

INSTRUCTOR'S SOLUTIONS MANUAL FOR

PRINCIPLES OF ELECTROMAGNETICS

ASIAN EDITION

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CHAPTER 1

P. E. 1.1

(a) $\mathbf{A} + \mathbf{B} = (1, 0, 3) + (5, 2, -6) = (6, 2, -3)$

$$|\mathbf{A} + \mathbf{B}| = \sqrt{36 + 4 + 9} = \underline{\underline{7}}$$

(b) $5\mathbf{A} - \mathbf{B} = (5, 0, 15) - (5, 2, -6) = (0, \underline{\underline{-2}}, \underline{\underline{21}})$

(c) The component of \mathbf{A} along \mathbf{a}_y is $A_y = \underline{\underline{0}}$

(d) $3\mathbf{A} + \mathbf{B} = (3, 0, 9) + (5, 2, -6) = (8, 2, 3)$

A unit vector parallel to this vector is

$$\begin{aligned}\mathbf{a}_{11} &= \frac{(8, 2, 3)}{\sqrt{64 + 4 + 9}} \\ &= \underline{\underline{\pm(0.9117\mathbf{a}_x + 0.2279\mathbf{a}_y + 0.3419\mathbf{a}_z)}}\end{aligned}$$

P. E. 1.2 (a) $\mathbf{r}_p = \underline{\underline{\mathbf{a}_x - 3\mathbf{a}_y + 5\mathbf{a}_z}}$

$$\mathbf{r}_R = \underline{\underline{3\mathbf{a}_y + 8\mathbf{a}_z}}$$

(b) The distance vector is

$$\mathbf{r}_{QR} = \mathbf{r}_R - \mathbf{r}_Q = (0, 3, 8) - (2, 4, 6) = \underline{\underline{-2\mathbf{a}_x - \mathbf{a}_y + 2\mathbf{a}_z}}$$

(c) The distance between Q and R is

$$|\mathbf{r}_{QR}| = \sqrt{4 + 1 + 4} = \underline{\underline{3}}$$

P. E. 1.3 Consider the figure shown on the next page:

$$\begin{aligned}\mathbf{u}_z &= \mathbf{u}_p + \mathbf{u}_w = -350\mathbf{a}_x + \frac{40}{\sqrt{2}}(-\mathbf{a}_x + \mathbf{a}_y) \\ &= -378.28\mathbf{a}_x + 28.28\mathbf{a}_y \text{ km/hr}\end{aligned}$$

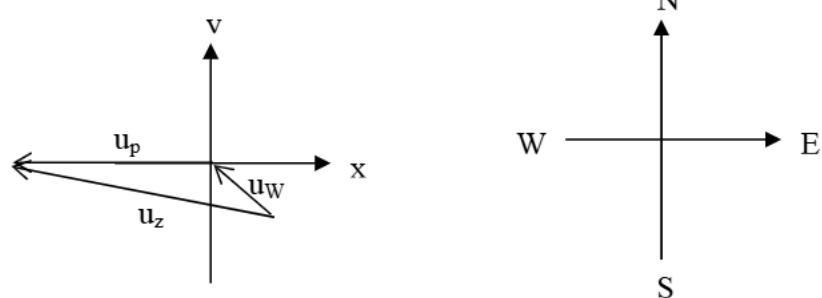
or

$$\mathbf{u}_z = 379.3 \angle 175.72^\circ \text{ km/hr}$$

Where \mathbf{u}_p = velocity of the airplane in the absence of wind

\mathbf{u}_w = wind velocity

\mathbf{u}_z = observed velocity

**P. E. 1.4**

Using the dot product,

$$\cos \theta_{AB} = \frac{\mathbf{A} \cdot \mathbf{B}}{AB} = \frac{-13}{\sqrt{10}\sqrt{65}} = -\sqrt{\frac{13}{50}}$$

$$\underline{\underline{\theta_{AB} = 120.66^\circ}}$$

P. E. 1.5

$$(a) \mathbf{E}_F = (\mathbf{E} \cdot \mathbf{a}_F) \mathbf{a}_F = \frac{(\mathbf{E} \cdot \mathbf{F}) \mathbf{F}}{|\mathbf{F}|^2} = \frac{-10(4, -10, 5)}{141}$$

$$= \underline{\underline{-0.2837\mathbf{a}_x + 0.7092\mathbf{a}_y - 0.3546\mathbf{a}_z}}$$

$$(b) \mathbf{E} \times \mathbf{F} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 0 & 3 & 4 \\ 4 & -10 & 5 \end{vmatrix} = (55, 16, -12)$$

$$\underline{\underline{\mathbf{a}_{E \times F} = \pm (0.9398, 0.2734, -0.205)}}$$

P. E. 1.6 $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$ showing that \mathbf{a} , \mathbf{b} , and \mathbf{c} form the sides of a triangle.

$$\mathbf{a} \cdot \mathbf{b} = 0,$$

hence it is a right angle triangle.

$$\text{Area} = \frac{1}{2} |\mathbf{a} \times \mathbf{b}| = \frac{1}{2} |\mathbf{b} \times \mathbf{c}| = \frac{1}{2} |\mathbf{c} \times \mathbf{a}|$$

$$\frac{1}{2} |\mathbf{a} \times \mathbf{b}| = \frac{1}{2} \begin{vmatrix} 4 & 0 & -1 \\ 1 & 3 & 4 \end{vmatrix} = \frac{1}{2} |(3, -17, 12)|$$

$$\text{Area} = \frac{1}{2} \sqrt{9 + 289 + 144} = \underline{\underline{10.51}}$$

P. E. 1.7

$$(a) P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\ = \sqrt{25 + 4 + 64} = \underline{\underline{9.644}}$$

$$(b) \mathbf{r}_P = \mathbf{r}_{P_1} + \lambda(\mathbf{r}_{P_2} - \mathbf{r}_{P_1}) \\ = (1, 2, -3) + \lambda(-5, -2, 8) \\ = \underline{\underline{(1 - 5\lambda, 2 - 2\lambda, -3 + 8\lambda)}}.$$

(c) The shortest distance is

$$\begin{aligned} d &= \mathbf{P}_1 \mathbf{P}_3 \sin \theta = |\mathbf{P}_1 \mathbf{P}_3 \times \mathbf{a}_{P_1 P_2}| \\ &= \frac{1}{\sqrt{93}} \begin{vmatrix} 6 & -3 & 5 \\ -5 & -2 & 8 \end{vmatrix} \\ &= \frac{1}{\sqrt{93}} |(-14, -73, -27)| = \underline{\underline{8.2}} \end{aligned}$$

P.E. 1.8

1.3 a) $A - 3B$

$$\begin{aligned} &= 4a_x - 2a_y + 6a_z - 3(12a_x + 18a_y - 8a_z) \\ &= -32a_x - 56a_y + 30a_z \end{aligned}$$

b) $(2A + 5B)/|B|$

$$\begin{aligned} &= \frac{[2(4a_x - 2a_y + 6a_z) + 5(12a_x + 18a_y - 8a_z)]}{(12^2 + 18^2 + 8^2)^{1/2}} \\ &= \frac{68a_x + 86a_y - 28a_z}{23.06} \\ &= 2.94a_x + 3.72a_y - 1.214a_z \end{aligned}$$

c) $a_x \times A$

$$\begin{aligned} &= a_x \times (4a_x - 2a_y + 6a_z) \\ &= 4(a_x \times a_x) - 2(a_x \times a_y) + 6(a_x \times a_z) \\ &= 0 - 2a_z - 6a_y = -6a_y - 2a_z \end{aligned}$$

d) $(B \times a_x) \cdot a_y$

$$\begin{aligned} &((12a_x + 18a_y - 8a_z) \times a_x) \cdot a_y \\ &(12(a_x \times a_x) + 18(a_y \times a_x) - 8(a_z \times a_x)) \cdot a_y \\ &(0 - 18a_z - 8a_y) \cdot a_y = 0 - 8 = -8 \end{aligned}$$

P.E. 1.9

$$\mathbf{P} \cdot \mathbf{Q} = (2, -6, 5) \cdot (0, 3, 1) = 0 - 18 + 5 = \underline{\underline{-13}}$$

$$\mathbf{P} \times \mathbf{Q} = \begin{vmatrix} 2 & -6 & 5 \\ 0 & 3 & 1 \end{vmatrix} = \underline{\underline{-21\mathbf{a}_x - 2\mathbf{a}_y + 6\mathbf{a}_z}}$$

$$\cos \theta_{PQ} = \frac{\mathbf{P} \cdot \mathbf{Q}}{PQ} = \frac{-13}{\sqrt{10} \sqrt{65}} = -0.51 \quad \longrightarrow \quad \theta_{PQ} = \underline{\underline{120.66^\circ}}$$

P.E. 1.10

$$\begin{aligned} \text{Area} &= \frac{1}{2} |\mathbf{D} \times \mathbf{E}| = \frac{1}{2} \begin{vmatrix} 4 & 1 & -5 \\ -1 & 2 & 3 \end{vmatrix} = \frac{1}{2} |(3+10)\mathbf{a}_x + (5-12)\mathbf{a}_y + (8+1)\mathbf{a}_z| \\ &= \frac{1}{2} |(13, -7, 9)| = \frac{1}{2} \sqrt{169 + 49 + 81} = \underline{\underline{8.646}} \end{aligned}$$

P.E. 1.11

$$\mathbf{A} \cdot \mathbf{B} = (4, -6, 1) \cdot (2, 0, 5) = 8 - 0 + 5 = 13$$

$$(a) |\mathbf{B}|^2 = 2^2 + 5^2 = 29$$

$$\mathbf{A} \cdot \mathbf{B} + 2|\mathbf{B}|^2 = 13 + 2 \times 29 = \underline{\underline{71}}$$

(b)

$$\mathbf{a}_\perp = \pm \frac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|}$$

$$\text{Let } \mathbf{C} = \mathbf{A} \times \mathbf{B} = \begin{vmatrix} 4 & -6 & 1 \\ 2 & 0 & 5 \end{vmatrix} = (-30, -18, 12)$$

$$\mathbf{a}_\perp = \pm \frac{\mathbf{C}}{|\mathbf{C}|} = \pm \frac{(-30, -18, 12)}{\sqrt{30^2 + 18^2 + 12^2}} = \underline{\underline{\pm(-0.8111\mathbf{a}_x - 0.4867\mathbf{a}_y + 0.3244\mathbf{a}_z)}}$$

Prob. 1.1

$$\mathbf{r}_{OP} = 4\mathbf{a}_x - 5\mathbf{a}_y + \mathbf{a}_z$$

$$\mathbf{a}_{r_{OP}} = \frac{\mathbf{r}_{OP}}{|\mathbf{r}_{OP}|} = \frac{(4, -5, 1)}{\sqrt{(16+25+1)}} = \underline{\underline{0.6172\mathbf{a}_x - 0.7715\mathbf{a}_y + 0.1543\mathbf{a}_z}}$$

Prob. 1.2

$$\mathbf{r} = (-3, 2, 2) - (2, 4, 4) = (-5, -2, -2)$$

$$\mathbf{a}_r = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{(-5, -2, -2)}{\sqrt{25+4+4}} = \underline{\underline{-0.8704\mathbf{a}_x - 0.3482\mathbf{a}_y - 0.3482\mathbf{a}_z}}$$

Prob. 1.3

$$(a) \quad A - 2B = (4, -6, 3) - 2(-1, 8, 5) = (4, -6, 3) - (-2, 16, 10) \\ = (6, \underline{-22}, \underline{-7})$$

$$(b) \quad A \cdot B = (4, -6, 3) \cdot (-1, 8, 5) = -4 - 48 + 15 = \underline{\underline{-37}}$$

$$(c) \quad A \times B = \begin{vmatrix} 4 & -6 & 3 \\ -1 & 8 & 5 \end{vmatrix} = (-30 - 24)\mathbf{a}_x + (-3 - 20)\mathbf{a}_y + (32 - 6)\mathbf{a}_z \\ = \underline{\underline{-54}}\mathbf{a}_x \underline{\underline{-23}}\mathbf{a}_y \underline{\underline{+26}}\mathbf{a}_z$$

Prob. 1.4

$$B \times C = \begin{vmatrix} 3 & 5 & 1 \\ 0 & 1 & -7 \end{vmatrix} = (-35 - 1)\mathbf{a}_x + (0 + 21)\mathbf{a}_y + (3 - 0)\mathbf{a}_z \\ = -36\mathbf{a}_x + 21\mathbf{a}_y + 3\mathbf{a}_z$$

$$A \cdot (B \times C) = (4, 2, 1) \cdot (-36, 21, 3) = -144 + 42 + 3 = \underline{\underline{-99}}$$

Prob. 1.5

$$(a) \quad B \times C = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} = \mathbf{a}_x - 2\mathbf{a}_y + \mathbf{a}_z$$

$$A \cdot (B \times C) = (1, 0, -1) \cdot (1, -2, 1) = 1 + 0 - 1 = \underline{\underline{0}}$$

$$(b) \quad A \times B = \begin{vmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{vmatrix} = \mathbf{a}_x - 2\mathbf{a}_y + \mathbf{a}_z$$

$$(A \times B) \cdot C = (1, -2, 1) \cdot (0, 1, 2) = 0 - 2 + 2 = \underline{\underline{0}}$$

$$(c) \quad A \times (B \times C) = \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \end{vmatrix} = \underline{\underline{-2}}\mathbf{a}_x \underline{\underline{-2}}\mathbf{a}_y \underline{\underline{-2}}\mathbf{a}_z$$

$$(d) \quad (A \times B) \times C = \begin{vmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = \underline{\underline{-5}}\mathbf{a}_x \underline{\underline{-2}}\mathbf{a}_y \underline{\underline{+1}}\mathbf{a}_z$$

Prob. 1.6

$$(a) \quad T = (\underline{3}, \underline{-2}, 1) \text{ and } S = (\underline{4}, \underline{6}, 2)$$

$$(b) \quad r_{TS} = r_s - r_t = (4, 6, 2) - (3, -2, 1) = \underline{\underline{\mathbf{a}_x + 8\mathbf{a}_y + \mathbf{a}_z}}$$

$$(c) \quad \text{distance} = |r_{TS}| = \sqrt{1 + 64 + 1} = \underline{\underline{8.124}} \text{ m}$$

Prob. 1.7

- (a) If \mathbf{A} and \mathbf{B} are parallel, $\mathbf{A}=k\mathbf{B}$, where k is a constant.

$$(\alpha, 3, -2) = k(4, \beta, 8)$$

Equating coefficients gives

$$-2 = 8k \longrightarrow k = -\frac{1}{4}$$

$$\alpha = 4k = \underline{\underline{-1}}$$

$$3 = \beta k \longrightarrow \beta = 3/k = \underline{\underline{-12}}$$

This can also be solved using $\mathbf{A} \times \mathbf{B} = 0$.

- (b) If \mathbf{A} and \mathbf{B} are perpendicular to each other,

$$\mathbf{A} \cdot \mathbf{B} = 0 \longrightarrow \underline{\underline{4\alpha + 3\beta - 16 = 0}}$$

Prob. 1.8

- (a) $\mathbf{A} \cdot \mathbf{B} = AB \cos \theta_{AB}$

$$\mathbf{A} \times \mathbf{B} = AB \sin \theta_{AB} \mathbf{a}_n$$

$$(\mathbf{A} \cdot \mathbf{B})^2 + |\mathbf{A} \times \mathbf{B}|^2 = (AB)^2 (\cos^2 \theta_{AB} + \sin^2 \theta_{AB}) = (AB)^2$$

- (b) $\mathbf{a}_x \cdot (\mathbf{a}_y \times \mathbf{a}_z) = \mathbf{a}_x \cdot \mathbf{a}_x = 1$. Hence,

$$\frac{\mathbf{a}_y \times \mathbf{a}_z}{\mathbf{a}_x \cdot \mathbf{a}_y \times \mathbf{a}_z} = \frac{\mathbf{a}_x}{1} = \mathbf{a}_x$$

$$\frac{\mathbf{a}_z \times \mathbf{a}_x}{\mathbf{a}_x \cdot \mathbf{a}_y \times \mathbf{a}_z} = \frac{\mathbf{a}_y}{1} = \mathbf{a}_y$$

$$\frac{\mathbf{a}_x \times \mathbf{a}_y}{\mathbf{a}_x \cdot \mathbf{a}_y \times \mathbf{a}_z} = \frac{\mathbf{a}_z}{1} = \mathbf{a}_z$$

Prob. 1.9

- (a) $\mathbf{P} + \mathbf{Q} = (6, 2, 0)$, $\mathbf{P} + \mathbf{Q} - \mathbf{R} = (7, 1, -2)$

$$|\mathbf{P} + \mathbf{Q} - \mathbf{R}| = \sqrt{49 + 1 + 4} = \sqrt{54} = \underline{\underline{7.3485}}$$

$$(b) \mathbf{P} \cdot \mathbf{Q} \times \mathbf{R} = \begin{vmatrix} 2 & -1 & -2 \\ 4 & 3 & 2 \\ -1 & 1 & 2 \end{vmatrix} = 2(6-2) + (8+2) - 2(4+3) = 8+10-14 = \underline{\underline{4}}$$

$$\mathbf{Q} \times \mathbf{R} = \begin{vmatrix} 4 & 3 & 2 \\ -1 & 1 & 2 \end{vmatrix} = (4, -10, 7)$$

$$\mathbf{P} \cdot \mathbf{Q} \times \mathbf{R} = (2, -1, -2) \cdot (4, -10, 7) = 8+10-14 = \underline{\underline{4}}$$

$$(c) \mathbf{Q} \times \mathbf{P} = \begin{vmatrix} 4 & 3 & 2 \\ 2 & -1 & -2 \end{vmatrix} = (-4, 12, -10)$$

$$\mathbf{Q} \times \mathbf{P} \cdot \mathbf{R} = (-4, 12, -10) \cdot (-1, 1, 2) = 4+12-20 = \underline{\underline{-4}}$$

$$\text{or } \mathbf{Q} \times \mathbf{P} \cdot \mathbf{R} = \mathbf{R} \cdot \mathbf{Q} \times \mathbf{P} = \begin{vmatrix} -1 & 1 & 2 \\ 4 & 3 & 2 \\ 2 & -1 & -2 \end{vmatrix} = -(-6+2) - (-8-4) + 2(-4-6) = \underline{\underline{-4}}$$

$$(d) (\mathbf{P} \times \mathbf{Q}) \cdot (\mathbf{Q} \times \mathbf{R}) = (4, -12, 10) \cdot (4, -10, 7) = 16+120+70 = \underline{\underline{206}}$$

$$(e) (\mathbf{P} \times \mathbf{Q}) \times (\mathbf{Q} \times \mathbf{R}) = \begin{vmatrix} 4 & -12 & 10 \\ 4 & -10 & 7 \end{vmatrix} = \underline{\underline{16a_x + 12a_y + 8a_z}}$$

$$(f) \cos \theta_{PR} = \frac{\mathbf{P} \cdot \mathbf{R}}{\|\mathbf{P}\| \|\mathbf{R}\|} = \frac{(-2-1-4)}{\sqrt{4+1+4} \sqrt{1+1+4}} = \frac{-7}{3\sqrt{6}} = -0.9526$$

$$\underline{\underline{\theta_{PR} = 162.3^\circ}}$$

$$(g) \sin \theta_{PQ} = \frac{|\mathbf{P} \times \mathbf{Q}|}{\|\mathbf{P}\| \|\mathbf{Q}\|} = \frac{\sqrt{16+144+100}}{3\sqrt{16+9+4}} = \frac{\sqrt{260}}{3\sqrt{29}} = 0.998$$

$$\underline{\underline{\theta_{PQ} = 86.45^\circ}}$$

Prob. 1.10

If \mathbf{A} and \mathbf{B} are parallel, then $\mathbf{B} = k\mathbf{A}$ and $\mathbf{A} \times \mathbf{B} = \mathbf{0}$. It is evident that $k = -2$ and that

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 1 & -2 & 3 \\ -2 & 4 & -6 \end{vmatrix} = \mathbf{0}$$

as expected.

Prob. 1.11

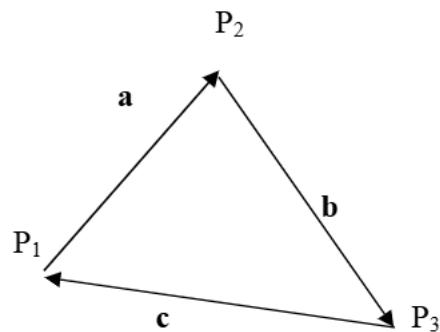
(a) Using the fact that

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{B} \cdot \mathbf{C})\mathbf{A},$$

we get

$$\mathbf{A} \times (\mathbf{A} \times \mathbf{B}) = -(\mathbf{A} \times \mathbf{B}) \times \mathbf{A} = \underline{(\mathbf{B} \cdot \mathbf{A})\mathbf{A} - (\mathbf{A} \cdot \mathbf{A})\mathbf{B}}$$

$$\begin{aligned} \text{(b)} \quad \mathbf{A} \times (\mathbf{A} \times (\mathbf{A} \times \mathbf{B})) &= \mathbf{A} \times [(\mathbf{B} \cdot \mathbf{A})\mathbf{A} - (\mathbf{A} \cdot \mathbf{A})\mathbf{B}] \\ &= (\mathbf{A} \cdot \mathbf{B}) - (\mathbf{A} \times \mathbf{A}) - (\mathbf{A} \cdot \mathbf{A}) - (\mathbf{A} \times \mathbf{B}) \\ &= -\mathbf{A}^2 (\mathbf{A} \times \mathbf{B}) \end{aligned}$$

since $\mathbf{A} \times \mathbf{A} = \mathbf{0}$ **Prob. 1.12**

$$\mathbf{a} \cdot \mathbf{r}_{P_2} - \mathbf{r}_{P_1} = (1, -2, 4) - (5, -3, 1) = (-4, 1, 3)$$

$$(a) \quad \mathbf{b} \cdot \mathbf{r}_{P_3} - \mathbf{r}_{P_2} = (3, 3, 5) - (1, -2, 4) = (2, 5, 1)$$

$$\mathbf{c} = \mathbf{r}_{P_1} - \mathbf{r}_{P_3} = (5, -3, 1) - (3, 3, 5) = (2, -6, -4)$$

Note that $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$

$$\mathbf{a} \cdot \mathbf{b} = -8 + 5 + 3 = 0 \quad \longrightarrow \quad \text{perpendicular}$$

$$\mathbf{b} \cdot \mathbf{c} = 4 - 30 - 4 \neq 0$$

$$\mathbf{c} \cdot \mathbf{a} = -8 - 6 - 12 \neq 0$$

Hence $\underline{\underline{P_2}}$ is a right angle.

$$\begin{aligned} \text{(b)} \quad \text{Area} &= \frac{1}{2} |\mathbf{a} \times \mathbf{b}| = \frac{1}{2} \begin{vmatrix} -4 & 1 & 3 \\ 2 & 5 & 1 \end{vmatrix} = \frac{1}{2} |(1-15)\mathbf{a}_x + (6+4)\mathbf{a}_y + (-20-2)\mathbf{a}_z| \\ &= \frac{1}{2} |(-14, 10, -22)| = \frac{1}{2} \sqrt{196+100+484} = \underline{\underline{13.96}} \end{aligned}$$

Prob. 1.13

Given $\mathbf{r}_P = (-1, 4, 8)$, $\mathbf{r}_Q = (2, -1, 3)$, $\mathbf{r}_R = (-1, 2, 3)$

(a) $|PQ| = \sqrt{9+25+25} = \underline{\underline{7.6811}}$

(b) $\mathbf{PR} = \underline{\underline{-2\mathbf{a}_y - 5\mathbf{a}_z}}$

(c) $\angle PQR = \cos^{-1}\left(\frac{\mathbf{QP} \cdot \mathbf{QR}}{|\mathbf{QP}| |\mathbf{QR}|}\right) = \underline{\underline{42.57^\circ}}$

(d) Area of triangle PQR = 11.023

(e) Perimeter = 17.31

Prob. 1.14

Let R be the midpoint of PQ.

$$\mathbf{r}_R = \frac{1}{2}\{(2, 4, -1) + (12, 16, 9)\} = (7, 10, 4)$$

$$OR = \sqrt{49+100+16} = \sqrt{165} = 12.845$$

$$t = \frac{OR}{v} = \frac{12.845}{300} = \underline{\underline{42.82 \text{ ms}}}$$

Prob. 1.15

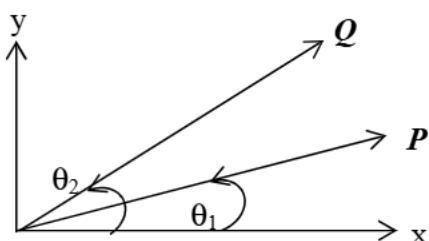
$$\mathbf{A} \cdot (\mathbf{A} \times \mathbf{B}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}, (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

Hence, $\boxed{\mathbf{A} \cdot (\mathbf{A} \times \mathbf{B}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}}$

Also, each equals the volume of the parallelopiped formed by the three vectors as sides.

Prob. 1.16

(a) Let \mathbf{P} and \mathbf{Q} be as shown below:



$$|\mathbf{P}| = \cos^2 \theta_1 + \sin^2 \theta_1 = 1, |\mathbf{Q}| = \cos^2 \theta_2 + \sin^2 \theta_2 = 1,$$

Hence \mathbf{P} and \mathbf{Q} are unit vectors.

$$(b) \mathbf{P} \cdot \mathbf{Q} = (1)(1)\cos(\theta_2 - \theta_1)$$

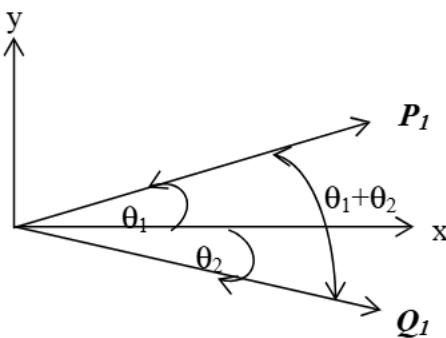
But $\mathbf{P} \cdot \mathbf{Q} = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2$. Thus,

$$\underline{\underline{\cos(\theta_2 - \theta_1) = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2}}$$

Let $\mathbf{P}_1 = \mathbf{P} = \cos \theta_1 \mathbf{a}_x + \sin \theta_1 \mathbf{a}_y$ and

$$\mathbf{Q}_1 = \cos \theta_2 \mathbf{a}_x - \sin \theta_2 \mathbf{a}_y.$$

\mathbf{P}_1 and \mathbf{Q}_1 are unit vectors as shown below:



$$\mathbf{P}_1 \cdot \mathbf{Q}_1 = (1)(1)\cos(\theta_1 + \theta_2)$$

But $\mathbf{P}_1 \cdot \mathbf{Q}_1 = \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2$,

$$\underline{\underline{\cos(\theta_2 + \theta_1) = \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2}}$$

Alternatively, we can obtain this formula from the previous one by replacing θ_2 by $-\theta_2$ in \mathbf{Q} .

(c)

$$\frac{1}{2} |\mathbf{P} - \mathbf{Q}| = \frac{1}{2} |(\cos \theta_1 - \cos \theta_2) \mathbf{a}_x + (\sin \theta_1 - \sin \theta_2) \mathbf{a}_y|$$

$$= \frac{1}{2} \sqrt{\cos^2 \theta_1 + \sin^2 \theta_1 + \cos^2 \theta_2 + \sin^2 \theta_2 - 2 \cos \theta_1 \cos \theta_2 - 2 \sin \theta_1 \sin \theta_2}$$

$$= \frac{1}{2} \sqrt{2 - 2(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2)} = \frac{1}{2} \sqrt{2 - 2 \cos(\theta_2 - \theta_1)}$$

Let $\theta_2 - \theta_1 = \theta$, the angle between \mathbf{P} and \mathbf{Q} .

$$\frac{1}{2} |\mathbf{P} - \mathbf{Q}| = \frac{1}{2} \sqrt{2 - 2 \cos \theta}$$

But $\cos 2A = 1 - 2 \sin^2 A$.

$$\frac{1}{2} |\mathbf{P} - \mathbf{Q}| = \frac{1}{2} \sqrt{2 - 2 + 4 \sin^2 \theta/2} = \sin \theta/2$$

Thus,

$$\frac{1}{2} |\mathbf{P} - \mathbf{Q}| = \sin \frac{\theta_2 - \theta_1}{2}$$

Prob. 1.17

$$\mathbf{w} = \frac{w(1, -2, 2)}{3} = (1, -2, 2), \quad \mathbf{r} = \mathbf{r}_p - \mathbf{r}_o = (1, 3, 4) - (2, -3, 1) = (-1, 6, 3)$$

$$\mathbf{u} = \mathbf{w} \times \mathbf{r} = \begin{vmatrix} 1 & -2 & 2 \\ -1 & 6 & 3 \end{vmatrix} = (-18, -5, 4)$$

$$\mathbf{u} = -18\mathbf{a}_x - 5\mathbf{a}_y + 4\mathbf{a}_z$$

Prob. 1.18

$$\mathbf{r}_1 = (1, 1, 1), \quad \mathbf{r}_2 = (1, 0, 1) - (0, 1, 0) = (1, -1, 1)$$

$$\cos \theta = \frac{\mathbf{r}_1 \cdot \mathbf{r}_2}{\|\mathbf{r}_1\| \|\mathbf{r}_2\|} = \frac{(1-1+1)}{\sqrt{3}\sqrt{3}} = \frac{1}{3} \quad \longrightarrow \quad \underline{\underline{\theta = 70.53^\circ}}$$

Prob. 1.19

$$(a) T_s = T \cdot \mathbf{a}_s = \frac{T \cdot S}{|S|} = \frac{(2, -6, 3) \cdot (1, 2, 1)}{\sqrt{6}} = \frac{-7}{\sqrt{6}} = \underline{\underline{-2.8577}}$$

$$(b) \mathbf{S}_T = (\mathbf{S} \cdot \mathbf{a}_T) \mathbf{a}_T = \frac{(\mathbf{S} \cdot \mathbf{T}) \mathbf{T}}{\mathbf{T}^2} = \frac{-7(2, -6, 3)}{7^2} \\ = -0.2857\mathbf{a}_x + 0.8571\mathbf{a}_y - 0.4286\mathbf{a}_z$$

$$(c) \sin \theta_{TS} = \frac{|\mathbf{T} \times \mathbf{S}|}{\|\mathbf{T}\| \|\mathbf{S}\|} = \frac{|(2, -6, 3) \times (1, 2, 1)|}{\sqrt{6} \sqrt{245}} = \frac{\sqrt{245}}{7\sqrt{6}} = 0.9129 \\ \Rightarrow \theta_{TS} = \underline{\underline{65.91^\circ}}$$

Prob. 1.20

$$\text{Let } \mathbf{A} = \mathbf{A}_{B\parallel} + \mathbf{A}_{B\perp}$$

$$\mathbf{A}_{B\parallel} = (\mathbf{A} \cdot \mathbf{a}_B) \mathbf{a}_B = \frac{\mathbf{A} \cdot \mathbf{B}}{\mathbf{B} \cdot \mathbf{B}} \mathbf{B}$$

Hence,

$$\mathbf{A}_{B\perp} = \mathbf{A} - \mathbf{A}_{B\parallel} = \mathbf{A} - \frac{\mathbf{A} \cdot \mathbf{B}}{\mathbf{B} \cdot \mathbf{B}} \mathbf{B}$$

Prob. 1.21

(a) $\mathbf{H}(1, 3, -2) = 6\mathbf{a}_x + \mathbf{a}_y + 4\mathbf{a}_z$

$$\mathbf{a}_H = \frac{(6, 1, 4)}{\sqrt{36+1+16}} = \underline{\underline{0.8242\mathbf{a}_x + 0.1374\mathbf{a}_y + 0.5494\mathbf{a}_z}}$$

(b) $|\mathbf{H}| = 10 = \sqrt{4x^2y^2 + (x+z)^2 + z^4}$

or

$$\underline{\underline{100 = 4x^2y^2 + x^2 + 2xz + z^2 + z^4}}$$

Prob. 1.22

$$\mathbf{C} = 5\mathbf{a}_x + \mathbf{a}_z$$

(a) $\mathbf{B} \times \mathbf{C} = \begin{vmatrix} 1 & 1 & 0 \\ 5 & 0 & 1 \end{vmatrix} = \mathbf{a}_x - \mathbf{a}_y - 5\mathbf{a}_z$

$$\underline{\underline{\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (4, -1, 1) \cdot (1, -1, -5) = 4 + 1 - 5 = 0}}$$

(b) $\mathbf{A}_B = (\mathbf{A} \cdot \mathbf{a}_B)\mathbf{a}_B = \frac{(\mathbf{A} \cdot \mathbf{B})\mathbf{B}}{|\mathbf{B}|^2} = \frac{(4-1)(1, 1, 0)}{1+1} = \underline{\underline{1.5\mathbf{a}_x + 1.5\mathbf{a}_y}}$

Prob. 1.23

(a) At (1, -2, 3), x = 1, y = -2, z = 3.

$$\mathbf{G} = \mathbf{a}_x + 2\mathbf{a}_y + 6\mathbf{a}_z, \quad \mathbf{H} = -6\mathbf{a}_x + 3\mathbf{a}_y - 3\mathbf{a}_z$$

$$G = \sqrt{1+4+36} = \underline{\underline{6.403}}$$

$$H = \sqrt{36+9+9} = \underline{\underline{7.348}}$$

(b) $\mathbf{G} \cdot \mathbf{H} = -6 + 6 - 18 = \underline{\underline{-18}}$

(c) $\cos \theta_{GH} = \frac{\mathbf{G} \cdot \mathbf{H}}{GH} = \frac{-18}{6.403 \times 7.348} = -0.3826$

$$\underline{\underline{\theta_{GH} = 112.5^\circ}}$$

Prob. 1.24

$$\mathbf{r}_{PQ} = \mathbf{r}_Q - \mathbf{r}_P = (-2, 1, 4) - (1, 0, 3) = (-3, 1, 1)$$

At P, $\mathbf{H} = 0\mathbf{a}_x - 1\mathbf{a}_z = -\mathbf{a}_z$

The scalar component of \mathbf{H} along \mathbf{r}_{PQ} is

$$D = \mathbf{H} \cdot \mathbf{a}_{\mathbf{r}_{PQ}} = \frac{\mathbf{H} \cdot \mathbf{r}_{PQ}}{|\mathbf{r}_{PQ}|} = \frac{-1}{\sqrt{9+1+1}} = \underline{\underline{-0.3015}}$$

Prob. 1.25

- (a) At P, x = -1, y = 2, z = 4

$$\mathbf{D} = 8\mathbf{a}_x - 4\mathbf{a}_y - 2\mathbf{a}_z, \quad \mathbf{E} = -10\mathbf{a}_x + 24\mathbf{a}_y + 128\mathbf{a}_z$$

$$\mathbf{C} = \mathbf{D} + \mathbf{E} = -2\mathbf{a}_x + 20\mathbf{a}_y + 126\mathbf{a}_z$$

$$(b) \mathbf{C} \cdot \mathbf{a}_x = C \cos \theta_x \quad \longrightarrow \quad \cos \theta_x = \frac{\mathbf{C} \cdot \mathbf{a}_x}{C} = \frac{-2}{\sqrt{2^2 + 20^2 + 126^2}} = -0.01575$$

$$\underline{\underline{\theta_x = 90.9^\circ}}$$

Prob. 1.26

- (a) At (1,2,3),
- $\mathbf{E} = (2,1,6)$

$$|\mathbf{E}| = \sqrt{4+1+36} = \sqrt{41} = \underline{\underline{6.403}}$$

- (b) At (1,2,3),
- $\mathbf{F} = (2,-4,6)$

$$\begin{aligned} \mathbf{E}_F &= (\mathbf{E} \cdot \mathbf{a}_F) \mathbf{a}_F = \frac{(\mathbf{E} \cdot \mathbf{F}) \mathbf{F}}{|\mathbf{F}|^2} = \frac{36}{56} (2, -4, 6) \\ &= \underline{\underline{1.286\mathbf{a}_x - 2.571\mathbf{a}_y + 3.857\mathbf{a}_z}} \end{aligned}$$

- (c) At (0,1,-3),
- $\mathbf{E} = (0,1,-3)$
- ,
- $\mathbf{F} = (0,-1,0)$

$$\mathbf{E} \times \mathbf{F} = \begin{vmatrix} 0 & 1 & -3 \\ 0 & -1 & 0 \end{vmatrix} = (-3, 0, 0)$$

$$\mathbf{a}_{E \times F} = \pm \frac{\mathbf{E} \times \mathbf{F}}{|\mathbf{E} \times \mathbf{F}|} = \underline{\underline{\pm \mathbf{a}_x}}$$

CHAPTER 2

P. E. 2.1

(a) At P(1,3,5), $x = 1$, $y = 3$, $z = 5$,

$$\rho = \sqrt{x^2 + y^2} = \sqrt{10}, \quad z = 5, \quad \phi = \tan^{-1} y/x = \tan^{-1} 3 = 71.6^\circ$$

$$P(\rho, \phi, z) = P(\sqrt{10}, \tan^{-1} 3, 5) = \underline{\underline{P(3.162, 71.6^\circ, 5)}}$$

Spherical system:

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{35} = 5.916$$

$$\theta = \tan^{-1} \sqrt{x^2 + y^2}/z = \tan^{-1} \sqrt{10}/5 = \tan^{-1} 0.6325 = 32.31^\circ$$

$$T(r, \theta, \phi) = \underline{\underline{T(5.916, 32.31^\circ, 71.57^\circ)}}$$

At T(0,-4,3), $x = 0$, $y = -4$, $z = 3$;

$$\rho = \sqrt{x^2 + y^2} = 4, z = 3, \phi = \tan^{-1} y/x = \tan^{-1} -4/0 = 270^\circ$$

$$T(\rho, \phi, z) = \underline{\underline{T(4, 270^\circ, 3)}}.$$

Spherical system:

$$r = \sqrt{x^2 + y^2 + z^2} = 5, \theta = \tan^{-1} \rho/z = \tan^{-1} 4/3 = 53.13^\circ.$$

$$T(r, \theta, \phi) = \underline{\underline{T(5, 53.13^\circ, 270^\circ)}}.$$

At S(-3-4-10), $x = -3$, $y = -4$, $z = -10$;

$$\rho = \sqrt{x^2 + y^2} = 5, \phi = \tan^{-1} \left(\frac{-4}{-3} \right) = 233.1^\circ$$

$$S(\rho, \phi, z) = \underline{\underline{S(5, 233.1, -10)}}.$$

Spherical system:

$$r = \sqrt{x^2 + y^2 + z^2} = 5\sqrt{5} = 11.18.$$

$$\theta = \tan^{-1} \rho/z = \tan^{-1} \frac{5}{-10} = 153.43^\circ;$$

$$S(r, \theta, \phi) = \underline{\underline{S(11.18, 153.43^\circ, 233.1^\circ)}}.$$

(b) In Cylindrical system, $\rho = \sqrt{x^2 + y^2}$; $yz = z\rho \sin \phi$,

$$Q_x = \frac{\rho}{\sqrt{\rho^2 + z^2}}; \quad Q_y = 0; \quad Q_z = -\frac{z\rho \sin \phi}{\sqrt{\rho^2 + z^2}};$$

$$\begin{bmatrix} Q_\rho \\ Q_\phi \\ Q_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Q_x \\ 0 \\ Q_z \end{bmatrix};$$

$$Q_\rho = Q_x \cos\phi = \frac{\rho \cos\phi}{\sqrt{\rho^2 + z^2}}, \quad Q_\phi = -Q_x \sin\phi = \frac{-\rho \sin\phi}{\sqrt{\rho^2 + z^2}}$$

Hence,

$$\underline{\underline{Q}} = \frac{\rho}{\sqrt{\rho^2 + z^2}} (\cos\phi \mathbf{a}_\rho - \sin\phi \mathbf{a}_\phi - z \sin\phi \mathbf{a}_z).$$

In Spherical coordinates:

$$Q_x = \frac{r \sin\theta}{r} = \sin\theta;$$

$$Q_z = -r \sin\phi \sin\theta r \cos\theta \frac{1}{r} = -r \sin\theta \cos\theta \sin\phi.$$

$$\begin{bmatrix} Q_r \\ Q_\theta \\ Q_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta \cos\phi & \sin\theta \sin\phi & \cos\theta \\ \cos\theta \cos\phi & \cos\theta \sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} Q_x \\ 0 \\ Q_z \end{bmatrix};$$

$$Q_r = Q_x \sin\theta \cos\phi + Q_z \cos\theta = \sin^2\theta \cos\phi - r \sin\theta \cos^2\theta \sin\phi.$$

$$Q_\theta = Q_x \cos\theta \cos\phi - Q_z \sin\theta = \sin\theta \cos\theta \cos\phi + r \sin^2\theta \cos\theta \sin\phi.$$

$$Q_\phi = -Q_x \sin\phi = -\sin\theta \sin\phi.$$

$$\therefore \underline{\underline{Q}} = \sin\theta (\sin\theta \cos\phi - r \cos^2\theta \sin\phi) \mathbf{a}_r + \sin\theta \cos\theta (\cos\phi + r \sin\theta \sin\phi) \mathbf{a}_\theta - \sin\theta \sin\phi \mathbf{a}_\phi.$$

At T :

$$\underline{\underline{Q}}(x, y, z) = \frac{4}{5} \mathbf{a}_x + \frac{12}{5} \mathbf{a}_z = 0.8 \mathbf{a}_x + 2.4 \mathbf{a}_z;$$

$$\begin{aligned} \underline{\underline{Q}}(\rho, \phi, z) &= \frac{4}{5} (\cos 270^\circ \mathbf{a}_\rho - \sin 270^\circ \mathbf{a}_\phi - 3 \sin 270^\circ \mathbf{a}_z) \\ &= 0.8 \mathbf{a}_\phi + 2.4 \mathbf{a}_z; \end{aligned}$$

$$\begin{aligned} \underline{\underline{Q}}(r, \theta, \phi) &= \frac{4}{5} (0 - \frac{45}{25}(-1)) \mathbf{a}_r + \frac{4}{5} (\frac{3}{5})(0 + \frac{20}{5}(-1)) \mathbf{a}_\theta - \frac{4}{5} (-1) \mathbf{a}_\phi \\ &= \frac{36}{25} \mathbf{a}_r - \frac{48}{25} \mathbf{a}_\theta + \frac{4}{5} \mathbf{a}_\phi = \underline{\underline{1.44 \mathbf{a}_r - 1.92 \mathbf{a}_\theta + 0.8 \mathbf{a}_\phi}}; \end{aligned}$$

Note, that the magnitude of vector $\mathbf{Q} = 2.53$ in all 3 cases above.

