Unit 4: Multivariate Functions

(**Book:** Advanced Engineering Mathematics By Jain and Iyengar, **Chapter-2**)

Learning Outcomes:

1. To know about functions of two variables.

2. To find limit of functions of two variables .

3. To check continuity of functions of two variables.

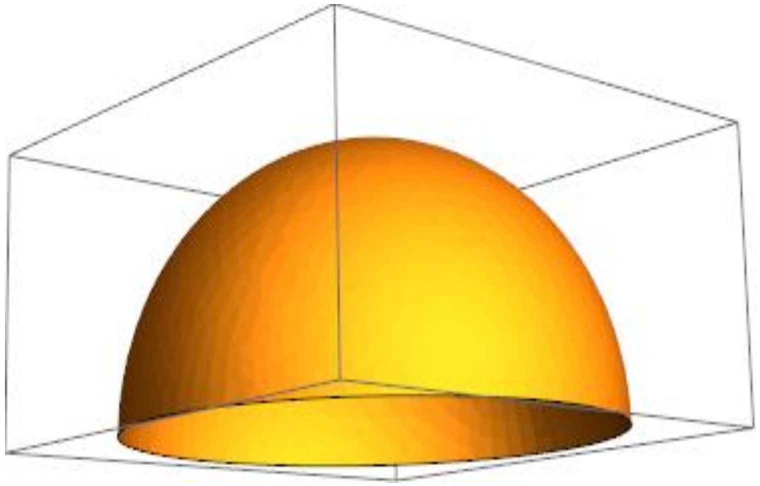
Functions of Two Variables

Consider the function of two variables $$z = f(x, y) \tag{1}$$

The set of points (x, y) in the $xy-$plane for which $f(x, y)$ id defined is called the *domain* of function and is demoted by D.

The collection of corresponding values of z is called *range* of the function.

For example: $z = \sqrt{1 - x^2 - y^2}$



Its domain is the region $x^2 + y^2 \leq 1$.

Its range is the set of all positive real numbers.

Limits

Let $z = f(x, y)$ be a function of two variables defined in domain D.

Let $P(x_0, y_0)$ be any point in domain D.

Then, the real, finite number L is called the limit of the function $f(x, y)$ as $(x, y) \longrightarrow (x_0, y_0)$.

Symbolically, we write it as:

$$\lim_{(x,y)\longrightarrow(x_0,y_0)} f(x, y) = L.$$

Note:

1. If $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ exists, it is unique.

2. Since the limit is unique, so it is same along all the paths, that it the limit is independent of path.

3. If the limit depends on the path, then limit does not exist.

Note:

If $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L_1$ and $\lim\limits_{(x,y)\to(x_0,y_0)} g(x,y) = L_2$, then:

1. $\lim\limits_{(x,y)\to(x_0,y_0)} k[f(x,y)] = kL_1$, where k is any real number.

2. $\lim\limits_{(x,y)\to(x_0,y_0)} [f(x,y) \pm g(x,y)] = L_1 \pm L_2$.

3. $\lim\limits_{(x,y)\to(x_0,y_0)} [f(x,y)g(x,y)] = L_1 L_2$.

4. $\lim\limits_{(x,y)\to(x_0,y_0)} \left[\frac{f(x,y)}{g(x,y)}\right] = \frac{L_1}{L_2}, L_2 \neq 0$.

Problem. What is the value of $\lim_{(x,y)\longrightarrow(2,1)}(3x+4y)$.

Solution. $\lim_{(x,y)\longrightarrow(2,1)}(3x+4y)$.

$$= 3(2) + 4(1)$$

$$= 6 + 4$$

$= 10$ **Answer.**

Polling Quiz

The value of $\lim_{(x,y)\to(1,1)} (x^2 + 2y)\ is$:

(A) 0

(B) 4

(C) 3

(D) 1.

