Topic:

Solution of Non-Homogeneous LDE with Constant coefficients.

Learning Outcomes:

To use Method of Variation of Parameters to solve Non-Homogeneous LDE with constant coefficients.

Method of Variation of Parameters:

Let us consider 2nd order Non-homogeneous LDE with constant coefficients as:

$$ay'' + by' + cy = r(x) \qquad (1)$$

Let two solutions of C.F. $(aD^2 + bD + c) = 0$ be y_1 and y_2

i.e. $y_c = c_1 y_1 + c_2 y_2$

Wronskian, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

P.I. $y_p = -y_1 \int \frac{y_2\, r(x)}{W} dx + y_2 \int \frac{y_1\, r(x)}{W} dx$

General solution is: $y = C.F. + P.I.$

i.e. $y = y_c + y_p$

Problem 1. Find the general solution of: $y'' + y = \sec x$

Solution: The given equation is:

$y'' + y = \sec x \qquad (1)$

S.F. : $(D^2 + 1)y = \sec x$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (D^2 + 1)$ and $r(x) = \sec x$

To find Complimentary Function (C.F.):

A.E. : $f(D) = 0 \quad \Rightarrow (D^2 + 1) = 0 \quad \Rightarrow D^2 = -1 \Rightarrow D = i, -i$

$\therefore$ Complimentary function is given by:

$y_c = e^{0x}(c_1 \cos x + c_2 \sin x)$

$\Rightarrow y_c = (c_1 \cos x + c_2 \sin x)$

To find Particular Integral (P.I.):

Comparing y_c with $y_c = c_1\, y_1 + c_2\, y_2$

Here $y_1 = \cos x$, $y_2 = \sin x$

Wronskian, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = (cos^2x + sin^2x) = 1$

P.I. is given by: $y_p = -y_1 \int \frac{y_2\, r(x)}{W} dx + y_2 \int \frac{y_1\, r(x)}{W} dx$

$\Rightarrow y_p = -\cos x \int \frac{\sin x(\sec x)}{1} dx + \sin x \int \frac{\cos x(\sec x)}{1} dx$

$\Rightarrow y_p = -\cos x \int \tan x\, dx + \sin x \int dx$ $\left(sec\ x = \frac{1}{\cos x}\right)$

$\Rightarrow y_p = -\cos x(-\log|\cos x|) + \sin x(x)$

$\therefore$ General solution is given by: y = C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = (c_1 \cos x + c_2 \sin x) + \cos x(\log|\cos x|) + x \sin x$ **Answer.**

Polling Question:

The Wronskian of $y'' + 5y' + 4y = 18e^{2x}$ is:

(A) $3e^{5x}$

(B) -$3e^{-5x}$

(C) $3e^{-5x}$

Problem 2. Find the general solution of: $y'' + y = \operatorname{cosec} x$

Solution: The given equation is:

$y'' + y = \operatorname{cosec} x$ (1)

S.F. : $(D^2 + 1)y = \operatorname{cosec} x$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (D^2 + 1)$ and $r(x) = \operatorname{cosec} x$

To find Complimentary Function (C.F.):

A.E. : $f(D) = 0$ $\Rightarrow (D^2 + 1) = 0$ $\Rightarrow D^2 = -1 \Rightarrow D = i, -i$

$\therefore$ Complimentary function is given by:

$y_c = e^{0x}(c_1 \cos x + c_2 \sin x)$

$\Rightarrow y_c = (c_1 \cos x + c_2 \sin x)$

To find Particular Integral (P.I.):

Comparing y_c with $y_c = c_1\, y_1 + c_2\, y_2$

Here $y_1 = \cos x$, $y_2 = \sin x$

Wronskian, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = (cos^2 x + sin^2 x) = 1$

P.I. is given by: $y_p = -y_1 \int \frac{y_2\, r(x)}{W} dx + y_2 \int \frac{y_1\, r(x)}{W} dx$

$\Rightarrow y_p = -\cos x \int \frac{\sin x(\operatorname{cosec} x)}{1} dx + \sin x \int \frac{\cos x(\operatorname{cosec} x)}{1} dx$

$\Rightarrow y_p = -\cos x \int dx + \sin x \int \cot x\, dx \qquad \left(cosec\ x = \frac{1}{\sin x}\right)$

$\Rightarrow y_p = -\cos x(x) + \sin x\ (\log|\sin x|)$

$\therefore$ General solution is given by: y = C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = (c_1 \cos x + c_2 \sin x) + \sin x(\log|\sin x|) - x \cos x$ **Answer.**

Problem 3. Find the general solution of: $y'' + y = \tan x$

Solution: The given equation is:

$y'' + y = \tan x$ (1)

S.F. : $(D^2 + 1)y = \tan x$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (D^2 + 1)$ and $r(x) = \tan x$

To find Complimentary Function (C.F.):

A.E. : $f(D) = 0$ $\Rightarrow (D^2 + 1) = 0$ $\Rightarrow D^2 = -1 \Rightarrow D = i, -i$

$\therefore$ Complimentary function is given by:

$y_c = e^{0x}(c_1 \cos x + c_2 \sin x)$

$\Rightarrow y_c = (c_1 \cos x + c_2 \sin x)$

To find Particular Integral (P.I.):

Comparing y_c with $y_c = c_1\, y_1 + c_2\, y_2$

Here $y_1 = \cos x$, $y_2 = \sin x$

Wronskian, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = (cos^2x + sin^2x) = 1$

