Determinant

Determinant of second order

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 \quad \text{or} \quad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

$$\begin{vmatrix} 4 & 6 \\ 2 & 5 \end{vmatrix} \qquad \begin{vmatrix} 8 & 5 \\ 3 & 1 \end{vmatrix} \qquad \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$$

$$\begin{vmatrix} -3 & 7 \\ 2 & 4 \end{vmatrix} \qquad \begin{vmatrix} 5 & -2 \\ 4 & 3 \end{vmatrix}$$

MINOR: The minor of an element is defined as a determinant obtained by deleting the row and column containing the element.
For ex.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Thus the minors a_1, b_1 and c_1 are respectively.

$$\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, \qquad \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} \text{ and } \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

So the determinant can be found as follows

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 (\text{minor of } a_1) - b_1 (\text{minor of } b_1) + c_1 (\text{minor of } c_1).$$

COFACTOR:

$$\text{Cofactor} = (-1)^{r+c} * \text{Minor}$$

where r is the number of rows of the element and c is the number of columns of the element. The cofactor of any element of jth row and ith column is $(-1)^{i+j}$ *minor

Ex. Find the Minors and cofactors of first row

$$\begin{vmatrix} 2 & 3 & 5 \\ 4 & 1 & 0 \\ 6 & 2 & 7 \end{vmatrix}$$

Ex. Find the determinant of $\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 2 & 0 \\ 2 & 0 & 1 & 3 \\ 1 & 2 & 1 & 0 \end{vmatrix}$

PROPERTIES OF DETERMINANTS

1. The value of a determinant remains unaltered, if the rows are interchanged into columns

2. If two rows (or two columns) of a determinant are interchanged, the sign of the value of the determinant changes.

3. If two rows (or columns) of a determinant are identical, the value of the determinant is zero.

4. If the elements of any row (or column) of a determinant be each multiplied by the same number, the determinant is multiplied by that number.

5. The value of the determinant remains unaltered if to the elements of one row (or column) be added any constant multiple of the corresponding elements of any other row (or column) respectively.

6. If each element of a row (or column) of a determinant consists of the algebraic sum of n terms, the determinant can be expressed as the sum of n determinants.

Application of determinant: To find the area of the triangle

Ex. Using determinants, find the area of the triangle with vertices (– 3, 5), (3, – 6) and (7, 2).

Ex. Using determinants, show that the points (11, 7), (5, 5) and (– 1, 3) are collinear

RANK OF A MATRIX

The rank of a matrix is said to be r if

(a) It has at least one non-zero minor of order r.

(b) Every minor of A of order higher than r is zero.

Find the rank of the matrix $\begin{pmatrix} 1 & 5 \\ 3 & 9 \end{pmatrix}$

Find the rank of the matrix $\begin{pmatrix} -5 & -7 \\ 5 & 7 \end{pmatrix}$

Find the rank of the matrix $\begin{pmatrix} 0 & -1 & 5 \\ 2 & 4 & -6 \\ 1 & 1 & 5 \end{pmatrix}$.

Find the rank of the matrix $\begin{pmatrix} 5 & 3 & 0 \\ 1 & 2 & -4 \\ -2 & -4 & 8 \end{pmatrix}$

Find the Rank of using echelon form

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{pmatrix}$$

Matrix A	Elementary Transformation
$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{pmatrix}$	$R_2 \to R_2 - 2R_1$ $R_3 \to R_3 - 3R_1$
$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$	$R_3 \to R_3 - R_2$
The above matrix is in echelon form	

Find the rank of the matrix $A = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{pmatrix}$

Matrix A	Elementary Transformation
$A = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{pmatrix}$	
$A = \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 3 & 1 & 1 & 3 \end{pmatrix}$	$R_1 \leftrightarrow R_2$
$\sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & -5 & -8 & -3 \end{pmatrix}$	$R_3 \rightarrow R_3 - 3R_1$
$\sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 2 \end{pmatrix}$	$R_3 \rightarrow R_3 + 5R_2$

Note: (i) Non-zero row is that row in which all the elements are not zero.

(ii) The rank of the product matrix AB of two matrices A and B is less than the rank of either of the matrices A and B.

Linearly dependence and independence of vectors:

Vectors (matrices) $X_1, X_2, \ldots . X_n$ are said to be dependent if

(1) all the vectors (row or column matrices) are of the same order.

(2) n scalars $C_1, C_2, \ldots C_n$ (not all zero) exist such that

$$C_1X_1 + C_2X_2 + \cdots + C_nX_n = 0$$

Otherwise they are linearly independent.

Find whether or not the following set of vectors are linearly dependent or independent:

(i) $(1, -2), (2, 1)$

(ii) $(1, -2), (2, 1), (3, 2)$

(iii)(1, 1, 1, 1), (0, 1, 1, 1), (0, 0, 1, 1), (0, 0, 0, 1).

LINEARLY DEPENDENCE AND INDEPENDENCE OF VECTORS BY DETERMINANT:

If the determinant of the these $X_1.X_2, X_3, \dots, X_n$ is zero then they are dependent otherwise independent.

Ex. (i) (1, −2), (2, 1)

(ii)(1,2,1),(2,1,1)(3,3,2)

(iii)(3,0,1)(1,2,1)(2,-2,0)

(iv) (1, 1, 1, 1), (0, 1, 1, 1), (0, 0, 1, 1), (0, 0, 0, 1).

LINEARLY DEPENDENCE AND INDEPENDENCE OF VECTORS BY RANK METHOD:

1. If the rank of the matrix of the given vectors is equal to number of vectors, then the vectors are linearly independent.

2. If the rank of the matrix of the given vectors is less than the number of vectors, then the vectors are linearly dependent.

Ex. The following vectors are linearly independent or not

$$X_1 = (2,2,1)^T, X_2 = (1,3,1)^T, X_3 = (1,2,2)^T$$

Ex. Show using a matrix that the set of vectors
X = [1, 2, – 3, 4], Y = [3, – 1, 2, 1] , Z = [1, –5, 8, –7] is linearly dependent.