Simultaneous Linear Differential Equations:

The system involving two first order linear differential equations in two dependent variables y_1 and y_2 and one independent variable x is called system of simultaneous linear differential equations.

Write the operator form of following system of LDE:

Problem1. $6\frac{dy_1}{dx} + 5\frac{dy_2}{dx} + 3y_1 + y_2 = 0, \frac{dy_2}{dx} - 5y_1 + 3y_2 = e^x$

Solution. The given system of simultaneous equations is:

$$6\frac{dy_1}{dx} + 5\frac{dy_2}{dx} + 3y_1 + y_2 = 0 \qquad (1)$$

$$\frac{dy_2}{dx} - 5y_1 + 3y_2 = e^x \qquad (2)$$

Let $D \equiv \frac{d}{dx}$, then operator form of given system can be written as:

$$(6D + 3)y_1 + (5D + 1)y_2 = 0 \qquad (3) \times 5$$

$$-5y_1 + (D + 3)y_2 = e^x \qquad (4) \times (6D + 3)$$

Solving these equations:

$$5(6D+3)y_1 - 5(5D+1)y_2 = 0$$

$$-5(6D+3)y_1 + (6D+3)(D+3)y_2 = (6D+3)e^x$$

On adding these equations, we get:

$$[(6D+3)(D+3) - 5(5D+1)]y_2 = (6D+3)e^x$$

$$\Longrightarrow (6D^2 - 4D + 4)y_2 = 6e^x + 3e^x = 9e^x \qquad (5)$$

<u>To find Complimentary Function (C.F.):</u>

$$(6D^2 - 4D + 4) = 0 \qquad \Longrightarrow D = \frac{1 \pm \sqrt{5}i}{3}$$

$$\Rightarrow y_c = e^{\frac{x}{3}}\left(c_1 \cos\frac{\sqrt{5}}{3}x + c_2 \sin\frac{\sqrt{5}}{3}x\right)$$

To find Particular Integral (P.I.):

$$y_p = \frac{1}{f(D)} r(x) = \frac{1}{(6D^2-4D+4)}(9e^x)$$

$$\Rightarrow y_p = 9\left[\frac{1}{(6D^2-4D+4)} e^x\right]$$

$$\Rightarrow y_p = 9\left[\frac{1}{(6(1)^2-4(1)+4)} e^x\right] \qquad (Put\ D = 1)$$

$$\Rightarrow y_p = 9\left[\frac{1}{6} e^{2x}\right] \qquad \Rightarrow y_p = \frac{3}{2} e^{2x}$$

∴ General solution is given by: y_2 = C.F. +P.I.

i.e. $y_2 = y_c + y_p$

$$\Rightarrow y_2 = e^{\frac{x}{3}}\left(c_1 \cos\frac{\sqrt{5}}{3}x + c_2 \sin\frac{\sqrt{5}}{3}x\right) + \frac{3}{2}e^{2x}$$

Again from equations (3) and (4):

$(6D + 3)y_1 + (5D + 1)y_2 = 0$ (3) $\times(D + 3)$

$-5y_1 + (D + 3)y_2 = e^x$ (4) $\times(5D + 1)$

Solving these equations:

$$(6D + 3)(D + 3)y_1 + (5\mathrm{D} + 1)(\mathrm{D} + 3)y_2 = 0$$

$$-5(5D + 1)y_1 + (5\mathrm{D} + 1)(\mathrm{D} + 3)y_2 = (5D + 1)e^x$$

Subtracting these equations, we get:

$$[(6D + 3)(D + 3) + 5(5D + 1)]y_1 = (5D + 1)e^x$$

$$\Rightarrow (6D^2 + 46D + 14)y_1 = 6\,e^x$$

Try it yourself.

Polling Question:

The operator form of $3\frac{dy_1}{dx} + 2y_1 + y_2 = e^{-x}$, $\frac{dy_1}{dx} + \frac{dy_2}{dx} - 2y_1 + 3y_2 = x$ is:

(A) $(3D + 2)y_1 + y_2 = 0$, $(D - 2)y_1 + (D + 3)y_2 = 0$

(B) $(3D + 2)y_1 + y_2 = e^{-x}$, $(D - 2)y_2 + (D + 3)y_1 = x$

(C) $(3D + 2)y_1 + y_2 = e^{-x}$, $(D - 2)y_1 + (D + 3)y_2 = x$

Problem2. $3\frac{dy_1}{dx}+2y_1+y_2=e^{-x}, \frac{dy_1}{dx}+\frac{dy_2}{dx}-2y_1+3y_2=x$

Solution. The given system of simultaneous equations is:

$3\frac{dy_1}{dx}+2y_1+y_2=e^{-x}$ (1)

$\frac{dy_1}{dx}+\frac{dy_2}{dx}-2y_1+3y_2=x$ (2)

