^2$.*

Above note 2 is also known as paired t-test

$$t = \frac{\bar{d}}{S/\sqrt{n}}$$

where $$\bar{d} = \frac{1}{n}\sum_{i=1}^{n} d_i \quad \text{and} \quad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(d_i - \bar{d})^2$$

follows Student's *t*-distribution with $(n - 1)$ d.f.

Example 14·8. *Samples of two types of electric light bulbs were tested for length of life and following data were obtained :*

	Type I	*Type II*
Sample No.	$n_1 = 8$	$n_2 = 7$
Sample Means	$\bar{x}_1 = 1{,}234$ *hrs.*	$\bar{x}_2 = 1{,}036$ *hrs.*
Sample S.D.'s	$s_1 = 36$ *hrs.*	$s_2 = 40$ *hrs.*

Is the difference in the means sufficient to warrant that type I is superior to type II regarding length of life ?

Solution. *Null Hypothesis,* $H_0 : \mu_X = \mu_Y$, *i.e.,* the two types I and II of electric bulbs are indentical.

Alternative Hypothesis, $H_1 : \mu_X > \mu_Y$, *i.e.,* type I is superior to type II.

Test Statistic. Under H_0, the test statistic is :

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{S^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \sim t_{n_1 + n_2 - 2} = t_{13},$$

where $$S^2 = \frac{1}{n_1 + n_2 - 2}\left[\Sigma(x_1 - \bar{x}_1)^2 + \Sigma(x_2 - \bar{x}_2)^2\right]$$

$$= \frac{1}{n_1 + n_2 - 2}\left[n_1 s_1^2 + n_2 s_2^2\right] = \frac{1}{13}\left[8 \times (36)^2 + 7 \times (40)^2\right] = 1659{\cdot}08$$

$$\therefore \quad t = \frac{1234 - 1036}{\sqrt{1659{\cdot}08\left(\frac{1}{8} + \frac{1}{7}\right)}} = \frac{198}{\sqrt{1659{\cdot}08 \times 0{\cdot}2679}} = 9{\cdot}39$$

Tabulated value of t for 13 *d.f.* at 5% level of significance for right (single) tailed test is 1·77. [This is the value of $t_{0.10}$ for 13 *d.f.* from two-tail tables given in Appendix].

Conclusion. Since calculated 't' is much greater than tabulated 't', it is highly significant and H_0 is rejected. Hence the two types of electric bulbs differ significantly. Further since $\bar{x}_1$ is much greater than $\bar{x}_2$, we conclude that type I is definitely superior to type II.

Example 7 *The following data represent the biological values of protein from cow's milk and buffalo's milk at a certain level.*

Cow's milk	:	*1.82*	*2.02*	*1.88*	*1.61*	*1.81*	*1.54*
Buffalo's milk	:	*2.00*	*1.83*	*1.86*	*2.03*	*2.19*	*1.88*

Examine if the average values of protein in the two samples significantly differ.

Solution $n = 6$

$$\bar{x}_1 = \frac{1}{6} \times 10.68 = 1.78$$

$$s_1^2 = \frac{1}{6} \times \Sigma x_1^2 - (\bar{x}_1)^2 = \frac{1}{6} \times 19.167 - (1.78)^2 = 0.0261$$

$$\bar{x}_2 = \frac{1}{6} \times 11.79 = 1.965$$

$$s_2^2 = \frac{1}{6} \times \Sigma x_2^2 - (\bar{x}_2)^2 = \frac{1}{6} \times 23.2599 - (1.965)^2 = 0.0154$$

As the two samples are independent, the test statistic is given by

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2 + s_2^2}{n-1}}}$$

with $\nu = 2n - 2$ (refer to Note 2 under Test 2)

$$t = \frac{1{\cdot}78 - 1{\cdot}965}{\sqrt{\dfrac{0{\cdot}0261 + 0{\cdot}0154}{5}}} = \frac{-0{\cdot}185}{\sqrt{{\cdot}0083}} = -\,2.03 \text{ and } \quad \nu = 10$$

H_0: $\mu_1 = \mu_2$ and H_1: $\mu_1 \neq \mu_2$.

Two-tailed test is to be used. Let LOS be 5%.

From t-table, $t_{0.05}$ ($\nu = 10$) = 2.23.

Since $|t| < t_{0.05}$ ($\nu = 10$). H_0 is accepted. That is, the difference between the mean protein values of the two varieties of milk is not significant at 5% level.