P.E. 2.2 (a)

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho z \sin\phi \\ 3\rho \cos\phi \\ \rho \cos\phi \sin\phi \end{bmatrix}$$

$$\mathbf{A} = (\rho z \cos\phi \sin\phi - 3\rho \cos\phi \sin\phi) \mathbf{a}_x + (\rho z \sin^2\phi + 3\rho \cos^2\phi) \mathbf{a}_y + \rho \cos\phi \sin\phi \mathbf{a}_z.$$

$$\text{But } \rho = \sqrt{x^2 + y^2}, \tan\phi = \frac{y}{x}, \cos\phi = \frac{x}{\sqrt{x^2 + y^2}}, \sin\phi = \frac{y}{\sqrt{x^2 + y^2}};$$

Substituting all this yields :

$$\mathbf{A} = \frac{1}{\sqrt{x^2 + y^2}} [(xyz - 3xy) \mathbf{a}_x + (zy^2 + 3x^2) \mathbf{a}_y + xy \mathbf{a}_z].$$

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \sin\theta \cos\phi & \cos\theta \cos\phi & -\sin\phi \\ \sin\theta \sin\phi & \cos\theta \sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} r^2 \\ 0 \\ \sin\theta \end{bmatrix}$$

$$\text{Since } r = \sqrt{x^2 + y^2 + z^2}, \tan\theta = \frac{\sqrt{x^2 + y^2}}{z}, \tan\phi = \frac{y}{z};$$

$$\text{and } \sin\theta = \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}}, \cos\theta = \frac{z}{\sqrt{x^2 + y^2 + z^2}};$$

$$\text{and } \sin\phi = \frac{y}{\sqrt{x^2 + y^2}}, \cos\phi = \frac{x}{\sqrt{x^2 + y^2}};$$

$$B_x = r^2 \sin\theta \cos\phi - \sin\theta \sin\phi = rx - \frac{y}{r} = \frac{1}{r}(r^2 x - y).$$

$$B_y = r^2 \sin\theta \sin\phi + \sin\theta \cos\phi = ry + \frac{x}{r} = \frac{1}{r}(r^2 y + x).$$

$$B_z = r^2 \cos\theta = rz = \frac{1}{r}(r^2 z).$$

Hence,

$$\mathbf{B} = \frac{1}{\sqrt{x^2 + y^2 + z^2}} [\{x(x^2 + y^2 + z^2) - y\} \mathbf{a}_x + \{y(x^2 + y^2 + z^2) + x\} \mathbf{a}_y + z(x^2 + y^2 + z^2) \mathbf{a}_z].$$

P.E.2.3 (a) At:

$$(1, \pi/3, 0), \quad \mathbf{H} = (0, 0.06767, 1)$$

$$\mathbf{a}_x = \cos \phi \mathbf{a}_\rho - \sin \phi \mathbf{a}_\phi = \frac{1}{2} (\mathbf{a}_\rho - \sqrt{3} \mathbf{a}_\phi)$$

$$\mathbf{H} \bullet \mathbf{a}_x = \underline{\underline{-0.0586}}$$

(b) At:

$$(1, \pi/3, 0), \quad \mathbf{a}_\theta = \cos \theta \mathbf{a}_\rho - \sin \theta \mathbf{a}_z = -\mathbf{a}_z.$$

$$\mathbf{H} \times \mathbf{a}_z = \begin{vmatrix} \mathbf{a}_\rho & \mathbf{a}_\phi & \mathbf{a}_z \\ 0 & 0.06767 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \underline{\underline{-0.06767 \mathbf{a}_\rho}}$$

$$(c) (\mathbf{H} \bullet \mathbf{a}_\rho) \mathbf{a}_\rho = \underline{\underline{0 \mathbf{a}_\rho}}.$$

$$(d) \mathbf{H} \times \mathbf{a}_z = \begin{vmatrix} \mathbf{a}_\rho & \mathbf{a}_\phi & \mathbf{a}_z \\ 0 & 0.06767 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0.06767 \mathbf{a}_\rho.$$

$$|\mathbf{H} \times \mathbf{a}_z| = \underline{\underline{0.06767}}$$

P.E. 2.4

(a)

$$\mathbf{A} \bullet \mathbf{B} = (3, 2, -6) \bullet (4, 0, 3) = \underline{\underline{-6}}.$$

$$(b) |\mathbf{A} \times \mathbf{B}| = \begin{vmatrix} 3 & 2 & -6 \\ 4 & 0 & 3 \end{vmatrix} = \left| 6\mathbf{a}_r - 33\mathbf{a}_\theta - 8\mathbf{a}_\phi \right|.$$

Thus the magnitude of $\mathbf{A} \times \mathbf{B} = \underline{\underline{34.48}}$.

(c)

At $(1, \pi/3, 5\pi/4)$, $\theta = \pi/3$,

$$\mathbf{a}_z = \cos \theta \mathbf{a}_r - \sin \theta \mathbf{a}_\theta = \frac{1}{2} \mathbf{a}_r - \frac{\sqrt{3}}{2} \mathbf{a}_\theta.$$

$$(\mathbf{A} \bullet \mathbf{a}_z) \mathbf{a}_z = \left(\frac{3}{2} - \sqrt{3} \right) \left(\frac{1}{2} \mathbf{a}_r - \frac{\sqrt{3}}{2} \mathbf{a}_\theta \right) = \underline{\underline{-0.116 \mathbf{a}_r + 0.201 \mathbf{a}_\theta}}$$

P.E. 2.5

In spherical coordinates, the distance between two points is given by eq. 2.33:

$$\begin{aligned} d^2 &= r_2^2 + r_1^2 - (2r_1 r_2 \cos \theta_2 \cos \theta_1) - (2r_1 r_2 \sin \theta_2 \sin \theta_1 \cos(\phi_2 - \phi_1)) \\ 7^2 &= 10^2 + 5^2 - (2 \times 5 \times 10 \times \cos(60) \times \cos(30)) \\ &\quad - (2 \times 5 \times 10 \times \sin(60) \times \sin(30) \times \cos(\phi - 50)) \end{aligned}$$

By solving the above equation, we obtain $\phi = 91^\circ$.

P.E. 2.6

In Cartesian system the dot product of two vectors $x_1\mathbf{a}_x + y_1\mathbf{a}_y + z_1\mathbf{a}_z$ and $x_2\mathbf{a}_x + y_2\mathbf{a}_y + z_2\mathbf{a}_z$ is given by

$$x_1 x_2 + y_1 y_2 + z_1 z_2 \quad (1)$$

Now using the eq. 2.21 to represent spherical coordinates in Cartesian system and the dot product is determined by using Eq. (1)

$$\begin{aligned} \mathbf{r}_A \cdot \mathbf{r}_B &= r_1 \sin \theta_1 \cos \phi_1 r_2 \sin \theta_2 \cos \phi_2 \\ &\quad + r_1 \sin \theta_1 \sin \phi_1 r_2 \sin \theta_2 \sin \phi_2 + r_1 \cos \theta_1 r_2 \cos \theta_2 \\ &= r_1 r_2 (\sin \theta_1 \sin \theta_2 (\cos \phi_1 \cos \phi_2 + \sin \phi_1 \sin \phi_2) + \cos \theta_1 \cos \theta_2) \end{aligned}$$

$$= r_1 r_2 (\sin \theta_1 \sin \theta_2 (\cos(\phi_1 - \phi_2)) + \cos \theta_1 \cos \theta_2)$$

Dot product of the given vectors is 3

Using the above equation

$$\begin{aligned} 3 &= 5.099r(\sin(78.69) \times \sin(158.19) \times \cos(53.13 - 90) + \cos(78.69) \\ &\quad \times \cos(158.19)) \\ &= 5.099r(0.1093) \\ &r = 5.381 \end{aligned}$$

Hence by using the above derived equation we can directly calculate dot product of vectors in spherical system

P.E. 2.7

At $P(0, 2, -5)$, $\phi = 90^\circ$;

$$\begin{aligned} \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} &= \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} B_\rho \\ B_\phi \\ B_z \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -5 \\ 1 \\ -3 \end{bmatrix} \end{aligned}$$

$$\mathbf{B} = -\mathbf{a}_x - 5\mathbf{a}_y - 3\mathbf{a}_z$$

$$(a) \mathbf{A} + \mathbf{B} = (2, 4, 10) + (-1, -5, -3)$$

$$= \underline{\underline{\mathbf{a}_x - \mathbf{a}_y + 7\mathbf{a}_z}}.$$

$$(b) \cos \theta_{AB} = \frac{\mathbf{A} \bullet \mathbf{B}}{AB} = \frac{-52}{\sqrt{4200}}$$

$$\theta_{AB} = \cos^{-1}\left(\frac{-52}{\sqrt{4200}}\right) = \underline{\underline{143.36^\circ}}.$$

$$(c) A_B = \mathbf{A} \bullet \mathbf{a}_B = \frac{\mathbf{A} \bullet \mathbf{B}}{B} = -\frac{52}{\sqrt{35}} = \underline{\underline{-8.789}}.$$

Prob. 2.1

$$\rho = \sqrt{x^2 + y^2} = \sqrt{4+36} = 6.324$$

$$(a) \phi = \tan^{-1} \frac{y}{x} = \tan^{-1} \frac{6}{2} = 71.56^\circ$$

$$P \text{ is } \underline{\underline{(6.324, 71.56^\circ, -4)}}$$

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{4+36+16} = 7.485$$

$$(b) \theta = \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z} = \tan^{-1} \frac{6.324}{-4} = 90^\circ + \tan^{-1} \frac{6.324}{4} = 147.69^\circ$$

$$P \text{ is } \underline{\underline{(7.483, 147.69^\circ, 71.56^\circ)}}$$

Prob.2.2

(a) Given P(1,-4,-3), convert to cylindrical and spherical values;

$$\rho = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-4)^2} = \sqrt{17} = 4.123.$$

$$\phi = \tan^{-1} \frac{y}{x} = \tan^{-1} \frac{-4}{1} = 284.04^\circ.$$

$$\therefore P(\rho, \phi, z) = \underline{\underline{(4.123, 284.04^\circ, -3)}}$$

Spherical :

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{1+16+9} = 5.099.$$

$$\theta = \tan^{-1} \frac{\rho}{z} = \tan^{-1} \frac{4.123}{-3} = 126.04^\circ.$$

$$P(r, \theta, \phi) = \underline{\underline{(5.099, 126.04^\circ, 284.04^\circ)}}.$$

(b) $\rho = 3, \phi = \tan^{-1} \frac{y}{x} = \tan^{-1} \frac{0}{3} = 0^\circ$
 $Q(\rho, \phi, z) = \underline{\underline{Q(3, 0^\circ, 5)}}$

$$r = \sqrt{9 + 0 + 25} = 5.831, \theta = \tan^{-1} \frac{\rho}{z} = \tan^{-1} \frac{3}{5} = 30.96^\circ$$

$$Q(r, \theta, \phi) = \underline{\underline{Q(5.831, 30.96^\circ, 0^\circ)}}$$

(c) $\rho = \sqrt{4 + 36} = 6.325, \phi = \tan^{-1} \frac{6}{-2} = 108.4^\circ$
 $R(\rho, \phi, z) = \underline{\underline{R(6.325, 108.4^\circ, 0)}}$

$$r = \rho = 6.325, \theta = \tan^{-1} \frac{\rho}{z} = \tan^{-1} \frac{6.325}{0} = 90^\circ$$

$$R(r, \theta, \phi) = \underline{\underline{R(6.325, 90^\circ, 108.4^\circ)}}$$

Prob. 2.3

(a)

$$x = \rho \cos \phi = 2 \cos 30^\circ = 1.732;$$

$$y = \rho \sin \phi = 2 \sin 30^\circ = 1;$$

$$z = 5;$$

$$P_1(x, y, z) = \underline{\underline{P_1(1.732, 1, 5)}}.$$

(b)

$$x = 1 \cos 90^\circ = 0; \quad y = 1 \sin 90^\circ = 1; \quad z = -3.$$

$$P_2(x, y, z) = \underline{\underline{P_2(0, 1, -3)}}.$$

(c)

$$x = r \sin \theta \cos \phi = 10 \sin(\pi/4) \cos(\pi/3) = 3.535;$$

$$y = r \sin \theta \sin \phi = 10 \sin(\pi/4) \sin(\pi/3) = 6.124;$$

$$z = r \cos \theta = 10 \cos(\pi/4) = 7.0711$$

$$P_3(x, y, z) = \underline{\underline{P_3(3.535, 6.124, 7.0711)}}.$$

(d)

$$x = 4 \sin 30^\circ \cos 60^\circ = 1$$

$$y = 4 \sin 30^\circ \sin 60^\circ = 1.7321$$

$$z = r \cos \theta = 4 \cos 30^\circ = 3.464$$

$$P_4(x, y, z) = \underline{\underline{P_4(1, 1.7321, 3.464)}}.$$

Prob. 2.4

$$x = \rho \cos \phi = 5 \cos 120^\circ = -2.5$$

(a) $y = \rho \sin \phi = 5 \sin 120^\circ = 4.33$
 $z = 1$

Hence $\underline{\underline{Q}} = (-2.5, 4.33, 1)$

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{\rho^2 + z^2} = \sqrt{25+1} = 5.099$$

(b) $\theta = \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z} = \tan^{-1} \frac{\rho}{z} = \tan^{-1} \frac{5}{1} = 78.69^\circ$
 $\phi = 120^\circ$

Hence $\underline{\underline{\underline{Q}}} = (5.099, 78.69^\circ, 120^\circ)$

Prob. 2.5

$$T(r, \theta, \phi) \longrightarrow r = 10, \theta = 60^\circ, \phi = 30^\circ$$

$$x = r \sin \theta \cos \phi = 10 \sin 60^\circ \cos 30^\circ = 7.5$$

$$y = r \sin \theta \sin \phi = 10 \sin 60^\circ \sin 30^\circ = 4.33$$

$$z = r \cos \theta = 10 \cos 60^\circ = 5$$

$$T(x, y, z) = \underline{\underline{(7.5, 4.33, 5)}}$$

$$\rho = r \sin \theta = 10 \sin 60^\circ = 8.66$$

$$T(\rho, \phi, z) = \underline{\underline{(8.66, 30^\circ, 5)}}$$

Prob. 2.6

(a)

$$x = \rho \cos \phi, \quad y = \rho \sin \phi,$$

$$V = \underline{\underline{\rho z \cos \phi - \rho^2 \sin \phi \cos \phi + \rho z \sin \phi}}$$

(b)

$$\begin{aligned} U &= x^2 + y^2 + z^2 + y^2 + 2z^2 \\ &= r^2 + r^2 \sin^2 \theta \sin^2 \phi + 2r^2 \cos^2 \theta \\ &= \underline{\underline{r^2 [1 + \sin^2 \theta \sin^2 \phi + 2 \cos^2 \theta]}} \end{aligned}$$

Prob. 2.7

(a)

$$\begin{bmatrix} F_\rho \\ F_\phi \\ F_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{x}{\sqrt{\rho^2+z^2}} \\ \frac{y}{\sqrt{\rho^2+z^2}} \\ \frac{4}{\sqrt{\rho^2+z^2}} \end{bmatrix}$$

$$F_\rho = \frac{1}{\sqrt{\rho^2+z^2}} [\rho \cos^2\phi + \rho \sin^2\phi] = \frac{\rho}{\sqrt{\rho^2+z^2}};$$

$$F_\phi = \frac{1}{\sqrt{\rho^2+z^2}} [-\rho \cos\phi \sin\phi + \rho \cos\phi \sin\phi] = 0;$$

$$F_z = \frac{4}{\sqrt{\rho^2+z^2}};$$

$$\bar{F} = \underline{\underline{\frac{1}{\sqrt{\rho^2+z^2}} (\rho a_\rho + 4 a_z)}}$$

In Spherical:

$$\begin{bmatrix} F_r \\ F_\theta \\ F_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta \cos\phi & \sin\theta \sin\phi & \cos\theta \\ \cos\theta \cos\phi & \cos\theta \sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \frac{x}{r} \\ \frac{y}{r} \\ \frac{4}{r} \end{bmatrix}$$

$$F_r = \frac{r}{r} \sin^2\theta \cos^2\phi + \frac{r}{r} \sin^2\theta \sin^2\phi + \frac{4}{r} \cos\theta = \sin^2\theta + \frac{4}{r} \cos\theta;$$

$$F_\theta = \sin\theta \cos\theta \cos^2\phi + \sin\theta \cos\theta \sin^2\phi - \frac{4}{r} \sin\theta = \sin\theta \cos\theta - \frac{4}{r} \sin\theta;$$

$$F_\phi = -\sin\theta \cos\phi \sin\phi + \sin\theta \sin\phi \cos\phi = 0;$$

$$\therefore \bar{F} = \underline{\underline{(\sin^2\theta + \frac{4}{r} \cos\theta) a_r + \sin\theta (\cos\theta - \frac{4}{r}) a_\theta}}$$

(b)

$$\begin{bmatrix} G_\rho \\ G_\phi \\ G_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{x\rho^2}{\sqrt{\rho^2+z^2}} \\ \frac{y\rho^2}{\sqrt{\rho^2+z^2}} \\ \frac{z\rho^2}{\sqrt{\rho^2+z^2}} \end{bmatrix}$$

$$G_\rho = \frac{\rho^2}{\sqrt{\rho^2+z^2}} [\rho \cos^2\phi + \rho \sin^2\phi] = \frac{\rho^3}{\sqrt{\rho^2+z^2}};$$

$$G_\phi = 0;$$

$$G_z = \frac{z\rho^2}{\sqrt{\rho^2+z^2}};$$

$$\underline{\underline{\mathbf{G}}} = \frac{\rho^2}{\sqrt{\rho^2+z^2}} (\rho \mathbf{a}_\rho + z \mathbf{a}_z)$$

Spherical :

$$\mathbf{G} = \frac{\rho^2}{r} (x \mathbf{a}_x + y \mathbf{a}_y + z \mathbf{a}_z) = \frac{r^2 \sin^2\theta}{r} r \mathbf{a}_r = \underline{\underline{r^2 \sin^2\theta \mathbf{a}_r}}$$

Prob. 2.8

$$\mathbf{B} = \rho \mathbf{a}_x + \frac{y}{\rho} \mathbf{a}_y + z \mathbf{a}_z$$

$$\begin{bmatrix} B_\rho \\ B_\phi \\ B_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho \\ y/\rho \\ z \end{bmatrix}$$

$$B_\rho = \rho \cos\phi + \frac{y}{\rho} \sin\phi$$

$$B_\phi = -\rho \sin\phi + \frac{y}{\rho} \cos\phi$$

$$B_z = z$$

$$\text{But } y = \rho \sin\phi$$

$$B_\rho = \rho \cos\phi + \sin^2\phi, B_\phi = -\rho \sin\phi + \sin\phi \cos\phi$$

Hence,

$$\underline{\underline{\mathbf{B}}} = (\rho \cos\phi + \sin^2\phi) \mathbf{a}_\rho + \sin\phi(\cos\phi - \rho) \mathbf{a}_\phi + z \mathbf{a}_z$$

Prob. 2.9

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

At P, $\rho = 2$, $\phi = \pi/2$, $z = -1$

$$A_x = 2 \cos\phi - 3 \sin\phi = 2 \cos 90^\circ - 3 \sin 90^\circ = -3$$

$$A_y = 2 \sin\phi + 3 \cos\phi = 2 \sin 90^\circ + 3 \cos 90^\circ = 2$$

$$A_z = 4$$

$$\text{Hence, } \underline{\underline{A}} = -3\underline{a}_x + 2\underline{a}_y + 4\underline{a}_z$$

Prob. 2.10

(a)

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho \sin\phi \\ \rho \cos\phi \\ -2z \end{bmatrix}$$

$$A_x = \rho \sin\phi \cos\phi - \rho \cos\phi \sin\phi = 0$$

$$A_y = \rho \sin^2\phi + \rho \cos^2\phi = \rho = \sqrt{x^2 + y^2}$$

$$A_z = -2z$$

Hence,

$$\underline{\underline{A}} = \underline{\underline{\sqrt{x^2 + y^2} \underline{a}_y - 2z \underline{a}_z}}$$

(b)

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \sin\theta \cos\phi & \cos\theta \cos\phi & -\sin\phi \\ \sin\theta \sin\phi & \cos\theta \sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} 4r \cos\phi \\ r \\ 0 \end{bmatrix}$$

$$B_x = 4r \sin\theta \cos^2\phi + r \cos\theta \cos\phi$$

$$B_y = 4r \sin\theta \sin\phi \cos\phi + r \cos\theta \sin\phi$$

$$B_z = 4r \cos\theta \cos\phi - r \sin\theta$$

$$\text{But } r = \sqrt{x^2 + y^2 + z^2}, \quad \sin\theta = \frac{\sqrt{x^2 + y^2}}{r}, \quad \cos\theta = \frac{z}{r}$$

$$\sin\phi = \frac{y}{\sqrt{x^2 + y^2}}, \quad \cos\phi = \frac{x}{\sqrt{x^2 + y^2}}$$

$$\begin{aligned}
 B_x &= 4\sqrt{x^2 + y^2} \frac{x^2}{x^2 + y^2} + \frac{zx}{\sqrt{x^2 + y^2}} \\
 B_y &= 4\sqrt{x^2 + y^2} \frac{xy}{x^2 + y^2} + \frac{zy}{\sqrt{x^2 + y^2}} \\
 B_z &= 4z \frac{x}{\sqrt{x^2 + y^2}} - \sqrt{x^2 + y^2} \\
 \mathbf{B} &= \frac{1}{\sqrt{x^2 + y^2}} \left[x(4x+z)\mathbf{a}_x + y(4x+z)\mathbf{a}_y + (4xz - x^2 - y^2)\mathbf{a}_z \right]
 \end{aligned}$$

Prob. 2.11

$$\begin{bmatrix} G_x \\ G_y \\ G_z \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho \sin \phi \\ -\rho \cos \phi \\ \rho \end{bmatrix}$$

$$G_x = \rho \cos \phi \sin \phi + \rho \sin \phi \cos \phi = 2\rho \sin \phi \cos \phi$$

$$G_y = \rho \sin^2 \phi - \rho \cos^2 \phi = \rho(1 - \cos^2 \phi) - \rho \cos^2 \phi = \rho - 2\rho \cos^2 \phi$$

$$G_z = \rho$$

$$\rho = \sqrt{x^2 + y^2}, \cos \phi = \frac{x}{\rho} = \frac{x}{\sqrt{x^2 + y^2}}, \sin \phi = \frac{y}{\rho} = \frac{y}{\sqrt{x^2 + y^2}}$$

$$G_x = 2\sqrt{x^2 + y^2} \frac{xy}{x^2 + y^2} = \frac{2xy}{\sqrt{x^2 + y^2}}$$

$$\text{But } G_y = \sqrt{x^2 + y^2} - 2\sqrt{x^2 + y^2} \frac{x^2}{x^2 + y^2} = \sqrt{x^2 + y^2} - \frac{2x^2}{\sqrt{x^2 + y^2}}$$

$$G_z = \sqrt{x^2 + y^2}$$

Thus,

$$\mathbf{G} = \frac{2xy}{\sqrt{x^2 + y^2}} \mathbf{a}_x + \left(\sqrt{x^2 + y^2} - \frac{2x^2}{\sqrt{x^2 + y^2}} \right) \mathbf{a}_y + \sqrt{x^2 + y^2} \mathbf{a}_z$$

Prob. 2.12

$$\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \phi \\ \sin \theta \sin \phi & \cos \theta \sin \phi & \cos \phi \\ \cos \theta & -\sin \theta & 0 \end{bmatrix} \begin{bmatrix} \cos \theta \\ \sin \theta \\ 0 \end{bmatrix}$$

$$H_x = \sin \theta \cos \theta \cos \phi + \sin \theta \cos \theta \cos \phi = 2 \sin \theta \cos \theta \cos \phi$$

$$H_y = \cos \theta \sin \theta \sin \phi + \sin \theta \cos \theta \sin \phi = 2 \sin \theta \cos \theta \sin \phi$$

$$H_z = \cos^2 \theta - \sin^2 \theta$$

But,

$$\sin \theta = \frac{\sqrt{x^2 + y^2}}{r}, \quad \cos \theta = \frac{z}{r},$$

$$\sin \phi = \frac{y}{\sqrt{x^2 + y^2}}, \quad \cos \phi = \frac{x}{\sqrt{x^2 + y^2}}$$

$$H_x = \frac{2z\sqrt{x^2 + y^2}}{x^2 + y^2 + z^2} \frac{x}{\sqrt{x^2 + y^2}} = \frac{2xz}{x^2 + y^2 + z^2}$$

$$H_y = \frac{2z\sqrt{x^2 + y^2}}{x^2 + y^2 + z^2} \frac{y}{\sqrt{x^2 + y^2}} = \frac{2yz}{x^2 + y^2 + z^2}$$

$$H_z = \frac{z^2 - x^2 - y^2}{x^2 + y^2 + z^2}$$

$$\underline{\underline{\boldsymbol{H}}} = \frac{1}{x^2 + y^2 + z^2} (2xz\boldsymbol{a}_x + 2yz\boldsymbol{a}_y + [z^2 - x^2 - y^2]\boldsymbol{a}_z)$$

Prob. 2.13

$$x = \rho \cos \phi$$

$$(a) \underline{\underline{\boldsymbol{B}}} = \rho \cos \phi \boldsymbol{a}_z$$

$$x = r \sin \theta \cos \phi$$

$$(b) \underline{\underline{\boldsymbol{B}}} = r \sin \theta \cos \phi \boldsymbol{a}_z, \quad B_x = 0 = B_y, B_z = r \sin \theta \cos \phi$$

$$\begin{bmatrix} B_r \\ B_\theta \\ B_\phi \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ r \sin \theta \cos \phi \end{bmatrix}$$

$$B_r = r \sin \theta \cos \theta \cos \phi = 0.5r \sin(2\theta) \cos \phi$$

$$B_\theta = -r \sin^2 \theta \cos \phi, \quad B_\phi = 0$$

$$\underline{\underline{\boldsymbol{B}}} = 0.5r \sin(2\theta) \cos \phi \boldsymbol{a}_r - r \sin^2 \theta \cos \phi \boldsymbol{a}_\theta$$

Prob. 2.14

(a)

$$\mathbf{a}_x \bullet \mathbf{a}_\rho = (\cos \phi \mathbf{a}_\rho - \sin \phi \mathbf{a}_\phi) \bullet \mathbf{a}_\rho = \cos \phi$$

$$\mathbf{a}_x \bullet \mathbf{a}_\phi = (\cos \phi \mathbf{a}_\rho - \sin \phi \mathbf{a}_\phi) \bullet \mathbf{a}_\phi = -\sin \phi$$

$$\mathbf{a}_y \bullet \mathbf{a}_\rho = (\sin \phi \mathbf{a}_\rho + \cos \phi \mathbf{a}_\phi) \bullet \mathbf{a}_\rho = \sin \phi$$

$$\bar{\mathbf{a}}_y \bullet \bar{\mathbf{a}}_\phi = (\sin \phi \mathbf{a}_\rho + \cos \phi \mathbf{a}_\phi) \bullet \mathbf{a}_\phi = \cos \phi$$

(b) and (c)

In spherical system :

$$\mathbf{a}_x = \sin \theta \cos \phi \mathbf{a}_r + \cos \theta \cos \phi \mathbf{a}_\theta - \sin \phi \mathbf{a}_\phi.$$

$$\mathbf{a}_y = \sin \theta \sin \phi \mathbf{a}_r + \cos \theta \sin \phi \mathbf{a}_\theta - \cos \phi \mathbf{a}_\phi.$$

$$\mathbf{a}_z = \cos \theta \mathbf{a}_r - \sin \theta \mathbf{a}_\theta.$$

Hence,

$$\mathbf{a}_x \bullet \mathbf{a}_r = \sin \theta \cos \phi;$$

$$\mathbf{a}_x \bullet \mathbf{a}_\theta = \cos \theta \cos \phi;$$

$$\mathbf{a}_y \bullet \mathbf{a}_r = \sin \theta \sin \phi;$$

$$\mathbf{a}_y \bullet \mathbf{a}_\theta = \cos \theta \sin \phi;$$

$$\bar{\mathbf{a}}_z \bullet \bar{\mathbf{a}}_r = \cos \theta;$$

$$\bar{\mathbf{a}}_z \bullet \bar{\mathbf{a}}_\theta = -\sin \theta;$$

Prob. 2.15

(a)

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{\rho^2 + z^2}.$$

$$\theta = \tan^{-1} \frac{\rho}{z}; \quad \phi = \phi.$$

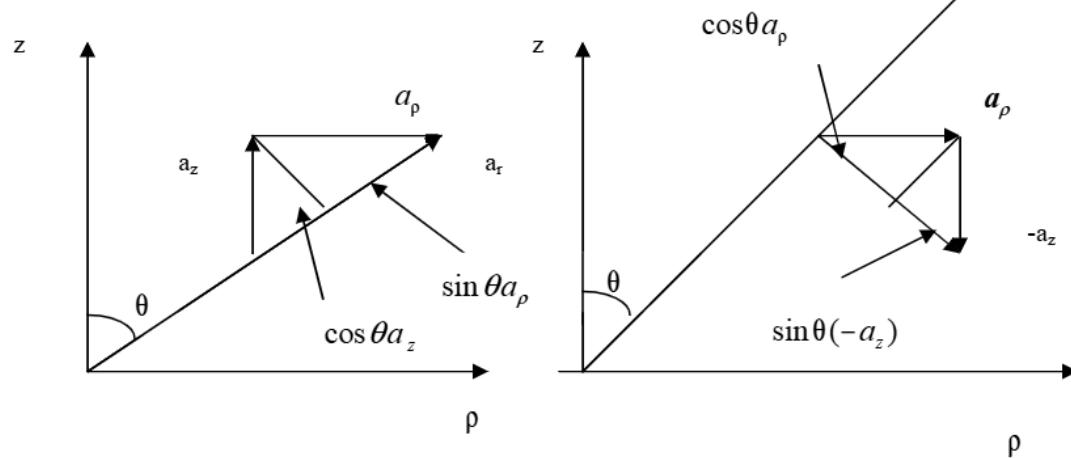
or

$$\rho = \sqrt{x^2 + y^2} = \sqrt{r^2 \sin^2 \theta \cos^2 \phi + r^2 \sin^2 \theta \sin^2 \phi}.$$

$$= r \sin \theta;$$

$$z = r \cos \theta; \quad \phi = \phi.$$

(b) From the figures below,



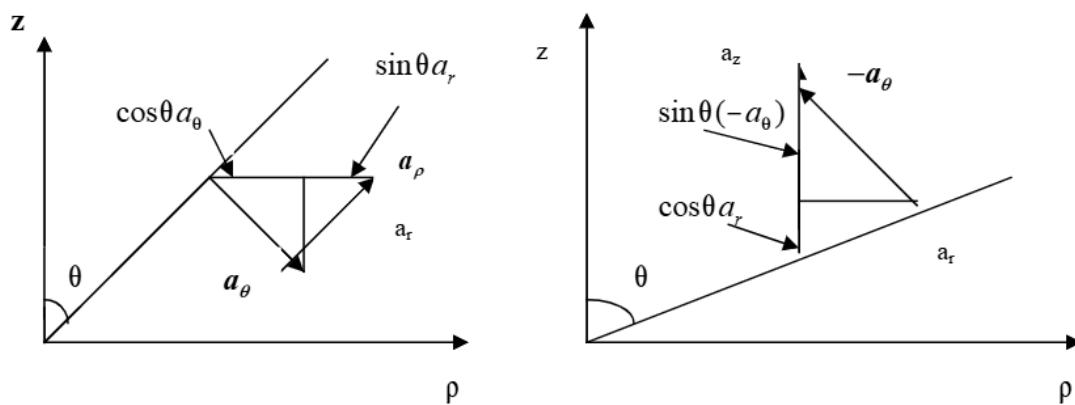
$$\mathbf{a}_r = \sin \theta \mathbf{a}_\rho + \cos \theta \mathbf{a}_z; \quad \mathbf{a}_\theta = \cos \theta \mathbf{a}_\rho - \sin \theta \mathbf{a}_z; \quad \mathbf{a}_\phi = \mathbf{a}_\phi.$$

Hence,

$$\begin{bmatrix} \mathbf{a}_r \\ \mathbf{a}_\theta \\ \mathbf{a}_\phi \end{bmatrix} = \begin{bmatrix} \sin \theta & 0 & \cos \theta \\ \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{a}_\rho \\ \mathbf{a}_\phi \\ \mathbf{a}_z \end{bmatrix}$$

From the figures below,

$$\mathbf{a}_\rho = \cos \theta \mathbf{a}_\theta + \sin \theta \mathbf{a}_r; \quad \mathbf{a}_z = \cos \theta \mathbf{a}_r - \sin \theta \mathbf{a}_\theta; \quad \mathbf{a}_\phi = \mathbf{a}_\phi.$$



$$\begin{bmatrix} \mathbf{a}_\rho \\ \mathbf{a}_\phi \\ \mathbf{a}_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \mathbf{a}_r \\ \mathbf{a}_\theta \\ \mathbf{a}_z \end{bmatrix}$$

Prob. 2.16

If \mathbf{A} and \mathbf{B} are perpendicular to each other, $\mathbf{A} \cdot \mathbf{B} = 0$

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= \rho^2 \sin^2 \phi + \rho^2 \cos^2 \phi - \rho^2 \\ &= \rho^2 (\sin^2 \phi + \cos^2 \phi) - \rho^2 \\ &= \rho^2 - \rho^2 \\ &= 0 \end{aligned}$$

As expected.

Prob. 2.17

$$(a) \mathbf{A} + \mathbf{B} = \underline{\underline{8\mathbf{a}_\rho + 2\mathbf{a}_\phi - 7\mathbf{a}_z}}$$

$$(b) \mathbf{A} \cdot \mathbf{B} = \underline{\underline{15 + 0 - 8}} = \underline{\underline{7}}$$

$$\begin{aligned} (c) \mathbf{A} \times \mathbf{B} &= \begin{vmatrix} 3 & 2 & 1 \\ 5 & 0 & -8 \end{vmatrix} \\ &= -16\mathbf{a}_\rho + (5+24)\mathbf{a}_\phi - 10\mathbf{a}_z \\ &= -16\mathbf{a}_\rho + 29\mathbf{a}_\phi - 10\mathbf{a}_z \end{aligned}$$

$$\begin{aligned} (d) \cos\theta_{AB} &= \frac{\mathbf{A} \cdot \mathbf{B}}{AB} = \frac{7}{\sqrt{9+4+1}\sqrt{25+64}} = \frac{7}{\sqrt{14}\sqrt{89}} \\ &= 0.19831 \end{aligned}$$

$$\underline{\underline{\theta_{AB} = 78.56^\circ}}$$

Prob. 2.18 (a)

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho \cos\phi \\ 0 \\ \rho z^2 \sin\phi \end{bmatrix}$$

$$A_x = \rho \cos^2 \phi = \sqrt{x^2 + y^2} \frac{x^2}{x^2 + y^2} = \frac{x^2}{\sqrt{x^2 + y^2}}$$

$$A_y = \rho \sin \phi \cos \phi = \sqrt{x^2 + y^2} \frac{xy}{x^2 + y^2} = \frac{xy}{\sqrt{x^2 + y^2}}$$

$$A_z = \rho z^2 \sin \phi = \rho z^2 \frac{y}{\rho} = yz^2$$

$$\underline{\underline{\boldsymbol{A}}} = \frac{1}{\sqrt{x^2 + y^2}} [x^2 \boldsymbol{a}_x + xy \boldsymbol{a}_y + yz^2 \sqrt{x^2 + y^2} \boldsymbol{a}_z]$$

At (3, -4, 0) x=3, y=-4, z=0;

$$\underline{\underline{\boldsymbol{A}}} = \frac{1}{5} [9 \boldsymbol{a}_x - 12 \boldsymbol{a}_y]$$

$$\underline{\underline{|\boldsymbol{A}|}} = 3$$

$$(b) \quad \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ -\cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \end{bmatrix} \begin{bmatrix} \frac{x^2}{\rho} \\ \frac{xy}{\rho} \\ yz^2 \end{bmatrix}$$

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta, \quad \rho = r \sin \theta.$$

$$A_r = \frac{r^2 \sin^2 \theta \cos^2 \phi}{r \sin \theta} \sin \theta \cos \phi + \frac{r^2 \sin^2 \theta \cos \phi \sin \phi}{r \sin \theta} \sin \theta \sin \phi + r^3 \sin \theta \cos^2 \theta \sin \phi \cos \theta$$

$$= r \sin^2 \theta \cos^3 \phi + r \sin^2 \theta \sin^2 \phi \cos \phi + r^3 \sin \theta \sin \phi \cos^3 \theta$$

$$A_\theta = r \sin \theta \cos^2 \phi \cos \theta \cos \phi + r \sin \theta \cos \phi \sin \phi \cos \theta \sin \phi - r^2 \cos^2 \theta \sin \phi \sin \theta$$

$$= r \sin \theta \cos \theta \cos \phi - r^2 \sin \theta \cos^2 \sin \phi$$

$$= r \sin \theta \cos \theta [\cos \phi - r \cos \theta \sin \phi]$$

$$A_\phi = -r \sin \theta \cos^2 \phi \sin \phi + r \sin \theta \cos \phi \sin \phi \cos \phi = 0.$$

∴

$$\underline{\underline{\boldsymbol{A}}} = r \sin \theta [\cos \phi \sin \theta + \sin \theta \cos \phi \sin^2 \phi + r^2 \cos^3 \theta \sin \phi] \boldsymbol{a}_r$$

$$+ r \sin \theta \cos \theta [\cos \phi - r^2 \cos \theta \sin \theta \sin \phi] \boldsymbol{a}_\theta$$

$$At (3-4, 0), r = 5, \theta = \pi/2, \phi = 306.83 \\ \cos \phi = 3/5, \sin \phi = -4/5.$$

$$\begin{aligned} A &= 5[1^2 * \frac{3}{5} + 5(0)(-4/5)]\mathbf{a}_r + 5(1)(0)\mathbf{a}_\theta \\ &= 3\mathbf{a}_r \\ \underline{\underline{|A|}} &= 3. \end{aligned}$$

Prob. 2.19

$$\begin{aligned} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} &= \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} \\ &= \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & -\frac{y}{\sqrt{x^2+y^2}} & 0 \\ \frac{y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \phi \\ \sin \theta \sin \phi & \cos \theta \sin \phi & \cos \phi \\ \cos \theta & -\sin \theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

$$= \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2+z^2}} & \frac{xz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} & \frac{-y}{\sqrt{x^2+y^2}} \\ \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{yz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} & \frac{x}{\sqrt{x^2+y^2}} \\ \frac{z}{\sqrt{x^2+y^2+z^2}} & -\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

Prob. 2.20 (a) Using the results in Prob. 2.14,

$$A_\rho = \rho z \sin \phi = r^2 \sin \theta \cos \theta \sin \phi$$

$$A_\phi = 3\rho \cos \phi = 3r \sin \theta \cos \phi$$

$$A_z = \rho \cos \phi \sin \phi = r \sin \theta \cos \phi \sin \phi$$

Hence,

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin \theta & 0 & \cos \theta \\ \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} r^2 \sin \theta \cos \theta \sin \phi \\ 3r \sin \theta \cos \phi \\ r \sin \theta \cos \phi \sin \phi \end{bmatrix}$$

$$A(r, \theta, \phi) = r \sin \theta \left[\underline{\sin \phi \cos \theta (r \sin \theta + \cos \phi) a_r + \sin \phi (r \cos^2 \theta - \sin \theta \cos \phi) a_\theta + 3 \cos \phi a_\phi} \right]$$

At $(10, \pi/2, 3\pi/4)$, $r = 10, \theta = \pi/2, \phi = 3\pi/4$

$$A = 10(0a_r + 0.5a_\theta - \frac{3}{\sqrt{2}}a_\phi) = \underline{\underline{5a_\theta - 21.21a_\phi}}$$

$$(b) \quad B_r = r^2 = (\rho^2 + z^2), \quad B_\theta = 0, \quad B_\phi = \sin \theta = \frac{\rho}{\sqrt{\rho^2 + z^2}}$$

$$\begin{bmatrix} B_\rho \\ B_\phi \\ B_z \end{bmatrix} = \begin{bmatrix} \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \\ \cos \theta & -\sin \theta & 0 \end{bmatrix} \begin{bmatrix} B_r \\ B_\theta \\ B_\phi \end{bmatrix}$$

$$B(\rho, \phi, z) = \sqrt{\rho^2 + z^2} \left(\rho a_\rho + \frac{\rho}{\rho^2 + z^2} a_\phi + z a_z \right)$$

At $(2, \pi/6, 1)$, $\rho = 2, \phi = \pi/6, z = 1$

$$B = \sqrt{5}(2a_\rho + 0.4a_\phi + a_z) = \underline{\underline{4.472a_\rho + 0.8944a_\phi + 2.236a_z}}$$

Prob. 2.21

$$(a) \quad d = \sqrt{(6-2)^2 + (-1-1)^2 + (2-5)^2} = \sqrt{29} = \underline{\underline{5.385}}$$

$$(b) \quad d^2 = 3^2 + 5^2 - 2(3)(5) \cos \pi + (-1-5)^2 = 100$$

$$d = \sqrt{100} = \underline{\underline{10}}$$

(c)

$$\begin{aligned}
 d^2 &= 10^2 + 5^2 - 2(10)(5) \cos \frac{\pi}{4} \cos \frac{\pi}{6} - 2(10)(5) \sin \frac{\pi}{4} \sin \frac{\pi}{6} \cos\left(7 \frac{\pi}{4} - \frac{3\pi}{4}\right) \\
 &= 125 - 100\left(\cos \frac{\pi}{4} \cos \frac{\pi}{6} - \sin \frac{\pi}{4} \sin \frac{\pi}{6}\right) = 125 - 100 \cos 75^\circ = 99.12 \\
 d &= \sqrt{99.12} = \underline{\underline{9.956}}.
 \end{aligned}$$

Prob. 2.22

We can convert Q to cylindrical system and then use equation 2.32

$$\text{At } Q, r = 4 \quad \theta = \frac{\pi}{2} \quad \phi = \frac{\pi}{2}$$

$$\rho = r \sin \theta = 4 \sin 90^\circ = 4$$

$$\phi = \frac{\pi}{2}$$

$$z = r \cos \theta = 4 \cos 90^\circ = 0$$

Q is $(4, \pi/2, 0)$.

$$\begin{aligned}
 d^2 &= \rho_2^2 + \rho_1^2 - 2\rho_1\rho_2 \cos(\phi_2 - \phi_1) + (z_2 - z_1)^2 \\
 &= 10^2 + 4^2 - 2(10)(4) \cos(\pi/4 - \pi/2) + 0 = 59.431
 \end{aligned}$$

$$\underline{\underline{d = 7.709}}$$

Prob. 2.23

- (a) An infinite line parallel to the z-axis.
- (b) Point $(2, -1, 10)$.
- (c) A circle of radius $r \sin \theta = 5$, i.e. the intersection of a cone and a sphere.
- (d) An infinite line parallel to the z-axis.
- (e) A semi-infinite line parallel to the x-y plane.
- (f) A semi-circle of radius 5 in the y-z plane.

