MTH174

UNIT 3

Method of Undeterminant Coefficients

Topic:

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Method of Undeterminant coefficients to solve Non-Homogeneous LDE with constant coefficients.

Method of Undeterminant Coefficients:

Let us consider 2nd order Non-homogeneous LDE with constant coefficients as:

$$ay'' + by' + cy = r(x) \qquad (1)$$

In this method, we guess the trial solution (P.I.) y_p by looking at the type of $r(x)$. As y_p is solution of equation (1), so it satisfies (1) from where we calculate the Undeterminant coefficients by comparing like coefficients on both sides.

For example:

1. If $\boldsymbol{r(x) = e^{\alpha x}}$, then assumed trial solution: $\boldsymbol{y_p = A\, e^{\alpha x}}$

2. If $\boldsymbol{r(x) = x^2}$, then assumed trial solution: $\boldsymbol{y_p = A\, x^2 + Bx + C}$

3. If $\boldsymbol{r(x) = \cos ax}$ *or* $(\boldsymbol{r(x) = sinax})$ then assumed trial solution: $\boldsymbol{y_p = A \cos x + B \sin x}$

where A, B, C are Undeterminant coefficients to be evaluated.

Problem 1. Find the general solution of: $4y'' - y = e^{3x}$

Solution: The given equation is:

$$4y'' - y = e^{3x} \qquad (1)$$

S.F. : $(4D^2 - 1)y = e^{3x}$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (4D^2 - 1)$ and $r(x) = e^{3x}$

To find Complimentary Function (C.F.):

A.E. : $f(D) = 0 \quad \Rightarrow (4D^2 - 1) = 0 \quad \Rightarrow D^2 = \frac{1}{4}$

$\Rightarrow D = \frac{1}{2}, -\frac{1}{2}$ (real and unequal roots)

$\therefore$ Complimentary function is given by:

$$\Rightarrow y_c = c_1 e^{\frac{1}{2}x} + c_2 e^{-\frac{1}{2}x}$$

To find Particular Integral (P.I.):

Since $r(x) = e^{3x}$, So, let trial solution be: $\boldsymbol{y_p = ae^{3x}}$

$\Rightarrow y_p' = 3ae^{3x}$ $\Rightarrow y_p'' = 9ae^{3x}$

Since y_p is a solution of equation (1)

So, $4y_p'' - y_p = e^{3x}$

$\Rightarrow 4(9ae^{3x}) - ae^{3x} = e^{3x}$ $\Rightarrow 35ae^{3x} = e^{3x}$ $\Rightarrow 35a = 1$ $\Rightarrow a = \frac{1}{35}$

$\therefore y_p = \frac{1}{35}e^{3x}$

General solution is given by: $y =$ C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = \left(c_1 e^{\frac{1}{2}x} + c_2 e^{-\frac{1}{2}x}\right) + \frac{1}{35}e^{3x}$ **Answer.**

Problem 2 . Find the particular integral of: $y'' - 3y' - 10y = x^2 + 1$

To find Particular Integral (P.I.):

Since $r(x) = x^2 + 1$, So, let trial solution be: $\boldsymbol{y_p = ax^2 + bx + c}$

$\Rightarrow y_p' = 2ax + b \qquad \Rightarrow y_p'' = 2a$

Since y_p is a solution of equation (1)

So, $y_p'' - 3y_p' - 10y_p = x^2 + 1$

$\Rightarrow 2a - 3(2ax + b) - 10(ax^2 + bx + c) = x^2 + 1$

$\Rightarrow -10ax^2 - (6a + 10b)x + (2a - 3b - 10c) = x^2 + 1$

Comparing the like coefficients:: Coeff. of x^2: $-10a = 1 \quad \Rightarrow a = \frac{1}{10}$

Coeff. of x : $-(6a + 10b) = 0 \quad \Rightarrow 10b = -6a \quad \Rightarrow b = -\frac{6}{100}$

Constant terms: $(2a - 3b - 10c) = 1 \quad \Rightarrow 10c = 2a - 3b - 1$

$$\Rightarrow c = \frac{1}{10}(2a - 3b - 1) = \frac{1}{10}\left(2\left(\frac{1}{10}\right) - 3\left(\frac{-6}{100}\right) - 1\right) = -\frac{62}{1000}$$

Putting back values of a, b, c in $y_p = ax^2 + bx + c$

$$y_p = \frac{1}{10}x^2 - \frac{6}{100}x - \frac{62}{1000}$$

MCQ-1

For $y'' - 3y' - 10y = x^3 + 1$, the assumed trial solution will be:

(A) $y_p = ax^3 + bx^2 + cx + d$

(B) $y_p = a + bx + cx^2 + dx^3$

(C) Both A and B

(D) None of these.

