Unit 1: Linear Algebra

(**Book:** Advanced Engineering Mathematics by Jain and Iyengar, **Chapter-3**)

Topic:

Solution of Linear System of Equations

Learning Outcomes:

1. Linear System of Equations- Homogeneous and Non Homogeneous.

2. Solution of Linear System of Equations using Cramer's rule (Determinant method).

3. Solution of Linear System of Equations using Gauss-Elimination method (Rank method).

Solution of Non-Homogeneous System of Equations:

(Cramer's Rule)

Let us consider the following system of equations:

$$a_1x + b_1y + c_1z = d_1 \qquad (1)$$

$$a_2x + b_2y + c_2z = d_2 \qquad (2)$$

$$a_3x + b_3y + c_3z = d_3 \qquad (3)$$

The given system can be written as:

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$ that is $AX = B$

$$\text{Where A} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \text{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \text{B} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\text{Let } D = |A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},$$

$$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Case 1. If $D \neq O$, the given system of equations is said to be *consistent* and has a *unique solution* given by:

$$x = \frac{D_x}{D}, y = \frac{D_y}{D}, z = \frac{D_z}{D}$$

Case 2. If $D = 0$ and $D_x = D_y = D_z = 0$, still the given system of equations is said to be *consistent* and has *infinitely many solutions*.

Case 3. If $D = 0$ but D_x, D_y, D_z are not all zero, then the given system of equations is said to be *inconsistent* and has *no solution*.

Problem 1. Show that the following system of equations is consistent:

$$x - y + z = 4 \qquad (1)$$

$$2x + y - 3z = 0 \qquad (2)$$

$$x + y + z = 2 \qquad (3)$$

Solution. The given system can be written as:

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$ that is $AX = B$

Where $\mathrm{A} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$, $\mathrm{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $\mathrm{B} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$

Let $D = |A| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{vmatrix} = 1(1+3) + 1(2+3) + 1(2-1) = 10 \neq 0.$

As $D \neq 0$, the system of equations is consistent and has a unique solution

$$D_x = \begin{vmatrix} 4 & -1 & 1 \\ 0 & 1 & -3 \\ 2 & 1 & 1 \end{vmatrix} = 4(1+3) + 1(0+6) + 1(0-2) = 20$$

$$D_y = \begin{vmatrix} 1 & 4 & 1 \\ 2 & 0 & -3 \\ 1 & 2 & 1 \end{vmatrix} = 1(0+6) - 4(2+3) + 1(4-0) = -10$$

$$D_z = \begin{vmatrix} 1 & -1 & 4 \\ 2 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix} = 1(2-0) + 1(4-0) + 4(2-1) = 10$$

$$x = \frac{D_x}{D} = \frac{20}{10} = 2,$$

$$y = \frac{D_y}{D} = \frac{-10}{10} = -1,$$

$$z = \frac{D_z}{D} = \frac{10}{10} = 1$$

Problem 2. Show that the following system of equations is inconsistent:

$$4x + 9y + 3z = 6 \qquad (1)$$

$$2x + 3y + z = 2 \qquad (2)$$

$$2x + 6y + 2z = 7 \qquad (3)$$

Solution. The given system can be written as:

$$\begin{bmatrix} 4 & 9 & 3 \\ 2 & 3 & 1 \\ 2 & 6 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 7 \end{bmatrix} \text{ that is } AX = B$$

Where $\mathrm{A} = \begin{bmatrix} 4 & 9 & 3 \\ 2 & 3 & 1 \\ 2 & 6 & 2 \end{bmatrix}$, $\mathrm{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $\mathrm{B} = \begin{bmatrix} 6 \\ 2 \\ 7 \end{bmatrix}$

Let $D = |A| = \begin{vmatrix} 4 & 9 & 3 \\ 2 & 3 & 1 \\ 2 & 6 & 2 \end{vmatrix} = 4(6-6) - 9(4-2) + 3(12-6) = 0.$

$$D_x = \begin{vmatrix} 6 & 9 & 3 \\ 2 & 3 & 1 \\ 7 & 6 & 2 \end{vmatrix} = 6(6-6) - 9(4-7) + 3(12-21) = 0$$

$$D_y = \begin{vmatrix} 4 & 6 & 3 \\ 2 & 2 & 1 \\ 2 & 7 & 2 \end{vmatrix} = 4(4-7) - 6(4-2) + 3(14-4) = 6$$

$$D_z = \begin{vmatrix} 4 & 9 & 6 \\ 2 & 3 & 2 \\ 2 & 6 & 7 \end{vmatrix} = 4(21 - 18) - 9(14 - 4) + 6(12 - 6) = -42$$

Since $D = 0$ and $D_x = 0, D_y = 6, D_z = -42$

So, the given system is inconsistent and has no solution.

The system of equation $\begin{bmatrix} 2 & -3 & 1 \\ 1 & -1 & 2 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}$ has ?