Prob. 2.24

At $(1, 60^\circ, -1)$, $\rho = 1, \phi = 60^\circ, z = -1$,

$$\begin{aligned} \text{(a)} \quad \mathbf{A} &= (-2 - \sin 60^\circ) \mathbf{a}_\rho + (4 + 2 \cos 60^\circ) \mathbf{a}_\phi - 3(1)(-1) \mathbf{a}_z \\ &= -2.866 \mathbf{a}_\rho + 5 \mathbf{a}_\phi + 3 \mathbf{a}_z \end{aligned}$$

$$\mathbf{B} = 1 \cos 60^\circ \mathbf{a}_\rho + \sin 60^\circ \mathbf{a}_\phi + \mathbf{a}_z = 0.5 \mathbf{a}_\rho + 0.866 \mathbf{a}_\phi + \mathbf{a}_z$$

$$\mathbf{A} \cdot \mathbf{B} = -1.433 + 4.33 + 3 = 5.897$$

$$AB = \sqrt{2.866^2 + 26 + 9} \sqrt{0.25 + 1 + 0.866^2} = 9.1885$$

$$\cos \theta_{AB} = \frac{\mathbf{A} \cdot \mathbf{B}}{AB} = \frac{5.897}{9.1885} = 0.6419 \quad \longrightarrow \quad \theta_{AB} = \underline{\underline{50.07^\circ}}$$

Let $\mathbf{D} = \mathbf{A} \times \mathbf{B}$. At $(1, 90^\circ, 0)$, $\rho = 1, \phi = 90^\circ, z = 0$

$$\text{(b)} \quad \mathbf{A} = -\sin 90^\circ \mathbf{a}_\rho + 4 \mathbf{a}_\phi = -\mathbf{a}_\rho + 4 \mathbf{a}_\phi$$

$$\mathbf{B} = 1 \cos 90^\circ \mathbf{a}_\rho + \sin 90^\circ \mathbf{a}_\phi + \mathbf{a}_z = \mathbf{a}_\phi + \mathbf{a}_z$$

$$\mathbf{D} = \mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_\rho & \mathbf{a}_\phi & \mathbf{a}_z \\ -1 & 4 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 4 \mathbf{a}_\rho + \mathbf{a}_\phi - \mathbf{a}_z$$

$$\mathbf{a}_D = \frac{\mathbf{D}}{D} = \frac{(4, 1, -1)}{\sqrt{16+1+1}} = \underline{\underline{0.9428 \mathbf{a}_\rho + 0.2357 \mathbf{a}_\phi - 0.2357 \mathbf{a}_z}}$$

Prob. 2.25

At $T(2, 3, -4)$

$$\theta = \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z} = \tan^{-1} \frac{\sqrt{13}}{-4} = 137.97$$

$$\cos \theta = \frac{-4}{\sqrt{29}} = -0.7428, \sin \theta = \frac{\sqrt{13}}{\sqrt{29}} = 0.6695$$

$$\phi = \tan^{-1} \frac{y}{x} = \tan^{-1} \frac{3}{2} = 56.31$$

$$\cos \phi = \frac{2}{\sqrt{13}}, \quad \sin \phi = \frac{3}{\sqrt{13}}$$

$$\mathbf{a}_r = \cos \theta \mathbf{a}_r - \sin \theta \mathbf{a}_\theta = \underline{\underline{-0.7428 \mathbf{a}_r - 0.6695 \mathbf{a}_\theta}}$$

$$\mathbf{a}_r = \sin \theta \cos \phi \mathbf{a}_x + \sin \theta \sin \phi \mathbf{a}_y + \cos \theta \mathbf{a}_z.$$

$$= \underline{\underline{0.3714 \mathbf{a}_x + 0.5571 \mathbf{a}_y - 0.7428 \mathbf{a}_z}}$$

Prob. 2.26

$$\begin{aligned}\mathbf{G} \cdot \mathbf{a}_y &= G_y = G_r \sin \theta \sin \phi + G_\theta \cos \theta \sin \phi + 0 \\ &= 6r^2 \sin \theta \sin^2 \phi + r^2 \cos \theta \sin \phi\end{aligned}$$

At $(2, -3, 1)$, $x = 2$, $y = -3$, $z = 1$

$$r^2 = x^2 + y^2 + z^2 = 4 + 9 + 1 = 14$$

$$\sin \theta = \frac{\rho}{r} = \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}} = \sqrt{\frac{13}{14}}$$

$$\cos \theta = \frac{z}{r} = \frac{1}{\sqrt{14}}, \quad \sin \phi = \frac{y}{\rho} = \frac{-3}{\sqrt{13}}$$

$$G_y = 6(14) \sqrt{\frac{13}{14}} \left(\frac{9}{13} \right) + 14 \frac{1}{\sqrt{14}} \left(\frac{-3}{\sqrt{13}} \right) = 56.04 - 3.1132 = \underline{\underline{52.925}}$$

Prob. 2.27

$$\bar{G} = \cos^2 \phi \bar{a}_x + \frac{2r \cos \theta \sin \phi}{r \sin \theta} \bar{a}_y + (1 - \cos^2 \phi) \bar{a}_z$$

$$= \cos^2 \phi \bar{a}_x + 2 \cot \theta \sin \phi \bar{a}_y + \sin^2 \phi \bar{a}_z$$

$$\begin{pmatrix} G_r \\ G_\theta \\ G_\phi \end{pmatrix} = \begin{bmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \end{bmatrix} \begin{bmatrix} \cos^2 \phi \\ 2 \cot \theta \sin \phi \\ \sin^2 \phi \end{bmatrix}$$

$$G_r = \sin \theta \cos^3 \phi + 2 \cos \theta \sin^2 \phi + \cos \theta \sin^2 \phi$$

$$= \sin \theta \cos^3 \phi + 3 \cos \theta \sin^2 \phi$$

$$G_\theta = \cos \theta \cos^3 \phi + 2 \cot \theta \cos \theta \sin^2 \phi - \sin \theta \sin^2 \phi$$

$$G_\phi = -\sin \phi \cos^2 \phi + 2 \cot \theta \sin \phi \cos \phi$$

$$\begin{aligned}\bar{G} &= [\sin \theta \cos^3 \phi + 3 \cos \theta \sin^2 \phi] \bar{a}_r \\ &\quad + [\cos \theta \cos^3 \phi + 2 \cot \theta \cos \theta \sin^2 \phi - \sin \theta \sin^2 \phi] \bar{a}_\theta \\ &\quad \underline{\underline{+ \sin \phi \cos \phi (2 \cot \theta - \cos \phi) \bar{a}_\phi}}$$

Prob. 2.28

(a) $\mathbf{J}_z = (\mathbf{J} \bullet \mathbf{a}_z) \mathbf{a}_z$.

At $(2, \pi/2, 3\pi/2)$, $\mathbf{a}_z = \cos \theta \mathbf{a}_r - \sin \theta \mathbf{a}_\theta = -\mathbf{a}_\theta$.

$$\mathbf{J}_z = -\cos 2\theta \sin \phi \mathbf{a}_\theta = -\cos \pi \sin(3\pi/2) \mathbf{a}_\theta = -\mathbf{a}_\theta.$$

$$(b) \mathbf{J}_\phi = \tan \frac{\theta}{2} \ln r \mathbf{a}_\phi = \tan \frac{\pi}{4} \ln 2 \mathbf{a}_\phi = \ln 2 \mathbf{a}_\phi = 0.6931 \mathbf{a}_\phi.$$

$$(c) \mathbf{J}_t = \mathbf{J} - \mathbf{J}_n = \mathbf{J} - \mathbf{J}_r = -\mathbf{a}_\theta + \ln 2 \mathbf{a}_\phi = \underline{-\mathbf{a}_\theta + 0.6931 \mathbf{a}_\phi}.$$

$$(d) \bar{\mathbf{J}}_P = (\bar{\mathbf{J}} \bullet \bar{\mathbf{a}}_x) \bar{\mathbf{a}}_x$$

$$\bar{\mathbf{a}}_x = \sin \theta \cos \phi \bar{\mathbf{a}}_r + \cos \theta \cos \phi \bar{\mathbf{a}}_\theta - \sin \phi \bar{\mathbf{a}}_\phi = \bar{\mathbf{a}}_\phi.$$

At $(2, \pi/2, 3\pi/2)$,

$$\bar{\mathbf{J}}_P = \underline{\ln 2 \bar{\mathbf{a}}_\phi}.$$

Prob. 2.29

$$\mathbf{H} \bullet \mathbf{a}_x = H_x$$

$$\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho^2 \cos \phi \\ -\rho \sin \phi \\ 0 \end{bmatrix}$$

$$H_x = \rho^2 \cos^2 \phi + \rho \sin^2 \phi$$

At P, $\rho = 2, \phi = 60^\circ, z = -1$

$$H_x = 4(1/4) + 2(3/4) = 1 + 1.5 = \underline{\underline{2.5}}$$

Prob. 2.30

$$(a) 5 = \mathbf{r} \cdot \mathbf{a}_x + \mathbf{r} \cdot \mathbf{a}_y = x + y \quad \underline{\underline{\text{a plane}}}$$

$$(b) 10 = |\mathbf{r} \times \mathbf{a}_z| = \begin{vmatrix} x & y & z \\ 0 & 0 & 1 \end{vmatrix} = |y \mathbf{a}_x - x \mathbf{a}_y| = \sqrt{x^2 + y^2} = \rho$$

a cylinder of infinite length

CHAPTER 3

P. E. 3.1

$$(a) DH = \int_{\phi=45^\circ}^{\phi=60^\circ} r \sin \theta d\phi \Big|_{r=3, 90^\circ} = 3(1)[\frac{\pi}{3} - \frac{\pi}{4}] = \frac{\pi}{4} = \underline{0.7854}.$$

$$(b) FG = \int_{\theta=60^\circ}^{\theta=90^\circ} r d\theta \Big|_{r=5} = 5(\frac{\pi}{2} - \frac{\pi}{3}) = \frac{5\pi}{6} = \underline{2.618}.$$

(c)

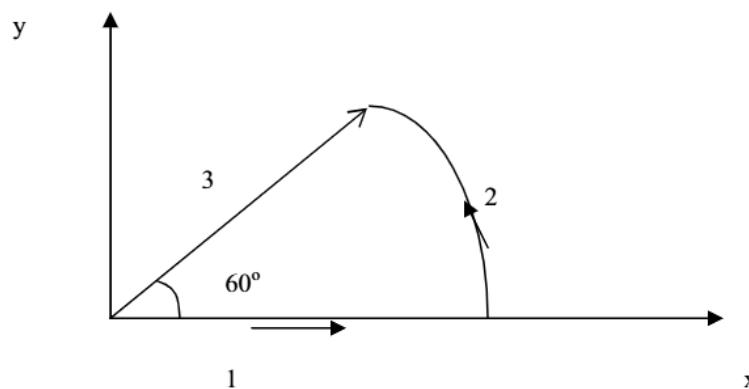
$$\begin{aligned} AEHD &= \int_{\theta=60^\circ}^{\theta=90^\circ} \int_{\phi=45^\circ}^{\phi=60^\circ} r^2 \sin \theta d\theta d\phi \Big|_{r=3} = 9(-\cos \theta) \Big|_{\theta=60^\circ}^{\theta=90^\circ} \Big|_{\phi=45^\circ}^{\phi=60^\circ} \\ &= 9\left(\frac{1}{2}\right)\left(\frac{\pi}{12}\right) = \frac{3\pi}{8} = \underline{1.178}. \end{aligned}$$

(d)

$$ABCD = \int_{r=3}^{r=5} \int_{\theta=60^\circ}^{\theta=90^\circ} r d\theta dr = \frac{r^2}{2} \Big|_{r=3}^r \Big|_{\theta=60^\circ}^{\theta=90^\circ} (\frac{\pi}{2} - \frac{\pi}{3}) = \frac{4\pi}{3} = \underline{4.189}.$$

(e)

$$\begin{aligned} \text{Volume} &= \int_{r=3}^{r=5} \int_{\phi=45^\circ}^{\phi=60^\circ} \int_{\theta=60^\circ}^{\theta=90^\circ} r^2 \sin \theta dr d\theta d\phi = \frac{r^3}{3} \Big|_{r=3}^r (-\cos \theta) \Big|_{\theta=60^\circ}^{\theta=90^\circ} \Big|_{\phi=45^\circ}^{\phi=60^\circ} = \frac{1}{3}(98)\left(\frac{1}{2}\right)\frac{\pi}{12} \\ &= \frac{49\pi}{36} = \underline{4.276}. \end{aligned}$$

P.E. 3.2


$$\oint_L \mathbf{A} \bullet d\mathbf{l} = (\int_1 + \int_2 + \int_3) \mathbf{A} \bullet d\mathbf{l} = C_1 + C_2 + C_3$$

$$\text{Along (1), } C_1 = \int_0^2 \rho \cos \phi d\rho |_{\phi=0} = \frac{\rho^2}{2} \Big|_0^2 = \frac{1}{2} = 2.$$

$$\text{Along (2), } d\mathbf{l} = \rho d\phi \mathbf{a}_\phi, \mathbf{A} \bullet d\mathbf{l} = 0, \quad C_2 = 0$$

$$\text{Along (3), } C_3 = \int_2^0 \rho \cos \phi d\rho |_{\phi=60^\circ} = -\frac{\rho^2}{2} \Big|_0^2 \left(\frac{1}{2}\right) = -1$$

$$\oint_L \mathbf{A} \bullet d\mathbf{l} = C_1 + C_2 + C_3 = 2 + 0 - 1 = 1$$

P.E. 3.3

$$(a) \quad \nabla U = \frac{\partial U}{\partial x} \mathbf{a}_x + \frac{\partial U}{\partial y} \mathbf{a}_y + \frac{\partial U}{\partial z} \mathbf{a}_z$$

$$= \underline{\underline{y(2x+z)\mathbf{a}_x + x(x+z)\mathbf{a}_y + xy\mathbf{a}_z}}$$

$$(b) \quad \nabla V = \frac{\partial V}{\partial \rho} \mathbf{a}_\rho + \frac{1}{\rho} \frac{\partial V}{\partial \phi} \mathbf{a}_\phi + \frac{\partial V}{\partial z} \mathbf{a}_z$$

$$= \underline{\underline{(z \sin \phi + 2\rho)\mathbf{a}_\rho + (z \cos \phi - \frac{z^2}{\rho} \sin 2\phi)\mathbf{a}_\phi + (\rho \sin \phi + 2z \cos^2 \phi)\mathbf{a}_z}}$$

(c)

$$\nabla f = \frac{\partial f}{\partial r} \mathbf{a}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \mathbf{a}_\theta + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \mathbf{a}_\phi$$

$$= \underline{\underline{(\frac{\cos \theta \sin \phi}{r} + 2r\phi)\mathbf{a}_r - \frac{\sin \theta \sin \phi \ln r}{r}\mathbf{a}_\theta + \frac{(\cos \theta \cos \phi \ln r + r^2)}{r \sin \theta}\mathbf{a}_\phi}}$$

$$= \underline{\underline{\left(\frac{\cos \theta \sin \phi}{r} + 2r\phi\right)\mathbf{a}_r - \frac{\sin \theta \sin \phi \ln r}{r}\mathbf{a}_\theta + \left(\frac{\cot \theta \cos \phi \ln r}{r} + r \operatorname{cosec} \theta\right)\mathbf{a}_\phi}}$$

P.E. 3.4

$$\nabla \Phi = (y+z)\mathbf{a}_x + (x+z)\mathbf{a}_y + (x+y)\mathbf{a}_z$$

$$\text{At } (1, 2, 3), \quad \nabla \Phi = \underline{\underline{(5, 4, 3)}}$$

$$\nabla \Phi \bullet \mathbf{a}_1 = (5, 4, 3) \bullet \frac{(2, 2, 1)}{3} = \frac{21}{3} = 7,$$

$$\text{where } (2, 2, 1) = (3, 4, 4) - (1, 2, 3)$$

P.E. 3.5

Let $f = x^2y + z - 3$, $g = x \log z - y^2 + 4$,

$$\nabla f = 2xy \mathbf{a}_x + x^2 \mathbf{a}_y + \mathbf{a}_z,$$

$$\nabla g = \log z \mathbf{a}_x - 2y \mathbf{a}_y + \frac{x}{z} \mathbf{a}_z$$

At $P(-1, 2, 1)$,

$$\mathbf{n}_f = \pm \frac{\nabla f}{|\nabla f|} = \pm \frac{(-4 \mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)}{\sqrt{18}}, \quad \mathbf{n}_g = \pm \frac{\nabla g}{|\nabla g|} = \pm \frac{(-4 \mathbf{a}_y - \mathbf{a}_z)}{\sqrt{17}}$$

$$\cos \theta = \mathbf{n}_f \cdot \mathbf{n}_g = \pm \frac{(-5)}{\sqrt{18 \times 17}}$$

Take positive value to get acute angle.

$$\theta = \cos^{-1} \frac{5}{17.493} = \underline{\underline{73.39^\circ}}$$

P.E. 3.6

$$(a) \nabla \bullet \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 0 + 4x + 0 = \underline{\underline{4x}}$$

At $(1, -2, 3)$, $\nabla \bullet \mathbf{A} = \underline{\underline{4}}$.

(b)

$$\begin{aligned} \nabla \bullet \mathbf{B} &= \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho B_\rho) + \frac{1}{\rho} \frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial \rho} \\ &= \frac{1}{\rho} 2\rho z \sin \phi - \frac{1}{\rho} 3\rho z^2 \sin \phi = 2z \sin \phi - 3z^2 \sin \phi \\ &= \underline{\underline{(2-3z)z \sin \phi}}. \end{aligned}$$

$$\text{At } (5, \frac{\pi}{2}, 1), \quad \nabla \bullet \mathbf{B} = (2-3)(1) = \underline{\underline{-1}}.$$

(c)

$$\begin{aligned} \nabla \bullet \mathbf{C} &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 C_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (C_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial C_\phi}{\partial \phi} \\ &= \frac{1}{r^2} 6r^2 \cos \theta \cos \phi \\ &= \underline{\underline{6 \cos \theta \cos \phi}} \end{aligned}$$

$$\text{At } (1, \frac{\pi}{6}, \frac{\pi}{3}), \quad \nabla \bullet \mathbf{C} = 6 \cos \frac{\pi}{6} \cos \frac{\pi}{3} = \underline{\underline{2.598}}.$$

P.E. 3.7 This is similar to Example 3.7.

$$\Psi = \oint_S \mathbf{D} \bullet d\mathbf{S} = \Psi_t + \Psi_b + \Psi_c$$

$\Psi_t = 0 = \Psi_b$ since \mathbf{D} has no z-component

$$\begin{aligned}\Psi_c &= \iint \rho^2 \cos^2 \phi \rho d\phi dz = \rho^3 \int_{\phi=0}^{\phi=2\pi} \cos^2 \phi d\phi \int_{z=0}^{z=1} dz \Big|_{\rho=4} \\ &= (4)^3 \pi (1) = 64\pi \\ \Psi &= 0 + 0 + 64\pi = \underline{\underline{64\pi}}\end{aligned}$$

By the divergence theorem,

$$\begin{aligned}\oint_S \mathbf{D} \bullet d\mathbf{S} &= \oint_V \nabla \bullet \mathbf{D} dV \\ \nabla \bullet \mathbf{D} &= \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho^3 \cos^2 \phi) + \frac{1}{\rho} \frac{\partial}{\partial \phi} z \sin \phi + \frac{\partial A_z}{\partial z} \\ &= 3\rho \cos^2 \phi + \frac{z}{\rho} \cos \phi. \\ \Psi &= \int_V \nabla \bullet \mathbf{D} dV = \int_V (3\rho \cos^2 \phi + \frac{z}{\rho} \cos \phi) \rho d\phi dz d\rho \\ &= 3 \int_0^4 \rho^2 d\rho \int_0^{2\pi} \cos^2 \phi d\phi \int_0^1 dz + \int_0^4 d\rho \int_0^{2\pi} \cos \phi d\phi \int_0^1 zdz \\ &= 3(\frac{4^3}{3})\pi(1) = \underline{\underline{64\pi}}.\end{aligned}$$

P.E. 3.8

(a)

$$\nabla \times \mathbf{A} = \mathbf{a}_x(1-0) + \mathbf{a}_y(y-0) + \mathbf{a}_z(4y-z)$$

$$= \underline{\underline{\mathbf{a}_x + y\mathbf{a}_y + (4y-z)\mathbf{a}_z}}$$

$$\text{At } (1, -2, 3), \nabla \times \mathbf{A} = \underline{\underline{\mathbf{a}_x - 2\mathbf{a}_y - 11\mathbf{a}_z}}$$

(b)

$$\nabla \times \mathbf{B} = \mathbf{a}_\rho (0 - 6\rho z \cos \phi) + \mathbf{a}_\phi (\rho \sin \phi - 0) + \mathbf{a}_z \frac{1}{\rho} (6\rho z^2 \cos \phi - \rho z \cos \phi)$$

$$= \underline{\underline{-6\rho z \cos \phi \mathbf{a}_\rho + \rho \sin \phi \mathbf{a}_\phi + (6z-1)z \cos \phi \mathbf{a}_z}}$$

$$\text{At } (5, \frac{\pi}{2}, -1), \nabla \times \mathbf{B} = \underline{\underline{5 \mathbf{a}_\phi}}$$

(c)

$$\begin{aligned} \nabla \times \mathbf{C} &= \mathbf{a}_r \frac{1}{r \sin \theta} (r^{-1/2} \cos \theta - 0) + \mathbf{a}_\theta \left(-\frac{2r \cos \theta \sin \phi}{\sin \theta} - \frac{3}{2} r^{1/2} \right) + \mathbf{a}_\phi \left(0 + 2r \sin \theta \cos \phi \right) \\ &= \underline{\underline{r^{-1/2} \cot \theta \mathbf{a}_r - (2 \cot \theta \sin \phi + \frac{3}{2} r^{-1/2}) \mathbf{a}_\theta + 2 \sin \theta \cos \phi \mathbf{a}_\phi}} \end{aligned}$$

$$\text{At } (1, \frac{\pi}{6}, \frac{\pi}{3}), \nabla \times \mathbf{C} = \underline{\underline{1.732 \mathbf{a}_r - 4.5 \mathbf{a}_\theta + 0.5 \mathbf{a}_\phi}}$$

P.E. 3.9

$$\oint_L \mathbf{A} \bullet d\mathbf{l} = \int_S (\nabla \times \mathbf{A}) \bullet d\mathbf{S}$$

$$\text{But } (\nabla \times \mathbf{A}) = \sin \phi \mathbf{a}_z + \frac{z \cos \phi}{\rho} \mathbf{a}_\rho \quad \text{and} \quad dS = \rho d\phi d\rho \mathbf{a}_z$$

$$\int_S (\nabla \times \mathbf{A}) \bullet d\mathbf{S} = \iint_S \rho \sin \phi \, d\phi \, d\rho$$

$$= \frac{\rho^2}{2} \Big|_0^2 (-\cos \phi) \Big|_0^{60^\circ}$$

$$= 2(-\frac{1}{2} + 1) = \underline{\underline{1}}$$

P.E. 3.10

$$\nabla \times \nabla V = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \end{vmatrix} = \\ = \left(\frac{\partial^2 V}{\partial y \partial z} - \frac{\partial^2 V}{\partial y \partial z} \right) \mathbf{a}_x + \left(\frac{\partial^2 V}{\partial x \partial z} - \frac{\partial^2 V}{\partial z \partial x} \right) \mathbf{a}_y + \left(\frac{\partial^2 V}{\partial x \partial y} - \frac{\partial^2 V}{\partial y \partial x} \right) \mathbf{a}_z = 0$$

P.E. 3.11

(a)

$$\nabla^2 U = \frac{\partial}{\partial x}(2xy + yz) + \frac{\partial}{\partial y}(x^2 + xz) + \frac{\partial}{\partial z}(xy) \\ = \underline{\underline{2y}}$$

(b)

$$\nabla^2 V = \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho(z \sin \phi + 2\rho) + \frac{1}{\rho^2} (-\rho z \sin \phi - 2z^2 \frac{\partial}{\partial \rho} \sin \phi \cos \phi) + \frac{\partial}{\partial z} (\rho \sin \phi + 2z \cos^2 \phi) \\ = \frac{1}{\rho} (z \sin \phi + 4\rho) - \frac{1}{\rho^2} (z \rho \sin \phi + 2z^2 \cos 2\phi) + 2 \cos^2 \phi. \\ = \underline{\underline{4 + 2 \cos^2 \phi - \frac{2z^2}{\rho^2} \cos 2\phi.}}$$

(c)

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} [r^2 \frac{1}{r} \cos \theta \sin \phi + 2r^3 \phi] + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} [-\sin^2 \theta \sin \phi \ln r] \\ + \frac{1}{r^2 \sin^2 \theta} [-\cos \theta \sin \phi \ln r] \\ = \frac{1}{r^2} \cos \theta \sin \phi (1 - 2 \ln r - \csc^2 \theta \ln r) + 6\phi \\ = \underline{\underline{\frac{1}{r^2} \cos \theta \sin \phi (1 - 2 \ln r - \csc^2 \theta \ln r) + 6\phi}}$$

P.E. 3.12

If \mathbf{B} is conservative, $\nabla \times \mathbf{B} = \mathbf{0}$ must be satisfied.

$$\nabla \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y + z \cos xz & x & x \cos xz \end{vmatrix}$$

$$= 0 \mathbf{a}_x + (\cos xz - xz \sin xz - \cos xz + xz \sin xz) \mathbf{a}_y + (1 - 1) \mathbf{a}_z = \mathbf{0}$$

Hence \mathbf{B} is a conservative field.

P.E. 3.13

$$\psi = \int_S \mathbf{B} \cdot d\mathbf{S}, \quad d\mathbf{S} = dx dy \mathbf{a}_z$$

$$\psi = \int_{y=0}^2 \int_{x=0}^1 3x^2 y dx dy = 3 \frac{x^3}{3} \left| \begin{matrix} 1 & y^2 \\ 0 & 2 \end{matrix} \right|_0^2 = (1)(2) = 2$$

P.E. 3.14

In Cartesian coordinates $\Rightarrow (\nabla \cdot \mathbf{A})_P = (6x)_P = 6 \times 3 = 18$.

For representing this vector field in cylindrical coordinates, we use eq. (2.13):

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3x^2 + 2y \\ x \\ 0 \end{bmatrix}$$

$$\mathbf{A} = (\cos \phi (3x^2 + 2y) + \sin \phi (x)) \mathbf{a}_\rho + ((3x^2 + 2y)(-\sin \phi) + (\cos \phi)x) \mathbf{a}_\phi$$

Using the following relationships, from eq. (2.8), $x = \rho \cos \phi$, $y = \sin \phi$, we obtain

$$\begin{aligned} &= (\cos \phi (3(\rho \cos \phi)^2 + 2(\rho \sin \phi)) + (\sin \phi)(\rho \cos \phi)) \mathbf{a}_\rho \\ &\quad + ((3(\rho \cos \phi)^2 + 2(\rho \sin \phi))(-\sin \phi) + (\cos \phi)(\rho \cos \phi)) \mathbf{a}_\phi \end{aligned}$$

$$\begin{aligned} \mathbf{A} &= (3\rho^2 \cos^3 \phi + 2\rho \sin \phi \cos \phi + \rho \sin \phi \cos \phi) \mathbf{a}_\rho \\ &\quad + (-3\rho^2 \cos^2 \phi \sin \phi - 2\rho \sin^2 \phi + \rho \cos^2 \phi) \mathbf{a}_\phi \end{aligned}$$

P in cylindrical coordinates is $(5, 53.13^\circ, 5)$

$$\begin{aligned}\nabla \cdot \mathbf{A} &= \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\rho) + \frac{1}{\rho} \frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z} \\ &= \frac{1}{\rho} (9\rho^2 \cos^3 \phi + 6\rho \sin \phi \cos \phi) \\ &\quad + \frac{1}{\rho} (6\rho^2 \cos \phi \sin^2 \phi - (3\rho^2 \cos^3 \phi) \\ &\quad - 4\rho \sin \phi \cos \phi - 2\rho \sin \phi \cos \phi) \\ &= 18.00 \text{ (after substituting the values of } \rho \text{ and } \phi)\end{aligned}$$

For representing the given vector field in spherical coordinates we use eq. (2.27):

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \end{bmatrix} \begin{bmatrix} 3x^2 + 2y \\ x \\ 0 \end{bmatrix}$$

$$\begin{aligned}\mathbf{A} &= ((\sin \theta \cos \phi)(3x^2 + 2y) + (\sin \theta \sin \phi)x)\mathbf{a}_r \\ &+ ((\cos \theta \cos \phi)(3x^2 + 2y) + (\cos \theta \sin \phi)x)\mathbf{a}_\theta \\ &+ (((-\sin \phi)(3x^2 + 2y)) + ((\cos \phi)x))\mathbf{a}_\phi\end{aligned}$$

Using the following relationships, from eq. (2.22), $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, we obtain

$$\begin{aligned}\mathbf{A} &= ((3r^2 \sin^3 \theta \cos^3 \phi) + (3r \sin^2 \theta \cos \phi \sin \phi))\mathbf{a}_r \\ &+ ((3r^2 \cos \theta \sin^2 \theta \cos^3 \phi) + (2r \sin \theta \cos \phi \cos \theta \sin \phi) \\ &+ (r \sin \theta \cos \phi \cos \theta \sin \phi))\mathbf{a}_\theta \\ &+ ((-3r^2 \sin^2 \theta \cos^2 \phi \sin \phi) + (-2r \sin \theta \sin^2 \phi) \\ &+ (r \sin \theta \cos^2 \phi))\mathbf{a}_\phi\end{aligned}$$

P in spherical system is $(7.071, 45^\circ, 53.13^\circ)$

$$\begin{aligned}\nabla \cdot \mathbf{A} &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} A_\theta \sin \theta + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi} \\ &\quad \frac{1}{r^2} (12r^3 \sin^3 \theta \cos^3 \phi + 9r^2 \sin^2 \theta \cos \phi \sin \phi) + \frac{1}{r \sin \theta} (-3r^2 \sin^4 \theta \cos^3 \phi + \\ &\quad 9r^2 \cos^2 \theta \sin^2 \theta \cos^3 \phi - 3r \sin^3 \theta \cos \phi \sin \phi + 6r \sin \theta \cos^2 \theta \cos \phi \sin \phi) + \\ &\quad \frac{1}{r \sin \theta} (6r^2 \sin^2 \theta \sin^2 \phi \cos \phi - 3r^2 \sin^2 \theta \cos^3 \phi - 4r \sin \theta \cos \phi \sin \phi - \\ &\quad 2r \sin \theta \cos \phi \sin \phi) = 18 \text{ (after substituting the values of } r, \theta \text{ and } \phi)\end{aligned}$$

Note: From this example, it is clear that the divergence of a vector field is the same irrespective of the coordinate system used.

P.E. 3.15

$$\nabla F = a_x - 2a_y + a_z$$

$$a_n = \frac{\nabla F}{|\nabla F|} = \frac{a_x - 2a_y + a_z}{\sqrt{1+4+1}} = \underline{\underline{0.4082a_x - 0.8165a_y + 0.4082a_z}}$$

Prob. 3.1

(a)

$$dS = \rho d\phi dz$$

$$S = \int dS = \rho \int \int d\phi dz = 2 \int_0^5 dz \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} d\phi = 2(5) \left[\frac{\pi}{2} - \frac{\pi}{3} \right] = \frac{10\pi}{6} = \underline{\underline{5.236}}$$

(b)

$$\text{In cylindrical, } dS = \rho d\rho d\phi$$

$$S = \int dS = \int_1^3 \rho d\rho \int_0^{\frac{\pi}{4}} d\phi = \frac{\pi}{4} \left(\frac{\rho^2}{2} \right) \Big|_1^3 = \underline{\underline{3.142}}$$

$$(c) \text{ In spherical, } dS = r^2 \sin \theta d\phi d\theta$$

$$S = \int dS = 100 \int_{\frac{\pi}{4}}^{\frac{2\pi}{3}} \sin \theta d\theta \int_0^{2\pi} d\phi = 100(2\pi)(-\cos \theta) \Big|_{\frac{\pi}{4}}^{\frac{2\pi}{3}} = 200\pi(0.5 + 0.7071) = \underline{\underline{758.4}}$$

(d)

$$dS = r dr d\theta$$

$$S = \int dS = \int_0^4 r dr \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} d\theta = \frac{r^2}{2} \Big|_0^4 \left(\frac{\pi}{2} - \frac{\pi}{3} \right) = \frac{8\pi}{6} = \underline{\underline{4.189}}$$

Prob. 3.2

(a)

$$dl = \rho d\phi; \quad \rho = 3$$

$$L = \int dl = 3 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\phi = 3\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{3\pi}{4} = \underline{\underline{2.356}}$$

(b)

$$dl = r \sin \theta d\phi; \quad r = 1, \quad \theta = 30^\circ;$$

$$L = \int dl = r \sin \theta \int_0^{\frac{\pi}{3}} d\phi = (1) \sin 30^\circ \left[\left(\frac{\pi}{3}\right) - 0\right] = \underline{\underline{0.5236}}$$

(c)

$$dl = rd\theta$$

$$L = \int dl = r \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} d\theta = 4\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \frac{4\pi}{3} = \underline{\underline{4.189}}$$

Prob. 3.3

$$\begin{aligned} S &= \int dS = \int_{z=0}^2 \int_{\phi=\pi/4}^{\pi/2} \rho d\phi dz \Big|_{\rho=10} \\ &= 10 \int_0^2 dz \int_{\pi/4}^{\pi/2} d\phi = 10(2)(\pi/2 - \pi/4) = 5\pi = \underline{\underline{15.71}} \end{aligned}$$

Prob. 3.4

$$L = \int dl = \int_{\phi=0}^{\pi/6} \rho d\phi \Big|_{\rho=4} = 4(\pi/6) = \underline{\underline{2.094}}$$

Prob. 3.5

$$(a) dV = dx dy dz$$

$$V = \int dx dy dz = \int_0^1 dx \int_1^2 dy \int_{-3}^3 dz = (1)(2-1)(3-(-3)) = \underline{\underline{6}}$$

$$(b) dV = \rho d\phi d\rho dz$$

$$V = \int_2^5 \rho d\rho \int_{-1}^4 dz \int_{\frac{\pi}{3}}^{\pi} d\phi = \frac{\rho^2}{2} \Big|_2^5 (4 - (-1))(\pi - \frac{\pi}{3}) = \frac{1}{2}(25 - 4)(5)(\frac{2\pi}{3}) = 35\pi = \underline{\underline{110}}$$

$$(c) \quad dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$\begin{aligned} V &= \int_I^3 r^2 dr \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \sin\theta \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} d\phi = \frac{r^3}{3} \Big|_I^3 (-\cos\theta) \Big|_{\pi/2}^{\pi/3} \left(\frac{\pi}{2} - \frac{\pi}{6}\right) \\ &= \frac{1}{3} (27 - I) \left(\frac{1}{2}\right) \left(\frac{\pi}{3}\right) = \frac{26\pi}{18} = \underline{\underline{4.538}} \end{aligned}$$

Prob. 3.6

$$\int_L \mathbf{H} \bullet d\mathbf{l} = \int (x^2 dx + y^2 dy)$$

But on L , $y = x^2$ $dy = 2x dx$

$$\int_L \mathbf{H} \bullet d\mathbf{l} = \int_0^1 (x^2 + x^4 \cdot 2x) dx = \frac{x^3}{3} + 2 \frac{x^6}{6} \Big|_0^1 = \frac{1}{3} + \frac{1}{3} = \underline{\underline{0.6667}}$$

Prob. 3.7

(a)

$$\begin{aligned} \int \mathbf{F} \bullet d\mathbf{l} &= \int_{y=0}^1 (x^2 - z^2) dy \Big|_{x=0, z=0} + \int_{x=0}^{x=2} 2xy dx \Big|_{y=1, z=0} + \int_{z=0}^{z=3} (-3xz^2) dz \Big|_{x=2, y=1} \\ &= 0 + 2(1) \frac{x^2}{2} \Big|_0^2 - 3(2) \frac{z^3}{3} \Big|_0^3 \\ &= 0 + 4 - 54 = \underline{\underline{-50}} \end{aligned}$$

(b)

Let $x = 2t$, $y = t$, $z = 3t$

$$dx = 2dt, \quad dy = dt, \quad dz = 3dt;$$

$$\begin{aligned} \int \mathbf{F} \bullet d\mathbf{l} &= \int_0^1 (8t^2 - 5t^2 - 162t^3) dt \\ &= (t^3 - 40.5t^4) \Big|_0^1 = \underline{\underline{-39.5}} \end{aligned}$$

Prob. 3.8

$$W = \int_L \mathbf{F} \bullet d\mathbf{l} = \int_{\phi=0}^{\pi/4} z\rho d\phi \Big|_{z=0, \rho=2} + \int_{z=0}^3 \rho \cos \phi dz \Big|_{\rho=2, \phi=\pi/4}$$

$$= 0 + 2 \cos(\pi/4)(3) = 6 \cos 45^\circ = \underline{\underline{4.243 \text{ J}}}$$

Prob. 3.9

$$\begin{aligned} \int H \bullet d\mathbf{l} &= \int_{x=1}^0 (x-y)dx \Big|_{y=0, z=0} + \int_{z=0}^1 5yzdz \Big|_{x=0, y=0} \\ &\quad + \int (x^2 + zy)dy + 5yzdz \Big|_{x=0, z=1-y/2} \\ &= \int_1^0 xdx + \int_0^2 \left(y - \frac{y^2}{2}\right)dy + \int_1^0 (10z - 10z^2)dz \\ &= \underline{\underline{-1.5}} \end{aligned}$$

Prob. 3.10*Method 1:*

$$\oint_L \mathbf{B} \bullet d\mathbf{l} = - \int_{y=0}^1 yzdy \Big|_{z=0} + \int_{z=0}^1 xzdz \Big|_{x=1} + \int (-yzdy + xzdz) \Big|_{x=1}$$

But $z = y \longrightarrow dz = dy$ on the last segment (or integral).

$$\begin{aligned} \oint_L \mathbf{B} \bullet d\mathbf{l} &= 0 + \frac{z^2}{2} \Big|_0^1 + \int_{y=1}^0 (-y^2 + y)dy = \frac{1}{2} + \left(-\frac{y^3}{3} + \frac{y^2}{2}\right) \Big|_1^0 \\ &= \frac{1}{2} + \frac{1}{3} - \frac{1}{2} = \frac{1}{3} = \underline{\underline{0.333}} \end{aligned}$$

Method 2:

$$\begin{aligned} \oint_L \mathbf{B} \bullet d\mathbf{l} &= \int_S \nabla \times \mathbf{B} \bullet d\mathbf{S} \\ \nabla \times \mathbf{B} &= \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & -yz & xz \end{vmatrix} = ya_x - za_y - xa_z, \quad d\mathbf{S} = dydz \mathbf{a}_x \\ \int_S \nabla \times \mathbf{B} \bullet d\mathbf{S} &= \int_{y=0}^1 \int_{z=0}^y ydz dy = \int_0^1 y^2 dy = \frac{y^3}{3} \Big|_0^1 = \frac{1}{3} = \underline{\underline{0.333}} \end{aligned}$$

Prob. 3.11

$$\psi = \int_S \mathbf{A} \cdot d\mathbf{S} = \int \int z dx dz = \int_0^1 dx \int_0^2 zdz = (1) \frac{z^2}{2} \Big|_0^2 = \underline{\underline{2}}$$

Prob. 3.12(a) $dv = dx dy dz$

$$\begin{aligned} \int_v xy dv &= \int_{z=0}^2 \int_{y=0}^1 \int_{x=0}^1 xy dx dy dz = \int_0^1 x dx \int_0^1 y dy \int_0^z dz \\ &= \frac{x^2}{2} \Big|_0^1 \frac{y^2}{2} \Big|_0^1 z \Big|_0^2 = (1/2)(1/2)(2) = \underline{\underline{0.5}} \end{aligned}$$

(b)

$$dv = \rho d\rho d\phi dz$$

$$\begin{aligned} \int_v \rho z dv &= \int_{\phi=0}^{\pi} \int_{z=0}^2 \int_{\rho=1}^3 \rho z \rho d\rho d\phi dz = \int_1^3 \rho^2 d\rho \int_0^2 z dz \int_0^{\pi} d\phi \\ &= \frac{\rho^3}{3} \Big|_1^3 \frac{z^2}{2} \Big|_0^2 (\pi) = (9 - \frac{1}{3})(2\pi) = \underline{\underline{54.45}} \end{aligned}$$

Prob. 3.13

$$\text{Let } I = \int_v A dv = \int_v r \sin \phi \mathbf{a}_r dv$$

$$\mathbf{a}_r = \sin \theta \cos \phi \mathbf{a}_x + \sin \theta \sin \phi \mathbf{a}_y + \cos \theta \mathbf{a}_z$$

$$A_r = r \sin \theta \sin \phi \cos \phi \mathbf{a}_x + r \sin \theta \sin^2 \phi \mathbf{a}_y + r \cos \theta \sin \phi \mathbf{a}_z$$

$$dv = r^2 \sin \theta d\theta d\phi dr$$

$$\begin{aligned}
 I &= \int_v A dv = \iiint r^3 \sin^2 \theta \sin \phi \cos \phi d\theta d\phi dr \mathbf{a}_x \\
 &\quad + \iiint r^3 \sin^2 \theta \sin^2 \phi d\theta d\phi dr \mathbf{a}_y \\
 &\quad + \iiint r^3 \sin \theta \cos \theta \sin \phi d\theta d\phi dr \mathbf{a}_z \\
 &= \mathbf{a}_x \int_{r=1}^1 r^3 dr \int_0^\pi \sin^2 \theta d\theta \int_0^{2\pi} \sin \phi \cos \phi d\phi \\
 &\quad + \mathbf{a}_y \int_{r=1}^1 r^3 dr \int_0^\pi \sin^2 \theta d\theta \int_0^{2\pi} \sin^2 \phi d\phi \\
 &\quad + \mathbf{a}_z \int_{r=1}^1 r^3 dr \int_0^\pi \sin \theta \cos \theta d\theta \int_0^{2\pi} \sin \phi d\phi \\
 &= 0\mathbf{a}_x + 0\mathbf{a}_z + \mathbf{a}_y \frac{r^4}{4} \left| \int_0^\pi \frac{1}{2} (1 - \cos 2\theta) d\theta \int_0^{2\pi} \frac{1}{2} (1 - \cos 2\phi) d\phi \right. \\
 &= \frac{\mathbf{a}_y}{4} (\pi/2)(\pi) = \underline{\underline{\underline{\frac{\pi^2}{8} \mathbf{a}_y}}} = 1.234 \mathbf{a}_y
 \end{aligned}$$

Prob. 3.14

$$\begin{aligned}
 (a) \quad \nabla V_1 &= \frac{\partial V_1}{\partial x} \mathbf{a}_x + \frac{\partial V_1}{\partial y} \mathbf{a}_y + \frac{\partial V_1}{\partial z} \mathbf{a}_z \\
 &= \underline{\underline{\underline{(6y - 2z)\mathbf{a}_x + 6x\mathbf{a}_y + (1 - 2x)\mathbf{a}_z}}}
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad \nabla V_2 &= \frac{\partial V_2}{\partial \rho} \mathbf{a}_\rho + \frac{1}{\rho} \frac{\partial V_2}{\partial \phi} \mathbf{a}_\phi + \frac{\partial V_2}{\partial z} \mathbf{a}_z \\
 &= \underline{\underline{\underline{(10 \cos \phi - z)\mathbf{a}_\rho - 10 \sin \phi \mathbf{a}_\phi - \rho \mathbf{a}_z}}}
 \end{aligned}$$

$$\begin{aligned}
 \nabla V_3 &= \frac{\partial V_3}{\partial r} \mathbf{a}_r + \frac{1}{r} \frac{\partial V_3}{\partial \theta} \mathbf{a}_\theta + \frac{1}{r \sin \theta} \frac{\partial V_3}{\partial \phi} \mathbf{a}_\phi \\
 (c) \quad &= -\frac{2}{r^2} \cos \phi \mathbf{a}_r + 0 + \frac{1}{r \sin \theta} \left(-\frac{2}{r} \sin \phi \right) \mathbf{a}_\phi \\
 &= \underline{\underline{\underline{-\frac{2}{r^2} \cos \phi \mathbf{a}_r - \frac{2 \sin \phi}{r^2 \sin \theta} \mathbf{a}_\phi}}}
 \end{aligned}$$

