10. Check the following system of equations for consistency.

$$4x - 2y = 3$$
$$6x - 3y = 5$$

Solution: The given system of equations can be written as

$AX = B$, where $A = \begin{bmatrix} 4 & -2 \\ 6 & -3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

Now, $|A| = \begin{vmatrix} 4 & -2 \\ 6 & -3 \end{vmatrix} = -12 + 12 = 0$

So, the given system of equations is inconsistent or it has infinitely many solutions according to $(\text{adj}\, A)B \neq 0$ or $(\text{adj}\, A)B = 0$, respectively.

The cofactors can be calculated as follows:

$$C_{11} = (-1)^{1+1}(-3) = -3$$
$$C_{12} = (-1)^{1+2}(6) = -6$$
$$C_{21} = (-1)^{2+1}(-2) = 2$$
$$C_{22} = (-1)^{2+2}(4) = 4$$
$$\text{adj}\, A = \begin{bmatrix} -3 & -6 \\ 2 & 4 \end{bmatrix}^T = \begin{bmatrix} -3 & 2 \\ -6 & 4 \end{bmatrix}$$

Thus,

$$(\text{adj}\, A)B = \begin{bmatrix} -3 & 2 \\ -6 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} -9+10 \\ -18+20 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq 0$$

Hence, the given system of equations is inconsistent.

PRACTICE EXERCISE

1. What is the rank of the matrix $A = \begin{bmatrix} 1 & 3 & 5 & 1 \\ 2 & 4 & 8 & 0 \\ 3 & 1 & 7 & 5 \end{bmatrix}$?

(a) 1 (b) 2
(c) 3 (d) 4

2. What is the rank of the matrix $A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$?

(a) 1 (b) 2
(c) 3 (d) 4

3. What is the rank of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$?

(a) 1 (b) 2
(c) 3 (d) 4

4. What is the rank of the matrix $A = \begin{bmatrix} 4 & 2 & 3 \\ 1 & 0 & 0 \\ 4 & 0 & 3 \end{bmatrix}$?

(a) 0 (b) 1
(c) 2 (d) 3

5. If $\begin{bmatrix} a+b & 3 \\ 5 & ab \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 5 & 3 \end{bmatrix}$, then what are the values of a and b?

(a) (2, 1) or (1, 2) (b) (2, 4) or (4, 2)
(c) (0, 3) or (3, 0) (d) (1, 3) or (3, 1)

6. If $A = \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ 7 & 1 \end{bmatrix}$, then what is the value of $4A - 3B$?

(a) $\begin{bmatrix} -5 & -20 \\ -9 & 17 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & 20 \\ 9 & -17 \end{bmatrix}$

(c) $\begin{bmatrix} -2 & -2 \\ -4 & 4 \end{bmatrix}$ (d) $\begin{bmatrix} -5 & 20 \\ 9 & 17 \end{bmatrix}$

7. What is the value of

$$\sin\theta\begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix} + \cos\theta\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}?$$

(a) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} \sin\theta\cos\theta & \sin\theta+\cos\theta \\ \sin\theta-\cos\theta & \sin\theta\cos\theta \end{bmatrix}$

(d) $\begin{bmatrix} \sin\theta\cos\theta & 0 \\ 0 & \sin\theta+\cos\theta \end{bmatrix}$

8. If $B = \begin{bmatrix} 1 & 7 \\ 3 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 4 & 1 \\ 6 & 8 \end{bmatrix}$, and $2A + 3B - 6C = 0$, then what is the value of A?

(a) $\begin{bmatrix} 21/2 & 27/2 \\ -15/2 & 45/2 \end{bmatrix}$ (b) $\begin{bmatrix} 21/4 & 27/4 \\ -15/4 & 45/4 \end{bmatrix}$

(c) $\begin{bmatrix} 21/2 & -15/2 \\ 27/2 & 45/2 \end{bmatrix}$ (d) $\begin{bmatrix} 21/4 & -15/4 \\ 27/4 & 45/4 \end{bmatrix}$

9. Find A and B, if

$$A + B = \begin{bmatrix} 8 & 5 \\ 8 & 13 \end{bmatrix} \text{ and } A - B = \begin{bmatrix} 6 & 1 \\ 2 & 3 \end{bmatrix}.$$

(a) $A = \begin{bmatrix} 7 & 3 \\ 5 & 8 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}, B = \begin{bmatrix} 7 & 3 \\ 5 & 8 \end{bmatrix}$

(c) $A = \begin{bmatrix} 5 & 2 \\ 5 & 6 \end{bmatrix}, B = \begin{bmatrix} 3 & 3 \\ 3 & 7 \end{bmatrix}$

(d) $A = \begin{bmatrix} 10 & 5 \\ 5 & 8 \end{bmatrix}, B = \begin{bmatrix} 4 & 4 \\ 3 & 5 \end{bmatrix}$

10. For what values of λ, the given set of equations has a unique solution?

$$2x + 3y + 5z = 9$$
$$7x + 3y - 2z = 8$$
$$2x + 3y + \lambda z = 9$$

(a) $\lambda = 15$
(b) $\lambda = 5$
(c) For all values except $\lambda = 15$
(d) For all values except $\lambda = 5$

11. For what values of λ, the given set of equations has a unique solution?

$$x + y + z = 3$$
$$x + 2y + 3z = 4$$
$$x + 4y + \lambda z = 6$$

(a) 5 (b) 7
(c) 9 (d) 0

12. How many solutions does the following system of equations have?

$$x + 2y + z = 6$$
$$2x + y + 2z = 6$$
$$x + y + z = 5$$

(a) One solution
(b) Infinite solutions
(c) No solutions
(d) None of the above

13. If $A = \begin{bmatrix} 1 & 2 & -7 \\ 3 & 1 & 5 \\ 4 & 7 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -5 & 1 \\ 4 & 8 & 5 \\ 1 & 2 & 6 \end{bmatrix}$, then what is the value of $(A \times B)$?