Determine the following limits, if they exist:

Problem 1. $\lim\limits_{(x,y)\longrightarrow(0,0)} \left[\frac{x+y}{x^2+y^2+1}\right]$

Solution. $\lim\limits_{(x,y)\longrightarrow(0,0)} \left[\frac{x+y}{x^2+y^2+1}\right]$

$$= \left[\frac{0+0}{0+0+1}\right]$$

$$= \left[\frac{0}{1}\right]$$

$= 0$ **Answer.**

Problem 2. $\lim\limits_{(x,y)\to(0,1)}\left[\frac{(y-1)tan^2x}{x^2(y^2-1)}\right]$

Solution. $\lim\limits_{(x,y)\to(0,1)}\left[\frac{(y-1)tan^2x}{x^2(y^2-1)}\right]$

$$= \lim_{x\to 0}\left[\frac{tan^2x}{x^2}\right]\lim_{y\to 1}\left[\frac{(y-1)}{(y^2-1)}\right]$$

$$= \lim_{x\to 0}\left[\frac{tanx}{x}\right]^2 \lim_{y\to 1}\left[\frac{(y-1)}{(y-1)(y+1)}\right]$$

$$= (1)^2\left(\frac{1}{1+1}\right) = \frac{1}{2}$$ **Answer.**

Problem 3. $\lim_{(x,y)\longrightarrow(0,0)} \left[y + x\cos\left(\frac{1}{y}\right)\right]$

Solution. $\lim_{(x,y)\longrightarrow(0,0)} \left[y + x\cos\left(\frac{1}{y}\right)\right]$

$= [0 + 0(finite)]$

$= 0$ **Answer.**

Show that the following limits do not exist:

Problem 1. $\lim_{(x,y)\longrightarrow(0,0)}\left[\frac{xy}{x^2+y^2}\right]$

Solution. The limit does not exist if it is not finite or it depends on a particular path.

Consider the path: $y = mx$ such that $y \to 0$ as $x \to 0$.

So, $\lim_{(x,y)\longrightarrow(0,0)}\left[\frac{xy}{x^2+y^2}\right] = \lim_{x\longrightarrow 0}\left[\frac{x(mx)}{x^2+(mx)^2}\right]$

$$= \lim_{x\longrightarrow 0}\left[\frac{x^2 m}{x^2(1+m^2)}\right]$$

$$= \frac{m}{1+m^2}$$

Which depends on *m*. For different values of *m*, we obtain different limits.

Hence the limit does not exist.

Alternative:

Problem 1. $\lim_{(x,y)\longrightarrow(0,0)} \left[\frac{xy}{x^2+y^2}\right]$

Solution. The limit does not exist if it is not finite or it depends on a particular path.

Let $x = r\cos\theta,\ y = r\sin\theta$.

So, $\lim_{(x,y)\longrightarrow(0,0)} \left[\frac{xy}{x^2+y^2}\right] = \lim_{x\longrightarrow 0} \left[\frac{(r\cos\theta)(r\sin\theta)}{(r\cos\theta)^2+(r\sin\theta)^2}\right]$

$$= \lim_{x\longrightarrow 0} \left[\frac{r^2\sin\theta\cos\theta}{r^2}\right]$$

$$= \sin\theta\cos\theta$$

Which depends on θ. For different values of θ, we obtain different limits.

Hence the limit does not exist.

Problem 2. $\lim\limits_{(x,y)\longrightarrow(0,0\,)} \left[\frac{x+\sqrt{y}}{x^2+y}\right]$

Solution. The limit does not exist if it is not finite or it depends on a particular path.

Consider the path: $y = mx^2$ such that $y \to 0$ as $x \to 0$.

So, $\lim\limits_{(x,y)\longrightarrow(0,0\,)} \left[\frac{x+y}{x^2+y}\right] = \lim\limits_{x\longrightarrow 0} \left[\frac{x+\sqrt{mx^2}}{x^2+mx^2}\right]$

$$= \lim_{x\longrightarrow 0} \left[\frac{x(1+\sqrt{m})}{x^2(1+m)}\right]$$

$$= \lim_{x\longrightarrow 0} \left[\frac{(1+\sqrt{m})}{x\,(1+m)}\right] = \infty$$

Since the limit is not finite, hence the limit does not exist.