P.I. is given by: $y_p = -y_1 \int \frac{y_2\, r(x)}{W} dx + y_2 \int \frac{y_1\, r(x)}{W} dx$

$\Rightarrow y_p = -\cos x \int \frac{\sin x(\tan x)}{1} dx + \sin x \int \frac{\cos x(\tan x)}{1} dx$

$\Rightarrow y_p = -\cos x \int \frac{sin^2x}{\cos x} dx + \sin x \int \sin x\, dx$ $\qquad \left(tan\, x = \frac{\sin x}{\cos x}\right)$

$\Rightarrow y_p = -\cos x \int \frac{1-cos^2x}{\cos x} dx + \sin x(-\cos x)$

$\Rightarrow y_p = -\cos x \int (\sec x - \cos x) dx - \sin x \cos x$

$\Rightarrow y_p = -\cos x \log|\sec x + \tan x| + \cos x \sin x - \sin x \cos x$

$\therefore$ General solution is given by: y = C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = (c_1 \cos x + c_2 \sin x) - \cos x(\log|\sec x + \tan x|)$ **Answer.**

Problem 4. Find the general solution of: $y'' + y = \cot x$

Solution: The given equation is:

$y'' + y = \cot x$ (1)

S.F. : $(D^2 + 1)y = \cot x$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (D^2 + 1)$ and $r(x) = \cot x$

To find Complimentary Function (C.F.):

A.E. : $f(D) = 0$ $\Rightarrow (D^2 + 1) = 0$ $\Rightarrow D^2 = -1 \Rightarrow D = i, -i$

$\therefore$ Complimentary function is given by:

$y_c = e^{0x}(c_1 \cos x + c_2 \sin x)$

$\Rightarrow y_c = (c_1 \cos x + c_2 \sin x)$

To find Particular Integral (P.I.):

Comparing y_c with $y_c = c_1\, y_1 + c_2\, y_2$

Here $y_1 = \cos x$, $y_2 = \sin x$

Wronskian, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = (cos^2 x + sin^2 x) = 1$

P.I. is given by: $y_p = -y_1 \int \frac{y_2\, r(x)}{W} dx + y_2 \int \frac{y_1\, r(x)}{W} dx$

$\Rightarrow y_p = -\cos x \int \frac{\sin x(\cot x)}{1} dx + \sin x \int \frac{\cos x(\cot x)}{1} dx$

$\Rightarrow y_p = -\cos x \int cos\, x\, dx + \sin x \int \frac{cos^2 x}{\sin x} dx$ $\left(cot\ x = \frac{\cos x}{\sin x}\right)$

$\Rightarrow y_p = -\cos x\,(sin\ x) + \sin x \int \frac{1 - sin^2 x}{\sin x} dx$

$\Rightarrow y_p = -\cos x\,(sin\ x) + \sin x \int (cosecx - \sin x) dx$

$\Rightarrow y_p = -\cos x(\sin x) + \sin x\,(-\log|cosecx + \cot x|) + \sin x(\cos x)$

$\therefore$ General solution is given by: $y =$ C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = (c_1 \cos x + c_2 \sin x) - \sin x(\log|\mathrm{cosec}\, x + \cot x|)$ **Answer.**

<u>Polling Question:</u>

The Wronskian of $y'' + 9y = \cos 3x$ is:

(A) 1

(B) 3

(C) -3

Problem 5. Find the general solution of: $y'' - 2y' - 3y = e^x$

Solution: The given equation is:

$$y'' - 2y' - 3y = e^x \qquad (1)$$

S.F. : $(D^2 - 2D - 3)y = e^x$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (D^2 - 2D - 3)$ and $r(x) = e^x$

To find Complimentary Function (C.F.):

A.E. : $f(D) = 0 \quad \Rightarrow (D^2 - 2D - 3) = 0 \quad \Rightarrow (D-3)(D+1) = 0$

$\Rightarrow D = 3, -1$ (real and distinct roots)

$\therefore$ Complimentary function is given by:

$y_c = c_1 e^{3x} + c_2 e^{-1x}$

To find Particular Integral (P.I.):

Comparing y_c with $y_c = c_1\, y_1 + c_2\, y_2$

Here $y_1 = e^{3x}$, $y_2 = e^{-x}$

Wronskian, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & e^{-x} \\ 3e^{3x} & -e^{-x} \end{vmatrix} = (-e^{3x}e^{-x} - 3e^{3x}e^{-x}) = -4e^{2x}$

P.I. is given by: $y_p = -y_1 \int \frac{y_2\, r(x)}{W} dx + y_2 \int \frac{y_1\, r(x)}{W} dx$

$\Rightarrow y_p = -\, e^{3x} \int \frac{e^{-x}(e^x)}{-4e^{2x}} dx + e^{-x} \int \frac{e^{3x}(e^x)}{-4e^{2x}} dx$

$\Rightarrow y_p = \frac{e^{3x}}{4} \int e^{-2x} dx - \frac{e^{-x}}{4} \int e^{2x}\, dx$

$\Rightarrow y_p = \frac{e^{3x}}{4} \left(\frac{e^{-2x}}{-2}\right) - \frac{e^{-x}}{4} \left(\frac{e^{2x}}{2}\right) = -\frac{1}{4} e^x$

$\therefore$ General solution is given by: y = C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = (c_1 e^{3x} + c_2 e^{-1x}) - \frac{1}{4} e^x$ **Answer.**