Example 10 *The following data relate to the marks obtained by 11 students in 2 tests, one held at the beginning of a year and the other at the end of the year after intensive coaching.*

Test 1:	*19*	*23*	*16*	*24*	*17*	*18*	*20*	*18*	*21*	*19*	*20*
Test 2:	*17*	*24*	*20*	*24*	*20*	*22*	*20*	*20*	*18*	*22*	*19*

Do the data indicate that the students have benefited by coaching?

H_0: $\bar{d} = 0$, i.e. the students have not benefitted by coaching; H_1: $\bar{d} < 0$ (i.e. $\bar{x}_1 < \bar{x}_2$).

One-tailed test is to be used. Let LOS be 5%.

Solution The given data relate to the marks obtained in 2 tests by the same set of students. Hence the marks in the 2 tests can be regarded as correlated and so the t-test for paired values should be used.

Let $d = x_1 - x_2$, where x_1, x_2 denote the marks in the 2 tests.

Thus the values of d are 2, – 1 – 4, 0, – 3, – 4, 0, – 2, 3, – 3, 1.

$$\Sigma d = -11 \quad \text{and} \quad \Sigma d^2 = 69$$

$$\therefore \qquad \bar{d} = \frac{1}{n}\Sigma d = \frac{1}{11} \times (-11) = -1$$

$$s^2 = s^2{}_d = \frac{1}{n}\Sigma d^2 - (\bar{d})^2 = \frac{1}{11} \times 69 - (-1)^2 = 5.27$$

$$\therefore \qquad s = 2.296$$

One-tailed test is to be used. Let LOS be 5%.

$$t = \frac{\bar{d}}{s/\sqrt{n-1}} = \frac{-1}{2 \cdot 296/\sqrt{10}} = -1.38 \quad \text{and} \quad \nu = 10$$

$t_{0.05}$ ($\nu = 10$) for one-tailed test = $t_{0.1}$ ($\nu = 10$) for two-tailed test = 1.81 from t-table).

Now $|t| < t_{10}$ ($\nu = 10$). Therefore, H_0 is accepted and H_1 is rejected, i.e., there is no significant difference between the two sets of marks. That is, the students have not benefitted by coaching.

Example 14·10. *A certain stimulus administered to each of the 12 patients resulted in the following increase of blood pressure :*

5, 2, 8, –1, 3, 0, –2, 1, 5, 0, 4 and 6

Can it be concluded that the stimulus will, in general, be accompanied by an increase in blood pressure ? **[Delhi Univ. B.Sc. 1989]**

Solution. Here we are given the increments in blood pressure *i.e.*, $d_i\ (= x_i - y_i)$.

Null Hypothesis, $H_0 : \mu_X = \mu_Y$, *i.e.*, there is no significant difference in the blood pressure readings of the patients before and after the drug. In other words, the given increments are just by chance (fluctuations of sampling) and not due to the stimulus.

Alternative Hypothesis, $H_1 : \mu_X < \mu_Y$, *i.e.*, the stimulus results is an increase in blood pressure.

Test Statistic. Under H_0, the test statistic is :

$$t = \frac{\bar{d}}{S/\sqrt{n}} \sim t_{(n-1)}$$

d	5	2	8	–1	3	0	–2	1	5	0	4	6	31
d^2	25	4	64	1	9	0	4	1	25	0	16	36	185

$$S^2 = \frac{1}{n-1}\Sigma(d-\bar{d})^2 = \frac{1}{n-1}\left[\Sigma d^2 - \frac{(\Sigma d)^2}{n}\right]$$

$$= \frac{1}{11}\left[185 - \frac{(31)^2}{12}\right] = \frac{1}{11}(185 - 80{\cdot}08) = 9{\cdot}5382$$

and $$\bar{d} = \frac{\Sigma d}{n} = \frac{31}{12} = 2{\cdot}58$$

$\therefore$ $$t = \frac{\bar{d}}{S/\sqrt{n}} = \frac{2{\cdot}58 \times \sqrt{12}}{\sqrt{9{\cdot}5382}} = \frac{2{\cdot}58 \times 3{\cdot}464}{3{\cdot}09} = 2{\cdot}89$$

Tabulated $t_{0{\cdot}05}$ for 11 d.f. for right-tail test is 1·80. [This is the value of $t_{0{\cdot}10}$ for 11 *d.f* in the Table for two-tailed test given in the Appendix].