(a) $\begin{bmatrix} 4 & -3 & -31 \\ 18 & 3 & 38 \\ 41 & 38 & 45 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & -3 & -38 \\ -3 & 3 & 38 \\ -31 & 31 & 45 \end{bmatrix}$

(c) $\begin{bmatrix} 11 & -3 & -31 \\ 18 & 9 & 18 \\ 41 & 38 & 35 \end{bmatrix}$ (d) $\begin{bmatrix} 4 & 3 & -31 \\ 18 & -3 & 38 \\ 45 & 38 & 41 \end{bmatrix}$

14. If $A = \begin{bmatrix} k & 0 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 \\ 6 & 16 \end{bmatrix}$, then what is the value of k for which $A^2 = B$?

(a) -1 (b) -2
(c) 1 (d) 2

15. If $A = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then what is the value of k for which $A^2 = 8A + kI$?

(a) 7 (b) -7
(c) 10 (d) 8

16. If $\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} A = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$, then what is the value of A?

(a) $\begin{bmatrix} 3 & 4 & 0 \\ 1 & 3 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 4 & 0 \\ 1 & -2 & -5 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -2 & -5 \\ 3 & 4 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 3 & 4 \\ 3 & 4 & 0 \end{bmatrix}$

17. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ is a matrix such that $AA^T = 9I_3$, then what are the values of a and b?

(a) $a = -1,\ b = -2$
(b) $a = -2,\ b = -1$
(c) $a = 1,\ b = 2$
(d) $a = 2,\ b = 1$

18. If $A = \begin{bmatrix} 8 & 4 & 6 \\ 4 & 0 & 2 \\ x & 6 & 0 \end{bmatrix}$ is singular, then what is the value of x?

(a) 12 (b) 8
(c) 4 (d) 1

19. If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$, then what is the value of A^{-1}?

(a) $\frac{1}{19}\begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix}$ (b) $\frac{1}{29}\begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix}$

(c) $\frac{1}{19}\begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$ (d) $\frac{1}{29}\begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$

20. What is the value of I^T, where I is an identity matrix of order 3?

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

21. If $A = \begin{bmatrix} 1 & 7 & -1 \\ 3 & 2 & 2 \\ 4 & 5 & 1 \end{bmatrix}$, then what is the first row of A^T?

(a) [1 7 −1] (b) [1 3 4]
(c) [3 2 2] (d) [4 5 1]

22. If $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$, then what is the value of A^{-1}?

(a) $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 7 & -1 & -1 \\ -3 & 0 & 1 \\ -3 & 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} -3 & -3 & 7 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

23. Calculate the adjoint of the matrix $A = \begin{bmatrix} 8 & 4 & 2 \\ 2 & 9 & 4 \\ 1 & 2 & 8 \end{bmatrix}$.

(a) $\begin{bmatrix} 64 & 28 & 2 \\ 12 & 62 & 18 \\ 6 & 36 & -64 \end{bmatrix}$ (b) $\begin{bmatrix} 62 & -28 & -2 \\ -12 & 62 & -38 \\ -2 & -12 & -62 \end{bmatrix}$

(c) $\begin{bmatrix} 64 & -28 & -2 \\ -12 & 62 & -28 \\ -5 & -12 & 64 \end{bmatrix}$ (d) $\begin{bmatrix} 64 & 28 & 24 \\ 10 & 62 & 48 \\ 6 & 36 & -64 \end{bmatrix}$

24. If $A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$, then what is the value of B such that $AB = C$?

(a) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 4 \\ 1 & 3 \end{bmatrix}$

25. If $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$ and if $ABC = I_2$, then what is the value of C?

(a) $\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

26. What is the value of $(AB)^{-1}B$?

(a) A^{-1} (b) B
(c) A (d) $A^{-1}B^{-1}$

27. If $A = \begin{bmatrix} x & 2 & 0 \\ 2 & 0 & 1 \\ 6 & 3 & 0 \end{bmatrix}$ is singular, then what is the value of x?

(a) 0 (b) 2
(c) 4 (d) 6

28. What is the nullity of the matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$?

(a) 0 (b) 1
(c) 2 (d) 3

29. What are the eigenvalues of $A = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$?

(a) 1, 4 (b) 2, 3
(c) 0, 5 (d) 1, 5

30. What are the eigenvalues of $A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$?

(a) 1, 4, 4 (b) 1, 4, − 4
(c) 3, 3, 3 (d) 1, 2, 6

31. What is the eigenvector of the matrix $A = \begin{bmatrix} 5 & -4 \\ -1 & 2 \end{bmatrix}$?

(a) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$

(c) $\begin{bmatrix} -2 \\ -1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

32. What are the eigenvectors of the matrix $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$?

(a) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

33. What are the eigenvalues of the matrix $A = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$?

(a) $\sqrt{2}, -\sqrt{2}, 1$ (b) $i, -i, 1$

(c) $2, -2, 1$ (d) $0, \frac{1}{2}, \frac{1}{2}$

34. What are the eigenvalues of the matrix $A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$?

(a) 1, 2, 3 (b) 4, 4, 5
(c) 3, 5, 6 (d) 3, 3, 7

35. Which one of the following options is not the eigenvector of matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$?