Prob. 3.15

$$(a) \nabla U = \frac{\partial U}{\partial x} \mathbf{a}_x + \frac{\partial U}{\partial y} \mathbf{a}_y + \frac{\partial U}{\partial z} \mathbf{a}_z \\ = \underline{\underline{e^{x+2y} \cosh z \mathbf{a}_x + 2e^{x+2y} \cosh z \mathbf{a}_y + e^{x+2y} \sinh z \mathbf{a}_z}}$$

$$(b) \nabla T = \frac{\partial T}{\partial \rho} \mathbf{a}_\rho + \frac{1}{\rho} \frac{\partial T}{\partial \phi} \mathbf{a}_\phi + \frac{\partial T}{\partial z} \mathbf{a}_z \\ = \underline{\underline{-\frac{3z}{\rho^2} \cos \phi \mathbf{a}_\rho - \frac{3z}{\rho^2} \sin \phi \mathbf{a}_\phi + \frac{3}{\rho} \cos \phi \mathbf{a}_z}}$$

$$(c) \nabla W = \frac{\partial W}{\partial r} \mathbf{a}_r + \frac{1}{r} \frac{\partial W}{\partial \theta} \mathbf{a}_\theta + \frac{1}{r \sin \theta} \frac{\partial W}{\partial \phi} \mathbf{a}_\phi \\ = \underline{\underline{\left(-\frac{5 \cos \theta}{r^2} + 4r \sin \phi \right) \mathbf{a}_r + \left[2r^2 \cos \phi - \frac{5 \sin \phi}{r} \right] \frac{1}{r \sin \theta} \mathbf{a}_\phi}}$$

Prob. 3.16

$$r = \sqrt{x^2 + y^2 + z^2}, \quad r^n = (x^2 + y^2 + z^2)^{n/2}$$

Method 1:

$$\nabla r^n = \frac{\partial r^n}{\partial x} \mathbf{a}_x + \frac{\partial r^n}{\partial y} \mathbf{a}_y + \frac{\partial r^n}{\partial z} \mathbf{a}_z = \frac{n}{2} (x^2 + y^2 + z^2)^{n/2-1} (2x) \mathbf{a}_x + \dots \\ = n(x^2 + y^2 + z^2)^{n/2-1} (x \mathbf{a}_x + y \mathbf{a}_y + z \mathbf{a}_z) = \underline{\underline{n r^{n-2} \mathbf{r}}}$$

Method 2:

$$\nabla r^n = \frac{\partial r^n}{\partial r} \mathbf{a}_r = n r^{n-1} \frac{\mathbf{r}}{r} = n r^{n-2} \mathbf{r}$$

Prob. 3.17

$$\nabla T = 2x \mathbf{a}_x + 2y \mathbf{a}_y - \mathbf{a}_z$$

At $(1, 1, 2)$, $\nabla T = (2, 2, -1)$. The mosquito should move in the direction of

$$\underline{\underline{2 \mathbf{a}_x + 2 \mathbf{a}_y - \mathbf{a}_z}}$$

Prob. 3.18**Method 1:**

$$\nabla T = \frac{\partial T}{\partial r} \mathbf{a}_r + \frac{1}{r} \frac{\partial T}{\partial \theta} \mathbf{a}_\theta + \frac{1}{r \sin \theta} \frac{\partial T}{\partial \phi} \mathbf{a}_\phi$$

$$= \sin \theta \cos \phi \mathbf{a}_r + \cos \theta \cos \phi \mathbf{a}_\theta - \sin \phi \mathbf{a}_\phi$$

At P, $r = 2, \theta = 60^\circ, \phi = 30^\circ$

$$\nabla T = \sin 60^\circ \cos 30^\circ \mathbf{a}_r + \cos 60^\circ \cos 30^\circ \mathbf{a}_\theta - \sin 30^\circ \mathbf{a}_\phi$$

$$= 0.75 \mathbf{a}_r - 0.433 \mathbf{a}_\theta - 0.5 \mathbf{a}_\phi$$

$$|\nabla T| = \sqrt{0.75^2 + 0.433^2 + 0.5^2} = 1$$

The magnitude of T is 1 and its direction is along ∇T .**Method 2:**

$$T = r \sin \theta \cos \phi = x$$

$$\nabla T = \mathbf{a}_x$$

$$|\nabla T| = 1$$

Prob. 3.19

$$\nabla f = \frac{\partial f}{\partial x} \mathbf{a}_x + \frac{\partial f}{\partial y} \mathbf{a}_y + \frac{\partial f}{\partial z} \mathbf{a}_z = (2xy - 2y^2) \mathbf{a}_x + (x^2 - 4xy) \mathbf{a}_y + 3z^2 \mathbf{a}_z$$

At point (2,4,-3), $x = 2, y = 4, z = -3$

$$\nabla f = (16 - 32) \mathbf{a}_x + (4 - 32) \mathbf{a}_y + 27 \mathbf{a}_z = -16 \mathbf{a}_x - 28 \mathbf{a}_y + 27 \mathbf{a}_z$$

$$\mathbf{a} = \frac{\mathbf{a}_x + 2\mathbf{a}_y - \mathbf{a}_z}{\sqrt{1+4+1}} = \frac{1}{\sqrt{6}} (1, 2, -1)$$

The directional derivative is

$$\nabla f \cdot \mathbf{a} = (-16, -28, 27) \cdot \frac{1}{\sqrt{6}} (1, 2, -1) = -\frac{99}{\sqrt{6}} = \underline{-40.42}$$

Prob. 3.20(a) Let $f = ax + by + cz - d = 0$

$$\nabla f = a\mathbf{a}_x + b\mathbf{a}_y + c\mathbf{a}_z$$

$$\mathbf{a}_{n1} = \frac{\nabla f}{|\nabla f|} = \frac{a\mathbf{a}_x + b\mathbf{a}_y + c\mathbf{a}_z}{\sqrt{a^2 + b^2 + c^2}}$$

Let $g = \alpha x + \beta y + \gamma z - \delta$

$$\mathbf{a}_{n2} = \frac{\nabla g}{|\nabla g|} = \frac{\alpha\mathbf{a}_x + \beta\mathbf{a}_y + \gamma\mathbf{a}_z}{\sqrt{\alpha^2 + \beta^2 + \gamma^2}}$$

$$\cos \theta = \mathbf{a}_{n1} \cdot \mathbf{a}_{n2} = \frac{a\alpha + b\beta + c\gamma}{\sqrt{a^2 + b^2 + c^2} \sqrt{\alpha^2 + \beta^2 + \gamma^2}}$$

$$\theta = \cos^{-1} \frac{a\alpha + b\beta + c\gamma}{\sqrt{(a^2 + b^2 + c^2)(\alpha^2 + \beta^2 + \gamma^2)}}$$

$$a = 1, b = 2, c = 3$$

(b) $\alpha = 1, \beta = 1, \gamma = 0$

$$\theta = \cos^{-1} \frac{1+2+0}{\sqrt{(1^2 + 2^2 + 3^2)(1^2 + 1^2 + 0^2)}} = \cos^{-1} \frac{3}{\sqrt{28}} = \cos^{-1} 0.5669 = \underline{\underline{55.46^\circ}}$$

Prob. 3.21

$$(a) \nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 3y - x$$

$$(b) \begin{aligned} \nabla \cdot \mathbf{B} &= \frac{1}{\rho} 2\rho z^2 + \frac{1}{\rho} \rho 2 \sin \phi \cos \phi + 2\rho \sin^2 \phi \\ &= 2z^2 + \sin 2\phi + 2\rho \sin^2 \phi \end{aligned}$$

$$(c) \nabla \cdot \mathbf{C} = \frac{1}{r^2} 3r^2 + 0 = \underline{\underline{3}}$$

Prob. 3.22

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 2xy + 0 + 2y = 2y(1+x)$$

(a) At $(-3, 4, 2), x = -3, y = 4$

$$\nabla \cdot \mathbf{A} = 2(4)(1-3) = \underline{\underline{-16}}$$

$$\nabla \cdot \mathbf{B} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho B_\rho) + \frac{1}{\rho} \frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (3\rho^2 \sin \phi) + 0 + 8z \cos^2 \phi$$

(b) $= 6 \sin \phi + 8z \cos^2 \phi$

At $(5, 30^\circ, 1), z = 1, \phi = 30^\circ$

$$\nabla \cdot \mathbf{B} = 6 \sin 30^\circ + 8(1) \cos^2 30^\circ = 3 + 6 = \underline{\underline{9}}$$

$$\nabla \cdot \mathbf{C} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 C_r) + 0 + \frac{1}{r \sin \theta} \frac{\partial C_\phi}{\partial \phi} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^4 \cos \theta) + 0 = 4r \cos \theta$$

(c) At $(2, \pi/3, \pi/2), r = 2, \theta = \pi/3$

$$\nabla \cdot \mathbf{C} = 4(2) \cos(\pi/3) = \underline{\underline{4}}$$

Prob. 3.23

$$\nabla \cdot \mathbf{H} = k \nabla \cdot \nabla T = k \nabla^2 T$$

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 50 \sin \frac{\pi x}{2} \cos h \frac{\pi y}{2} \left(-\frac{\pi^2}{4} + \frac{\pi^2}{4} \right) = 0$$

Hence, $\nabla \cdot \mathbf{H} = 0$

Prob. 3.24

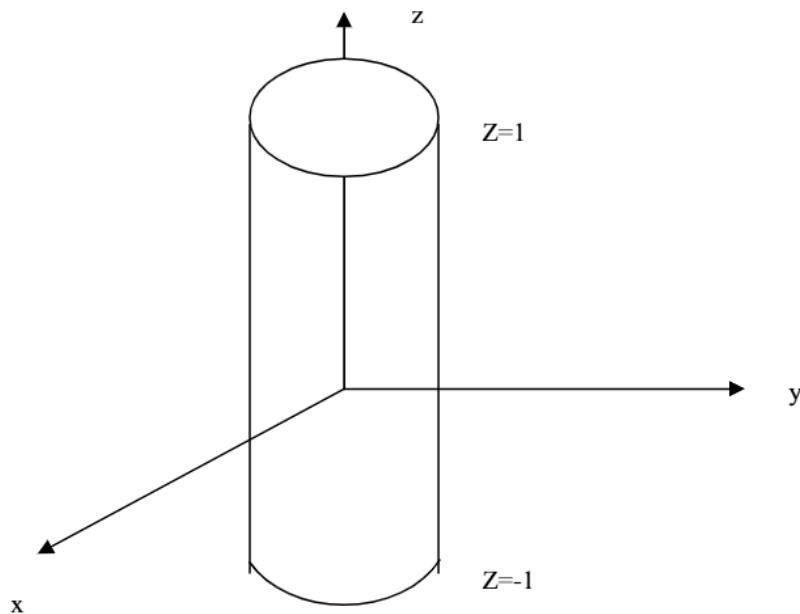
We convert A to cylindrical coordinates; only the ρ -component is needed.

$$A_\rho = A_x \cos \phi + A_y \sin \phi = 2x \cos \phi - z^2 \sin \phi$$

But $x = \rho \cos \phi$,

$$A_\rho = 2\rho \cos^2 \phi - z^2 \sin \phi$$

$$\begin{aligned} \Psi &= \int_S \mathbf{A} \cdot d\mathbf{S} = \iint_A A_\rho \rho d\phi dz = \iint \left[2\rho^2 \cos^2 \phi - \rho z^2 \sin \phi \right] d\phi dz \\ &= 2(2)^2 \int_0^{\pi/2} \frac{1}{2} (1 + \cos 2\phi) d\phi \int_0^1 dz - 2 \int_0^1 z^2 dz \int_0^{\pi/2} \sin \phi d\phi \\ &= 4\left(\phi + \frac{1}{2} \sin 2\phi\right) \Big|_0^{\pi/2} - 2 \frac{z^3}{3} \Big|_0^1 (-\cos \phi) \Big|_0^{\pi/2} = 2\pi - 2/3 = \underline{\underline{5.6165}} \end{aligned}$$

Prob. 3.25

(a)

$$\oint \mathbf{D} \bullet d\mathbf{S} = \left[\iint_{z=-1} + \iint_{z=1} + \iint_{\rho=5} \right] \mathbf{D} \bullet d\mathbf{S}$$

$$= - \iint \rho^2 \cos^2 \phi d\phi d\rho + \iint \rho^2 \cos^2 \phi d\phi d\rho + \iint 2\rho^2 z^2 d\phi dz \Big|_{\rho=5}$$

$$= 2(5)^2 \int_0^{2\pi} d\phi \int_{-1}^1 z^2 dz = +50(2\pi) \left(\frac{z^3}{3}\right) \Big|_{-1}^1$$

$$= \frac{200\pi}{3} = \underline{\underline{209.44}}$$

$$(b) \nabla \bullet \mathbf{D} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (2\rho^2 z^2) = 4z^2$$

$$\int \nabla \bullet \mathbf{D} dv = \iiint 4z^2 \rho d\rho d\phi dz = 4 \int_{-1}^1 z^2 dz \int_0^5 \rho d\rho \int_0^{2\pi} d\phi$$

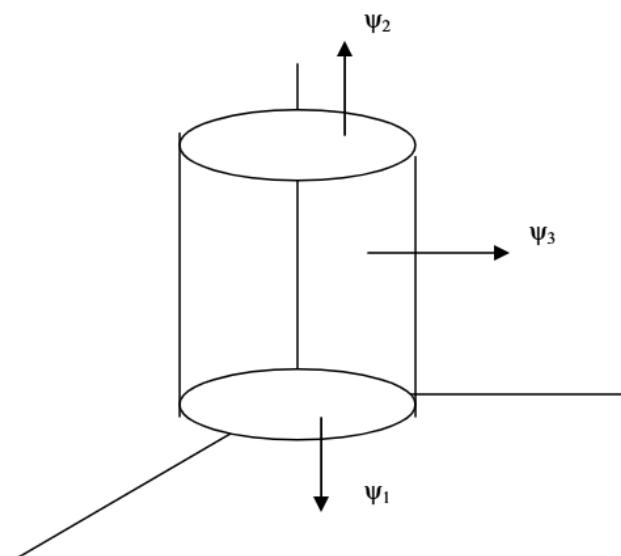
$$= 4x \left. \frac{z^3}{3} \right|_{-1}^1 \left. \frac{\rho^2}{2} \right|_0^5 (2\pi) = \frac{200\pi}{3} = \underline{\underline{209.44}}$$

Prob. 3.26

$$\begin{aligned}\oint_S \mathbf{H} \cdot d\mathbf{S} &= \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} 10 \cos \theta r^2 \sin \theta d\theta d\phi \Big|_{r=1} \\ &= 10(1)^2 \int_0^{2\pi} d\phi \int_0^{\pi/2} \sin \theta \cos \theta d\theta = 10(2\pi) \int_0^{\pi/2} \sin \theta d(\sin \theta) \\ &= 20\pi \left(\frac{\sin^2 \theta}{2} \right) \Big|_0^{\pi/2} = 10\pi = \underline{\underline{31.416}}\end{aligned}$$

Prob. 3.27

$$\begin{aligned}\oint_S \mathbf{H} \cdot d\mathbf{S} &= \int_V \nabla \cdot \mathbf{H} dv \\ \oint_S \mathbf{H} \cdot d\mathbf{S} &= - \iint_{x=0} 2xy dy dz + \iint_{x=1} 2xy dy dz - \iint_{y=1} (x^2 + z^2) dx dz \\ &\quad + \iint_{y=2} (x^2 + z^2) dx dz - \iint_{z=-1} 2yz dx dy + \iint_{z=3} 2yz dx dy \\ &= 0 + 2 \int_1^2 y dy \int_{-1}^3 dz + 2 \int_0^1 dx \int_1^2 y dy + 6 \int_0^1 dx \int_1^2 y dy \\ &= 12 + 3 + 9 = \underline{\underline{24}} \\ \nabla \cdot \mathbf{H} &= \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 2y + 0 + 2y = 4y \\ \int_V \nabla \cdot \mathbf{H} dv &= \iiint 4y dx dy dz = 4 \int_0^1 dx \int_1^2 y dy \int_{-1}^3 dz \\ &= 4(1) \frac{y^2}{2} \Big|_1^2 (3+1) = \underline{\underline{\underline{24}}}\end{aligned}$$

Prob. 3.28Side 1:

$$\begin{aligned}\psi &= \oint_S \mathbf{D} \cdot d\mathbf{S} = \psi_1 + \psi_2 + \psi_3 \\ &= 0 + \int_{\phi=0}^{2\pi} \int_{\rho=0}^3 10z \times \rho d\phi d\rho \Big|_{z=4} + \int_{z=0}^4 \int_{\phi=0}^{2\pi} 5\rho \times \rho d\phi dz \Big|_{\rho=3} \\ &= 10(4)(2\pi) \left(\frac{\rho^2}{2} \right) \Big|_0^3 + 5(9)(2\pi)(4) = 360\pi + 360\pi = \underline{\underline{2261.95}}\end{aligned}$$

Side 2:

$$\begin{aligned}\psi &= \int_v \nabla \cdot \mathbf{B} dv, \quad \nabla \cdot \mathbf{B} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (5\rho^2) + 0 + 10 = 10 + 10 = 20 \\ \psi &= \iiint 20 dv = 20 \int_{\phi=0}^{2\pi} \int_{z=0}^4 \int_{\rho=0}^3 \rho d\phi d\rho dz = 20(2\pi)(4) \left(\frac{\rho^2}{2} \Big|_0^3 \right) \\ &= 720\pi = \underline{\underline{2261.95}}\end{aligned}$$

Prob. 3.29

$$\text{Let } \psi = \oint_S \mathbf{A} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{A} dv$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 2(x + y + z)$$

$$\nabla \cdot \mathbf{A} = 2(\rho \cos \phi + \rho \sin \phi + z), \quad dv = \rho d\rho d\phi dz$$

$$\psi = \iiint 2(\rho \cos \phi + \rho \sin \phi + z) \rho d\rho d\phi dz$$

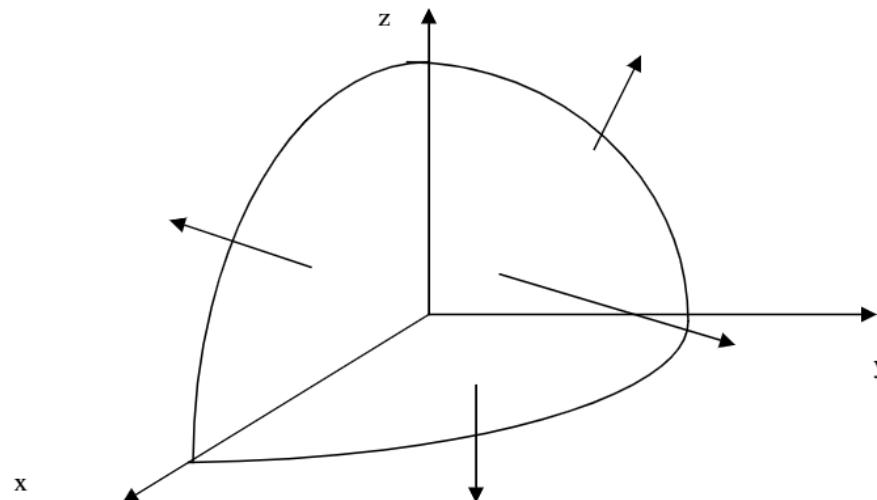
$$= 0 + 0 + 2 \int_0^1 \rho d\rho \int_2^4 zdz \int_0^{2\pi} d\phi = \frac{2\rho^2}{2} \Big|_0^1 \frac{z^2}{2} \Big|_2^4 (2\pi) = \frac{1}{2}(16 - 4)(2\pi)$$

$$= 12\pi = \underline{\underline{37.7}}$$

Prob. 3.30

$$\begin{aligned} \nabla \cdot \mathbf{A} &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^4) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (r \sin^2 \theta \cos \phi) \\ &= 4r + 2 \cos \theta \cos \phi \end{aligned}$$

$$\begin{aligned} \int \nabla \cdot \mathbf{A} dv &= \iiint 4r^3 \sin \theta d\theta d\phi dr + \iiint 2r^2 \sin \theta \cos \theta \cos \phi d\theta d\phi dr \\ &= 4 \frac{r^4}{4} \Big|_0^3 (-\cos \theta) \Big|_0^{\pi/2} \left(\frac{\pi}{2} \right) + \frac{2r^3}{3} \Big|_0^3 \left(-\frac{\cos^2 \theta}{2} \right) \Big|_0^{\pi/2} \sin \phi \Big|_0^{\pi/2} \\ &= 81(1)\left(\frac{\pi}{2}\right) + 18\left(0 + \frac{1}{2}\right)(1 - 0) \\ &= \frac{81\pi}{2} + 9 = \underline{\underline{136.23}} \end{aligned}$$



$$\int \mathbf{A} \cdot d\mathbf{S} = \left[\iint_{\phi=0} + \iint_{\phi=\pi/2} + \iint_{r=3} + \iint_{\theta=\pi/2} \right] \mathbf{A} \cdot d\mathbf{S}$$

Since \mathbf{A} has no ϕ -component, the first two integrals on the right hand side vanish.

$$\begin{aligned}\int \mathbf{A} \bullet d\mathbf{S} &= \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} r^4 \sin \theta d\theta d\phi \Big|_{r=3} + \int_{r=0}^3 \int_{\phi=0}^{\pi/2} r^2 \sin^2 \theta \cos \phi dr d\phi \Big|_{\theta=\pi/2} \\ &= 81 \left(\frac{\pi}{2}\right) (-\cos \theta) \Big|_0^{\pi/2} + 9(1) \sin \phi \Big|_0^{\pi/2} \\ &= \frac{81\pi}{2} + 9 = \underline{\underline{136.23}}\end{aligned}$$

Prob. 3.31

Let $\psi = \oint \mathbf{F} \bullet d\mathbf{S} = \psi_t + \psi_b + \psi_o + \psi_i$

where $\psi_t, \psi_b, \psi_o, \psi_i$ are the fluxes through the top surface, bottom surface, outer surface ($\rho = 3$), and inner surface respectively.

For the top surface, $d\mathbf{S} = \rho d\phi d\rho \mathbf{a}_z, z = 5$;

$$\mathbf{F} \bullet d\mathbf{S} = \rho^2 z d\phi dz. \text{ Hence:}$$

$$\psi_t = \int_{\rho=2}^3 \int_{\phi=0}^{2\pi} \rho^2 z d\phi dz \Big|_{z=5} = \frac{190 \pi}{3} = 198.97$$

For the bottom surface, $z = 0, d\mathbf{S} = \rho d\phi d\rho (-\mathbf{a}_z)$

$$\mathbf{F} \bullet d\mathbf{S} = -\rho^2 z d\phi d\rho = 0. \text{ Hence, } \psi_b = 0.$$

For the outer curved surface, $\rho = 3, d\mathbf{S} = \rho d\phi dz \mathbf{a}_\rho$

$$\mathbf{F} \bullet d\mathbf{S} = \rho^2 \sin \phi d\phi dz. \text{ Hence,}$$

$$\psi_a = \int_{z=0}^5 dz \int_{\phi=0}^{2\pi} \sin \phi d\phi \Big|_{\rho=3} = 0$$

For the inner curved surface, $\rho = 2, d\mathbf{S} = \rho d\phi dz (-\mathbf{a}_\rho)$

$$\mathbf{F} \bullet d\mathbf{S} = -\rho^3 \sin \phi d\phi dz. \text{ Hence,}$$

$$\psi_a = - \int_{z=0}^5 dz \int_{\phi=0}^{2\pi} \sin \phi d\phi \Big|_{\rho=2} = 0$$

$$\psi = \frac{190 \pi}{3} + 0 + 0 + 0 = \frac{190 \pi}{3} = \underline{\underline{198.97}}$$

$$\psi = \oint \mathbf{F} \bullet d\mathbf{S} = \int \nabla \bullet \mathbf{F} dV$$

$$\begin{aligned}\nabla \bullet \mathbf{F} &= \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho^3 \sin \phi) + \frac{1}{\rho} \frac{\partial}{\partial \phi} (z \cos \phi) + \rho \\ &= 3\rho \sin \phi - \frac{z}{\rho} \sin \phi + \rho\end{aligned}$$

$$\begin{aligned}\int_V \nabla \bullet \mathbf{F} dV &= \iiint (3\rho \sin \phi - \frac{z}{\rho} \sin \phi + \rho) \rho d\phi d\rho dz \\ &= 0 + 0 + \int_0^5 dz \int_0^{2\pi} d\phi \int_0^3 \rho^2 d\rho \\ &= \frac{190\pi}{3} = 198.97\end{aligned}$$

Prob. 3.32

$$(a) \quad \nabla_x \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & y^2 & -xz \end{vmatrix} = z\mathbf{a}_y - x\mathbf{a}_z$$

$$\nabla_x \mathbf{B} = \left(\frac{1}{\rho} 2\rho z 2 \sin \phi \cos \phi - 0 \right) \mathbf{a}_\rho + (2\rho z - 2z \sin^2 \phi) \mathbf{a}_\phi + \frac{1}{\rho} (2\rho \sin^2 \phi - 0) \mathbf{a}_z$$

$$\begin{aligned}(b) \quad &= 4z \sin \phi \cos \phi \mathbf{a}_\rho + 2(\rho z - z \sin^2 \phi) \mathbf{a}_\phi + 2 \sin^2 \phi \mathbf{a}_z \\ &= 2z \sin 2\phi \mathbf{a}_\rho + 2z(\rho - \sin^2 \phi) \mathbf{a}_\phi + 2 \sin^2 \phi \mathbf{a}_z\end{aligned}$$

$$\nabla_x \mathbf{C} = \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (r \cos^2 \theta \sin \theta) \right] \mathbf{a}_r - \frac{1}{r} \left[\frac{\partial}{\partial r} (r^2 \cos^2 \theta) \right] \mathbf{a}_\theta$$

$$\begin{aligned}(c) \quad &= \frac{r}{r \sin \theta} \left[(2 \cos \theta)(-\sin \theta) \sin \theta + \cos \theta (\cos^2 \theta) \right] \mathbf{a}_r - \frac{\cos^2 \theta}{r} (2r) \mathbf{a}_\theta \\ &= \frac{(\cos^3 \theta - 2 \sin^2 \theta \cos \theta)}{\sin \theta} \mathbf{a}_r - 2 \cos^2 \theta \mathbf{a}_\theta\end{aligned}$$

Prob. 3.33

(a)

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & y^2 z & -2xz \end{vmatrix} = \underline{\underline{-y^2 \mathbf{a}_x + 2z \mathbf{a}_y - x^2 \mathbf{a}_z}}$$

$$\nabla \bullet \nabla \times \mathbf{A} = \underline{\underline{0}}$$

(b)

$$\begin{aligned} \nabla \times \mathbf{A} &= \left(\frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) \mathbf{a}_\rho + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right) \mathbf{a}_\phi + \frac{1}{\rho} \left(\frac{\partial (\rho A_\rho)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi} \right) \mathbf{a}_z \\ &= (0 - 0) \mathbf{a}_\rho + (\rho^2 - 3z^2) \mathbf{a}_\phi + \frac{1}{\rho} (4\rho^3 - 0) \mathbf{a}_z \\ &= \underline{\underline{(\rho^2 - 3z^2) \mathbf{a}_\phi + 4\rho^2 \mathbf{a}_z}} \end{aligned}$$

$$\nabla \bullet \nabla \times \mathbf{A} = \underline{\underline{0}}$$

$$\begin{aligned} \nabla \times \mathbf{A} &= \frac{1}{r \sin \theta} \left[0 - \frac{\sin \phi}{r^2} \right] \mathbf{a}_r + \frac{1}{r} \left[\frac{-1}{\sin \theta} \frac{\cos \phi}{r^2} - 0 \right] \mathbf{a}_\theta + \frac{1}{r} \left[\frac{\partial}{\partial r} \left(\frac{\cos \phi}{r} \right) - 0 \right] \mathbf{a}_\phi \\ (c) \quad &= -\frac{\sin \phi}{r^3 \sin \theta} \mathbf{a}_r + \frac{\cos \phi}{r^3 \sin \theta} \mathbf{a}_\theta + \frac{\cos \phi}{r^3} \mathbf{a}_\phi \end{aligned}$$

$$\nabla \bullet \nabla \times \mathbf{A} = \frac{-\sin \phi}{r^4 \sin \theta} + 0 + \frac{\sin \phi}{r^4 \sin \theta} = 0$$

$$\nabla \bullet \nabla \times \mathbf{A} = \underline{\underline{0}}$$

Prob. 3.34

$$\nabla \times \mathbf{H} = 0 \mathbf{a}_\rho + 1 \mathbf{a}_\phi + \frac{1}{\rho} (2\rho \cos \phi - \rho \cos \phi) \mathbf{a}_z = \underline{\underline{\mathbf{a}_\phi + \cos \phi \mathbf{a}_z}}$$

$$\nabla \times \nabla \times \mathbf{H} = \left(-\frac{1}{\rho} \sin \phi - 0 \right) \mathbf{a}_\rho + 0 \mathbf{a}_\phi + \frac{1}{\rho} (1 - 0) \mathbf{a}_z = \underline{\underline{-\frac{1}{\rho} \sin \phi \mathbf{a}_\rho + \frac{1}{\rho} \mathbf{a}_z}}$$

Prob. 3.35

Method 1: We can express \mathbf{A} in spherical coordinates.

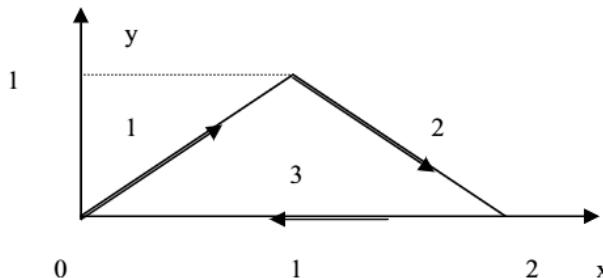
$$\mathbf{A} = \frac{r}{r^3} \mathbf{a}_r = \frac{\mathbf{a}_r}{r^2},$$

$$\nabla \times \mathbf{A} = \nabla \times \left(\frac{\mathbf{a}_r}{r^2} \right) = \nabla \left(\frac{1}{r^2} \right) \times \mathbf{a}_r = \frac{-2}{r^3} \mathbf{a}_r \times \mathbf{a}_r = \mathbf{0}$$

Method 2:

$$\mathbf{A} = \frac{x}{r^3} \mathbf{a}_x + \frac{y}{r^3} \mathbf{a}_y + \frac{z}{r^3} \mathbf{a}_z$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{r^3} & \frac{y}{r^3} & \frac{z}{r^3} \end{vmatrix} = \left\{ -\frac{3}{2} z(x^2 + y^2 + z^2)^{-5/2} (2y) - \frac{3}{2} y(x^2 + y^2 + z^2)^{-5/2} (2z) \right\} \mathbf{a}_x + \dots = \mathbf{0}$$

Prob. 3.36

(a)

$$\oint_L \mathbf{F} \bullet d\mathbf{l} = (\int_1 + \int_2 + \int_3) \mathbf{F} \bullet d\mathbf{l}$$

For 1, $y = x$, $dy = dx$, $d\mathbf{l} = dx \hat{\mathbf{a}}_x + dy \hat{\mathbf{a}}_y$,

$$\int_1 \mathbf{F} \bullet d\mathbf{l} = \int_0^1 x^3 dx - x dx = -\frac{1}{4}$$

For 2, $y = -x + 2$, $dy = -dx$, $d\mathbf{l} = dx \hat{\mathbf{a}}_x + dy \hat{\mathbf{a}}_y$,

$$\int_2 \mathbf{F} \bullet d\mathbf{l} = \int_1^2 (-x^3 + 2x^2 - x + 2) dx = \frac{17}{12}$$

For 3,

$$\int_3 \mathbf{F} \bullet d\mathbf{l} = \int_2^0 x^2 y dx \Big|_{y=0} = 0$$

$$\oint_L \mathbf{F} \bullet d\mathbf{l} = -\frac{1}{4} + \frac{17}{12} + 0 = \underline{\underline{\frac{7}{6}}}$$

(b)

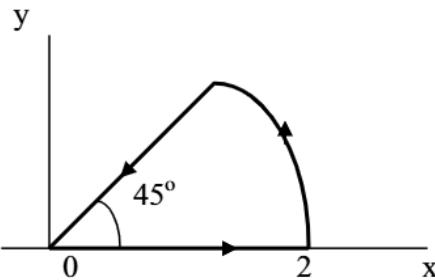
$$\nabla \times \mathbf{F} = -x^2 \mathbf{a}_z ; \quad d\mathbf{S} = dx dy (-\mathbf{a}_z)$$

$$\begin{aligned} \int (\nabla \times \mathbf{F}) \bullet d\mathbf{S} &= - \iint (-x^2) dx dy = \int_0^1 \int_0^x x^2 dy dx + \int_1^2 \int_{-x+2}^{x+2} x^2 dy dx \\ &= \int_0^1 x^2 y \Big|_0^x dx + \int_1^2 x^2 y \Big|_{-x+2}^{x+2} dx = \frac{x^3}{4} \Big|_0^1 + \int_1^2 x^2 (-x+2) dx = \underline{\underline{\frac{7}{6}}} \end{aligned}$$

(c) Yes

Prob. 3.37

$$\begin{aligned} \oint A \bullet d\mathbf{l} &= \int_{\rho=2}^1 \rho \sin \phi d\rho \Big|_{\phi=0} + \int_{\phi=0}^{\pi/2} \rho^2 \rho d\phi \Big|_{\rho=1} + \int_{\rho=1}^2 \rho \sin \phi d\rho \Big|_{\phi=90^\circ} + \int_{\phi=\pi/2}^0 \rho^3 d\phi \Big|_{\rho=2} \\ &= \frac{\pi}{2} + \frac{1}{2}(4-1) + 8(-\frac{\pi}{2}) = \underline{\underline{-9.4956}} \end{aligned}$$

Prob. 3.38

$$\begin{aligned}
 \oint \mathbf{F} \bullet d\mathbf{l} &= \int_0^2 2\rho z d\rho \Big|_{z=1} + \int_0^{\pi/4} 3z \sin \phi \rho d\phi \Big|_{\rho=2, z=1} + \int_2^0 2\rho z d\rho \Big|_{z=1} \\
 &= \rho^2 \Big|_0^2 + (-6 \cos \phi) \Big|_0^{\pi/4} + \rho^2 \Big|_2^0 = (4 - 0) + 6(-\cos \pi/4 + 1) + (0 - 4) = \underline{\underline{1.757}} \\
 \nabla_x \mathbf{F} &= \frac{1}{\rho} [3z \sin \phi - 0] \mathbf{a}_z + \dots \\
 \int (\nabla_x \mathbf{F}) \cdot d\mathbf{S} &= \int_{\rho=0}^2 \int_{\phi=0}^{\pi/4} \frac{3z}{\rho} \sin \phi \rho d\phi d\rho \Big|_{z=1} = 3(2)(-\cos \phi) \Big|_0^{\pi/4} \\
 &= 6(-\cos \pi + 1) = \underline{\underline{1.757}}
 \end{aligned}$$

Prob. 3.39

$$\nabla \cdot \mathbf{A} = 8xe^{-y} + 8xe^{-y} = 16xe^{-y}$$

$$\nabla(\nabla \cdot \mathbf{A}) = 16e^{-y} \mathbf{a}_x - 16xe^{-y} \mathbf{a}_y$$

$$\nabla \times \nabla(\nabla \cdot \mathbf{A}) = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 16e^{-y} & -16xe^{-y} & 0 \end{vmatrix} = (-16e^{-y} + 16e^{-y}) \mathbf{a}_z = \underline{\underline{0}}$$

Should be expected since $\nabla \times \nabla V = 0$.

Prob. 3.40

$$(a) \nabla V = -\frac{\sin \theta \cos \phi}{r^2} \mathbf{a}_r + \frac{\cos \theta \cos \phi}{r^2} \mathbf{a}_\theta - \frac{\sin \phi}{r^2} \mathbf{a}_\phi$$

$$(b) \nabla_x \nabla V = \underline{\underline{0}}$$

(c)

$$\begin{aligned}\nabla \cdot \nabla V &= \nabla^2 V = \frac{1}{r^2} \frac{\partial}{\partial r} (-\sin \theta \cos \phi) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\cos \theta \cos \phi}{r}) + \frac{1}{r^2 \sin^2 \theta} (-\frac{\sin \theta \cos \phi}{r}) \\ &= 0 + \frac{\cos \phi}{r^3 \sin \theta} (1 - 2 \sin^2 \theta) - \frac{\cos \phi}{r^3 \sin \theta} \\ &= -\frac{2 \sin \theta \cos \phi}{r^3}\end{aligned}$$

Prob. 3.41

$$\begin{aligned}\mathbf{Q} &= \frac{r}{r \sin \theta} r \sin \theta [(\cos \phi - \sin \phi) \mathbf{a}_x + (\cos \phi + \sin \phi) \mathbf{a}_y] \\ &= r(\cos \phi - \sin \phi) \mathbf{a}_x + r(\cos \phi + \sin \phi) \mathbf{a}_y\end{aligned}$$

$$\begin{bmatrix} Q_r \\ Q_\theta \\ Q_\phi \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \end{bmatrix} \begin{bmatrix} Q_x \\ Q_y \\ Q_z \end{bmatrix}$$

$$\mathbf{Q} = r \sin \theta \mathbf{a}_r + r \cos \theta \mathbf{a}_\theta + r \mathbf{a}_\phi$$

(a)

$$dl = \rho d\phi \mathbf{a}_\phi, \quad \rho = r \sin 30^\circ = 2\left(\frac{1}{2}\right) = 1$$

$$z = r \cos 30^\circ = \sqrt{3}$$

$$Q_\phi = r = \sqrt{\rho^2 + z^2}$$

$$\oint \mathbf{Q} \bullet dl = \int_0^{2\pi} \sqrt{\rho^2 + z^2} \rho d\phi = 2(1)(2\pi) = \underline{\underline{4\pi}}$$

(b)

$$\nabla \times \mathbf{Q} = \cot \theta \mathbf{a}_r - 2 \mathbf{a}_\theta + \cos \theta \mathbf{a}_\phi$$

$$\text{For } S_1, \quad dS = r^2 \sin \theta d\theta d\phi \mathbf{a}_r$$

$$\int_{S_1} (\nabla \times \mathbf{Q}) \bullet dS = \int r^2 \sin \theta \cot \theta d\theta d\phi \Big|_{r=2}$$

$$= 4 \int_0^{2\pi} d\phi \int_0^{30^\circ} \cos \theta d\theta = \underline{\underline{4\pi}}$$

(c)

$$\text{For } S_2, \quad d\mathbf{S} = r \sin \theta d\theta dr \mathbf{a}_\theta$$

$$\begin{aligned}\int_{S_2} (\nabla \times \mathbf{Q}) \cdot d\mathbf{S} &= -2 \int_{S_2} r \sin \theta d\phi dr \Big|_{\theta=30^\circ} \\ &= -2 \sin 30 \int_0^2 r dr \int_0^{2\pi} d\phi \\ &= -\frac{4\pi}{2}\end{aligned}$$

(d)

$$\text{For } S_1, \quad d\mathbf{S} = r^2 \sin \theta d\phi d\theta \mathbf{a}_r$$

$$\begin{aligned}\int_{S_1} \mathbf{Q} \cdot d\mathbf{S} &= r^3 \int_{S_1} \sin^2 \theta d\theta d\phi \Big|_{r=2} \\ &= 8 \int_0^{2\pi} d\phi \int_0^{30^\circ} \sin^2 \theta d\theta \\ &= 4\pi \left[\frac{\pi}{3} - \frac{\sqrt{3}}{2} \right] = \underline{\underline{2.2767}}\end{aligned}$$

(e)

$$\text{For } S_2, \quad d\mathbf{S} = r \sin \theta d\phi dr \mathbf{a}_\theta$$

$$\begin{aligned}\int_{S_2} \mathbf{Q} \cdot d\mathbf{S} &= \int_{S_2} r^2 \sin \theta \cos \theta d\phi dr \Big|_{\theta=30^\circ} \\ &= \frac{4\pi\sqrt{3}}{3} = \underline{\underline{7.2552}}\end{aligned}$$

(f)

$$\begin{aligned}\nabla \cdot \mathbf{Q} &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^3 \sin \theta) + \frac{r}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \cos \theta) + 0 \\ &= 2 \sin \theta + \cos \theta \cot \theta\end{aligned}$$

$$\begin{aligned}\int \nabla \cdot \mathbf{Q} dv &= \int (2 \sin \theta + \cos \theta \cot \theta) r^2 \sin \theta d\theta d\phi dr \\ &= \frac{r^3}{3} \Big|_0^2 (2\pi) \int_0^{30^\circ} (1 + \sin^2 \theta) d\theta \\ &= \frac{4\pi}{3} \left(\pi - \frac{\sqrt{3}}{2} \right) = \underline{\underline{9.532}}\end{aligned}$$

$$\begin{aligned}
 \text{Check: } \int \nabla \cdot \mathbf{Q} dv &= \left(\int_{S_1} + \int_{S_2} \right) \mathbf{Q} \cdot d\mathbf{S} \\
 &= 4\pi \left[\frac{\pi}{3} - \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{3} \right] \\
 &= \frac{4\pi}{3} \left[\pi - \frac{\sqrt{3}}{2} \right] \quad (\text{It checks!})
 \end{aligned}$$

Prob. 3.42

Since $\mathbf{u} = \boldsymbol{\omega} \times \mathbf{r}$, $\nabla \times \mathbf{u} = \nabla \times (\boldsymbol{\omega} \times \mathbf{r})$. From Appendix A.10,

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$$

$$\nabla \times \mathbf{u} = \nabla \times (\boldsymbol{\omega} \times \mathbf{r})$$

$$\begin{aligned}
 \nabla \times (\boldsymbol{\omega} \times \mathbf{r}) &= \boldsymbol{\omega}(\nabla \cdot \mathbf{r}) - \mathbf{r}(\nabla \cdot \boldsymbol{\omega}) + (\mathbf{r} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\mathbf{r} \\
 &= \boldsymbol{\omega}(3) - \boldsymbol{\omega} = 2\boldsymbol{\omega}
 \end{aligned}$$

$$\text{or } \boldsymbol{\omega} = \frac{1}{2} \nabla \times \mathbf{u}.$$

Alternatively, let $x = r \cos \omega t$, $y = r \sin \omega t$

$$\mathbf{u} = \frac{\partial x}{\partial t} \mathbf{a}_x + \frac{\partial y}{\partial t} \mathbf{a}_y$$

$$= -\omega r \sin \omega t \mathbf{a}_x + \omega r \cos \omega t \mathbf{a}_y$$

$$= -\omega y \mathbf{a}_x + \omega x \mathbf{a}_y$$

$$\nabla \times \mathbf{u} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = 2\omega \mathbf{a}_z = 2\boldsymbol{\omega}$$

$$\text{i.e., } \boldsymbol{\omega} = \frac{1}{2} \nabla \times \mathbf{u}$$

Note that we have used the fact that $\nabla \cdot \boldsymbol{\omega} = 0$, $(\mathbf{r} \cdot \nabla)\boldsymbol{\omega} = 0$, $(\boldsymbol{\omega} \cdot \nabla)\mathbf{r} = \boldsymbol{\omega}$