Problem 3. Find P.I. of: $y'' + y' - 6y = 39\cos 3x$

To find Particular Integral (P.I.):

Since $r(x) = 39\cos 3x$, So, let trial solution be: $\boldsymbol{y_p = a\cos 3x + b\sin 3x}$

$\Rightarrow y_p{}' = -3a\sin 3x + 3b\cos 3x \qquad \Rightarrow y_p{}'' = -9a\cos 3x - 9b\sin 3x$

So, $y_p{}'' + y_p{}' - 6y_p = 39\cos 3x$

$\Rightarrow (-9a\cos 3x - 9b\sin 3x) + (-3a\sin 3x + 3b\cos 3x) - 6(a\cos 3x +$

$b\sin 3x) = 39\cos 3x$

$\Rightarrow (-15a + 3b)\cos 3x + (-3a - 15b)\sin 3x = 39\cos 3x + 0\sin 3x$

Comparing the like coefficients:

Coeff. of $\cos 3x$: $\quad -15a + 3b = 39 \qquad (2)$

Coeff. of $\sin 3x$: $\quad -3a - 15b = 0 \qquad (3)$

Solving equations (2) and (3):

$-15a + 3b = 39$ (2) $\times 3$

$-3a - 15b = 0$ (3) $\times 15$ and subtracting, we get:

$234b = 127 \Rightarrow b = \frac{1}{2}$

Put value of b in equation (3): $3a = -15b = -\frac{15}{2} \quad \Rightarrow a = -\frac{5}{2}$

Putting back values of a, b in $y_p = a\cos 3x + b\sin 3x$

$$y_p = -\frac{5}{2}\cos 3x + \frac{1}{2}\sin 3x$$

General solution is given by: $y =$ C.F. +P.I.

i.e. $y = y_c + y_p$

$$\Rightarrow y = (c_1 e^{2x} + c_2 e^{-3x}) - \frac{5}{2}\cos 3x + \frac{1}{2}\sin 3x$$ **Answer.**

Problem 4. Find the general solution of: $y'' + y' - 6y = e^{2x}$

Solution: The given equation is:

$$y'' + y' - 6y = e^{2x} \qquad (1)$$

S.F. : $(D^2 + D - 6)y = e^{2x}$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (D^2 + D - 6)$ and $r(x) = e^{2x}$

To find Complimentary Function (C.F.):

A.E. : $f(D) = 0 \quad \Rightarrow (D^2 + D - 6) = 0 \quad \Rightarrow (D - 2)(D + 3) = 0$

$\Rightarrow D = 2, -3$ (real and unequal roots)

Let $m_1 = 2$ and $m_2 = -3$

$\therefore$ Complimentary function is given by:

$$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$\Rightarrow y_c = c_1 e^{2x} + c_2 e^{-3x}$$

To find Particular Integral (P.I.):

Since $r(x) = e^{2x}$, So, let trial solution be: $\boldsymbol{y_p = axe^{2x}}$

$\Rightarrow y_p' = a[x(2e^{2x}) + e^{2x}(1)] = 2axe^{2x} + ae^{2x}$

$\Rightarrow y_p'' = 2a[x(2e^{2x}) + e^{2x}(1)] + 2ae^{2x} = 4axe^{2x} + 4ae^{2x}$

Since y_p is a solution of equation (1)

So, $y_p'' + y_p' - 6y_p = e^{2x}$

$\Rightarrow (4axe^{2x} + 4ae^{2x}) + (2axe^{2x} + ae^{2x}) - 6(axe^{2x}) = e^{2x}$

$\Rightarrow 5ae^{2x} = e^{2x}$ $\quad \Rightarrow 5a = 1$ $\quad \Rightarrow a = \frac{1}{5}$ $\quad \therefore y_p = \frac{1}{5}xe^{2x}$

General solution is given by: $y =$ C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = (c_1e^{2x} + c_2e^{-3x}) + \frac{1}{5}xe^{3x}$ **Answer.**

