<u>Unit 2:</u> Differential Equations of Higher Order

(**Book:** Advanced Engineering Mathematics by R.K.Jain and S.R.K Iyengar,**Chapter-5**)

<u>Topic:</u>

Linear Differential Equations (LDE)

<u>Learning Outcomes:</u>

1. Identification of Linear Differential Equations (LDE).

2. Necessary and Sufficient condition for LDE to be Normal on an interval

Linear Differential Equations (LDE):

A linear differential equation of order n is written as:

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + --- + a_{n-1} \frac{dy}{dx} + a_n y = r(x) \qquad (1)$$

or

$$a_0 y^n + a_1 y^{n-1} + ---- + a_{n-1} y' + a_n y = r(x) \qquad (1)$$

For example, a second order LDE is written as:

$$a_0 y'' + a_1 y' + a_2 y = r(x) \qquad (2)$$

* If $r(x) = 0$, then LDE is called Homogeneous LDE.

* If $r(x) \neq 0$, then LDE is called Non-Homogeneous LDE.

* If $a_0, a_1, ---, a_n$ are all constants, then LDE is called LDE with constant coefficients.

* If $a_0, a_1, ---, a_n$ are not all constants, then it is called LDE with variable coefficients.

Classify the following LDE:

1. $y'' + 4y' + 3y = x^2 e^x$

It is a 2nd order Non-homogeneous LDE with constant coefficients.

2. $y'' + 2y' + y = \sin x$

It is a 2nd order Non-homogeneous LDE with constant coefficients.

3. $x^2 y'' + xy' + (x^2 - 4)y = 0$

It is a 2nd order Homogeneous LDE with variable coefficients.

4. $(1 - x^2)y'' - 2xy' + 20y = 0$

It is a 2nd order Homogeneous LDE with variable coefficients.

Polling Question

The equation: $y'' + 4y' + xy = x^2 e^x$ **is:**

(A) 1^{st} order Homogeneous LDE with variable coefficients.

(B) 2^{nd} order Homogeneous LDE with constant coefficients.

(C) 2^{nd} order Non-homogeneous LDE with variable coefficients.

(D) 2^{nd} order Non-homogeneous LDE with constant coefficients.

Necessary and Sufficient condition for LDE to be Normal on an interval:

A linear differential equation of order n:

$$a_0 y^n + a_1 y^{n-1} + ----- + a_{n-1} y' + a_n y = r(x) \qquad (1)$$

is said to be normal on an interval I if:

1. $a_0, a_1, ---, a_n$ and $r(x)$ are all continuous on an interval I

2. $a_0 \neq 0$

* In homogeneous LDE ($r(x)$=0), the problem arises only because of $a_0 \neq 0$.

* In non-homogeneous LDE ($r(x) \neq 0$), the problem arises due to $a_0 \neq 0$ and also due to domain of $r(x)$.

* In almost all numerical $a_0, a_1, ---, a_n$ are all continuous, so first condition automatically holds. We are focused on second condition only.

<u>Find the intervals on which the following differential equations are normal.</u>

<u>Problem 1.</u> $\boldsymbol{(1-x^2)y''-2xy'+3y=0}$

<u>Solution:</u> $(1-x^2)y''-2xy'+3y=0$ (1)

Comparing with: $a_0y''+a_1y'+a_2y=0$

1. $\left.\begin{aligned} a_0 &= (1-x^2)\\ a_1 &= -2x\\ a_2 &= 3\end{aligned}\right\}$ Being polynomials, a_0, a_1, a_2 are all continuous on $(-\infty,\infty)$

2. $a_0 \neq 0$ $\Rightarrow (1-x^2)\neq 0$ $\Rightarrow x^2 \neq 1$ $\Rightarrow x \neq \pm 1$

Thus, LDE (1) is normal on subintervals: $(-\infty,-1),(-1,1),(1,\infty)$.

<u>Problem 2.</u> $\boldsymbol{x^2y'' + xy' + \left(n^2 - x^2\right)y = 0}$**; $\boldsymbol{n}$ is real.**

<u>Solution:</u> $x^2y'' + xy' + (n^2 - x^2)y = 0$ (1)

Comparing with: $a_0y'' + a_1y' + a_2y = 0$

1. $\left.\begin{aligned} a_0 &= x^2 \\ a_1 &= x \\ a_2 &= (n^2 - x^2) \end{aligned}\right\}$ Being polynomials, a_0, a_1, a_2 are all continuous on $(-\infty, \infty)$

2. $a_0 \neq 0 \qquad \Rightarrow x^2 \neq 0 \qquad \Rightarrow x \neq 0$

Thus, LDE (1) is normal on subintervals: $(-\infty, 0), (0, \infty)$.

