Function of several Real variables

2.2 Functions of Two Variables

Consider the function of two variables

$$z = f(x, y). \tag{2.3}$$

The set of points (x, y) in the x-y plane for which $f(x, y)$ is defined is called the *domain* of definition of the function and is denoted by D. This domain may be the entire x-y plane or a part of the x-y plane. The collection of the corresponding values of z is called the *range* of the function. The following are some examples

https://c3d.libretexts.org/CalcPlot3D/index.html

$z = \sqrt{1 - x^2 - y^2}$: z is real. Therefore, we have $1 - x^2 - y^2 \geq 0$, or $x^2 + y^2 \leq 1$, that is, the domain is the region $x^2 + y^2 \leq 1$. The range is the set of all real, positive numbers.

$z = 1/(x^2 - y^2)$: The domain is the set of all points (x, y) such that $x^2 - y^2 \neq 0$, that is $y \neq \pm x$. The range is $\mathbb{R}$.

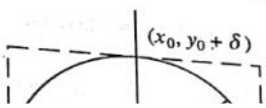
$z = \log(x + y)$: The domain is the set of all points (x, y) such that $x + y > 0$. The range is $\mathbb{R}$.

Distance between two points Let $P(x_0, y_0)$ and $Q(x_1, y_1)$ be any two points in $\mathbb{R}^2$. Then

$$d(P, Q) = |PQ| = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$

is called the distance between the points P and Q.

Neighborhood of a point Let $P(x_0, y_0)$ be a point in $\mathbb{R}^2$. Then the δ-neighborhood of the point $P(x_0, y_0)$ is the set of all points (x, y) which lie inside a circle of radius δ with centre at the point (x_0, y_0), (Fig. 2.2). We usually denote this neighborhood by $N_\delta(P)$ or by $N(P, \delta)$.



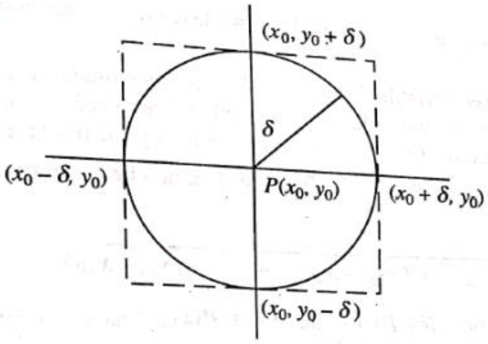


Fig. 2.2. Neighborhood of a point $P(x_0, y_0)$.

Therefore,

$$N_\delta(P) = \left\{(x, y): \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta\right\}. \tag{2.5}$$

Open domain A domain D is open, if for every point P in D, there exists a $\delta > 0$ such that all points in the δ-neighborhood of P are in D.

Connected domain A domain D is connected, if any two points $P, Q \in D$ can be joined by finitely many number of line segments all of which lie entirely in D.

Bounded domain A domain D is bounded, if there exists a real finite positive number M (no matter how large) such that D can be enclosed within a circle with radius M and centre at the origin. That is, the distance of any point P in D from the origin is less than M, $|OP| < M$.

Closed region A closed region is a bounded domain together with its boundary.

Bounded function A function $f(x, y)$ defined in some domain D in $\mathbb{R}^2$ is bounded, if there exists a real finite positive number M such that $|f(x, y)| \le M$ for all $(x, y) \in D$.

2.2.1 Limits

Let $z = f(x, y)$ be a function of two variables defined in a domain D. Let $P(x_0, y_0)$ be a point in D. If for a given real number $\varepsilon > 0$, however small, we can find a real number $\delta > 0$ such that for every point (x, y) in the δ-neighborhood of $P(x_0, y_0)$

$$|f(x, y) - L| < \varepsilon, \quad \text{whenever} \quad \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \tag{2.8}$$

then the real, finite number L is called the limit of the function $f(x, y)$ as $(x, y) \to (x_0, y_0)$. Symbolically, we write it as

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = L.$$

Note that for the limit to exist, the function $f(x, y)$ may or may not be defined at (x_0, y_0). If $f(x, y)$ is not defined at $P(x_0, y_0)$, then we write

Example 2.3 Show that the following limits

(i) $\displaystyle\lim_{(x, y) \to (0, 0)} \frac{xy}{x^2 + y^2}$, (ii) $\displaystyle\lim_{(x, y) \to (0, 0)} \frac{x + \sqrt{y}}{x^2 + y^2}$,

(iii) $\displaystyle\lim_{(x, y) \to (0, 0)} \frac{x^3 y}{x^6 + y^2}$. (iv) $\displaystyle\lim_{(x, y) \to (0, 1)} \tan^{-1}\left(\frac{y}{x}\right)$.

do not exist.

