Unit 1: Linear Algebra

(**Book:** Advanced Engineering Mathematics by Jain and Iyengar, **Chapter-3**)

Topic:

Eigen values and Eigen vectors

Learning Outcomes:

1. Characteristic Equation.

2. Eigen values/Characteristic values/Latent values/Spectral values

3. Eigen vectors/Latent vectors

Eigen values:

Let A be a square matrix (Singular or Non-singular).

A scalar λ is said to be an Eigen value of matrix A if:

$$AX = \lambda X$$

Characteristic Equation:

$AX = \lambda X$

$\Rightarrow (A - \lambda I)X = 0$ (1) where I is the Identity matrix.

This homogeneous system will have non-trivial solution if:

$|A - \lambda I| = 0$ (2)

Equation (2) is called characteristic equation.

Solving characteristic equation $|A - \lambda I| = 0$, we get different values of λ which are called Eigen values.

Putting these values back in equation (1) i.e. $(A - \lambda I)X = 0$, we get different values of column vectors X which are called Eigen vectors.

Note: 1. There can be more than one Eigen vectors for one Eigen value.

2. Distinct Eigen values have distinct eigen vectors.

3. Repeated (same) Eigen values may have same or distinct Eigen vectors.

Problem 1. Find the Eigen values and the corresponding Eigen vectors of:

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$$

Solution. The characteristic equation is: $|A - \lambda I| = 0$

$$\Rightarrow \left| \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \left| \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda) - 12 = 0$$

$$\Rightarrow 2 - \lambda - 2\lambda + \lambda^2 - 12 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda - 10 = 0$$

$$\Rightarrow (\lambda - 5)(\lambda + 2) = 0$$

$$\Rightarrow \lambda = 5, \lambda = -2$$

Which are the required Eigen values.

For $\boldsymbol{\lambda = 5}$;

$$[A - \lambda I]X = 0$$

$$\Rightarrow \left[\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 1-5 & 4 \\ 3 & 2-5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -4 & 4 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow -4x_1 + 4x_2 = 0 \qquad \Rightarrow x_1 = x_2$$

And $\Rightarrow 3x_1 - 3x_2 = 0 \qquad \Rightarrow x_1 = x_2$

Thus, $\frac{x_1}{1} = \frac{x_2}{1} = \mu$(Say)

If $\mu = 1$, then $x_1 = 1, x_2 = 1$

So, corresponding Eigen vector is $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Note: For different values of μ, we get different multiples of Eigen vector X. That is why we say, a single Eigen value may have many Eigen vectors.

For $\boldsymbol{\lambda = -2}$;

$$[A - \lambda I]Y = 0$$

$$\Rightarrow \left[\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 1+2 & 4 \\ 3 & 2+2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0$$

$$\Longrightarrow 3y_1 + 4y_2 = 0 \qquad \Longrightarrow 3y_1 = -4y_2$$

And $\Longrightarrow 3y_1 + 4y_2 = 0 \qquad \Longrightarrow 3y_1 = -4y_2$

Thus, $\frac{y_1}{4} = \frac{y_2}{-3} = \mu$(Say)

If $\mu = 1$, then $y_1 = 4, y_2 = -3$

So, corresponding Eigen vector is Y $= \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$

Note: For different values of μ, we get different multiples of Eigen vector Y. That is why we say, a single Eigen value may have many Eigen vectors.

Problem 2. Find the Eigen values and the corresponding Eigen vectors of:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

Solution. The characteristic equation is: $|A - \lambda I| = 0$

$$\Rightarrow \left| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda)(3-\lambda) = 0$$

$$\Rightarrow \lambda = 1,2,3$$

Which are the required Eigen values.