Conclusion. Since calculated $t > t_{0.05}$, H_0 is rejected at 5% level of significance. Hence we conclude that the stimulus will, in general, be accompanied by an increase in blood pressure.

F-test

Use of F-Distribution

F-distribution is used to test the equality of the variance of the populations from which two small samples have been drawn.

F-test

F-test of significance of the difference between population variances and F-table.

$$F = \frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2}$$

follows a F-distribution with ν_1 and ν_2 degrees of freedom. If $\hat{\sigma}_1^2 = \hat{\sigma}_2^2$, then $F = 1$.

Hence our aim is to find how far any observed value of F can differ from unity as a result of fluctuations of sampling.

Note

1. *We should always make $F > 1$. This is done by taking the larger of the two estimates of σ^2 as σ_1^2 and by assuming that the corresponding degree of freedom as ν_1.*
2. *To test if two small samples have been drawn from the same normal population, it is not enough to test if their means differ significantly or not, because in this test we assumed that the two samples came from the same population or from populations with equal variance. So, before applying the t-test for the significance of the difference of two sample means, we should satisfy ourselves about the equality of the population variances by F-test.*

To test the significance of the difference between population variances, we shall first find their estimates, $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$ based on the sample variances s_1^2 and s_2^2 and then test their equality. It is known that $\hat{\sigma}_1^2 = \dfrac{n_1 s_1^2}{n_1 - 1}$ with the number of degrees of freedom $\nu_1 = n_1 - 1$ and $\hat{\sigma}_2^2 = \dfrac{n_2 s_2^2}{n_2 - 1}$ with the number of degrees of freedom $\nu_2 = n_2 - 1$.

OR

Let $x_1, x_2, \ldots, x_{n_1}$ and $y_1, y_2, \ldots, y_{n_2}$ be the values of two independent random samples drawn from two normal populations having equal variances i.e., $\sigma_1^2 = \sigma_2^2 = \sigma^2$.

Under the Null Hypothesis

F-statistics is defined by the relation

$$\boxed{F = \frac{S_1^2}{S_2^2} \quad \text{when } S_1^2 > S_2^2}$$

Where

$$S_1^2 = \frac{\sum (x_i - \bar{x})^2}{n_1 - 1}$$

$$S_2^2 = \frac{\sum (y - \bar{y})^2}{n_2 - 1}$$

are the unbiased estimators of common population variance σ^2.

If s_1^2, s_2^2 are the variance of the sample X and Y respectively then we have

$$s_1^2 = \frac{\sum (x - \bar{x})^2}{n_1} \quad ; \quad s_2^2 = \frac{\sum (y - \bar{y})^2}{n_2}$$

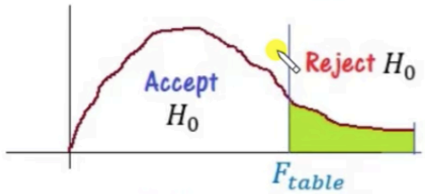
Hence, unbiased estimators can also be expressed as

$$S_1^2 = \frac{n_1 s_1^2}{n_1 - 1} \quad ; \quad S_2^2 = \frac{n_2 s_2^2}{n_2 - 1}$$

F – Distribution Table

The available F tables give the critical values of F

for the right-tailed test.



The significant value $F_{(\nu_1,\nu_2)}(\alpha)$ or $F_\alpha(\nu_1,\nu_2)$

at level of significance α with

(ν_1,ν_2) degree of freedom

is determined by

$$P\left(F > F_{(\nu_1,\nu_2)}(\alpha)\right) = \alpha$$

Critical values of F for the 0.05 significance level:

Denominator Degrees of Freedom	Numerator Degrees of Freedom							
	1	2	3	4	5	6	7	8
1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37
3	10.13	9.552	9.277	9.117	9.013	8.941	8.887	8.845
4	7.709	6.944	6.591	6.388	6.256	6.163	6.094	6.041
5	6.608	5.786	5.409	5.192	5.050	4.950	4.876	4.818
6	5.987	5.143	4.757	4.534	4.387	4.284	4.207	4.147
7	5.591	4.737	4.347	4.120	3.972	3.866	3.787	3.726
8	5.318	4.459	4.066	3.838	3.688	3.581	3.500	3.438
9	5.117	4.256	3.863	3.633	3.482	3.374	3.293	3.230
10	4.965	4.103	3.708	3.478	3.326	3.217	3.135	3.072
11	4.844	3.982	3.587	3.357	3.204	3.095	3.012	2.948
12	4.747	3.885	3.490	3.259	3.106	2.996	2.913	2.849
13	4.667	3.806	3.411	3.179	3.025	2.915	2.832	2.767
14	4.600	3.739	3.344	3.112	2.958	2.848	2.764	2.699
15	4.543	3.682	3.287	3.056	2.901	2.790	2.707	2.641
16	4.494	3.634	3.239	3.007	2.852	2.741	2.657	2.591
17	4.451	3.592	3.197	2.965	2.810	2.699	2.614	2.548

