Symmetric Matrix: (1)

A Square matrix A is said to be Symmetric if $A^T = A$

i.e. If $A = [a_{ij}]_{n\times n}$

Then, $A^T = [a_{ji}]_{n\times n}$ where $a_{ij} = a_{ji}$

e.g. (1) $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a Symmetric matrix

(2) $\begin{bmatrix} 1 & -1 & 4 \\ -1 & 2 & 0 \\ 4 & 0 & 3 \end{bmatrix}$ is a Symmetric matrix.

Skew Symmetric Matrix:-

A Square matrix A is said to be Skew Symmetric if $A^T = -A$

i.e. If $A = [a_{ij}]_{n\times n}$ Then $A^T = [a_{ji}]_{n\times n}$

Where $a_{ij} = -a_{ji}$

e.g. $\begin{bmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{bmatrix}$ is a Skew-Symmetric matrix

Note that ~~the~~ all the principle diagonal elements of a Skew Symmetric matrix are always zero.

As $a_{ij} = -a_{ji}$ put $i=j$, we have,
$a_{ii} = -a_{ii} \Rightarrow 2a_{ii} = 0 \Rightarrow \boxed{a_{ii} = 0}$.

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Remarks:

Properties of Symmetric and Skew Symmetric matrices

(1) The matrix which is both symmetric and skew symmetric is a null matrix.

(2) For any real square matrix A,
$A+A^T$, AA^T, A^TA are symmetric matrices, and $A-A^T$, is skew symmetric.

(3) Every square matrix A can be uniquely expressed as a sum of symmetric and skew symmetric matrices. As $A = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T)$.

(4) If A and B are symmetric matrices of same order then, $A \pm B$ ~~symmetric~~ $AB+BA$, are symmetric.
Also, AB is symmetric iff $AB=BA$.
and $AB-BA$ is a skew-symmetric matrix.

(5) If A and B are skew symmetric matrices of same order then $A \pm B$, $AB-BA$ are skew-symmetric and $AB+BA$ is symmetric

(6) If A is a symmetric matrix and k is scalar then kA is also a symmetric matrix.

(7) If A & B are symmetric matrices and commute i.e. $AB=BA$ then $A^{-1}B$, AB^{-1} and $A^{-1}B^{-1}$ are symmetric.

(8) If A is symmetric then A^n, $n \in N$ is symmetric.

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Ex:- Express the matrix $A = \begin{bmatrix} 2 & -4 & 9 \\ 14 & 7 & 13 \\ 9 & 5 & 11 \end{bmatrix}$ as a Sum of a Symmetric and Skew Symmetric matrices.

Sol We have, $A = \begin{bmatrix} 2 & -4 & 9 \\ 14 & 7 & 13 \\ 9 & 5 & 11 \end{bmatrix}$.

$$\therefore A^T = \begin{bmatrix} 2 & 14 & 9 \\ -4 & 7 & 5 \\ 9 & 13 & 11 \end{bmatrix}$$

Now, $$A + A^T = \begin{bmatrix} 2 & -4 & 9 \\ 14 & 7 & 13 \\ 9 & 5 & 11 \end{bmatrix} + \begin{bmatrix} 2 & 14 & 9 \\ -4 & 7 & 5 \\ 9 & 13 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 10 & 18 \\ 10 & 14 & 18 \\ 18 & 18 & 22 \end{bmatrix}$$

$$\Rightarrow P = \frac{1}{2}(A + A^T) = \frac{1}{2}\begin{bmatrix} 4 & 10 & 18 \\ 10 & 14 & 18 \\ 18 & 18 & 22 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 9 \\ 5 & 7 & 9 \\ 9 & 9 & 11 \end{bmatrix}$$

which is Symmetric.

Also, $$A - A^T = \begin{bmatrix} 2 & -4 & 9 \\ 14 & 7 & 13 \\ 9 & 5 & 11 \end{bmatrix} - \begin{bmatrix} 2 & 14 & 9 \\ -4 & 7 & 5 \\ 9 & 13 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -18 & 0 \\ 18 & 0 & 8 \\ 0 & -8 & 0 \end{bmatrix}$$

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$\Rightarrow Q = \frac{1}{2}(A - A^T)$

$$= \frac{1}{2}\begin{bmatrix} 0 & -18 & 0 \\ 18 & 0 & 8 \\ 0 & -8 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -9 & 0 \\ 9 & 0 & 4 \\ 0 & -4 & 0 \end{bmatrix}$$

Which is Skew Symmetric.

