UNIT 4 - (i)

Fourier series

$$\left(\begin{array}{c}\text{Euler's Formulae}\\ \text{Change of Interval}\\ \text{Functions Having Points of Discontinuity}\end{array}\right)$$

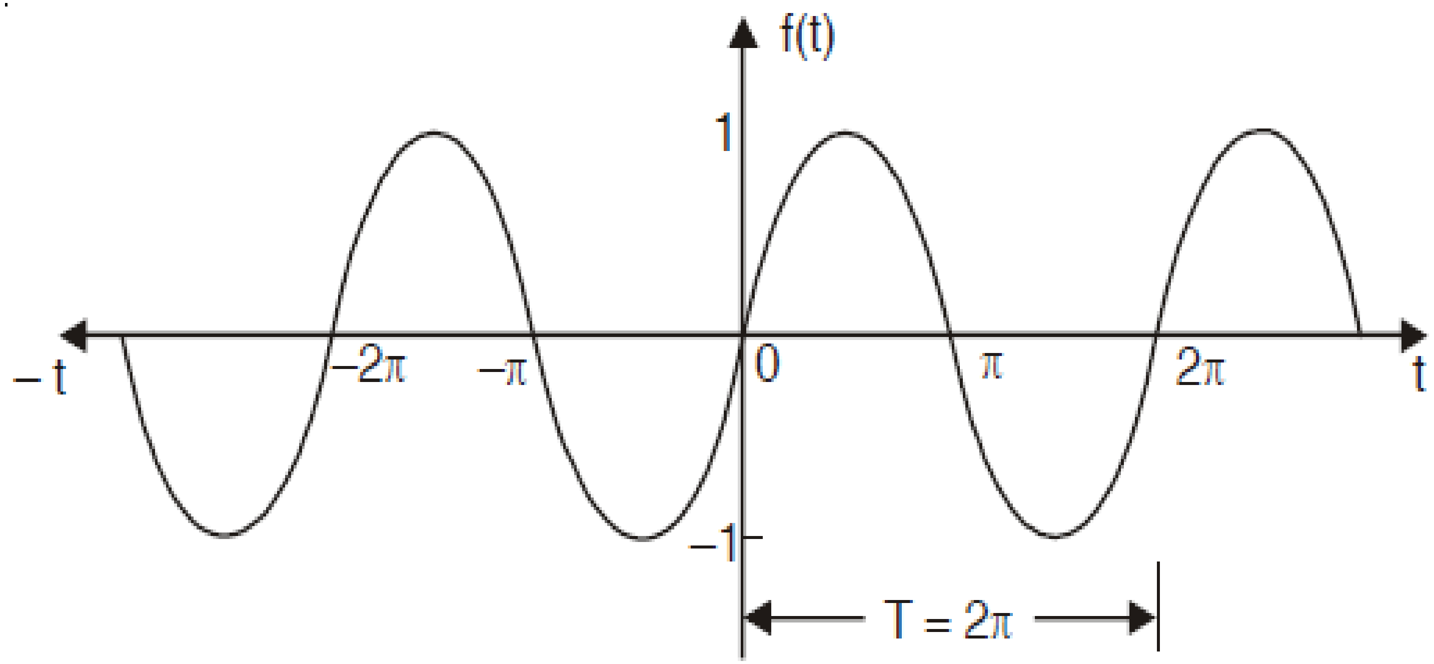
PERIODIC FUNCTIONS

If the value of each ordinate $f(t)$ repeats itself at equal intervals in the abscissa, then $f(t)$ is said to be a periodic function.

If $f(t) = f(t + T) = f(t + 2T) =$ .... then T is called the period of the function $f(t)$.

For example :

$\sin x = \sin(x + 2\pi) = \sin(x + 4\pi) = \ldots$ so $\sin x$ is a periodic function with the period 2π. This is also called sinusoidal periodic function.



DIRICHLET'S CONDITIONS FOR A FOURIER SERIES

If the function $f(x)$ for the interval $(-\pi, \pi)$

(1) is single-valued (2) is bounded

(3) has at most a finite number of maxima and minima.

