Linear Differential equations

A linear ordinary differential equation of order n, is written as:

$$a_0(x)y^n + a_1(x)y^{n-1} + \dots + a_{n-1}(x)y' + a_n(x)y = r(x)$$

If r(x) = 0 then it is called homogeneous equation, otherwise its is called a non-homogeneous equation.

Theorem: if the function $a_0(x), a_1(x), ..., a_n(x)$ and r(x) are continuous over I and $a_0(x) \neq 0$ on I, then there exist a unique solution to intial value problem

$$a_0(x)y^n + a_1(x)y^{n-1} + \dots + a_{n-1}(x)y' + a_n(x)y = r(x)$$

$$y(x_0) = c_1, y'(x_0) = c_2, ..., y^{n-1}(x_0) = c_n$$

Where $x_0 \in I$ and $c_1, c_2, ..., c_n$ are n known constants.

Example: Find the interval on which the following differential equation

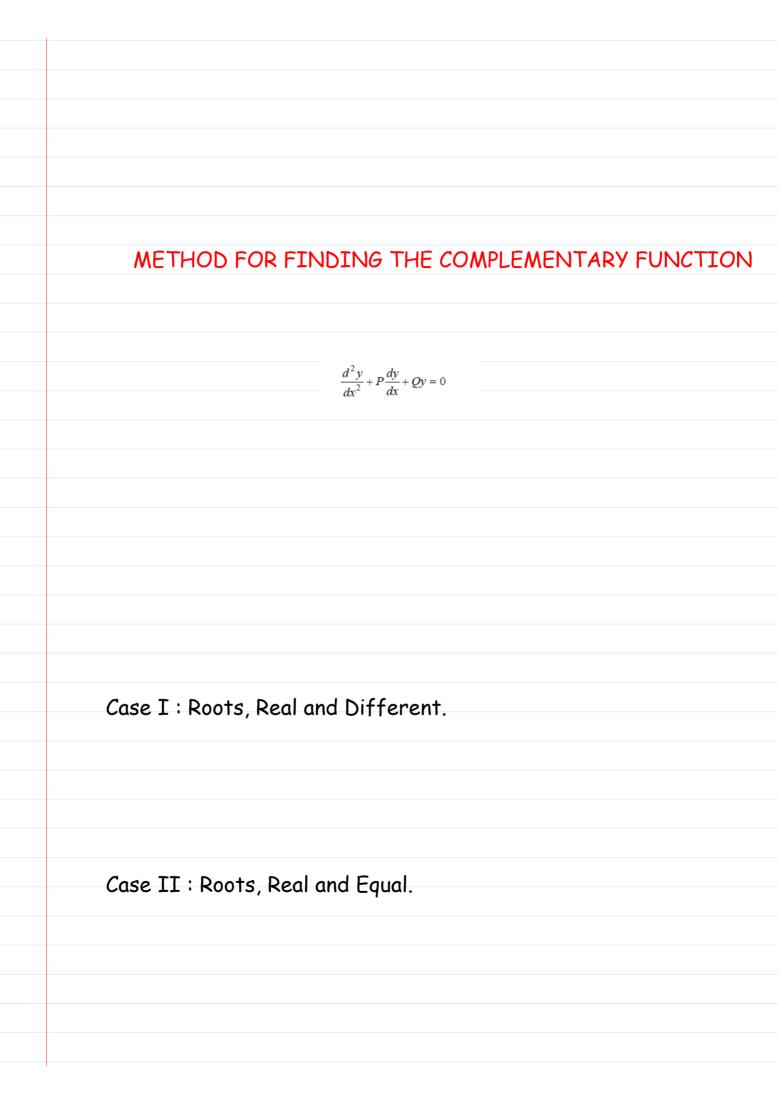
(a)
$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

Auxiliary equation

(B)
$$x^2y'' + xy' + (n^2 - x^2)y = 0$$

$$m^2 + Pm + Q = 0.$$

(C)
$$\sqrt{x}$$
 y''+6xy'+15y=ln($x^4 - 256$)

