MTH174

UNIT 3

Topic:

Solution of Non-Homogeneous LDE with Constant coefficients Using Operator Method

Non-Homogeneous LDE Using operator method applicable only when:

1. Function is of the form: $r(x) = e^{\alpha x}$

2. Function is of the form: $r(x) = \cos \alpha x$ or $r(x) = \sin \alpha x$

3.Function is of the form: $r(x) = x^{m}$

4. Function is of the form: $r(x) = e^{\alpha x} g(x)$

Solution of Non-homogeneous LDE with constant coefficients Using Operator Method:

Let us consider 2nd order Non-homogeneous LDE with constant coefficients as:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = r(x) \tag{1}$$

or

$$ay'' + by' + cy = r(x) \tag{1}$$

Let $\boldsymbol{\frac{d}{dx}} \equiv \boldsymbol{D}$ be Differential operator (An algebraic operator like $+, -, \times, \div$)

Equation (1) becomes:

$$aD^2y + bDy + cy = r(x)$$

Symbolic Form (S.F.): $(aD^2 + bD + c)y = r(x)$

$$\Rightarrow f(D)y = r(x)$$

To find Complimentary Function (C.F.):

A.E.: $f(D) = 0$

$\Rightarrow \ (aD^2 + bD + c) = 0$

$\Rightarrow D = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = m_1, m_2$ (Say) (Suppose here $m_1 \neq m_2$ are real roots)

Complementary function C.F. is given by:

$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

To find Particular Integral (P.I.):

P.I.: $y_p = \frac{1}{f(D)} r(x)$ (There are different methods to evaluate it)

General Solution: $y =$ C.F. +P.I.

i.e. $\boldsymbol{y = y_c + y_p}$

Operator Method to find Particular Integral (P.I.):

Case 1: If $r(x) = e^{\alpha x}$, then P.I. $y_p = \frac{1}{f(D)} r(x)$

$\Rightarrow y_p = \frac{1}{f(D)} e^{\alpha x} = \frac{1}{f(\alpha)} e^{\alpha x}$, (i.e. Put $D = \alpha$), provided $f(\alpha) \neq 0$

If $f(\alpha) = 0$, then:

$y_p = x \frac{1}{f'(D)} e^{\alpha x} = x \frac{1}{f'(\alpha)} e^{\alpha x}$, provided $f'(\alpha) \neq 0$

If $f'(\alpha) = 0$, then:

$y_p = x^2 \frac{1}{f''(D)} e^{\alpha x} = x^2 \frac{1}{f''(\alpha)} e^{\alpha x}$, provided $f''(\alpha) \neq 0$

and so on…

Problem 1. Find the general solution of: $y'' + 5y' + 4y = 18e^{2x}$

Solution: The given equation is:

$$y'' + 5y' + 4y = 18e^{2x} \qquad (1)$$

S.F. : $(D^2 + 5D + 4)y = 18e^{2x}$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (D^2 + 5D + 4)$ and $r(x) = 18e^{2x}$

To find Complimentary Function (C.F.):

A.E. : $f(D) = 0 \quad \Rightarrow (D^2 + 5D + 4) = 0 \quad \Rightarrow (D + 1)(D + 4) = 0$

$\Rightarrow D = -1, -4$ (real and unequal roots)

Let $m_1 = -1$ and $m_2 = -4$

$\therefore$ Complimentary function is given by:

$$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$\Rightarrow y_c = c_1 e^{-1x} + c_2 e^{-4x}$$

To find Particular Integral (P.I.):

$$y_p = \frac{1}{f(D)} r(x) = \frac{1}{(D^2+5D+4)} (18e^{2x})$$

$$\Rightarrow y_p = 18\left[\frac{1}{(D^2+5D+4)} e^{2x}\right]$$

$$\Rightarrow y_p = 18\left[\frac{1}{\left((2)^2+5(2)+4\right)} e^{2x}\right] \qquad (Put\ D = 2)$$

$$\Rightarrow y_p = 18\left[\frac{1}{18} e^{2x}\right] \qquad \Rightarrow y_p = e^{2x}$$

