Unit 1: Linear Algebra

(**Book:** Advanced Engineering Mathematics by Jain and Iyengar, **Chapter-3**)

Topic:

Cayley-Hamilton Theorem

Learning Outcomes:

1. Verification of Cayley-Hamilton Theorem.

2. Using Cayley-Hamilton Theorem to Find the Inverse of a matrix.

Cayley-Hamilton Theorem:

Definition: *Every square matrix A satisfies its own characteristic equation.*

For instance, let A be a square matrix of order 3.

The characteristic equation is: $|A - \lambda I| = 0$

$$\Rightarrow \lambda^3 + a\lambda^2 + b\lambda + c = 0 \qquad (1)$$

By Cayley-Hamilton Theorem, matrix A must satisfy characteristic equ. (1)

i.e.

$$\Rightarrow A^3 + aA^2 + bA + cI = 0$$

Finding Inverse using Cayley-Hamilton Theorem:

Since $$A^3 + aA^2 + bA + cI = 0$$

$$\Rightarrow cI = -(A^3 + aA^2 + bA)$$

$$\Rightarrow I = -\frac{1}{c}(A^3 + aA^2 + bA)$$

Pre-multiplying both sides by A^{-1}, we get:

$$\Rightarrow (A^{-1})I = -\frac{1}{c}(A^{-1})(A^3 + aA^2 + bA)$$

$$\Rightarrow A^{-1} = -\frac{1}{c}(A^2 + aA + bI)$$

Problem 1. Verify Cayley-Hamilton Theorem for matrix $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$. If possible, find A^{-1}.

Solution. The characteristic equation is: $|A - \lambda I| = 0$

$$\Rightarrow \left| \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \left| \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda) - 12 = 0$$

$$\Rightarrow 2 - \lambda - 2\lambda + \lambda^2 - 12 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda - 10 = 0 \qquad (1)$$

By Cayley-Hamilton Theorem, matrix A must satisfy characteristic equ. (1)

i.e. $$\Rightarrow A^2 - 3A - 10I = 0 \qquad (2)$$

Here $A^2 = A.A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1+12 & 4+8 \\ 3+6 & 12+4 \end{bmatrix} = \begin{bmatrix} 13 & 12 \\ 9 & 16 \end{bmatrix}$

From equation (2):

L.H.S. $A^2 - 3A - 10I$

$$= \begin{bmatrix} 13 & 12 \\ 9 & 16 \end{bmatrix} - 3\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} - 10\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 13-3-10 & 12-12-0 \\ 9-9-0 & 16-6-10 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$ **R.H.S.**

Hence, Cayley-Hamilton Theorem is verified.

To find A^{-1}:

From equation (2): $A^2 - 3A - 10I = 0$

$$\Longrightarrow 10I = A^2 - 3A$$

$$\Longrightarrow I = \frac{1}{10}[A^2 - 3A]$$

Pre-multiplying both sides by A^{-1}, we get:

$$(A^{-1})I = \frac{1}{10}(A^{-1})[A^2 - 3A]$$

$$\Longrightarrow A^{-1} = \frac{1}{10}[A - 3I]$$

$$\Rightarrow A^{-1} = \frac{1}{10}[A - 3I]$$

$$\Rightarrow A^{-1} = \frac{1}{10}\left[\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]$$

$$\Rightarrow A^{-1} = \frac{1}{10}\begin{bmatrix} 1-3 & 4-0 \\ 3-0 & 2-3 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{10}\begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix}$$

* We calculated the inverse of the same matrix in **Lecture-6** using Gauss-Jordan method (Slides 4-5).

Problem 2. *If* $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$, *then use Cayley Hamilton Theorem to find the matrix represented by* A^5.

Sol. Characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = 0 \text{ or } \lambda^2 - 7\lambda + 1 = 0$$

By Cayley Hamilton Theorem $A^2 - 7A + I = 0$

$\therefore$

$$A^2 = 7A - I \qquad \ldots$$

$$A^4 = 49A^2 - 14A + I$$

$$= 49(7A - I) - 14A + I \qquad [\text{Using (}$$

$$= 329A - 48I$$

$$A^5 = A^4 \cdot A = (329A - 48I)A$$

$$= 329A^2 - 48A = 329(7A - I) - 48A = 2255A - 329I$$

$$= 2255\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} - 329\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4181 & 6765 \\ 6765 & 10946 \end{bmatrix}.$$

Quiz-Time

If $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ then what is the value of $A^6 - 6A^5 + 9A^4 - 2A^3 - 12A^2 + 23A - 9I$

A) $5A - I$ B) $6A - I$ C) $5A + I$ D) $6A + I$

Problem 2. Verify Cayley-Hamilton Theorem for matrix $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. If possible, find A^{-1}.