Prob. 3.43

$$\nabla \cdot \mathbf{B} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho B_\rho) + \frac{1}{\rho} \frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z} = \frac{1}{\rho} 2\rho \cos \phi + \rho \cos \phi - 4 \\ = \underline{\underline{(2+\rho)\cos \phi - 4}}$$

$$(a) \quad \nabla \times \mathbf{B} = \left[\frac{1}{\rho} \frac{\partial B_z}{\partial \phi} - \frac{\partial B_\phi}{\partial z} \right] \mathbf{a}_\rho + \left[\frac{\partial B_\rho}{\partial z} - \frac{\partial B_z}{\partial \rho} \right] \mathbf{a}_\phi + \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} (\rho B_\phi) - \frac{\partial B_\rho}{\partial \phi} \right] \mathbf{a}_z \\ = 0\mathbf{a}_\rho + 0\mathbf{a}_\phi + \frac{1}{\rho} \left[3\rho^2 \sin \phi + \rho \sin \phi \right] \mathbf{a}_z \\ = \underline{\underline{(3\rho+1)\sin \phi \mathbf{a}_z}}$$

(b)

$$\nabla \cdot \mathbf{F} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 F_r) + 0 + \frac{1}{r \sin \theta} \frac{\partial F_\phi}{\partial \phi} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^4 \sin \theta) + \frac{1}{r \sin \theta} (-2r \sin \phi) \\ = 4r \sin \theta - \underline{\underline{\frac{2 \sin \phi}{\sin \theta}}}$$

$$\nabla \times \mathbf{F} = \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (F_\phi \sin \theta) - \frac{\partial F_\theta}{\partial \phi} \right] \mathbf{a}_r + \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial F_r}{\partial \phi} - \frac{\partial}{\partial r} (r F_\phi) \right] \mathbf{a}_\theta \\ + \frac{1}{r} \left[\frac{\partial}{\partial r} (r F_\theta) - \frac{\partial F_r}{\partial \theta} \right] \mathbf{a}_\phi \\ = \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (2r \sin \theta \cos \phi) - 0 \right] \mathbf{a}_r + \frac{1}{r} \left[\frac{1}{\sin \theta} r^2 \cos \phi - 4r \cos \phi \right] \mathbf{a}_\theta + 0\mathbf{a}_\phi \\ = 2 \cot \theta \cos \phi \mathbf{a}_r + \left(\frac{r \cos \phi}{\sin \theta} - 4 \cos \phi \right) \mathbf{a}_\theta \\ = \underline{\underline{2 \cot \theta \cos \phi \mathbf{a}_r + \left(\frac{r \cos \phi}{\sin \theta} - 4 \cos \phi \right) \mathbf{a}_\theta}}$$

Prob. 3.44

$$(a) \quad \nabla \cdot (\nabla V) = \nabla \cdot \left(V \frac{\partial V}{\partial x} \mathbf{a}_x + V \frac{\partial V}{\partial y} \mathbf{a}_y + V \frac{\partial V}{\partial z} \mathbf{a}_z \right) \\ = \frac{\partial}{\partial x} \left(V \frac{\partial V}{\partial x} \right) + \frac{\partial}{\partial y} \left(V \frac{\partial V}{\partial y} \right) + \frac{\partial}{\partial z} \left(V \frac{\partial V}{\partial z} \right) \\ = V \frac{\partial^2 V}{\partial x^2} + V \frac{\partial^2 V}{\partial y^2} + V \frac{\partial^2 V}{\partial z^2} + \left(\frac{\partial V}{\partial x} \right)^2 + \left(\frac{\partial V}{\partial y} \right)^2 + \left(\frac{\partial V}{\partial z} \right)^2 \\ = V \nabla^2 V + |\nabla V|^2$$

$$\nabla \times V\mathbf{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ VA_x & VA_y & VA_z \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(VA_z) - \frac{\partial}{\partial z}(VA_y) \right] \mathbf{a}_x + \left[\frac{\partial}{\partial z}(VA_x) - \frac{\partial}{\partial x}(VA_z) \right] \mathbf{a}_y + \left[\frac{\partial}{\partial x}(VA_y) - \frac{\partial}{\partial y}(VA_x) \right] \mathbf{a}_z$$

(b)

$$= \left[A_z \frac{\partial V}{\partial y} + V \frac{\partial A_z}{\partial y} - A_y \frac{\partial V}{\partial z} - V \frac{\partial A_y}{\partial z} \right] \mathbf{a}_x$$

$$+ \left[A_x \frac{\partial V}{\partial z} + V \frac{\partial A_x}{\partial z} - A_z \frac{\partial V}{\partial x} - V \frac{\partial A_z}{\partial x} \right] \mathbf{a}_y$$

$$+ \left[A_y \frac{\partial V}{\partial x} + V \frac{\partial A_y}{\partial x} - A_x \frac{\partial V}{\partial y} - V \frac{\partial A_x}{\partial y} \right] \mathbf{a}_z$$

$$\nabla \times V\mathbf{A} = V \left[\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \mathbf{a}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \mathbf{a}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \mathbf{a}_z \right]$$

$$+ \left(A_z \frac{\partial V}{\partial y} - A_y \frac{\partial V}{\partial z} \right) \mathbf{a}_x + \left(A_x \frac{\partial V}{\partial z} - A_z \frac{\partial V}{\partial x} \right) \mathbf{a}_y + \left(A_y \frac{\partial V}{\partial x} - A_x \frac{\partial V}{\partial y} \right) \mathbf{a}_z$$

$$= V \nabla \times \mathbf{A} + \nabla V \times \mathbf{A}$$

Prob. 3.45

(a)

$$\nabla \cdot \mathbf{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 2xy + 1 + 1 = \underline{\underline{2 + 2xy}}$$

(b)

$$\nabla \times \mathbf{B} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & (2x^2 + y) & (z - y) \end{vmatrix} = (-1 + 0)\mathbf{a}_x + (0 - 0)\mathbf{a}_y + (4x + x^2)\mathbf{a}_z$$

$$= \underline{\underline{-\mathbf{a}_x + x(4-x)\mathbf{a}_z}}$$

(c)

$$\nabla(\nabla \cdot \mathbf{B}) = \nabla(2 + 2xy) = \underline{\underline{2y\mathbf{a}_x + 2x\mathbf{a}_y}}$$

(d)

$$\nabla \times \nabla \times \mathbf{B} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -1 & 0 & (4x - x^2) \end{vmatrix} = 0\mathbf{a}_x - (4 - 2x)\mathbf{a}_y + 0\mathbf{a}_z \\ = \underline{\underline{2(x-2)\mathbf{a}_y}}$$

Prob. 3.46

(a)

$$V_1 = x^3 + y^3 + z^3 \\ \nabla^2 V_1 = \frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} + \frac{\partial^2 V_1}{\partial z^2} \\ = \frac{\partial}{\partial x}(3x^2) + \frac{\partial}{\partial y}(3y^2) + \frac{\partial}{\partial z}(3z^2) \\ = 6x + 6y + 6z = \underline{\underline{6(x+y+z)}}$$

(b)

$$V_2 = \rho z^2 \sin 2\phi \\ \nabla^2 V_2 = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho z^2 \sin 2\phi) - \frac{4z^2}{\rho} \sin 2\phi + \frac{\partial}{\partial z} (2\rho z \sin 2\phi) \\ = \frac{z^2}{\rho} \sin 2\phi - \frac{4z^2}{\rho} \sin 2\phi + 2\rho \sin 2\phi \\ = \underline{\underline{\left(\frac{-3z^2}{\rho} + 2\rho\right) \sin 2\phi}}$$

(c)

$$V_3 = r^2(1 + \cos \theta \sin \phi) \\ \nabla^2 V_3 = \frac{1}{r^2} \frac{\partial}{\partial r} [2r^3(1 + \cos \theta \sin \phi)] \\ + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (-\sin^2 \theta \sin \phi) r^2 + \frac{1}{r^2 \sin^2 \theta} r^2 (-\cos \theta \sin \phi) \\ = 6(1 + \cos \theta \sin \phi) - \frac{2 \sin \theta}{\sin \theta} \cos \theta \sin \phi - \frac{\cos \theta \sin \phi}{\sin^2 \theta} \\ = \underline{\underline{6 + 4 \cos \theta \sin \phi - \frac{\cos \theta \sin \phi}{\sin^2 \theta}}}$$

Prob. 3.47

(a)

$$\begin{aligned}
 U &= x^3 y^2 e^{xz} \\
 \nabla^2 U &= \frac{\partial}{\partial x} (3x^2 y^2 e^{xz} + x^3 y^2 z e^{xz}) + \frac{\partial}{\partial y} (2x^3 y e^{xz}) + \frac{\partial}{\partial z} (x^4 y^2 e^{xz}) \\
 &= 6xy^2 e^{xz} + 3x^2 yze^{xz} + 3x^2 y^2 z e^{xz} + x^3 y^2 z^2 e^{xz} + 2x^3 e^{xz} + x^5 y^2 e^{xz} \\
 &\underline{= e^{xz}(6xy^2 + 3x^2 y^2 z + 3x^2 y^2 z + x^3 y^2 z^2 + 2x^3 + x^5 y^2)}
 \end{aligned}$$

At $(1, -1, 1)$,

$$\nabla^2 U = e^1 (6 + 3 + 3 + 1 + 2 + 1) = 16e = \underline{\underline{43.493}}$$

(b)

$$\begin{aligned}
 V &= \rho^2 z (\cos \phi + \sin \phi) \\
 \nabla^2 V &= \frac{1}{\rho} \frac{\partial}{\partial \rho} [2\rho^2 z (\cos \phi + \sin \phi)] - z(\cos \phi + \sin \phi) + 0 \\
 &= 4z(\cos \phi + \sin \phi) - z(\cos \phi + \sin \phi) \\
 &\underline{= 3z(\cos \phi + \sin \phi)}
 \end{aligned}$$

$$\text{At } (5, \frac{\pi}{6}, -2), \quad \nabla^2 V = -6(0.866 + 0.5) = \underline{\underline{-8.196}}$$

(c)

$$\begin{aligned}
 W &= e^{-r} \sin \theta \cos \phi \\
 \nabla^2 W &= \frac{1}{r^2} \frac{\partial}{\partial r} (-r^2 e^{-r} \sin \theta \cos \phi) + \frac{e^{-r}}{r^2 \sin \theta} \cos \phi \frac{\partial}{\partial \theta} (\sin \theta \cos \theta) \\
 &\quad - \frac{e^{-r} \sin \theta \cos \phi}{r^2 \sin^2 \theta} \\
 &= \frac{1}{r^2} (-2re^{-r} \sin \theta \cos \phi) + e^{-r} \sin \theta \cos \phi \\
 &\quad + \frac{e^{-r} \cos \phi}{r^2 \sin \theta} (1 - 2 \sin^2 \theta) - \frac{e^{-r} \cos \phi}{r^2 \sin \theta}
 \end{aligned}$$

$$\nabla^2 W = e^{-r} \sin \theta \cos \phi \left(1 - \frac{2}{r} - \frac{2}{r^2}\right)$$

At $(1, 60^\circ, 30^\circ)$,

$$\nabla^2 W = e^{-1} \sin 60 \cos 30 \left(1 - 2 - 2\right) = -2.25e^{-1} = \underline{\underline{-0.8277}}$$

Prob. 3.48(a) Let $V = \ln r = \ln \sqrt{x^2 + y^2 + z^2}$

$$\frac{\partial V}{\partial x} = \frac{1}{r} \cdot \frac{1}{2} (2x) (x^2 + y^2 + z^2)^{-1/2} = \frac{x}{r^2}$$

$$\nabla V = \frac{\partial V}{\partial x} \mathbf{a}_x + \frac{\partial V}{\partial y} \mathbf{a}_y + \frac{\partial V}{\partial z} \mathbf{a}_z = \frac{x \mathbf{a}_x + y \mathbf{a}_y + z \mathbf{a}_z}{r^2} = \underline{\underline{\frac{\mathbf{r}}{r^2}}}$$

(b) Let $\nabla V = \mathbf{A} = \frac{\mathbf{r}}{r^2} = \frac{1}{r} \mathbf{a}_r$ in spherical coordinates.

$$\begin{aligned}\nabla^2(1nr) &= \nabla \cdot \nabla (1nr) = \nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) = \frac{1}{r^2} \frac{\partial}{\partial r} (r) \\ &= \underline{\underline{\frac{1}{r^2}}}\end{aligned}$$

Prob. 3.49

$$\begin{aligned}\nabla V &= \frac{\partial V}{\partial x} \mathbf{a}_x + \frac{\partial V}{\partial y} \mathbf{a}_y + \frac{\partial V}{\partial z} \mathbf{a}_z \\ &= y^2 z^3 \mathbf{a}_x + 2xyz^3 \mathbf{a}_y + 3xy^2 z^2 \mathbf{a}_z\end{aligned}$$

At P(1,2,3,) $x = 1, y = 2, z = 3$

$$\begin{aligned}\nabla V &= 4(27)\mathbf{a}_x + 2(2)(27)\mathbf{a}_y + 3(4)(9)\mathbf{a}_z \\ &= \underline{\underline{108(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)}}\end{aligned}$$

$$\begin{aligned}\nabla^2 V &= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \\ &= \frac{\partial}{\partial x} (y^2 z^3) + \frac{\partial}{\partial y} (2xyz^3) + \frac{\partial}{\partial z} (3xy^2 z^2) \\ &= 0 + 2xz^3 + 6xy^2 z \\ &= 2xz(z^2 + 3y^2)\end{aligned}$$

At P(1,2,3,) $x = 1, y = 2, z = 3$.

$$\begin{aligned}\nabla^2 V &= 2(1)(3)(9 + 3 \times 4) = 6(9 + 12) \\ &= \underline{\underline{126}}\end{aligned}$$

Prob. 3.50

$$\nabla V = \frac{\partial V}{\partial \rho} \mathbf{a}_\rho + \frac{1}{\rho} \frac{\partial V}{\partial \phi} \mathbf{a}_\phi + \frac{\partial V}{\partial z} \mathbf{a}_z = \underline{2\rho z \cos \phi \mathbf{a}_\rho - \rho z \sin \phi \mathbf{a}_\phi + \rho^2 \cos \phi \mathbf{a}_z}$$

$$\begin{aligned}\nabla^2 V &= \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial V}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (2\rho^2 z \cos \phi) - \frac{1}{\rho^2} \rho^2 z \cos \phi + 0 \\ &= (4-1) z \cos \phi = \underline{\underline{3z \cos \phi}}\end{aligned}$$

Prob. 3.51

$$\begin{aligned}\nabla V &= \frac{\partial V}{\partial r} \mathbf{a}_r + \frac{1}{r} \frac{\partial V}{\partial \theta} \mathbf{a}_\theta + \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \mathbf{a}_\phi \\ \text{(a)} \quad &= \underline{\underline{-\frac{10}{r^3} \cos \phi \mathbf{a}_r - \frac{5 \sin \phi}{r^3 \sin \theta} \mathbf{a}_\phi}}$$

$$\begin{aligned}\nabla \cdot \nabla V &= \nabla^2 V = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial V}{\partial r}) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial V}{\partial \theta}) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2} \\ \text{(b)} \quad &= \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \left(-\frac{10 \cos \phi}{r^3} \right) + 0 + \frac{1}{r^2 \sin^2 \theta} \left(-\frac{5 \cos \phi}{r^2} \right) \\ &= \underline{\underline{\frac{10 \cos \phi}{r^4} - \frac{5 \cos \phi}{r^4 \sin^2 \theta}}}\end{aligned}$$

$$\text{(c)} \quad \nabla \times \nabla V = 0, \text{ see Example 3.10.}$$

Prob. 3.52

$$\nabla U = \frac{\partial U}{\partial x} \mathbf{a}_x + \frac{\partial U}{\partial y} \mathbf{a}_y + \frac{\partial U}{\partial z} \mathbf{a}_z = 4yz^2 \mathbf{a}_x + (4xz^2 + 10z) \mathbf{a}_y + (8xyz + 10y) \mathbf{a}_z$$

$$\nabla \cdot \nabla U = \frac{\partial}{\partial x} (\nabla U_x) + \frac{\partial}{\partial y} (\nabla U_y) + \frac{\partial}{\partial z} (\nabla U_z) = 0 + 0 + 8xy = 8xy$$

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0 + 0 + 8xy = 8xy$$

Hence, $\nabla^2 U = \nabla \cdot \nabla U$

Prob. 3.53Method 1

$$\begin{aligned}\nabla^2 \mathbf{G} \Big|_{\rho} &= \nabla^2 G_{\rho} - \frac{2}{\rho^2} \frac{\partial G_{\phi}}{\partial \phi} - \frac{G_{\phi}}{\rho^2} \\ &= \frac{1}{\rho} \frac{\partial}{\partial \rho} (2\rho \sin \phi) - \frac{2\rho \sin \phi}{\rho^2} + 0 + \frac{8\rho \sin \phi}{\rho^2} - \frac{2\rho \sin \phi}{\rho^2} \\ &= \frac{2 \sin \phi}{\rho} - \frac{2 \sin \phi}{\rho} + \frac{8 \sin \phi}{\rho} - \frac{2 \sin \phi}{\rho} = \frac{6 \sin \phi}{\rho}\end{aligned}$$

$$\begin{aligned}\nabla^2 \mathbf{G} \Big|_{\phi} &= \nabla^2 G_{\phi} + \frac{2}{\rho^2} \frac{\partial G_{\rho}}{\partial \phi} - \frac{G_{\rho}}{\rho^2} \\ &= \frac{1}{\rho} \frac{\partial}{\partial \rho} (4\rho \cos \phi) - \frac{1}{\rho} 4\rho \cos \phi + 0 + \frac{4\rho \cos \phi}{\rho^2} - \frac{4\rho \cos \phi}{\rho^2} \\ &= \frac{4 \cos \phi}{\rho} - \frac{4 \cos \phi}{\rho} + \frac{4 \cos \phi}{\rho} - \frac{4 \cos \phi}{\rho} = 0\end{aligned}$$

$$\begin{aligned}\nabla^2 \mathbf{G} \Big|_z &= \nabla^2 G_z = \frac{1}{\rho} \frac{\partial}{\partial \rho} [\rho(z^2 + 1)] + 0 + \frac{\partial}{\partial z} (2z\rho) \\ &= \frac{1}{\rho} (z^2 + 1) + 2\rho\end{aligned}$$

Adding the components together gives

$$\underline{\underline{\nabla^2 \mathbf{G} = \frac{6 \sin \phi}{\rho} \mathbf{a}_{\rho} + \left[2\rho + \frac{1}{\rho} (z^2 + 1) \right] \mathbf{a}_z}}$$

Method 2:

$$\nabla^2 \mathbf{G} = \nabla(\nabla \cdot \mathbf{G}) - \nabla \times (\nabla \times \mathbf{G})$$

$$\text{Let } V = \nabla \cdot \mathbf{G} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (2\rho^2 \sin \phi) + \frac{1}{\rho} (-4\rho \sin \phi) + 2z\rho = 2z\rho$$

$$\nabla(V) = \nabla V = 2z\mathbf{a}_{\rho} + 2\rho\mathbf{a}_z$$

Let $\mathbf{A} = \nabla \times \mathbf{G} = \left[\frac{1}{\rho} 0 - 0 \right] \mathbf{a}_\rho + \left[0 - (z^2 + 1) \right] \mathbf{a}_\phi + \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} (4\rho^2 \cos \phi) - 2\rho \cos \phi \right] \mathbf{a}_z$

$$= -(z^2 + 1) \mathbf{a}_\phi + 6 \cos \phi \mathbf{a}_z$$

$$\nabla \times \nabla \times \mathbf{G} = \nabla \times \mathbf{A} = \left[-\frac{6}{\rho} \sin \phi + 2z \right] \mathbf{a}_\rho + (0 - 0) \mathbf{a}_\phi + \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} (-\rho(z^2 + 1)) - 0 \right] \mathbf{a}_z$$

$$= \left[2z - \frac{6}{\rho} \sin \phi \right] \mathbf{a}_\rho - \frac{1}{\rho} (z^2 + 1) \mathbf{a}_z$$

$$\nabla^2 \mathbf{G} = \nabla V - \nabla \times \mathbf{A}$$

$$= 2z \mathbf{a}_\rho + 2\rho \mathbf{a}_z - \left[2z - \frac{6}{\rho} \sin \phi \right] \mathbf{a}_\rho + \frac{1}{\rho} (z^2 + 1) \mathbf{a}_z$$

$$= \underline{\underline{\frac{6}{\rho} \sin \phi \mathbf{a}_\rho + \left[2\rho + \frac{1}{\rho} (z^2 + 1) \right] \mathbf{a}_z}}$$

Prob. 3.54

$$\nabla \cdot \mathbf{A} = \frac{\partial}{\partial x} (xz) + \frac{\partial}{\partial y} (z^2) + \frac{\partial}{\partial z} (yz) = z + y$$

$$\nabla(\nabla \cdot \mathbf{A}) = \mathbf{a}_y + \mathbf{a}_z$$

$$\nabla^2 \mathbf{A} = \nabla^2 A_x \mathbf{a}_x + \nabla^2 A_y \mathbf{a}_y + \nabla^2 A_z \mathbf{a}_z = 0 + 2\mathbf{a}_y + 0 = 2\mathbf{a}_y$$

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = -\mathbf{a}_y + \mathbf{a}_z \quad (1)$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & z^2 & yz \end{vmatrix} = -z\mathbf{a}_x + x\mathbf{a}_y$$

$$\nabla \times \nabla \times \mathbf{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -z & x & 0 \end{vmatrix} = -\mathbf{a}_y + \mathbf{a}_z \quad (2)$$

From (1) and (2),

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

Prob. 3.55

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 1+1+1=3 \neq 0$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \mathbf{0}$$

$$\nabla \cdot \mathbf{B} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho B_\rho) + \frac{1}{\rho} \frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z} = 4 \cos \phi - 4 \cos \phi = 0$$

$$\begin{aligned} \nabla \times \mathbf{B} &= \left[\frac{1}{\rho} \frac{\partial B_z}{\partial \phi} - \frac{\partial B_\phi}{\partial z} \right] \mathbf{a}_\rho + \left[\frac{\partial B_\rho}{\partial z} - \frac{\partial B_z}{\partial \rho} \right] \mathbf{a}_\phi + \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} (\rho B_\rho) - \frac{\partial B_\rho}{\partial \phi} \right] \mathbf{a}_z \\ &= 0 \mathbf{a}_\rho + 0 \mathbf{a}_\phi + \frac{1}{\rho} [-8\rho \sin \phi + 2\rho \sin \phi] \mathbf{a}_z = -6 \sin \phi \mathbf{a}_z \neq \mathbf{0} \end{aligned}$$

$$\nabla \cdot \mathbf{C} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \sin \theta) + 0 + 0 = \frac{2 \sin \theta}{r} \neq 0$$

$$\begin{aligned} \nabla \times \mathbf{C} &= \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (r \sin^2 \theta) - 0 \right] \mathbf{a}_r + \frac{1}{r} \left[0 - \frac{\partial}{\partial r} (r^2 \sin \theta) \right] \mathbf{a}_\theta \\ &\quad + \frac{1}{r} [0 - \cos \theta] \mathbf{a}_\phi \\ &= 2 \cos \theta \mathbf{a}_r - 2 \sin \theta \mathbf{a}_\theta - \frac{\cos \theta}{r} \mathbf{a}_\phi \neq \mathbf{0} \end{aligned}$$

(a) **B** is solenoidal.

(b) **A** is irrotational.

Prob. 3.56 (a)

$$\nabla \times \mathbf{G} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 16xy-z & 8x^2 & -x \end{vmatrix}$$

$$= 0 \mathbf{a}_x + (-1+1) \mathbf{a}_y + (16x-16x) \mathbf{a}_z = 0$$

Thus, **G** is irrotational.

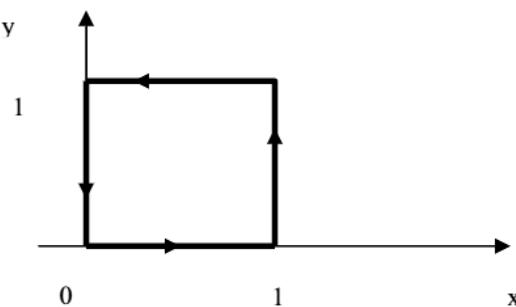
(b) Assume that ψ represents the net flux.

$$\psi = \oint \mathbf{G} \bullet d\mathbf{S} = \int \nabla \bullet \mathbf{G} dv$$

$$\nabla \bullet \mathbf{G} = 16y + 0 + 0 = 16y$$

$$\psi = \iiint 16y dx dy dz = 16 \int_0^1 dx \int_0^1 dz \int_0^1 y dy = 16(1)(1)\left(\frac{y^2}{2}\Big|_0^1\right) = \underline{\underline{8}}$$

(c)



$$\begin{aligned} \oint_L \mathbf{G} \bullet d\mathbf{l} &= \int_{x=0}^{x=1} (16xy - z) dx \Big|_{y=0}^{y=1} + \int_{y=0}^{y=1} 8x^2 dy \Big|_{z=0}^{z=1} + \int_{z=0}^{z=1} (16xy - z) dx \Big|_{y=1}^{y=0} + \int_{x=1}^{x=0} 8x^2 dy \Big|_{z=0}^{z=1} \\ &= 0 + 8(1)y \Big|_0^1 + 16(1)\frac{x^2}{2} \Big|_1^0 + 0 \\ &= 8 - 8 = \underline{\underline{0}} \end{aligned}$$

This is expected since \mathbf{G} is irrotational, i.e.

$$\oint \mathbf{G} \bullet d\mathbf{l} = \int (\nabla \times \mathbf{G}) \bullet d\mathbf{S} = 0$$

$$\nabla \bullet \mathbf{T} = -6 + 0 = \underline{\underline{-6}}$$

Prob. 3.57

$$\nabla \cdot \mathbf{F} = 0$$

$$\nabla \cdot x\mathbf{F} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = (x-x)\mathbf{a}_x + (y-y)\mathbf{a}_y + (z-z)\mathbf{a}_z = \mathbf{0}$$

Hence \mathbf{F} is both solenoidal and conservative.

Prob. 3.58

$$\nabla \times \mathbf{H} = \mathbf{0}$$

$$\oint_L H \cdot dl = \int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} = 0$$

Prob. 3.59

From Appendix A.10,

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

If \mathbf{A} and \mathbf{B} are irrotational,

$$\nabla \times \mathbf{A} = \mathbf{0} = \nabla \times \mathbf{B}$$

$$\text{i.e. } \nabla \cdot (\mathbf{A} \times \mathbf{B}) = 0$$

which implies that $\mathbf{A} \times \mathbf{B}$ is solenoidal.

CHAPTER 4

P. E. 4.1

$$(a) \quad \mathbf{F} = \frac{1 \times 10^{-9}}{4\pi \left(\frac{10^{-9}}{36\pi} \right)} \left[\frac{5 \times 10^{-9}[(1, -3, 7) - (2, 0, 4)]}{[(1, -3, 7) - (2, 0, 4)]^3} + \frac{(-2 \times 10^{-9})[(1, -3, 7) - (-3, 0, 5)]}{[(1, -3, 7) - (-3, 0, 5)]^3} \right]$$

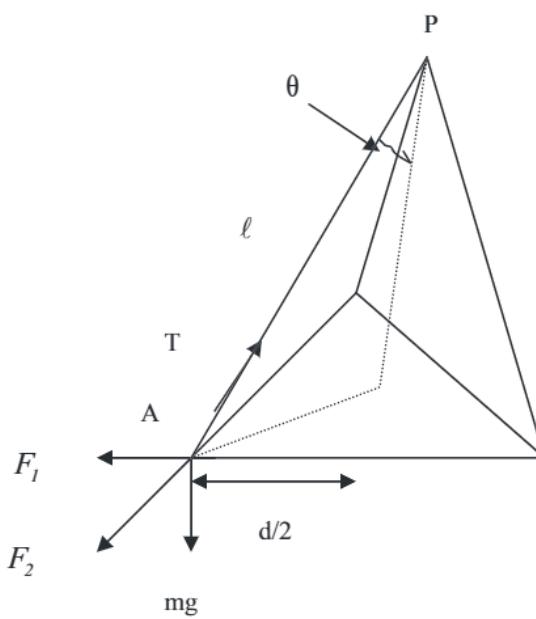
$$= \left[\frac{45(-1, -3, 3)}{19^{3/2}} - \frac{18(4, -3, 2)}{29^{3/2}} \right] \text{ nN}$$

$$= \underline{\underline{-1.004\mathbf{a}_x - 1.284\mathbf{a}_y + 1.4\mathbf{a}_z \text{ nN}}}$$

$$(b) \quad \mathbf{E} = \frac{\mathbf{F}}{Q} = \underline{\underline{-1.004\mathbf{a}_x - 1.284\mathbf{a}_y + 1.4\mathbf{a}_z \text{ V/m}}}$$

P. E. 4.2

Let q be the charge on each sphere, i.e. $q=Q/3$. The free body diagram below helps us to establish the relationship between various forces.



At point A,

$$\begin{aligned} T \sin \theta \cos 30^\circ &= F_1 + F_2 \cos 60^\circ \\ &= \frac{q^2}{4\pi\epsilon_0 d^2} + \frac{q^2}{4\pi\epsilon_0 d^2} \left(\frac{1}{2}\right) \\ &= \frac{3q^2}{8\pi\epsilon_0 d^2} \end{aligned}$$

$$T \cos \theta = mg$$

$$\text{Hence, } \tan \theta \cos 30^\circ = \frac{3q^2}{8\pi\epsilon_0 d^2 mg}$$

$$\text{But } \sin \theta = \frac{h}{l} = \frac{d}{\sqrt{3}l} \quad \tan \theta = \frac{\frac{d}{\sqrt{3}}}{\sqrt{l^2 - \frac{d^2}{3}}}$$

$$\text{Thus, } \frac{\frac{d}{\sqrt{3}} \left(\frac{\sqrt{3}}{2}\right)}{\sqrt{l^2 - \frac{d^2}{3}}} = \frac{3q^2}{8\pi\epsilon_0 d^2 mg}$$

$$\text{or } q^2 = \frac{4\pi\epsilon_0 d^3 mg}{3\sqrt{l^2 - \frac{d^2}{3}}}$$

$$\text{but } q = \frac{Q}{3} \longrightarrow q^2 = \frac{Q^2}{9}. \text{ Hence,}$$

$$Q^2 = \frac{12\pi\epsilon_0 d^3 mg}{\sqrt{l^2 - \frac{d^2}{3}}}$$

P.E. 4.3

$$e\bar{E} = m \frac{d^2 \bar{l}}{dt^2}$$

$$eE_0(-2\bar{a}_x + \bar{a}_y) = m \left(\frac{d^2 x}{dt^2} \bar{a}_x + \frac{d^2 y}{dt^2} \bar{a}_y + \frac{d^2 z}{dt^2} \bar{a}_z \right)$$

where $E_0 = 200 \text{ kV/m}$

$$\frac{d^2 z}{dt^2} = 0 \quad \longrightarrow \quad z = ct + c_2$$

$$m \frac{d^2 x}{dt^2} = -2eE_0 \quad \longrightarrow \quad x = \frac{-2eE_0 t^2}{2m} + c_3 t + c_4$$

$$m \frac{d^2 y}{dt^2} = eE_0 \quad \longrightarrow \quad y = \frac{eE_0 t^2}{2m} + c_5 t + c_6$$

At $t = 0$, $(x, y, z) = (0, 0, 0)$ $c_1 = 0 = c_4 = c_6$

$$\text{Also, } \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right) = (0, 0, 0)$$

$$\text{At } t = 0 \quad \longrightarrow \quad c_1 = 0 = c_3 = c_5$$

$$\text{Hence, } (x, y) = \frac{eE_0 t^2}{2m} (-2, 1)$$

i.e. $2 |y| = |x|$

Thus the largest value of is

$$80 \text{ cm} = \underline{\underline{0.8 \text{ m}}}$$

P.E. 4.4

(a)

Consider an element of area dS of the disk.

The contribution due to $dS = \rho d\phi d\rho$ is

$$dE = \frac{\rho_s dS}{4\pi\epsilon_0 r^2} = \frac{\rho_s dS}{4\pi\epsilon_0 (\rho^2 + h^2)}$$

The sum of the contribution along ρ gives zero.

$$\begin{aligned} E_z &= \frac{\rho_s}{4\pi\epsilon_0} \int_{\rho=0}^a \int_{\phi=0}^{2\pi} \frac{h\rho d\rho d\phi}{(\rho^2 + h^2)^{3/2}} = \frac{h\rho_s}{2\epsilon_0} \int_{\rho=0}^a \frac{\rho d\rho}{(\rho^2 + h^2)^{3/2}} \\ &= \frac{h\rho_s}{4\epsilon_0} \int_0^a (\rho^2 + h^2)^{-3/2} d(\rho^2) = \frac{h\rho_s}{4\epsilon_0} \left(-2(\rho^2 + h^2)^{-1/2} \right) \Big|_0^a \\ &= \frac{\rho_s}{2\epsilon_0} \left[1 - \frac{h}{(h^2 + a^2)^{1/2}} \right] \end{aligned}$$

(b)

As $a \longrightarrow \infty$,

$$\underline{\underline{E}} = \frac{\rho_s}{2\epsilon_0} \underline{\underline{a}_z}$$

(c) Let us recall that if $a/h \ll 1$ then $(1+a/h)^n$ can be approximated by $(1+na/h)$. Thus the expression for E_z from (a) can be modified for $a \ll h$ as follows.

$$\begin{aligned} E_z &= \frac{\rho_s}{2\epsilon_0} \left[1 - \frac{1}{\sqrt{1 + \frac{a^2}{h^2}}} \right] = \frac{\rho_s}{2\epsilon_0} \left[1 - \left(1 + \frac{a^2}{h^2} \right)^{-\frac{1}{2}} \right] \xrightarrow{a \rightarrow 0, \text{ but } \rho_s \pi a^2 = Q} \frac{\rho_s}{2\epsilon_0} \left[\frac{a^2}{2h^2} \right] \\ &= \frac{\rho_s}{2\epsilon_0} \left[\frac{\pi a^2}{2\pi h^2} \right] = \frac{Q}{4\pi\epsilon_0 h^2} \end{aligned}$$

This is in keeping with original Coulomb's law.

P. E. 4.5

$$Q_S = \int \rho_s dS = \int_{-2}^2 \int_{-2}^2 12|y| dx dy$$

$$= 12(4) \int_0^2 2y dy = \underline{\underline{192 \text{ mC}}}$$

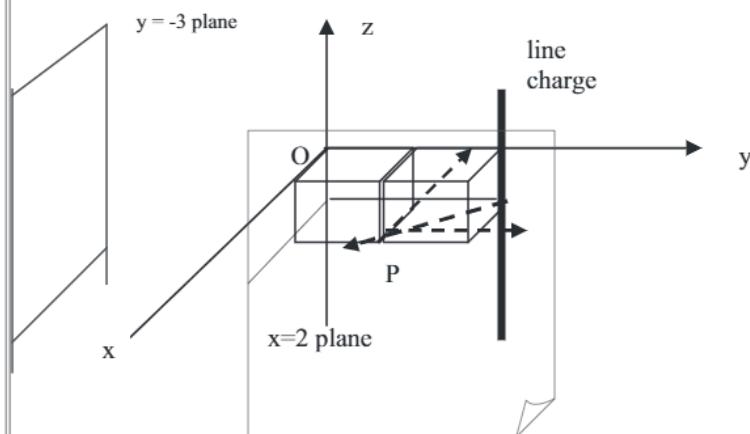
$$\underline{\underline{E}} = \int \frac{\rho_s dS}{4\pi\epsilon r^2} \underline{\underline{a}_r} = \int \frac{\rho_s dS |\mathbf{r} - \mathbf{r}'|}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^3}$$

where $\mathbf{r} - \mathbf{r}' = (0, 0, 10) - (x, y, z) = (-x, -y, 10)$.

$$\begin{aligned}
 E &= \int_{x=-2}^2 \int_{y=-2}^2 \frac{12 |y| 10^{-3}(-x, -y, 10)}{4\pi \left(\frac{10^{-9}}{36\pi}\right) (x^2 + y^2 + 100)^{3/2}} \\
 &= 108(10^6) \left[\int_{-2}^2 |y| \int_{-2}^2 \frac{-x dx dy \mathbf{a}_x}{(x^2 + y^2 + 100)^{3/2}} + \int_{-2}^2 \int_{-2}^2 \frac{-y |y| dy dx \mathbf{a}_y}{(x^2 + y^2 + 100)^{3/2}} \right. \\
 &\quad \left. + 10 \mathbf{a}_z \int_{-2}^2 \int_{-2}^2 \frac{|y| dx dy}{(x^2 + y^2 + 100)^{3/2}} \right] \\
 E &= 108(10^7) \mathbf{a}_z \int_{-2}^2 \left[2 \int_0^2 \frac{\frac{1}{2} d(y^2)}{(x^2 + y^2 + 100)^{3/2}} \right] dx \\
 &= -216(10^7) \mathbf{a}_z \int_{-2}^2 \left[\frac{1}{(x^2 + 104)^{1/2}} - \frac{1}{(x^2 + 100)^{1/2}} \right] dx \\
 &= -216(10^7) \mathbf{a}_z \ln \left| \frac{x + \sqrt{x^2 + 104}}{x + \sqrt{x^2 + 100}} \right| \Big|_{-2}^2 \\
 &= -216(10^7) \mathbf{a}_z \left(\ln \left(\frac{2 + \sqrt{108}}{2 + \sqrt{104}} \right) - \ln \left(\frac{-2 + \sqrt{108}}{-2 + \sqrt{104}} \right) \right) \\
 &= -216(10^7) \mathbf{a}_z (-7.6202 \cdot 10^{-3})
 \end{aligned}$$

$$E = \underline{\underline{16.46 \mathbf{a}_z \text{ MV/m}}}$$

P.E. 4.6



\mathbf{E}_1 and \mathbf{E}_2 remain the same as in Example 4.6.

$$\mathbf{E}_3 = \frac{\rho_L}{2\pi\epsilon_0\rho} \mathbf{a}_\rho$$

This expression, which represents the field due to a line charge, is modified as follows. To get \mathbf{a}_ρ , consider the $z = -1$ plane. $\rho = \sqrt{2}$

$$\mathbf{a}_\rho = \mathbf{a}_x \cos 45^\circ - \mathbf{a}_y \sin 45^\circ$$

$$= \frac{1}{\sqrt{2}}(\mathbf{a}_x - \mathbf{a}_y)$$

$$\mathbf{E}_3 = \frac{10(10^{-9})}{2\pi(\frac{10^{-9}}{36\pi})} \frac{1}{2}(\mathbf{a}_x - \mathbf{a}_y)$$

$$= 90\pi(\mathbf{a}_x - \mathbf{a}_y). \quad \text{Hence,}$$

$$\begin{aligned}\mathbf{E} &= \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 \\ &= -180\pi\mathbf{a}_x + 270\pi\mathbf{a}_y + 90\pi\mathbf{a}_x - 90\pi\mathbf{a}_y \\ &= \underline{\underline{-282.7\mathbf{a}_x + 565.5\mathbf{a}_y \text{ V/m}}}\end{aligned}$$

P.E. 4.7

$$\begin{aligned}\mathbf{D} &= \mathbf{D}_Q + \mathbf{D}_\rho = \frac{Q}{4\pi r^2} \mathbf{a}_r + \frac{\rho_s}{2} \mathbf{a}_n \\ &= \frac{30 \times 10^{-9}}{4\pi(5)^2} \frac{[(0,4,3) - (0,0,0)]}{5} + \frac{10 \times 10^{-9}}{2} \mathbf{a}_y \\ &= \frac{30}{500\pi}(0,4,3) + 5\mathbf{a}_y \text{ nC/m}^2 \\ &= \underline{\underline{5.076\mathbf{a}_y + 0.0573\mathbf{a}_z \text{ nC/m}^2}}\end{aligned}$$

P.E. 4.8

$$(a) \rho_v = \nabla \bullet \mathbf{D} = 4x$$

$$\rho_v(-1,0,3) = \underline{\underline{-4 \text{ C/m}^3}}$$

$$\begin{aligned}(b) \Psi &= Q = \int \rho_v dV = \int_0^1 \int_0^1 \int_0^1 4x dx dy dz \\ &= 4(1)(1)(1/2) = \underline{\underline{2 \text{ C}}}\end{aligned}$$

$$(c) Q = \Psi = \underline{\underline{2 \text{ C}}}$$

P.E. 4.9

$$Q = \int \rho v dv = \psi = \oint \mathbf{D} \bullet d\mathbf{S}$$

For $0 \leq r \leq 10$,

$$D_r(4\pi r^2) = \iiint 2r (r^2) \sin \theta d\theta dr d\phi$$

$$D_r(4\pi r^2) = 4\pi \left(\frac{2r^4}{4}\right) \Big|_0^r = 2\pi r^4$$

$$D_r = \frac{r^2}{2} \quad \mathbf{E} = \frac{r^2}{2\epsilon_0} \mathbf{a}_r \text{ nV/m}$$

$$\mathbf{E}(r=2) = \frac{4(10^{-9})}{2\left(\frac{10^{-9}}{36\pi}\right)} \mathbf{a}_r = 72\pi \mathbf{a}_r = \underline{\underline{226 \mathbf{a}_r \text{ V/m}}}$$

For $r \geq 10$,

$$D_r(4\pi r^2) = 2\pi r_0^4, \quad r_0 = 10 \text{ m}$$

$$D_r = \frac{r_0^4}{2r^2} \longrightarrow \mathbf{E} = \frac{r_0^4}{2\epsilon_0 r^2} \mathbf{a}_r \text{ nV/m}$$

$$\begin{aligned} \mathbf{E}(r=12) &= \frac{10^4(10^{-9})}{2\left(\frac{10^{-9}}{36\pi}\right)(144)} \mathbf{a}_r = 1250\pi \mathbf{a}_r \\ &= \underline{\underline{3.927 \mathbf{a}_r \text{ kV/m}}} \end{aligned}$$

P. E. 4.10

$$V(\mathbf{r}) = \sum_{k=1}^3 \frac{Q_k}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}_k|} + C$$

At $V(\infty) = 0$, $C = 0$

$$|\mathbf{r} - \mathbf{r}_1| = |(-1, 5, 2) - (2, -1, 3)| = \sqrt{46}$$

$$|\mathbf{r} - \mathbf{r}_2| = |(-1, 5, 2) - (0, 4, -2)| = \sqrt{18}$$

$$|\mathbf{r} - \mathbf{r}_3| = |(-1, 5, 2) - (0, 0, 0)| = \sqrt{30}$$

$$\begin{aligned} V(-1, 5, 2) &= \frac{10^{-6}}{4\pi\left(\frac{10^{-9}}{36\pi}\right)} \left[\frac{-4}{\sqrt{46}} + \frac{5}{\sqrt{18}} + \frac{3}{\sqrt{30}} \right] \\ &= \underline{\underline{10.23 \text{ kV}}} \end{aligned}$$

P.E. 4.11

$$V = \frac{Q}{4\pi\epsilon_0 r} + C$$

If $V(0, 6, -8) = V(r=10) = 2$;

$$2 = \frac{5(10^{-9})}{4\pi(\frac{10^{-9}}{36\pi})(10)} + C \quad \longrightarrow \quad C = -2.5$$

(a)

$$\begin{aligned} V_A &= \frac{5(10^{-9})}{4\pi(\frac{10^{-9}}{36\pi})|(-3, 2, 6) - (0, 0, 0)|} - 2.5 \\ &= \underline{\underline{3.929 \text{ V}}} \end{aligned}$$