(a) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

36. What are the eigenvectors of the matrix $A = \begin{bmatrix} 2 & 5 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix}$?

(a) $\begin{bmatrix} k \\ 0 \\ k \end{bmatrix}, \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ k \\ 2k \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix}, \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ k \\ 2k \end{bmatrix}$

(c) $\begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2k \end{bmatrix}$ (d) $\begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix}, \begin{bmatrix} k \\ k \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2k \\ k \end{bmatrix}$

37. What is the sum of eigenvalues of $A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 5 & 4 \\ 1 & 6 & 2 \end{bmatrix}$?

(a) 8 (b) 10
(c) 4 (d) 5

38. What is the value of x and y if $A = \begin{bmatrix} x & y \\ -4 & 10 \end{bmatrix}$ and eigenvalues of A are 4 and 8?

(a) $x = 3, y = 2$ (b) $x = 2, y = 4$
(c) $x = 4, y = 2$ (d) $x = 2, y = 3$

39. What are the eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$?

(a) $1, -1$ (b) $1, i$
(c) $i, -i$ (d) $0, 1$

40. What are the values of x and y, if $A = \begin{bmatrix} 3 & 0 & 0 \\ 5 & x & 0 \\ 3 & 6 & y \end{bmatrix}$ and eigenvalues of A are 3, 4 and 1?

(a) $x = 2, y = 2$ (b) $x = 3, y = 2$
(c) $x = 2, y = 3$ (d) $x = 4, y = 1$

ANSWERS

1. (c)	**8.** (c)	**15.** (b)	**22.** (a)	**29.** (c)	**36.** (b)
2. (b)	**9.** (a)	**16.** (c)	**23.** (c)	**30.** (a)	**37.** (a)
3. (b)	**10.** (d)	**17.** (b)	**24.** (b)	**31.** (b)	**38.** (d)
4. (d)	**11.** (b)	**18.** (a)	**25.** (d)	**32.** (c)	**39.** (c)
5. (d)	**12.** (c)	**19.** (c)	**26.** (a)	**33.** (a)	**40.** (d)
6. (a)	**13.** (a)	**20.** (d)	**27.** (c)	**34.** (b)	
7. (b)	**14.** (d)	**21.** (b)	**28.** (b)	**35.** (d)	

EXPLANATIONS AND HINTS

1. (c) Matrix $A = \begin{vmatrix} 1 & 3 & 5 & 1 \\ 2 & 4 & 8 & 0 \\ 3 & 1 & 7 & 5 \end{vmatrix}$

Maximum possible rank = 3

Now, consider 3 × 3 minors

$$\begin{vmatrix} 1 & 3 & 5 \\ 2 & 4 & 8 \\ 3 & 1 & 7 \end{vmatrix} = (28-8) - 2\ (21-5) + 3\ (24-20)$$

$$= 20 - 32 + 12 = 0$$

$$\begin{vmatrix} 3 & 5 & 1 \\ 4 & 8 & 0 \\ 1 & 7 & 5 \end{vmatrix} = 3(40 - 0) - 4(25 - 7) + 1(0 - 8)$$

$$= 120 - 72 - 8 = 40 \neq 0$$

Hence, rank of $A = 3$.

2. (b) Matrix $A = \begin{vmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{vmatrix}$

Maximum possible rank = 3

Now, consider 3 × 3 minors

$$\begin{vmatrix} 4 & 2 & 1 \\ 6 & 3 & 4 \\ 2 & 1 & 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} 2 & 1 & 3 \\ 3 & 4 & 7 \\ 1 & 0 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 4 & 1 & 3 \\ 6 & 3 & 7 \\ 2 & 0 & 1 \end{vmatrix} = 0$$

and
$$\begin{vmatrix} 4 & 2 & 3 \\ 6 & 3 & 7 \\ 2 & 1 & 1 \end{vmatrix} = 0$$

Now, because all 3 × 3 minors are zero, let us consider 2 × 2 minors

$$\begin{vmatrix} 4 & 2 \\ 6 & 3 \end{vmatrix} = 0$$

$$\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 8 - 3 = 5 \neq 0$$

Hence, rank of A is 2.

3. (b) Matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

Maximum rank = 3

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = (-1-0) - 1(1-1) + 1(0+1)$$

$$= -1 + 1 = 0$$

Hence, rank $\neq 3$.

Now, let us consider 2 × 2 minors

$$\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = (-1-1) = -2 \neq 0$$

Hence, rank of A is 2.

4. (d) Matrix $A = \begin{bmatrix} 4 & 2 & 3 \\ 1 & 0 & 0 \\ 4 & 0 & 3 \end{bmatrix}$

Maximum rank = 3

$$\begin{bmatrix} 4 & 2 & 3 \\ 1 & 0 & 0 \\ 4 & 0 & 3 \end{bmatrix} = 4(0) - 1(6-0) + 4(0) = -6 \neq 0$$

Hence, rank of $A = 3$.

