Unit 1: **Linear Algebra**

(**Book:** Advanced Engineering Mathematics by Jain and Iyengar,**Chapter-3**)

Topic:

Linear Dependence and Linear Independence of Vectors

Learning Outcomes:

1. Definition of Linear Dependence and Linear Independence of Vectors.

2. Calculation of L.D. and L.I. using determinants.

3. Calculation of L.D. and L.I. using rank.

Linear Dependence and Linear Independence of Vectors:

A finite set of vectors $(v_1, v_2, v_3, -- -v_n)$ is said to be *linearly dependent*

If $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \cdots + \alpha_n v_n = 0$ (1)

Such that $\alpha_1, \alpha_2, \alpha_3, -- -\alpha_n$ are *not all zero.*

A finite set of vectors $(v_1, v_2, v_3, -- -v_n)$ is said to be *linearly independent*

If $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \cdots + \alpha_n v_n = 0$ (1)

Such that $\alpha_1, \alpha_2, \alpha_3, -- -\alpha_n$ *are all zero.*

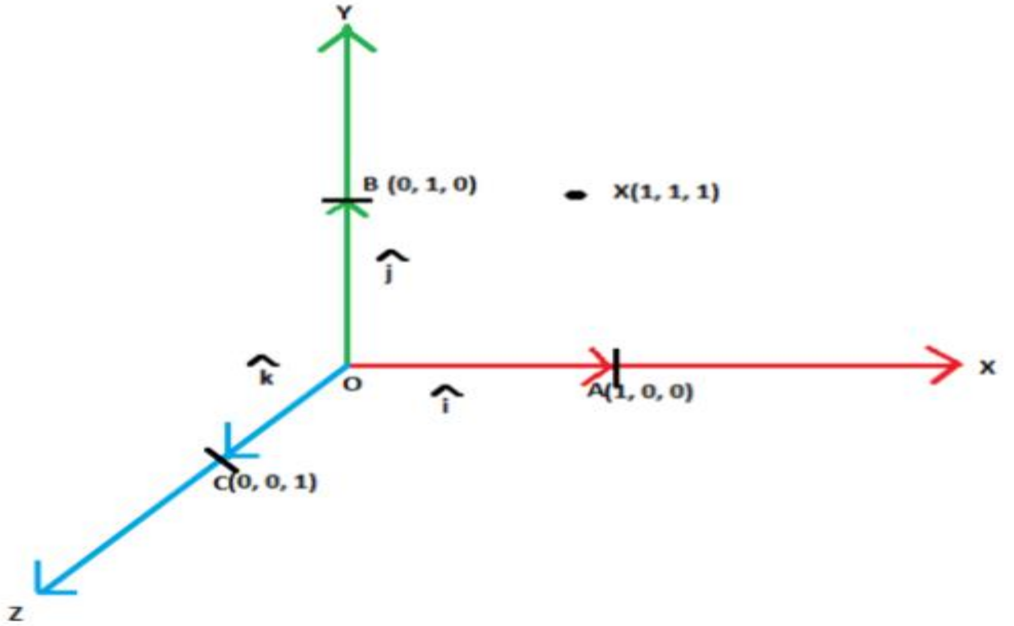
To calculate L.D. and L.I. using Determinants:

If A be any $n \times n$ matrix $(n = 3 \; or \; 2)$, obtained by replacing rows and columns of the matrix by set of n vectors, then the vectors are said to be:

(I) Linearly dependent (L.D.) if and only if $|A| = 0$

(II) Linearly independent (L.I.) if and only if $|A| \neq 0$

Problem: Show that the vectors $v_1 = (1,0,0), v_2 = (0,1,0)$, $v_3 = (0,0,1)$ are linearly independent.



Solution. Let $A = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

Expanding by first row,

$|A| = 1(1 - 0) - 0(0 - 0) + 0(0 - 0) = 1 \neq 0$

Since $|A| \neq 0$, so the given vectors v_1, v_2, v_3 are linearly independent.

Problem: Show that the vectors $v_1 = (3,1,6), v_2 = (2,0,4)$, $v_3 = (2,1,4)$ are linearly dependent.

Solution. Let $A = \begin{vmatrix} 3 & 1 & 6 \\ 2 & 0 & 4 \\ 2 & 1 & 4 \end{vmatrix}$

Expanding by first row,

$|A| = 3(0-4) - 1(8-8) + 6(2-0) = 0$

Since $|A| = 0$, so the given vectors v_1, v_2, v_3 are linearly dependent.

<u>To calculate L.D. and L.I. using Rank method:</u>

1. If the rank of the matrix of given vectors is equal to the number of vectors, then the vectors are *Linearly independent.*
2. If the rank of the matrix of given vectors is less than the number of vectors, then the vectors are *Linearly dependent.*

Problem: Show that the vectors $v_1 = (1,2,3), v_2 = (3,4,5)$, $v_3 = (6,7,8)$ are linearly dependent.