Example 11 *A sample of size 13 gave an estimated population variance of 3.0, while another sample of size 15 gave an estimate of 2.5. Could both samples be from populations with the same variance?*

Solution Here, $n_1 = 13$, $\hat{\sigma}_1^2 = 3.0$ and $\nu_1 = 12$, $n_2 = 15$, $\hat{\sigma}_2^2 = 2.5$ and $\nu_2 = 14$.

H_0: $\hat{\sigma}_1^2 = \hat{\sigma}_2^2$, i.e. the two samples have been drawn from populations with the same variance. H_1 : $\hat{\sigma}_1^2 \neq \hat{\sigma}_2^2$.

Let LOS. be 5%.

$$F = \frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2} = \frac{3\cdot0}{2\cdot5} = 1.2$$

$$\nu_1 = 12 \quad \text{and} \quad \nu_2 = 14.$$

$F_{0.05\%}$ ($\nu_1 =$ 12, $\nu_2 = 14$) = 2.53, from the F-table. Since $F <$ $F_{0.05}$, H_0 is accepted. That is the two samples could have come from two normal populations with the same variance.

Example 12 *Two samples of sizes 9 and 8 gave the sums of squares of deviations from their respective means equal to 160 and 91 respectively. Can they be regarded as drawn from the same normal population?*

Solution $n_1 = 9$, $\Sigma(x_i - \bar{x})^2 = 160$, i.e. $n_1 s_1^2 = 160$

$n_2 = 8$, $\Sigma(y_i - \bar{y})^2 = 91$, i.e. $n_2 s_2^2 = 91$

$$\hat{\sigma}_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = \frac{1}{8} \times 160 = 20\,; \quad \hat{\sigma}_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = \frac{1}{7} \times 91 = 13$$

Since $\hat{\sigma}_1^2 > \hat{\sigma}_2^2$, $\nu_1 = n_1 - 1 = 8$ and $\nu_2 = n_2 - 1 = 7$

H_0: $\hat{\sigma}_1^2 = \hat{\sigma}_2^2$ and H_1: $\hat{\sigma}_1^2 \neq \hat{\sigma}_2^2$.

Let the LOS be 5%.

$$F = \frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2} = \frac{20}{13} = 1.54$$

$F_{0.05}\,(\nu_1 = 8, \nu_2 = 7) = 3.73$, from the F-table. Since $F < F_{0.05\%}$, H_0 is accepted.

That is, the two samples could have come from two normal populations with the same variance.

We cannot say that the samples have come from the same population, as we are unable to test whether the means of the samples differ significantly or not.

OR

Example: Two samples of sizes 9 and 8 give the sum of squares of deviations from their respective means equal to 160 inches square and 91 inches squares respectively. Can they be regarded as drawn from the two normal populations with the same variance.

Solution: Step 1: Define the hypothesis

$$H_0: \sigma_1^2 = \sigma_2^2$$
$$H_1: \sigma_1^2 \neq \sigma_2^2$$

Step 2: Compute F –statistics.

Given that $n_1 = 9; n_2 = 8;$

$$\sum(x_i - \bar{x})^2 = 160\,; \sum(y_i - \bar{y})^2 = 91$$

$$S_1^2 = \frac{\sum(x_i - \bar{x})^2}{n_1 - 1} = \frac{160}{8} = 20$$

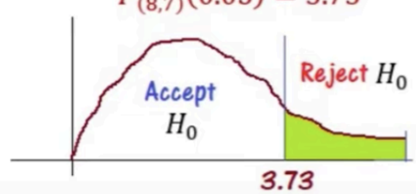
$$S_2^2 = \frac{\sum(y - \bar{y})^2}{n_2 - 1} = \frac{90}{7} = 13$$

$$\therefore \quad \boxed{F = \frac{S_1^2}{S_2^2} = \frac{20}{13} = 1.54}$$

Step 3: Conclusion:

The degree of freedom is $(8,7)$

$$F_{(8,7)}(0.05) = 3.73$$



Since $1.54 < 3.73$ and thus we fail to reject H_0 or we MAY accept H_0.