Solution. The characteristic equation is: $|A - \lambda I| = 0$

$$\Rightarrow \left| \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \left| \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 1 - \lambda & 1 \\ -1 & 1 - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(1-\lambda) + 1 = 0$$

$$\Rightarrow 1 - \lambda - \lambda + \lambda^2 + 1 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda + 2 = 0 \qquad (1)$$

By Cayley-Hamilton Theorem, matrix A must satisfy characteristic equ. (1)

i.e. $$\Rightarrow A^2 - 2A + 2I = 0 \qquad (2)$$

Here $A^2 = A.A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1-1 & 1+1 \\ -1-1 & -1+1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

From equation (2):

L.H.S. $A^2 - 2A + 2I$

$$= \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} - 2\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0-2+2 & 2-2+0 \\ -2+2+0 & 0-2+2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$ **R.H.S.**

Hence, Cayley-Hamilton Theorem is verified.

To find A^{-1}:

From equation (2): $A^2 - 2A + 2I = 0$

$$\Longrightarrow 2I = -A^2 + 2A$$

$$\Longrightarrow I = \frac{1}{2}[-A^2 + 2A]$$

Pre-multiplying both sides by A^{-1}, we get:

$$(A^{-1})I = \frac{1}{2}(A^{-1})[-A^2 + 2A]$$

$$\Longrightarrow A^{-1} = \frac{1}{10}[-A + 2I]$$

$$\Longrightarrow A^{-1} = \tfrac{1}{2}[-A + 2I]$$

$$\Longrightarrow A^{-1} = \tfrac{1}{2}\left[-\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]$$

$$\Longrightarrow A^{-1} = \tfrac{1}{2}\begin{bmatrix} -1+2 & -1+0 \\ 1+0 & -1+2 \end{bmatrix}$$

$$\Longrightarrow A^{-1} = \tfrac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Problem 3. Verify Cayley-Hamilton Theorem for matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$. If possible, find A^{-1}.

Solution. The characteristic equation is: $|A - \lambda I| = 0$

$$\Rightarrow \left| \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \left| \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 1 - \lambda & 2 \\ 3 & 6 - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 3 & 6-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(6-\lambda) - 6 = 0$$

$$\Rightarrow 6 - \lambda - 6\lambda + \lambda^2 - 6 = 0$$

$$\Rightarrow \lambda^2 - 7\lambda = 0 \qquad (1)$$

By Cayley-Hamilton Theorem, matrix A must satisfy characteristic equ. (1)

i.e. $$\Rightarrow A^2 - 7A = 0 \qquad (2)$$

Here $A^2 = A.A = \begin{bmatrix}1 & 2\\3 & 6\end{bmatrix}\begin{bmatrix}1 & 2\\3 & 6\end{bmatrix} = \begin{bmatrix}1+6 & 2+12\\3+18 & 6+36\end{bmatrix} = \begin{bmatrix}7 & 14\\21 & 42\end{bmatrix}$

From equation (2):

L.H.S. $A^2 - 7A$

$$= \begin{bmatrix}7 & 14\\21 & 42\end{bmatrix} - 7\begin{bmatrix}1 & 2\\3 & 6\end{bmatrix}$$

$$= \begin{bmatrix}7-7 & 14-14\\21-21 & 42-42\end{bmatrix}$$

$$= \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = 0$$ **R.H.S.**

Hence, Cayley-Hamilton Theorem is verified.

To find $\boldsymbol{A^{-1}}$:

From equation (2): $A^2 - 7A = 0$

We **can not** find A^{-1} as there is **no constant term** in **characteristic equation.**

Constant term in Characteristic equation corresponds to determinant of matrix A.

i.e. $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = 6 - 6 = 0$

So, A is a singular matrix and A^{-1} does not exist. $\left[\boldsymbol{A^{-1} = \frac{adj\ A}{|A|}}\right]$

Problem 4. Verify Cayley-Hamilton Theorem for matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$.

If possible, find A^{-1}.

Solution. The characteristic equation is: $|A - \lambda I| = 0$

$$\Rightarrow \left| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda)(3-\lambda) = 0$$

$$\Rightarrow (1-\lambda)(6-5\lambda+\lambda^2) = 0$$

$$\Rightarrow 6-11\lambda+6\lambda^2-\lambda^3 = 0$$

$$\Rightarrow \lambda^3-6\lambda^2+11\lambda-6 = 0 \qquad (1)$$

By Cayley-Hamilton Theorem, matrix A must satisfy characteristic equ. (1)

i.e. $$\Rightarrow A^3-6A^2+11A-6I = 0 \qquad (2)$$

Try to verify the theorem yourself.

To find A^{-1}:

From equation (2): $A^3 - 6A^2 + 11A - 6I = 0$

$$\Rightarrow 6I = A^3 - 6A^2 + 11A$$

$$\Rightarrow I = \frac{1}{6}[A^3 - 6A^2 + 11A]$$

Pre-multiplying both sides by A^{-1}, we get:

$$(A^{-1})I = \frac{1}{6}(A^{-1})[A^3 - 6A^2 + 11A]$$

$$\Rightarrow A^{-1} = \frac{1}{6}[A^2 - 6A + 11I]$$

Try it yourself.