Solution of Non-homogeneous Differential equations

A linear ordinary differential equation of order n, is written as:

$$a_0y^n + a_1y^{n-1} + \cdots + a_{n-1}y' + a_ny = r(x)$$

$$(a_0D^n + a_1D^{n-1} + \cdots + a_{n-1}D' + a_n)y = r(x)$$

$$F(D)y = r(x)$$

$$y_p = \frac{1}{F(D)}r(x)$$

Case 1: $r(x) = e^{ax},\ For\ P.I.\ Put\ D = a$

1. $y'' - 2y' - 3y = 3e^{2x}$

2. $y''' - 2y'' - 5y' + 6y = 4e^{-x} - e^{2x}$

3. $y'' + y' - 6y = 5e^{-3x}$

$y'' - 4y' + y = e^{\frac{x}{2}}$

Case 2: $r(x) = \cos ax \quad or \sin ax\ \ For\ P.I.\ Put\ D^2 = -a^2$

$$y'' + 4y = 6\cos x$$

$$2y'' + y' - y = 16\cos 2x$$

$$y'' - 5y' + 4y = 65\sin 2x$$

$$y''' - y'' + 4y' - 4y = \sin 3x$$

$$y'' + y = 6\sin x$$

Case 3: $r(x) = x^n$

$$y'' + 5y' + 4y = 3x + 2$$

$y'' + 16y = 64x^2$
$y^{iv} + 3y'' = x^2$

Case 4: $r(x) = e^{ax}h(x)$

$y'' + 2y' + 5y = e^{-x}\cos 2x$

$y'' - y = 6xe^{x}$

$2y'' + 7y' - 4y = xe^{-4x}$

Case 5: $r(x) = x^n \sin ax \; or \; x^n \cos ax$

$y' + y = x\cos x$

$y'' + 4y' + 3y = x\sin 2x$

$y'' + 3y' + 2y = x\,e^{x}\sin x$

Solution of Euler-Cauchy equations

$$2x^2y'' + 3xy' - 3y = x^3$$

$$x^2y'' + 5xy' + 3y = \ln x$$
$$x^3y'' + 5x^2\,y'' + 5xy' + y = x^2 + \ln x$$
$$x^2y'' + 5xy' - 5y = 24\,x\ln x$$

$$4x^2y'' + y = \log x, \qquad y(1) = 0, y(e) = 5$$

$$x^2y'' - xy' + 2y = 6, y(1) = 1, y'(1) = 2$$
$$2x^2y'' + 3xy' - y = x, y(1) = 1, y(4) = \frac{41}{16}$$
$$x^2y'' - 3xy' + 3y = 5x^2 - x, y(1) = 1, y'(1) = \frac{3}{2}$$

Method of Undetermined Coefficients:

………………r(x)………………	…………Initial Guess for Y_p ……..

P(x)	Q(x)
$c\ e^{ax}$	de^{ax}
$\alpha \cos(bx)\ \ or\ \beta \sin(bx)$	$A \cos(bx)\ \ or\ B \sin(bx)$
$P(x)\ \cos(bx)\ \ or\ P(x) \sin(bx)$	$Q(x)\ \cos(bx)\ \ or R(x) \sin(bx)$
$P(x)\ \mathrm{e}^{\mathrm{ax}}\ \cos(bx)\ \ or\ P(x)\ \mathrm{e}^{\mathrm{ax}}$ si	$Q(x)\ \mathrm{e}^{\mathrm{ax}}\ \cos(bx)\ \ or\ R(x)\, \mathrm{e}^{\mathrm{ax}} \sin(bx$

Note: whenever the term(or terms) of C.F. and P.I. is (are) same in that case we have to multiply the P.I.'s term with x.

$$y'' - 3y' - 10y = x$$

$$2y'' - y' - 3y = x^2 + 1$$

$$4y'' - y = e^x$$

$$4y'' - y = e^x + e^{3x}$$

$$y'' - y = e^x$$

$$y'' + y = \sin 2x$$
$$y'' + y = \sin x$$
$$y'' + y = e^x \sin x$$

Method of variation of Parameters

Consider the second order non-homogeneous linear equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = r(x), \qquad a_0(x) \neq 0$$

C.F= $y = C_1 y_1(x) + C_2 y_2(x)$

$$y(x) = A(x)y_1(x) + B(x)y_2(x)$$

$$A(x) = -\int \frac{g(x)y_2(x)}{W(x)}dx + c_1$$

$$B(x) = \int \frac{g(x)y_1(x)}{W(x)}dx + c_2$$

$$g(x) = \frac{r(x)}{a_0(x)}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix}$$

$$y'' + 3y' + 2y = 2e^x$$

$$y'' + 16y = 32\sec 2x$$

$$x^2y'' + xy' - y = x$$

$$y'' + 4y' + 4y = e^{-2x}\sin x$$

$$y'' + 2y' + 2y = e^{-x}\cos x$$

$$y'' + 6y' + 9y = \frac{e^{-3x}}{x}$$

$$x^2y'' + xy' - 4y = x^2\ln x, \qquad y_1 = x^2, \qquad y_2 = \frac{1}{x^2}$$

Solution of systems of equations

$$y_1' = 2y_1 + y_2, \qquad y_2' = y_1 + 2y_2$$

$$y_1' = y_1 + y_2, \qquad y_2' = 9y_1 + y_2$$

$$y_1' = y_2, \qquad y_2' = -9y_1$$

$$y_1' = 2y_1 + y_2, \qquad y_2' = -18y_1 - 7y_2$$

$$y_1' + y_2 = 4\sin t, \qquad y_2' + y_1 = 8\cos t$$

$$y_1' + y_1 + 3y_2 = 4e^{-t}, \qquad y_2' + 4y_1 - 3y_2 = 8t$$

$$y_1' + 3y_1 + y_2 = 6e^{t}, \qquad y_2' - 5y_1 - 3y_2 = 3e^{-t}$$

$$y_1' + 4y_1 - 5y_2 = 16\sin t, \qquad y_2' + 5y_1 - 4y_2 = e^{t}$$

$$y_1' + y_1 - 3y_2 = 6e^{-t}, \qquad y_2' + 2y_1 - 14y_2 = 12e^{t}$$

$$(D-2)y_1 + (D+3)y_2 = t+4,$$
$$(3D+5)y_1 + (2D+6)y_2 = 5t+3$$