Now, $P + Q = \begin{bmatrix} 2 & 5 & 9 \\ 5 & 7 & 9 \\ 9 & 9 & 11 \end{bmatrix} + \begin{bmatrix} 0 & -9 & 0 \\ 9 & 0 & 4 \\ 0 & -4 & 0 \end{bmatrix}$

H/W

Q. Express the matrix $A = \begin{bmatrix} 2 & 4 & 6 \\ 2 & 2 & 0 \\ 0 & 4 & 8 \end{bmatrix}$ Ans.

As a Sum of Symmetric and a Skew Symmetric matrix.

Orthogonal Matrix

A Square matrix A is Said to be Orthogonal If $AA^T = A^TA = I$.

Properties:

1. If A and B are Orthogonal matrices Then AB and BA are orthogonal.
2. Inverse and transpose of an Orthogonal matrix is orthogonal.
3. A Square matrix A is Orthogonal iff
   → Sum of Squares of elements in each row (Column) is equal to 1
   → Sum of the products of Corresponding elements in two distinct rows (Columns) is zero.

Q. Show that $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is orthogonal.

Sol We have, $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$\therefore A^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

$\Rightarrow AA^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \because \cos^2\theta + \sin^2\theta = 1.$

$= I.$

Also, $A^TA = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= I.$

Thus, $AA^T = A^TA = I$.

$\Rightarrow A$ is orthogonal matrix.

Q H/W Show that $A = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ is Orthogonal.

(6)

Complex matrix

A matrix whose elements are Complex numbers is called a Complex matrix.

Conjugate of a Complex matrix:

A matrix obtained from matrix A on replacing its elements by the Corresponding Conjugate Complex numbers is called Conjugate of a Complex matrix. It is denoted by $\bar{A}$.

i.e. If $A = [a_{ij}]_{m\times n}$,

$\bar{A} = [\bar{a}_{ij}]_{m\times n}$ where $\bar{a}_{ij}$ is the Conjugate of a_{ij}.

e.g. If $A = \begin{bmatrix} 2+3i & 5 \\ 6-7i & -5+i \end{bmatrix}$

Then $\bar{A} = \begin{bmatrix} 2-3i & 5 \\ 6+7i & -5-i \end{bmatrix}$

Properties:

If $\bar{A}$ and $\bar{B}$ are the Conjugate matrices of A and B respectively, Then

① $\overline{(\bar{A})} = A$ ② $\overline{A+B} = \bar{A}+\bar{B}$ ③ $\overline{(kA)} = \bar{k}\bar{A}$

where k is some Complex Constant.

Tranjugate of a matrix:

The transpose of a Conjugate matrix is called tranjugate or transposed Conjugate of matrix. It is denoted by A^θ or A^*.

Properties: If A^θ and B^θ are the Transposed Conjugates of A and B respectively, Then

(1) $(A^\theta)^\theta = A$ (2) $(A+B)^\theta = A^\theta + B^\theta$

(3) $(A-B)^\theta = A^\theta - B^\theta$ (4) $(kA)^\theta = \bar{k} A^\theta$ (5) $(AB)^\theta = B^\theta A^\theta$

Hermitian Matrix

A Square matrix A is said to be Hermitian If $A^\theta = A$.

e.g $A = \begin{bmatrix} 4 & 1+3i \\ 1-3i & 7 \end{bmatrix}$.

Then, $\bar{A} = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$

and so, $A^\theta = (\bar{A})^T$
$= \begin{bmatrix} 4 & 1+3i \\ 1-3i & 7 \end{bmatrix} = A$.

$\Rightarrow$ A is Hermitian matrix.

Properties: (1) In Hermitian matrix, the principal diagonal elements are real.

(2) The Hermitian matrix over the field of real numbers is nothing but real symmetric matrix

(3) In Hermitian matrix, $A = [a_{ij}]_{n\times n}$, $a_{ij} = \overline{a_{ji}}$ $\forall ij$

(4) If A is Θ Hermitian matrix, Then A^n is always Hermitian matrix.

(5) AB is Hermitian iff A & B Comutes

(6) $A + A^\theta$ is Hermitian matrix.

Skew- Hermitian Matrix:

A Square matrix A is said to be Skew Hermitian if $A^\theta = -A$

e.g let $A = \begin{bmatrix} -3i & 2+i \\ -2+i & -i \end{bmatrix}$

Then $\bar{A} = \begin{bmatrix} 3i & 2-i \\ -2-i & i \end{bmatrix}$

and so, $A^\theta = (\bar{A})^T = \begin{bmatrix} 3i & -2-i \\ 2-i & i \end{bmatrix}$

$= - \begin{bmatrix} -3i & 2+i \\ -2+i & -i \end{bmatrix}$

$= -A.$

$\Rightarrow$ A is Skew- Hermitian matrix.