15. $\lim_{(x,y)\to(0,0)} \dfrac{x^2}{x^3 + y^3}$.

16. $\lim_{(x,y)\to(0,0)} \dfrac{x^4 y^2}{(x^4 + y^2)^2}$.

17. $\lim_{(x,y,z)\to(0,0,0)} \log\left(\dfrac{z}{xy}\right)$.

18. $\lim_{(x,y,z)\to(0,0,0)} \dfrac{xy + z}{x + y + z^2}$.

19. $\lim_{(x,y,z)\to(0,0,0)} \dfrac{x y^2 z^2}{x^4 + y^4 + z^8}$.

20. $\lim_{(x,y,z)\to(0,0,0)} \dfrac{x(x + y + z)}{x^2 + y^2 + z^2}$.

$$\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2 + y^2}}.$$

$$\lim_{(x,y)\to(1,-1)} \frac{x^3 - y^3}{x - y}$$

$$\lim_{(x,y)\to(\alpha,0)} \left(1 + \frac{x}{y}\right)^y.$$

$$\lim_{(x,y)\to(0,0)} \cot^{-1}\left(\frac{1}{\sqrt{x^2+y^2}}\right).$$

$$\lim_{(x,y)\to(0,1)} \frac{(y-1)\tan^2 x}{x^2(y^2-1)}.$$

$$\lim_{(x,y)\to(1,0)} \frac{(x-1)\sin y}{y \ln x}.$$

Example 2.1 Using the δ-ε approach, show that

(i) $\lim_{(x,y)\to(2,1)}(3x+4y)=10$, (ii) $\lim_{(x,y)\to(1,1)}(x^2+2y)=3$.

Solution

(i) Here $f(x, y) = 3x + 4y$ is defined at (2, 1). We have

$|f(x,y)-10| = |3x+4y-10| = |3(x-2)+4(y-1)| \le 3|x-2| + 4|y-1|$.

If we take $|x-2| < \delta$ and $|y-1| < \delta$, we get $|f(x,y)-10| < 7\delta < \varepsilon$, which is satisfied when $\delta < \varepsilon/7$.

Hence, $\lim_{(x,y)\to(2,1)} f(x,y) = 10$.

Example 2.2 Using δ-ε approach, show that

(i) $\lim_{(x,y)\to(0,0)}\left(\frac{xy}{\sqrt{x^2+y^2}}\right)=0$, (ii) $\lim_{(x,y,z)\to(0,0,0)}\left(\frac{xy+xz+yz}{\sqrt{x^2+y^2+z^2}}\right)=0$.

2.2.2 Continuity

A function $z=f(x,y)$ is said to be *continuous* at a point (x_0, y_0), if

(i) $f(x,y)$ is defined at the point (x_0, y_0), (ii) $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists, and

(iii) $\lim_{(x,y)\to(x_0,y_0)} f(x,y)=f(x_0,y_0)$.

Example 2.4 Show that the following functions are continuous at the point (0, 0).

(i) $f(x,y)=\begin{cases}\dfrac{2x^4+3y^4}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$ (ii) $f(x,y)=\begin{cases}\dfrac{2x(x^2-y^2)}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0),\end{cases}$

(iii) $f(x,y)=\begin{cases}\dfrac{\sin^{-1}(x+2y)}{\tan^{-1}(2x+4y)}, & (x,y)\neq(0,0)\\ 1/2, & (x,y)=(0,0).\end{cases}$

Example 2.5 Show that the following functions are discontinuous at the given points

(i) $f(x,y)=\begin{cases}\dfrac{x-y}{x+y}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$ at the point (0, 0). (ii) $f(x,y)=\begin{cases}\dfrac{x^2-x\sqrt{y}}{x^2+y}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$ at the point (0, 0).

(iii) $f(x,y)=\begin{cases}\dfrac{x^2+xy+x+y}{x+y}, & (x,y)\neq(2,2)\\ 4, & (x,y)=(2,2)\end{cases}$ at the point (2, 2).

Example 2.6 Let $f(x,y)=\begin{cases}\dfrac{x^4y-3x^2y^3+y^5}{(x^2+y^2)^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0).\end{cases}$

Find a $\delta>0$ such that $|f(x,y)-f(0,0)|<0.01$, whenever $\sqrt{x^2+y^2}<\delta$.