(4) has only a finite number of discontinuous

(5) is $f(x + 2\pi) = f(x)$ for values of x outside $[-\pi, \pi]$, then

$$S_p(x) = \frac{a_0}{2} + \sum_{n=1}^{P} a_n \cos nx + \sum_{n=1}^{P} b_n \sin nx$$

converges to $f(x)$ as $P \to \infty$ at values of x for which $f(x)$ is continuous and to $\frac{1}{2}[f(x+0) + f(x-0)]$ at points of discontinuity.

FOURIER SERIES (EULER'S FORMULAE)

Definition Let f be a piecewise continuous function on $[-\pi, \pi]$. Then the **Fourier series** of f is the series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where the coefficients a_n and b_n in this series are defined by

$$a_0 = \frac{1}{\cancel{2}\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

and are called the **Fourier coefficients** of f.

FOURIER SERIES (EULER'S FORMULAE)

Definition If f is a piecewise continuous function on $[-L, L]$, its **Fourier series** is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

where

$$a_0 = \frac{1}{\cancel{2}L} \int_{-L}^{L} f(x)\, dx$$

and, for $n \geqslant 1$,

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \qquad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

USEFUL INTEGRALS

The following integrals are useful in Fourier Series.

(i) $\int_0^{2\pi} \sin nx \, dx = 0$

(ii) $\int_0^{2\pi} \cos nx \, dx = 0$

(iii) $\int_0^{2\pi} \sin^2 nx \, dx = \pi$

(iv) $\int_0^{2\pi} \cos^2 nx \, dx = \pi$

(v) $\int_0^{2\pi} \sin nx \cdot \sin mx \, dx = 0$

(vi) $\int_0^{2\pi} \cos nx \cos mx \, dx = 0$

(vii) $\int_0^{2\pi} \sin nx \cdot \cos mx \, dx = 0$

(viii) $\int_0^{2\pi} \sin nx \cdot \cos nx \, dx = 0$

(ix) $\int uv \, dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots$

where $v_1 = \int v \, dx, v_2 = \int v_1 \, dx$ and so on $u' = \frac{du}{dx}$, $u'' = \frac{d^2u}{dx^2}$ and so on and

(x) $\sin n\pi = 0$, $\cos n\pi = (-1)^n$ where $n \in I$

Question Find the Fourier coefficients and Fourier series of the **square-wave function** f defined by

$$f(x) = \begin{cases} 0 & \text{if } -\pi \leqslant x < 0 \\ 1 & \text{if } 0 \leqslant x < \pi \end{cases} \qquad \text{and} \qquad f(x + 2\pi) = f(x)$$

SOLUTION Using the formulas for the Fourier coefficients in Definition 7, we have

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{2\pi}\int_{0}^{\pi} 1\,dx = 0 + \frac{1}{2\pi}(\pi) = \frac{1}{2}$$

and, for $n \geqslant 1$,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi} \cos nx\,dx$$

$$= 0 + \frac{1}{\pi}\frac{\sin nx}{n}\Bigg]_0^{\pi} = \frac{1}{n\pi}(\sin n\pi - \sin 0) = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi} \sin x\,dx$$

$$= -\frac{1}{\pi}\frac{\cos nx}{n}\Bigg]_0^{\pi} = -\frac{1}{n\pi}(\cos n\pi - \cos 0) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{2}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

Fourier series in sigma notation as

$$f(x) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2}{(2k-1)\pi} \sin(2k-1)x$$

