MTH165

Lecture-42

Fourier Series of Discontinuous functions
And
Parseval's Identity

Unit 6: Fourier Series

(**Book:** Advanced Engineering Mathematics By Jain and Iyengar, **Chapter-9**)

Learning Outcomes:

1. Dirichlet’s conditions for Fourier series

2. Parseval’s identity.

3. Fourier series of discontinuous functions.

Dirichlet's Conditions for Fourier Series Expansion of a function $f(x)$

1. $f(x)$ must be single valued and absolutely integrable over a period.
2. $f(x)$ must have a finite number of extrema in any given interval, i.e. there must be a finite number of maxima and minima in the interval.
3. $f(x)$ must have a finite number of discontinues in any given interval, however the discontinuity cannot be infinite.
4. $f(x)$ must be bounded.

Parseval's Identity

If Fourier series for a function $f(x)$ converges uniformly in $(-l, l)$, then

$$\int_{-l}^{l} [f(x)]^2 dx = l\left[\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right]$$

This result is known as Parseval's identity.

Polling Quiz

In the Fourier series expansion of a periodic function $f(x) = \cos x$ in the interval $[-\pi, \pi]$, the value of b_n is given by:

(A) $b_n = 0$

(B) $b_n = \pi$

(C) $b_n = 2\pi$

(D) $b_n = \frac{2}{\pi}\int_0^\pi x dx$

Fourier Series at a Point of Discontinuity

Let $f(x)$ and $f'(x)$ be piecewise continuous on the interval $[-l, l]$.

Then, Fourier series of $f(x)$ on this interval converges to $f(x)$ at a point of continuity.

At a point of discontinuity, the Fourier series converges to:

$$\frac{1}{2}[f(x+) + f(x-)]$$

where $f(x+)$ and $f(x-)$ are the right and left hand limits respectively.

Problem 1. Find the Fourier series expansion of the function if:

$$f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

Solution. Here$f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$

$$a_0 = \frac{1}{l} \int_{-l}^{l} f(x)dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)dx$$

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi dx + \int_{0}^{\pi} x dx\right]$$

$$= \frac{1}{\pi}\left[-\pi^2 + \frac{\pi^2}{2}\right] = -\frac{\pi}{2}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\left(\frac{n\pi x}{\pi}\right)dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos(nx)\,dx + \int_{0}^{\pi} f(x)\cos(nx)\,dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\cos(nx)\,dx + \int_{0}^{\pi} x\cos(nx)\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left[\frac{\sin(nx)}{n}\right]_{-\pi}^{0} + \left[\frac{x\sin(nx)}{n} + \frac{\cos(nx)}{n^2}\right]_{0}^{\pi}\right]$$

$$= \frac{1}{n^2\pi}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin\left(\frac{n\pi x}{\pi}\right) dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x) \sin(nx)\, dx + \int_{0}^{\pi} f(x) \sin(nx)\, dx \right]$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \sin(nx)\, dx + \int_{0}^{\pi} x \sin(nx)\, dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left[\frac{-\cos(nx)}{n}\right]_{-\pi}^{0} + \left[\frac{-x\cos(nx)}{n} + \frac{\sin(nx)}{n^2}\right]_{0}^{\pi}\right]$$

$$= \frac{1}{n}[1 - 2(-1)^n]$$

The required Fourier series is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \, cos\left(\frac{n\pi x}{l}\right) + b_n \, sin\left(\frac{n\pi x}{l}\right) \right]$$

$$f(x) = \frac{-\pi/2}{2} + \sum_{n=1}^{\infty} \left[\frac{1}{n^2\pi} [(-1)^n - 1] \, cos\left(\frac{n\pi x}{l}\right) + \frac{1}{n} [1 - 2(-1)^n] \, sin\left(\frac{n\pi x}{l}\right) \right]$$

Which is the required Fourier series.

Polling Quiz

The Fourier coefficient a_2 of $f(x) = x$ in the interval $[-2,2]$is:

(A) -2π

(B) π

(C) 1

(D) 0

Problem 2. Find the Fourier series expansion of the function if:

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ \sin x, & 0 < x < \pi \end{cases}$$

Solution. *Try it yourself.*