Partial derivatives

$$\lim_{\Delta x \to 0} \frac{\Delta_x z}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}.$$

$$\lim_{\Delta y \to 0} \frac{\Delta_y z}{\Delta y} = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

Example 2.7 Find the first order partial derivatives of the following functions (i) $f(x, y) = x^2 + y^2 + x$, (ii) $f(x, y) = y\, e^{-x}$, (iii) $f(x, y) = \sin(2x + 3y)$ at the point (x, y) from the first principles.

Example 2.9 Show that the function

$$f(x, y) = \begin{cases} \dfrac{x^2 + y^2}{|x| + |y|}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

is continuous at (0, 0) but its partial derivatives f_x and f_y do not exist at (0, 0).

Example 2.10 Show that the function

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + 2y^2}, & (x, y) \neq (0, 0) \\ 0 & , (x, y) = (0, 0) \end{cases}$$

is not continuous at (0, 0) but its partial derivatives f_x and f_y exist at (0, 0).

Total derivatives

For a function of n variables $z = f(x_1, x_2, \ldots, x_n)$, we write the total differential as

$$dz = f_{x_1} dx_1 + f_{x_2} dx_2 + \ldots + f_{x_n} dx_n.$$

Example 2.11 Find the total differential of the following functions

(i) $z = \tan^{-1}(x/y)$, $(x, y) \neq (0, 0)$, (ii) $u = \left(xz + \dfrac{x}{z}\right)^y$, $z \neq 0$.

Derivative of composite and implicit function(Chain Rule)

Suppose the function
$z = f(x_1, x_2, \dots . x_n)$ and $x_1, x_2, \dots x_n$ are functions of t

$$\frac{dz}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt} + \dots + \frac{\partial f}{\partial x_n}\frac{dx_n}{dt}.$$

Example 2.17 Find df/dt at $t = 0$, where

(i) $f(x, y) = x \cos y + e^x \sin y$, $x = t^2 + 1$, $y = t^3 + t$.

(ii) $f(x, y, z) = x^3 + x z^2 + y^3 + xyz$, $x = e^t$, $y = \cos t$, $z = t^3$.

Example 2.18 If $z = f(x, y)$, $x = e^{2u} + e^{-2v}$, $y = e^{-2u} + e^{2v}$, then show that

$$\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} = 2\left[x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y}\right].$$

Solution Using the chain rule, we obtain

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = 2e^{2u}\frac{\partial f}{\partial x} - 2e^{-2u}\frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} = -2e^{-2v}\frac{\partial f}{\partial x} + 2e^{2v}\frac{\partial f}{\partial y}.$$

Therefore,

$$\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} = 2(e^{2u} + e^{-2v})\frac{\partial f}{\partial x} - 2(e^{-2u} + e^{2v})\frac{\partial f}{\partial y}$$

$$= 2x\frac{\partial f}{\partial x} - 2y\frac{\partial f}{\partial y}.$$

Change of Variable

Suppose that $f(x, y)$ is a function of two independent variables x, y and x, y are functions of two new independent variables u, v given by $x = \phi(u, v)$, $y = \psi(u, v)$. By chain rule, we have

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

We want to determine $\partial f/\partial x$, $\partial f/\partial y$ in terms of $\partial f/\partial u$ and $\partial f/\partial v$. Solving the above system of equations by Cramer's rule, we get

$$\frac{\partial f/\partial x}{\frac{\partial f}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial f}{\partial v}\frac{\partial y}{\partial u}} = \frac{\partial f/\partial y}{\frac{\partial f}{\partial v}\frac{\partial x}{\partial u} - \frac{\partial f}{\partial u}\frac{\partial x}{\partial v}} = \frac{1}{\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}}.$$

The determinant

$$J = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \frac{\partial(x, y)}{\partial(u, v)}$$

is called the *Jacobian* of the variables of transformation.