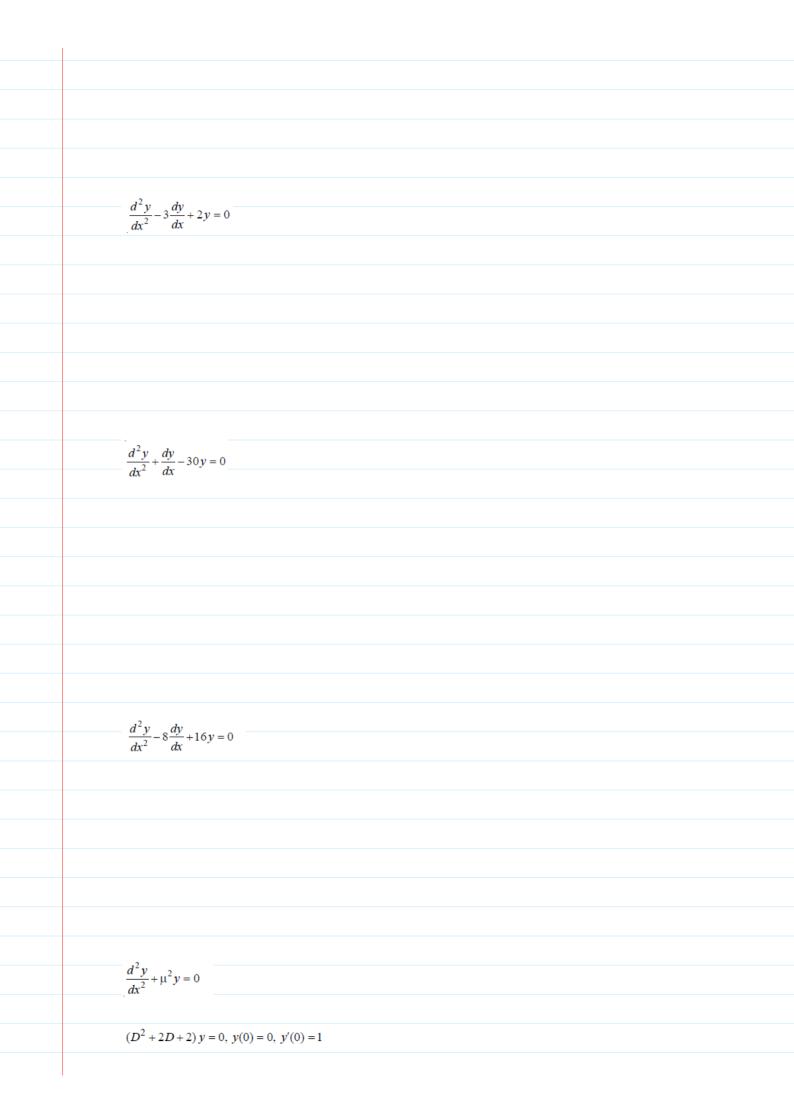




Solve:
$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0.$$

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$$

Solve:
$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0$$
,



$$\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 8y = 0$$

$$\frac{d^4y}{dx^4} - 32\frac{d^2y}{dx^2} + 256 = 0$$

Examples:

$$y''' - 2y'' - 5y' + 6y = 0$$

$$y''' - y'' - 4y' + 4y = 0$$

$$y^{iv} - 5y^{\prime\prime} + 4y = 0$$

$$4y^{iv} - 12y''' - y'' + 27y' - 18y = 0$$

$$y^{\prime\prime\prime} - 3y^{\prime} - 2y = 0$$

$$8y''' - 12y'' + 6y' - y = 0$$

$$y''' + 3y'' - 4y = 0$$
, $y(0) = 1$, $y'(0) = 0$, $y''(0) = 1/2$

Linearly Independent and Dependent

$$(1)$$
 2x, 6x+3, 3x+2

(2)
$$x^2 - x$$
, $3x^2 + x + 1$, $9x^2 - x + 2$

- (3) $1, \cos x, \sin x$
- (4) e^x , $\sinh x$, $\cosh x$
- (5) $x^2, \frac{1}{x^2}$

(6)
$$(x-1)$$
, $(x+1)$, $(x-1)^2$

Euler-Cauchy equation

$$a_0 x^n y^n + a_1 x^{n-1} y^{n-1} + \dots + a_{n-1} x y' + a_n y = r(x)$$

Examples:

$$x^2y'' + 2xy' - 2y = 0$$

$$2x^2y'' + xy' - 6y = 0$$

$$4x^2y^{\prime\prime} + y = 0$$

$$4x^2y'' + 8xy' + 17y = 0$$

Example: It is known that 1/x is a solution of the differential equation $x^2y'' + 4xy' + 2y = 0$. Find the second linearly independent solution and write the general solution.

$$y_2(x) = u(x)y_1(x)$$

$$u(x) = \int v(x) dx$$

$$v(x) = \frac{1}{y^2} e^{-\int p(x)dx}$$

$$p(x) = \frac{a_1(x)}{a_0(x)}$$

Examples: Find the solution of the following differential equations, if one of its solutions is known

$$y'' - y' - 6y = 0$$
, $y_1 = e^{-2x}$

$$y'' + 3y' - 4y = 0, y_1 = e^x$$

$$(x^2 - 1)y'' - 2xy' + 2y = 0,$$
 $y_1 = x,$ $x \neq \pm 1$

$$x^2y'' + xy' + (x^2 - 1/4)y = 0,$$
 $y_1 = x^{-\frac{1}{2}}\sin x$

$$(x-2)y'' - xy' + 2y = 0,$$
 $x \ne 2,$ $y_1 = e^x$