Problem 5. Find the general solution of: $y'' - 6y' + 9y = 14e^{3x}$

Solution: The given equation is:

$$y'' - 6y' + 9y = 14e^{3x} \qquad (1)$$

S.F. : $(D^2 - 6D + 9)y = 14e^{3x}$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (D^2 - 6D + 9)$ and $r(x) = 14e^{3x}$

To find Complimentary Function (C.F.):

A.E. : $f(D) = 0 \quad \Rightarrow (D^2 - 6D + 9) = 0 \quad \Rightarrow (D - 3)(D - 3) = 0$

$\Rightarrow D = 3,3$ (real and equal roots)

Let $m_1 = 3$ and $m_2 = 3$

$\therefore$ Complimentary function is given by:

$y_c = (c_1 + c_2x)e^{m_2x}$

$\Rightarrow y_c = (c_1 + c_2x)e^{3x} = c_1\, e^{3x} + c_2x\, e^{3x}$

To find Particular Integral (P.I.):

Since $r(x) = e^{3x}$, So, let trial solution be: $y_p = ax^2e^{3x}$

$\Rightarrow y_p{}' = a[x^2(3e^{3x}) + e^{3x}(2x)] = 3ax^2e^{3x} + 2axe^{3x}$

$\Rightarrow y_p{}'' = 3a[x^2(3e^{3x}) + e^{3x}(2x)] + 2a[e^{3x}(1) + x(3e^{3x})] = 9ax^2e^{3x} + 12axe^{3x} + 2ae^{3x}$

Since y_p is a solution of equation (1)

So, $y_p{}'' - 6y_p{}' + 9y_p = 14e^{3x}$

$\Rightarrow (9ax^2e^{3x} + 12axe^{3x} + 2ae^{3x}) - 6(3ax^2e^{3x} + 2axe^{3x}) + 9(ax^2e^{3x}) = 14e^{3x}$

$\Rightarrow 2ae^{3x} = 14e^{3x}$ $\quad\Rightarrow 2a = 14$ $\quad\Rightarrow a = 7$ $\qquad\therefore y_p = 7x^2e^{2x}$

General solution is given by: $y =$ C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = c_1\, e^{3x} + c_2 x\, e^{3x} + 7x^2e^{2x}$ **Answer.**

Problem 5. Find the general solution of: $y'' + 4y = \cos x$

Solution: The given equation is:

$$y'' + 4y = \cos x \qquad (1)$$

S.F. : $(D^2 + 4)y = \cos x$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (D^2 + 4)$ and $r(x) = \cos x$

To find Complimentary Function (C.F.):

A.E. : $f(D) = 0 \quad \Rightarrow (D^2 + 4) = 0$

$\Rightarrow D = 2i, -2i$ (Complex roots)

$\therefore$ Complimentary function is given by:

$\Rightarrow y_c = c_1 \cos 2x + c_2 \sin 2x$

To find Particular Integral (P.I.):

Since $r(x) = \cos x$, So, let trial solution be: $y_p = a\cos x + b\sin x$

$\Rightarrow y_p' = -a\sin x + b\cos x$ $\Rightarrow y_p'' = -a\cos x - b\sin x$

Since y_p is a solution of equation (1)

So, $y_p'' + 4y_p = \cos x$

$\Rightarrow (-a\cos x - b\sin x) + 4(a\cos x + b\sin x) = \cos x$

$\Rightarrow 3a\cos x + 3b\sin x = \cos x + 0\sin x$

Comparing the like coefficients:

Coeff. of $\cos 3x$: $3a = 1$ $\Rightarrow a = \frac{1}{3}$

Coeff. of $\sin 3x$: $3b = 0$ $\Rightarrow b = 0$

Putting back values of a, b in $y_p = \frac{1}{3}\cos x$

General solution is given by: $y =$ C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{3}\cos x$ **Answer.**

MCQ-2

For $y'' + y' - 6y = 16\sin 2x$, the assumed trial solution will be:

(A) $y_p = a\cos 2x$

(B) $y_p = a\sin 2x$

(C) $y_p = a\cos 2x + b\sin 2x$

MCQ-3

For $y'' + y = \cos x$, the assumed trial solution will be:

(A) $y_p = x(a\cos x + b\sin x)$

(B) $y_p = (a\cos x + b\sin x)$

(C) $y_p = x(a\cos 2x + b\sin 2x)$