Unit 1: Linear Algebra

(**Book:** Advanced Engineering Mathematics by Jain and Iyengar,**Chapter-3**)

Topic:

Rank of a Matrix

Learning Outcomes:

1. Definition of Rank of a matrix.

2. Calculation of rank using determinants .

3. Calculation of rank using elementary operations.

<u>Rank of a Matrix:</u>

Definition (Determinant/Minor based) - The rank of a matrix A is the order of the largest non-zero minor of A and is denoted by $\rho(A)$ or $r(A)$.

In other words, a positive integer r is said to be the rank of a non-zero matrix A, if

(I) There is at least one minor of order r which is not zero.

(II) Every other minor of matrix A of higher order is zero.

Note- If A is a matrix of order $m \times n$ that is $[A]_{m\times n}$, then rank of A

$$r(A) \leq \min(m, n)$$

Find the rank of the matrix $\begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{pmatrix}$

Solution:

Let $A = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{pmatrix}$

Order of A is 3×4

$\therefore \rho(A) \leq 3$.

Consider the third order minors

$$\begin{vmatrix} 1 & 2 & -1 \\ 2 & 4 & 1 \\ 3 & 6 & 3 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & -1 & 3 \\ 2 & 1 & -2 \\ 3 & 3 & -7 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & -2 \\ 3 & 6 & -7 \end{vmatrix} = 0, \quad \begin{vmatrix} 2 & -1 & 3 \\ 4 & 1 & -2 \\ 6 & 3 & -7 \end{vmatrix} = 0$$

Since all third order minors vanishes, $\rho(A) \neq 3$.

Now, let us consider the second order minors,

Now, let us consider the second order minors,

Consider one of the second order minors $\begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix} = 6 \neq 0$

There is a minor of order 2 which is not zero.

$\therefore \rho\ (A) = 2$.

Find the rank of the matrix $\begin{pmatrix} 5 & 3 & 0 \\ 1 & 2 & -4 \\ -2 & -4 & 8 \end{pmatrix}$

Solution:

Let A= $\begin{pmatrix} 5 & 3 & 0 \\ 1 & 2 & -4 \\ -2 & -4 & 8 \end{pmatrix}$

Order Of A is 3x3

$\therefore \rho(A) \leq 3$

Consider the third order minor $\begin{vmatrix} 5 & 3 & 0 \\ 1 & 2 & -4 \\ -2 & -4 & 8 \end{vmatrix} = 0$

Since the third order minor vanishes, therefore $\rho(A) \neq 3$

Consider a second order minor $\begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = 7 \neq 0$

There is a minor of order 2, which is not zero.

$\therefore \rho(A) = 2.$

Find rank of A

Let A= $\begin{pmatrix} 0 & -1 & 5 \\ 2 & 4 & -6 \\ 1 & 1 & 5 \end{pmatrix}$

Order Of A is 3x3

$\therefore \rho(A) \leq 3$

Consider the third order minor $\begin{vmatrix} 0 & -1 & 5 \\ 2 & 4 & -6 \\ 1 & 1 & 5 \end{vmatrix} = 6 \neq 0$

There is a minor of order 3, which is not zero

$\therefore \rho(A) = 3$.

Elementary transformations of a matrix

(i) Interchange of any two rows (or columns): $R_i \leftrightarrow R_j$ $(or\ C_i \leftrightarrow C_j)$.

(ii) Multiplication of each element of a row (or column) by any non-zero scalar k : $R_i \to kR_i$ $(or\ C_i \to kC_i)$

(iii) Addition to the elements of any row (or column) the same scalar multiples of corresponding elements of any other row (or column):

$R_i \to R_i + kR_i$.$(or\ C_i \to C_i + kC_i)$

Equivalent Matrices

Two matrices A and B are said to be equivalent if one is obtained from the another by applying a finite number of elementary transformations and we write it as $A \sim B$ or $B \sim A$.

Echelon form of a matrix:

A matrix A of order $m \times n$ is said to be in echelon form if

(i) Every row of A which has all its entries 0 occurs below every row which has a non-zero entry.

(ii) The number of zeros before the first non-zero element in a row is less then the number of such zeros in the next row.

Examples of Echelon form:

$$\begin{bmatrix} 3 & 2 & 0 & 7 & 9 \\ 0 & 4 & 5 & 10 & 0 \\ 0 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rank of a Matrix:

Definition (Echelon form based) – The number of non-zero rows (columns) in the Echelon form of a given matrix.

Note- If A is a matrix of order $m \times n$ that is $[A]_{m\times n}$, then rank of A

$$r(A) \leq \min(m, n)$$

Find the rank of the matrix A= $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{pmatrix}$

Solution :

The order of A is 3×3.

$\therefore \rho(A) \leq 3$.

Let us transform the matrix A to an echelon form by using elementary transformations.

Matrix A	Elementary Transformation
$\begin{array}{cccc} & 2 & 4 & 6 \\ A = & \multicolumn{3}{c}{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{pmatrix}} \end{array}$ 3 6 9	
$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{pmatrix}$	$R_2 \to R_2 - 2R_1$ $R_3 \to R_3 - 3R_1$
$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$	$R_3 \to R_3 - R_2$
The above matrix is in echelon form	

The number of non zero rows is 2

$\therefore$Rank of A *is* 2.

$\rho\,(A) = 2.$

Note

A row having atleast one non -zero element is called as non-zero row.

Find the rank of the matrix A= $\begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{pmatrix}$

Solution:

The order of A is 3×4.

$\therefore \rho(A) \leq 3$.

Let us transform the matrix A to an echelon form

Matrix A	Elementary Transformation
$A = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{pmatrix}$	
$\begin{array}{c} \begin{array}{cccc} \scriptstyle 3 & \scriptstyle 6 & \scriptstyle 9 & \scriptstyle 6 \end{array} \\ A = \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 3 & 1 & 1 & 3 \end{pmatrix} \end{array}$	$R_1 \leftrightarrow R_2$
$\sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ \scriptstyle 0 & \scriptstyle 5 & \scriptstyle 10 & \scriptstyle 5 \\ 0 & 1 & 2 & 1 \\ 0 & -5 & -8 & -3 \end{pmatrix}$	$R_3 \rightarrow R_3 - 3R_1$
$\sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 2 \end{pmatrix}$	$R_3 \rightarrow R_3 + 5R_2$

The number of non zero rows is 3. $\therefore \rho(A) = 3$.

Find the rank of the matrix A= $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 4 & 5 & 2 \\ 2 & 3 & 4 & 0 \end{pmatrix}$

Solution:

The order of A is 3×4.

$\therefore \rho(A) \leq 3$.

Let us transform the matrix A to an echelon form

Matrix A	Elementary Transformation
$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 4 & 5 & 2 \\ 2 & 3 & 4 & 0 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & -2 \end{pmatrix}$	$R_2 \rightarrow R_2 - 3R_1$ $R_3 \rightarrow R_3 - 2R_1$
$\sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$	$R_3 \rightarrow R_3 - R_2$

The number of non zero rows is 3. $\therefore \rho(A) = 3.$