<u>Unit 1:</u> Linear Algebra

(**Book:** Advanced Engineering Mathematics by Jain and Iyengar, **Chapter-3**)

<u>Topic:</u>

Eigen values and Eigen vectors

<u>Learning Outcomes:</u>

1. To know about properties of Eigen values and Eigen vectors

2. Use of properties of Eigen values in various types of matrices.

Properties of Eigen Values

For instance, let A be a square matrix of order 3.

Let $\lambda_1, \lambda_2, \lambda_3$ be the corresponding three Eigen values.

Then, the following properties always hold:

1. Product of the Eigen values is always equal to the determinant of matrix A.

i.e. $det(A) = |A| = \lambda_1 . \lambda_2 . \lambda_3$

2. Sum of the Eigen values is always equal to the trace (Sum of diagonal elements) of matrix A i.e. $Trace(A) = \lambda_1 + \lambda_2 + \lambda_3$

Properties of Eigen Values

3. $\alpha\lambda_1, \alpha\lambda_2, \alpha\lambda_3$ are the Eigen values of αA.

4. ${\lambda_1}^{-1}, {\lambda_2}^{-1}, {\lambda_3}^{-1}$ are the Eigen values of A^{-1}.

5. ${\lambda_1}^{m}, {\lambda_2}^{m}, {\lambda_3}^{m}$ are the Eigen values of A^{m}.

6. A and A^T have the same eigen values.

7. For a real matrix A, if $\lambda = \alpha + i\beta$ is an eigen value, then its conjugate $\bar{\lambda} = \alpha - i\beta$ is also its Eigen value. This result does not hold if A is a complex matrix.

Problem 1. Find the Eigen values and the corresponding Eigen vectors of:

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$$

Show that: (I) Sum of the Eigen values is always equal to the trace of matrix A.

(II) Product of the Eigen values is always equal to the determinant of matrix A

Solution. The characteristic equation is: $|A - \lambda I| = 0$

$$\Rightarrow \left| \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \left| \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 1 - \lambda & 4 \\ 3 & 2 - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda) - 12 = 0$$

$$\Rightarrow 2 - \lambda - 2\lambda + \lambda^2 - 12 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda - 10 = 0$$

$$\Rightarrow (\lambda - 5)(\lambda + 2) = 0$$

$$\Rightarrow \lambda = 5, \lambda = -2$$

Let $\lambda_1 = 5, \lambda_2 = -2$ (Say)

Which are the required Eigen values.

(I) Sum of Eigen values $= \lambda_1 + \lambda_2 = 5 - 2 = 3$

Trace(A) $= 1 + 2 = 3$

Hence, Trace(A) = Sum of Eigen values.

(II) Product of Eigen values $= \lambda_1 . \lambda_2 = 5(-2) = -10$

det(A) $= |A| = \begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix} = 2 - 12 = -10$

Hence, det(A) = Product of Eigen values.

Problem 2. Find the Eigen values and the corresponding Eigen vectors of:

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Show that: (I) Sum of the Eigen values is always equal to the trace of matrix A.

(II) Product of the Eigen values is always equal to the determinant of matrix A.

Solution. The characteristic equation is: $|A - \lambda I| = 0$

$$\Rightarrow \left| \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \left| \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(1-\lambda) + 1 = 0$$

$$\Rightarrow 1 - \lambda - \lambda + \lambda^2 + 1 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda + 2 = 0$$

$$\Rightarrow \lambda = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} = \frac{2(1 \pm i)}{2} = 1 \pm i$$

Let $\lambda_1 = 1 + i, \lambda_2 = 1 - i$ (Say)

Which are the required Eigen values.

(I) Sum of Eigen values $= \lambda_1 + \lambda_2 = (1+i) + (1-i) = 2$

Trace(A) $= 1 + 1 = 2$

Hence, Trace(A) $=$ Sum of Eigen values.

(II) Product of Eigen values $= \lambda_1 . \lambda_2 = (1+i)(1-i) = 1 - i^2 = 1 + 1 = 2$

det(A) $= |A| = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 1 - (-1) = 2$

Hence, det(A) $=$ Product of Eigen values.

Problem 3. Find the Eigen values and the corresponding Eigen vectors of:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

Show that: (I) Sum of the Eigen values is always equal to the trace of matrix A.

(II) Product of the Eigen values is always equal to the determinant of matrix A.

Solution. The characteristic equation is: $|A - \lambda I| = 0$

$$\Rightarrow \left| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda)(3-\lambda) = 0$$

$$\Rightarrow \lambda = 1,2,3$$

(I) Sum of Eigen values $= \lambda_1 + \lambda_2 + \lambda_3 = 1 + 2 + 3 = 6$

Trace(A) $= 1 + 2 + 3 = 6$

Hence, Trace(A) $=$ Sum of Eigen values.

(II) Product of Eigen values $= \lambda_1 . \lambda_2 . \lambda_3 = 1(2)(3) = 6$

$$\det(\mathrm{A}) = |A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{vmatrix} = 1(6-0) - 0 + 0 = 6$$

Hence, $\det(\mathrm{A}) =$ Product of Eigen values.

Problem 4. Let a 4×4 matrix A have Eigen values $(1, -1, 2, -2)$. Find the value of determinant and trace of matrix $B = 2A + A^{-1} - I$.

Solution. Eigen values of $A = (1, -1, 2, -2)$

Eigen values of $A^{-1} = (1, -1, \frac{1}{2}, -\frac{1}{2})$

Eigen values of $I = (1,1,1,1,)$

Eigen values of $B = 2$(Eigen values of A) +(Eigen values of A^{-1})

−(Eigen values of I)

Eigen values of $B = 2(1,-1,2,-2) + (1,-1,\frac{1}{2},-\frac{1}{2}) - (1,1,1,1)$

$$= (2+1-1, -2-1-1, 4+\frac{1}{2}-1, -4-\frac{1}{2}-1)$$

$$= (2, -4, \frac{7}{2}, -\frac{11}{2})$$

det(B) $= |B| =$ Product of Eigen values of matrix B

$$= (2)(-4)\left(\frac{7}{2}\right)\left(-\frac{11}{2}\right) = 154$$

Trace(B) $=$ Sum of Eigen values of matrix B

$$= 2 - 4 + \frac{7}{2} - \frac{11}{2} = -4$$

Problem 5. Let a 3×3 matrix A have Eigen values $(1,2,-1)$. Find the value of determinant and trace of matrix $B = A - A^{-1} + A^2$.

Solution. Eigen values of $A = (1,2,-1)$

Eigen values of $A^{-1} = (1, \frac{1}{2}, -1)$

Eigen values of $A^2 = (1,4,1)$

Eigen values of B =(Eigen values of A) $-$(Eigen values of A^{-1})

+(Eigen values of A^2)

Eigen values of B $=(1,2,-1)-(1,\frac{1}{2},-1)+(1,4,1)$

$$= \left(1 - 1 + 1, 2 - \frac{1}{2} + 4, -1 + 1 + 1\right)$$

$$= \left(1, \frac{11}{2}, 1\right)$$

det(B) $= |B| =$ Product of Eigen values of matrix B

$$= (1)\left(\frac{11}{2}\right)(1) = \frac{11}{2}$$

Trace(B) $=$ Sum of Eigen values of matrix B

$$= 1 + \frac{11}{2} + 1 = \frac{15}{2}$$