For $\boldsymbol{\lambda = 1}$;

$$[A - \lambda I]X = 0$$

$$\Rightarrow \left| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} - 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 2 \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

From 2nd and 3rd row:

$$0x_1 + 1x_2 + 1x_3 = 0$$

$$2x_1 + 0x_2 + 2x_3 = 0$$

Solving these equations, we get;

$$\begin{cases} 0x_1 + 1x_2 + 1x_3 = 0 \\ 2x_1 + 0x_2 + 2x_3 = 0 \end{cases}$$

Coeff. of :	x_2	x_3	x_1	x_2
	2	3	1	2
	1	1	0	1
	0	2	2	0

$$\frac{x_1}{2-0} = \frac{x_2}{2-0} = \frac{x_3}{0-2} = \mu(Say)$$

$$\Longrightarrow \frac{x_1}{2} = \frac{x_2}{2} = \frac{x_3}{-2} = \mu$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{-1} = \mu$$

If $\mu = 1$, then $x_1 = 1, x_2 = 1, x_3 = -1$

So, corresponding Eigen vector is $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

For $\boldsymbol{\lambda = 2}$;

$$[A - \lambda I]Y = 0$$

$$\Rightarrow \left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 0$$

From 1st and 2nd row:

$$-1y_1 + 0y_2 + 0y_3 = 0$$

$$0y_1 + 0y_2 + 1y_3 = 0$$

Solving these equations, we get;

Coeff. of :

$$\begin{array}{cccc} y_2 & y_3 & y_1 & y_2 \\ \underline{2} & \underline{3} & \underline{1} & \underline{2} \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{array}$$

$$\begin{cases} -1y_1 + 0y_2 + 0y_3 = 0 \\ 0y_1 + 0y_2 + 1y_3 = 0 \end{cases}$$

$$\frac{y_1}{0-0} = \frac{y_2}{0+1} = \frac{y_3}{0-0} = \mu(Say)$$

$$\Rightarrow \frac{y_1}{0} = \frac{y_2}{1} = \frac{y_3}{0} = \mu$$

If $\mu = 1$, then $y_1 = 0, y_2 = 1, y_3 = 0$

So, corresponding Eigen vector is Y$= \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

For $\boldsymbol{\lambda = 3}$;

$$[A - \lambda I]Z = 0$$

$$\Rightarrow \left| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} -2 & 0 & 0 \\ 0 & -1 & 1 \\ 2 & 0 & 0 \end{vmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = 0$$

From 1st and 2nd row:

$$-2z_1 + 0z_2 + 0z_3 = 0$$

$$0z_1 - 1z_2 + 1z_3 = 0$$

Solving these equations, we get;

$$\begin{cases} -2z_1 + 0z_2 + 0z_3 = 0 \\ 0z_1 - 1z_2 + 1z_3 = 0 \end{cases}$$

Coeff. of :	z_2	z_3	z_1	z_2
	2	3	1	2
	0	0	-2	0
	-1	1	0	-1

$$\frac{z_1}{0-0} = \frac{z_2}{0+2} = \frac{z_3}{2-0} = \mu(Say)$$

$$\Longrightarrow \frac{z_1}{0} = \frac{z_2}{2} = \frac{z_3}{2} = \mu$$

$$\Rightarrow \frac{z_1}{0} = \frac{z_2}{1} = \frac{z_3}{1} = \mu$$

If $\mu = 1$, then $z_1 = 0, z_2 = 1, z_3 = 1$

So, corresponding Eigen vector is $Z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Diagonalizable: For three distinct Eigen values, we have three distinct linearly independent Eigen vectors, so the matrix A is diagonalizable.

Problem 3. Find the Eigen values and the corresponding Eigen vectors of:

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$$

Solution. The characteristic equation is: $|A - \lambda I| = 0$

$$\Rightarrow \left| \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 2 \\ 0 & 2-\lambda & 1 \\ -1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)[(2-\lambda)^2 - 2] - 2[0+1] + 2[0 + 2(2-\lambda)] = 0$$

$$\Rightarrow \lambda = 1,2,2$$

Which are the required Eigen values.