Thus, we MAY conclude that two samples are drawn from two normal populations with the same variance.

Example: In sampling experiment of two samples composed of 6 and 11 elements and $S_1 = 3.6; S_2 = 2.0.$

Is the difference between S_1 and S_2 significant at the 5% level?

Solution: Step 1: Define the hypothesis

$$H_0: \sigma_1^2 = \sigma_2^2$$
$$H_1: \sigma_1^2 \neq \sigma_2^2$$

Step 2: Compute F –statistics.

Given that $n_1 = 6$; $n_2 = 11$;

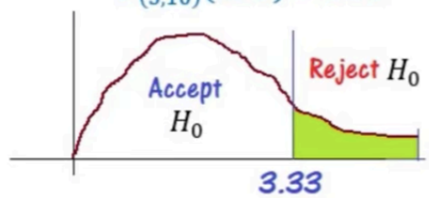
$S_1 = 3.6$; $S_2 = 2.0$

$$\therefore \quad F = \frac{S_1^2}{S_2^2}$$
$$= \frac{(3.6)^2}{(2)^2}$$
$$= 3.24$$

Step 3: Conclusion:

The degree of freedom $(5,10)$ and

$F_{(5,10)}(0.05) = 3.33$



Since $3.24 < 3.33$ and thus we fail to reject H_0

or we MAY accept H_0.

Thus, we MAY conclude that

the different between variances

is NOT significant.

Example: Two samples are composed of 7 and 9 individuals respectively and have variances 9.6 and 4.8 respectively. Is the variance 9.6 significantly greater than the variance 4.8?

Solution: Step 1: Define the hypothesis

$$\boxed{H_0: \sigma_1^2 = \sigma_2^2}$$
$$H_1: \sigma_1^2 > \sigma_2^2$$

Step 2: Compute F –statistics.

$n_1 = 7$; $n_2 = 9$;

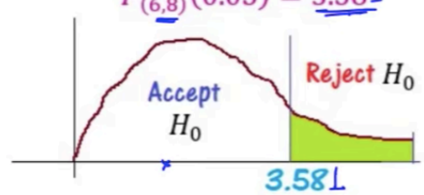
$s_1^2 = 9.6$; $s_2^2 = 4.8$

$$S_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = \frac{7(9.6)}{6} = 11.2$$
$$S_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = \frac{9(4.8)}{8} = 5.4$$
$$\therefore F = \frac{S_1^2}{S_2^2} = \frac{11.2}{5.4} = 2.074$$

Step 3: Conclusion:

The degree of freedom is $(6,8)$ and

$F_{(6,8)}(0.05) = 3.581$



As $2.074 < 3.581$ and thus we fail to reject H_0

or we MAY accept H_0.

Thus, the first variance CANNOT be regarded as significantly greater than the second.

Uses of χ^2-Distribution

1. χ^2-distribution is used to test the goodness of fit. i.e., it is used to judge whether a given sample may be reasonably regarded as a simple sample from a certain hypothetical population.
2. It is used to test the independence of attributes. That is, if a population is known to have two attributes (or traits), then χ^2-distribution is used to test whether the two attributes are associated or independent, based on a sample drawn from the population.

χ^2-Test of Goodness of Fit

On the basis of the hypothesis assumed about the population, we find the expected frequencies $E_i (i = 1, 2, \ldots, n)$, corresponding to the observed frequencies O_i $(i = 1, 2, \ldots, n)$ such that $\sum E_i = \sum O_i$. It is known that

$$\chi^2 = \sum_{i=1}^{n} \frac{\left(O_i - E_i\right)^2}{E_i}$$

The critical value of χ^2 for ν degrees of freedom at α LOS, denoted by $\chi^2_\nu(\alpha)$, is given by

$$P\,[\chi^2 > \chi^2_\nu(\alpha)] = \alpha$$

Critical values of the χ^2-distribution corresponding to a few important LOS and a range of values of ν are available in the form of a table called χ^2-table, which is given at the end of the chapter.

