Exercise 5.1

In the following linear differential equations, find the constant coefficient and variable coefficient equations.

1. $y'' - a^2y = 0.$
2. $y' = y/x.$
3. $y''' + 3y'' + 6y' + 12y = x^2.$
4. $x^3y''' + 9x^2y'' + 18xy' + 6y = 0.$
5. $(1 - x)y'' + xy' - y = 0.$
6. $y'' - (1 + x^2)y = 0.$

Find the intervals on which the following differential equations are normal.

7. $y' = 3y/x.$
8. $(1 + x^2)y'' + 2xy' + y = 0.$
9. $x^2y'' - 4xy' + 6y = x.$
10. $y'' + 3y' + \sqrt{x}\, y = \sin x.$
11. $y''' + 9y' + y = \log(x^2 - 9).$
12. $y'' + |x|y' + y = x \ln x.$

13. $x(1-x)y'' - 3xy' - y = 0$.

14. $y'' + xy' + 6y = \ln \sin(\pi x/4)$.

15. Verify that $y = x^2$ is a solution of $x^2y'' + xy' - 4y = 0$, $x \in (0, \infty)$ and satisfies the conditions $y(0) = 0$, $y'(0) = 0$. Does Theorem 5.1 guarantee the existence and uniqueness of such a solution? Is the Remark 1 applicable in this case?

16. By inspection find a solution of $x^2y'' + xy' - y = 0$, $x \in (-\infty, \infty)$ which satisfies the conditions $y(0) = 0$, $y'(0) = 2$. Does Theorem 5.1 guarantee the existence and uniqueness of such a solution?

17. Show that

$$y_1(x) = x^3 - x^2, \; -3 \le x \le 3, \quad \text{and} \quad y_2(x) = \begin{cases} x^2 - x^3, & -3 \le x \le 0, \\ x^3 - x^2, & 0 \le x \le 3 \end{cases}$$

both satisfy the differential equation $x^2y'' - 4xy' + 6y = 0$ and the conditions $y(2) = 4$, $y'(2) = 8$. But $y_1(x)$, $y_2(x)$ are different. Does this contradict Theorem 5.1?

Verify that the given functions are solutions of the associated differential equation. Verify also that a linear combination of these functions is also a solution.

18. $1, x, e^x$; $y''' - y'' = 0$.

19. e^x, e^{-2x}; $y'' + y' - 2y = 0$.

20. $e^{-x}\cos 2x, e^{-x}\sin 2x$; $y'' + 2y' + 5y = 0$.

Examine whether the following functions are linearly independent for $x \in (0, \infty)$.

21. $2x, 6x + 3, 3x + 2$.

22. $x^2 - x, 3x^2 + x + 1, 9x^2 - x + 2$.

23. $x^2 - 2x, 3x^2 + x + 2, 4x^2 - x + 1$.

24. $\sin x, \sin 2x, \sin 3x$.

25. $1, \cos x, \sin x$.

26. $e^x, \sinh x, \cosh x$.

27. $x^2, 1/x^2$.

28. $\ln x, \ln x^2, \ln x^3$.

29. $x - 1, x + 1, (x-1)^2$.

30. $e^{-x}, \sinh x, \cosh x$.

31. Find the intervals on which the three functions $1, \cos x, \sec x$, $x > 0$ are linearly independent.

32. Determine how many of the given functions are linearly independent on $[0, 1]$.

(i) $1, 1 + x, x^2, x(1-x), x$; (ii) $1 + x, 1 - x, 1, x^2, 1 + x^2$.

33. Show that $y_1(x) = \sin x$, and $y_2(x) = 4\sin x - 2\cos x$ are linearly independent solutions of $y'' + y = 0$. Write the solution $y_3(x) = \cos x$ as a linear combination of y_1 and y_2.

34. Let $a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$ be a second order differential equation. Let $a_0(x)$, $a_1(x)$, $a_2(x)$ be continuous and $a_0(x) \ne 0$ on I and $y_1(x)$, $y_2(x)$ be two linearly independent solutions. Show that the Wronskian of $y_1(x)$, $y_2(x)$ satisfies the differential equation $a_0(x)W'(x) + a_1(x)W(x) = 0$. Also, show that the Wronskian is given by

$$W(x) = c\, e^{-\int [a_1(x)/a_0(x)]dx}.$$

(This is called the *Abel's formula*).

35. Show that $\cos at$, $\sin at$ are solutions of the equation $y'' + a^2y = 0$, $a \ne 0$ on any interval. Show that they are independent. Use the result (Abel's formula) given in Problem 34 and find the Wronskian. Are the two Wronskians same?

36. Show that e^{2x} and xe^{2x} are solutions of the equation $y'' - 4y' + 4y = 0$ on any interval. Show that they are independent. Use the result given in problem 34 and find the Wronskian. Are the two Wronskians same?

Show that in the following problems, $\{y_i(x)\}$ forms a set of fundamental solutions (basis) to the corresponding differential equation.

37. $x^{1/4}, x^{5/4}$; $16x^2y'' - 8xy' + 5y = 0$, $x > 0$.