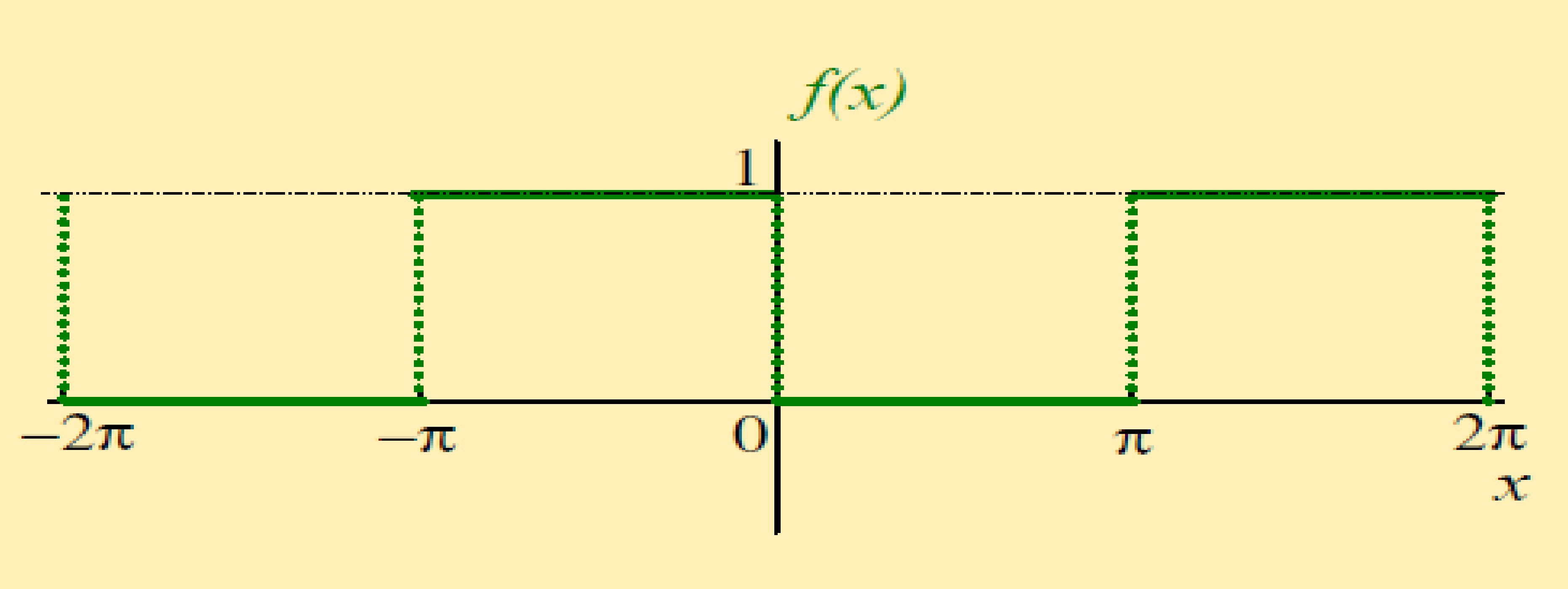
Fourier Convergence Theorem If f is a periodic function with period 2π and f and f' are piecewise continuous on $[-\pi, \pi]$, then the Fourier series (7) is convergent. The sum of the Fourier series is equal to $f(x)$ at all numbers x where f is continuous. At the numbers x where f is discontinuous, the sum of the Fourier series is the average of the right and left limits, that is

$$\tfrac{1}{2}[f(x^+) + f(x^-)]$$

Question

$$f(x) = \begin{cases} 1, & -\pi < x < 0 \\ 0, & 0 < x < \pi, \end{cases} \quad \text{and has period } 2\pi$$

a) <u>Sketch a graph of $f(x)$ in the interval $-2\pi < x < 2\pi$</u>



STEP ONE

$$\begin{aligned}
a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x &= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} f(x)\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{0} 1\cdot\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} 0\cdot\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{0} \mathrm{d}x \\
&= \frac{1}{\pi}\left[x\right]_{-\pi}^{0} \\
&= \frac{1}{\pi}\left(0-(-\pi)\right) \\
&= \frac{1}{\pi}\cdot(\pi) \\
\text{i.e. } a_0 &= 1\,.
\end{aligned}$$

$$\boxed{f(x) = \begin{cases} 1, & -\pi < x < 0 \\ 0, & 0 < x < \pi\,. \end{cases}}$$

STEP TWO

$$
\begin{aligned}
a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x &= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{0} 1\cdot\cos nx\,\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} 0\cdot\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{0} \cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} = \frac{1}{n\pi}\left[\sin nx\right]_{-\pi}^{0} \\
&= \frac{1}{n\pi}\left(\sin 0 - \sin(-n\pi)\right) \\
&= \frac{1}{n\pi}(0 + \sin n\pi) \\
\text{i.e. } a_n &= \frac{1}{n\pi}(0+0) = 0.
\end{aligned}
$$

$$
\boxed{f(x) = \begin{cases} 1, & -\pi < x < 0 \\ 0, & 0 < x < \pi\,. \end{cases}}
$$

