Fourier Series

EULER'S FORMULAE

The Fourier series for the function on

interval $(\alpha, \alpha + 2c)$ *is given by*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c}$$

where

$$\left.\begin{aligned} a_0 &= \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\,dx \\ a_n &= \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\cos\frac{n\pi x}{c}\,dx \\ b_n &= \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\sin\frac{n\pi x}{c}\,dx \end{aligned}\right\}$$

To establish these formulae, the following definite integrals will be required :

1. $$\int_{\alpha}^{\alpha+2\pi} \cos nx\,dx = \left|\frac{\sin nx}{n}\right|_{\alpha}^{\alpha+2\pi} = 0$$

2. $$\int_{\alpha}^{\alpha+2\pi} \sin nx\,dx = -\left|\frac{\cos nx}{n}\right|_{\alpha}^{\alpha+2\pi} = 0$$

3. $$\int_{\alpha}^{\alpha+2\pi} \cos mx \cos nx\,dx$$

$$= \frac{1}{2}\int_{\alpha}^{\alpha+2\pi} [\cos (m+n)x + \cos (m-n)x]\,dx$$

$$= \frac{1}{2}\left|\frac{\sin (m+n)x}{m+n} + \frac{\sin (m-n)x}{m-n}\right|_{\alpha}^{\alpha+2\pi} = 0$$

4. $$\int_{\alpha}^{\alpha+2\pi} \cos^2 nx\,dx = \left|\frac{x}{2} + \frac{\sin 2nx}{4n}\right|_{\alpha}^{\alpha+2\pi} = \pi$$

5. $$\int_{\alpha}^{\alpha+2\pi} \sin mx \cos nx\,dx = -\frac{1}{2}\left[\frac{\cos (m-n)x}{m-n} + \frac{\cos (m+n)x}{m+n}\right] = 0$$

6. $$\int_{\alpha}^{\alpha+2\pi} \sin nx \cos nx\,dx = \left|\frac{\sin^2 nx}{2n}\right|_{\alpha}^{\alpha+2\pi} = 0$$

7. $$\int_{\alpha}^{\alpha+2\pi} \sin mx \sin nx\,dx = \frac{1}{2}\left|\frac{\sin (m-n)x}{m-n} - \frac{\sin (m+n)x}{m+n}\right|_{\alpha}^{\alpha+2\pi} = 0$$

8. $$\int_{\alpha}^{\alpha+2\pi} \sin^2 nx\,dx = \left|\frac{x}{2} - \frac{\sin 2nx}{4n}\right|_{\alpha}^{\alpha+2\pi} = \pi.$$

interval $(\alpha, \alpha + 2c)$ *is given by*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c}$$

where

$$\left.\begin{aligned} a_0 &= \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\,dx \\ a_n &= \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\cos\frac{n\pi x}{c}\,dx \\ b_n &= \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\sin\frac{n\pi x}{c}\,dx \end{aligned}\right\}$$

Ex. Find a Fourier series to represent $f(x) = x$ from $-\pi$ to π

Ex. Find a Fourier series to represent $f(x) = x^2$ from $-\pi$ to π

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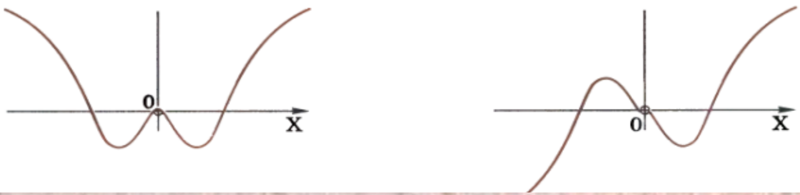
EVEN AND ODD FUNCTIONS

A function $f(x)$ *is said to be* **even** *if* $f(-x) = f(x)$,
e.g., $\cos x$, $\sec x$, x^2 are all even functions. Graphically an even function is symmetrical about the y-axis.
A function $f(x)$ *is said to be* **odd** *if* $f(-x) = -f(x)$,



Example 10.7. *Find the Fourier series for the function*

$$f(t) = \begin{cases} -1 & \text{for } -\pi < t < -\pi/2 \\ 0 & \text{for } -\pi/2 < t < \pi/2 \\ 1 & \text{for } \pi/2 < t < \pi \end{cases}$$

1. Obtain the Fourier series for $f(x) = \pi x$ in $0 \leq x \leq 2$.

Ex. Find a Fourier series to represent $f(x) = x^2$ from $0\ to\ 3$

Find the Fourier series for $f(x) = \begin{cases} x & \text{in } 0 \le x \le 3 \\ 6 - x & \text{in } 3 \le x \le 6 \end{cases}$

Example 10.10. *Obtain Fourier series for the function*

$$f(x) = \begin{cases} \pi x, & 0 \le x \le 1 \\ \pi(2 - x), & 1 \le x \le 2 \end{cases}$$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \infty = \frac{\pi^2}{8}$.

