<u>Unit 1:</u> Linear Algebra

(**Book:** Advanced Engineering Mathematics by Jain and Iyengar, **Chapter-3**)

<u>Topic:</u>

Solution of Linear System of Equations-Homogeneous and Non-Homogeneous

<u>Learning Outcomes:</u>

Solution of Linear System of Equations using Gauss-Elimination method (Rank method).

Solution of Non-Homogeneous System of Equations:

(Gauss Elimination Method-Rank Method)

Let us consider the following system of equations:

$$a_1x + b_1y + c_1z = d_1 \qquad (1)$$

$$a_2x + b_2y + c_2z = d_2 \qquad (2)$$

$$a_3x + b_3y + c_3z = d_3 \qquad (3)$$

The given system can be written as:

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$ that is $AX = B$

Where A $= \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, X $= \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, B $= \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Let us consider the Augmented matrix: $[A \quad |B]$

$$[A \quad |B] = \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$

Case 1. If $r(A) = r(A|B) = n$ (Number of unknowns),

then given system of equations is said to be *consistent* and has a *unique solution*

Case 2. If $r(A) = r(A|B) < n$ (Number of unknowns),

still the given system of equations is said to be *consistent* and has *infinitely many solutions*.

Case 3. If $r(A) \neq r(A|B)$,

then the given system of equations is said to be *inconsistent* and has *no solution*.

Quiz-Time

The given system of equations is said to be

consistent

A. If $r(A) = r(A|B)$

B. If $r(A) \neq r(A|B)$

Problem 1. Show that the following system of equations is consistent:

$$2x - 3y + z = -2 \qquad (1)$$

$$x - y + 2z = 3 \qquad (2)$$

$$2x + y - 3z = -2 \qquad (3)$$

Solution. The given system can be written as:

$$\begin{bmatrix} 2 & -3 & 1 \\ 1 & -1 & 2 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix} \text{ that is } AX = B$$

Where $\mathrm{A} = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -1 & 2 \\ 2 & 1 & -3 \end{bmatrix}$, $\mathrm{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $\mathrm{B} = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}$

Let us consider the Augmented matrix: $[A \quad |B]$

$$[A \quad |B] = \left[\begin{array}{ccc|c} 2 & -3 & 1 & -2 \\ 1 & -1 & 2 & 3 \\ 2 & 1 & -3 & -2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 2 & -3 & 1 & -2 \\ 2 & 1 & -3 & -2 \end{array}\right] R_1 \leftrightarrow R_2$$

$$[A \quad |B] \sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & -1 & -3 & -8 \\ 0 & 3 & -7 & -8 \end{array}\right] \begin{array}{l} R_2 - 2R_1 \\ R_3 - 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & -1 & -3 & -8 \\ 0 & 0 & -16 & -32 \end{array}\right] R_3 + 3R_2$$

Here $r(A) = r(A|B) = 3 =$ (Number of unknowns)

Quiz-Time

If $r(A) = r(A|B) = 3 =$ (Number of unknowns)

A). *Unique solution*

B). Infinitely many solutions

C). *No solution*

D) I don't Know

So, the given system of equations is *consistent* and has a *unique solution*

From the 3rd row of Augmented matrix:

$$0x + 0y - 16z = -32 \Longrightarrow z = 2$$

From the 2nd row of Augmented matrix:

$$0x - 1y - 3z = -8 \Longrightarrow y = 8 - 3z \Longrightarrow y = 2$$

From the 1st row of Augmented matrix:

$$1x - 1y + 2z = 3 \Longrightarrow x = 3 + y - 2z \Longrightarrow x = 1$$

So, $(x, y, z) = (1,2,2)$ **Answer.**

Problem 2. Show that the following system of equations is inconsistent:

$x - 4y + 7z = 8$ (1)

$3x + 8y - 2z = 6$ (2)

$7x - 8y + 26z = 3$ (3)

Solution. The given system can be written as:

$$\begin{bmatrix} 1 & -4 & 7 \\ 3 & 8 & -2 \\ 7 & -8 & 26 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ 3 \end{bmatrix} \text{ that is } AX = B$$

Where $A = \begin{bmatrix} 1 & -4 & 7 \\ 3 & 8 & -2 \\ 7 & -8 & 26 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 8 \\ 6 \\ 3 \end{bmatrix}$

