Topic:

Solution of Higher order Homogeneous LDE with Constant coefficients

Learning Outcomes:

1. Solution of 3^{rd} order (Cubic) homogeneous LDE with constant coefficients.

2. Solution of 4^{th} order (Biquadratic) homogeneous LDE with constant coefficients.

Find the general solution of the following differential equations:

Problem 1. $y''' - 9y' = 0$

Solution: The given equation is:

$$y''' - 9y' = 0 \tag{1}$$

S.F. : $(D^3 - 9D)y = 0$ where $D \equiv \frac{d}{dx}$

A.E. : $(D^3 - 9D) = 0 \quad \Rightarrow D(D^2 - 9) = 0 \Rightarrow D(D-3)(D+3) = 0$

$\Rightarrow D = 0,3,-3$ (Real and distinct roots)

Let $m_1 = 0, m_2 = 3, m_3 = -3$

$\therefore$ General Solution of equation (1) is given by:

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} \quad \Rightarrow y = c_1 e^{0x} + c_2 e^{3x} + c_3 e^{-3x}$

$\Rightarrow y = c_1 + c_2 e^{3x} + c_3 e^{-3x}$ **Answer.**

Problem 2. $3y''' - 2y'' - 3y' + 2y = 0$

Solution: The given equation is:

$$3y''' - 2y'' - 3y' + 2y = 0 \qquad (1)$$

S.F. : $(3D^3 - 2D^2 - 3D + 2)y = 0$ where $D \equiv \frac{d}{dx}$

A.E. : $(3D^3 - 2D^2 - 3D + 2) = 0 \Rightarrow D^2(3D - 2) - 1(3D - 2) = 0$

$\Rightarrow (3D - 2)(D^2 - 1) = 0 \quad \Rightarrow D = 1, -1, {}^2/_3$ (Real and distinct roots)

Let $m_1 = 1, m_2 = -1, m_3 = {}^2/_3$

$\therefore$ General Solution of equation (1) is given by:

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} \qquad \Rightarrow y = c_1 e^{1x} + c_2 e^{-1x} + c_3 e^{{}^2/_3 x}$

$\Rightarrow y = c_1 e^{x} + c_2 e^{-x} + c_3 e^{{}^2/_3 x}$ **Answer.**

Problem 3. $y''' - 2y'' + y' = 0$

Solution: The given equation is:

$$y''' - 2y'' + y' = 0 \qquad (1)$$

S.F. : $(D^3 - 2D^2 + D)y = 0$ where $D \equiv \frac{d}{dx}$

A.E. :$(D^3 - 2D^2 + D) = 0 \Rightarrow D(D^2 - 2D + 1) = 0$

$\Rightarrow D(D-1)^2 = 0 \quad \Rightarrow D = 0,1,1$ (Two equal and one distinct real roots)

Let $m_1 = 0, m_2 = 1, m_3 = 1$

$\therefore$ General Solution of equation (1) is given by:

$y = c_1 e^{m_1 x} + (c_2 + c_3 x)e^{m_2 x} \qquad \Rightarrow y = c_1 e^{0x} + (c_2 + c_3 x)e^{1x}$

$\Rightarrow y = c_1 + (c_2 + c_3 x)e^{x}$ **Answer.**

<u>Polling Question</u>

The roots of the equation: $y''' - 16y' = 0$ are:

(A) 4,4

(B) 4,-4

(C) 0,4,~~4~~ -4

(D) 0,16

Problem 4. $27y''' - 27y'' + 9y' - y = 0$

Solution: The given equation is:

$$27y''' - 27y'' + 9y' - y = 0 \qquad (1)$$

S.F. : $(27D^3 - 27D^2 + 9D - 1)y = 0$ where $D \equiv \frac{d}{dx}$

A.E. :$(27D^3 - 27D^2 + 9D - 1) = 0$ $\quad [a^3 - b^3 - 3ab(a-b) = (a-b)^3]$

$\Rightarrow (3D-1)^3 = 0 \quad \Rightarrow D = {}^1/_3, {}^1/_3, {}^1/_3$ (All real and equal roots)

Let $m_1 = {}^1/_3, m_2 = {}^1/_3, m_3 = {}^1/_3$

∴ General Solution of equation (1) is given by:

$y = (c_1 + c_2x + c_3x^2)e^{m_1x}$

$\Rightarrow y = (c_1 + c_2x + c_3x^2)e^{{}^1/_3 x}$ **Answer.**

Problem 5. $y''' - 2y'' + 4y' - 8y = 0$

Solution: The given equation is:

$$y''' - 2y'' + 4y' - 8y = 0 \qquad (1)$$

S.F. : $(D^3 - 2D^2 + 4D - 8)y = 0$ where $D \equiv \frac{d}{dx}$

A.E. :$(D^3 - 2D^2 + 4D - 8) = 0 \Rightarrow D^2(D-2) + 4(D-2) = 0$

$\Rightarrow (D-2)(D^2+4) = 0 \quad \Rightarrow D = 2, \pm 2i$ (one real and two complex roots)

Let $m_1 = 2, m_2 = 0 + 2i, m_3 = 0 - 2i$ [Complex roots:$(\alpha \pm i\beta)$]