$y'' , 2y - y$

$-2 \pm \sqrt{4+4}$

$-2 \pm 2\sqrt{8}$

38. $e^{2x}\cos 3x$, $e^{2x}\sin 3x$; $2y'' - 8y' + 26y = 0$.
39. $1, x^2$; $x^2y'' - xy' = 0$, $x > 0$.
40. e^x, e^{2x}, e^{-3x}; $y''' - 7y' + 6y = 0$.
41. $e^x, e^x\cos x$, $e^x\sin x$; $y''' - 3y'' + 4y' - 2y = 0$.
42. $e^{2x}, e^{-x}\cos(\sqrt{3}x)$, $e^{-x}\sin(\sqrt{3}x)$; $y''' - 8y = 0$.
43. $\sin(\ln x^2)$, $\cos(\ln x^2)$; $x^2y'' + xy' + 4y = 0$, $x > 0$.
44. Let the coefficients $a_0(x)$, $a_1(x)$, $a_2(x)$ in the equation $a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$ be continuous and $a_0(x) \neq 0$ on I. Let $\{y_1(x), y_2(x)\}$ be the basis (set of fundamental solutions) of the equation. Show that the set $\{u(x), v(x)\}$ such that $u = ay_1(x) + by_2(x)$, $v = cy_1(x) + dy_2(x)$, is also a basis of the equation if $ad - bc \neq 0$. If $y_1(x) = \cosh kx$, $y_2 = \sinh kx$, obtain a simple form of u and v.
45. Let $y_1(x)$, $y_2(x)$ be the linearly independent solutions of the equation $y'' + a(x)y' + b(x)y = 0$ on I. Show that there is no point $x_0 \in I$ at which (i) both $y_1(x)$, $y_2(x)$ vanish, (ii) both $y_1(x)$, $y_2(x)$ take extreme values.
46. Let $\{y_1(x), y_2(x)\}$ be the basis of the equation $y'' + a(x)y' + b(x)y = 0$. Show that the equation can be written as the Wronskian $W(y, y_1, y_2) = 0$.
47. Let $y_1(x)$ be a solution of the homogeneous equation $y'' + a(x)y' + b(x)y = 0$, on the interval $I : \alpha \leq x \leq \beta$. The coefficients $a(x)$ and $b(x)$ are continuous on I. If the curve $y = y_1(x)$ is tangential to the x-axis at a point x_1 in I, then prove that $y_1(x) \equiv 0$.

...ing the problem 46, find a differential equation of the form $y'' + a(x)y' + b(x)y = 0$ for which the following ...ctions are solutions.

48. e^{3x}, e^{-2x}.
49. $e^{-(\alpha + i\omega)x}, e^{-(\alpha - i\omega)x}$.
50. e^{5x}, xe^{5x}.

3 Methods fo...

The general solution is $y(x) = Ay_1(x) + By_2(x) = \frac{A}{x} + \frac{B}{x^2}$.

Exercise 5.2

Show that the given set of functions $\{y_1(x), y_2(x)\}$ forms a basis of the equation and hence solve the ini[tial value]
problem.

1. e^x, e^{4x}, $y'' - 5y' + 4y = 0$, $y(0) = 2$, $y'(0) = 1$.
2. e^{2x}, e^{-2x}, $y'' - 4y = 0$, $y(0) = 1$, $y'(0) = 4$.
3. e^{-3x}, xe^{-3x}, $y'' + 6y' + 9y = 0$, $y(0) = 1$, $y'(0) = 2$.
4. $x^2, 1/x^2$, $x^2y'' + xy' - 4y = 0$, $y(1) = 2$, $y'(1) = 6$.
5. $x, x\ln x$, $x^2y'' - xy' + y = 0$, $y(1) = 3$, $y'(1) = 4$.

Find a general solution of the following differential equations.

6. $y'' - 4y = 0$.
7. $y'' - y' - 2y = 0$.
8. $y'' + y' - 2y = 0$.
9. $y'' - 4y' - 12y = 0$.

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10. $y'' + 4y' + y = 0$.

11. $4y'' - 9y' + 2y = 0$.

12. $4y'' + 8y' - 5y = 0$.

13. $y'' + 2y' + y = 0$.

14. $y'' + 2\pi y' + \pi^2 y = 0$.

15. $9y'' - 12y' + 4y = 0$.

16. $4y'' + 4y' + y = 0$.

17. $25y'' - 20y' + 4y = 0$.

18. $y'' + 25y = 0$.

19. $y'' + 4y' + 5y = 0$.

20. $y'' - 2y' + 2y = 0$.

21. $(4D^2 - 4D + 17)y = 0$.

22. $(D^2 - 6D + 18)y = 0$.

23. $(D^2 + 9D)y = 0$.

24. $[D^2 - 2aD + (a^2 + b^2)]y = 0$.

Find a differential equation of the form $ay'' + by' + cy = 0$, for which the following functions are solutions

25. e^{3x}, e^{-2x}.

26. $e^{x/4}, e^{-(3x)/4}$.

