**Topic:**

Solution of Non-Homogeneous LDE with Variable coefficients.

**Learning Outcomes:**

To convert equation with variable coefficients (Euler-Cauchy Equation) to an equation with constant coefficients and then solve it with standard known methods.

**Euler-Cauchy Equation:**

Let us consider 2nd order Non-homogeneous LDE with variable coefficients as:

$x^2y'' + xy' + y = r(x)$ (1)

Equation of type (1) called Euler-Cauchy Equation.

S.F. : $(x^2D^2 + xD + 1)y = r(x)$ (2) where $D \equiv \frac{d}{dx}$

The very first job is to convert this equation with variable coefficients to an equation with constant coefficients using an appropriate transformation.

Let transformation be: $x = e^t \quad \Rightarrow t = \log x$

Let $\theta \equiv \frac{d}{dt}$ ( Another differential operator)

then $xD = \theta,\ x^2D^2 = \theta(\theta - 1),\ x^3D^3 = \theta(\theta - 1)(\theta - 2)$ and so on…

Equation (2) becomes: $[\theta(\theta - 1) + \theta + 1]y = r(t)$ (3)

which is an equation with constant coefficients and we know the methods to solve equ.(3)

**Problem 1.** Find the general solution of: $x^2y'' + xy' - 4y = 0$

**Solution:** The given equation is:

$x^2y'' + xy' - 4y = 0$ (1)

S.F. : $(x^2D^2 + xD - 4)y = 0$ (2) where $D \equiv \frac{d}{dx}$

Let transformation be: $x = e^t \quad \Rightarrow t = \log x$

then $xD = \theta,\ x^2D^2 = \theta(\theta - 1)$ where $\theta \equiv \frac{d}{dt}$

Equation (2) becomes: $[\theta(\theta - 1) + \theta - 4]y = 0$

A.E. : $[\theta(\theta - 1) + \theta - 4] = 0 \quad \Rightarrow [\theta^2 - \theta + \theta - 4] = 0$

$\Rightarrow (\theta^2 - 4) = 0 \quad \Rightarrow \theta = 2, -2$ (real and unequal roots)

$\therefore$ General solution is given by:

$\Rightarrow y = c_1e^{2t} + c_2e^{-2t} \quad \Rightarrow y = c_1x^2 + c_2x^{-2}$ **Answer.**

**Problem 2.** Find the general solution of: $9x^2y'' + 15xy' + y = 0$

**Solution:** The given equation is:

$9x^2y'' + 15xy' + y = 0$ (1)

S.F. : $(9x^2D^2 + 15xD + 1)y = 0$ (2) where $D \equiv \frac{d}{dx}$

Let transformation be: $x = e^t \quad \Rightarrow t = \log x$

then $xD = \theta,\ x^2D^2 = \theta(\theta - 1)$ where $\theta \equiv \frac{d}{dt}$

Equation (2) becomes: $[9\theta(\theta - 1) + 15\theta + 1]y = 0$

A.E. : $[9\theta(\theta - 1) + 15\theta + 1] = 0 \quad \Rightarrow [9\theta^2 - 9\theta + 15\theta + 1] = 0$

$\Rightarrow (9\theta^2 + 6\theta + 1) = 0 \quad \Rightarrow (3\theta + 1)(3\theta + 1) = 0 \quad \Rightarrow \theta = -\frac{1}{3}, -\frac{1}{3}$ (equal roots)

$\therefore$ General solution is given by:

$\Rightarrow y = (c_1 + c_2 t)e^{-\frac{1}{3}t} \quad \Rightarrow y = (c_1 + c_2 \log x)\, x^{-\frac{1}{3}}$ **Answer.**

## Polling Quiz

The transformation used to convert a LDE with variable coefficients to LDE with constant coefficients is:

(A) $t = e^x$

(B) $x = e^t$

(C) None of these.

