Exercise 5.3

Find the general solution of the following differential equations.

1. $y''' - 2y'' - 3y' + 2y = 0.$

2. $2y''' + y'' - 13y' + 6y = 0.$

3. $3y''' - 12y'' + 7y'' + 3y' - 2y = 0.$

4. $y^{iv} - 13y'' + 36y = 0.$

5. $4y^{iv} - 12y''' - 7y'' + 6y' - y = 0.$

6. $y^{iv} + y''' - 4y'' - 4y' = 0.$

7. $8y^{iv} - 6y''' - 7y'' + 6y' - y = 0.$

8. $144y^{iv} - 25y'' + y = 0.$

9. $y''' - 2y'' + y' = 0.$

10. $y''' + 4y'' + 5y' + 2y = 0.$

11. $y''' - 2y'' - 4y' + 8y = 0.$

12. $27y''' - 27y'' + 9y' - y = 0.$

13. $y^{iv} - 11y''' + 35y'' - 25y' = 0.$

14. $y^{iv} - 3y''' + 3y'' - y' = 0.$

15. $4y^{iv} + 4y''' - 3y'' - 2y' + y = 0.$

16. $9y^{iv} - 66y''' + 157y'' - 132y' + 36y = 0.$

17. $y''' + y' = 0.$

18. $y''' - 2y'' + 4y' - 8y = 0.$

19. $y''' + 5y'' + 8y' + 6y = 0.$

20. $y''' - 7y'' + 19y' - 13y = 0.$

21. $y^{iv} + 8y'' - 9y = 0.$

22. $y^{iv} + y''' + 14y'' + 16y' - 32y = 0.$

23. $4y^{iv} + 101y'' + 25y = 0.$

24. $y^{iv} + 2y''' - 9y'' - 10y' + 50y = 0.$

25. $y^{iv} + 50y'' + 625y = 0.$

26. $y^{iv} + 2y'' + y = 0.$

Find a homogeneous linear differential equation with real constant coefficients of lowest order which has the following particular solution.

27. $5 + e^x + 2e^{3x}.$

28. $e^{-x} + \cos 5x + 3 \sin 5x.$

29. $xe^{-x} + e^{2x}.$

30. $1 + x + e^x - 3e^{3x}.$

31. $x^2 e^{2x} + 2e^{-2x}.$

32. $3\cos 2x + 5 \sinh 3x.$

Solve the following initial value problems.

33. $y'' - 2y' + 6y = 0, \ y(0) = 0, \ y'(0) = 0, \ y''(0) = 0, \ y'''(0) = 1.$

34. $4y''' - 4y'' - 9y' + 9y = 0, \ y(0) = 1, \ y'(0) = 0, \ y''(0) = 0.$

35. $y'' - 5y'' + 7y' - 3y = 0, \ y(0) = 1, \ y'(0) = 0, \ y''(0) = -5.$

36. $y^{iv} - 2y''' - 3y'' + 4y' + 4y = 0, \ y(0) = 3, \ y'(0) = 3, \ y''(0) = 3, \ y'''(0) = 6.$

37. $y^{iv} + y'' = 0, \ y(0) = 1, \ y'(0) = 2, \ y''(0) = -1, \ y'''(0) = -1.$

38. $y''' - y'' + 4y' - 4y = 0, \ y(0) = 0, \ y'(0) = 3, \ y''(0) = -5.$

39. $y''' + y'' - 2y = 0, \ y(0) = 2, \ y'(0) = 2, \ y''(0) = -3.$

40. $y^{iv} - 3y''' = 0, \ y(0) = 2, \ y'(0) = 5, \ y''(0) = 15, \ y'''(0) = 27.$

Find the solution of the following differential equations satisfying the given conditions.