5. (d) Since both the matrices are equal

$$\begin{bmatrix} a+b & 3 \\ 5 & ab \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 5 & 3 \end{bmatrix}$$

$$\Rightarrow a + b = 4 \qquad (1)$$

$$ab = 3 \qquad (2)$$

From Eq. (1), we have

$$a = 4 - b$$

Substituting the value of a in Eq. (2), we get

$$(4 - b)\ b = 3$$

$$\Rightarrow 4b - b^2 = 3$$

$$\Rightarrow b^2 - 4b + 3 = 0$$

$$\Rightarrow (b - 3)\ (b - 1) = 0$$

$$\Rightarrow b = 3, 1$$

For values of b, $a = 4 - 3, 4 - 1 = 1, 3$

Therefore, the values of a and $b = (1, 3)$ or $(3, 1)$

6. (a) $A = \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 4 \\ 7 & 1 \end{bmatrix}$

$$4A = \begin{bmatrix} 4 & -8 \\ 12 & 20 \end{bmatrix}, \quad 3B = \begin{bmatrix} 9 & 12 \\ 21 & 3 \end{bmatrix}$$

$$4A - 3B = \begin{bmatrix} 4-9 & -8-12 \\ 12-21 & 20-3 \end{bmatrix} = \begin{bmatrix} -5 & -20 \\ -9 & 17 \end{bmatrix}$$

7. (b) We have

$$\cos\theta \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ -\sin\theta\cos\theta & \cos^2\theta \end{bmatrix} + \begin{bmatrix} \sin^2\theta & -\sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \sin\theta\cos\theta - \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & \cos^2\theta + \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

8. (c) We have

$$2A + 3B - 6C = 0$$

$$A = \frac{1}{2}(6C - 3B)$$

Also, $B = \begin{bmatrix} 1 & 7 \\ 3 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & 1 \\ 6 & 8 \end{bmatrix}$

$$A = \frac{1}{2}\left(6\begin{bmatrix} 4 & 1 \\ 6 & 8 \end{bmatrix} - 3\begin{bmatrix} 1 & 7 \\ 3 & 1 \end{bmatrix}\right)$$

$$= \frac{1}{2}\left(\begin{bmatrix} 24 & 6 \\ 36 & 48 \end{bmatrix} - \begin{bmatrix} 3 & 21 \\ 9 & 3 \end{bmatrix}\right)$$

$$= \frac{1}{2}\left(\begin{bmatrix} 21 & -15 \\ 27 & 45 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 21/2 & -15/2 \\ 27/2 & 45/2 \end{bmatrix}$$

9. (a) We have

$$A + B = \begin{bmatrix} 8 & 5 \\ 8 & 13 \end{bmatrix} \text{ and } A - B = \begin{bmatrix} 6 & 1 \\ 2 & 3 \end{bmatrix}$$

$$\therefore (A+B) + (A-B) = \begin{bmatrix} 8 & 5 \\ 8 & 13 \end{bmatrix} + \begin{bmatrix} 6 & 1 \\ 2 & 3 \end{bmatrix}$$

$$2A = \begin{bmatrix} 14 & 6 \\ 10 & 16 \end{bmatrix} \Rightarrow A = \frac{1}{2}\begin{bmatrix} 14 & 6 \\ 10 & 16 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 5 & 8 \end{bmatrix}$$

Also, $(A+B) - (A-B) = \begin{bmatrix} 8 & 5 \\ 8 & 13 \end{bmatrix} - \begin{bmatrix} 6 & 1 \\ 2 & 3 \end{bmatrix}$

$$2B = \begin{bmatrix} 2 & 4 \\ 6 & 10 \end{bmatrix} \Rightarrow B = \frac{1}{2}\begin{bmatrix} 2 & 4 \\ 6 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$$

Thus, $A = \begin{bmatrix} 7 & 3 \\ 5 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$.

10. (d) The given set of equations can be written as

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ 9 \end{bmatrix}$$

The system will have a unique solution if the rank of coefficient matrix is 3.

Thus,

$$\begin{vmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{vmatrix} = 2(3\lambda + 6) - 7(3\lambda - 15) + 2(-6 - 15)$$

$$= 6\lambda + 12 - 21\lambda + 105 - 12 - 30$$
$$= -15\lambda + 75$$
$$= 15(5 - \lambda)$$

For rank = 3,

$$\begin{vmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{vmatrix} \neq 0$$

$\therefore 15(5 - \lambda) \neq 0$

$\lambda \neq 5$

11. (b) The given set of equations can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & k \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

Thus, $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & k \end{vmatrix} = 1(2k - 12) - 1(k - 4) + 1(3 - 2)$

$$= 2k - 12 - k + 4 + 1$$
$$= k - 7$$

Now, for a system to be unique

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & k \end{vmatrix} \neq 0$$

$$\Rightarrow k - 7 \neq 0$$
$$\Rightarrow k \neq 7$$

Thus, the value of k for which the given set of equations does not have a unique solution is 7.

12. (c) The given system can be written as

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 5 \end{bmatrix}$$

Now, $\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{vmatrix} = 1(1-2) - 2\ (2-1) + 1(4-1)$

$$= -1 - 2 + 3 = 0$$

Hence, rank of matrix is not 3.

Now, taking a minor from the matrix

$$\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1 - 4 = -3 \neq 0$$

Hence, rank of matrix = 2.

Now, rank of matrix is less than the number of variables.

Hence, the system is inconsistent or has no solution.