STEP THREE

$$f(x) = \begin{cases} 1, & -\pi < x < 0 \\ 0, & 0 < x < \pi\ . \end{cases}$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x \\ &= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,\mathrm{d}x \\ &= \frac{1}{\pi}\int_{-\pi}^{0} 1\cdot\sin nx\,\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} 0\cdot\sin nx\,\mathrm{d}x \end{aligned}$$

$$\begin{aligned} \text{i.e.}\quad b_n &= \frac{1}{\pi}\int_{-\pi}^{0}\sin nx\,\mathrm{d}x = \frac{1}{\pi}\left[\frac{-\cos nx}{n}\right]_{-\pi}^{0} \\ &= -\frac{1}{n\pi}[\cos nx]_{-\pi}^{0} = -\frac{1}{n\pi}\left(\cos 0 - \cos(-n\pi)\right) \\ &= -\frac{1}{n\pi}(1-\cos n\pi) = -\frac{1}{n\pi}\left(1-(-1)^n\right), \end{aligned}$$

$$\text{i.e.}\quad b_n = \begin{cases} 0 & ,\ n \text{ even} \\ -\frac{2}{n\pi} & ,\ n \text{ odd} \end{cases} \quad ,\ \text{since } (-1)^n = \begin{cases} 1 & ,\ n \text{ even} \\ -1 & ,\ n \text{ odd} \end{cases}$$

$$f(x) = \begin{cases} 1, & -\pi < x < 0 \\ 0, & 0 < x < \pi\,. \end{cases}$$

We now have that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

with the three steps giving

$$a_0 = 1,\;\; a_n = 0\,,\text{ and }\; b_n = \begin{cases} 0 & , n \text{ even} \\ -\frac{2}{n\pi} & , n \text{ odd} \end{cases}$$

It may be helpful to construct a table of values of b_n

n	1	2	3	4	5
b_n	$-\frac{2}{\pi}$	0	$-\frac{2}{\pi}\left(\frac{1}{3}\right)$	0	$-\frac{2}{\pi}\left(\frac{1}{5}\right)$

Substituting our results now gives the required series

$$f(x) = \frac{1}{2} - \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\right]$$

c) Pick an appropriate value of x, to show that

$$f(x) = \begin{cases} 1, & -\pi < x < 0 \\ 0, & 0 < x < \pi \,. \end{cases}$$

$$\boxed{\tfrac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \ldots}$$

Comparing this series with

$$f(x) = \frac{1}{2} - \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\right],$$

we need to introduce a minus sign in front of the constants $\frac{1}{3}, \frac{1}{7}, \ldots$

So we need $\sin x = 1, \ \sin 3x = -1, \ \sin 5x = 1, \ \sin 7x = -1$, etc

The first condition of $\sin x = 1$ suggests trying $x = \frac{\pi}{2}$.

This choice gives	$\sin\frac{\pi}{2}$	$+$	$\frac{1}{3}\sin 3\frac{\pi}{2}$	$+$	$\frac{1}{5}\sin 5\frac{\pi}{2}$	$+$	$\frac{1}{7}\sin 7\frac{\pi}{2}$
i.e.	1	$-$	$\frac{1}{3}$	$+$	$\frac{1}{5}$	$-$	$\frac{1}{7}$

Looking at the graph of $f(x)$, we also have that $f(\frac{\pi}{2}) = 0$.

$$f(x) = \begin{cases} 1, & -\pi < x < 0 \\ 0, & 0 < x < \pi\,. \end{cases}$$

Picking $x = \frac{\pi}{2}$ thus gives

$$\begin{aligned} 0 = \tfrac{1}{2} - \tfrac{2}{\pi}\Big[\sin\tfrac{\pi}{2} + \tfrac{1}{3}\sin\tfrac{3\pi}{2} + \tfrac{1}{5}\sin\tfrac{5\pi}{2} \\ + \tfrac{1}{7}\sin\tfrac{7\pi}{2} + \ldots\Big] \end{aligned}$$

$$\begin{aligned} \text{i.e. } 0 = \tfrac{1}{2} - \tfrac{2}{\pi}\Big[1 - \tfrac{1}{3} + \tfrac{1}{5} \\ - \tfrac{1}{7} + \ldots\Big] \end{aligned}$$

