Unit 1: Linear Algebra

(Book: Advanced Engineering Mathematics by Jain and Iyengar, Chapter-3)

Topic:

Rank of a Matrix

Learning Outcomes:

- 1. Definition of Rank of a matrix.
- 2. Calculation of rank using determinants.
- 3. Calculation of rank using elementary operations.

Rank of a Matrix:

Definition (Determinant/Minor based) - The rank of a matrix A is the order of the largest non-zero minor of A and is denoted by $\rho(A)$ or r(A).

In other words, a positive integer r is said to be the rank of a non-zero matrix A, if

- (I) There is at least one minor of order r which is not zero.
- (II) Every other minor of matrix A of higher order is zero.

Note- If A is a matrix of order $m \times n$ that is $[A]_{m \times n}$, then rank of A

$$r(A) \le \min(m, n)$$

Find the rank of the matrix
$$\begin{bmatrix}
1 & 2 & -1 & 3 \\
2 & 4 & 1 & -2 \\
3 & 6 & 3 & -7
\end{bmatrix}$$
Consider the third order minors
$$\begin{vmatrix}
1 & 2 & -1 & 3 \\
2 & 4 & 1 & -2 \\
3 & 6 & 3 & -7
\end{vmatrix} = 0, \quad \begin{vmatrix}
1 & -1 & 3 \\
2 & 1 & -2 \\
3 & 3 & -7
\end{vmatrix} = 0$$
Solution:

$$Let A = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{pmatrix}$$

Order of A is 3×4

$$\rho(A) \leq 3$$
.

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & 1 \\ 3 & 6 & 3 \end{bmatrix} = 0, \quad \begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & -2 \\ 3 & 3 & -7 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -2 \\ 3 & 6 & -7 \end{bmatrix} = 0, \quad \begin{bmatrix} 2 & -1 & 3 \\ 4 & 1 & -2 \\ 6 & 3 & -7 \end{bmatrix} = 0$$

Since all third order minors vanishes, $\rho(A) \neq 3$.

Now, let us consider the second order minors,

Now, let us consider the second order minors,

Consider one of the second order minors
$$\begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix} = 6 \neq 0$$

There is a minor of order 2 which is not zero.

$$\therefore \rho(A) = 2.$$

Find the rank of the matrix
$$\begin{pmatrix} 5 & 3 & 0 \\ 1 & 2 & -4 \\ -2 & -4 & 8 \end{pmatrix}$$

Let
$$A = \begin{pmatrix} 5 & 3 & 0 \\ 1 & 2 & -4 \\ -2 & -4 & 8 \end{pmatrix}$$

Order Of A is 3x3

$$\rho(A) \leq 3$$

Consider the third order minor
$$\begin{vmatrix} 5 & 3 & 0 \\ 1 & 2 & -4 \\ -2 & -4 & 8 \end{vmatrix} = 0$$

Since the third order minor vanishes, therefore $\rho(A) \neq 3$

Consider a second order minor
$$\begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = 7 \neq 0$$

There is a minor of order 2, which is not zero.

$$\rho(A) = 2.$$

Find rank of A

$$Let A = \begin{pmatrix} 0 & -1 & 5 \\ 2 & 4 & -6 \\ 1 & 1 & 5 \end{pmatrix}$$

Order Of A is 3x3

$$\therefore \rho(A) \leq 3$$

Consider the third order minor
$$\begin{vmatrix} 0 & -1 & 5 \\ 2 & 4 & -6 \\ 1 & 1 & 5 \end{vmatrix} = 6 \neq 0$$

There is a minor of order 3, which is not zero

$$\therefore \rho(A) = 3.$$

Elementary transformations of a matrix

- (i) Interchange of any two rows (or columns): $R_i \leftrightarrow R_j$ (or $C_i \leftrightarrow C_j$).
- (ii) Multiplication of each element of a row (or column) by any non-zero scalar $k : R_i \rightarrow kR_i$ (or $C_i \rightarrow kC_i$)
- (iii) Addition to the elements of any row (or column) the same scalar multiples of corresponding elements of any other row (or column):

$$R_i \rightarrow R_i + kR_i$$
 (or $C_i \rightarrow C_i + kC_i$)