Let us consider the Augmented matrix: $[A \quad |B]$

$$[A \quad |B] = \left[\begin{array}{ccc|c} 1 & -4 & 7 & 8 \\ 3 & 8 & -2 & 6 \\ 7 & -8 & 26 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -4 & 7 & 8 \\ 0 & 20 & -23 & -18 \\ 0 & 20 & -23 & -53 \end{array}\right] \begin{array}{l} R_2 - 3R_1 \\ R_3 - 7R_1 \end{array}$$

$$[A \quad |B] \sim \begin{bmatrix} 1 & -4 & 7 & | & 8 \\ 0 & 20 & -23 & | & -18 \\ 0 & 0 & 0 & | & -35 \end{bmatrix} R_3 - R_2$$

Here $r(A) = 2,\ r(A|B) = 3$

Since $r(A) \neq r(A|B)$,

So, the given system of equations is *inconsistent* and has *no solution*.

Quiz-Time

A). *Unique solution*

B). Infinitely many solutions

C). *No solution*

D) I don't Know

Problem 3. Show that the following system of equations is consistent:

$$x + 4y + 7z = 1 \qquad (1)$$

$$2x + 5y + 8z = 2 \qquad (2)$$

$$x + 2y + 3z = 1 \qquad (3)$$

Solution. The given system can be written as:

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \text{ that is } AX = B$$

Where $\mathrm{A} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 1 & 2 & 3 \end{bmatrix}$, $\mathrm{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $\mathrm{B} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Let us consider the Augmented matrix: $[A \quad |B]$

$$[A \quad |B] = \left[\begin{array}{ccc|c} 1 & 4 & 7 & 1 \\ 2 & 5 & 8 & 2 \\ 1 & 2 & 3 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 4 & 7 & 1 \\ 0 & -3 & -6 & 0 \\ 0 & -2 & -4 & 0 \end{array}\right] \begin{array}{c} R_2 - 2R_1 \\ R_3 - R_1 \end{array}$$

$$[A \quad |B] \sim \left[\begin{array}{ccc|c} 1 & 4 & 7 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \begin{array}{l} R_2/-3 \\ R_3/-2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 4 & 7 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] R_3 - R_2$$

Here $r(A) = r(A|B) = 2 < 3$ (Number of unknowns),

Quiz-Time

If $r(A) = r(A|B) = 2 < 3 =$ (Number of unknowns)

A). *Unique solution*

B). Infinitely many solutions

C). *No solution*

D) I don't Know

So, the given system of equations is *consistent* and has *infinitely many solution*

From the 2nd row of Augmented matrix:

$0x + 1y + 2z = 0 \Longrightarrow y = -2z$

From the 1st row of Augmented matrix:

$x + 4y + 7z = 1 \Longrightarrow x = 1 - 4y - 7z \Longrightarrow x = 1 + z$

Let $z = \alpha$

$\Longrightarrow y = -2\alpha$ and $x = 1 + \alpha$

So, $(x, y, z) = (1 + \alpha, -2\alpha, \alpha)$ **Answer.**

For different values of α, we get different values of x, y, z.

Problem 4. Determine the values of λ and μ for which the system of equations:

$x + 2y + z = 6$ (1)

$x + 4y + 3z = 10$ (2)

$x + 4y + \lambda z = \mu$ (3)

has (I) Unique solution (II) Infinitely many solutions (III) No solution

Solution. The given system can be written as:

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 4 & 3 \\ 1 & 4 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix} \text{ that is } AX = B$$

Where $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 4 & 3 \\ 1 & 4 & \lambda \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$

Let us consider the Augmented matrix: $[A \quad |B]$

$$[A \quad |B] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 1 & 4 & 3 & 10 \\ 1 & 4 & \lambda & \mu \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 2 & 2 & 4 \\ 0 & 2 & \lambda - 1 & \mu - 6 \end{array}\right] \begin{array}{l} R_2 - R_1 \\ R_3 - R_1 \end{array}$$

$$[A \quad |B] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & \lambda - 3 & \mu - 10 \end{array}\right] R_3 - R_2$$

(I) For unique solution: $r(A) = r(A|B) = 3$ (Number of unknowns)

$\Rightarrow \lambda - 3 \neq 0$ i.e. $\lambda \neq 3, \mu$ can have any value.

