Linear Differential equations

A linear ordinary differential equation of order n, is written as:

$$a_0(x)y^n + a_1(x)y^{n-1} + \cdots + a_{n-1}(x)y' + a_n(x)y = r(x)$$

If $r(x) = 0$ then it is called homogeneous equation, otherwise its is called a non-homogeneous equation.

Theorem: if the function $a_0(x), a_1(x), \dots, a_n(x)$ and $r(x)$ are continuous over I and $a_0(x) \neq 0$ on I, then there exist a unique solution to intial value problem

$$a_0(x)y^n + a_1(x)y^{n-1} + \cdots + a_{n-1}(x)y' + a_n(x)y = r(x)$$

$$y(x_0) = c_1, y'(x_0) = c_2, \dots, y^{n-1}(x_0) = c_n$$

Where $x_0 \in I$ and $c_1, c_2, \dots, c_n$ are n known constants.

Example: Find the interval on which the following differential equation

(a) $(1 - x^2)y'' - 2xy' + n(n+1)y = 0$

Auxiliary equation

(B) $x^2y'' + xy' + (n^2 - x^2)y = 0$

$m^2 + Pm + Q = 0.$

(C) $\sqrt{x}\, y'' + 6xy' + 15y = \ln(x^4 - 256)$

METHOD FOR FINDING THE COMPLEMENTARY FUNCTION

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$$

Case I : Roots, Real and Different.

Case II : Roots, Real and Equal.

Case II : Complex roots

Solve: $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0.$

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$$

Solve: $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0,$

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 30y = 0$$

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0$$

$$\frac{d^2y}{dx^2} + \mu^2 y = 0$$

$$(D^2 + 2D + 2)\,y = 0,\ y(0) = 0,\ y'(0) = 1$$

$$\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 8y = 0$$

$$\frac{d^4y}{dx^4} - 32\frac{d^2y}{dx^2} + 256 = 0$$

Examples:

$$y''' - 2y'' - 5y' + 6y = 0$$

$$y''' - y'' - 4y' + 4y = 0$$

$$y^{iv} - 5y'' + 4y = 0$$

$$4y^{iv} - 12y''' - y'' + 27y' - 18y = 0$$

$$y''' - 3y' - 2y = 0$$

$$8y''' - 12y'' + 6y' - y = 0$$

$$y''' + 3y'' - 4y = 0, \qquad y(0) = 1, \qquad y'(0) = 0, \qquad y''(0) = 1/2$$

Linearly Independent and Dependent

(1) 2x, 6x+3, 3x+2

(2) $x^2 - x, 3x^2 + x + 1, 9x^2 - x + 2$

(3) $1, \cos x, \sin x$

(4) $e^x, \sinh x, \cosh x$

(5) $x^2, \frac{1}{x^2}$

(6) $(x-1), (x+1), (x-1)^2$

Euler-Cauchy equation

$$a_0 x^n y^n + a_1 x^{n-1} y^{n-1} + \cdots + a_{n-1} x y' + a_n y = r(x)$$

Examples:

$$x^2y'' + 2xy' - 2y = 0$$

$$2x^2y'' + xy' - 6y = 0$$

$$4x^2y'' + y = 0$$

$$4x^2y'' + 8xy' + 17y = 0$$

Example: It is known that 1/x is a solution of the differential equation $x^2y'' + 4xy' + 2y = 0$. Find the second linearly independent solution and write the general solution.

$$y_2(x) = u(x)y_1(x)$$

$$u(x) = \int v(x)dx$$

$$v(x) = \frac{1}{y^2}\, e^{-\int p(x)dx}$$

$$p(x) = \frac{a_1(x)}{a_0(x)}$$

Examples : Find the solution of the following differential equations, if one of its solutions is known

$$y'' - y' - 6y = 0, \quad y_1 = e^{-2x}$$

$$y'' + 3y' - 4y = 0, \quad y_1 = e^{x}$$

$$(x^2 - 1\,)y'' - 2xy' + 2y = 0, \quad y_1 = x, \quad x \neq \pm 1$$

$$x^2y'' + xy' + (x^2 - 1/4)y = 0, \quad y_1 = x^{-\frac{1}{2}} \sin x$$

$$(x-2)y'' - xy' + 2y = 0, \quad x \neq 2, \quad y_1 = e^{x}$$