(a) No solution (b) Infinite solution (c) Unique solution (d) None of these

Problem 3. Show that the following system of equations is consistent and has infinitely many solutions:

$x - y + 3z = 3$ (1)

$2x + 3y + z = 2$ (2)

$3x + 2y + 4z = 5$ (3)

Solution. The given system can be written as:

$$\begin{bmatrix} 1 & -1 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} \text{ that is } AX = B$$

Where $A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$

$$\text{Let } D = |A| = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 4 \end{vmatrix} = 1(12-2) + 1(8-3) + 3(4-9) = 0.$$

$$D_x = \begin{vmatrix} 3 & -1 & 3 \\ 2 & 3 & 1 \\ 5 & 2 & 4 \end{vmatrix} = 3(12-2) + 1(8-5) + 3(4-15) = 0$$

$$D_y = \begin{vmatrix} 1 & 3 & 3 \\ 2 & 2 & 1 \\ 3 & 5 & 4 \end{vmatrix} = 1(8-5) - 3(8-3) + 3(10-6) = 0$$

$$D_z = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 3 & 2 \\ 3 & 2 & 5 \end{vmatrix} = 1(15-4) + 1(10-6) + 3(4-9) = 0$$

Since $D = 0$ and $D_x = D_y = D_z = 0$

So, the given system is consistent and has infinitely many solutions.

From equations (1) and (2):

$x - y = 3 - 3z$ (4)

$2x + 3y = 2 - z$ (5)

Solving equations (4) and (5), we get:

$$x = \frac{1}{5}(11 - 10z)$$

$$y = \frac{1}{5}(5z - 5)$$

For different values of z, we get different values of x and y.

So, the system has infinitely many solutions.

Solution of Homogeneous System of Equations:

(Cramer's Rule)

Let us consider the following system of equations:

$a_1x + b_1y + c_1z = 0$ (1)

$a_2x + b_2y + c_2z = 0$ (2)

$a_3x + b_3y + c_3z = 0$ (3)

The given system can be written as:

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ that is } AX = 0$$

Where A $= \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, X $= \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, 0 $= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Case 1. If $|A| \neq O$ (Non-singular matrix), the given system of equations has *trivial (Zero/unique) solution* $(x = 0, y = 0, z = 0)$.

Case 2. If $|A| = 0$ (Singular matrix), the given system of equations has *non-trivial (non-Zero/infinitely many) solutions.*

Problem 1. Solve the following system of homogeneous equations:

$x + 2y + 3z = 0$ (1)

$2x + 3y - 2z = 0$ (2)

$4x + 7y + 4z = 0$ (3)

Solution. The given system can be written as:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & -2 \\ 4 & 7 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ that is $AX = O$

Where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & -2 \\ 4 & 7 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & -2 \\ 4 & 7 & 4 \end{vmatrix} = 1(12+14) - 2(8+8) + 3(14-12) = 0.$$

So, the given system has non-trivial (infinitely many) solutions.

From equations (1) and (2):

$x + 2y = -3z$ (4)

$2x + 3y = 2z$ (5)

Solving these, we get $x = 13z, y = -8z$

For different values of z, we get different values of x and y.

So, the system has non-trivial (infinitely many) solutions.

Problem 2. Solve the following system of homogeneous equations:

$x + 2y - 3z = 0 \qquad (1)$

$x + y - z = 0 \qquad (2)$

$x - y + z = 0 \qquad (3)$

Solution. The given system can be written as:

$$\begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ that is } AX = O$$

Where $A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 & -3 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} = 1(1-1) - 2(1+1) - 3(-1-1) = 2 \neq 0.$$

So, the given system has trivial (zero) solution that is

$$x = 0, y = 0, z = 0.$$

Problem 3. Determine the values of k for which the system of equations:

$x - ky + z = 0$ (1)

$kx + 3y - kz = 0$ (2)

$3x + y - z = 0$ (3)

has (I) Only trivial solution (II) Non-trivial solution

Solution. The given system can be written as:

$$\begin{bmatrix} 1 & -k & 1 \\ k & 3 & -k \\ 3 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ that is } AX = O$$

Where $A = \begin{bmatrix} 1 & -k & 1 \\ k & 3 & -k \\ 3 & 1 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & -k & 1 \\ k & 3 & -k \\ 3 & 1 & -1 \end{vmatrix} = 1(-3+k) + k(-k+3k) + 1(k-9).$$

$$|A| = 2k^2 + 2k - 12$$

(I) For trivial solution: $|A| \neq 0$

$$\Longrightarrow 2k^2 + 2k - 12 \neq 0 \Longrightarrow k \neq 2 \; and \; k \neq -3$$

(II) For non-trivial solution: $|A| = 0$

$$\Longrightarrow 2k^2 + 2k - 12 = 0 \Longrightarrow k = 2 \; or \; k = -3$$