(b)

$$V_B = \frac{45}{\sqrt{7^2 + 1^2 + 5^2}} - 2.5 = \underline{\underline{2.696 \text{ V}}}$$

$$(c) \quad V_{AB} = V_B - V_A = 2.696 - 3.929 = \underline{\underline{-1.233 \text{ V}}}$$

P.E. 4.12

(a)

$$\begin{aligned} \frac{-W}{Q} &= \int \mathbf{E} \bullet d\mathbf{l} = \int (3x^2 + y)dx + xdy \\ &= \int_0^2 (3x^2 + y)dx \Big|_{y=5} + \int_5^{-1} x dy \Big|_{x=2} \\ &= 18 - 12 = 6 \text{ kV} \end{aligned}$$

$$W = -6Q = \underline{\underline{12 \text{ mJ}}}$$

(b)

$$dy = -3dx$$

$$\begin{aligned} \frac{-W}{Q} &= \int \mathbf{E} \bullet d\mathbf{l} = \int_0^2 (3x^2 + 5 - 3x)dx + x(-3)dx \\ &= \int_0^2 (3x^2 - 6x + 5)dx = 8 - 12 + 10 = 6 \end{aligned}$$

$$W = \underline{\underline{12 \text{ mJ}}}$$

P.E. 4.13

(a)

$$(0,0,10) \longrightarrow (r=10, \theta=0, \phi=0)$$

$$V = \frac{100 \cos 0}{4\pi \epsilon_0 (10^2)} (10^{-12}) = \frac{10^{-12}}{4\pi (\frac{10^{-9}}{36\pi})} = 9 \text{ mV}$$

$$\begin{aligned} \mathbf{E} &= \frac{100(10^{-12})}{4\pi(\frac{10^{-9}}{36\pi})10^3} [2 \cos 0 \mathbf{a}_r + \sin 0 \mathbf{a}_\theta] \\ &= \underline{\underline{1.8 \mathbf{a}_r \text{ mV/m}}} \end{aligned}$$

(b)

$$\text{At } (1, \frac{\pi}{3}, \frac{\pi}{2}),$$

$$V = \frac{100 \cos \frac{\pi}{3} (10^{-12})}{4\pi (\frac{10^{-9}}{36\pi}) (1)^2} = \underline{\underline{0.45 \text{ V}}}$$

$$\begin{aligned} \mathbf{E} &= \frac{100(10^{-12})}{4\pi(\frac{10^{-9}}{36\pi})(1)^2} (2 \cos \frac{\pi}{3} \mathbf{a}_r + \sin \frac{\pi}{3} \mathbf{a}_\theta) \\ &= \underline{\underline{0.9 \mathbf{a}_r + 0.7794 \mathbf{a}_\theta \text{ V/m}}} \end{aligned}$$

P.E. 4.14After Q_1 , $W_1 = 0$

$$\begin{aligned} \text{After } Q_2, \quad W_2 &= Q_2 V_{21} = \frac{Q_2 Q_1}{4\pi \epsilon_0 |(1,0,0) - (0,0,0)|} \\ &= \frac{I(-2)(10^{-18})}{4\pi (10^{-9}) \frac{I}{36\pi}} = \underline{\underline{-18 \text{ nJ}}} \end{aligned}$$

After Q_3 ,

$$\begin{aligned} W_3 &= Q_3 (V_{31} + V_{32}) + Q_2 V_{21} \\ &= 3(9)(10^{-9}) \left\{ \frac{I}{|(0,0,-I) - (0,0,0)|} + \frac{-2}{|(0,0,-I) - (1,0,0)|} \right\} - 18 \text{ nJ} \\ &= 27(I - \frac{2}{\sqrt{2}}) - 18 \\ &= \underline{\underline{-29.18 \text{ nJ}}} \end{aligned}$$

After Q_4 ,

$$\begin{aligned}
 W_4 &= Q_4(V_{41} + V_{42} + V_{43}) + Q_3(V_{31} + V_{32}) + Q_2 V_{21} \\
 &= -4(9)(10^{-9}) \left\{ \frac{1}{|(0,0,1) - (0,0,0)|} + \frac{-2}{|(0,0,1) - (1,0,0)|} + \frac{3}{|(0,0,1) - (0,0,-1)|} \right\} + W_3 \\
 &= -36(1 - \frac{2}{\sqrt{2}} + \frac{3}{2}) + W_3 \\
 &= -39.09 - 29.18 \text{ nJ} = \underline{\underline{-68.27 \text{ nJ}}}
 \end{aligned}$$

P.E. 4.15

$$\mathbf{E} = -\nabla V = -(y+1)\mathbf{a}_x + (1-x)\mathbf{a}_y - 2\mathbf{a}_z$$

At (1,2,3), $\mathbf{E} = \underline{\underline{-3\mathbf{a}_x - 2\mathbf{a}_z \text{ V/m}}}$

$$\begin{aligned}
 W &= \frac{1}{2} \epsilon_0 \int \mathbf{E} \bullet \mathbf{E} dV = \frac{1}{2} \epsilon_0 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 (x^2 + y^2 - 2x + 2y + 6) dx dy dz \\
 &= \frac{1}{2} \epsilon_0 \left[\int_{-1}^1 x^2 dx \iint dy dz + \int_{-1}^1 y^2 dy \iint dx dz - 2 \int_{-1}^1 x dx \iint dy dz + 2 \int_{-1}^1 y dy \iint dx dz + 6(2)(2)(2) \right] \\
 &= \frac{1}{2} \epsilon_0 \left[2 \frac{x^3}{3} \Big|_{-1}^1 (2)(2) + 0 + 0 + 6(8) \right] = \frac{80\epsilon_0}{3} \\
 &= \underline{\underline{0.2358 \text{ nJ}}}
 \end{aligned}$$

P.E. 4.16 (a)

$$\begin{aligned}
 \mathbf{E}(5,0,6) &= \frac{Q_1}{4\pi\epsilon_0} \frac{[(5,0,6) - (4,0,-3)]}{|(5,0,6) - (4,0,-3)|^3} + \frac{Q_2}{4\pi\epsilon_0} \frac{[(5,0,6) - (2,0,1)]}{|(5,0,6) - (2,0,1)|^3} \\
 &= \frac{Q_1}{4\pi\epsilon_0} \frac{(1,0,9)}{(\sqrt{82})^3} + \frac{Q_2}{4\pi\epsilon_0} \frac{(3,0,5)}{(34)^{3/2}}
 \end{aligned}$$

If $E_z = 0$, then

$$\begin{aligned}
 \frac{9Q_1}{4\pi\epsilon_0} \frac{1}{(82)^{3/2}} + \frac{5Q_2}{4\pi\epsilon_0} \frac{1}{(34)^{3/2}} &= 0 \\
 Q_1 &= -\frac{5}{9} Q_2 \left(\frac{82}{34} \right)^{3/2} = -\frac{5}{9} 4 \left(\frac{82}{34} \right)^{3/2} \text{ nC} \\
 &= \underline{\underline{-8.3232 \text{ nC}}}
 \end{aligned}$$

(b)

$$\mathbf{F}(5, 0, 6) = q\mathbf{E}(5, 0, 6)$$

If $F_x = 0$, then

$$\frac{qQ_1}{4\pi\epsilon_0(82)^{3/2}} + \frac{3qQ_2}{4\pi\epsilon_0(34)^{3/2}} = 0$$

$$Q_1 = -3Q_2 \left(\frac{82}{34}\right)^{3/2} = -12 \left(\frac{82}{34}\right)^{3/2} nC$$

$$Q_1 = \underline{\underline{-44.945 nC}}$$

P.E. 4.17

$$\mathbf{F}_e = \frac{e^2}{4\pi\epsilon_0 r^2} \hat{\mathbf{e}}_r$$

$$\begin{aligned} \frac{F_e}{F_g} &= \frac{e^2}{4\pi\epsilon_0} \frac{1}{Gm^2} = \frac{1}{4\pi \times 10^{-9} / 36\pi \times 6.67 \times 10^{-11}} \left(\frac{1.6 \times 10^{-19}}{9.1 \times 10^{-31}} \right)^2 \\ &= \underline{\underline{4.17 \times 10^{42}}} \end{aligned}$$

P.E. 4.18

(a)

$$\mathbf{E} = \frac{QR_1}{4\pi\epsilon_0 R_1^3} - \frac{QR_2}{4\pi\epsilon_0 R_2^3}$$

A point on the x-axis is $(x, 0, 0)$.

$$\mathbf{R}_1 = (x, 0, 0) - (0, 0, d) = (x, 0, -d)$$

$$R_1^3 = (x^2 + d^2)^{3/2}$$

$$\mathbf{R}_2 = (x, 0, 0) - (0, 0, -d) = (x, 0, d)$$

$$R_2^3 = (x^2 + d^2)^{3/2}$$

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0(x^2 + d^2)^{3/2}} [(x, 0, -d) - (x, 0, d)] = \underline{\underline{\frac{-2Qda_z}{4\pi\epsilon_0(x^2 + d^2)^{3/2}}}}$$

(b) A point along the z-axis is $(0, 0, z)$.

$$\mathbf{R}_1 = (0, 0, z) - (0, 0, d) = (z - d)\mathbf{a}_z$$

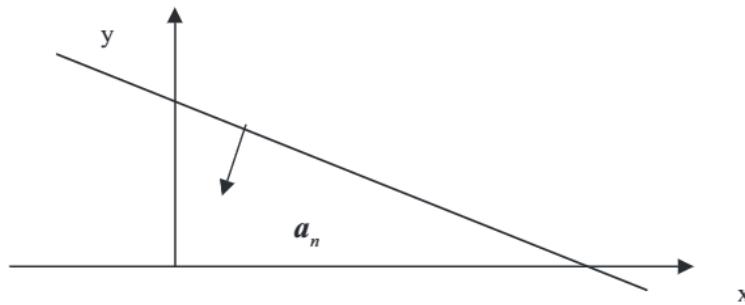
$$R_1^3 = (z - d)^3$$

$$\mathbf{R}_2 = (0, 0, z) - (0, 0, -d) = (z + d)\mathbf{a}_z$$

$$R_2^3 = (z + d)^3$$

$$\begin{aligned} \mathbf{E} &= \frac{Q(z-d)\mathbf{a}_z}{4\pi\epsilon_0(z-d)^3} - \frac{Q(z+d)\mathbf{a}_z}{4\pi\epsilon_0(z+d)^3} = \frac{Q\mathbf{a}_z}{4\pi\epsilon_0} \left[\frac{1}{(z-d)^2} - \frac{1}{(z+d)^2} \right] \\ &= \underline{\underline{\frac{Qdz\mathbf{a}_z}{\pi\epsilon_0(z^2-d^2)^2}}} \end{aligned}$$

P.E. 4.19



Let $f(x, y) = x + 2y - 5$; $\nabla f = \mathbf{a}_x + 2\mathbf{a}_y$

$$\mathbf{a}_n = \pm \frac{\nabla f}{|\nabla f|} = \pm \frac{(\mathbf{a}_x + 2\mathbf{a}_y)}{\sqrt{5}}$$

Since point $(-1, 0, 1)$ is below the plane,

$$\mathbf{a}_n = -\frac{(\mathbf{a}_x + 2\mathbf{a}_y)}{\sqrt{5}}.$$

$$\mathbf{E} = \frac{\rho_s}{2\epsilon_0} \mathbf{a}_n = \frac{6(10^{-9})}{2(10^{-9}/36\pi)} \left(-\frac{(\mathbf{a}_x + 2\mathbf{a}_y)}{\sqrt{5}} \right)$$

$$= \underline{\underline{-151.7\mathbf{a}_x - 303.5\mathbf{a}_y \text{ V/m}}}$$

Prob. 4.1

$$\begin{aligned}\mathbf{F}_{Q_1} &= \frac{Q_1 Q_2 (\mathbf{r}_{Q_1} - \mathbf{r}_{Q_2})}{4\pi\epsilon_0 |\mathbf{r}_{Q_1} - \mathbf{r}_{Q_2}|^3} = \frac{-20(10^{-12})[(3, 2, 1) - (-4, 0, 6)]}{4\pi \frac{10^{-9}}{36\pi} |(3, 2, 1) - (-4, 0, 6)|^3} = -180 \frac{(7, 2, -5)}{688.88} \times 10^{-3} \\ &= \underline{\underline{-1.8291 \mathbf{a}_x - 0.5226 \mathbf{a}_y + 1.3065 \mathbf{a}_z \text{ mN}}}\end{aligned}$$

Prob. 4.2

$$\begin{aligned}(a) \quad \mathbf{E} &= \sum_{k=1}^2 \frac{Q(\mathbf{r} - \mathbf{r}'_k)}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'_k|^3} = \frac{Q[(0, 0, 0) - (a, 0, 0)]}{4\pi\epsilon_0 |(0, 0, 0) - (a, 0, 0)|^3} - \frac{Q[(0, 0, 0) - (-a, 0, 0)]}{4\pi\epsilon_0 |(0, 0, 0) - (-a, 0, 0)|^3} \\ &= \frac{Q(-a, 0, 0)}{4\pi\epsilon_0 a^3} - \frac{Q(a, 0, 0)}{4\pi\epsilon_0 a^3} = \underline{\underline{\frac{Q}{2\pi\epsilon_0 a^2} \mathbf{a}_x}}\end{aligned}$$

$$\begin{aligned}(b) \quad \mathbf{E} &= \frac{Q[(0, a, 0) - (a, 0, 0)]}{4\pi\epsilon_0 |(0, a, 0) - (a, 0, 0)|^3} - \frac{Q[(0, a, 0) - (-a, 0, 0)]}{4\pi\epsilon_0 |(0, a, 0) - (-a, 0, 0)|^3} \\ &= \frac{Q(-a, a, 0)}{4\pi\epsilon_0 (2a^2)^{3/2}} - \frac{Q(a, a, 0)}{4\pi\epsilon_0 (2a^2)^{3/2}} = \underline{\underline{\frac{-Q}{4\sqrt{2}\pi\epsilon_0 a^2} \mathbf{a}_x}}\end{aligned}$$

(c)

$$\begin{aligned}\mathbf{E} &= \frac{Q[(a, 0, a) - (a, 0, 0)]}{4\pi\epsilon_0 |(a, 0, a) - (a, 0, 0)|^3} - \frac{Q[(a, 0, a) - (-a, 0, 0)]}{4\pi\epsilon_0 |(a, 0, a) - (-a, 0, 0)|^3} \\ &= \frac{Q(0, 0, a)}{4\pi\epsilon_0 a^3} - \frac{Q(2a, 0, a)}{4\pi\epsilon_0 (5a^2)^{3/2}} = \underline{\underline{\frac{-Q}{10\sqrt{5}\pi\epsilon_0 a^2} \mathbf{a}_x + \frac{Q}{4\pi\epsilon_0 a^2} \left[1 - \frac{1}{5\sqrt{5}}\right] \mathbf{a}_z}}\end{aligned}$$

Prob. 4.3

$$F = qE = mg \quad \longrightarrow \quad E = \frac{mg}{q} = \frac{2 \times 9.8}{4 \times 10^{-3}} = \underline{\underline{4.9 \text{ kV/m}}}$$

Prob. 4.4

$$(a) \quad Q = \int \rho_L dl = \int_0^5 12x^2 dx = 4x^3 \Big|_0^5 mC = \underline{\underline{0.5 C}}$$

(b)

$$Q = \int \rho_s dS = \int_{z=0}^4 \int_{\phi=0}^{2\pi} \rho z^2 \rho d\phi dz \Big|_{\rho=3} = 9(2\pi) \frac{z^3}{3} \Big|_0^4 nC$$

$$= \underline{\underline{1.206 \mu C}}$$

(c)

$$Q = \int \rho_v dV = \iiint \frac{10}{r \sin \theta} r^2 \sin \theta d\theta d\phi dr$$

$$= 10 \int_0^{2\pi} d\phi \int_0^\pi d\theta \int_0^4 r dr = 10(2\pi)(\pi) \frac{4^2}{2}$$

$$= \underline{\underline{1579.1 C}}$$

Prob. 4.5

$$Q = \int \rho_v dV = \int_0^a \int_0^a \int_0^a \frac{\rho_o x}{a} dx dy dz = (a)(a)\rho_o \left(\frac{x^2}{2a} \Big|_0^a \right) = \underline{\underline{\frac{a^3 \rho_o}{2}}}$$

Prob. 4.6

$$Q = \int \rho_v dV = \int_{\rho=0}^2 \int_{z=0}^1 \int_{\phi=\pi/6}^{\pi/2} 5\rho^2 z \rho d\phi d\rho dz \text{ mC}$$

$$= 5 \frac{\rho^4}{4} \Big|_0^2 \frac{z^2}{2} \Big|_0^1 \phi \Big|_{\pi/6}^{\pi/2} = \frac{5}{8}(16)(1)(\pi/2 - \pi/6) = \frac{10\pi}{3}$$

$$Q = \underline{\underline{10.472 \text{ mC}}}$$

Prob. 4.7

$$Q = \int \rho_s dS = \int 6xy dx dy$$

$$= \int_{x=0}^2 \int_{y=0}^x 6xy dx dy + \int_{x=2}^4 \int_{y=2}^{-x+4} 6xy dx dy$$

$$= 6 \int_{x=0}^2 x \frac{y^2}{2} \Big|_0^x dx + \int_{x=2}^4 6x \frac{y^2}{2} \Big|_0^{-x+4} dx$$

$$= 6 \int_{x=0}^2 x \left(\frac{x^2}{2} - 0 \right) dx + 3 \int_{x=2}^4 x \left[(4-x)^2 - 0 \right] dx$$

$$= 6 \int_0^2 \frac{x^3}{2} dx + 3 \int_2^4 (16x - 8x^2 + x^3) dx$$

$$\begin{aligned}
 &= 3 \frac{x^4}{4} \Big|_0^2 + 6(8x^2 - \frac{8x^3}{3} + \frac{x^4}{4}) \Big|_2^4 \\
 &= 12 + 3(128 - 32 - \frac{512}{3} + \frac{64}{3} + 64 - 4) \\
 &= 12 + 3(96 - \frac{448}{3} + 60) \\
 Q &\underline{\underline{= 32 \text{ C}}}
 \end{aligned}$$

Prob. 4.8

$$\begin{aligned}
 Q &= \int_v \rho_v dv \\
 &= \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 6x^2y^2 dx dy dz \\
 &= 6 \int_{-1}^1 dz \int_{-1}^1 x^2 dx \int_{-1}^1 y^2 dy \\
 &= 6(2) \frac{x^3}{3} \Big|_{-1}^1 \frac{y^3}{3} \Big|_{-1}^1 \\
 &= \frac{12}{9}(1 - (-1))(1 - (-1)) = \frac{48}{9} \text{ nC} \\
 &\underline{\underline{= 5.33 \text{ nC}}}
 \end{aligned}$$

Prob. 4.9

$$\begin{aligned}
 Q &= \int_v \rho_v dv = \iiint 4\rho^2 z \cos\phi d\rho d\phi dz \text{ nC} \\
 &= 4 \int_0^2 \rho^3 d\rho \int_0^1 zdz \int_0^{\pi/4} \cos\phi d\phi = \rho^4 \Big|_0^2 \frac{z^2}{2} \Big|_0^1 (\sin\phi) \Big|_0^{\pi/4} \\
 &= (16)(0.5)(\sin\pi/4) = \underline{\underline{5.657 \text{ nC}}}
 \end{aligned}$$

Prob. 4.10For $2 < r < 4\text{cm}$

$$\begin{aligned} Q &= \int_v \rho_v dv = \rho_v \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=2\text{cm}}^{4\text{cm}} 5r^2 \sin \theta dr d\theta d\phi \\ &= 5(4\pi)\rho_v \int_{2\text{cm}}^{4\text{cm}} r^2 dr \\ &= 20\pi\rho_v \left. \frac{r^3}{3} \right|_{2\text{cm}}^{4\text{cm}} \\ &= 20\pi \times 5 \times \frac{1}{3} (4^3 - 2^3) \times 10^{-6} \text{ mC} \end{aligned}$$

$$\begin{aligned} &= \frac{100\pi}{3} (64 - 8) \text{nC} \\ &= 5864.3 \text{ nC} \end{aligned}$$

$$Q = \underline{\underline{5.864 \mu\text{C}}}$$

Prob. 4.11

$$E = \int_L \frac{\rho_L dl}{4\pi\epsilon_o R^2} \mathbf{a}_R$$

$$\mathbf{R} = -a\mathbf{a}_\rho + h\mathbf{a}_z, R = |\mathbf{R}| = \sqrt{a^2 + h^2}, dl = ad\phi$$

$$E = \frac{\rho_L}{4\pi\epsilon_o} \int \frac{(-a\mathbf{a}_\rho + h\mathbf{a}_z)}{(a^2 + h^2)^{3/2}} ad\phi$$

Due to symmetry, the ρ -component cancels

$$E = \frac{\rho_L}{4\pi\epsilon_o} \int_0^{2\pi} \frac{h a a_z d\phi}{(a^2 + h^2)^{3/2}} = \frac{\rho_L h a a_z}{4\pi\epsilon_o (a^2 + h^2)^{3/2}} (2\pi)$$

out.

$$F = QE = \frac{4 \times 10^{-3} \times 12 \times 10^{-6} \times 4 \times 3 a_z}{4\pi \times \frac{10^{-9}}{36\pi} \times 5^3} (2\pi) = \underline{\underline{260.58 a_z \text{ N}}}$$

Prob. 4.12

(a) At P(5,-1,4),

$$\begin{aligned} \mathbf{E} &= \sum_{k=1}^3 \frac{\rho_{sk}}{2\epsilon_0} \mathbf{a}_{nk} = \frac{10 \times 10^{-6}}{2 \times \frac{10^{-9}}{36\pi}} (\mathbf{a}_x) + \frac{-20 \times 10^{-6}}{2 \times \frac{10^{-9}}{36\pi}} (\mathbf{a}_y) + \frac{30 \times 10^{-6}}{2 \times \frac{10^{-9}}{36\pi}} (-\mathbf{a}_z) \\ &= 36\pi (5, -10, -15) \times 10^3 = \underline{\underline{565.5\mathbf{a}_x - 1131\mathbf{a}_y - 1696.5\mathbf{a}_z \text{ kV/m}}} \end{aligned}$$

(b) At R(0,-2,1)

$$\mathbf{E} = 36\pi [5(-\mathbf{a}_x) - 10(\mathbf{a}_y) + 15(-\mathbf{a}_z)] \times 10^3 = \underline{\underline{-565.5\mathbf{a}_x - 1131\mathbf{a}_y - 1696.5\mathbf{a}_z \text{ kV/m}}}$$

(c) At Q(3,-4,10),

$$\mathbf{E} = 36\pi [5\mathbf{a}_x - 10(-\mathbf{a}_y) + 15\mathbf{a}_z] \times 10^3 = \underline{\underline{565.5\mathbf{a}_x + 1131\mathbf{a}_y + 1696.5\mathbf{a}_z \text{ kV/m}}}$$

Prob. 4.13

We apply $\mathbf{E} = \frac{\rho_s}{2\epsilon_0} \mathbf{a}_n$

For $z < 0$,

$$\begin{aligned} \mathbf{E} &= \frac{1}{2\epsilon_0} [-10(-\mathbf{a}_z) + 5(-\mathbf{a}_z) + 20(-\mathbf{a}_z)] \times 10^{-9} = \frac{1}{2 \times \frac{10^{-9}}{36\pi}} (-10 + 5 + 20)(-\mathbf{a}_z) 10^{-9} \\ &= 18\pi \times 15(-\mathbf{a}_z) = -848.23\mathbf{a}_z \text{ V/m} \end{aligned}$$

For $0 < z < 1$,

$$\begin{aligned} \mathbf{E} &= \frac{1}{2\epsilon_0} [-10(\mathbf{a}_z) + 5(-\mathbf{a}_z) + 20(-\mathbf{a}_z)] \times 10^{-9} = \frac{1}{2 \times \frac{10^{-9}}{36\pi}} (-10 - 5 - 20)(\mathbf{a}_z) 10^{-9} \\ &= -18\pi \times 35(\mathbf{a}_z) = -1.979\mathbf{a}_z \text{ kV/m} \end{aligned}$$

For $1 < z < 2$,

$$\begin{aligned} \mathbf{E} &= \frac{1}{2\epsilon_0} [-10(\mathbf{a}_z) + 5(\mathbf{a}_z) + 20(-\mathbf{a}_z)] \times 10^{-9} = \frac{1}{2 \times \frac{10^{-9}}{36\pi}} (-10 + 5 - 20)(\mathbf{a}_z) 10^{-9} = -18\pi(25)\mathbf{a}_z \\ &= -1.4137\mathbf{a}_z \text{ kV/m} \end{aligned}$$

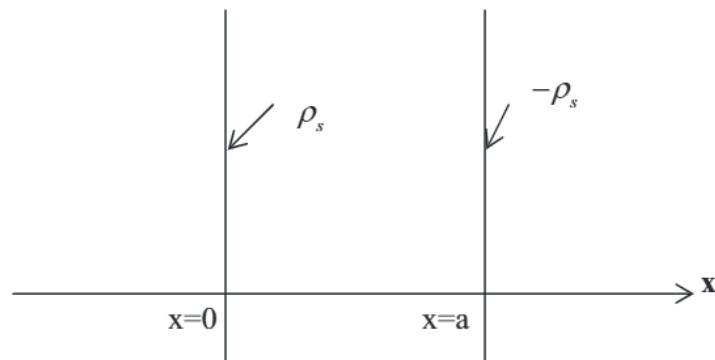
For $z > 2$,

$$\begin{aligned} \mathbf{E} &= \frac{1}{2\epsilon_0} [-10(\mathbf{a}_z) + 5(\mathbf{a}_z) + 20\mathbf{a}_z] \times 10^{-9} = 18\pi(-10 + 5 + 20)(\mathbf{a}_z) \\ &= 848.23\mathbf{a}_z \text{ V/m} \end{aligned}$$

Thus,

$$\mathbf{E} = \begin{cases} -848.23\mathbf{a}_z \text{ V/m, } z < 0 \\ -1.979\mathbf{a}_z \text{ kV/m, } 0 < z < 1 \\ -1.4137\mathbf{a}_z \text{ kV/m, } 1 < z < 2 \\ 848.23\mathbf{a}_z \text{ V/m, } z > 2 \end{cases}$$

Prob. 4.14



(a) For $x < 0$,

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \frac{\rho_s}{2\epsilon_0}(-\mathbf{a}_x) + \frac{(-\rho_s)}{2\epsilon_0}(-\mathbf{a}_x) = \underline{\underline{0}}$$

(b) For $0 < x < a$,

$$\mathbf{E} = \frac{\rho_s}{2\epsilon_0}\mathbf{a}_x + \frac{(-\rho_s)}{2\epsilon_0}(-\mathbf{a}_x) = \frac{\rho_s}{\epsilon_0}\mathbf{a}_x = \underline{\underline{0}}$$

(c) For $x > a$,

$$\mathbf{E} = \frac{\rho_s}{2\epsilon_0}\mathbf{a}_x + \frac{(-\rho_s)}{2\epsilon_0}(\mathbf{a}_x) = \underline{\underline{0}}$$

Prob. 4.15

$$\mathbf{E} = \int_S \frac{\rho_s dS}{4\pi\epsilon_0 R^3} \mathbf{R}, \quad \mathbf{R} = \rho(-\mathbf{a}_\rho) + h\mathbf{a}_z, \quad dS = \rho d\phi d\rho$$

$$\mathbf{E} = \frac{\rho_s}{4\pi\epsilon_0} \int_S \frac{\rho d\phi d\rho}{(\rho^2 + h^2)^{3/2}} (-\rho\mathbf{a}_\rho + h\mathbf{a}_z)$$

Due to symmetry, the ρ -component vanishes.

$$\mathbf{E} = \frac{\rho_s h \mathbf{a}_z}{4\pi\epsilon_0} \int_a^b \rho (\rho^2 + h^2)^{-3/2} d\rho \int_0^{2\pi} d\phi$$

$$\text{Let } u = \rho^2 + h^2, du = 2\rho d\rho$$

$$\mathbf{E} = \frac{\rho_s h \mathbf{a}_z}{4\pi\epsilon_0} (2\pi) \int \frac{1}{2} u^{-3/2} du = \frac{\rho_s h \mathbf{a}_z}{2\epsilon_0} \frac{1/2 u^{-1/2}}{-1/2} = \frac{\rho_s h \mathbf{a}_z}{2\epsilon_0} \left(-\frac{1}{\sqrt{\rho^2 + h^2}} \right|_a^b$$

$$\mathbf{E} = \frac{\rho_s h}{4\pi\epsilon_0} \left[\frac{1}{\sqrt{a^2 + h^2}} - \frac{1}{\sqrt{b^2 + h^2}} \right] \mathbf{a}_z$$

Prob. 4.16

Let Q_1 be located at the origin. At the spherical surface of radius r ,

$$Q_1 = \oint \mathbf{D} \cdot d\mathbf{S} = \epsilon E_r (4\pi r^2)$$

Or

$$\mathbf{E} = \frac{Q_1}{4\pi\epsilon r^2} \mathbf{a}_r \quad \text{by Gauss's law}$$

If a second charge Q_2 is placed on the spherical surface, Q_2 experiences a force

$$\mathbf{F} = Q_2 \mathbf{E} = \frac{Q_1 Q_2}{4\pi\epsilon r^2} \mathbf{a}_r$$

which is Coulomb's law.

Prob. 4.17

$$\text{For a point charge, } \mathbf{D} = \frac{Q}{4\pi R^3} \mathbf{R}$$

For the given three point charges,

$$\mathbf{D} = \frac{1}{4\pi} \left[\frac{Q\mathbf{R}_1}{R_1^3} + \frac{Q\mathbf{R}_2}{R_2^3} - \frac{2Q\mathbf{R}_3}{R_3^3} \right]$$

$$\mathbf{R}_1 = (0, 0) - (-1, 0) = (1, 0), R_1 = 1$$

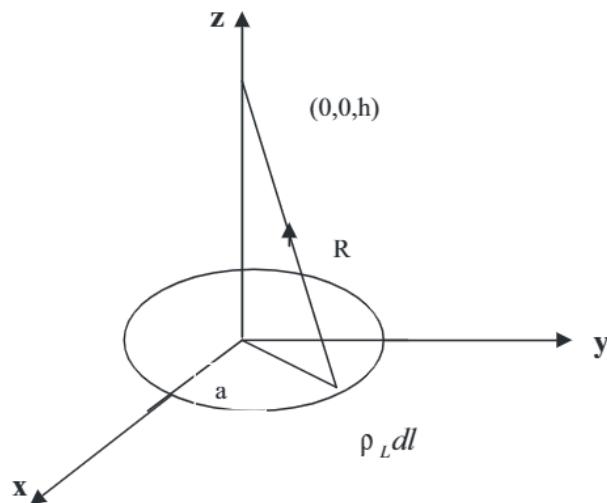
$$\mathbf{R}_2 = (0, 0) - (1, 0) = (-1, 0), R_2 = 1$$

$$\mathbf{R}_3 = (0, 0) - (0, 1) = (0, -1), R_3 = 1$$

$$\mathbf{D} = \frac{Q}{4\pi} [(1, 0) + (-1, 0) - 2(0, -1)] = \frac{Q}{4\pi} (0, 2) = \underline{\underline{\frac{Q}{2\pi} \mathbf{a}_y}}$$

Prob. 4.18

(a) Assume for now that the ring is placed on the z=0 plane.



$$\mathbf{D} = \int \frac{\rho_L dl \mathbf{R}}{4\pi R^3}, \mathbf{R} = -a \mathbf{a}_\rho + h \mathbf{a}_z$$

$$\mathbf{D} = \frac{\rho_L}{4\pi} \int_{\phi=0}^{\phi=2\pi} \frac{a d\phi (-a \mathbf{a}_\rho + h \mathbf{a}_z)}{(a^2 + h^2)^{3/2}}$$

Due to symmetry, the ρ component vanishes.

$$\mathbf{D} = \frac{\rho_L a (2\pi h) \mathbf{a}_z}{4\pi (a^2 + h^2)^{3/2}} = \frac{\rho_L a h \mathbf{a}_z}{2(a^2 + h^2)^{3/2}}$$

$$a = 2, h = 3, \rho_L = 5 \mu\text{C/m}$$

Since the ring is actually placed in $x = 0$, \mathbf{a}_z becomes \mathbf{a}_x .

$$\mathbf{D} = \frac{(6)(5) \mathbf{a}_x}{2(4+9)^{3/2}} = \underline{\underline{0.32 \mathbf{a}_x \mu\text{C/m}^2}}$$

(b)

$$\begin{aligned}\mathbf{D}_Q &= \frac{Q}{4\pi} \frac{[(3,0,0)-(0,-3,0)]}{|(3,0,0)-(0,-3,0)|^3} + \frac{Q}{4\pi} \frac{[(3,0,0)-(0,3,0)]}{|(3,0,0)-(0,3,0)|^3} \\ &= \frac{Q(3,3,0)}{4\pi(18)^{3/2}} + \frac{Q(3,-3,0)}{4\pi(18)^{3/2}} = \frac{6Q(1,0,0)}{4\pi(18)^{3/2}}\end{aligned}$$

$$\mathbf{D} = \mathbf{D}_R + \mathbf{D}_Q = 0$$

$$0.32(10^{-6}) + \frac{6Q}{4\pi(18)^{3/2}} = 0$$

$$\therefore Q = -0.32(4\pi)(18^{3/2})10^{-6} \frac{1}{6} = \underline{\underline{\underline{-51.182\mu C}}}$$

Prob. 4.19

$$(a) \rho_v = \nabla \cdot \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \underline{\underline{\underline{8y \text{ C/m}^2}}}$$

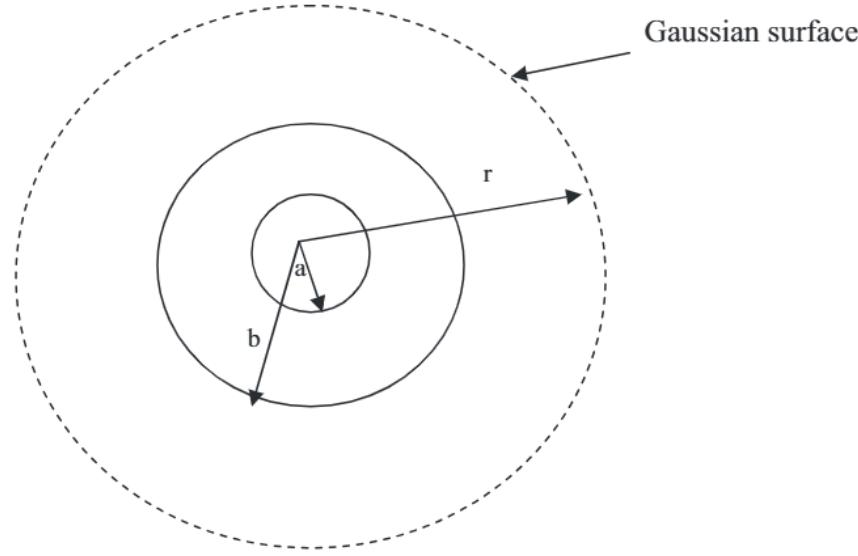
$$\begin{aligned}(b) \rho_v &= \nabla \cdot \mathbf{D} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho D_\rho) + \frac{1}{\rho} \frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z} \\ &= 8 \sin \phi - 2 \sin \phi + 4z \\ &= \underline{\underline{\underline{6 \sin \phi + 4z \text{ C/m}^3}}}\end{aligned}$$

$$\begin{aligned}(c) \rho_v &= \nabla \cdot \mathbf{D} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 D_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (D_\theta \sin \theta) + 0 \\ &= -\frac{2}{r^4} \cos \theta + \frac{2 \cos \theta}{r^4} \\ &= \underline{\underline{\underline{0}}}\end{aligned}$$

Prob. 4.20

$$\begin{aligned}(a) \rho_v &= \nabla \cdot \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = 2(1+z^2) + 0 + 2x^2 \\ &= \underline{\underline{\underline{2(1+x^2+z^2) \text{ nC/m}^3}}}\end{aligned}$$

$$\begin{aligned}(b) \psi &= \int_S \mathbf{D} \cdot d\mathbf{S} = \int_{y=0}^3 \int_{x=0}^2 2x^2 z dx dy \Big|_{z=1} = 2(1) \int_0^2 x^2 dx \int_0^3 dy \\ &= 2 \frac{x^3}{3} \Big|_0^2 (3) = \underline{\underline{\underline{16 \text{ nC}}}}\end{aligned}$$

Prob. 4.21

Apply Gauss's law,

$$\oint \mathbf{D} \cdot d\mathbf{S} = Q_{enc}$$

$$D_r 4\pi r^2 = \rho_{s1} 4\pi a^2 + \rho_{s2} 4\pi b^2 = 8 \times 10^{-9} \times 4\pi(1)^2 + (-6 \times 10^{-3}) \times 4\pi(2)^2 = -0.3016$$

$$D_r = \frac{-0.3016}{4\pi r^2} = \frac{0.3016}{4\pi(3)^2} = -0.0027$$

$$\underline{\underline{\mathbf{D} = -2.7a_r \text{ mC/m}^2}}$$

Prob. 4.22

For $r < a$.

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q_{enc} = \int_V \rho_v dv$$

$$\begin{aligned} D_r (4\pi r^2) &= \iiint 5r^{1/2} r^2 \sin \theta d\theta d\phi dr = 5 \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi \int_0^r r^{5/2} dr \\ &= 5(2)(2\pi) \frac{r^{7/2}}{7/2} \Big|_0^r = \frac{40\pi r^{7/2}}{7} \end{aligned}$$

$$D_r = \epsilon_o E_r = \frac{\frac{40\pi r^{7/2}}{7}}{4\pi r^2} = \frac{10}{7} r^{3/2}$$

$$E_r = \frac{10}{7\epsilon_o} r^{3/2}, 0 < r < a$$

For $r > a$,

$$D_r(4\pi r^2) = \frac{40}{7}\pi a^{7/2}$$

$$D_r = \epsilon_0 E_r = \frac{\frac{40}{7}\pi a^{7/2}}{4\pi r^2}$$

$$E_r = \frac{10a^{7/2}}{7\epsilon_0 r^2}, r > a$$

Thus,

$$\underline{\underline{E}} = \begin{cases} \frac{10}{7\epsilon_0} r^{3/2} \underline{\underline{a}_r}, & 0 < r < a \\ \frac{10a^{7/2}}{7\epsilon_0 r^2} \underline{\underline{a}_r}, & r > a \end{cases}$$

Prob. 4.23

(a) $\rho_v = \nabla \cdot \underline{\underline{D}} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \underline{\underline{2y \text{ C/m}^3}}$

(b) $\Psi = \int \underline{\underline{D}} \cdot d\underline{S} = \iint x^2 dx dz \Big|_{y=1} = \int_0^1 x^2 dx \int_0^1 dz = \underline{\underline{\frac{1}{3} \text{ C}}}$

(c) $Q = \int_v \rho_v dv = \iiint 2y dx dy dz = 2 \int_0^1 dx \int_0^1 y dy \int_0^1 dz = \underline{\underline{1 \text{ C}}}$

Prob. 4.24

(a)

$$\rho_v = \nabla \cdot \underline{\underline{D}} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho D_\rho) + \frac{1}{\rho} \frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}$$

$$\rho_v = 4(z+1)\cos\phi - (z+1)\cos\phi + 0$$

$$\rho_v = \underline{\underline{3(z+1)\cos\phi \text{ } \mu\text{C/m}^3}}$$

(b)

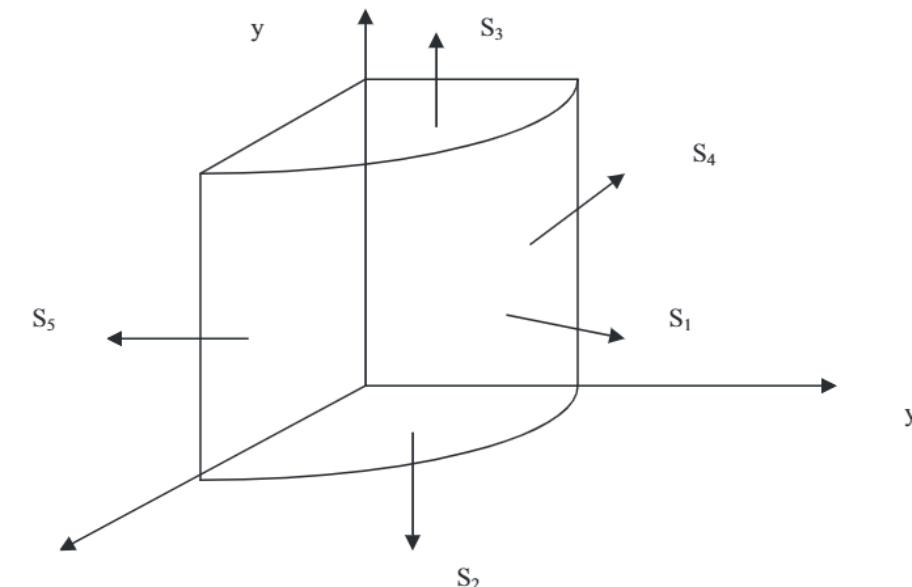
$$\begin{aligned} Q_{enc} &= \int \rho_v dv = \iiint 3(z+1)\cos\phi \rho d\phi d\rho dz \\ &= 3 \int_0^2 \rho d\rho \int_0^4 (z+1) \int_0^{\pi/2} \cos\phi d\phi = 3(2) \left(\frac{z^2}{2} + z\right) \Big|_0^4 (\sin\phi \Big|_0^{\pi/2}) \\ &= 6(8+4)(1-0) = \underline{\underline{72\mu\text{C}}} \end{aligned}$$

(c)

$$\text{Let } \psi = \psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5 = \oint \mathbf{D} \bullet d\mathbf{S}$$

where $\psi_1, \psi_2, \psi_3, \psi_4, \psi_5$ respectively correspond with surfaces S_1, S_2, S_3, S_4, S_5

(in the figure below) respectively.



$$\text{For } S_1, \rho = 2, d\mathbf{S} = \rho d\phi dz \mathbf{a}_\rho$$

$$\begin{aligned}\psi_1 &= \iint 2\rho(z+1) \cos \phi dS \Big|_{\rho=2} = 2(2)^2 \int_0^4 (z+1) dz \int_0^{\pi/2} \cos \phi d\phi \\ &= 8(12)(1) = 96\end{aligned}$$

$$\text{For } S_2, z = 0, d\mathbf{S} = \rho d\phi d\rho (-\mathbf{a}_z)$$

$$\begin{aligned}\psi_2 &= - \iint \rho^2 \cos \phi \rho d\phi d\rho = - \int_0^2 \rho^3 d\rho \int_0^{\pi/2} \cos \phi d\phi \\ &= - \frac{\rho^4}{4} \Big|_0^2 (1) = -4\end{aligned}$$

$$\text{For } S_3, z = 4, d\mathbf{S} = \rho d\phi d\rho \mathbf{a}_z, \psi_3 = +4$$

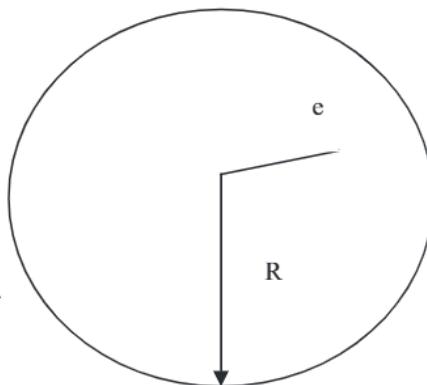
$$\text{For } S_4, \phi = \pi/2, d\mathbf{S} = d\rho dz \mathbf{a}_\phi$$

$$\begin{aligned}\psi_4 &= - \iint \rho(z+1) \sin \phi d\rho dz \Big|_{\phi=\pi/2} = (1) \int_0^2 \rho d\rho \int_0^4 (z+1) dz \\ &= - \frac{\rho^2}{2} \Big|_0^2 (12) = -(2)(12) = -24\end{aligned}$$

$$\text{For } S_5, \phi = 0, d\mathbf{S} = d\rho dz (-\mathbf{a}_\phi), \psi_5 = \iint \rho(z+1) \sin \phi d\rho dz \Big|_{\phi=0} = 0$$

$$\psi = 96 - 4 + 4 - 24 + 0 = \underline{\underline{72 \mu C}}$$

This is exactly the answer obtained in part (b).