A little manipulation then gives a series representation of $\frac{\pi}{4}$

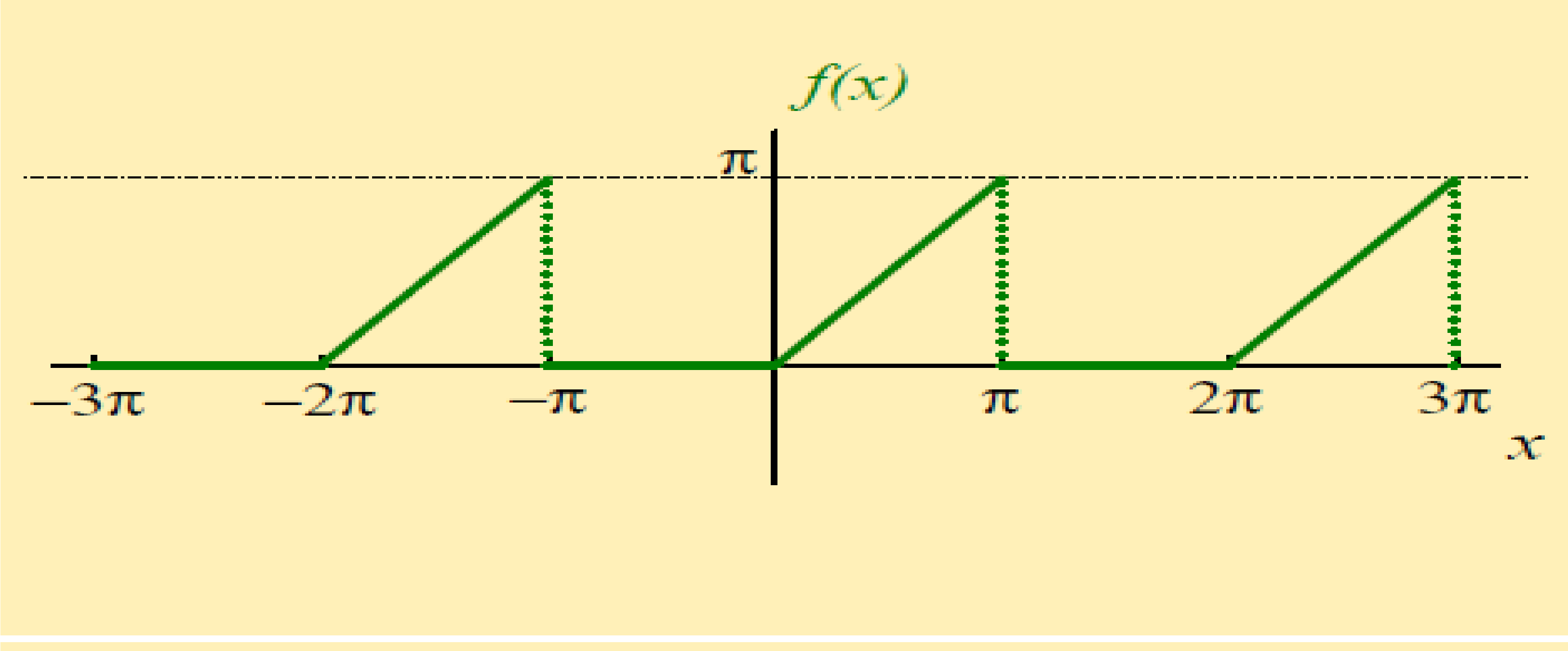
$$\frac{2}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right] = \frac{1}{2}$$

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}.$$

Question

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi, \end{cases} \quad \text{and has period } 2\pi$$

a) Sketch a graph of $f(x)$ in the interval $-3\pi < x < 3\pi$



STEP ONE

$$\begin{aligned}
a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x &= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} f(x)\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{0} 0\cdot \mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} x\,\mathrm{d}x \\
&= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} \\
&= \frac{1}{\pi}\left(\frac{\pi^2}{2} - 0\right) \\
\text{i.e. } a_0 &= \frac{\pi}{2}\,.
\end{aligned}$$

$$\boxed{f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi\,. \end{cases}}$$