$\therefore$ General Solution of equation (1) is given by:

$$y = c_1 e^{m_1 x} + e^{\alpha x}(c_2 \cos\beta x \ + c_3 \sin\beta x)$$

$$\Rightarrow y = c_1 e^{2x} + e^{0x}(c_2 \cos 2x \ + c_3 \sin 2x)$$

$$\Rightarrow y = c_1 e^{2x} + \ (c_2 \cos 2x \ + c_3 \sin 2x)$$ **Answer.**

Problem 6. $y^{IV} - 13y'' + 36y = 0$

Solution: The given equation is:

$$y^{IV} - 13y'' + 36y = 0 \qquad (1)$$

S.F. : $(D^4 - 13D^2 + 36)y = 0$ where $D \equiv \frac{d}{dx}$

A.E. :$(D^4 - 13D^2 + 36) = 0$

$\Rightarrow (D^2 - 4)(D^2 - 9) = 0 \quad \Rightarrow D = 2, -2, 3, -3$ (real and distinct roots)

Let $m_1 = 2, m_2 = -2, m_3 = 3, m_4 = -3$

$\therefore$ General Solution of equation (1) is given by:

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + c_4 e^{m_4 x}$$

$\Rightarrow y = c_1 e^{2x} + c_2 e^{-2x} + c_3 e^{3x} + c_4 e^{-3x}$ **Answer.**

Problem 7. $y^{IV} + 8y'' - 9y = 0$

Solution: The given equation is:

$$y^{IV} + 8y'' - 9y = 0 \tag{1}$$

S.F. : $(D^4 + 8D^2 - 9)y = 0$ where $D \equiv \frac{d}{dx}$

A.E. :$(D^4 + 8D^2 - 9) = 0$

$\Rightarrow (D^2 - 1)(D^2 + 9) = 0 \quad \Rightarrow D = 1, -1, 3i, -3i$ (Mix of real and complex roots)

Let $m_1 = 1, m_2 = -1, m_3 = 0 + 3i, m_4 = 0 - 3i$

$\therefore$ General Solution of equation (1) is given by:

$y = c_1 e^{1x} + c_2 e^{-1x} + e^{0x}(c_3 \cos 3x + c_4 \sin 3x)$

$\Rightarrow y = c_1 e^{x} + c_2 e^{-x} + (c_3 \cos 3x + c_4 \sin 3x)$ **Answer.**

Problem 8. $4y^{IV} + 101y'' + 25y = 0$

Solution: The given equation is:

$$4y^{IV} + 101y'' + 25y = 0 \qquad (1)$$

S.F. : $(4D^4 + 101D^2 + 25)y = 0$ where $D \equiv \frac{d}{dx}$

A.E. : $(4D^4 + 101D^2 + 25) = 0$

$\Rightarrow (4D^2 + 1)(D^2 + 25) = 0 \quad \Rightarrow D = \frac{1}{2}i, -\frac{1}{2}i, 5i, -5i$ (Two sets of complex roots)

Let $m_1 = 0 + \frac{1}{2}i, m_2 = 0 - \frac{1}{2}i, m_3 = 0 + 5i, m_4 = 0 - 5i \quad [(\alpha \pm i\beta), (\gamma \pm i\delta)]$

$\therefore$ General Solution of equation (1) is given by:

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) + e^{\gamma x}(c_1 \cos \delta x + c_2 \sin \delta x)$$

$$\Rightarrow y = e^{0x}\left(c_1 \cos \frac{1}{2}x + c_2 \sin \frac{1}{2}x\right) + e^{0x}(c_1 \cos 5x + c_2 \sin 5x)$$

$$\Rightarrow y = \left(c_1 \cos \frac{1}{2}x + c_2 \sin \frac{1}{2}x\right) + (c_1 \cos 5x + c_2 \sin 5x)$$ **Answer.**

Polling Question

The roots of the equation: $y^{iv} - 25y'' + 144y = 0$ are:

(A) 3,3,4,4

(B) 3,-3,4,-4

(C) 0,3,4,6

(D) 5,5,12,12

Problem 9. $y^{IV} + 50y'' + 625y = 0$

Solution: The given equation is:

$$y^{IV} + 50y'' + 625y = 0 \tag{1}$$

S.F. : $(D^4 + 50D^2 + 625)y = 0$ where $D \equiv \frac{d}{dx}$

A.E. :$(D^4 + 50D^2 + 625) = 0 \quad \Rightarrow (D^2 + 25)^2 = 0$

$\Rightarrow (D^2 + 25)(D^2 + 25) = 0 \quad \Rightarrow D = 5i, -5i, 5i, -5i$ (Repeated complex roots)

Let $m_1 = 0 + 5i, m_2 = 0 - 5i, m_3 = 0 + 5i, m_4 = 0 - 5i \quad [(\alpha \pm i\beta), (\alpha \pm i\beta)]$

∴ General Solution of equation (1) is given by:

$y = e^{\alpha x}[(c_1 + c_2 x)\cos\beta x \ + (c_3 + c_4 x)\sin\beta x)]$

$\Rightarrow \ y = e^{0x}[(c_1 + c_2 x)\cos 5x \ + (c_3 + c_4 x)\sin 5x)]$

$\Rightarrow y = [(c_1 + c_2 x)\cos 5x \ + (c_3 + c_4 x)\sin 5x)]$**Answer.**