13. (a) We have

$$A = \begin{bmatrix} 1 & 2 & -7 \\ 3 & 1 & 5 \\ 4 & 7 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & -5 & 1 \\ 4 & 8 & 5 \\ 1 & 2 & 6 \end{bmatrix}$$

$$A \times B = \begin{bmatrix} 3+8+(-7) & -5+16-14 & 1+10-42 \\ 9+4+5 & -15+8+10 & 3+5+30 \\ 12+28+1 & -20+56+2 & 4+35+6 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -3 & -31 \\ 18 & 3 & 38 \\ 41 & 38 & 45 \end{bmatrix}$$

14. (d) We have

$$A = \begin{bmatrix} k & 0 \\ 1 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & 0 \\ 6 & 16 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} k & 0 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} k & 0 \\ 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} k^2+0 & 0+0 \\ k+4 & 0+16 \end{bmatrix} = \begin{bmatrix} k^2 & 0 \\ k+4 & 16 \end{bmatrix}$$

Now, since $A^2 = B$

$$\begin{bmatrix} k^2 & 0 \\ k+4 & 16 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 6 & 16 \end{bmatrix}$$

$$k^2 = 4 \text{ and } k + 4 = 6$$

$$k = \pm 2 \text{ and } k = 2$$

Hence, value of $k = 2$.

15. (b) We have

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix} \quad \text{and} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^2 \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1+0 & 0 \\ -1-7 & 0+49 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix}$$

$$= 8A = 8\begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ -8 & 56 \end{bmatrix}$$

$$kI = k\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

Now,

$$A^2 = 8A + kI$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ -8 & 56 \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix} = \begin{bmatrix} 8+k & 0 \\ -8 & 56+k \end{bmatrix}$$

or $\quad 8 + k = 1 \quad \text{and} \quad 56 + k = 49$

$$\Rightarrow k = -7$$

16. (c) As product is a 3×3 matrix and one of the matrix is 3×2, the order of A is 2×3.

Consider $A = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix}$, then

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 - x_2 & 2y_1 - y_2 & 2z_1 - z_2 \\ x_1 & y_1 & z_1 \\ -3x_1 + 4x_2 & -3y_1 + 4y_2 & -3z_1 + 4z_2 \end{bmatrix} = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$$

Now, $2x_1 - x_2 = -1$, $x_1 = 1$

$$\Rightarrow 2 - x_2 = -1$$

$$\Rightarrow x_2 = 3$$

Also, $2y_1 - y_2 = -8$, $y_1 = -2$

$$\Rightarrow -4 - y_2 = -8$$

$$\Rightarrow y_2 = 4$$

Also, $2z_1 - z_2 = -10$, $z_1 = -5$

$$\Rightarrow z_2 = 0$$

Thus, $A = \begin{bmatrix} 1 & -2 & -5 \\ 3 & 4 & 0 \end{bmatrix}$.

17. (b) We have

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix}$$

$AA^T = 9I_3$

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix} = 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 9 & 0 & a+2b+4 \\ 0 & 9 & 2a+2-2b \\ a+2b+4 & 2a+2-2b & a^2+4+b^2 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$a + 2b + 4 = 0 \quad \Rightarrow a + 2b = -4$$

$$2a + 2 - 2b = 0 \quad \Rightarrow a - b = -1$$

Solving above equations,

$$3b = -3 \Rightarrow b = -1$$

$$a = -4 + 2 = -2$$

Hence, $a = -2$ and $b = -1$.

18. (a) We know that $A = \begin{bmatrix} 8 & 4 & 6 \\ 4 & 0 & 2 \\ x & 6 & 0 \end{bmatrix}$ is singular.

Hence, $\begin{vmatrix} 8 & 4 & 0 \\ 4 & 0 & 2 \\ x & 6 & 0 \end{vmatrix} = 0$

$$8(0 - 12) - 4(0) + x(8 - 0) = 0$$

$$-96 + 8x = 0$$

$$8x = 96$$

$$x = 12$$

19. (c) We have

$$A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$$

$$|A| = \{[2 \times (-2)] - (3 \times 5)\} = -4 - 15 = -19 \neq 0$$

Hence, A is invertible.

Now, cofactors of the matrix A are

$$C_{11} = -2$$

$$C_{12} = -5$$

$$C_{21} = -3$$

$$C_{22} = 2$$

$$\text{adj}\, A = \begin{bmatrix} -2 & -5 \\ -3 & 2 \end{bmatrix}^T = \begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj A} = \frac{-1}{19}\begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 2/19 & 3/19 \\ 5/19 & -2/19 \end{bmatrix}$$

20. (d) $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $I^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

21. (b) $A = \begin{bmatrix} 1 & 7 & -1 \\ 3 & 2 & 2 \\ 4 & 5 & 1 \end{bmatrix}$

$$A^T = \begin{bmatrix} 1 & 3 & 4 \\ 7 & 2 & 5 \\ -1 & 2 & 1 \end{bmatrix}$$

The first row of transport of $A = [1 \quad 3 \quad 4]$.

22. (a) $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

$$\begin{aligned} |A| &= 1(16-9) - 1(12-9) + 1(9-12) \\ &= 7 - 3 - 3 = 1 \neq 0 \end{aligned}$$

Hence, A is invertible.