Solution. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -4 \\ 0 & -5 & -10 \end{bmatrix} \begin{matrix} R_2 - 3R_1 \\ R_3 - 6R_1 \end{matrix}$$

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -10 \end{bmatrix} R_2/-2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} R_3 + 5R_2$$

Here $r(A) = 2 < 3$ (Number of vectors)

So, the given vectors v_1, v_2, v_3 are linearly dependent.

* (Try Using Determinant)

Problem: Show that the vectors $v_1 = (2,2,1), v_2 = (1,-1,1)$, $v_3 = (1,0,1)$ are linearly independent.

Solution. Let $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 2 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$A \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 4 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 - R_1 \end{matrix}$$

Here $r(A) = 3 =$ (Number of vectors)

So, the given vectors v_1, v_2, v_3 are linearly independent.

* (Try Using Determinant)

Problem: Check if the vectors $v_1 = (1,3,4), v_2 = (1,1,0)$, $v_3 = (1,4,2)$, $v_4 = (1,-2,1)$ are linearly dependent or independent.

Solution. Let $A = \begin{bmatrix} 1 & 3 & 4 \\ 1 & 1 & 0 \\ 1 & 4 & 2 \\ 1 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 \\ 0 & -2 & -4 \\ 0 & 1 & -2 \\ 0 & -5 & -3 \end{bmatrix} \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{matrix}$

$$\sim \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & -2 \\ 0 & -2 & -4 \\ 0 & -5 & -3 \end{bmatrix} R_2 \leftrightarrow R_3$$

$$A \sim \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & -8 \\ 0 & 0 & -13 \end{bmatrix} \begin{matrix} \\ R_3 + 2R_2 \\ R_4 + 5R_2 \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & -8 \\ 0 & 0 & 0 \end{bmatrix} R_4 - \frac{13}{8} R_3$$

Here $r(A) = 3 < 4$ (Number of vectors)

So, the given vectors $v_1, v_2, v_3,$ are linearly dependent.

Problem: Check if the vectors $v_1 = (1,2,3,-1), v_2 = (0,1,-1,2)$, $v_3 = (1,5,1,8)$, $v_4 = (-1,7,8,3)$ are linearly dependent or independent.

Solution. Let $A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & -1 & 2 \\ 1 & 5 & 1 & 8 \\ -1 & 7 & 8 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 3 & -2 & 9 \\ 0 & 9 & 11 & 2 \end{bmatrix} \begin{matrix} R_3 - R_1 \\ R_4 + R_1 \end{matrix}$

$$\sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 20 & -16 \end{bmatrix} \begin{matrix} R_3 - 3R_2 \\ R_4 - 9R_2 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -76 \end{bmatrix} R_4 - 20R_3$$

Here $r(A) = 4 =$ (Number of vectors)

So, the given vectors v_1, v_2, v_3, v_4 are linearly independent.

Problem: Show that the vectors $v_1 = (1, -2), v_2 = (2,1), v_3 = (3,2)$ are linearly dependent and find the relation between them.

Solution. Let us consider the relation:

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0 \qquad (1)$$

Where $\alpha_1, \alpha_2, \alpha_3$ are arbitrary constants.

$$\Longrightarrow \alpha_1(1, -2) + \alpha_2(2,1) + \alpha_3(3,2) = (0,0)$$

$$\Longrightarrow (\alpha_1 + 2\alpha_2 + 3\alpha_3, -2\alpha_1 + \alpha_2 + 2\alpha_3) = (0,0)$$

$$\Rightarrow \alpha_1 + 2\alpha_2 + 3\alpha_3 = 0 \qquad (2)$$

and $-2\alpha_1 + \alpha_2 + 2\alpha_3 = 0 \qquad (3)$

Solving equations (2) and (3):

Coeff. Of	$\boldsymbol{\alpha_2}$	$\boldsymbol{\alpha_3}$	$\boldsymbol{\alpha_1}$	$\boldsymbol{\alpha_2}$
	2	3	1	2
	1	2	-2	1

$$\frac{\alpha_1}{4-3} = \frac{\alpha_2}{-6-2} = \frac{\alpha_3}{1+4} = \lambda(say) \quad \Rightarrow \alpha_1 = \lambda, \alpha_2 = -8\lambda, \alpha_3 = 5\lambda$$

Since $\alpha_1, \alpha_2, \alpha_3$ are not all zero, so the vectors v_1, v_2, v_3 are linearly dependent.

Relation between the vectors v_1, v_2, v_3:

Since $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$ (1)

Also $\alpha_1 = \lambda, \alpha_2 = -8\lambda, \alpha_3 = 5\lambda$

So, from equation (1)

$\lambda v_1 - 8\lambda v_2 + 5\lambda v_3 = 0$

$\Longrightarrow \lambda(v_1 - 8v_2 + 5v_3) = 0$

$\Longrightarrow v_1 - 8v_2 + 5v_3 = 0$ which is the required relation.