$$\frac{\partial f}{\partial x} = \frac{1}{J}\left[\frac{\partial(f, y)}{\partial(u, v)}\right] \quad \text{and} \quad \frac{\partial f}{\partial y} = -\frac{1}{J}\left[\frac{\partial(f, x)}{\partial(u, v)}\right].$$

$$\frac{\partial f}{dx} = \frac{1}{J}\left[\frac{\partial(f, y, z)}{\partial(u, v, w)}\right] = \frac{1}{J}\begin{vmatrix} \partial f/\partial u & \partial f/\partial v & \partial f/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix}$$

$$\frac{\partial f}{dy} = \frac{1}{J}\left[\frac{\partial(x, f, z)}{\partial(u, v, w)}\right] = -\frac{1}{J}\left[\frac{\partial(f, x, z)}{\partial(u, v, w)}\right] = -\frac{1}{J}\begin{vmatrix} \partial f/\partial u & \partial f/\partial v & \partial f/\partial w \\ \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix}$$

$$\frac{\partial f}{dz} = \frac{1}{J}\left[\frac{\partial(x, y, f)}{\partial(u, v, w)}\right] = \frac{1}{J}\left[\frac{\partial(f, x, y)}{\partial(u, v, w)}\right] = \frac{1}{J}\begin{vmatrix} \partial f/\partial u & \partial f/\partial v & \partial f/\partial w \\ \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \end{vmatrix}$$

$$J = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix}$$

Example 2.19 If $z = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$, then show that

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2$$

Remark

The variables of transformation $u = f(x, y, z)$, $v = g(x, y, z)$, $w = h(x, y, z)$ are functionally related if

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = 0,$$

that is, there exists a relationship between the variables u, v, w and the transformation is not independent.

Example 2.20(b) Show that the variables $u = x - y + z$, $v = x + y - z$, $w = x^2 + xz - xy$, are functionally related. Find the relationship between them.

$u = x^2 - y^2 - z^2$, $v = x^2 - y^2 + z^2$, $w = x^4 + y^4 + z^4 - 2x^2 y^2$;
$u = x + 3z$, $v = x - y - z$, $w = y^2 + 16z^2 + 8yz$.
$u = x + y + z$, $v = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$, $w = x^3 + y^3 + z^3 - 3xyz$.

2.4.1 Homogeneous Functions

A function $f(x, y)$ is said to be *homogeneous* of degree n in x and y, if it can be written in any one of the following forms

(i) $f(\lambda x, \lambda y) = \lambda^n f(x, y)$. (2.46)

(ii) $f(x, y) = x^n g(y/x)$. (2.47)

(iii) $f(x, y) = y^n g(x/y)$. (2.48)

Similarly, a function $f(x, y, z)$ of three variables is said to be homogeneous, of degree n, if it can be written as $f(\lambda x, \lambda y, \lambda z) = \lambda^n f(x, y, z)$, or $f(x, y, z) = x^n g\left(\frac{y}{x}, \frac{z}{x}\right)$ etc.

Some examples of homogeneous functions are the following:

Some examples of homogeneous functions are the following:

f	degree
$x^2 + xy$	
$\tan^{-1}(y/x)$	
$1/(x + y)$	
$1/(x^4 + y^4 + z^4)$	
$xyz/(x^4 + y^4 + z^4)$	
$\sqrt{x}/\sqrt{x^2 + y^2 + z^2}$	

The function $f(x, y)$

An important result concerning homogeneous functions is the following

Theorem 2.4 (Euler's theorem) If $f(x, y)$ is a homogeneous function of degree n in x and y and has continuous first and second order partial derivatives, then

(i)
$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf. \tag{2.49}$$

(ii)
$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x\,\partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f. \tag{2.50}$$

Example 2.26 If $u(x, y) = \cos^{-1}\left(\dfrac{x + y}{\sqrt{x} + \sqrt{y}}\right)$, $0 < x, y < 1$, then prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u.$$

Using Euler's theorem, establish the following results.

29. If $u = \sin^{-1}\left(\dfrac{x^2 + y^2}{x + y}\right)$, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \tan u$.

30. If $u = \log\left[\dfrac{\sqrt{x^2 + y^2}}{x}\right]$, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$.

31. If $u = \sqrt{y^2 - x^2}\,\sin^{-1}\left(\dfrac{x}{y}\right)$, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = u$.

2.5 Maximum and Minimum Values of a Function

Let a function $f(x, y)$ be defined and continuous in some closed and bounded region R. Let (a, b) be an interior point of R and $(a + h, b + k)$ be a point in its neighborhood and lies inside R. We define the following.

(i) The point (a, b) is called a point of *relative* (or *local*) *minimum*, if

$$f(a + h, b + k) \geq f(a, b) \tag{2.67a}$$

for all h, k. Then, $f(a, b)$ is called the *relative* (or *local*) *minimum* value.