Problem 3. $\lim\limits_{(x,y)\longrightarrow(0,1\ )} tan^{-1}\left[\frac{y}{x}\right]$

Solution. So, $\lim\limits_{(x,y)\longrightarrow(0,1\ )} tan^{-1}\left[\frac{y}{x}\right] = \lim\limits_{(x,y)\longrightarrow(0,1\ )} tan^{-1}\left[\frac{1}{0}\right]$

$$= tan^{-1}(\pm\infty)$$

$$= \pm\frac{\pi}{2}$$

Since the limit is not unique, hence the limit does not exist.

Problem 4. $\lim_{(x,y)\longrightarrow(0,0)}\left[\frac{x}{\sqrt{x^2+y^2}}\right]$

Solution. The limit does not exist if it is not finite or it depends on a particular path.

Consider the path: $y = mx$ such that $y \to 0$ as $x \to 0$.

So, $\lim_{(x,y)\longrightarrow(0,0)}\left[\frac{x}{\sqrt{x^2+y^2}}\right] = \lim_{x\longrightarrow 0}\left[\frac{x}{\sqrt{x^2+(mx)^2}}\right]$

$$= \lim_{x\longrightarrow 0}\left[\frac{x}{x\sqrt{1+m^2}}\right]$$

$$= \frac{1}{\sqrt{1+m^2}}$$

Which depends on *m*. For different values of *m*, we obtain different limits.

Hence the limit does not exist.

Polling Quiz

The value of $\lim\limits_{(x,y)\longrightarrow(1,-1)} \dfrac{x^3-y^3}{x-y}$:

(A) 0

(B) 4

(C) 3

(D) 1.

Continuity

A function $z = f(x, y)$ is said to be *continuous* at a point $P(x_0, y_0)$, if:

1. $f(x, y)$ is defined at the point (x_0, y_0).

2. $\lim_{(x,y) \to (x_0,y_0)} f(x, y)$ exists.

3. $\lim_{(x,y) \to (x_0,y_0)} f(x, y) = f(x_0, y_0)$.

If any one of the above conditions is not satisfied, then the function is said to be discontinuous at the point (x_0, y_0).

Discuss the continuity of the following functions at the given points:

Problem 1. $f(x,y) = \begin{cases} \frac{(x-y)^2}{x^2+y^2}, & (x,y) \neq 0 \\ 0 & ,(x,y) = 0 \end{cases}$ at point $(0,0)$.

Solution. Here $f(0,0) = 0$ that is the function exists at $(0,0)$.

Now $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} \frac{(x-y)^2}{x^2+y^2}$

Consider the path: $y = mx$ such that $y \to 0$ as $x \to 0$.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{(x-mx)^2}{x^2+(mx)^2}$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2(1-m)^2}{x^2(1+m^2)}$$

$$= \frac{(1-m)^2}{(1+m^2)}$$

which is path dependent. So, the limit does not exist.

As one of the conditions of continuity is not satisfied, so the given function is *discontinuous* at (0,0).

Problem 2. $f(x,y) = \begin{cases} \frac{2x^4+3y^4}{x^2+y^2}, (x,y) \neq 0 \\ 0 \qquad\quad , (x,y) = 0 \end{cases}$ at point (0,0).

Solution. Here $f(0,0) = 0$ that is the function exists at $(0,0)$.

Now $\lim_{(x,y)\longrightarrow(0,0)} f(x,y) = \lim_{(x,y)\longrightarrow(0,0)} \frac{2x^4+3y^4}{x^2+y^2}$

Consider the path: $y = mx$ such that $y \rightarrow 0$ as $x \rightarrow 0$.

$$\lim_{(x,y)\longrightarrow(0,0)} f(x,y) = \lim_{(x,y)\longrightarrow(0,0)} \frac{2x^4+3(mx)^4}{x^2+(mx)^2}$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^4(2+3m^4)}{x^2(1+m^2)}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^2(2+3m^4)}{(1+m^2)}$$

$= 0$ which exists and is finite

Also $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$

Hence, all the conditions of continuity are satisfied.

So, the given function is continuous at point (0,0).