Now, cofactors of the matrix A are given as

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} = 7$$

$$C_{12} = (1)^{1+2}\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -1$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = -1$$

$$C_{21} = (-1)^{2+1}\begin{vmatrix} 3 & 3 \\ 3 & 4 \end{vmatrix} = -3$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1$$

$$C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} 3 & 3 \\ 4 & 3 \end{vmatrix} = -3$$

$$C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1$$

Thus, $\text{adj } A = \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Hence, $A^{-1} = \dfrac{1}{|A|}\text{adj}\,A$

$$= \frac{1}{1}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

23. (c) We have

$$A = \begin{bmatrix} 8 & 4 & 2 \\ 2 & 9 & 4 \\ 1 & 2 & 8 \end{bmatrix}$$

Now, cofactors of the matrix A are given as

$$C_{11} = \begin{vmatrix} 9 & 4 \\ 2 & 8 \end{vmatrix}(-1)^{1+1} = 64$$

$$C_{12} = \begin{vmatrix} 2 & 4 \\ 1 & 8 \end{vmatrix}(-1)^{1+2} = -12$$

$$C_{13} = \begin{vmatrix} 2 & 9 \\ 1 & 2 \end{vmatrix}(-1)^{1+3} = -5$$

$$C_{21} = \begin{vmatrix} 4 & 2 \\ 2 & 8 \end{vmatrix}(-1)^{2+1} = -28$$

$$C_{22} = \begin{vmatrix} 8 & 2 \\ 1 & 8 \end{vmatrix}(-1)^{2+2} = 62$$

$$C_{23} = \begin{vmatrix} 8 & 4 \\ 1 & 2 \end{vmatrix}(-1)^{2+3} = -12$$

$$C_{31} = \begin{vmatrix} 4 & 2 \\ 9 & 4 \end{vmatrix}(-1)^{3+1} = -2$$

$$C_{32} = \begin{vmatrix} 8 & 2 \\ 2 & 4 \end{vmatrix}(-1)^{3+2} = -28$$

$$C_{33} = \begin{vmatrix} 8 & 4 \\ 2 & 9 \end{vmatrix}(-1)^{3+3} = 64$$

Thus,

$$\text{adj } A = \begin{bmatrix} 64 & -12 & -5 \\ -28 & 62 & -12 \\ -2 & -28 & 64 \end{bmatrix}^T = \begin{bmatrix} 64 & -28 & -2 \\ -12 & 62 & -28 \\ -5 & -12 & 64 \end{bmatrix}$$

24. (b) We know

$$A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

Now, $\quad AB = C$

$$B = CA^{-1}$$

Now, we can calculate A^{-1} as follows:

$$|A| = 4 + 2 = 6 \neq 0$$

Hence, A is invertible. Now, cofactor of the matrix A are given as

$$C_{11} = (-1)^{1+1}\,(1) = 1$$
$$C_{12} = (-1)^{2+1}\,(-1) = 1$$
$$C_{21} = (-1)^{2+1}\,(2) = -2$$
$$C_{22} = (-1)^{2+2}\,(4) = 4$$

$$\text{adj}\,A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{6}\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \times \frac{1}{6}\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$$

25. (d) We know that

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$$

Now, $|A| = (4 - 3) = 1 \neq 0$

$$|B| = (9 - 10) = -1 \neq 0$$

Hence, both A and B are invertible.

Now, $ABC = I$

$$C = A^{-1}B^{-1}I$$

or $C = A^{-1}B^{-1}$

Calculating cofactors of A,

$$C_{A_{11}} = (-1)^{1+1}(2) = 2$$

$$C_{A_{12}} = (-1)^{1+2}(3) = -3$$

$$C_{A_{21}} = (-1)^{2+1}(1) = -1$$

$$C_{A_{22}} = (-1)^{2+2}(2) = 2$$

$$\text{adj}\,A = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}^T = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{1}\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

Calculating cofactors of B,

$$C_{B_{11}} = (-1)^{1+1}(-3) = -3$$

$$C_{B_{12}} = (-1)^{1+2}(5) = -5$$

$$C_{B_{21}} = (-1)^{2+1}(2) = -2$$

$$C_{B_{22}} = (-1)^{2+2}(-3) = -3$$

$$\text{adj}\,B = \begin{bmatrix} -3 & -5 \\ -2 & -3 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} -3 & -2 \\ -5 & -3 \end{bmatrix}$$

$$B^{-1} = \frac{1}{|B|}\text{adj}\,B = \frac{1}{(-1)}\begin{bmatrix} -3 & -2 \\ -5 & -3 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$

Now,

$$C = A^{-1}B^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 6-5 & -9+10 \\ 4-3 & -6+6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

26. (a) $(AB)^{-1} = (A^{-1}B^{-1})\ B$

$$= A^{-1}[(B^{-1}) \cdot B]$$

$$= \mathrm{A}^{-1}I = A^{-1}$$

27. (c) We know

$$A = \begin{bmatrix} x & 2 & 0 \\ 2 & 0 & 1 \\ 6 & 3 & 0 \end{bmatrix}$$

For A to be singular,

$$|A| = 0$$

$$x(0-3) - 2(0-0) + 6(2) = 0$$

$$-3x + 12 = 0 \Rightarrow x = \frac{-12}{-3}$$

$$\Rightarrow x = 4$$

28. (b) We know

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$$

$|A| = 0(1-0) - 1(-1-0) + (-1)\ (0+1)$

$= 0 + 1 - 1 = 0$

Hence, rank is not 3.

Choosing a minor from A, we get

$$\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = 0 - 1 = -1 \neq 0$$

Therefore, rank $(A) = 2$

Now, nullity = Number of columns − Rank of matrix

$= 3 - 2$

$= 1$

29. (c) We know

$$A = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$$

Now,

$$[A - \lambda I] = \begin{vmatrix} 4-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(1-\lambda) - (-2)(-2) = 0$$

$$\lambda^2 - 5\lambda + 4 - 4 = 0$$

$$\lambda(\lambda - 5) = 0$$

$$\lambda = 0, 5$$

Hence, eigenvalues are 0 and 5.