$\therefore$ General solution is given by: y = C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = (c_1 e^{-1x} + c_2 e^{-4x}) + e^{2x}$ **Answer.**

Problem 2. Find the Particular Integral of: $y'' + y' - 6y = e^{2x}$

Solution:

To find Particular Integral (P.I.):

$$y_p = \frac{1}{f(D)} r(x) = \frac{1}{(D^2+D-6)} (e^{2x})$$

$$\Rightarrow y_p = \left[\frac{1}{\left((2)^2+(2)-6\right)} e^{2x}\right] \quad (Put\ D = 2) \qquad \Rightarrow y_p = \left[\frac{1}{(6-6)} e^{2x}\right] \text{ (Case of failure)}$$

$$\therefore y_p = x \frac{1}{f'(D)} r(x) = x \frac{1}{(2D+1)} (e^{2x})$$

$$\Rightarrow y_p = x\left[\frac{1}{(2(2)+1)} e^{2x}\right] \quad (Put\ D = 2) \qquad \Rightarrow y_p = \frac{x}{5} e^{2x}$$

Problem 3. Find the Particular Integral of: $y'' - 6y' + 9y = 14e^{3x}$

To find Particular Integral (P.I.):

$$y_p = \frac{1}{f(D)} r(x) = \frac{1}{(D^2-6D+9)}(14e^{3x})$$

$$\Rightarrow y_p = 14\left[\frac{1}{\left((3)^2-6(3)+9\right)} e^{2x}\right] \quad (Put\ D = 3) \ \Rightarrow y_p = 14\left[\frac{1}{(18-18)} e^{3x}\right] \text{ (Case of failure)}$$

$$\therefore y_p = 14x \frac{1}{f'(D)} r(x) = 14x \frac{1}{(2D-6)}(e^{3x}) = 14x\left[\frac{1}{(6-6)} e^{3x}\right] (Put\ D = 3) \text{ (Case of failure)}$$

$$\therefore y_p = 14x^2\left[\frac{1}{f''(D)} r(x)\right] = 14x^2\left[\frac{1}{2} e^{3x}\right] = 7x^2 e^{3x}$$

Operator Method to find Particular Integral (P.I.):

Case 2: If $r(x) = \cos \alpha x$ or $r(x) = \sin \alpha x$

Then P.I is: $y_p = \frac{1}{f(D)} r(x) = \frac{1}{f(D)} \cos \alpha x$

$y_p = \frac{1}{f(D^2 = -\alpha^2)} \cos \alpha x$, provided $f(D^2 = -\alpha^2) \neq 0$

If $f(D^2 = -\alpha^2) = 0$, then:

$y_p = x \frac{1}{f'(D)} \cos \alpha x = x \frac{1}{f'(D^2 = -\alpha^2)} \cos \alpha x$, provided $f'(D^2 = -\alpha^2) \neq 0$

Note: **1.** $D[r(x)] = \frac{d}{dx}[r(x)]$

2. $\frac{1}{D}[r(x)] = \int r(x) dx$

3. $\frac{1}{D-a}[r(x)] = \frac{1}{D-a} \times \frac{D+a}{D+a}[r(x)]$ (Rationalize to create D^2 in denominator)

Problem 1. Find the general solution of: $y'' - 16y = \cos 2x$

Solution: The given equation is:

$$y'' - 16y = \cos 2x \qquad (1)$$

S.F. : $(D^2 - 16)y = \cos 2x$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (D^2 - 16)$ and $r(x) = \cos 2x$

To find Complimentary Function (C.F.):

A.E. : $f(D) = 0 \quad \Rightarrow (D^2 - 16) = 0 \quad \Rightarrow (D-4)(D+4) = 0$

$\Rightarrow D = 4, -4$ (real and unequal roots)

Let $m_1 = 4$ and $m_2 = -4$

$\therefore$ Complimentary function is given by:

$$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$\Rightarrow y_c = c_1 e^{4x} + c_2 e^{-4x}$$