STEP TWO

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 0\cdot\cos nx\,\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,\mathrm{d}x$$

$$\text{i.e. } a_n = \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,\mathrm{d}x = \frac{1}{\pi}\left\{\left[x\frac{\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}\,\mathrm{d}x\right\}$$

(using integration by parts)

$$\text{i.e. } a_n = \frac{1}{\pi}\left\{\left(\pi\frac{\sin n\pi}{n} - 0\right) - \frac{1}{n}\left[-\frac{\cos nx}{n}\right]_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{(\quad 0 \quad - \quad 0) + \frac{1}{n^2}[\cos nx]_0^{\pi}\right\}$$

$$= \frac{1}{\pi n^2}\{\cos n\pi - \cos 0\} = \frac{1}{\pi n^2}\{(-1)^n - 1\}$$

$$\text{i.e. } a_n = \begin{cases} 0 & , n \text{ even} \\ -\frac{2}{\pi n^2} & , n \text{ odd} \end{cases}$$, see TRIG.

$$\boxed{f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi\,. \end{cases}}$$

STEP THREE

$$
\begin{aligned}
b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x &= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{0} 0\cdot\sin nx\,\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} x\sin nx\,\mathrm{d}x \\
\text{i.e. } b_n = \frac{1}{\pi}\int_{0}^{\pi} x\sin nx\,\mathrm{d}x &= \frac{1}{\pi}\left\{\left[x\left(-\frac{\cos nx}{n}\right)\right]_0^{\pi} - \int_0^{\pi}\left(-\frac{\cos nx}{n}\right)\mathrm{d}x\right\} \\
&\qquad\qquad \text{(using integration by parts)} \\
&= \frac{1}{\pi}\left\{-\frac{1}{n}\left[x\cos nx\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\,\mathrm{d}x\right\} \\
&= \frac{1}{\pi}\left\{-\frac{1}{n}(\pi\cos n\pi - 0) + \frac{1}{n}\left[\frac{\sin nx}{n}\right]_0^{\pi}\right\} \\
&= -\frac{1}{n}(-1)^n + \frac{1}{\pi n^2}(0-0), \text{ see } \textsc{Trig} \\
&= -\frac{1}{n}(-1)^n
\end{aligned}
$$

$$
f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi\,. \end{cases}
$$

i.e. $\quad b_n = \begin{cases} -\frac{1}{n} & , n \text{ even} \\ +\frac{1}{n} & , n \text{ odd} \end{cases}$

$$\boxed{f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi\,. \end{cases}}$$

We now have

$$f(x) \quad = \quad \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

where $a_0 = \dfrac{\pi}{2}$, $\quad a_n = \begin{cases} 0 & , n \text{ even} \\ -\frac{2}{\pi n^2} & , n \text{ odd} \end{cases} \quad , \quad b_n = \begin{cases} -\frac{1}{n} & , n \text{ even} \\ \frac{1}{n} & , n \text{ odd} \end{cases}$

Constructing a table of values gives

n	1	2	3	4	5
a_n	$-\frac{2}{\pi}$	0	$-\frac{2}{\pi} \cdot \frac{1}{3^2}$	0	$-\frac{2}{\pi} \cdot \frac{1}{5^2}$
b_n	1	$-\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{4}$	$\frac{1}{5}$

This table of coefficients gives

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi\,. \end{cases}$$

$$\begin{aligned} f(x) = \frac{1}{2}\left(\frac{\pi}{2}\right) \quad &+ \left(-\frac{2}{\pi}\right)\cos x + 0\cdot\cos 2x \\ &+ \left(-\frac{2}{\pi}\cdot\frac{1}{3^2}\right)\cos 3x + 0\cdot\cos 4x \\ &+ \left(-\frac{2}{\pi}\cdot\frac{1}{5^2}\right)\cos 5x + \ldots \\ &+ \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \ldots \end{aligned}$$

$$\begin{aligned} \text{i.e. } f(x) = \frac{\pi}{4} \quad &- \frac{2}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right] \\ &+ \left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \ldots\right] \end{aligned}$$

and we have found the required series!

c) Pick an appropriate value of x, to show that

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \,. \end{cases}$$

$$\boxed{\text{(i) } \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots}$$