30. (a) We know

$$A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

Now,

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & -1 & -1 \\ -1 & 3-\lambda & -1 \\ -1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\left[(3-\lambda)^2 - 1\right] - (-1)[-(3-\lambda)-1] + (-1)[1+(3-\lambda)] = 0$$

$$[(3-\lambda)+1]\{(3-\lambda)[(3-\lambda)-1] - (1) - 1\} = 0$$

$$(4-\lambda)\left[\lambda^2 - 5\lambda + 4\right] = 0 \Rightarrow (4-\lambda)(\lambda-1)(\lambda-4) = 0$$

$$\lambda = 1, 4, 4$$

31. (b) We have

$$A = \begin{bmatrix} 5 & -4 \\ -1 & 2 \end{bmatrix}$$

The characteristic equation is given by

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 5-\lambda & -4 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)(2-\lambda) - 4 = 0$$

$$\lambda^2 - 7\lambda + 6 = 0$$

$$(\lambda - 6)(\lambda - 1) = 0$$

$$\lambda = 6, 1$$

$\therefore$ Eigenvalues of $A = 1, 6$

Now, using $|A - \lambda|\widehat{X} = 0$

and substituting $\lambda = 1$, we get

$$\begin{bmatrix} 4 & -4 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$4X_1 - 4X_2 = 0$$

$$-X_1 + X_2 = 0$$

or $\qquad X_1 = X_2$

Now, the solution is $X_1 = X_2 = k$.

Hence, from the given options, the solution is $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

32. (c) We have

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The characteristic equation is given by

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)^2 - 1 = 0$$

$$[(1-\lambda)+1][(1-\lambda)-1] = 0$$

$$\Rightarrow (2-\lambda)(-\lambda) = 0$$

$$\lambda = 2, 0$$

Hence, eigenvalues of $A = 2, 0$

Now, using $|A - \lambda I|\widehat{X} = 0$

Putting $\lambda = 0$, we have

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$x_1 - x_2 = 0$$

$$-x_1 + x_2 = 0$$

$$x_1 = x_2$$

Putting $\lambda = 2$, we have

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$-x_1 - x_2 = 0$$

$$x_1 = -x_2$$

Hence, from the given options, eigenvectors for the corresponding eigenvalues can be $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

33. (a) We have

$$A = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Now,

$$|A - \lambda I| = \begin{vmatrix} 0-\lambda & -1 & 1 \\ -1 & 0-\lambda & 0 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (0-\lambda)[(0-\lambda)(1-\lambda) - 0] + 1[(\lambda-1) - 1]$$
$$+ 1[0 - (0-\lambda)] = 0$$

$$\Rightarrow -\lambda\left[-\lambda + \lambda^2\right] + 1[\lambda - 2] + 1[\lambda] = 0$$

$$\Rightarrow -\lambda^3 + \lambda^2 + \lambda - 2 + \lambda = 0$$

$$\Rightarrow \lambda^3 - \lambda^2 - 2\lambda + 2 = 0$$

$$\Rightarrow \lambda^2(\lambda - 1) - 2(\lambda - 1) = 0$$

$$\Rightarrow (\lambda^2 - 2)(\lambda - 1) = 0$$

$$\Rightarrow \left(\lambda + \sqrt{2}\right)\left(\lambda - \sqrt{2}\right)(\lambda - 1) = 0$$

Hence, eigenvalues are $\lambda = \sqrt{2}, -\sqrt{2}, 1$.

34. (b) We have

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Now,

$$(A-\lambda I)=\begin{bmatrix}4-\lambda & 0 & 0\\ 1 & 4-\lambda & 0\\ 0 & 0 & 5-\lambda\end{bmatrix}=0$$

$$\Rightarrow (4-\lambda)[(4-\lambda)(5-\lambda)-0]-1[0-0]+0[0-0]=0$$

$$\Rightarrow (4-\lambda)(4-\lambda)(5-\lambda)=0$$

$\therefore$ Eigenvalues of the matrix are $\lambda = 4, 4, 5$.

35. (d) We have

$$A=\begin{bmatrix}1 & 1 & 0\\ 0 & 2 & 2\\ 0 & 6 & 3\end{bmatrix}$$

The characteristic equation is given by

$$|A-\lambda I|=0$$

$$\begin{vmatrix}1-\lambda & 1 & 0\\ 0 & 2-\lambda & 2\\ 0 & 0 & 3-\lambda\end{vmatrix}=0$$

$$\Rightarrow (1-\lambda)[(2-\lambda)(3-\lambda)-0]-0[(3-\lambda)(2-\lambda)-0]$$
$$+0[2-0]=0 \Rightarrow (1-\lambda)(2-\lambda)(3-\lambda)=0$$

$$\lambda = 1,\ 2,\ 3$$

Using $|A-\lambda I|\widehat{X}=0$

and putting $\lambda = 1$, we get

$$\begin{bmatrix}0 & 1 & 0\\ 0 & 1 & 2\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$y=0$$
$$y-2z=0$$
$$2z=y$$
$$2z=0$$

Hence, $x=k$, $y=0$, $z=0$.

Therefore, the eigenvector can be of the form $\begin{bmatrix}k\\ 0\\ 0\end{bmatrix}$.