Equivalent Matrices

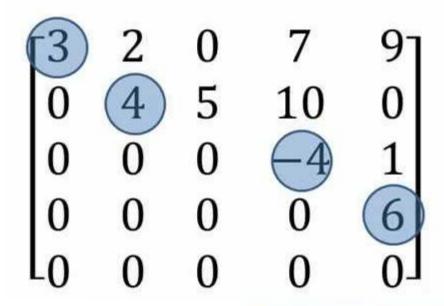
Two matrices A and B are said to be equivalent if one is obtained from the another by applying a finite number of elementary transformations and we write it as $A \sim B$ or $B \sim A$.

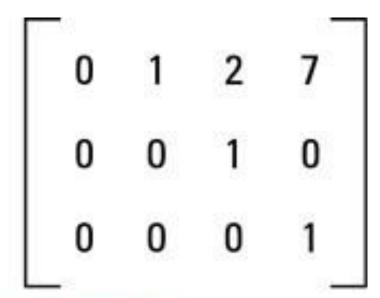
Echelon form of a matrix:

A matrix A of order $m \times n$ is said to be in echelon form if

- (i) Every row of A which has all its entries 0 occurs below every row which has a non-zero entry.
- (ii) The number of zeros before the first non-zero element in a row is less then the number of such zeros in the next row.

Examples of Echelon form:







Rank of a Matrix:

Definition (Echelon form based) – The number of non-zero rows (columns) in the Echelon form of a given matrix.

Note- If A is a matrix of order $m \times n$ that is $[A]_{m \times n}$, then rank of A $r(A) \leq \min(m, n)$

Find the rank of the matrix
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$$

The order of A is 3×3 .

$$\therefore \rho(A) \leq 3.$$

Let us transform the matrix A to an echelon form by using elementary transformations.

Matrix A	Elementary Transformation
$A = \begin{pmatrix} 2 & 4 & 6 & 3 & 6 & 9 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{pmatrix}$	
$A = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$	
3 5 7)	
~ 0 -1 -2	$R_2 \rightarrow R_2 - 2R_1$ $R_3 \rightarrow R_3 - 3R_1$
(0 -1 -2)	$R_3 \rightarrow R_3 - 3R_1$
(1 2 3)	
$ \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix} $	
(0 0 0)	$R_3 \rightarrow R_3 - R_2$
The above matrix is in echelon form	

The number of non zero rows is 2

 \therefore Rank of A is 2.

$$\rho(A) = 2.$$

Note

A row having atleast one non -zero element is called as non-zero row.

Find the rank of the matrix
$$A = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{pmatrix}$$

The order of A is 3×4 .

Let us transform the matrix A to an echelon form

Matrix A	Elementary Transformation
$A = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \\ 3 & 6 & 9 & 6 \\ 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 3 & 1 & 1 & 3 \end{pmatrix}$ $A = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 3 & 1 & 1 & 3 \end{pmatrix}$	
A = 1 2 3 2	
3 1 1 3	
(1 2 3 2)	
A = 0 1 2 1	$R_1 \leftrightarrow R_2$
3 1 1 3	
$ \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 5 & 10 & 5 \\ 0 & 1 & 2 & 1 \\ 0 & -5 & -8 & -3 \end{bmatrix} $	$R_3 \rightarrow R_3 - 3R_1$
$ \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix} $	$R_3 \rightarrow R_3 + 5R_2$

The number of non zero rows is 3. $\therefore \rho(A) = 3$.

Find the rank of the matrix
$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 4 & 5 & 2 \\ 2 & 3 & 4 & 0 \end{pmatrix}$$

The order of A is 3×4 .

$$\rho(A) \leq 3.$$

Let us transform the matrix A to an echelon form

Matrix A	Elementary Transformation
$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 4 & 5 & 2 \\ 2 & 3 & 4 & 0 \end{pmatrix}$ $\sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & -2 \end{pmatrix}$	$R_2 \rightarrow R_2 - 3R_1$ $R_3 \rightarrow R_3 - 2R_1$
$ \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} $	$R_3 \rightarrow R_3 - R_2$

The number of non zero rows is 3.

$$\rho(A) = 3.$$