Properties:- (1) In Skew Hermitian matrix, the principal diagonal elements are either zero or purely imaginary

(2) The Skew Hermitian matrix Over the field of real numbers is nothing but real Skew-Symmetric matrix.

(3) In Skew-Symmetric matrix, $A = [a_{ij}]_{n\times n}$, $a_{ij} = -\overline{a_{ji}}$ $\forall i,j$

(4) If A is Skew-Hermitian Then kA is Skew Hermitian.

(5) $A - A^\theta$ is Skew-Hermitian matrix.

(6) Every Square matrix Can be Uniquely expressed as a Sum of Hermitian and Skew-Hermitian matrices.

As $A = \frac{1}{2}(A+A^\theta) + \frac{1}{2}(A-A^\theta)$

(7) If A is Hermitian matrix, Then iA is Skew Hermitian matrix.

(8) If A and B are Hermitian matrix, Then $AB-BA$ is a Skew Hermitian matrix.

Q If $A = \begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$ Then Show That A is Hermitian and iA is Skew Hermitian.

Sol. We have, $A = \begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$

Then, $\bar{A} = \begin{bmatrix} 3 & 7+4i & -2-5i \\ 7-4i & -2 & 3-i \\ -2+5i & 3+i & 4 \end{bmatrix}$

and so, $A^\theta = (\bar{A})^T = \begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix} = A$

$\Rightarrow A$ is Hermitian matrix.

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Also, Let $B = iA$

$$= i\begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3i & 4+7i & -5-2i \\ -4+7i & -2i & -1+3i \\ 5-2i & 1+3i & 4i \end{bmatrix}$$

Then, $$\bar{B} = \begin{bmatrix} -3i & 4-7i & -5+2i \\ -4-7i & 2i & -1-3i \\ 5+2i & 1-3i & -4i \end{bmatrix}$$

$$\Rightarrow B^\theta = (\bar{B})^T = \begin{bmatrix} -3i & -4-7i & 5+2i \\ 4-7i & 2i & 1-3i \\ -5+2i & -1-3i & -4i \end{bmatrix}$$

$$= (-1)\begin{bmatrix} 3i & 4+7i & -5-2i \\ -4+7i & -2i & -1+3i \\ 5-2i & 1+3i & 4i \end{bmatrix}$$

$$= -B$$

$\Rightarrow B = iA$ is Skew Hermitian matrix.

Note: Every Square matrix Can be Uniquely expressed as $P+iQ$ where P and Q are Hermitian matrices.

Unitary Matrix.

A Square matrix A is said to be Unitary matrix if $AA^{\theta} = A^{\theta}A = I$

e.g. $A = \frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}$

We have, $\bar{A} = \frac{1}{6}\begin{bmatrix} -4 & -2+4i \\ 2+4i & -4 \end{bmatrix}$.

$A^{\theta} = (\bar{A})^T = \frac{1}{6}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}$.

Now, $AA^{\theta} = \frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix} \cdot \frac{1}{6}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}$.

$$= \frac{1}{36}\begin{bmatrix} (-4)(-4)+(-2-4i)(-2+4i) & (-4)(2+4i)+(-2-4i)(-4) \\ (2-4i)(-4)+(-4)(-2+4i) & (2-4i)(2+4i)+(-4)(-4) \end{bmatrix}$$

$$= \frac{1}{36}\begin{bmatrix} 16+4-8i+8i+16 & -8-16i+8+16i \\ -8+16i+8-16i & 4+16+16 \end{bmatrix}$$

$$= \frac{1}{36}\begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I$$

IIIy, $A^{\theta}A = I$.

Hence, A is Unitary matrix.

Q. Show that $A = \begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}$ is Unitary iff $a^2+b^2+c^2+d^2=1$.

Sol We have, $A = \begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}$

Then, $\bar{A} = \begin{bmatrix} a-ic & -b-id \\ b-id & a+ic \end{bmatrix}$

Hence, $A^{\theta} = (\bar{A})^T = \begin{bmatrix} a-ic & b-id \\ -b-id & a+ic \end{bmatrix}$

$\therefore AA^{\theta} = \begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix} \begin{bmatrix} a-ic & b-id \\ -b-id & a+ic \end{bmatrix}$

$= \begin{bmatrix} a^2+b^2+c^2+d^2 & 0 \\ 0 & a^2+b^2+c^2+d^2 \end{bmatrix}$.

Now, $AA^{\theta} = I$

ie, $\begin{bmatrix} a^2+b^2+c^2+d^2 & 0 \\ 0 & a^2+b^2+c^2+d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

iff $a^2+b^2+c^2+d^2=0$.

Properties: (1) The inverse of a Unitary matrix is Unitary.

(2) If A is Unitary matrix, then $A^{\theta} = A^{-1}$.