Let $D\equiv\frac{d}{dx}$, then operator form of given system can be written as:

$(3D+2)y_1+y_2= e^{-x}$ (3)$\times(D+3)$

$(D-2)y_1+(D+3)y_2=x$ (4)$\times 1$

Solving these equations:

$$(3D+2)(D+3)y_1+(D+3)y_2=(D+3)e^{-x}$$

$$(D-2)y_1+(D+3)y_2=x$$

Subtracting these equations, we get:

$$[(3D+2)(D+3)-(D-2)]y_1=(D+3)e^{-x}-x$$

$$\Rightarrow (3D^2+10D+8)y_1=2e^{-x}-x \qquad (5)$$

<u>To find Complimentary Function (C.F.):</u>

$$(3D^2+10D+8)=0 \qquad \Longrightarrow D=-2,-\frac{4}{3}$$

$$\Rightarrow y_c=\left(c_1e^{-2x}+c_2e^{-\frac{4}{3}x}\right)$$

To find Particular Integral (P.I.):

$$y_p = \frac{1}{f(D)} r(x) = \frac{1}{(3D^2+10D+8)} (2e^{-x} - x)$$

$$\Rightarrow y_p = 2\left[\frac{1}{(3D^2+10D+8)} e^{-x}\right] - \left[\frac{1}{(3D^2+10D+8)} x\right]$$

$$\Rightarrow y_p = 2\left[\frac{1}{(3(-1)^2+10(-1)+8)} e^{-x}\right] - \left[\frac{1}{8\left(1+\left(\frac{10D+3D^2}{8}\right)\right)} x\right]$$

$$\Rightarrow y_p = 2\left[\frac{1}{1} e^{-x}\right] - \frac{1}{8}\left[\left(1 + \left(\frac{10D+3D^2}{8}\right)\right)^{-1} x\right]$$

$$\Rightarrow y_p = 2e^{-x} - \frac{1}{8}\left(x - \frac{10}{8}\right)$$

$\therefore$ General solution is given by: y_1 = C.F. +P.I.

i.e. $y_1 = y_c + y_p$

$$\Rightarrow y_1 = \left(c_1 e^{-2x} + c_2 e^{-\frac{4}{3}x}\right) + 2e^{-x} + \frac{1}{8}\left(x - \frac{10}{8}\right)$$

Again from equations (3) and (4):

$(3D+2)y_1 + y_2 = e^{-x}$ (3)$\times(D-2)$

$(D-2)y_1 + (D+3)y_2 = x$ (4)$\times(3D+2)$

Solving these equations:

$$(3D+2)(D-2)y_1 + (D-2)y_2 = (D-2)e^{-x}$$

$$(3D+2)(D-2)y_1 + (3D+2)(D+3)y_2 = (3D+2)x$$

Subtracting these equations, we get:

$$[(3D+2)(D+3)-(D-2)]y_2 = (3D+2)x-(D-2)e^{-x}$$

$$\Rightarrow (3D^2+10D+8)y_2 = (3+2x)+3e^{-x} \qquad (6)$$

<u>To find Complimentary Function (C.F.):</u>

$$(3D^2+10D+8)=0 \qquad \Longrightarrow D=-2,-\frac{4}{3}$$

$$\Rightarrow y_c = \left(c_3e^{-2x}+c_4e^{-\frac{4}{3}x}\right)$$

To find Particular Integral (P.I.):

$$y_p = \frac{1}{f(D)} r(x) = \frac{1}{(3D^2+10D+8)} \left(3e^{-x} + (2x+3)\right)$$

$$\Rightarrow y_p = 3\left[\frac{1}{(3D^2+10D+8)} e^{-x}\right] + \left[\frac{1}{(3D^2+10D+8)} (2x+3)\right]$$

$$\Rightarrow y_p = 3\left[\frac{1}{\left(3(-1)^2+10(-1)+8\right)} e^{-x}\right] + \left[\frac{1}{8\left(1+\left(\frac{10D+3D^2}{8}\right)\right)} (2x+3)\right]$$

$$\Rightarrow y_p = 3\left[\frac{1}{1} e^{-x}\right] + \frac{1}{8}\left[\left(1+\left(\frac{10D+3D^2}{8}\right)\right)^{-1} (2x+3)\right]$$

$$\Rightarrow y_p = 3e^{-x} + \frac{1}{8}\left((2x+3) - \frac{10}{4}\right)$$

$\therefore$ General solution is given by: $y_2 = \text{C.F.} + \text{P.I.}$

i.e. $y_2 = y_c + y_p$

$$\Rightarrow y_2 = \left(c_3 e^{-2x} + c_4 e^{-\frac{4}{3}x}\right) + 2e^{-x} + \frac{1}{8}\left((2x+3) - \frac{10}{4}\right)$$