To find Particular Integral (P.I.):

$$y_p = \frac{1}{f(D)} r(x) = \frac{1}{(D^2-16)} (\cos 2x \ )$$

$$\Rightarrow y_p = \left[\frac{1}{(-(2)^2-16)} \cos 2x\right] \qquad (Put\ D^2 = -(2)^2)$$

$$\Rightarrow y_p = \left[\frac{1}{(-4-16)} \cos 2x\right]$$

$$\Rightarrow y_p = -\frac{1}{20} \cos 2x$$

∴ General solution is given by: y = C.F. +P.I.

i.e. $y = y_c + y_p$

$$\Rightarrow y = (c_1 e^{4x} + c_2 e^{-4x}) - \frac{1}{20} \cos 2x$$ **Answer.**

Problem 2. Find the Particular Integral of: $y'' + 9y = \sin 3x$

To find Particular Integral (P.I.):

$$y_p = \frac{1}{f(D)} r(x) = \frac{1}{(D^2+9)} (\sin 3x\ )$$

$$\Rightarrow y_p = \left[\frac{1}{(-(3)^2+9)} \sin 3x\right] \qquad (Put\ D^2 = -(3)^2)$$

$$\Rightarrow y_p = \left[\frac{1}{(-9+9)} \sin 3x\right] \quad \text{(Case of Failure)}$$

$$\therefore y_p = x\left[\frac{1}{f'(D)} r(x)\right] = x\left[\frac{1}{2D} (\sin 3x\ )\right] = \frac{x}{2}\int \sin 3x\, dx = \frac{x}{2}\left[\frac{-\cos 3x}{3}\right]$$

Problem 3. Find the general solution of: $2y'' - 5y' + 3y = \sin x$

Solution: The given equation is:

$2y'' - 5y' + 3y = \sin x$ (1)

S.F. : $(2D^2 - 5D + 3)y = \sin x$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (2D^2 - 5D + 3)$ and $r(x) = \sin x$

To find Complimentary Function (C.F.):

A.E. : $f(D) = 0$ $\Rightarrow (2D^2 - 5D + 3) = 0$ $\Rightarrow (2D - 3)(D - 1) = 0$

$\Rightarrow D = 1, {}^3/_2$ (real and unequal roots)

Let $m_1 = 1$ and $m_2 = {}^3/_2$

$\therefore$ Complimentary function is given by:

$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$\Rightarrow y_c = c_1 e^{1x} + c_2 e^{{}^3/_2 x}$

To find Particular Integral (P.I.):

$$y_p = \frac{1}{f(D)} r(x) = \frac{1}{(2D^2-5D+3)} (\sin x\ )$$

$$\Rightarrow y_p = \left[\frac{1}{(2(-(1)^2)-5D+3)} \sin x\right] \qquad (Put\ D^2 = -(1)^2)$$

$$\Rightarrow y_p = \left[\frac{1}{(1-5D)} \sin x\right] = \left[\frac{1}{(1-5D)} \times \frac{(1+5D)}{(1+5D)} \sin x\right] = \left[\frac{(1+5D)}{(1-25D^2)} \sin x\right]$$

$$\Rightarrow y_p = \frac{1}{26}[(1+5D)\sin x] = \frac{1}{26}\left[\sin x + 5\frac{d}{dx}(\sin x)\right] = \frac{1}{26}(\sin x + 5\cos x)$$

∴ General solution is given by: $y =$ C.F. +P.I.

i.e. $y = y_c + y_p$

$$\Rightarrow y = \left(c_1 e^{1x} + c_2 e^{3/2 x}\right) + \frac{1}{26}(\sin x + 5\cos x)$$ **Answer.**

Operator Method to find Particular Integral (P.I.):

Case 3: If $r(x) = x^m$

Then P.I. is: $y_p = \frac{1}{f(D)} r(x)$

$\Rightarrow y_p = \frac{1}{f(D)} x^m = \frac{1}{[1 \pm h(D)]} x^m$ (By taking least degree term common from $f(D)$)

$\Rightarrow y_p = [1 \pm h(D)]^{-1} x^m$ and we expand this expression by Binomial expansion.