If the calculated $\chi^2 < \chi^2_\nu(\alpha)$, we will accept the null hypothesis H_0 which assumes that the given sample is one drawn from the hypothetical population, i.e. we will conclude that the difference between the observed and expected frequencies is not significant at α % LOS. If $\chi^2 > \chi^2_\nu(\alpha)$, we will reject H_0 and conclude that the difference is significant.

Conditions for the Validity of χ^2-Test

1. The number of observations N in the sample must be reasonably large, say ≥ 50.
2. Individual frequencies must not be too small, i.e. $O_i \geq 10$. In case $O_i < 10$, it is combined with the neighbouring frequencies, so that the combined frequency is ≥ 10.
3. The number of classes n must be neither too small nor too large, i.e. $4 \leq n \leq 16$.

OR

Chi - Square Test

If $O_i (i = 1,2, \ldots n)$ be the set of observed (experimental) frequencies and E_i is the corresponding set of expected (theoretical or Hypothetical) frequencies, i.e.,

Events	1	2	…	n
Observed frequencies	O_1	O_2	…	O_n
Expected frequencies	E_1	E_2	…	E_n

then Karl Pearson's chi-square, given by

$$\boxed{\chi^2 = \sum \frac{(O-E)^2}{E}}$$

where $\sum O = \sum E$ follows chi-square distribution with $(n-1)$ degree of freedom

Condition for the validity of χ^2 −test

- The sample observation should be independent.
- Constraints on the cell frequencies, if any, should be linear, eg. $\sum O = \sum E$
- The total frequency should be greater than 50.
- No theoretical cell frequency should be LESS than 5.

ILLUSTRATIONS

Example 1: The numbers of scooter accidents per month in a certain town were as follows:

12	8	20	2	14	10	15	6	9	4

Are these frequencies in agreement with the belief that accident conditions were the same during this 10 month period?

Solution: **Step 1:** Null Hypothesis H_0: Given frequencies (no. of accidents per month in a certain town) are CONSISTENT with the belief that accident conditions were same during the 10- month period.

Step 2: Compute "E" and χ^2

Under Null hypothesis, the accidents are uniformly distributed over the period, so the expected frequency (E) = 100/10 = 10

$$\chi^2 = \sum \frac{(O-E)^2}{E}$$

$$= 26.6$$

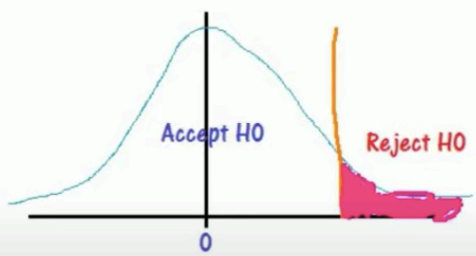
Month	O	E	$(O-E)^2$	$\frac{(O-E)^2}{E}$
1	12	10	4	0.4
2	8	10	4	0.4
3	20	10	100	10
4	2	10	64	6.4
5	14	10	16	1.6
6	10	10	0	0
7	15	10	25	2.5
8	6	10	16	1.6
9	9	10	1	0.1
10	4	10	36	3.6
Total	100	100		26.6

Step 3: $df = n-$

Step 2: Compute "E" and χ^2

Under Null hypothesis, the accidents are uniformly distributed over the period, so the expected frequency (E) = 100/10 = 10

$$\chi^2 = \sum \frac{(O-E)^2}{E} = 26.6$$



Step 3: degree of freedom (d.f.) = 10 - 1 = 9

Tabulated $\chi^2(0.05)$ for 9 d.f. = 16.92

Example 2 *The following data give the number of air-craft accidents that occurred during the various days of a week.*

Day:	*Mon*	*Tues*	*Wed*	*Thu*	*Fri*	*Sat*
No. of accidents:	*15*	*19*	*13*	*12*	*16*	*15*

Test whether the accidents are uniformly distributed over the week.

Solution H_0: Accidents occur uniformly over the week.

Total number of accidents = 90

Based on H_0, the expected number of accidents on any day = $\dfrac{90}{6} = 15$.

O_i :	15	19	13	12	16	15
E_i :	15	15	15	15	15	15

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} = \frac{1}{15}\,(0 + 16 + 4 + 9 + 1 + 0) = 2$$

Since $\sum E_i = \sum O_i$, $\quad \nu = 6 - 1 = 5$.

From the χ^2-table, $\chi^2_{0.05}\,(\nu = 5) = 11.07$.

Since $\chi^2 < \chi^2_{0.05}$, H_0 is accepted. That is, accidents may be regarded to occur uniformly over the week.