Prob. 4.25

$$F = eE$$

$$\rho_0 = \frac{e}{4\pi \frac{R^3}{3}} = \frac{3e}{4\pi R^3}$$

$$\rho_V = \begin{cases} \rho_0, & 0 < r < R \\ 0, & \text{elsewhere} \end{cases}$$

$$\oint \mathbf{D} \bullet d\mathbf{S} = Q_{enc} = \int \rho_V dV = \frac{3e}{4\pi R^3} \frac{4\pi r^3}{3} = D_r(4\pi r^2)$$

$$E_r = \frac{3e r}{12\pi\epsilon_0 R^3}$$

$$F = eE = \underline{\underline{\frac{e^2 r}{4\pi\epsilon_0 R^3}}}$$

Prob. 4.26

(a)

$$\psi = Q_{enc} \quad \text{at } r = 2$$

$$Q_{enc} = \int \rho_V dV = \iiint \frac{10}{r^2} r^2 \sin\theta d\theta dr d\phi$$

$$= 10 \int_{r=1}^2 \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin\theta d\theta dr d\phi$$

$$= 10(1)(2\pi)(2) = (40\pi) \text{ mC}$$

$$\text{Thus, } \psi = \underline{\underline{125.7 \text{ mC}}}$$

At $r = 6$:

$$Q_{enc.} = 10 \int_{r=1}^4 dr \int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi} \sin\theta d\theta$$

$$= 10(3)(2\pi)(2) = 120\pi \text{ mC}$$

$$\psi = \underline{\underline{377 \text{ mC}}}$$

(b)

$$\psi = Q_{enc}$$

$$\text{But } \psi = \oint \mathbf{D} \cdot d\mathbf{S} = D_r \oint dS = D_r (4\pi r^2)$$

At $r = 1$,

$$Q_{enc} = 0 \longrightarrow \underline{\underline{\mathbf{D} = 0}}$$

At $r = 5$, $Q_{enc} = 120 \pi$

$$D_r = \frac{Q_{enc}}{4\pi r^2} = \frac{120\pi}{4\pi(5)^2} = 1.2$$

$$\mathbf{D} = \underline{\underline{1.2 \mathbf{a}_r \text{ mC/m}^2}}$$

Prob. 4.27

$$\rho_v = \frac{Q}{\text{volume}} = \frac{Q}{4\pi a^3 / 3} = \frac{3Q}{4\pi a^3}$$

$$\text{For } r < a, \oint \mathbf{D} \cdot d\mathbf{S} = Q_{enc} = \int \rho_v dv$$

$$D_r 4\pi r^2 = \frac{3Q}{4\pi a^3} \frac{4\pi r^3}{3} = \frac{Qr^3}{a^3} \longrightarrow D_r = \frac{Qr}{4\pi a^3}$$

$$\text{For } r > a, \oint \mathbf{D} \cdot d\mathbf{S} = Q$$

$$D_r 4\pi r^2 = Q \longrightarrow D_r = \frac{Q}{4\pi r^2}$$

Hence,

$$\mathbf{D} = \begin{cases} \frac{Qr}{4\pi a^3} \mathbf{a}_r, & r < a \\ \frac{Q}{4\pi r^2} \mathbf{a}_r, & r > a \end{cases}$$

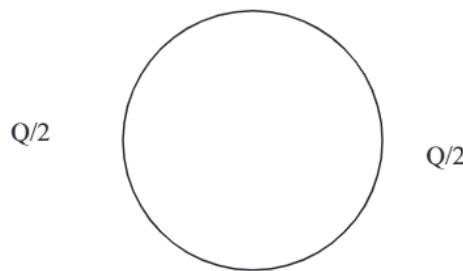
Prob. 4.28

$$\begin{aligned} V_p &= \frac{Q_1}{4\pi\epsilon_0 r_1} + \frac{Q_2}{4\pi\epsilon_0 r_2} = \frac{10^{-9}}{4\pi \times \frac{10^{-9}}{36\pi}} \left[\frac{2}{|(1,-2,3)-(1,0,3)|} - \frac{4}{|(1,-2,3)-(-2,1,5)|} \right] \\ &= 9 \left[\frac{2}{2} - \frac{4}{\sqrt{9+9+4}} \right] = \underline{\underline{1.325 \text{ V}}} \end{aligned}$$

Prob. 4.29

$$V = 4 \frac{Q}{4\pi\epsilon_0 r}, \quad r = \sqrt{a^2 + a^2 + h^2} = \sqrt{2^2 + 2^2 + 3^2} = \sqrt{17} \text{ cm}$$

$$V = \frac{4 \times 8 \times 10^{-9}}{4\pi \times \frac{10^{-9}}{36\pi} \times \sqrt{17} \times 10^{-2}} = \underline{\underline{6.985 \text{ kV}}}$$

Prob. 4.30 (a)

$$\begin{aligned} V &= \frac{2 \frac{Q}{2}}{4\pi\epsilon_0 r} = \frac{Q}{4\pi\epsilon_0 r} \\ &= \frac{60(10^{-6})}{4\pi \times \frac{10^{-9}}{36\pi} \times 4} = \underline{\underline{135 \text{ kV}}} \end{aligned}$$

(b)

$$V = \frac{3(\frac{Q}{3})}{4\pi\epsilon_0 r} = \underline{\underline{135 \text{ kV}}}$$

(c)

$$V = \int \frac{\rho_L dl}{4\pi\epsilon_0 r} = \frac{\frac{Q}{8\pi} 2\pi(4)}{4\pi\epsilon_0 r} \frac{Q}{4\pi\epsilon_0 r} = \underline{\underline{135 \text{ kV}}}$$

Prob. 4.31

(a)

$$V_p = \sum \frac{Q_k}{4\pi |\mathbf{r}_p - \mathbf{r}_k|}$$

$$4\kappa\epsilon_0 V_p = \frac{10^{-3}}{|(-1,1,2) - (0,0,4)|} + \frac{-2(10^{-3})}{|(-1,1,2) - (-2,5,1)|} + \frac{3(10^{-3})}{|(-1,1,2) - (3, -4,6)|}$$

$$4\pi\epsilon_0(10^3) V_p = \frac{1}{|(-1,1,-2)|} - \frac{2}{|(1,-4,1)|} + \frac{3}{|(-4,5,4)|} = \frac{1}{\sqrt{6}} - \frac{2}{\sqrt{18}} + \frac{3}{\sqrt{57}}$$

$$4\pi \frac{10^{-9}}{36\pi}(10^3) V_p = 0.3542$$

$$\therefore V_p = \underline{\underline{3.008 \times 10^6 \text{ V}}}$$

(b)

$$V_Q = \sum \frac{Q_k}{4\pi\epsilon_0 |\mathbf{r}_p - \mathbf{r}_k|}$$

$$4\pi\epsilon_0 V_Q = \frac{10^{-3}}{|(1,2,3) - (0,0,4)|} + \frac{-2(10^{-3})}{|(1,2,3) - (-2,5,1)|} + \frac{3(10^{-3})}{|(1,2,3) - (3,-4,6)|}$$

$$4\pi\epsilon_0(10^3) V_p = \frac{1}{|(1,2,-1)|} - \frac{2}{|(3,-3,2)|} + \frac{3}{|(-2,6,-3)|} = \frac{1}{\sqrt{6}} - \frac{2}{\sqrt{22}} + \frac{3}{7}$$

$$4\pi \frac{10^{-9}}{36\pi}(10^3) V_p = 0.410$$

$$V_Q = \underline{\underline{3.694 (10^6) \text{ V}}}$$

$$\therefore V_{PQ} = V_Q - V_p = 0.686(10^6) = \underline{\underline{686 \text{ kV}}}$$

Prob. 4.32

$$V = \int_S \frac{\rho_s dS}{4\pi\epsilon_0 r}; \quad \rho_s = \frac{1}{\rho}; \quad dS = \rho d\phi d\rho; \quad r = \sqrt{\rho^2 + h^2}$$

$$V = \frac{1}{4\pi\epsilon_0} \iint \frac{1}{(\rho^2 + h^2)^{1/2}} (\rho d\phi d\rho) = \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} d\phi \int_{\rho=0}^a \frac{d\rho}{(\rho^2 + h^2)}$$

$$= \frac{2\pi}{4\pi\epsilon_0} \ln(\rho + \sqrt{\rho^2 + h^2}) \Big|_{\rho=0}^a = \frac{1}{2\epsilon_0} [\ln(a + \sqrt{a^2 + h^2}) - \ln h]$$

$$= \frac{1}{2\epsilon_0} \ln \frac{a + \sqrt{a^2 + h^2}}{h}$$

Prob. 4.33

(a)

$$\begin{aligned} \mathbf{E} &= -\left(\frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z\right) \\ &= -2xy(z+3)\mathbf{a}_x - x^2(z+3)\mathbf{a}_y - x^2y\mathbf{a}_z \\ \text{At } (3, 4, -6), \quad x &= 3, \quad y = 4, \quad z = -6, \\ \mathbf{E} &= -2(3)(4)(-3)\mathbf{a}_x - 9(-3)\mathbf{a}_y - 9(4)\mathbf{a}_z \\ &= \underline{\underline{72\mathbf{a}_x + 27\mathbf{a}_y - 36\mathbf{a}_z \text{ V/m}}} \end{aligned}$$

(b)

$$\begin{aligned} \rho_v &= \nabla \bullet \mathbf{D} = \epsilon_0 \nabla \bullet \mathbf{E} = -\epsilon_0 (2y)(z+3) \\ \psi &= Q_{enc} = \int \rho_v dV = -2\epsilon_0 \iiint y(z+3) dx dy dz \\ &= -2\epsilon_0 \int_0^1 dx \int_0^1 y dy \int_0^1 (z+3) dz = -2\epsilon_0 (1)(1/2) \left(\frac{z^2}{2} + 3z\right) \Big|_0^1 \\ &= -\epsilon_0 \left(\frac{1}{2} + 3\right) = \frac{-7}{2} \left(\frac{10^{-9}}{36\pi}\right) \\ Q_{enc} &= \underline{\underline{-30.95 \text{ pC}}} \end{aligned}$$

Prob. 4.34

(a)

$$\begin{aligned} Q &= \int_v \rho_v dv = \iiint \rho_o \left(1 - \frac{r^2}{a^2}\right) r^2 \sin \theta d\theta d\phi dr \\ &= \rho_o \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta \int_0^a \left(r^2 - \frac{r^4}{a^2}\right) dr = \rho_o (2\pi)(2) \left(\frac{a^3}{3} - \frac{a^3}{5}\right) = \underline{\underline{\frac{8\pi}{15} a^3 \rho_o}} \end{aligned}$$

(b) Outside the nucleus, $r > a$,

$$\begin{aligned} \oint_S \mathbf{D} \bullet d\mathbf{S} &= Q_{enc} \quad \longrightarrow \quad \mathbf{E} = \frac{Q_{enc}}{4\pi\epsilon_o r^2} \mathbf{a}_r \\ \mathbf{E} &= \frac{8\pi a^3 \rho_o}{4\pi\epsilon_o r^2} \mathbf{a}_r = \underline{\underline{\frac{2a^3 \rho_o}{15\epsilon_o r^2} \mathbf{a}_r}} \\ V &= - \int \mathbf{E} \bullet d\mathbf{l} = - \int E_r dr = \frac{2a^3 \rho_o}{15\epsilon_o r} + C_1 \\ \text{Since } V(\infty) &= 0, C_1 = 0. \\ V &= \underline{\underline{\frac{2a^3 \rho_o}{15\epsilon_o r}}} \end{aligned}$$

(c) Inside the nucleus, $r < a$

$$Q_{enc} = 4\pi r \rho_o \left(\frac{r^3}{3} - \frac{r^5}{5a^2} \right)$$

$$\underline{\underline{\mathbf{E}}} = \frac{Q_{enc}}{4\pi \epsilon_o r^2} \mathbf{a}_r = \frac{\rho_o}{\epsilon_o} \left(\frac{r}{3} - \frac{r^3}{5a^2} \right) \mathbf{a}_r$$

$$V = - \int E_r dr = - \frac{\rho_o}{\epsilon_o} \left(\frac{r^2}{6} - \frac{r^4}{20a^2} \right) + C_2$$

$$V(r = a) = \frac{2a^2 \rho_o}{15\epsilon_o} = \frac{\rho_o}{\epsilon_o} \left(\frac{a^2}{20} - \frac{a^2}{6} \right) + C_2$$

$$C_2 = \frac{2a^2 \rho_o}{15\epsilon_o} + \frac{7a^2 \rho_o}{60\epsilon_o} = \frac{a^2 \rho_o}{4\epsilon_o}$$

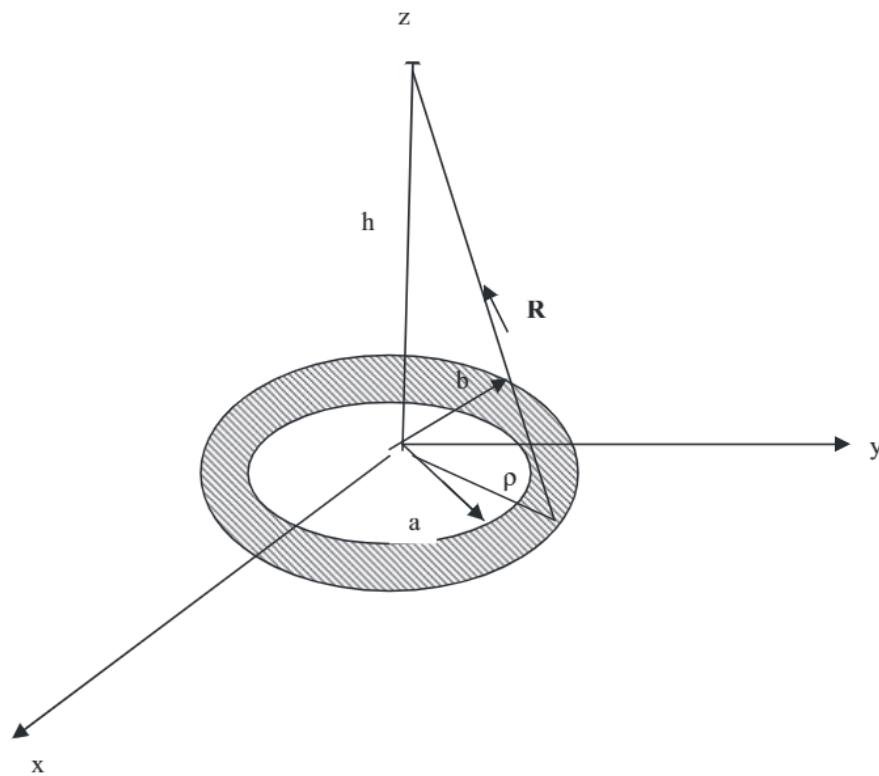
$$\underline{\underline{\mathbf{V}}} = \frac{\rho_o}{\epsilon_o} \left(\frac{r^4}{20a^2} - \frac{r^2}{6} \right) + \frac{a^2 \rho_o}{4\epsilon_o}$$

(d) E is maximum when

$$\frac{dE}{dr} = 0 = \frac{\rho_o}{\epsilon_o} \left(\frac{1}{3} - \frac{3r^2}{5a^2} \right) \longrightarrow 9r^2 = 5a^2$$

$$r = \frac{\sqrt{5}}{3} a = \underline{\underline{0.7454a}}$$

We are able to say maximum because $\frac{d^2E}{dr^2} = -\frac{6r}{5a^2} < 0$.

Prob. 4.35

$$\mathbf{D} = \int \frac{\rho_s dS \mathbf{R}}{4\pi R^3}, \quad \mathbf{R} = -\rho \mathbf{a}_\rho + h \mathbf{a}_z, \quad R = |\mathbf{R}| = \sqrt{\rho^2 + h^2}, \quad dS = \rho d\phi d\rho$$

$$\rho_s = \frac{Q}{S} = \frac{Q}{\pi(b^2 - a^2)}$$

$$\mathbf{D} = \frac{\rho_s}{4\pi} \int \frac{\rho d\phi d\rho (-\rho \mathbf{a}_\rho + h \mathbf{a}_z)}{(\rho^2 + h^2)^{3/2}}$$

Due to symmetry, the component along \mathbf{a}_ρ vanishes.

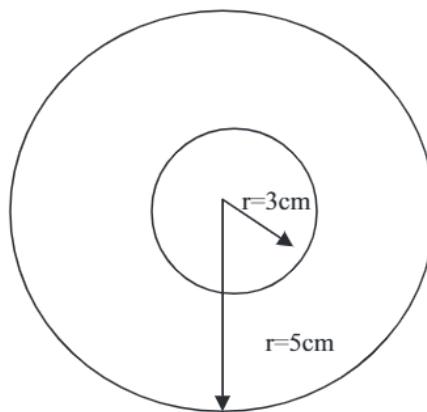
$$\begin{aligned} D_z &= \frac{\rho_s h}{4\pi} \int_{\rho=a}^b \int_{\phi=0}^{2\pi} \frac{\rho d\phi d\rho}{(\rho^2 + h^2)^{3/2}} = \frac{\rho_s h}{4\pi} (2\pi) \int_a^b (\rho^2 + h^2)^{-3/2} \rho d\rho \\ &= \frac{\rho_s h}{2} \left[\frac{-1}{\sqrt{\rho^2 + h^2}} \right]_a^b = \frac{\rho_s h}{2} \left[\frac{1}{\sqrt{a^2 + h^2}} - \frac{1}{\sqrt{b^2 + h^2}} \right] \\ \mathbf{D} &= \underline{\underline{\frac{Qh}{2\pi(b^2 - a^2)} \left[\frac{1}{\sqrt{a^2 + h^2}} - \frac{1}{\sqrt{b^2 + h^2}} \right] \mathbf{a}_z}} \end{aligned}$$

Prob. 4.36

$$\rho_v = \nabla \cdot D = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = 4 - 20y + 2z$$

At P(1,2,3), x=1, y= 2, z=3

$$\rho_v = 4 - 20(2) + 2(3) = \underline{\underline{-30 \text{ C/m}^3}}$$

Prob. 4.37

For $r < 3\text{cm}$, $Q_{enc} = 0 \longrightarrow D = 0$

For $3 < r < 5\text{cm}$,

$$\oint D \cdot dS = Q_{enc} = 10 \text{ nC}$$

$$D_r 4\pi r^2 = 10 \text{ nC} \longrightarrow D_r = \frac{10}{4\pi r^2} \text{ nC/m}^2$$

For $r > 5 \text{ cm}$,

$$\oint D \cdot dS = Q_{enc} = 10 - 5 = 5 \text{ nC} \longrightarrow D_r = \frac{5}{4\pi r^2} \text{ nC/m}^2$$

Thus,

$$D = \begin{cases} 0, & r < 3 \text{ cm} \\ \frac{10}{4\pi r^2} a_r \text{ nC/m}^2, & 3 < r < 5 \text{ cm} \\ \frac{5}{4\pi r^2} a_r \text{ nC/m}^2, & r > 5 \text{ cm} \end{cases}$$

Prob. 4.38

$$\rho_v = \nabla \cdot \mathbf{D} = \nabla \cdot \epsilon_o \mathbf{E} = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\frac{\epsilon_o E_o \rho^2}{a} \right) = \underline{\underline{\frac{2\epsilon_o E_o}{a}}}, 0 < \rho < a$$

Prob. 4.39

Let us choose the following path of two segments.
 $(2, 1, -1) \rightarrow (5, 1, -1) \rightarrow (5, 1, 2)$

$$W = -q \int \mathbf{E} \bullet d\mathbf{l}$$

$$\begin{aligned} -\frac{W}{q} &= \int \mathbf{E} \bullet d\mathbf{l} = \int_{x=2}^5 2xyz dx \Big|_{z=-1, y=1} + \int_{z=-1}^2 x^2 y dz \Big|_{x=5, y=1} \\ &= 2(1)(-1) \frac{x^2}{2} \Big|_2^5 + (5)^2 (1) z \Big|_{-1}^2 = -21 + 75 = 54 \end{aligned}$$

$$W = -54q = \underline{\underline{-108 \mu J}}$$

Prob. 4.40

(a)

From A to B , $d\mathbf{l} = rd\theta \mathbf{a}_\theta$,

$$W_{AB} = -Q \int_{\theta=30^\circ}^{90^\circ} 10r \cos \theta r d\theta \Big|_{r=5} = \underline{\underline{-1250 \text{ nJ}}}$$

(b)

From A to C , $d\mathbf{l} = dr \mathbf{a}_r$,

$$W_{AC} = -Q \int_{r=5}^{10} 20r \sin \theta dr \Big|_{\theta=30^\circ} = \underline{\underline{-3750 \text{ nJ}}}$$

(c)

From A to D , $d\mathbf{l} = r \sin \theta d\phi \mathbf{a}_\phi$,

$$W_{AD} = -Q \int 0(r \sin \theta) d\phi = \underline{\underline{0 \text{ J}}}$$

(d)

$$W_{AE} = W_{AD} + W_{DF} + W_{FE}$$

where F is $(10, 30^\circ, 60^\circ)$. Hence,

$$\begin{aligned} W_{AE} &= -Q \left\{ \int_{r=5}^{10} 20r \sin \theta dr \Big|_{\theta=30^\circ} + \int_{\theta=30^\circ}^{90^\circ} 10r \cos \theta r d\theta \Big|_{r=10} \right\} \\ &= -100 \left[\frac{75}{2} + \frac{100}{2} \right] \text{ nJ} = \underline{\underline{-8750 \text{ nJ}}} \end{aligned}$$

Prob. 4.41

$$\begin{aligned} V_{AB} &= - \int_A^B \mathbf{E} \cdot d\mathbf{l} = - \int_1^5 \frac{10}{r^2} dr \\ &= \frac{10}{r} \Big|_1^5 = 10 \left(\frac{1}{5} - 1 \right) = \underline{\underline{-8V}} \end{aligned}$$

Prob. 4.42

$$A = (2, 3, -1)$$

$$\downarrow d\mathbf{l} = dx \mathbf{a}_x$$

$$A'(8, 3, -1) \xrightarrow{dl = dy \mathbf{a}_y} B(8, 0, -1)$$

$$W = -Q \int E \cdot d\mathbf{l}$$

$$-\frac{W}{Q} = \int \mathbf{E} \cdot d\mathbf{l} = \int_{x=2}^8 2xy^2 dx \Big|_{y=3} + \int_{y=3}^0 2y^2(x^2 + 1) dy \Big|_{x=8}$$

$$= 2(3)^2 \frac{x^2}{2} \Big|_2^8 + 2(8^2 + 1) \frac{y^3}{3} \Big|_3^0$$

$$= 9(64 - 4) + \frac{2}{3} 65(0 - 81)$$

$$= 540 - 3510 = -2970$$

$$W = 2970 Q = 5940 \text{ nJ}$$

$$\underline{\underline{= 5.94 \mu J}}$$

Prob. 4.43*Method 1:*

$$W = -Q \int_L \mathbf{E} \cdot d\mathbf{l}, \quad d\mathbf{l} = \rho d\phi \mathbf{a}_\phi$$

$$\begin{bmatrix} E_\rho \\ E_\phi \\ E_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

$$E_\phi = -E_x \sin\phi + E_y \cos\phi = -20x \sin\phi + 40y \cos\phi$$

$$x = \rho \cos\phi, y = \rho \sin\phi$$

$$E_\phi = -20\rho \cos\phi \sin\phi + 40\rho \sin\phi \cos\phi = 20\rho \cos\phi \sin\phi$$

$$\begin{aligned} W &= -Q \int_L \mathbf{E} \cdot d\mathbf{l} = -2 \times 10^{-3} \int 20\rho \cos\phi \sin\phi \rho d\phi \Big|_{\rho=2} \\ &= -2(20)(2)^2 \int_0^\pi \sin\phi d(\sin\phi) \text{ mJ} = 160 \frac{\sin^2\phi}{2} \Big|_0^{\pi/2} = \underline{\underline{-80 \text{ mJ}}} \end{aligned}$$

Method 2:

$$-\frac{W}{Q} = \int_L \mathbf{E} \cdot d\mathbf{l} = \int 20x dx + 40y dy$$

$$y = 2 - x, dy = -dx$$

$$\begin{aligned} -\frac{W}{Q} &= \int 20x dx + 40(2-x)(-dx) = \int_{x=2}^0 (60x - 80) dx \\ &= \frac{60x^2}{2} - 80x \Big|_2^0 = 40 \end{aligned}$$

$$W = -40Q = \underline{\underline{-80 \text{ mJ}}}$$

Method 3:

$$\nabla \times \mathbf{E} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 20x & 40y & -10z \end{vmatrix} = 0$$

$$V = - \int_L \mathbf{E} \cdot d\mathbf{l} = -10x^2 - 20y^2 + 5z^2 + C$$

$$W = Q(V_2 - V_1) = Q(-20 \times 4 + 10 \times 4) = -40Q$$

$$W = -40Q = \underline{\underline{-80 \text{ mJ}}}$$

Prob. 4.44

$$W = -Q \int_L \mathbf{E} \cdot d\mathbf{l}$$

$$\mathbf{E} = \frac{\rho_s}{2\epsilon_0} \mathbf{a}_n = \frac{\rho_s}{2\epsilon_0} \mathbf{a}_x, \quad d\mathbf{l} = dx \mathbf{a}_x$$

$$W = -\frac{Q\rho_s}{2\epsilon_0} \int_3^1 dx = -\frac{Q\rho_s}{2\epsilon_0} (-2) = \frac{Q\rho_s}{\epsilon_0}$$

$$= \frac{10 \times 10^{-6} \times 40 \times 10^{-9}}{10^9} = 400 \times 36\pi \times 10^{-6} = \underline{\underline{45.24 \text{ mJ}}}$$

$$= \frac{36\pi}{3}$$

Prob. 4.45

(a) $\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial x} \mathbf{a}_x - \frac{\partial V}{\partial y} \mathbf{a}_y - \frac{\partial V}{\partial z} \mathbf{a}_z = \underline{\underline{-4x\mathbf{a}_x - 8y\mathbf{a}_y}}$

$$\rho_v = \nabla \cdot D = \epsilon_0 \nabla \cdot E = \epsilon_0 (-4 - 8) = -12\epsilon_0 = \underline{\underline{-106.25 \text{ pC/m}^3}}$$

(b) $\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial \rho} \mathbf{a}_\rho - \frac{1}{\rho} \frac{\partial V}{\partial \phi} \mathbf{a}_\phi - \frac{\partial V}{\partial z} \mathbf{a}_z$

$$= \underline{\underline{-(20\rho \sin \phi + 6z)\mathbf{a}_\rho - 10\rho \cos \phi \mathbf{a}_\phi - 6\rho \mathbf{a}_z}}$$

$$\rho_v = \nabla \cdot D = \epsilon_0 \nabla \cdot E = \epsilon_0 \left(-\frac{1}{\rho} [40\rho \sin \phi + 6z] + 10 \sin \phi \right)$$

$$= \underline{\underline{-\left(30 \sin \phi + \frac{6z}{\rho} \right) \epsilon_0 \text{ C/m}^3}}$$

(c) $\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial r} \mathbf{a}_r - \frac{1}{r} \frac{\partial V}{\partial \theta} \mathbf{a}_\theta - \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \mathbf{a}_\phi$

$$= \underline{\underline{-10r \cos \theta \sin \phi \mathbf{a}_r + 5r \sin \theta \sin \phi \mathbf{a}_\theta - 5r \cot \theta \cos \phi \mathbf{a}_\phi}}$$

$$\rho_v = \nabla \cdot D = \epsilon_0 \nabla \cdot E$$

$$\frac{\rho_v}{\epsilon_0} = \frac{1}{r^2} (-30r^2 \cos \theta \sin \phi) + \frac{5r \sin \phi}{r \sin \theta} 2 \sin \theta \cos \theta + \frac{5r \cot \theta \sin \phi}{r \sin \theta}$$

$$\rho_v = \underline{\underline{\epsilon_0 (5 \sin \phi \csc^2 \theta \cos \theta - 20 \cos \theta \sin \phi) \text{ C/m}^3}}$$

Prob. 4.46

$$\begin{aligned}
 V &= -r^{-3} \sin \theta \cos \phi \\
 -\mathbf{E} &= \nabla V = \frac{\partial V}{\partial r} \mathbf{a}_r + \frac{1}{r} \frac{\partial V}{\partial \theta} \mathbf{a}_\theta + \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \mathbf{a}_\phi \\
 &= -3r^{-4} \sin \theta \cos \phi \mathbf{a}_r + r^{-4} \cos \theta \cos \phi \mathbf{a}_\theta + \frac{r^{-4} \sin \theta}{\sin \theta} (-\sin \phi) \mathbf{a}_\phi \\
 \mathbf{E} &= \frac{3}{r^4} \sin \theta \cos \phi \mathbf{a}_r - \frac{1}{r^4} \cos \theta \cos \phi \mathbf{a}_\theta + \frac{\sin \phi}{r^4} \mathbf{a}_\phi \\
 \text{At } (1, 30^\circ, 60^\circ), r=1, \theta=30^\circ, \phi=60^\circ \\
 \mathbf{E} &= 3 \sin 30^\circ \cos 60^\circ \mathbf{a}_r - \cos 30^\circ \cos 60^\circ \mathbf{a}_\theta + \sin 60^\circ \mathbf{a}_\phi \\
 &= 0.75 \mathbf{a}_r - 0.433 \mathbf{a}_\theta + 0.866 \mathbf{a}_\phi \\
 \mathbf{D} &= \epsilon_0 \mathbf{E} = \frac{10^{-9}}{36\pi} (0.75 \mathbf{a}_r - 0.433 \mathbf{a}_\theta + 0.866 \mathbf{a}_\phi) \\
 &= 6.635 \mathbf{a}_r - 3.829 \mathbf{a}_\theta + 7.657 \mathbf{a}_\phi \text{ pC/m}^2
 \end{aligned}$$

Prob. 4.47

For $a < r < b$, we apply Gauss's law.

$$\begin{aligned}
 \oint_S D \cdot dS &= Q_{enc} = Q \\
 D_r (4\pi r^2) &= Q \quad \longrightarrow \quad E_r = \frac{D_r}{\epsilon_0} = \frac{Q}{4\pi \epsilon_0 r^2} \\
 V_{ab} &= - \int_a^b E \cdot dl = - \frac{Q}{4\pi \epsilon_0} \int_a^b \frac{1}{r^2} dr = \frac{Q}{4\pi \epsilon_0} \left. \frac{1}{r} \right|_a^b = - \frac{Q}{4\pi \epsilon_0} \left(\frac{1}{a} - \frac{1}{b} \right)
 \end{aligned}$$

Prob. 4.48

$$\begin{aligned}
 V &= \int_S \frac{\rho_s dS}{4\pi \epsilon_0 r} = \frac{\rho_s}{4\pi \epsilon_0} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} \frac{r^2 \sin \theta d\theta d\phi}{r} = \frac{\rho_s}{4\pi \epsilon_0 r} (2\pi) \int_0^{\pi/2} \sin \theta d\theta \\
 &= \frac{\rho_s}{2\epsilon_0 r} \left(-\cos \theta \Big|_0^{\pi/2} \right) \Big|_{r=a} \\
 V &= \frac{\rho_s}{2\epsilon_0 a}
 \end{aligned}$$

Prob. 4.49

$$\nabla \times \mathbf{E} = 0 \longrightarrow \nabla \times \mathbf{D} = 0$$

$$\begin{aligned}\nabla \times \mathbf{D} &= \left[\frac{1}{\rho} \frac{\partial D_z}{\partial \phi} - \frac{\partial D_\phi}{\partial z} \right] \mathbf{a}_\rho + \left[\frac{\partial D_\rho}{\partial z} - \frac{\partial D_z}{\partial \rho} \right] \mathbf{a}_\phi + \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} (\rho D_\phi) - \frac{\partial D_\rho}{\partial \phi} \right] \mathbf{a}_z \\ &= 0 \mathbf{a}_\rho - 0 \mathbf{a}_\phi - \frac{1}{\rho} 2\rho \cos \phi \mathbf{a}_z \neq 0\end{aligned}$$

Hence D is not a genuine EM field.

$$\begin{aligned}\psi &= \int_S \mathbf{D} \cdot d\mathbf{S} = \int_{\phi=0}^{\pi/4} \int_{z=0}^1 2\rho \sin \phi \rho d\phi dz = 2 \int_0^{\pi/4} \sin \phi d\phi \int_0^1 dz \rho^2 \Big|_{\rho=1} \\ &= -2 \cos \phi \Big|_0^{\pi/4} (1)(1)^2 = -2(\cos \pi/4 - 1) = \underline{\underline{0.5858 \text{ C}}}\end{aligned}$$

Prob. 4.50

(a)

$$m \frac{d^2 y}{dt^2} = eE; \text{ divide by } m, \text{ and integrate once, one obtains :}$$

$$\begin{aligned}u &= \frac{dy}{dt} = \frac{eEt}{m} + c_0 \\ y &= \frac{eEt^2}{2m} + c_0 t + c_1 \quad (1)\end{aligned}$$

"From rest" implies $c_1 = 0 = c_0$

$$\text{At } t = t_0, \quad y = d, \quad E = \frac{V}{d} \text{ or } V = Ed.$$

Substituting this in (1) yields :

$$t^2 = \frac{2m}{eE} d$$

Hence :

$$u = \frac{eE}{m} \sqrt{\frac{2md}{eE}} = \sqrt{\frac{2eEd}{m}} = \sqrt{\frac{2eV}{m}}$$

that is, $u \propto \sqrt{V}$

$$\text{or } u = k \sqrt{V}$$

(b)

$$\begin{aligned}k &= \sqrt{\frac{2e}{m}} = \sqrt{\frac{2(1.603)10^{-19}}{9.1066(10^{-31})}} \\ &= \underline{\underline{5.933 \times 10^5}}\end{aligned}$$

(c)

$$V = \frac{u^2 m}{2e} = \frac{9(10^{16}) \frac{1}{100}}{2(1.76)(10^{11})} = \underline{\underline{2.557 \text{ kV}}}$$

Prob. 4.51

(a)

This is similar to Example 4.3.

$$u_y = \frac{eEt}{m}, \quad u_x = u_0$$

$$y = \frac{eEt^2}{2m}, \quad x = u_0 t$$

$$t = \frac{x}{u_0} = \frac{10(10^{-2})}{10^7} = 10 \text{ ns}$$

Since $x = 10 \text{ cm}$ when $y = 1 \text{ cm}$,

$$E = \frac{2my}{et^2} = \frac{2(10^{-2})}{1.76(10^{11})(10^{-16})} = 1.136 \text{ kV/m}$$

$$\underline{\underline{E = -1.136 \mathbf{a}_y \text{ kV/m}}}$$

(b)

$$u_x = u_0 = 10^7,$$

$$u_y = \frac{2000}{1.76}(1.76)10^{11}(10^{-8}) = 2(10^6)$$

$$\underline{\underline{\mathbf{u} = (\mathbf{a}_x + 0.2\mathbf{a}_y)(10^7) \text{ m/s}}}$$

Prob. 4.52

$$\nabla \cdot \mathbf{E} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(\frac{20 \cos \theta}{r} \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left(\frac{10 \sin^2 \theta}{r^3} \right) + 0$$

$$= -\frac{20 \cos \theta}{r^4} + \frac{20 \cos \theta}{r^4} = 0$$

$$\nabla \times \mathbf{E} = \frac{1}{r} \left[\frac{\partial}{\partial r} \left(\frac{10 \sin \theta}{r^2} \right) + \frac{20 \sin \theta}{r^3} \right] \mathbf{a}_\phi$$

$$= \frac{1}{r} \left[-\frac{20 \sin \theta}{r^3} + \frac{20 \sin \theta}{r^3} \right] \mathbf{a}_\phi = \mathbf{0}$$

At $r = 0$, $\nabla \cdot \mathbf{E}$ and $\nabla \times \mathbf{E}$ are not defined. So they are zero every where except at the origin.

Prob. 4.53

The dipole is oriented along y -axis.

$$V = \frac{\mathbf{p} \cdot \mathbf{r}}{4\pi\epsilon_0 r^2}; \mathbf{p} \cdot \mathbf{r} = Qd \mathbf{a}_y \cdot \mathbf{a}_r = Qd \sin\theta \sin\phi$$

$$V = \frac{Qd \sin\theta \sin\phi}{4\pi\epsilon_0 r^2}$$

$$\begin{aligned} \mathbf{E} &= -\nabla V = -\frac{\partial V}{\partial r} \mathbf{a}_r - \frac{1}{r} \frac{\partial V}{\partial \theta} \mathbf{a}_\theta - \frac{1}{r \sin\theta} \frac{\partial V}{\partial \phi} \mathbf{a}_\phi \\ &= \frac{Qd}{4\pi\epsilon_0} \left\{ \frac{2 \sin\theta \sin\phi}{r^3} \mathbf{a}_r - \frac{\cos\theta \sin\phi}{r^3} \mathbf{a}_\theta - \frac{\cos\phi}{r^3} \mathbf{a}_\phi \right\} \\ \mathbf{E} &= \underline{\underline{\frac{Qd}{4\pi\epsilon_0 r^3} (2 \sin\theta \sin\phi \mathbf{a}_r - \cos\theta \sin\phi \mathbf{a}_\theta - \cos\phi \mathbf{a}_\phi)}} \end{aligned}$$

Prob. 4.54

$$V = \frac{p \cos\theta}{4\pi\epsilon_0 r^2} = \frac{k \cos\theta}{r^2}$$

At (0,1 nm), $\theta = 0$, $r = 1$ nm, $V = 9$;

$$\text{that is, } 9 = \frac{k(1)}{1(10^{-18})}, \therefore k = 9(10^{-18})$$

$$V = 9(10^{-18}) \frac{\cos\theta}{r^2}$$

At (1,1) nm, $r = \sqrt{2}$ nm, $\theta = 45^\circ$,

$$V = \frac{9(10^{-18}) \cos 45^\circ}{10^{-18} (\sqrt{2})^2} = \frac{9}{2\sqrt{2}} = \underline{\underline{3.182 \text{ V}}}$$

Prob. 4.55

$$\begin{aligned} W &= W_1 + W_2 = 0 + Q_2 V_{21} = Q_2 \frac{Q_1}{4\pi\epsilon_0 |(2,0,0) - (0,0,1)|} \\ &= \frac{40 \times 10^{-9} \times (-50) \times 10^{-9}}{4\pi \times \frac{10^{-9}}{36\pi} |(2,0,-1)|} = \frac{40 \times 9 \times (-50) \times 10^{-9}}{\sqrt{4+1}} \\ &= \underline{\underline{-8.05 \mu\text{J}}} \end{aligned}$$

Prob. 4.56

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial x} \mathbf{a}_x - \frac{\partial V}{\partial y} \mathbf{a}_y = -4x\mathbf{a}_x - 12y\mathbf{a}_y \text{ V/m}$$

$$\begin{aligned} W &= \frac{1}{2} \epsilon_0 \int_V |\mathbf{E}|^2 dv = \frac{1}{2} \epsilon_0 \int_{z=-1}^1 \int_{y=-1}^1 \int_{x=-1}^1 (16x^2 + 144y^2) dx dy dz \\ &= \frac{1}{2} \epsilon_0 \left[16(4) \frac{x^3}{3} \Big|_{-1}^1 + 144(4) \frac{y^3}{3} \Big|_{-1}^1 \right] = \frac{1}{2} \frac{10^{-9}}{36\pi} (160)(4) \frac{1}{3} (1+1) \\ &= \underline{\underline{1.886 \text{ nJ}}} \end{aligned}$$

Prob. 4.57

Given that $\mathbf{E} = 2r \sin \theta \cos \phi \mathbf{a}_r + r \cos \theta \cos \phi \mathbf{a}_\theta - r \sin \phi \mathbf{a}_\phi$

$$\begin{aligned} E^2 &= 4r^2 \sin^2 \theta \cos^2 \phi + r^2 \cos^2 \theta \cos^2 \phi + r^2 \sin^2 \phi \\ &= r^2 \cos^2 \phi (4 \sin^2 \theta + \cos^2 \theta) + r^2 \sin^2 \phi \\ &= r^2 \cos^2 \phi + 3r^2 \cos^2 \phi \sin^2 \theta + r^2 \sin^2 \phi \\ &= r^2 (1 + 3 \cos^2 \phi \sin^2 \theta) \end{aligned}$$

$$\begin{aligned} W &= \frac{\epsilon}{2} \iiint E^2 r^2 \sin \theta dr d\theta d\phi \\ &= \frac{\epsilon}{2} \int_0^2 r^4 dr \int_{\theta=0}^{\pi} \int_{\phi=0}^{\pi} (1 + 3 \cos^2 \phi \sin^2 \theta) \sin \theta d\theta d\phi \\ &= \frac{16\epsilon}{5} \int_0^{\pi} (\pi \sin \theta + \frac{3\pi}{2} \sin^2 \theta) d\theta \\ &= \frac{16}{5} x \frac{10^{-9}}{36\pi} (4\pi) = \frac{16}{45} \text{ nJ} = \underline{\underline{0.36 \text{ nJ}}} \end{aligned}$$

Prob. 4.58

Method 1:

$$W = \frac{1}{2} \int_S \rho_s V dS = \frac{V}{2} \int_S \rho_s dS = \frac{1}{2} QV$$

$$\text{But } V = \frac{Q}{4\pi\epsilon_0 a}$$

$$W = \frac{Q^2}{8\pi\epsilon_0 a}$$

Method 2:

$$\begin{aligned}
 W &= \frac{1}{2} \int_v \mathbf{D} \cdot \mathbf{E} dv = \frac{1}{2} \epsilon_o \int_v E^2 dv \\
 &= \frac{1}{2} \epsilon_o \iiint \left(\frac{Q}{4\pi\epsilon_o r^2} \right)^2 r^2 \sin\theta d\theta dr d\phi \\
 &= \frac{1}{2} \epsilon_o \int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta \int_{r=a}^{\infty} \frac{Q^2}{16\pi^2 \epsilon_o^2 r^2} dr = \frac{\epsilon_o}{2} (2\pi)(2) \frac{Q^2}{16\pi^2 \epsilon_o^2 a} \\
 W &= \underline{\underline{\frac{Q^2}{8\pi\epsilon_o a}}}
 \end{aligned}$$