(ii) The point (a, b) is called a point of *relative* (or *local*) *maximum*, if

$$f(a + h, b + k) \leq f(a, b) \tag{2.67b}$$

for all h, k. Then $f(a, b)$ is called the *relative* (or *local*) *maximum* value.

A function $f(x, y)$ may also attain its minimum or maximum values on the boundary of the region. The smallest and the largest values attained by a function over the entire region including the boundary are called the *absolute* (or *global*) *minimum* and *absolute* (or *global*) *maximum* values respectively.

The points at which minimum / maximum values of the function occur are also called *points of extrema* or the *stationary points* and the minimum and the maximum values taken together are called the *extreme values* of the function.

Theorem 2.7 (Sufficient conditions for a function to have a minimum/maximum) Let a function $f(x, y)$ be continuous and possess first and second order partial derivatives at a point $P(a, b)$. If $P(a, b)$ is a critical point, then the point P is a point of

$$\textit{relative minimum} \text{ if } rt - s^2 > 0 \text{ and } r > 0 \tag{2.73a}$$

$$\textit{relative maximum} \text{ if } rt - s^2 > 0 \text{ and } r < 0 \tag{2.73b}$$

where $r = f_{xx}(a, b)$, $s = f_{xy}(a, b)$ and $t = f_{yy}(a, b)$.

No conclusion about an extremum can be drawn if $rt - s^2 = 0$ and further investigation is needed. If $rt - s^2 < 0$, then the function f has no minimum or maximum at this point. In this case, the point P is called a *saddle point*.

6. Discuss minimum value of f(x,y)=$x^2 + y^2 + 6x + 12$.

a) 3

b) 3

c) -9

d) 9

2

Test the following functions for relative maximum and minimum.

1. $xy + (9/x) + (3/y)$.
2. $\sqrt{a^2 - x^2 - y^2}$ $a > 0$.
3. $x^2 + 2bxy + y^2$.
4. $x^2 + xy + y^2 + (1/x) + (1/y)$.

Example 2.34 Find the relative maximum and minimum values of the function

$$f(x, y) = 2(x^2 - y^2) - x^4 + y^4.$$

This alternative statement of the Theorem 2.7 is useful when we consider the extreme values of the functions of three or more variables. For example, for the function $f(x, y, z)$ of three variables, we have

$$\mathbf{A} = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix}$$

where $f_{yx} = f_{xy}, f_{zx} = f_{xz}, f_{zy} = f_{yz}$. The matrix **A** or the matrix **B** = – **A** can be tested whether it is positive definite, to find the points of minimum/maximum. Therefore, a critical point (a point at which $f_x = 0 = f_y = f_z$)

(i) is a point of relative minimum if **A** is positive definite and f_{xx}, f_{yy}, f_{zz} are all positive.

(ii) is a point of relative maximum if **B** = – **A** is positive definite (that is, the leading minors of **A** are alternately negative and positive) and f_{xx}, f_{yy}, f_{zz} are all negative.

1. $x^2 + y^2 + z^2 - 2x - 2y - 2z$

2. $(x-1)^2 + (y-1)^2 + (z-1)^2$

3. $2x + 2y + 2z - x^2 - y^2 - z^2$

Lagrange method of Multiplier

2.5.1 Lagrange Method of Multipliers

We want to find the extremum of the function $f(x_1, x_2, \ldots, x_n)$ under the conditions

$$\phi_i(x_1, x_2, \ldots, x_n) = 0, \quad i = 1, 2, \ldots, k. \qquad (2.75)$$

We construct an auxiliary function of the form

$$F(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_k) = f(x_1, x_2, \ldots, x_n) + \sum_{i=1}^{k} \lambda_i \phi_i(x_1, x_2, \ldots, x_n) \qquad (2.76)$$

where λ_i's are undetermined parameters and are known as *Lagrange multipliers*. Then, to determine the stationary points of F, we have the necessary conditions

$$\frac{\partial F}{\partial x_1} = 0 = \frac{\partial F}{\partial x_2} = \ldots = \frac{\partial F}{\partial x_n}$$

which give the equations

22. Find the smallest and the largest value of xy on the line segment $x + 2y = 2, x \geq 0, y \geq 0$.

23. Find the smallest and the largest value of $x + 2y$ on the circle $x^2 + y^2 = 1$.

Ex.

Find the extreme values of $f(x, y, z) = 2x + 3y + z$ such that $x^2 + y^2 = 5$ and $x + z = 1$.