Problem 3. $\boldsymbol{x(1-x)y''-3xy'-y=0}$

Solution: $x(1-x)y''-3xy'-y=0$ (1)

Comparing with: $a_0y''+a_1y'+a_2y=0$

1. $\left.\begin{aligned} a_0 &= x(1-x) \\ a_1 &= -3x \\ a_2 &= -1 \end{aligned}\right\}$ Being polynomials, a_0, a_1, a_2 are all continuous on $(-\infty,\infty)$

2. $a_0 \neq 0 \quad \Rightarrow x(1-x) \neq 0 \quad \Rightarrow x \neq 0, \;\; x \neq 1$

Thus, LDE (1) is normal on subintervals: $(-\infty,0),(0,1),(1,\infty)$.

Problem 4. $\boldsymbol{y'' + 9y' + y = \log(x^2 - 9)}$

Solution: $y'' + 9y' + y = \log(x^2 - 9)$ (1)

Comparing with: $a_0 y'' + a_1 y' + a_2 y = r(x)$

1. $\left.\begin{aligned} a_0 &= 1 \\ a_1 &= 9 \\ a_2 &= 1 \end{aligned}\right\}$ Being constants, a_0, a_1, a_2 are all continuous on $(-\infty, \infty)$

Also $r(x) = \log(x^2 - 9)$ will be defined if $(x^2 - 9) > 0$ i.e. $x^2 > 9$

$\Rightarrow |x| > 3 \quad \Rightarrow -\infty < x < -3$ and $3 < x < \infty$

2. $a_0 \neq 0 \quad \Rightarrow 1 \neq 0$ which is true.

Thus, LDE (1) is normal on subintervals: $(-\infty, -3), (3, \infty)$.

<u>Problem 5.</u> $\boldsymbol{\sqrt{x}y'' + 6xy' + 15y = \log\left(x^4 - 256\right)}$

<u>Solution:</u> $\sqrt{x}y'' + 6xy' + 15y = \log(x^4 - 256)$ (1)

Comparing with: $a_0 y'' + a_1 y' + a_2 y = r(x)$

1. $\left.\begin{aligned} a_0 &= \sqrt{x} \\ a_1 &= 6x \\ a_2 &= 15 \end{aligned}\right\}$ a_1, a_2 are all continuous on $(-\infty, \infty)$, but a_0 is continuous on $(0, \infty)$

Also $r(x) = \log(x^4 - 256)$ will be defined if $(x^4 - 256) > 0$ i.e. $x^4 > 256$

$\Rightarrow |x| > 4 \qquad \Rightarrow -\infty < x < -4$ and $4 < x < \infty$

2. $a_0 \neq 0 \qquad \Rightarrow \sqrt{x} \neq 0 \Rightarrow x > 0$ (Square root of a negative number is not defined)

Thus, LDE (1) is normal on subintervals: $(4, \infty)$.

<u>Polling Question</u>

A linear differential equation: $\boldsymbol{a_0 y''' + a_1 y'' + a_2 y' + a_3 y = 0}$ is said to be normal on an interval *I* if:

(A) a_0, a_1, a_2, a_3 and $r(x)$ are all continuous on an interval I and $a_0 \neq 0$

(B) a_0, a_1, a_2, a_3 are all continuous on an interval I and $a_0 \neq 0$

(C) $a_0 \neq 0$

Problem 6. $\boldsymbol{y''+3y'+\sqrt{x}y=\sin x}$

Solution: $y''+3y'+\sqrt{x}y=\sin x$ (1)

Comparing with: $a_0y''+a_1y'+a_2y=r(x)$

1. $\left.\begin{array}{r} a_0=1 \\ a_1=3 \\ a_2=\sqrt{x} \end{array}\right\}$ a_0, a_1 are all continuous on $(-\infty,\infty)$, but a_2 is continuous on $(0,\infty)$

Also $r(x)=\sin x$ is continuous on $(-\infty,\infty)$.

2. $a_0\neq 0$ $\Rightarrow 1\neq 0$ which is true.

Thus, LDE (1) is normal on subintervals: $(0,\infty)$.