Unit 6: Fourier Series

(**Book:** Advanced Engineering Mathematics By Jain and Iyengar, **Chapter-9**)

Learning Outcomes:

1. To know about even and odd functions.

2. To find Fourier series of even and odd functions.

Even functions:

Let $f(x)$ be a function defined on $[-l, l]$.

Then $f(x)$ is said to be an *even function* if

$$f(-x) = f(x), \qquad -l \leq x \leq l$$

For example, $f(x) = x^2, f(x) = x^4, f(x) = k, f(x) = \cos x$ are all even functions as:

Odd functions:

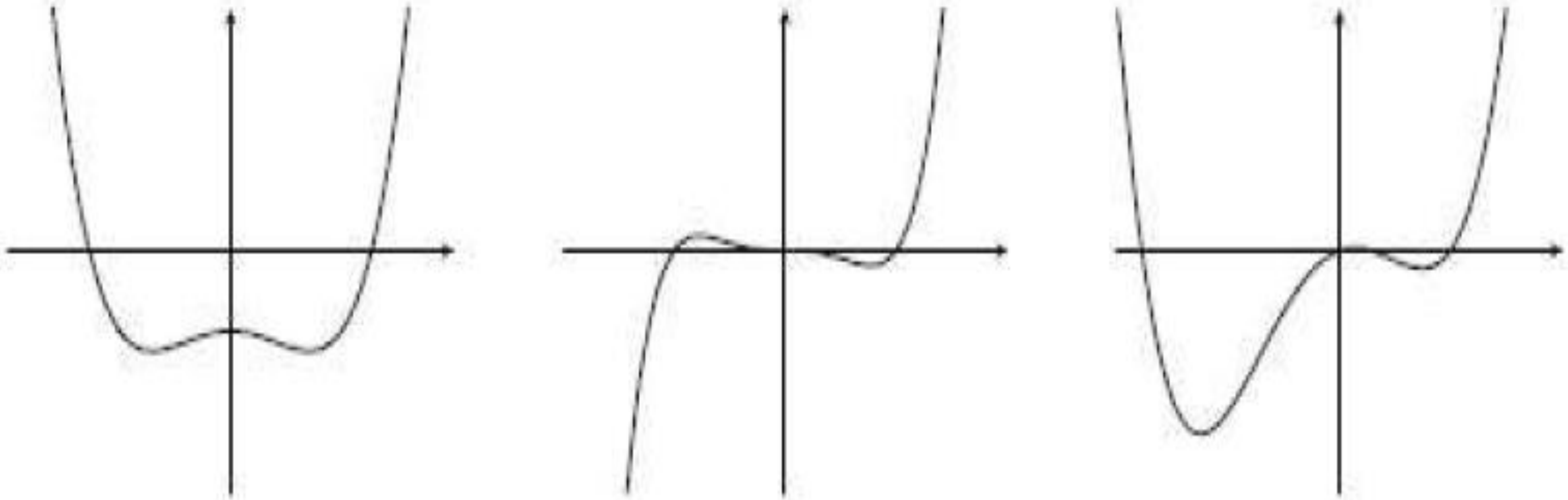
Let $f(x)$ be a function defined on $[-l, l]$.

Then $f(x)$ is said to be an *odd function* if

$$f(-x) = -f(x), \qquad -l \leq x \leq l$$

For example, $f(x) = x, f(x) = x^3, f(x) = \sin x, f(x) = \tan x,$ are all odd functions as:

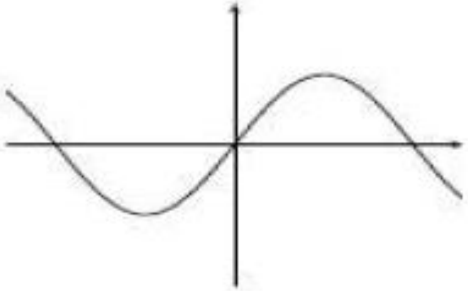
Graphically, even functions have symmetry about the y-axis, whereas odd functions have symmetry around the origin.



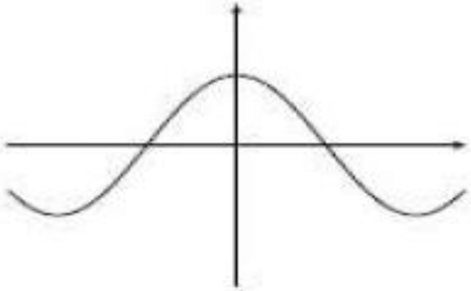
Even Odd Neither

Examples:

- Sums of odd powers of x are odd: $5x^3 - 3x$
- Sums of even powers of x are even: $-x^6 + 4x^4 + x^2 - 3$
- $\sin x$ is odd, and $\cos x$ is even



$\sin x$ (odd)



$\cos x$ (even)

- The product of two odd functions is even: $x \sin x$ is ever
- The product of two even functions is even: $x^2 \cos x$ is ev
- The product of an even function and an odd function is odd: $\sin x \cos x$ is odd

Polling Quiz

The function $f(x) = x\sin x + \cos x + x^2 + x^3 \tan x + 3$ is:

(A) An even function

(B) An odd function

(C) Neither even nor odd

(D) I don't Know.