41. $y'' + \pi^2 y' = 0, \ y(0) = 0, \ y(1) = 0, \ y'(0) + y'(1) = 0.$

42. $y'' - 36y' = 0, \ y(0) = 2, \ y'(0) = 12, \ y'(1) = 6 \sinh(6) + 12 \cosh(6).$

43. $y^{iv} + 13y'' + 36y = 0, \ y(0) = 0, \ y''(0) = -1, \ y(\pi/2) = -1, \ y'(\pi/2) = -4.$

44. $y^{iv} + 13y'' + 36y = 0, \ y(0) = 0, \ y''(0) = 0, \ y(\pi/2) = 0, \ y''(\pi/2) = 0.$

45. $y^{iv} - \omega^4 y = 0, \ \omega \neq 0, \ y(0) = 0, \ y'(0) = 0, \ y''(0) = 0, \ y(\pi) = 0, \ y'(\pi) = 0, \ y''(\pi) = -53.$

5.4 Solution of Non-Homogeneous Linear Equations

In the previous section we discussed methods for finding the general and particular solutions

$$= \text{Im} \left\{ -\frac{1}{65} [(\cos 2x - 8 \sin 2x) + i(\sin 2x + 8 \cos 2x)] \left[\left(x - \frac{28}{65} \right) + \frac{36}{65} i \right] \right\}$$

$$= -\frac{1}{4225} [65x(8 \cos 2x + \sin 2x) - 28(8 \cos 2x + \sin 2x) + 36(\cos 2x - 8 \sin 2x)]$$

$$= -\frac{1}{4225} [65x(8 \cos 2x + \sin 2x) - 188 \cos 2x - 316 \sin 2x].$$

The general solution is

$$y(x) = Ae^{-x} + Be^{-3x} - \frac{1}{4225} [65x(8 \cos 2x + \sin 2x) - 188 \cos 2x - 316 \sin 2x].$$

Example 5.66 Find the general solution of the equation $y^{iv} + 3y'' = 108x^2$.

Solution The characteristic equation of the homogeneous equation is

$$m^4 + 3m^2 = 0, \quad \text{or} \quad m^2(m^2 + 3) = 0. \text{ Its roots are } m = 0, 0, \pm \sqrt{3}i.$$

The complementary function is $y_c(x) = A + Bx + (C \cos \sqrt{3}x + D \sin \sqrt{3}x)$.

We have $F(D) = D^4 + 3D^2 = D^2(D^2 + 3)$. The particular integral is given by

$$y_p(x) = 108[D^2(D^2 + 3)]^{-1}(x^2) = 108[D^{-2}] \frac{1}{3} \left[1 + \frac{D^2}{3} \right]^{-1} (x^2)$$

$$= 36[D^{-2}] \left[1 - \frac{D^2}{3} + \frac{D^4}{9} - \cdots \right] (x^2) = 36 D^{-2} \left[x^2 - \frac{2}{3} \right]$$

$$= 36 \left[\frac{x^4}{12} - \frac{x^2}{3} \right] = 3x^4 - 12x^2.$$

The general solution is $y(x) = A + Bx + (C \cos \sqrt{3}x + D \sin \sqrt{3}x) + 3x^4 - 12x^2$.

Exercise 5.7

Find the general solution of the following differential equations.

1. $(D^2 + 5D + 4)y = 18e^{2x}$.

2. $(D^2 - 1)y = 8e^{3x}$.

3. $(D^2 - 3D - 4)y = e^x + 6e^{5x}$.

4. $(D^2 + D + 2)y = e^{x/2}$.

5. $(D^2 + 3D + 3)y = 7e^x$.

6. $(D^2 - 2D + 1)y = 5e^{4x} + 4e^{2x}$.

7. $(9D^2 - 6D + 1)y = 4e^{-x}$.

8. $(D^2 - 6D + 9)y = 14e^{3x}$.

9. $(D^2 + D - 6)y = e^{2x}$.

10. $(2D^2 - 3D - 2)y = xe^{-x/2}$.

11. $(D^2 - 1)y = 6xe^x$.

12. $(4D^2 + 9D + 2)y = xe^{-2x}$.

13. $(9D^2 + 6D + 1)y = e^{-x/3}$.

14. $(2D^2 + 7D - 4)y = xe^{-4x}$.

15. $(D^3 + 2D^2 - 5D - 6)y = 4e^x$.

16. $(2D^3 + 3D^2 - 3D - 2)y = 10e^{2x}$.

17. $(D^3 - 2D^2 - D + 2)y = e^{3x}$.

18. $(D^3 - 6D^2 + 12D - 8)y = 18e^{2x}$.