For $\boldsymbol{\lambda = 1}$;

$$[A - \lambda I]X = 0$$

$$\Rightarrow \left| \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix} - 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 0 & 2 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

From 2nd and 3rd row:

$$0x_1 + 1x_2 + 1x_3 = 0$$

$$-1x_1 + 2x_2 + 1x_3 = 0$$

Solving these equations, we get;

$$\begin{cases} 0x_1 + 1x_2 + 1x_3 = 0 \\ -1x_1 + 2x_2 + 1x_3 = 0 \end{cases}$$

Coeff. of :	x_2	x_3	x_1	x_2
	2	3	1	2
	1	1	0	1
	2	1	-1	2

$$\frac{x_1}{2-1} = \frac{x_2}{-1-0} = \frac{x_3}{0+1} = \mu(Say)$$

$$\Longrightarrow \frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1} = \mu$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1} = \mu$$

If $\mu = 1$, then $x_1 = 1, x_2 = -1, x_3 = -1$

So, corresponding Eigen vector is $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

For $\boldsymbol{\lambda} = \mathbf{2}$;

$$[A - \lambda I]Y = 0$$

$$\Rightarrow \left| \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{vmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 0$$

From 1st and 2nd row:

$$-1y_1 + 2y_2 + 2y_3 = 0$$

$$0y_1 + 0y_2 + 1y_3 = 0$$

Solving these equations, we get;

Coeff. of :

	y_2	y_3	y_1	y_2
	2	3	1	2
	2	2	-1	2
	0	1	0	0

$$\begin{cases} -1y_1 + 2y_2 + 2y_3 = 0 \\ 0y_1 + 0y_2 + 1y_3 = 0 \end{cases}$$

$$\frac{y_1}{2-0} = \frac{y_2}{0+1} = \frac{y_3}{0-0} = \mu(Say)$$

$$\Longrightarrow \frac{y_1}{2} = \frac{y_2}{1} = \frac{y_3}{0} = \mu$$

If $\mu = 1$, then $y_1 = 2, y_2 = 1, y_3 = 0$

So, corresponding Eigen vector is $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

Not Diagonalizable: For three Eigen values, we have only two distinct linearly independent Eigen vectors, so the matrix A is not diagonalizable.

Problem 4. Find the Eigen values and the corresponding Eigen vectors of:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

Solution. The characteristic equation is: $|A - \lambda I| = 0$

$$\Rightarrow \left| \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)[(1-\lambda)^2 - 0] - 2[3(1-\lambda) - 0] + 3[0 + 2(1-\lambda)] = 0$$

$$\Rightarrow \lambda = 1,1,1$$

Which are the required Eigen values.

For $\boldsymbol{\lambda = 1}$;

$$[A - \lambda I]X = 0$$

$$\Rightarrow \left|\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} - 1\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right| \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 0 & 2 & 3 \\ 3 & 0 & 0 \\ -2 & 0 & 0 \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

From 1st and 2nd row:

$$0x_1 + 2x_2 + 3x_3 = 0$$

$$3x_1 + 0x_2 + 0x_3 = 0$$

Solving these equations, we get;

Coeff. of :

x_2	x_3	x_1	x_2
$\underline{2}$	$\underline{3}$	$\underline{1}$	$\underline{2}$
2	3	0	2
0	0	3	0

$$\begin{cases} 0x_1 + 2x_2 + 3x_3 = 0 \\ 3x_1 + 0x_2 + 0x_3 = 0 \end{cases}$$

$$\frac{x_1}{0-0} = \frac{x_2}{9-0} = \frac{x_3}{0-6} = \mu(Say)$$

$$\Longrightarrow \frac{x_1}{0} = \frac{x_2}{9} = \frac{x_3}{-6} = \mu$$

$$\Rightarrow \frac{x_1}{0} = \frac{x_2}{3} = \frac{x_3}{-2} = \mu$$

If $\mu = 1$, then $x_1 = 0, x_2 = 3, x_3 = -2$

So, corresponding Eigen vector is $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$

Not Diagonalizable: For three Eigen values, we have only one linearly independent Eigen vector, so the matrix A is not diagonalizable.