Change of interval:

Let periodic function $f(x)$ be defined on the interval $[c, 2l + c]$.

If $c = -l$, then interval becomes $[-l, l]$

Then, Euler's coefficients a_0, a_n, b_n are:

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\, cos\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\, sin\left(\frac{n\pi x}{l}\right) dx$$

Two Important Definite integrals

If $f(x)$ is an odd function, then:

$$\int_{-l}^{l} f(x)dx = 0$$

If $f(x)$ is an even function, then:

$$\int_{-l}^{l} f(x)dx = 2\int_{0}^{l} f(x)dx$$

Fourier series of an Even function

If $f(x)$ is an even function on $[-l, l]$, then:

$$\boldsymbol{b_n = 0}$$

Required Fourier series is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right)\right]$$

Fourier series of an Odd function

If $f(x)$ is an odd function on $[-l, l]$, then:

$$\boldsymbol{a_o = 0}, \qquad \boldsymbol{a_n = 0}$$

Required Fourier series is:

$$f(x) = \sum_{n=1}^{\infty} \left[b_n \, sin\left(\frac{n\pi x}{l}\right) \right]$$

Problem 1. Find the Fourier series expansion of the function:

$$f(x) = x, \qquad x \in [-\pi, \pi].$$

Solution. Here $f(x) = x,\ \ x \in [-\pi, \pi]$

Since $f(-x) = -x = -f(x)$

So, $f(x)$ is an odd function.

Hence, $\boldsymbol{a_o = 0},\ \boldsymbol{a_n = 0}$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\, sin\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, sin\left(\frac{n\pi x}{\pi}\right) dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin(nx)\, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x \sin(nx)\, dx$$ { $x \sin(nx)$ is an even function}

$$= \frac{2}{\pi}\left[x\left(-\frac{\cos(nx)}{n}\right) + 1\left(\frac{\sin(nx)}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\pi\left(\frac{\cos(n\pi)}{n}\right) + 0\right] = \frac{2}{n}(-1)^{n+1}$$

The required Fourier series is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n\, cos\left(\frac{n\pi x}{l}\right) + b_n\, sin\left(\frac{n\pi x}{l}\right)\right]$$

$$f(x) = \frac{0}{2} + \sum_{n=1}^{\infty} \left[(0)\, cos\left(\frac{n\pi x}{\pi}\right) + \frac{2}{n}(-1)^{n+1} sin\left(\frac{n\pi x}{\pi}\right)\right]$$

$$x = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin(nx)$$ **Answer.**

Polling Quiz

If $f(x)$ is an odd function, which of the following is true:

(A) $a_0 = 0$

(B) $a_n = 0$

(C) $b_n = 0$

(D) (A) and (B)

Problem 2. Find the Fourier series expansion of the function:

$$f(x) = x^2, \qquad x \in [-2,2].$$

Solution. Here $f(x) = x^2$, $x \in [-2,2]$

Since $f(-x) = (-x)^2 = x^2 = f(x)$

So, $f(x)$ is an even function.

Hence, $\boldsymbol{b_n = 0}$

$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$$

$$= \frac{1}{2}\int_{-2}^{2} x^2 dx$$

$$= \int_{0}^{2} x^2 dx = \frac{8}{3}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\, cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{1}{2}\int_{-2}^{2} x^2 cos\left(\frac{n\pi x}{2}\right) dx$$

$$= \int_{0}^{2} x^2 cos\left(\frac{n\pi x}{2}\right) dx \quad \left\{x^2 cos\left(\frac{n\pi x}{2}\right) \text{ is an even function}\right\}$$

$$a_n = \frac{16}{n^2\pi^2}\cos(n\pi) = \frac{16}{n^2\pi^2}(-1)^n$$

The required Fourier series is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\, cos\left(\frac{n\pi x}{l}\right) + b_n\, sin\left(\frac{n\pi x}{l}\right)\right]$$

$$f(x) = \frac{8/3}{2} + \sum_{n=1}^{\infty}\left[\frac{16}{n^2\pi^2}(-1)^n cos\left(\frac{n\pi x}{2}\right) + (0)\, sin\left(\frac{n\pi x}{2}\right)\right]$$

$$x^2 = \frac{4}{3} + \sum_{n=1}^{\infty}\left[\frac{16}{n^2\pi^2}(-1)^n cos\left(\frac{n\pi x}{2}\right)\right]$$ **Answer.**