Example 2: The theory predicts the proportion of beans, in the four groups A, B, C and D should be 9:3:3:1. In an experiment among 1600 beams, the number of four groups was 882, 313, 287 and 118. Does the experiment result support the theory?

Step 2: Calculate E and χ^2:

$$\boxed{\chi^2 = \sum \frac{(O - E)^2}{E} = 4.726}$$

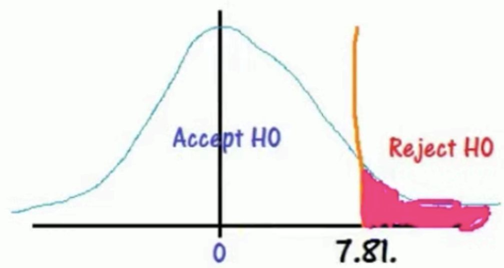
Group	O	E	$(O-E)^2$	$\frac{(O-E)^2}{E}$
A	882	$\frac{9}{16} \times 1600 = 900$	324	0.360
B	313	$\frac{3}{16} \times 1600 = 300$	169	0.563
C	287	$\frac{3}{16} \times 1600 = 300$	169	0.563
D	118	$\frac{1}{16} \times 1600 = 100$	324	3.240
Total	1600	1600		4.726

Step 3: d.f. = 4 - 1 = 3

Tabulated $\chi^2(0.05)$ for 3 d.f. is 7.81.

Step 2: Calculate E and χ^2:

$$\boxed{\chi^2 = \sum \frac{(O-E)^2}{E} = 4.726}$$



Step 3: d.f. = 4 - 1 = 3

Tabulated $\chi^2(0.05)$ for 3 d.f. is 7.81.

Since 4.726 < 7.81, so it is not significant. Hence, null hypothesis MAY be accepted at 5% level of significance and we may conclude that the experimental result support the theory

Example 3 *The following data show defective articles produced by 4 machines:*

Machine:	*A*	*B*	*C*	*D*
Production time:	*1*	*1*	*2*	*3*
No. of detectives:	*12*	*30*	*63*	*98*

Do the figures indicate a significant difference in the performance of the machines?

Solution H_0: Production rates of the machines are the same.

Total number of detectives = 203.

Based on H_0, the expected numbers of defectives produced by the machines are

	E_i:	$\frac{1}{7}\times 203$	$\frac{1}{7}\times 203$	$\frac{2}{7}\times 203$	$\frac{3}{7}\times 203$
i.e.	E_i:	29	29	58	87
	O_i:	12	30	63	98

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} = \frac{17^2}{29} + \frac{1^2}{29} + \frac{5^2}{58} + \frac{11^2}{87} = 11.82$$

Since $\sum E_i = \sum O_i$, $v = 4 - 1 = 3$.

From the χ^2-table, $\chi^2_{0.05}$ $(v = 3) = 7.815$.

Since $\chi^2 > \chi^2_{0.05}$, H_0 is rejected. That is, there is significant difference in the performance of machines.

Example 4: A survey of 800 families with four children each revealed the following distribution

No. of boys	0	1	2	3	4
No. of girls	4	3	2	1	0
No. of families	32	178	290	236	64

Is this result consistent with the hypothesis that male and female births are equally probable?

Solution:

Step 1: Null hypothesis H_0: Data are consistent with the hypothesis of equal probability for male and female births.

Thus, probability of male birth, $p = \frac{1}{2}$

Step 2: Calculate E and χ^2

Now, by binomial distribution, we have the probability of 'x' male births in a family is

$$^4C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{4-x}; \quad x = 0,1,2,3,4$$

$$= \; ^4C_x \left(\frac{1}{2}\right)^4$$

Hence, the theoretical frequency (E) of male birth is

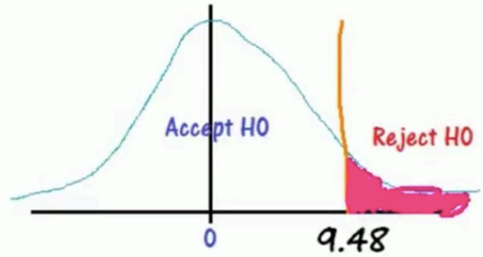
$$E = 800 \times {}^4C_x \left(\frac{1}{2}\right)^4$$

No. of male births	O	E	$(O-E)^2$	$\frac{(O-E)^2}{E}$
0	32	50	324	6.48
1	178	200	484	2.42
2	290	300	100	0.33
3	236	200	1296	6.48
4	64	50	196	3.92
Total	800	800		19.63

$$\chi^2 = \sum \frac{(O-E)^2}{E}$$

$$= 19.63$$

$$\chi^2 = \sum \frac{(O-E)^2}{E} = 19.63$$



Step 3: d.f = 5 - 1 = 4

Tabulated value of $\chi^2(0.05)$ for 4 d.f. = 9.48

Since 19.63> 9.48, so we REJECT the H_0 and conclude that male and female births are not equally probable.

