Topic:

Solution of 2nd order Homogeneous LDE with Constant coefficients

Learning Outcomes:

1. Formulation of 2nd order homogeneous LDE when roots are given.

2. Solution of Initial Value Problems (IVP) and Boundary Value Problems (BVP).

Formulation of LDE of the form: $ay'' + by' + cy = 0$ when Roots are given:

Let the two given roots be: m_1 and m_2.

Then required 2nd order homogeneous LDE is:

$$y'' - (\text{sum of roots})\, y' + (\text{Product of roots})y = 0$$

i.e. $y'' - (m_1 + m_2)y' + (m_1 m_2)y = 0$

or

Then required 2nd order homogeneous LDE is:

$(D - m_1)(D - m_2)y = 0$ where $D \equiv \frac{d}{dx}$

Find a LDE of the form: $ay'' + by' + cy = 0$ for which the following functions are solutions:

Problem 1. (e^{3x}, e^{-2x})

Solution: Comparing with: $(e^{m_1 x}, e^{m_2 x})$

We have: $m_1 = 3, m_2 = -2$

Then required 2nd order homogeneous LDE is:

$y'' - (\text{sum of roots})\, y' + (\text{Product of roots})y = 0$

i.e. $y'' - (m_1 + m_2)y' + (m_1 m_2)y = 0$

$\Rightarrow y'' - (3 - 2)y' + (3)(-2)y = 0$

$\Rightarrow y'' - y' - 6y = 0$ **Answer.**

Problem 2. $(1, e^{-2x})$

Solution: Here $(1, e^{-2x}) = (e^{0x}, e^{-2x})$

Comparing with: $(e^{m_1 x}, e^{m_2 x})$

We have: $m_1 = 0, m_2 = -2$

Then required 2nd order homogeneous LDE is:

$y'' -$ (sum of roots) $y' +$ (Product of roots)$y = 0$

i.e. $y'' - (m_1 + m_2)y' + (m_1 m_2)y = 0$

$\Rightarrow y'' - (0 - 2)y' + (0)(-2)y = 0$

$\Rightarrow y'' + 2y' = 0$ **Answer.**

Problem 3. (e^{2x}, xe^{2x})

Solution: Comparing with: $(e^{m_1 x}, xe^{m_2 x})$

We have: $m_1 = 2, m_2 = 2$

Then required 2nd order homogeneous LDE is:

$y'' -$ (sum of roots) $y' +$ (Product of roots)$y = 0$

i.e. $y'' - (m_1 + m_2)y' + (m_1 m_2)y = 0$

$\Rightarrow y'' - (2 + 2)y' + (2)(2)y = 0$

$\Rightarrow y'' - 4y' + 4y = 0$ **Answer.**

Problem 4. $\left(e^{-3ix}, e^{3ix}\right)$

Solution: Comparing with: $(e^{m_1 x}, e^{m_2 x})$

We have: $m_1 = -3i, m_2 = 3i$

Then required 2nd order homogeneous LDE is:

$y'' -$ (sum of roots) $y' +$ (Product of roots)$y = 0$

i.e. $y'' - (m_1 + m_2)y' + (m_1 m_2)y = 0$

$\Rightarrow y'' - (-3i + 3i)y' + (-3i)(3i)y = 0$

$\Rightarrow y'' + 9y = 0$ **Answer.** $(i^2 = -1)$

Problem 5. $\left(e^{(5+3i)x}, e^{(5-3i)x}\right)$

Solution: Comparing with: $(e^{m_1 x}, e^{m_2 x})$

We have: $m_1 = 5 + 3i, m_2 = 5 - 3i$

Then required 2nd order homogeneous LDE is:

$y'' -$ (sum of roots) $y' +$ (Product of roots)$y = 0$

i.e. $y'' - (m_1 + m_2)y' + (m_1 m_2)y = 0$

$\Rightarrow y'' - [(5 + 3i) + (5 - 3i\ )]y' + (5 + 3i)(5 - 3i)y = 0$

$\Rightarrow y'' + 10y' + 34y = 0$ **Answer.** $(i^2 = -1)$

Polling Question

If (e^{-3x}, e^{2x}) are the roots, then the corresponding LDE is:

(A) $y'' + y' + 6y = 0$

(B) $y'' + y' - 6y = 0$

(C) $y'' - y' - 6y = 0$

Problem: Solve the Initial value problem: $y'' - y = 0, \quad y(0) = 0, y'(0) = 2.$

Solution: The given equation is:

$$y'' - y = 0 \tag{1}$$

Such that: $y(0) = 0, y'(0) = 2$

S.F. : $(D^2 - 1)y = 0$ where $D \equiv \frac{d}{dx}$

A.E. : $(D^2 - 1) = 0 \quad \Rightarrow D^2 = 1 \quad \Rightarrow D = \pm 1$ (real and unequal roots)

Let $m_1 = 1$ and $m_2 = -1$

$\therefore$ General Solution of equation (1) is given by:

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$\Rightarrow y = c_1 e^{1x} + c_2 e^{-1x}$

$$\Rightarrow y(x) = c_1 e^{x} + c_2 e^{-x} \tag{2}$$

$$\Rightarrow y'(x) = c_1 e^{x} - c_2 e^{-x} \tag{3}$$

Using $y(0) = 0$ in equation (2), we get:

$$y(0) = c_1 e^{0} + c_2 e^{-0} \qquad \Rightarrow 0 = c_1 + c_2 \tag{4}$$

Using $y'(0) = 2$ in equation (3), we get:

$$y'(0) = c_1 e^{0} - c_2 e^{-0} \qquad \Rightarrow 2 = c_1 - c_2 \tag{5}$$

Solving equations (4) and (5), we get: $c_1 = 1, c_2 = -1$

Putting these values of c_1 and c_2 in equation (2), we get:

$y(x) = e^{x} - e^{-x}$ **Answer.**

Problem: Solve the Boundary value problem: $y'' - 4y' + 3y = 0$ such that $y(0) = 1, y(1) = 0$.

Solution: The given equation is:

$$y'' - 4y' + 3y = 0 \tag{1}$$

Such that: $y(0) = 1, y(1) = 0$

S.F. : $(D^2 - 4D + 3)y = 0$ where $D \equiv \frac{d}{dx}$

A.E. : $(D^2 - 4D + 3) = 0 \quad \Rightarrow (D-1)(D-3) = 0$

$\Rightarrow D = 1, 3$ (real and unequal roots)

Let $m_1 = 1$ and $m_2 = 3$

$\therefore$ General Solution of equation (1) is given by:

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$\Rightarrow y = c_1 e^{1x} + c_2 e^{3x}$

$$y(x) = c_1 e^{1x} + c_2 e^{3x} \tag{2}$$

Using $y(0) = 1$, we get:

$$y(0) = c_1 e^0 + c_2 e^0 \qquad \Rightarrow 1 = c_1 + c_2 \tag{3}$$

Using $y(1) = 0$, we get:

$$y(1) = c_1 e^1 + c_2 e^3 \qquad \Rightarrow 0 = c_1 e^1 + c_2 e^3 \tag{4}$$

Solving equations (3) and (4), we get: $c_1 = \frac{e^2}{e^2-1}$ and $c_2 = \frac{1}{e^2-1}$

Putting these values of c_1 and c_2 in equation (2), we get:

$y(x) = \frac{e^2}{e^2-1} e^x + \frac{1}{e^2-1} e^{3x}$ **Answer.**

Polling Question

A Linear differential equation with conditions given as:

$y(a) = 0$ and $y(b) = 1$ (say) is called:

(A) Initial value problem

(B) Boundary value problem