Note:

1. $[1 + h(D)]^{-1} = 1 - h(D) + \left(h(D)\right)^2 - \left(h(D)\right)^3 + - - -$

2. $[1 - h(D)]^{-1} = 1 + h(D) + \left(h(D)\right)^2 + \left(h(D)\right)^3 + - - -$

Problem 1. Find the general solution of: $y'' + 25y = 4x^2$

Solution: The given equation is:

$y'' + 25y = 4x^2$ (1)

S.F. : $(D^2 + 25)y = 4x^2$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (D^2 + 25)$ and $r(x) = 4x^2$

To find Complimentary Function (C.F.):

A.E. : $f(D) = 0$ $\Rightarrow (D^2 + 25) = 0$ $\Rightarrow D^2 = -25$

$\Rightarrow D = 5i, -5i$ (Complex roots)

Let $m_1 = 5i$ and $m_2 = -5i$

$\therefore$ Complimentary function is given by:

$y_c = e^{0x}(c_1 \cos 5x + c_2 \sin 5x)$

$\Rightarrow y_c = (c_1 \cos 5x + c_2 \sin 5x)$

To find Particular Integral (P.I.):

$$y_p = \frac{1}{f(D)} r(x) = \frac{1}{(D^2+25)} (4x^2)$$

$$\Rightarrow y_p = 4\left[\frac{1}{25\left(1+\frac{D^2}{25}\right)} x^2\right] = \frac{4}{25}\left[\left(1+\frac{D^2}{25}\right)^{-1} x^2\right]$$

$$\Rightarrow y_p = \frac{4}{25}\left[\left(1-\left(\frac{D^2}{25}\right)^1+\left(\frac{D^2}{25}\right)^2 ----\right) x^2\right] = \frac{4}{25}\left[x^2-\frac{2}{25}+0\right] \qquad (D^2(x^2)=2)$$

$$\Rightarrow y_p = \frac{4}{625}(25x^2-2)$$

$\therefore$ General solution is given by: y = C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = (c_1 \cos 5x + c_2 \sin 5x) + \frac{4}{625}(25x^2-2)$ **Answer.**

Problem 2. Find the Particular Integral of: $y'' - 6y' + 9y = 4x^2 - 1$

To find Particular Integral (P.I.):

$$y_p = \frac{1}{f(D)} r(x) = \frac{1}{(D^2-6D+9)}(4x^2-1)$$

$$\Rightarrow y_p = \left[\frac{1}{9\left(1-\frac{(6D-D^2)}{9}\right)}(4x^2-1)\right] = \frac{1}{9}\left[\left(1-\frac{(6D-D^2)}{9}\right)^{-1}(4x^2-1)\right]$$

$$\Rightarrow y_p = \frac{1}{9}\left[\left(1+\left(\frac{(6D-D^2)}{9}\right)^1+\left(\frac{(6D-D^2)}{9}\right)^2+----\right)(4x^2-1)\right]$$

$$\Rightarrow y_p = \frac{1}{9}\left[(4x^2-1)+\left(\frac{(6D-D^2)}{9}\right)(4x^2-1)+\frac{36}{81}D^2(4x^2-1)+0\right]$$

$$\Rightarrow y_p = \frac{1}{9}\left[(4x^2-1)+\frac{6}{9}(8x)-\frac{1}{9}(8)+\frac{36}{81}(8)\right]$$

Operator Method to find Particular Integral (P.I.):

Case 4: If $r(x) = e^{\alpha x} g(x)$

Then P.I. is: $y_p = \frac{1}{f(D)} r(x)$

$\Rightarrow y_p = \frac{1}{f(D)} e^{\alpha x} g(x)$

$\Rightarrow y_p = e^{\alpha x} \left[\frac{1}{f(D+\alpha)} g(x) \right]$

Either $g(x) = x^m$ or $g(x) = \cos \alpha x$

Then we proceed with the rules that we already know

Problem 1. Find the general solution of: $y'' - 4y' + 5y = 24e^{2x} \sin x$

Solution: The given equation is:

$$y'' - 4y' + 5y = 24e^{2x} \sin x \qquad (1)$$

S.F. : $(D^2 - 4D + 5)y = 24e^{3x} \sin x$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (D^2 - 4D + 5)$ and $r(x) = 24e^{3x} \sin x$

To find Complimentary Function (C.F.):