27. $1, e^{-2x}$.

28. e^{2x}, xe^{2x}.

29. e^{-x}, xe^{-x}.

30. e^{-3ix}, e^{3ix}.

31. $e^{-(a+ib)x}, e^{-(a-ib)x}$.

32. $e^{(5+3i)x}, e^{(5-3i)x}$.

Solve the following initial value problems.

33. $y'' - y = 0,\ y(0) = 0,\ y'(0) = 2$.

34. $y'' - y' - 12y = 0,\ y(0) = 4,\ y'(0) = -5$.

35. $y'' + y' - 2y = 0,\ y(0) = 0,\ y'(0) = 3$.

36. $\dfrac{d^2\theta}{dt^2} + g\theta = 0$, g constant, $\theta(0) = a$, constant, $\dfrac{d\theta}{dt}(0) = 0$.

37. $y'' - 4y' + 5y = 0,\ y(0) = 2,\ y'(0) = -1$.

38. $25y'' - 10y' + 2y = 0,\ y(0) = 1,\ y'(0) = 0$.

39. $4y'' + 12y' + 9y = 0,\ y(0) = -1,\ y'(0) = 2$.

40. $9y'' + 6y' + y = 0,\ y(0) = 0,\ y'(0) = 1$.

Solve the following boundary value problems.

41. $y'' + 25y = 0,\ y(0) = 1,\ y(\pi) = -1$.

42. $y'' - 36y = 0,\ y(0) = 2,\ y(1/6) = 1/e$.

43. $y'' + 2y' + 2y = 0,\ y(0) = 1,\ y(\pi/2) = e^{-\pi/2}$.

44. $9y'' - 6y' + y = 0,\ y(1) = e^{1/3},\ y(2) = 1$.

45. $y'' - 4y' + 3y = 0,\ y(0) = 1,\ y(1) = 0$.

46. Verify that $(D - 2)(D + 3) \sin x = (D + 3)(D - 2) \sin x = (D^2 + D - 6) \sin x$.

47. Show that $x^2Dy \neq D(x^2y)$.

48. Find the conditions under which the following equations hold.

(i) $(D + a)[D + b(x)]f(x) = [D + b(x)][D + a]f(x)$, a constant.

(ii) $[D + a(x)][D + b(x)]f(x) = [D + b(x)][D + a(x)]f(x)$.

Factorize the operator and find the solution of the following differential equations using the method of reduction of order or by the direct method.

49. $(D^2 + 5D + 4)y = 0$.

50. $(4D^2 + 8D + 3)y = 0$.

51. $(4D^2 + 12D + 9)y = 0.$

52. $(D^2 + 6D + 9)y = 0.$

53. $(D^2 - 4)y = 0.$

54. $(9D^2 + 6D + 1)y = 0.$

55. The displacement $x(t)$ of a particle is governed by the differential equation $\ddot{x} + \dot{x} + bx = c\dot{x}$, $b > 0$. For what values of b and c is the motion of the particle oscillatory?

56. Find all non-trivial solutions of the boundary value problem

$$y'' + \omega^2 y = 0,\ y(0) = 0,\ y(\pi) = 0.$$

57. Find all the non-trivial solutions of the boundary value problem

$$y'' + \omega^2 y = 0,\ y'(0) = 0,\ y'(\pi) = 0.$$

58. Find all non-trivial solutions of the boundary value problem

$$y'' + \omega^2 y = 0,\ y(0) = 0,\ y'(\pi) = 0.$$

59. If $a^2 > 4b$, then show that the solution of the differential equation $y'' + ay' + by = 0$ can be expressed as $y(x) = e^{px}(A \cosh qx + B \sinh qx)$ where $p = -a/2$ and $q = \sqrt{a^2 - 4b}/2$.

60. The motion of a damped mechanical system is governed by the linear differential equation $m\ddot{y} + c\dot{y} + ky = 0$ in which m (mass), k (spring modulus), c (damping factor) are positive constants and dot denotes derivative with respect to time t. Discuss the behaviour of the general solution when $t \to \infty$ in the following three cases: (i) $c^2 > 4mk$ (over damping), (ii) $c^2 < 4mk$ (under damping), (iii) $c^2 = 4mk$ (critical damping). In each case, obtain the solution subject to the initial conditions $y(0) = 0$, $\dot{y}(0) = v_0$.

Find the solution of the following differential equations, if one of its solutions is known.

61. $y'' - y' - 6y = 0,\ y_1 = e^{-2x}.$

62. $y'' + 3y' - 4y = 0,\ y_1 = e^x.$

63. $(x^2 - 1)y'' - 2xy' + 2y = 0,\ y_1 = x,\ x \neq \pm 1.$

64. $x^2y'' + xy' + (x^2 - 1/4)y = 0,\ x > 0,\ y_1 = x^{-1/2} \sin x.$

65. $(x - 2)y'' - xy' + 2y = 0,\ x \neq 2,\ y_1 = e^x.$

5.3.4 Solution of Higher Order Homogeneous Linear Equations with Constant Coefficients

In this ...