Example 8 *Fit a binomial distribution for the following data and also test the goodness of fit.*

x:	0	1	2	3	4	5	6	Total
f:	5	18	28	12	7	6	4	80

To find the binomial distribution $N(q + p)^n$, which fits the given data, we require p.

Solution We know that the mean of the binomial distribution is np, from which we can find p. Now the mean of the given distribution is found out and is equated to np.

x :	0	1	2	3	4	5	6	Total
f:	5	18	28	12	7	6	4	80
fx :	0	18	56	36	28	30	24	192

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{192}{80} = 2.4$$

i.e. $np = 2.4$ or $6p = 2.4$, since the maximum value taken by x is n.

$\therefore$ $p = 0.4$ and hence $q = 0.6$

Therefore, the expected frequencies are given by the successive terms in the expansion of $80\,(0.6 + 0.4)^6$.

Thus	E_i:	3.73	14.93	24.88	22.12	11.06	2.95	0.33

Converting the E_i's into whole number such that $\sum E_i = \sum O_i = 80$, we get

E_i:	4	15	25	22	11	3	0

Let us now proceed to test the goodness of binomial fit.

O_i:	5	18	28	12	7	6	4

The first class is combined with the second and the last 2 classes are combined with the last but second class in order to make the expected frequency in each class greater than or equal to 10. Thus, after regrouping, we have

E_i :	19	25	22	14
O_i :	23	28	12	17

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} = \frac{4^2}{19} + \frac{3^2}{25} + \frac{10^2}{22} + \frac{3^2}{14} = 6.39$$

We have used the given sample to find $\sum E_i$ (= $\sum O_i$) and p through its mean.

Hence $n = n - k = 4 - 2 = 2$ and $\chi^2_{0.05}$ ($\nu = 2$) = 5.99, from the χ^2-table.

Since $\chi^2 > \chi^2_{0.05}$, H_0, which assumes that the given distribution is approximately a binomial distribution, is rejected, i.e. the binomial fit for the given distribution is not satisfactory.

Example 5: When the first proof of 392 pages of a book of 1200 pages were read, the distribution of printing mistakes were found to be as follows:

No. of mistakes in a page	0	1	2	3	4	5	6
No. of pages	275	72	30	7	5	2	1

Fit a Poisson distribution to the above data and test the goodness of fit.

Solution: The Poisson distribution is

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}; \quad x = 0,1,2,\ldots$$

To fit the Poisson distribution, we need the value of λ.

We know that the mean of Poisson distribution is λ.

Mean $= \frac{\Sigma fx}{\Sigma f}$

$= \frac{189}{392}$

$= 0.482$

λ

No. of mistakes in a page (x)	No. of pages (f)	fx
0	275	0
1	72	72
2	30	60
3	7	21
4	5	20
5	2	10
6	1	6
Total	392	189

For calculation of mean

Step 1: Null Hypothesis H_0: The Poisson distribution is the best fit to the given information.

Step 2: Calculate E and χ^2

The expected frequency by Poisson distribution is

$$E = 392 \times \frac{e^{-\lambda}\lambda^x}{x!};$$

$$\boxed{\chi^2 = \sum \frac{(O-E)^2}{E} = 40.937}$$

Mistakes in a page	O	E	$(O-E)^2$	$\frac{(O-E)^2}{E}$
0	275	242.1	1082.41	4.471
1	72	116.7	1998.09	17.121
2	30	28.1	3.61	0.128
3	7	4.5		
4	5 15	0.5 5.1	98.01	19.217
5	2	0.1		
6	1	0		
Total	392	392		40.937

Step 3: d.f. = 4 - 1 - 1 = 2

Tabulated value $\chi^2(0.05)$ for 2 d.f = 5.99

Since 40.937> 5.99, so it is highly significant and hence we reject H_0.

Therefore, we conclude that the Poisson distribution is not a good fit to the given data.