Comparing this series with

$$f(x) = \frac{\pi}{4} \quad - \quad \frac{2}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right]$$
$$+ \quad \left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \ldots\right],$$

the required series of constants does <u>not</u> involve terms like $\frac{1}{3^2}, \frac{1}{5^2}, \frac{1}{7^2}, \ldots$. So we need to pick a value of x that sets the $\cos nx$ terms to zero. The TRIG section shows that $\cos n\frac{\pi}{2} = 0$ when n is odd, and note also that $\cos nx$ terms in the Fourier series all have odd n

i.e. $\quad \cos x = \cos 3x = \cos 5x = \ldots = 0 \quad$ when $x = \frac{\pi}{2}$,

i.e. $\quad \cos\frac{\pi}{2} = \cos 3\frac{\pi}{2} = \cos 5\frac{\pi}{2} = \ldots = 0$

Setting $x = \frac{\pi}{2}$ in the series for $f(x)$ gives

$$\begin{aligned}
f\left(\frac{\pi}{2}\right) &= \frac{\pi}{4} - \frac{2}{\pi}\left[\cos\frac{\pi}{2} + \frac{1}{3^2}\cos\frac{3\pi}{2} + \frac{1}{5^2}\cos\frac{5\pi}{2} + \ldots\right] \\
&\quad + \left[\sin\frac{\pi}{2} - \frac{1}{2}\sin\frac{2\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{2} - \frac{1}{4}\sin\frac{4\pi}{2} + \frac{1}{5}\sin\frac{5\pi}{2} - \ldots\right] \\
&= \frac{\pi}{4} - \frac{2}{\pi}[0 \;+\; 0 \;+\; 0 \;+ \ldots] \\
&\quad + \left[1 - \frac{1}{2}\underbrace{\sin\pi}_{=0} + \frac{1}{3}\cdot(-1) - \frac{1}{4}\underbrace{\sin 2\pi}_{=0} + \frac{1}{5}\cdot(1) - \ldots\right]
\end{aligned}$$

The graph of $f(x)$ shows that $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$, so that

$$\begin{aligned}
\frac{\pi}{2} &= \frac{\pi}{4} + 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \\
\text{i.e. } \frac{\pi}{4} &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots
\end{aligned}$$

$$\boxed{f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi\,. \end{cases}}$$

Pick an appropriate value of x, to show that

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \,. \end{cases}$$

$$\boxed{\text{(ii)}\ \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots}$$

Compare this series with

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right]$$
$$+ \left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \ldots\right].$$

This time, we want to use the coefficients of the $\cos nx$ terms, and the same choice of x needs to set the $\sin nx$ terms to zero

Picking $x = 0$ gives

$$\sin x = \sin 2x = \sin 3x = 0 \quad \text{and} \quad \cos x = \cos 3x = \cos 5x = 1$$

Note also that the graph of $f(x)$ gives $f(x) = 0$ when $x = 0$

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi\,. \end{cases}$$

So, picking $x = 0$ gives

$$\begin{aligned} 0 &= \frac{\pi}{4} - \frac{2}{\pi}\left[\cos 0 + \frac{1}{3^2}\cos 0 + \frac{1}{5^2}\cos 0 + \frac{1}{7^2}\cos 0 + \ldots\right] \\ &\quad + \sin 0 - \frac{\sin 0}{2} + \frac{\sin 0}{3} - \ldots \\ \text{i.e. } 0 &= \frac{\pi}{4} - \frac{2}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\right] + 0 - 0 + 0 - \ldots \end{aligned}$$

We then find that

$$\begin{aligned} \frac{2}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\right] &= \frac{\pi}{4} \\ \text{and} \qquad 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots &= \frac{\pi^2}{8}. \end{aligned}$$

Some important series deduction from Fourier series

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots \qquad = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

$$\frac{\pi}{\sinh \pi} = 1 + 2\left[-\frac{1}{2} + \frac{1}{5} - \frac{1}{10} + \ldots.\right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

$$\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} \ldots \infty = \frac{1}{4}(\pi - 2)$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$$

$$\sum \frac{1}{n^4} = \frac{\pi^4}{90} \qquad \sum \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$