**Problem 3.** Find the general solution of: $2x^2y'' + 2xy' + y = 0$

**Solution:** The given equation is:

$$2x^2y'' + 2xy' + y = 0 \qquad (1)$$

S.F. : $(2x^2D^2 + 2xD + 1)y = 0 \qquad (2) \qquad$ where $D \equiv \frac{d}{dx}$

Let transformation be: $x = e^t \quad \Rightarrow t = \log x$

then $xD = \theta,\ x^2D^2 = \theta(\theta - 1) \qquad$ where $\theta \equiv \frac{d}{dt}$

Equation (2) becomes: $[2\theta(\theta - 1) + 2\theta + 1]y = 0$

A.E. : $[2\theta(\theta - 1) + 2\theta + 1] = 0 \quad \Rightarrow [2\theta^2 - 2\theta + 2\theta + 1] = 0$

$\Rightarrow (2\theta^2 + 1) = 0 \quad \Rightarrow \theta^2 = -\frac{1}{2} \quad \Rightarrow \theta = \frac{1}{\sqrt{2}}i, -\frac{1}{\sqrt{2}}i$ (complex roots)

$\therefore$ General solution is given by:

$\Rightarrow y = e^{0t}(c_1 \cos\frac{1}{\sqrt{2}}t + c_2 \sin\frac{1}{\sqrt{2}}t) \quad \Rightarrow y = (c_1 \cos\frac{1}{\sqrt{2}}\log x + c_2 \sin\frac{1}{\sqrt{2}}\log x)$ **Ans.**

**Problem 4.** Find the general solution of: $x^2y'' - 2y = 2x$

**Solution:** The given equation is:

$x^2y'' - 2y = 2x$ (1)

S.F. : $(x^2D^2 - 2)y = 2x$ (2) where $D \equiv \frac{d}{dx}$

Let transformation be: $x = e^t \quad \Rightarrow t = \log x$

then $xD = \theta,\ x^2D^2 = \theta(\theta - 1)$ where $\theta \equiv \frac{d}{dt}$

Equation (2) becomes: $[\theta(\theta - 1) - 2]y = 2e^t$

$\Rightarrow f(\theta)y = r(t)$ where $f(\theta) = (\theta^2 - \theta - 2)$ and $r(t) = 2e^t$

To find Complimentary Function (C.F.):

A.E. : $f(\theta) = 0 \quad \Rightarrow (\theta^2 - \theta - 2) = 0 \quad \Rightarrow (\theta - 2)(\theta + 1) = 0$

$\Rightarrow \theta = 2, -1$ (real and unequal roots)

$\therefore$ Complimentary function is given by:

$\Rightarrow y_c = c_1e^{2t} + c_2e^{-t} \quad \Rightarrow y_c = c_1x^2 + c_2x^{-1}$

To find Particular Integral (P.I.):

$$y_p = \frac{1}{f(\theta)} r(t) = \frac{1}{(\theta^2-\theta-2)}(2e^t)$$

$$\Rightarrow y_p = 2\left[\frac{1}{(\theta^2-\theta-2)} e^t\right] \quad \Rightarrow y_p = 2\left[\frac{1}{((1)^2-(1)-2)} e^t\right] \quad (Put\ \theta = 1)$$

$$\Rightarrow y_p = 2\left[\frac{1}{-2} e^t\right] \quad \Rightarrow y_p = -e^t = -x$$

∴ General solution is given by: $y$ = C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = (c_1x^2 + c_2x^{-1}) - x$ **Answer.**

**Problem 5.** Find the general solution of: $x^2y'' + 2xy' = \cos(\log x)$

**Solution:** The given equation is:

$x^2y'' + 2xy' = \cos(\log x)$ (1)

S.F. : $(x^2D^2 + 2xD)y = \cos(\log x)$ (2) where $D \equiv \frac{d}{dx}$

Let transformation be: $x = e^t \quad \Rightarrow t = \log x$

then $xD = \theta,\ x^2D^2 = \theta(\theta - 1)$ where $\theta \equiv \frac{d}{dt}$

Equation (2) becomes: $[\theta(\theta - 1) + 2\theta]y = \cos t$

$\Rightarrow f(\theta)y = r(t)$ where $f(\theta) = (\theta^2 + \theta)$ and $r(t) = \cos t$

To find Complimentary Function (C.F.):

A.E. : $f(\theta) = 0 \quad \Rightarrow (\theta^2 + \theta) = 0 \quad \Rightarrow \theta(\theta + 1) = 0$

$\Rightarrow \theta = 0, -1$ (real and unequal roots)

$\therefore$ Complimentary function is given by:

$\Rightarrow y_c = c_1e^{0t} + c_2e^{-t} \quad \Rightarrow y_c = c_1 + c_2x^{-1}$

To find Particular Integral (P.I.):

$$y_p = \frac{1}{f(\theta)} r(t) = \frac{1}{(\theta^2+\theta)}(\cos t)$$

$$\Rightarrow y_p = \left[\frac{1}{(-(1)^2+\theta)}(\cos t)\right] \qquad (Put\ \theta^2 = -(1)^2)$$

$$\Rightarrow y_p = \left[\frac{1}{(\theta-1)}(\cos t)\right] = \left[\frac{1}{(\theta-1)} \times \frac{(\theta+1)}{(\theta+1)}(\cos t)\right] = \left[\frac{(\theta+1)}{(\theta^2-1)}(\cos t)\right]$$

$$\Rightarrow y_p = \left[\frac{(\theta+1)}{(-1-1)}(\cos t)\right] = -\frac{1}{2}\left[\frac{d}{dt}(\cos t) + (\cos t)\right] = -\frac{1}{2}(-\sin t + \cos t)$$

$$\Rightarrow y_p = -\frac{1}{2}[-\sin(\log x) + \cos(\log x)]$$

$\therefore$ General solution is given by: $y =$ C.F. +P.I.

i.e. $y = y_c + y_p$

$$\Rightarrow y = c_1 + c_2 x^{-1} - \frac{1}{2}[-\sin(\log x) + \cos(\log x)]$$ **Answer.**

# Polling Quiz

Which of the following is not an Euler-Cauchy differential equation:

(A) $x^2y'' + xy' - 4y = 0$

(B) $x^2y'' - 2xy = 2x$

(C) $x^2y'' - 3xy' + 3y = 2 + 3\log x$

**Problem 6.** Find the general solution of: $x^2y'' - 3xy' + 3y = 2 + 3\log x$

**Solution:** The given equation is:

$x^2y'' - 3xy' + 3y = 2 + 3\log x$ (1)

S.F. : $(x^2D^2 - 3xD + 3)y = 2 + 3\log x$ (2) where $D \equiv \frac{d}{dx}$

Let transformation be: $x = e^t \quad \Rightarrow t = \log x$

then $xD = \theta,\ x^2D^2 = \theta(\theta - 1)$ where $\theta \equiv \frac{d}{dt}$

Equation (2) becomes: $[\theta(\theta - 1) - 3\theta + 3]y = 2 + 3t$

$\Rightarrow f(\theta)y = r(t)$ where $f(\theta) = (\theta^2 - 4\theta + 3)$ and $r(t) = 2 + 3t$

To find Complimentary Function (C.F.):

A.E. : $f(\theta) = 0 \quad \Rightarrow (\theta^2 - 4\theta + 3) = 0 \quad \Rightarrow (\theta - 1)(\theta - 3) = 0$

$\Rightarrow \theta = 1, 3$ (real and unequal roots)

$\therefore$ Complimentary function is given by:

$\Rightarrow y_c = c_1e^t + c_2e^{3t} \quad \Rightarrow y_c = c_1x + c_2x^3$

To find Particular Integral (P.I.):

$$y_p = \frac{1}{f(\theta)} r(t) = \frac{1}{(\theta^2-4\theta+3)}(2+3t)$$

$$\Rightarrow y_p = \left[\frac{1}{3\left[1+\left(\frac{\theta^2-4\theta}{3}\right)\right]}(2+3t)\right] = \frac{1}{3}\left[\left(1+\left(\frac{\theta^2-4\theta}{3}\right)\right)^{-1}(2+3t)\right]$$

$$\Rightarrow y_p = \frac{1}{3}\left[\left(1-\left(\frac{\theta^2-4\theta}{3}\right)+\left(\frac{\theta^2-4\theta}{3}\right)^2 - - -\right)(2+3t)\right]$$

$$\Rightarrow y_p = \frac{1}{3}\left[(2+3t)+\frac{4}{3}\frac{d}{dt}(2+3t)+0\right] = \frac{1}{3}[(2+3t)+4] = 2+t$$

$$\Rightarrow y_p = 2+\log x$$

$\therefore$ General solution is given by: $y =$ C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = c_1 x + c_2 x^3 + 2 + \log x$ **Answer.**