UNIT 4 . Fourier Series

(**Book:** Advanced Engineering Mathematics By Jain and Iyengar, **Chapter-9**)

Learning Outcomes:

1. To know about Fourier Series

2. To Know about Euler’s coefficients

3. To find Fourier series for certain functions.

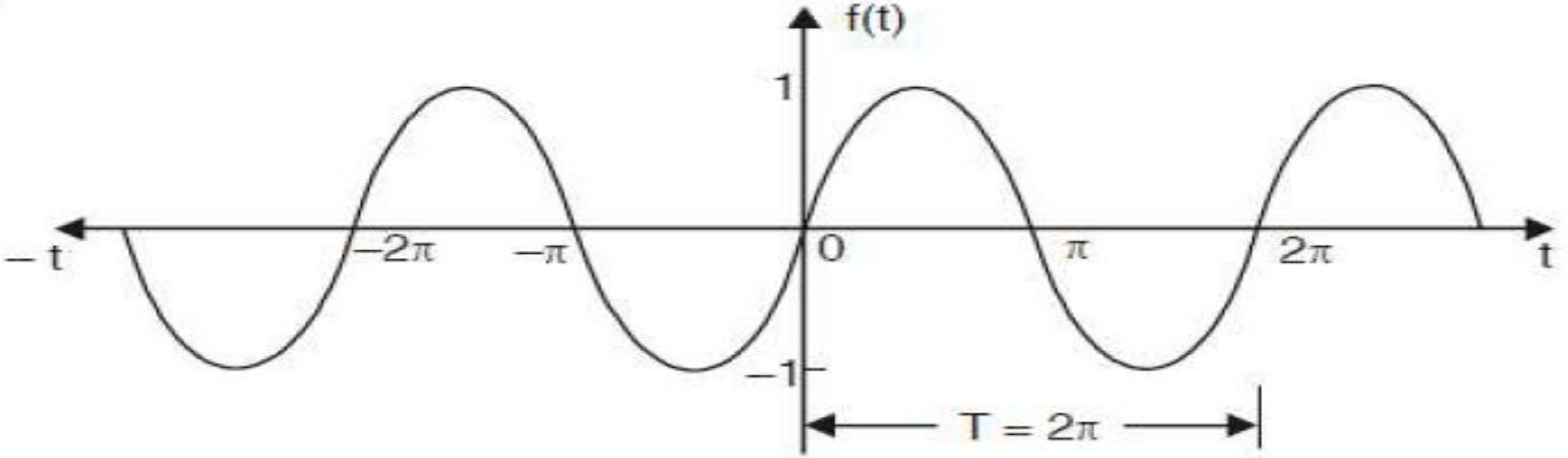
PERIODIC FUNCTIONS

If the value of each ordinate $f(t)$ repeats itself at equal intervals in the abscissa, then said to be a periodic function.

If $f(t) = f(t + T) = f(t + 2T) =$ .... then T is called the period of the function

For example :

$\sin x = \sin(x + 2\pi) = \sin(x + 4\pi) = \ldots$ so $\sin x$ is a periodic function with the period 2π is also called sinusoidal periodic function.



Polling Quiz

Which of the following is a periodic function with period π.

(A) $\sin x$

(B) $\cos x$

(C) $\tan x$

(D) I don’t Know.

Fourier Series

A Fourier series is an expansion of a periodic function in terms of an infinite sum of sines and cosines.

Let $f(x)$ be a periodic function of period $2l$ defined on the interval $[c, 2l+c]$ that is $f(x+2l) = f(x)$.

Then, the corresponding Fourier series is written as:

$$f(x) = \frac{a_0}{2} + \left[a_1 \cos\left(\frac{\pi x}{l}\right) + a_2 \cos\left(\frac{2\pi x}{l}\right) + \cdots\right] + \left[b_1 \sin\left(\frac{\pi x}{l}\right) + b_2 \sin\left(\frac{2\pi x}{l}\right) + \cdots\right]$$

$$\boldsymbol{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]}$$

Euler’s coefficients:

Fourier series of periodic function $f(x)$ defined on the interval $[c, 2l + c]$ is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]$$

Where a_0, a_n, b_n are called Euler’s coefficients.

$$a_0 = \frac{1}{l}\int_{c}^{2l+c} f(x)dx$$

$$a_n = \frac{1}{l}\int_{c}^{2l+c} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_{c}^{2l+c} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Problem 1. Find the Fourier series expansion of the function:

$$f(x) = k, x \in [0,2l].$$

Solution. Here $f(x) = k, x \in [0,2l]$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)dx$$

$$= \frac{1}{l}\int_0^{2l} kdx$$

$$= 2k$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\, cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{1}{l}\int_0^{2l} k\, cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{k}{l}\left[\frac{sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)}\right]_0^{2l}$$

$$= \frac{k}{n\pi}\left[\sin(2n\pi) - sin0\right]$$

$$= 0$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\, sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{1}{l}\int_0^{2l} k\, sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{k}{l}\left[-\frac{cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)}\right]_0^{2l}$$

$$= \frac{k}{n\pi}\left[-\cos(2n\pi) + \cos 0\right]$$

$$= 0$$

The required Fourier series is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \, cos\left(\frac{n\pi x}{l}\right) + b_n \, sin\left(\frac{n\pi x}{l}\right) \right]$$

$$f(x) = \frac{2k}{2} + \sum_{n=1}^{\infty} \left[(0) \, cos\left(\frac{n\pi x}{l}\right) + (0) \, sin\left(\frac{n\pi x}{l}\right) \right]$$

$f(x) = k$ **Answer.**

Polling Quiz

Which of the following is the correct value of Euler's coefficients for the function $f(x)$ over the interval $[c, 2l + c]$

(A) $a_0 = \frac{1}{l}\int_c^{2l+c} f(x)dx$

(B) $a_n = \frac{1}{l}\int_c^{2l+c} f(x)\, cos\left(\frac{n\pi x}{l}\right) dx$

(C) $b_n = \frac{1}{l}\int_c^{2l+c} f(x)\, sin\left(\frac{n\pi x}{l}\right) dx$

(D) All are correct.

Problem 2. Find the Fourier series expansion of the function:

$$f(x) = x, \qquad x \in [0,2\pi].$$

Solution. Here $f(x) = x, x \in [0,2\pi]$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} xdx$$

$$= 2\pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\, cos\left(\frac{n\pi x}{\pi}\right) dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\, cos(nx)\, dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{\sin(nx)}{n}\right) - 1\left(\frac{\cos(nx)}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{n\pi}\left[-\left(\frac{\cos(2n\pi)}{n^2}\right) + \left(\frac{\cos 0}{n^2}\right)\right]$$

$$= 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\, cos\left(\frac{n\pi x}{\pi}\right) dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x \sin(nx)\, dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos(nx)}{n}\right) + 1\left(\frac{\sin(nx)}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-2\pi\left(\frac{\cos(2n\pi)}{n}\right) + 0\right]$$

$$= -\frac{2}{n}$$

The required Fourier series is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \, cos\left(\frac{n\pi x}{l}\right) + b_n \, sin\left(\frac{n\pi x}{l}\right) \right]$$

$$f(x) = \frac{2\pi}{2} + \sum_{n=1}^{\infty} \left[(0) \, cos\left(\frac{n\pi x}{\pi}\right) + (-\frac{2}{n}) \, sin\left(\frac{n\pi x}{\pi}\right) \right]$$

$$f(x) = \pi - 2\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$$ **Answer.**