HALF RANGE SERIES

Sine series. If it be required to expand $f(x)$ as a sine series in $0 < x < c$; then we extend the function reflecting it in the origin, so that $f(x) = -f(-x)$.

Then the extended function is odd in $(-c, c)$ and the expansion will give the desired Fourier sine series :

$$\left.\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{c} \\ \text{where} \quad b_n &= \frac{2}{c} \int_0^c f(x) \sin \frac{n\pi x}{c}\, dx \end{aligned}\right\} \qquad \text{...(1)}$$

Cosine series. If it be required to express $f(x)$ as a cosine series in $0 < x < c$, we extend the function reflecting it in the y-axis, so that $f(-x) = f(x)$.

Then the extended function is even in $(-c, c)$ and its expansion will give the required Fourier cosine series :

$$\left.\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{c} \\ \text{where} \quad a_0 &= \frac{2}{c} \int_0^c f(x)\, dx \end{aligned}\right\} \qquad \text{...(2)}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{c}$$

where $a_0 = \frac{2}{c} \int_0^c f(x)\, dx$ ...(2)

and $a_n = \frac{2}{c} \int_0^c f(x) \cos \frac{n\pi x}{c}\, dx$

Example 10.16. *Express $f(x) = x$ as a half-range sine series in $0 < x < 2$.*

Example 10.17. *Express $f(x) = x$ as a half-range cosine series in $0 < x < 2$.*

Example 10.20. *Expand* $f(x) = \frac{1}{4} - x$, *if* $0 < x < \frac{1}{2}$,

$= x - \frac{3}{4}$, *if* $\frac{1}{2} < x < 1$,

as the Fourier series of sine terms.

PARSEVAL'S FORMULA

$$\int_{-l}^{l} [f(x)]^2\, dx = l \left\{ \frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right\},$$

Cor. 1. *If* $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$ *in* $(0, 2l)$, **then**

$$\int_0^{2l} [f(x)]^2 \, dx = l \left\{ \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right\}$$

Cor. 2. *If the half-range cosine series is* $(0, l)$ *for* $f(x)$ *is*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right), \text{ then}$$

$$\int_0^{l} [f(x)]^2 \, dx = \frac{l}{2} \left(\frac{a_0^2}{2} + a_1^2 + a_2^2 + a_3^2 + \ldots \infty \right)$$

Cor. 3. *If the half-range sine series in* $(0, l)$ *for* $f(x)$ *is* $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$, **then**

$$\int_0^{l} [f(x)]^2 = \frac{l}{2} (b_1^2 + b_2^2 + b_3^2 + \ldots \infty)$$

$$y = x^2 \ in \ -\pi < x < \pi.$$

(2) Root mean square (rms) value. *The root mean square value of the function* $f(x)$ *over an interval* (a, b) *is defined as*

$$[f(x)]_{\text{rms}} = \sqrt{\left\{ \frac{\int_a^b [f(x)]^2 \, dx}{b - a} \right\}} \qquad \ldots(7)$$

(2) Root mean square (rms) value. *The root mean square value of the function $f(x)$ over an interval (a, b) is defined as*

$$[f(x)]_{\text{rms}} = \sqrt{\left\{\frac{\int_a^b [f(x)]^2\,dx}{b-a}\right\}} \qquad ...(7)$$

The use of root mean square value of a periodic function is frequently made in the theory of mechanical vibrations and in electric circuit theory. The r.m.s. value is also known as the effective value of the function.

COMPLEX FORM OF FOURIER SERIES

The Fourier series of a periodic function $f(x)$ of period $2l$, is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

Since $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$,

Then the series (2) can be compactly written as :

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}$$

which is the *complex form of Fourier series* and its coefficients are given by (3).

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x) e^{-in\pi x/l}\,dx$$

Obs. The complex form of a Fourier series is especially useful in problems on electrical circuits having impressed periodic voltage.

Example 10.22. *Find the complex form of the Fourier series of $f(x) = e^{-x}$ in $-1 \le x \le 1$.*

Ex. Find the complex form of Fourier series of $f(x) = x \quad in\ -l \leq x \leq l$

Ex. Find the complex form of Fourier series of $f(x) = \cos ax \quad in\ -\pi \leq x \leq \pi$

Ex. Find the complex form of Fourier series of $f(x) = \sin x \quad in\ 0 \leq x \leq \pi$