(II) For infinitely many solutions: $r(A) = r(A|B) = 2 < 3$ (unknowns),

$\Rightarrow \lambda - 3 = 0$ and $\mu - 10 = 0$ i.e. $\lambda = 3$ and $\mu = 10$.

(III) For no solution: $r(A) \neq r(A|B)$ i.e. $r(A) = 2$ and $r(A|B) = 3$

$\Rightarrow \lambda - 3 = 0$ and $\mu - 10 \neq 0$ i.e. $\lambda = 3$ and $\mu \neq 10$.

Solution of Homogeneous System of Equations:

(Gauss Elimination Method-Rank Method)

Let us consider the following system of equations:

$$a_1x + b_1y + c_1z = 0 \qquad (1)$$

$$a_2x + b_2y + c_2z = 0 \qquad (2)$$

$$a_3x + b_3y + c_3z = 0 \qquad (3)$$

The given system can be written as:

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ that is $AX = 0$

Where $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Case 1. If $r(A) = n$ (Number of unknowns), then the given system of equations has *trivial (Zero/unique) solution* $(x = 0, y = 0, z = 0)$.

Case 2. If $r(A) < n$ (Number of unknowns), then the given system of equations has *non-trivial (non-Zero/infinitely many) solutions*.

Quiz-Time

The given Homogeneous system of equations is said to be in*consistent*

A. If $r(A) = n$ (Number of unknowns)

B. If $r(A) < n$ (Number of unknowns)

C. Never

D. *I don't Know*

Problem 1. Solve the following system of homogeneous equations:

$$x + 2y - 3z = 0 \qquad (1)$$

$$x + y - z = 0 \qquad (2)$$

$$x - y + z = 0 \qquad (3)$$

Solution. The given system can be written as:

$$\begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ that is $AX = O$

Where $A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 2 \\ 0 & -3 & 4 \end{bmatrix} \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix} \qquad \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix} R_3 - 3R_1$$

Since $r(A) = 3 = n$ (Number of unknowns)

So, the given system of equations has *trivial (Zero/unique) solution* that is

$x = 0, y = 0, z = 0.$

Problem 2. Solve the following system of homogeneous equations:

$3x + y + 2z = 0 \qquad (1)$

$x - 2y + 3z = 0 \qquad (2)$

$x + 5y - 4z = 0 \qquad (3)$

Solution. The given system can be written as:

$$\begin{bmatrix} 3 & 1 & 2 \\ 1 & -2 & 3 \\ 1 & 5 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ that is } AX = O$$

Where $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & -2 & 3 \\ 1 & 5 & -4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & -2 & 3 \\ 1 & 5 & -4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 3 \\ 3 & 1 & 2 \\ 1 & 5 & -4 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & 7 & -7 \\ 0 & 7 & -7 \end{bmatrix} \begin{matrix} R_2 - 3R_1 \\ R_3 - R_1 \end{matrix} \quad \sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \end{bmatrix} R_3 - R_2$$

Since $r(A) = 2 < 3$ (Number of unknowns),

Quiz-Time

Since $r(A) = 2 < 3$ (Number of unknowns) in the given Homogeneous system then the system of equations will have

A. *Trivial (Zero/unique) solution*

B. *Non-trivial (non-Zero/infinitely many) solutions*

C. *I don't Know*

$$A \sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \end{bmatrix} R_3 - R_2$$

Since $r(A) = 2 < 3$ (Number of unknowns),

So, the given system of equations has *non-trivial (non-Zero/infinitely many) solutions.*

From the 2nd row of Augmented matrix:

$$0x + 7y - 7z = 0 \Rightarrow y = z$$

From the 1st row of Augmented matrix:

$x - 2y + 3z = 0 \Longrightarrow x = 2y - 3z \Longrightarrow x = -z$

Let $z = \alpha$

$\Longrightarrow y = \alpha$ and $x = -\alpha$

So, $(x, y, z) = (-\alpha, \alpha, \alpha)$ **Answer.**

For different values of α, we get different values of x, y, z.