A.E. : $f(D) = 0 \quad \Rightarrow (D^2 - 4D + 5) = 0$

$\Rightarrow D = 2 \pm i$ (Complex roots)

Let $m_1 = 2 + i$ and $m_2 = 2 - i$

$\therefore$ Complimentary function is given by:

$y_c = e^{2x}(c_1 \cos x + c_2 \sin x)$

To find Particular Integral (P.I.):

$$y_p = \frac{1}{f(D)} r(x) = \frac{1}{(D^2-4D+5)} (24e^{2x} \sin x \ )$$

$$\Rightarrow y_p = 24e^{2x} \left[\frac{1}{\left((D+2)^2-4(D+2)+5\right)} \sin x\right] \Rightarrow y_p = 24e^{2x} \left[\frac{1}{(D^2+1)} \sin x\right]$$

$$\Rightarrow y_p = 24e^{2x} \left[\frac{1}{(-1+1)} \sin x\right] \quad (\text{Put } D^2 = -(1)^2) \quad (\text{Case of failure})$$

$$\therefore y_p = 24e^{2x} x \left[\frac{1}{f'(D)} \sin x\right] = 24e^{2x} x \left[\frac{1}{2D} \sin x\right] = 12e^{2x} x \int \sin x dx = -12xe^{2x} \cos x$$

General solution is given by: $y =$ C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = e^{2x}(c_1 \cos x + c_2 \sin x) - 12xe^{2x} \cos x$ **Answer.**

Problem 2. Find the general solution of: $y'' - y' - 6y = xe^{-2x}$

Solution: The given equation is:

$$y'' - y' - 6y = xe^{-2x} \qquad (1)$$

S.F. : $(D^2 - D - 6)y = xe^{-2x}$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (D^2 - D - 6)$ and $r(x) = xe^{-2x}$

To find Complimentary Function (C.F.):

A.E. : $f(D) = 0$ $\Rightarrow (D^2 - D - 6) = 0$ $\Rightarrow (D-3)(D+2) = 0$

$\Rightarrow D = 3, -2$ (real and distinct roots)

Let $m_1 = 3$ and $m_2 = -2$

$\therefore$ Complimentary function is given by:

$y_c = c_1 e^{3x} + c_2 e^{-2x}$

To find Particular Integral (P.I.):

$$y_p = \frac{1}{f(D)} r(x) = \frac{1}{(D^2-D-6)} (xe^{-2x})$$

$$\Rightarrow y_p = e^{-2x}\left[\frac{1}{\left((D-2)^2-(D-2)-6\right)} x\right] \Rightarrow y_p = e^{-2x}\left[\frac{1}{\left(D^2-5D\right)} x\right]$$

$$\Rightarrow y_p = e^{-2x}\left[\frac{1}{-5D\left(1-\frac{D^2}{5D}\right)} x\right] = -\frac{e^{-2x}}{5}\left[\frac{1}{D}\left(1-\frac{D}{5}\right)^{-1} x\right] = -\frac{e^{-2x}}{5}\left[\frac{1}{D}\left(1+\frac{D}{5}+\left(\frac{D}{5}\right)^2+\cdots\right) x\right]$$

$$\therefore y_p = -\frac{e^{-2x}}{5}\int\left(x+\frac{1}{5}\right) dx = -\frac{e^{-2x}}{5}\left(\frac{x^2}{2}+\frac{x}{5}\right) = -\frac{e^{-2x}}{50}(5x^2+2x)$$

General solution is given by: $y =$ C.F. +P.I.

i.e. $y = y_c + y_p$

$$\Rightarrow y = c_1 e^{3x} + c_2 e^{-2x} - \frac{e^{-2x}}{50}(5x^2+2x)$$ **Answer.**

MCQ

The Particular Integral of $y'' + 5y' + 4y = 18e^{2x}$ is:

(A) e^{2x}

(B) xe^{2x}

(C) e^{3x}

(D) None of these

MCQ

The Particular Integral of $y'' + 9y = \sin 3x$ is:

(A) $-\frac{x}{6}\sin 3x$

(B)$-\frac{x}{6}\cos 3x$

(C) $\frac{x}{6}\cos 3x$

(D) None of these