Now putting $\lambda = 2$, we get

$$\begin{bmatrix}-1 & 1 & 0\\ 0 & 0 & 2\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$-x+y=0 \qquad \Rightarrow x=y$$
$$2z=0 \qquad \Rightarrow z=0$$
$$z=0$$

Hence, $x=k$, $y=k$, $z=0$.

Therefore, the eigenvector can be of the form $\begin{bmatrix}k\\ k\\ 0\end{bmatrix}$.

Now putting $\lambda = 3$, we get

$$\begin{bmatrix}-2 & 1 & 0\\ 0 & -1 & 2\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$-2x+y=0 \Rightarrow x=y/2$$
$$-y+2z=0 \Rightarrow y=2z$$

Hence, $x=2k$, $y=k$, $z=2k$.

Therefore, the eigenvector can be of the form $\begin{bmatrix}2k\\ k\\ 2k\end{bmatrix}$.

Hence, the eigenvector which is not of the matrix A is $\begin{bmatrix}1\\ 2\\ 0\end{bmatrix}$.

36. (b) We have

$$A=\begin{bmatrix}2 & 5 & 0\\ 0 & 3 & 0\\ 0 & 1 & 1\end{bmatrix}$$

The characteristic equation is given by

$$|A-\lambda I|=0$$

$$\begin{vmatrix}2-\lambda & 5 & 0\\ 0 & 3-\lambda & 0\\ 0 & 1 & 1-\lambda\end{vmatrix}$$

$$\Rightarrow (2-\lambda)(3-\lambda)(1-\lambda)=0$$

$$\lambda = 1, 2, 3$$

Now, using $|A-\lambda I|\widehat{X}=0$

Putting $\lambda = 1$, we get

$$\begin{bmatrix}1 & 5 & 0\\ 0 & 2 & 0\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$x+5y=0$$
$$2y=0$$
$$y=0$$

Thus, the eigenvector can be of the form $\begin{bmatrix}0\\ 0\\ k\end{bmatrix}$.

Putting $\lambda = 2$, we get

$$\begin{bmatrix}0 & 5 & 0\\ 0 & 1 & 0\\ 0 & 1 & -1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$5y=0$$
$$y=0$$
$$y-\mathrm{z}=0 \Rightarrow y=z$$

Thus, the eigenvector can be of the form $\begin{bmatrix}k\\ 0\\ 0\end{bmatrix}$.

Putting $\lambda = 3$, we get

$$\begin{bmatrix} -1 & 5 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x = 0 \Rightarrow x = 0$$

$$y - 2z = 0 \Rightarrow y = 2z$$

Thus, the eigenvector can be of the form $\begin{bmatrix} 0 \\ k \\ 2k \end{bmatrix}$.

Hence, the eigenvectors for the corresponding eigenvalues are given as

$$\begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix}, \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ k \\ 2k \end{bmatrix}$$

37. (a) We have

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 5 & 4 \\ 1 & 6 & 2 \end{bmatrix}$$

Sum of eigenvalues = sum of diagonal elements of matrix A

$$= 1 + 5 + 2$$

$$= 8$$

38. (d) We know that

$$A = \begin{bmatrix} x & y \\ -4 & 10 \end{bmatrix}$$

Also, we know that the eigenvalues of A are 4 and 8.

Now, sum of eigenvalues = sum of diagonal elements

$$\therefore 4 + 8 = x + 10$$

$$\Rightarrow x = 12 - 10 = 2$$

Also, product of eigenvalues = $|A|$

$$\therefore 4 \times 8 = 10x + 4y$$

$$\Rightarrow 4 \times 8 = 10 \times 2 + 4y$$

$$\Rightarrow 4y = 32 - 20$$

$$\Rightarrow y = \frac{12}{4} = 3$$

Hence, $x = 2$ and $y = 3$.

39. (c) We have

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

The characteristic equation is given by

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 0-\lambda & 1 \\ -1 & 0-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + 1 = 0$$

$$\Rightarrow \lambda^2 = -1$$

$$\Rightarrow \lambda = \pm\, i$$

40. (d) We know

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 5 & x & 0 \\ 3 & 6 & y \end{bmatrix}$$

Now, eigenvalues of A are 1, 3 and 4.

Sum of eigenvalues = Sum of diagonal

$$\therefore 3 + x + y = 1 + 3 + 4$$

$$\Rightarrow x + y = 5$$

Also, product of eigenvalues = |A|

$$\therefore 3(xy - 0) - 5(0 - 0) + 3(0 - 0) = 1 \times 3 \times 4$$

$$\Rightarrow xy = 4$$

Now, $y = 5 - x \Rightarrow (5 - x)x = 4$

$$\Rightarrow 5x - x^2 = 4$$

$$\Rightarrow x^2 - 5x + 4 = 0$$

$$\Rightarrow (x - 4)(x - 1) = 0$$

$$\Rightarrow x = 1, 4$$

$$\therefore y = 4, 1$$

Hence, from the given options

$$x = 4,\ y = 1$$

SOLVED GATE PREVIOUS YEARS' QUESTIONS

1. Given a matrix $[A] = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$, the rank of the matrix is

(a) 4 (b) 3
(c) 2 (d) 1

(GATE 2003, 1 Mark)

Solution: Consider four 3×3 minors because maximum possible rank is 3.

$$\begin{vmatrix} 4 & 2 & 1 \\ 6 & 3 & 4 \\ 2 & 1 & 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} 2 & 1 & 3 \\ 3 & 4 & 7 \\ 1 & 0 & 1 \end{vmatrix} = 0$$