Prob. 4.59

$$\begin{aligned}
 W &= \frac{1}{2} \int_v \epsilon_o |\mathbf{E}|^2 dv \\
 &= \frac{\epsilon_o}{2} \int_{z=0}^3 \int_{y=0}^2 \int_{x=0}^1 (9x^2 + 25z^2) dx dy dz \\
 &= \frac{1}{2} \frac{10^{-9}}{36\pi} \left[9 \int_0^3 dz \int_0^2 dy \int_0^1 x^2 dx + 25 \int_0^3 z^2 dz \int_0^2 dy \int_0^1 dx \right] \\
 &= \frac{1}{72\pi} \left[9(3)(2) \frac{x^3}{3} \Big|_0^1 + 25(1)(2) \frac{z^3}{3} \Big|_0^3 \right] nJ = \frac{1}{72\pi} (18 + 50 \times 9) = \frac{468}{72\pi} nJ
 \end{aligned}$$

$$W = \underline{\underline{2.069 \text{ nJ}}}$$

CHAPTER 5

P. L. 5.1 $d\mathbf{S} = \rho d\phi dz \mathbf{a}_\rho$

$$I = \int J \bullet dS = \int_{\phi=0}^{2\pi} \int_{z=1}^5 10z \sin^2 \phi \rho dz d\phi |_{\rho=2} = 10(2) \frac{z^2}{2} \Big|_1^5 \int_0^{2\pi} \frac{1}{2}(1 - \cos 2\phi) d\phi = 240\pi$$

I = 754 A

P. E. 5.2

$$I = \rho_s w u = 0.5 \times 10^{-6} \times 0.1 \times 10 = 0.5 \mu A$$

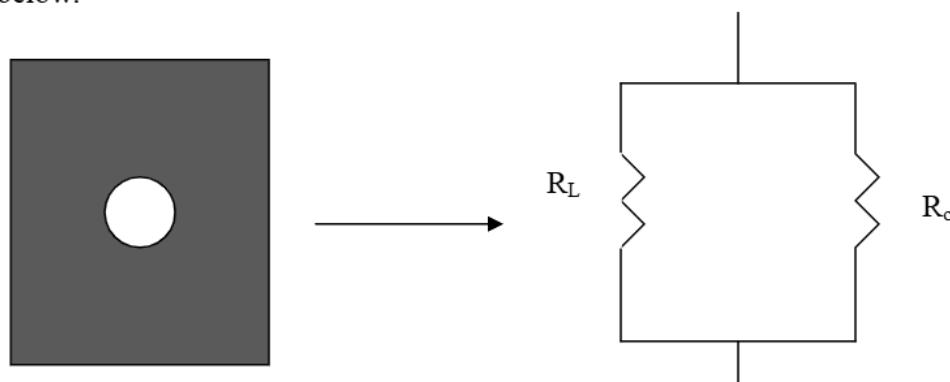
$$V = IR = 10^{14} \times 0.5 \times 10^{-6} = \underline{\underline{50 \text{ MV}}}$$

P. E. 5.3 $\sigma = 5.8 \times 10^7 \text{ S/m}$

$$J = \sigma E \quad \longrightarrow \quad E = \frac{J}{\sigma} = \frac{8 \times 10^6}{5.8 \times 10^7} = \underline{\underline{0.138 \text{ V/m}}}$$

$$J = \rho_v u \quad \longrightarrow \quad u = \frac{J}{\rho_v} = \frac{8 \times 10^6}{1.81 \times 10^{10}} = \underline{\underline{4.42 \times 10^{-4} \text{ m/s}}}$$

P. E. 5.4 The composite bar can be modeled as a parallel combination of resistors as shown below.



For the lead, $R_L = \frac{l}{\sigma_L S_L}$, $S_L = d^2 - \pi r^2 = 9 - \frac{\pi}{4} \text{ cm}^2$

$$R_L = 0.974 \text{ m } \Omega$$

For copper, $R_c = \frac{l}{\sigma_c S_c}$, $S_c = \pi r^2 = \frac{\pi}{4} \text{ cm}^2$

$$R_c = \frac{4}{5.8 \times 10^7 \times \frac{\pi}{4} \times 10^{-4}} = 0.8781 \text{ m}\Omega$$

$$R = \frac{R_L R_c}{R_L + R_c} = \frac{0.974 \times 0.8781}{0.974 + 0.8781} = \underline{\underline{461.7 \mu\Omega}}$$

P. E. 5.5 $\rho_{ps} = \mathbf{P} \bullet \mathbf{a}_x = ax^2 + b$

$$\rho_{ps} \Big|_{x=0} = \mathbf{P} \bullet (-\mathbf{a}_x) \Big|_{x=0} = \underline{\underline{-b}}$$

$$\rho_{ps} \Big|_{x=L} = \mathbf{P} \bullet \mathbf{a}_x \Big|_{x=L} = \underline{\underline{aL^2 + b}}$$

$$Q_s = \int \rho_{ps} dS = -bA + (aL^2 + b)A = AaL^2$$

$$\rho_{pv} = -\nabla \bullet \mathbf{P} = -\frac{d}{dx}(ax^2 + b) = -2ax$$

$$\rho_{pv} \Big|_{x=0} = \underline{\underline{0}}, \quad \rho_{pv} \Big|_{x=L} = \underline{\underline{-2aL}}$$

$$Q_v = \int \rho_{pv} dv = \int_0^L (-2ax)Adx = -AaL^2$$

Hence,

$$Q_T = Q_v + Q_s = -AaL^2 + AaL^2 = 0$$

P. E. 5.6

$$E = \frac{V}{d} \mathbf{a}_x = \frac{10^3}{2 \times 10^{-3}} \mathbf{a}_x = 500 \mathbf{a}_x \text{ kV/m}$$

$$\mathbf{P} = \chi_e \epsilon_o \mathbf{E} = (2.55 - 1) \times \frac{10^{-9}}{36\pi} \times 0.5 \times 10^6 \mathbf{a}_x = \underline{\underline{6.853 \mathbf{a}_x \mu C / m^2}}$$

$$\rho_{ps} = \mathbf{P} \bullet \mathbf{a}_x = \underline{\underline{6.853 \mu C / m^2}}$$

P. E. 5.7 (a) Since $P = \epsilon_o \chi_e E$, $P_x = \epsilon_o \chi_e E_x$

$$\chi_e = \frac{P_x}{\epsilon_o E_x} = \frac{3 \times 10^{-9}}{10\pi} \frac{1}{5} \times 36\pi \times 10^9 = \underline{\underline{2.16}}$$

$$(b) E = \frac{P}{\chi_e \epsilon_0} = \frac{36\pi \times 10^9}{2.16} \frac{1}{10\pi} (3, -1, 4) 10^{-9} = \underline{\underline{5a_x - 1.67a_y + 6.67a_z}} \text{ V/m}$$

(c)

$$D = \epsilon_0 \epsilon_r E = \frac{\epsilon_r P}{\chi_e} = \frac{3.16}{2.16} \left(\frac{1}{10\pi} \right) (3, -1, 4) \text{ nC/m}^2 = \underline{\underline{139.7a_x - 46.6a_y + 186.3a_z}} \text{ pC/m}^2$$

P. E. 5.8 From Example 5.8,

$$F = \frac{\rho_s^2 S}{2\epsilon_0} \longrightarrow \rho_s^2 = \frac{2\epsilon_0 F}{S}$$

But $\rho_s = \epsilon_0 E = \epsilon_0 \frac{V_d}{d}$. Hence

$$\rho_s^2 = \frac{2\epsilon_0 F}{S} = \frac{\epsilon_0^2 V_d^2}{d^2} \longrightarrow V_d^2 = \frac{2Fd^2}{\epsilon_0 S}$$

i.e.

$$V_d = V_1 - V_2 = \sqrt{\frac{2Fd^2}{\epsilon_0 S}}$$

as required.

P. E. 5.9 (a) Since $a_n = a_x$,

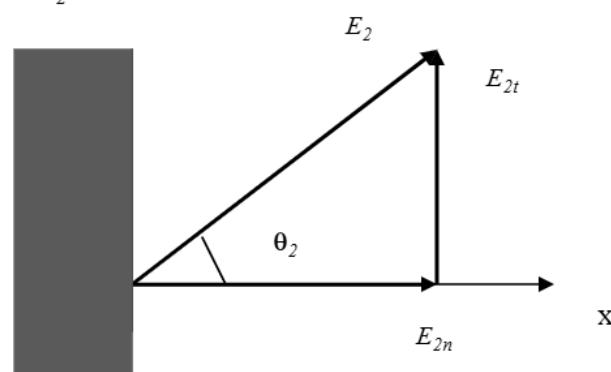
$$D_{1n} = 12a_x, \quad D_{1t} = -10a_x + 4a_z, \quad D_{2n} = D_{1n} = 12a_x$$

$$E_{2t} = E_{1t} \longrightarrow D_{2t} = \frac{\epsilon_2 D_{1t}}{\epsilon_1} = \frac{1}{2.5} (-10a_y + 4a_z) = -4a_y + 1.6a_z$$

$$D_2 = D_{2n} + D_{2t} = \underline{\underline{12a_x - 4a_y + 1.6a_z}} \text{ nC/m}^2.$$

$$\tan \theta_2 = \frac{D_{2t}}{D_{2n}} = \frac{\sqrt{(-4)^2 + (1.6)^2}}{12} = 0.359 \longrightarrow \underline{\underline{\theta_2 = 19.75^\circ}}$$

$$(b) E_{1t} = E_{2t} = E_2 \sin \theta_2 = 12 \sin 60^\circ = 10.392$$



$$E_{1n} = \frac{\epsilon_{r2}}{\epsilon_{r1}} E_{2n} = \frac{1}{2.5} 12 \cos 60^\circ = 2.4$$

$$E_I = \sqrt{E_{I_t}^2 + E_{I_n}^2} = \underline{\underline{10.67}}$$

$$\tan \theta_1 = \frac{\epsilon_{r1}}{\epsilon_{r2}} \tan \theta_2 = \frac{2.5}{1} \tan 60^\circ = 4.33 \quad \longrightarrow \quad \underline{\underline{\theta_1 = 77^\circ}}$$

Note that $\theta_1 > \theta_2$.

P. E. 5.10

$$D = \epsilon_0 E = \frac{10^{-9}}{36\pi} (60, 20, -30) \times 10^{-3} = \underline{\underline{0.531a_x + 0.177a_y - 0.265a_z \text{ pC/m}^2}}$$

$$\rho_s = D_n = |D| = \frac{10^{-9}}{36\pi} (10) \sqrt{36+4+9} (10^{-3}) = \underline{\underline{0.619 \text{ pC/m}^2}}$$

Prob. 5.1

$$I = \int \mathbf{J} \bullet d\mathbf{S}, \quad d\mathbf{S} = dy dz \mathbf{a}_x$$

$$\begin{aligned} I &= \iint e^{-x} \cos(4y) dy dz \Big|_{x=2} = e^{-2} \int_0^{\pi/3} \cos(4y) dy \int_0^4 dz \\ &= 4e^{-2} \left(\frac{\sin 4y}{4} \Big|_0^{\pi/3} \right) = e^{-2} \left(\sin(\frac{4\pi}{3}) - 0 \right) = \underline{\underline{-0.1172 \text{ A}}} \end{aligned}$$

Prob. 5.2

Method 1:

$$\begin{aligned} I &= \int \mathbf{J} \bullet d\mathbf{S} = \iint \frac{10}{r} e^{-10^3 t} r^2 \sin \theta d\theta d\phi \Big|_{t=2ms, r=4m} \\ &= 10(4) e^{-10^3 \times 2 \times 10^{-3}} \int_{\theta=0}^{\pi} \sin \theta d\theta \int_{\phi=0}^{2\pi} d\phi = 40e^{-2} (2)(2\pi) = 160\pi e^{-2} \\ &= \underline{\underline{68.03 \text{ A}}} \end{aligned}$$

Method 2:

$$I = \int \mathbf{J} \bullet d\mathbf{S} = \iint \frac{10}{r} e^{-10^3 t} dS = \frac{10}{r} e^{-10^3 t} (4\pi r^2)$$

since r is constant on the surface.

$$I = 40\pi r e^{-2} = 160\pi e^{-2} = \underline{\underline{68.03 \text{ A}}}$$

Prob. 5.3

$$I = -\frac{dQ}{dt} = 3 \times 10^{-4} e^{-3t}$$

$$I(t=0) = \underline{\underline{0.3 \text{ mA}}, \quad I(t=2.5) = 0.3 e^{-7.5} = \underline{\underline{166 \text{ nA}}}}$$

Prob. 5.4

$$\begin{aligned} I &= \int \mathbf{J} \bullet d\mathbf{S} = 5 \int_{\rho=0}^a \int_{\phi=0}^{2\pi} e^{-10\rho} \rho d\phi d\rho = 5 \int_0^{2\pi} d\phi \int_{\rho=0}^a \rho e^{-10\rho} d\rho \\ &= 5(2\pi) \left(\frac{e^{-10\rho}}{100} (-10\rho - 1) \right) \Big|_0^a = \frac{10\pi}{100} [e^{-10a}(-10a - 1) - 1(-0 - 1)] \\ &= \frac{\pi}{10} [e^{-0.04}(-0.04 - 1) + 1] = \frac{\pi}{10} (0.00078) = \underline{\underline{244.7 \mu\text{A}}} \end{aligned}$$

Prob. 5.5

$$\begin{aligned} I &= \int \mathbf{J} \bullet d\mathbf{S} = \iint \frac{10}{\rho} \sin \phi \rho d\phi dz = 10 \int_0^5 dz \int_0^\pi \sin \phi d\phi \\ &= 10(5)(-\cos \phi) \Big|_0^\pi = \underline{\underline{100 \text{ A}}} \end{aligned}$$

Prob. 5.6

$$R = \frac{l}{\sigma S} \longrightarrow \sigma = \frac{l}{RS} = \frac{2 \times 10^{-2}}{10^6 (\pi) (4 \times 10^{-3})^2} = \underline{\underline{3.978 \times 10^{-4} \text{ S/m}}}$$

$$\text{Prob. 5.7 (a)} \quad R = \frac{l}{\sigma S} = \frac{8 \times 10^{-2}}{3 \times 10^4 \pi (25) 10^{-6}} = \frac{8}{75\pi} = \underline{\underline{33.95 \text{ m}\Omega}}$$

$$(b) \quad I = V / R = 9 \times \frac{75\pi}{8} = \underline{\underline{265.1 \text{ A}}}$$

(c) $P = IV = \underline{2.386 \text{ kW}}$

Prob. 5.8

If R and S are the same,

$$R_1 = \frac{\ell_1}{\sigma_1 S} = R_2 = \frac{\ell_2}{\sigma_2 S} \longrightarrow \ell_1 = \ell_2 \frac{\sigma_1}{\sigma_2}$$

If 1 corresponds to copper and 2 to silver,

$$\sigma_1 = 5.8 \times 10^7 \text{ S/m}, \quad \sigma_2 = 6.1 \times 10^7 \text{ S/m}$$

$$\ell_1 = \ell_2 \frac{5.8}{6.1} = 0.951 \ell_2$$

That is, the copper wire is shorter than silver wire or the silver wire is longer.

Prob. 5.9 (a) $S_i = \pi r_i^2 = \pi(1.5)^2 \times 10^{-4} = 7.068 \times 10^{-4}$

$$S_o = \pi(r_o^2 - r_i^2) = \pi(4 - 2.25) \times 10^{-4} = 5.498 \times 10^{-4}$$

$$R_i = \frac{\rho_i l}{S_i} = \frac{11.8 \times 10^{-8} \times 10}{7.068 \times 10^{-4}} = 16.69 \times 10^{-4}$$

$$R_o = \frac{\rho_o l}{S_o} = \frac{1.77 \times 10^{-8} \times 10}{5.498 \times 10^{-4}} = 3.219 \times 10^{-4}$$

$$R = R_i // R_o = \frac{R_i R_o}{R_i + R_o} = \frac{16.69 \times 3.219 \times 10^{-4}}{16.69 + 3.219} = \underline{\underline{0.27 \text{ m}\Omega}}$$

(b) $V = I_i R_i = I_o R_o \longrightarrow \frac{I_i}{I_o} = \frac{R_o}{R_i} = \frac{0.3219}{1.669} = 0.1929$

$$I_i + I_o = 1.1929 I_o = 60 \text{ A}$$

$$I_o = \underline{\underline{50.3 \text{ A}}} \quad (\text{copper}), \quad I_i = \underline{\underline{9.7 \text{ A}}} \quad (\text{steel})$$

Alternatively, using the principle of current division,

$$I_o = 60 \frac{R_i}{R_i + R_o} = 50.3 \text{ A}$$

$$I_i = 60 \frac{R_o}{R_i + R_o} = 9.7 \text{ A}$$

(c) $R = \frac{10 \times 1.77 \times 10^{-8}}{1.75 \pi \times 10^{-4}} = \underline{\underline{0.322 \text{ m}\Omega}}$

Prob. 5.10

From eq. (5.16),

$$R_1 = \frac{\rho_1 \ell}{S_1} = \frac{\rho_1 \ell}{ab}, \quad R_2 = \frac{\rho_2 \ell}{S_2} = \frac{\rho_2 \ell}{ac}$$

$$R = R_1 \parallel R_2 = \frac{R_1 R_2}{R_1 + R_2} = \frac{\frac{\rho_1 \ell}{ab} \frac{\rho_2 \ell}{ac}}{\frac{\rho_1 \ell}{ab} + \frac{\rho_2 \ell}{ac}} = \frac{\rho_1 \rho_2 \ell^2}{ac \rho_1 \ell + ab \rho_2 \ell}$$

$$R = \frac{\rho_1 \rho_2 \ell}{\underline{a(c\rho_1 + b\rho_2)}}$$

Prob. 5.11

$$|\mathbf{P}| = n |\mathbf{p}| = n Q d = 2 n e d = \chi_e \epsilon_o E \quad (Q = 2e)$$

$$\chi_e = \frac{2 n e d}{\epsilon_o E} = \frac{2 \times 5 \times 10^{25} \times 1.602 \times 10^{-19} \times 10^{-18}}{\frac{10^{-9}}{36\pi} \times 10^4} = 0.000182$$

$$\epsilon_r = 1 + \chi_e = \underline{\underline{1.000182}}$$

Prob. 5.12

$$\mathbf{P} = \frac{\sum_{i=1}^N q_i \mathbf{d}_i}{v} = \frac{\sum_{i=1}^N \mathbf{p}_i}{v}$$

$$|\mathbf{P}| = \frac{N}{v} |\mathbf{p}| = 2 \times 10^{19} \times 1.8 \times 10^{-27} = 3.6 \times 10^{-8}$$

$$\mathbf{P} = |\mathbf{P}| \mathbf{a}_x = \underline{\underline{3.6 \times 10^{-8} \mathbf{a}_x \text{ C/m}^2}}$$

$$\text{But } P = \chi_e \epsilon_o E \quad \text{or} \quad \chi_e = \frac{P}{\epsilon_o E} = \frac{3.6 \times 36\pi \times 10^9 \times 10^{-18}}{10^5} = 0.0407$$

$$\epsilon_r = 1 + \chi_e = \underline{\underline{1.0407}}$$

Prob. 5.13

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0\epsilon_r r^2} \mathbf{a}_r$$

$$\mathbf{P} = \chi_e \epsilon_0 \mathbf{E} = \frac{\chi_e Q}{4\pi\epsilon_r r^2} \mathbf{a}_r = \frac{3(10)10^{-3}}{4\pi(4)\ell^2} \mathbf{a}_r = \underline{\underline{596.8 \mathbf{a}_r \text{ } \mu\text{C/m}^2}}$$

Prob. 5.14

$$\mathbf{P} = \chi_e \epsilon_0 \mathbf{E} \quad \longrightarrow \quad \mathbf{E} = \frac{\mathbf{P}}{\chi_e \epsilon_0} = \frac{100 \times 10^{-9}}{2.5 \frac{10^{-9}}{36\pi}(2)} \mathbf{a}_\rho = \underline{\underline{2.261 \mathbf{a}_\rho \text{ kV/m}}}$$

$$\mathbf{D} = \epsilon_0 \epsilon_r \mathbf{E} = 3.5 \times \frac{10^{-9}}{36\pi} 2.261 \times 10^3 \mathbf{a}_\rho = \underline{\underline{70 \mathbf{a}_\rho \text{ nC/m}^2}}$$

Prob. 5.15

(a)

$$\begin{aligned} Q_{s1} &= \int_S \mathbf{P} \cdot d\mathbf{S}, d\mathbf{S} = r^2 \sin \theta d\theta d\phi (-\mathbf{a}_r) \\ &= - \iint 4r r^2 \sin \theta d\theta d\phi \Big|_{r=1.2\text{cm}} \\ &= -4(1.2)^3 (10^{-6}) \int_0^{2\pi} d\phi \int \sin \theta d\theta (10^{-12}) \\ &= -6.912(2\pi)(2) \times 10^{-18} \\ &= \underline{\underline{-86.86 \times 10^{-18} \text{ C}}} \end{aligned}$$

(b)

$$\begin{aligned} Q_{s2} &= \int_S \mathbf{P} \cdot d\mathbf{S}, d\mathbf{S} = r^2 \sin \theta d\theta d\phi (-\mathbf{a}_r) \\ &= - \iint 4r r^2 \sin \theta d\theta d\phi \Big|_{r=2.6\text{cm}} \\ &= -4(2.6)^3 (10^{-6}) \int_0^{2\pi} d\phi \int \sin \theta d\theta (10^{-12}) \\ &= -4(2.6)^3 (2\pi)(2) \times 10^{-18} = \underline{\underline{883.5 \times 10^{-18} \text{ C}}} \end{aligned}$$

(c)

$$\rho_{pv} = -\nabla \cdot \mathbf{P} = -\frac{1}{r^2} \frac{\partial}{\partial r} (4r^3) \text{ pC/m}^3 = -12 \text{ pC/m}^3$$

$$Q_v = \int_v \rho_{pv} dv = -12 \iiint dv = -12 \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi \int_{1.2}^{2.6} r^2 dr (10^{-18})$$

$$= -12(2)(2\pi) \frac{r^3}{3} \Big|_{1.2}^{2.6} (10^{-18}) = -16\pi(2.6^3 - 1.2^3)(10^{-18})$$

$$= \underline{\underline{-796.61 \times 10^{-18} \text{ C}}}$$

Prob. 5.16

$$\mathbf{D} = \epsilon_0 \epsilon_r \mathbf{E} = 2.1x \frac{10^{-9}}{36\pi} (6, 12, -20) = \underline{\underline{0.1114 \mathbf{a}_x + 0.2228 \mathbf{a}_y - 0.3714 \mathbf{a}_z \text{ nC/m}^2}}$$

$$\mathbf{P} = \chi_e \epsilon_0 \mathbf{E} = 1.1x \frac{10^{-9}}{36\pi} (6, 12, -20) = \underline{\underline{0.0584 \mathbf{a}_x + 0.1167 \mathbf{a}_y - 0.1945 \mathbf{a}_z \text{ nC/m}^2}}$$

Prob. 5.17

$$\mathbf{E} = -\nabla V = -\left(\frac{\partial V}{\partial \rho} \mathbf{a}_\rho + \frac{1}{\rho} \frac{\partial V}{\partial \phi} \mathbf{a}_\phi + \frac{\partial V}{\partial z} \mathbf{a}_z \right)$$

$$= (10z \sin \phi \mathbf{a}_\rho + 10z \cos \phi \mathbf{a}_\phi + 10\rho \sin \phi \mathbf{a}_z)$$

$$\mathbf{D} = \epsilon \mathbf{E} = 5\epsilon_0 \mathbf{E}$$

$$= \underline{\underline{-50\epsilon_0 (z \sin \phi \mathbf{a}_\rho + z \cos \phi \mathbf{a}_\phi + \rho \sin \phi \mathbf{a}_z) \text{ C/m}^2}}$$

Prob. 5.18

$$(a) \quad \mathbf{E} = -\nabla V = -\left(\frac{\partial V}{\partial x} \mathbf{a}_x + \frac{\partial V}{\partial y} \mathbf{a}_y + \frac{\partial V}{\partial z} \mathbf{a}_z \right) = \underline{\underline{-20xyz \mathbf{a}_x - 10x^2z \mathbf{a}_y - 10(x^2y - z) \mathbf{a}_z \text{ V/m}}}$$

$$(b) \quad \mathbf{D} = \epsilon \mathbf{E} = 5\epsilon_0 \mathbf{E} = \underline{\underline{-0.8842xyz \mathbf{a}_x - 0.4421x^2z \mathbf{a}_y - 0.4421(x^2y - z) \mathbf{a}_z \text{ nC/m}^2}}$$

$$(c) \quad \mathbf{P} = \chi_e \epsilon_0 \mathbf{E} = 4\epsilon_0 \mathbf{E} = \underline{\underline{-0.7073xyz \mathbf{a}_x - 0.3537x^2z \mathbf{a}_y - 0.3537(x^2y - z) \mathbf{a}_z \text{ nC/m}^2}}$$

$$(d) \quad \rho_v = -\epsilon \nabla^2 V$$

$$\nabla^2 V = \frac{\partial}{\partial x} (20xyz) + \frac{\partial}{\partial y} (10x^2z) + \frac{\partial}{\partial z} (10x^2y - 10z) = 20yz - 10$$

$$\rho_v = -5\epsilon_0 10(2yz - 1) = \underline{\underline{-0.8854yz + 0.4427 \text{ nC/m}^3}}$$

Prob. 5.19

$$\underline{\underline{P}} = \chi_e \epsilon_o \underline{\underline{E}} = \chi_e \epsilon_o \frac{\underline{\underline{D}}}{\epsilon_o \epsilon_r} = \frac{(\epsilon_r - 1) \underline{\underline{D}}}{\epsilon_r} = \frac{1.4}{2.4} \times 450 \underline{\underline{a}_x} \text{ nC/m}^2$$

$$\underline{\underline{P}} = 262.5 \underline{\underline{a}_x} \text{ nC/m}^2$$

Prob. 5.20 (a) Applying Coulomb's law,

$$E_r = \begin{cases} \frac{D_r}{\epsilon_o} = \frac{Q}{4\pi\epsilon_o r^2}, & r > b \\ \frac{D_r}{\epsilon} = \frac{Q}{4\pi\epsilon r^2}, & a < r < b \\ P = \frac{\epsilon_r - 1}{\epsilon_r} D & (= D - \epsilon_o E) \end{cases}$$

Hence

$$\underline{\underline{P}_r} = \frac{\epsilon_r - 1}{\epsilon_r} \cdot \frac{Q}{4\pi r^2}, \quad a < r < b$$

$$(b) \quad \rho_{pv} = -\nabla \bullet \underline{\underline{P}} = -\frac{1}{r^2} \frac{d}{dr} (r^2 P_r) = 0$$

(c)

$$\underline{\underline{\rho_{ps}}} = \underline{\underline{P}} \bullet (-\underline{\underline{a}_r}) = -\frac{Q}{4\pi a^2} \left(\frac{\epsilon_r - 1}{\epsilon_r} \right), \quad r = a$$

$$\underline{\underline{\rho_{ps}}} = \underline{\underline{P}} \bullet (\underline{\underline{a}_r}) = -\frac{Q}{4\pi b^2} \left(\frac{\epsilon_r - 1}{\epsilon_r} \right), \quad r = b$$

Prob. 5.21

$$F_1 = \frac{Q_1 Q_2}{4\pi\epsilon_o d^2} = 2.6 \text{ nN}, \quad F_2 = \frac{Q_1 Q_2}{4\pi\epsilon_o \epsilon_r d^2} = 1.5 \text{ nN}$$

$$\frac{F_1}{F_2} = \frac{2.6}{1.5} = \epsilon_r = \underline{\underline{1.733}}$$

Prob. 5.22

(a) By Gauss's law,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q_{enc} \longrightarrow D_r = \frac{Q}{4\pi r^2}$$

$$E_r = \frac{D_r}{\epsilon} = \frac{Q}{4\pi\epsilon r^2}$$

$$W = \int_v \frac{1}{2} \epsilon |E|^2 dv, \quad dv = r^2 \sin\theta dr d\phi d\theta$$

$$W = \frac{1}{2} \epsilon \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=a}^{\infty} \frac{Q^2}{16\pi^2 \epsilon^2 r^4} r^2 \sin\theta dr d\phi d\theta = \frac{Q^2}{8\pi\epsilon a}$$

(b) D_r remains the same but

$$E_r = \frac{D_r}{\epsilon} = \frac{Q}{4\pi r^2 \epsilon_o \left(1 + \frac{a}{r}\right)^2} = \frac{Q}{4\pi\epsilon_o (r+a)^2}$$

$$W = \int_v \frac{1}{2} \epsilon |E|^2 dv = \frac{1}{2} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=a}^{\infty} \frac{Q^2 r^2 \sin\theta dr d\theta d\phi}{16\pi^2 \epsilon^2 (r+a)^4} \epsilon_o \left(\frac{r+a}{r}\right)^2$$

$$= \frac{Q^2}{32\pi^2 \epsilon_o} (4\pi) \int_a^{\infty} \frac{dr}{(r+a)^2} = \frac{Q^2}{8\pi\epsilon_o} \left(-\frac{1}{r+a}\Big|_a^{\infty}\right) = \frac{Q^2}{8\pi\epsilon_o} \frac{1}{2a}$$

$$W = \frac{Q^2}{16a\pi\epsilon_o}$$

Prob. 5.23

(a)

$$\rho_v = \begin{cases} \rho_o, & 0 < r < a \\ 0, & r > a \end{cases}$$

$$\text{For } r < a, \quad \epsilon E_r (4\pi r^2) = \rho_o \frac{4\pi r^3}{3} \longrightarrow E_r = \frac{\rho_o r}{3\epsilon}$$

$$V = - \int \mathbf{E} \bullet d\mathbf{l} = - \frac{\rho_o r^2}{6\epsilon} + c_1$$

$$\text{For } r > a, \quad \epsilon_o E_r (4\pi r^2) = \rho_o \frac{4\pi a^3}{3} \longrightarrow E_r = \frac{\rho_o a^3}{3\epsilon_o r^2}$$

$$V = - \int \mathbf{E} \bullet d\mathbf{l} = \frac{\rho_o a^3}{3\epsilon_o r} + c_2$$

As $r \rightarrow \infty, \quad V = 0$ and $c_2 = 0$ At $r = a, \quad V(a^+) = V(a^-)$

$$-\frac{\rho_o a^2}{6\epsilon_o \epsilon_r} + c_1 = \frac{\rho_o a^2}{3\epsilon_o} \longrightarrow c_1 = \frac{\rho_o a^2}{6\epsilon_o \epsilon_r} (2\epsilon_r + 1)$$

$$V(r=0) = c_1 = \frac{\rho_o a^2 (2\epsilon_r + 1)}{6\epsilon_o \epsilon_r}$$

$$(b) \quad V(r=a) = \frac{\rho_o a^2}{3\epsilon_o}$$

Prob. 5.24

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \epsilon_o E_o \begin{bmatrix} 4 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$D_x = \epsilon_o E_o (4+1-1) = 4\epsilon_o E_o$$

$$D_y = \epsilon_o E_o (1+3-1) = 3\epsilon_o E_o$$

$$D_z = \epsilon_o E_o (1+1-2) = 0$$

$$D = \underline{\epsilon_o E_o (4a_x + 3a_y) \text{ C/m}^2}$$

Prob. 5.25

Since $\frac{\partial \rho_v}{\partial t} = 0$, $\nabla \bullet J = 0$ must hold.

$$(a) \quad \nabla \bullet J = 6x^2y + 0 - 6x^2y = 0 \longrightarrow \text{This is possible.}$$

$$(b) \quad \nabla \bullet J = y + (z+1) \neq 0 \longrightarrow \text{This is not possible.}$$

$$(c) \quad \nabla \bullet J = \frac{1}{\rho} \frac{\partial}{\partial \rho} (z^2) + \cos \phi \neq 0 \longrightarrow \text{This is not possible.}$$

$$(d) \quad \nabla \bullet J = \frac{1}{r^2} \frac{\partial}{\partial r} (\sin \theta) = 0 \longrightarrow \text{This is possible.}$$

Prob. 5.26

$$\nabla \bullet J = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = 2e^{-2y} \cos 2x - 2e^{-2y} \cos 2x + 1 = 1 = -\frac{\partial \rho_v}{\partial t}$$

Hence, $\frac{\partial \rho_v}{\partial t} = \underline{\underline{-1 \text{ C/m}^3 \cdot s}}$

Prob. 5.27

$$(a) \quad \nabla \bullet J = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\frac{100}{\rho} \right) = -\frac{100}{\rho^3}$$

$$-\frac{\partial \rho_v}{\partial t} = \nabla \bullet J = -\frac{100}{\rho^3} \quad \longrightarrow \quad \frac{\partial \rho_v}{\partial t} = \underline{\underline{\frac{100}{\rho^3} \text{ C/m}^3 \cdot s}}$$

$$(b) \quad I = \int J \bullet dS = \iint \frac{100}{\rho^2} \rho d\phi dz \Big|_{\rho=2} = \frac{100}{2} \int_0^{2\pi} d\phi \int_0^1 dz = 100\pi = \underline{\underline{314.16 \text{ A}}}$$

Prob. 5.28

$$T_r = \frac{\epsilon}{\sigma} = \frac{2.5 \times 10^{-9}}{5 \times 10^{-6} \times 36\pi} = 4.42 \mu\text{s}$$

$$\rho_{vo} = \frac{Q}{V} = \frac{1}{\frac{4\pi}{3} \times 10^{-6} \times 8} = \underline{\underline{29.84 \text{ kC/m}^3}}$$

$$\rho_v = \rho_{vo} e^{-t/T_r} = 29.84 e^{-2/4.42} = \underline{\underline{18.98 \text{ kC/m}^3}}$$

Prob. 5.29

$$-\frac{\partial \rho_v}{\partial t} = \nabla \bullet J = \frac{\partial J_x}{\partial x} = 0.5\pi \cos \pi x$$

At P(2,4,-3), x = 2

$$\frac{\partial \rho_v}{\partial t} = -0.5\pi \cos(2\pi) = -0.5\pi = \underline{\underline{-1.571 \text{ C/m}^2 \cdot s}}$$

Prob. 5.30

(a)

$$\frac{\epsilon}{\sigma} = \frac{3.1 \times \frac{10^{-9}}{36\pi}}{10^{-15}} = \underline{\underline{2.741 \times 10^4 \text{ s}}}$$

$$(b) \quad \frac{\epsilon}{\sigma} = \frac{6 \times \frac{10^{-9}}{36\pi}}{10^{-15}} = \underline{\underline{5.305 \times 10^4 \text{ s}}}$$

$$(c) \quad \frac{\epsilon}{\sigma} = \frac{80 \times \frac{10^{-9}}{36\pi}}{10^{-4}} = \underline{\underline{7.07 \mu s}}$$

Prob. 5.31

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} \quad \longrightarrow \quad \rho_v = -\int \nabla \cdot \mathbf{J} dt$$

$$\nabla \cdot \mathbf{J} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 J_r) = \frac{1}{r^2} \frac{\partial}{\partial r} (0.5r \exp(-10^4 t)) = \frac{0.5}{r^2} \exp(-10^4 t)$$

$$\rho_v = -\int \nabla \cdot \mathbf{J} dt = \frac{0.5}{r^2 10^4} \exp(-10^4 t) + C$$

If $\rho_v \rightarrow 0$ as $t \rightarrow \infty$, $C = 0$.

$$\rho_v = \underline{\underline{\frac{50}{r^2} \exp(-10^4 t) \mu C/m^3}}$$

Prob. 5.32

$$\begin{aligned} P_1 &= \chi_{el} \epsilon_o E_1 = \chi_{el} \epsilon_o \frac{D_1}{\epsilon_o \epsilon_{r1}} = \frac{4-1}{4} D_1 = \frac{3}{4} D_1 \\ &= 12a_x + 22.5a_y - 15a_z \text{ nC/m}^2 \end{aligned}$$

$$D_{2n} = D_{1n} = -20a_z$$

$$E_{2t} = E_{1t} \quad \longrightarrow \quad \frac{D_{2t}}{\epsilon_2} = \frac{D_{1t}}{\epsilon_1}$$

$$D_{2t} = \frac{\epsilon_2}{\epsilon_1} D_{1t} = \frac{6.5\epsilon_o}{4\epsilon_o} (16a_x + 30a_y) = 26a_x + 48.75a_y$$

$$D_2 = D_{2n} + D_{2t} = \underline{\underline{26a_x + 48.75a_y - 20a_z \text{ nC/m}^2}}$$

Prob. 5.33

Let $x > 0$ be region 1 and $x < 0$ be region 2.

$$D_{1n} = 50a_x, \quad D_{1t} = 80a_y - 30a_z$$

$$D_{2n} = D_{1n} = 50a_x$$

$$E_{2t} = E_{1t} \quad \longrightarrow \quad \frac{D_{2t}}{\epsilon_2} = \frac{D_{1t}}{\epsilon_1}$$

$$D_{2t} = \frac{\epsilon_2}{\epsilon_1} D_{1t} = \frac{7.6}{2.1} (80a_y - 30a_z) = 289.5a_y - 108.6a_z$$

$$D_2 = D_{2t} + D_{2n} = \underline{\underline{50a_x + 289.5a_y - 108.6a_z \text{ nC/m}^2}}$$

Prob. 5.34

$$f(x,y) = 4x + 3y - 10 = 0$$

$$\nabla f = 4\mathbf{a}_x + 3\mathbf{a}_y \quad \longrightarrow \quad \mathbf{a}_n = -\frac{\nabla f}{|\nabla f|} = \frac{-(4\mathbf{a}_x + 3\mathbf{a}_y)}{5} = -0.8\mathbf{a}_x - 0.6\mathbf{a}_y$$

The minus sign is chosen for \mathbf{a}_n because it is directed toward the origin.

$$\mathbf{D}_{1n} = (\mathbf{D}_1 \cdot \mathbf{a}_n) \mathbf{a}_n = (1.6 - 2.4) \mathbf{a}_n = -0.64\mathbf{a}_x - 0.48\mathbf{a}_y$$

$$\mathbf{D}_{1t} = \mathbf{D}_1 - \mathbf{D}_{1n} = 2.64\mathbf{a}_x - 3.52\mathbf{a}_y + 6.5\mathbf{a}_z$$

$$\mathbf{D}_{2n} = \mathbf{D}_{1n} = -0.64\mathbf{a}_x - 0.48\mathbf{a}_y$$

$$\mathbf{E}_{2t} = \mathbf{E}_{1t} \quad \longrightarrow \quad \frac{\mathbf{D}_{2t}}{\epsilon_2} = \frac{\mathbf{D}_{1t}}{\epsilon_2}$$

$$\mathbf{D}_{2t} = \frac{\epsilon_2}{\epsilon_1} \mathbf{D}_{1t} = \frac{2.5}{1} (2.64, -3.52, 6.5) = (6.6, -8.8, 16.25)$$

$$\mathbf{D}_2 = \mathbf{D}_{2n} + \mathbf{D}_{2t} = \underline{\underline{5.96\mathbf{a}_x - 9.28\mathbf{a}_y + 16.25\mathbf{a}_z \text{ nC/m}^2}}$$

$$\theta_2 = \cos^{-1} \frac{\mathbf{D}_2 \cdot \mathbf{a}_n}{|\mathbf{D}_2|} = \underline{\underline{87.66^\circ}}$$

Prob. 5.35

$$(a) \quad \mathbf{P}_1 = \epsilon_0 \chi_{e1} \mathbf{E}_1 = 2 \times \frac{10^{-9}}{36\pi} (10, -6, 12) = \underline{\underline{0.1768\mathbf{a}_x - 0.1061\mathbf{a}_y + 0.2122\mathbf{a}_z \text{ nC/m}^2}}$$

$$(b) \quad \mathbf{E}_{1n} = -6\mathbf{a}_y, \quad \mathbf{E}_{2t} = \mathbf{E}_{1t} = 10\mathbf{a}_x + 12\mathbf{a}_z$$

$$\mathbf{D}_{2n} = \mathbf{D}_{1n} \quad \longrightarrow \quad \epsilon_2 \mathbf{E}_{2n} = \epsilon_1 \mathbf{E}_{1n}$$

$$\text{or} \quad \mathbf{E}_{2n} = \frac{\epsilon_1}{\epsilon_2} \mathbf{E}_{1n} = \frac{3\epsilon_0}{4.5\epsilon_0} (-6\mathbf{a}_z) = -4\mathbf{a}_y$$

$$\mathbf{E}_2 = \underline{\underline{10\mathbf{a}_x - 4\mathbf{a}_y + 12\mathbf{a}_z \text{ V/m}}}$$

$$\tan \theta_2 = \frac{E_{2t}}{E_{2n}} = \frac{\sqrt{10^2 + 12^2}}{4} = 3.905 \quad \longrightarrow \quad \underline{\underline{\theta_2 = 75.64^\circ}}$$

$$(c) \quad w_E = \frac{1}{2} \mathbf{D} \bullet \mathbf{E} = \frac{1}{2} \epsilon_0 |\mathbf{E}|^2$$

$$w_{E1} = \frac{1}{2} \epsilon_0 |\mathbf{E}_1|^2 = \frac{1}{2} \times 3 \times \frac{10^{-9}}{36\pi} (10^2 + 6^2 + 12^2) = \underline{\underline{3.7136 \text{ nJ/m}^3}}$$

$$w_{E2} = \frac{1}{2} \epsilon_0 |\mathbf{E}_2|^2 = \frac{1}{2} \times 4.5 \times \frac{10^{-9}}{36\pi} (10^2 + 4^2 + 12^2) = \underline{\underline{5.1725 \text{ nJ/m}^3}}$$

Prob. 5.36 (a) $\mathbf{D}_{2n} = 12\mathbf{a}_\rho = \mathbf{D}_{1n}$, $\mathbf{D}_{2t} = -6\mathbf{a}_\phi + 9\mathbf{a}_z$

$$\mathbf{E}_{2t} = \mathbf{E}_{2t} \quad \longrightarrow \quad \frac{\mathbf{D}_{1t}}{\epsilon_0} = \frac{\mathbf{D}_{2t}}{\epsilon_2}$$

$$\mathbf{D}_{1t} = \frac{\epsilon_0}{\epsilon_2} \mathbf{D}_{2t} = \frac{3.5\epsilon_0}{1.5\epsilon_0} (-6\mathbf{a}_\phi + 9\mathbf{a}_z) = -14\mathbf{a}_\phi + 21\mathbf{a}_z$$

$$\mathbf{D}_1 = 12\mathbf{a}_\rho - 14\mathbf{a}_\phi + 21\mathbf{a}_z \quad \text{nC/m}^2$$

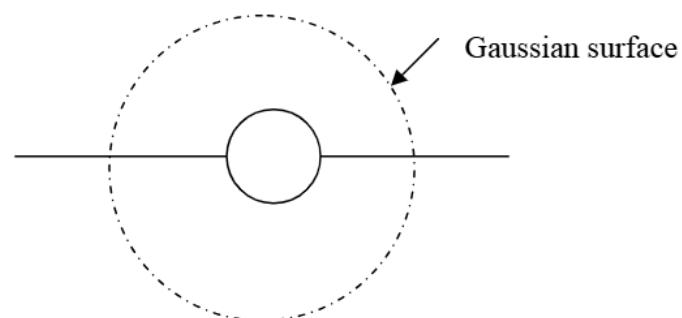
$$\mathbf{E}_1 = \mathbf{D}_1 / \epsilon_0 = \frac{(12, -14, 21) \times 10^{-9}}{3.5 \times \frac{10^{-9}}{36\pi}} = 387.8\mathbf{a}_\rho - 452.4\mathbf{a}_\phi + 678.6\mathbf{a}_z$$

$$(b) \quad \mathbf{P}_2 = \epsilon_0 \chi_{e2} \mathbf{E}_2 = 0.5\epsilon_0 \frac{\mathbf{D}_2}{\epsilon_2} = \frac{0.5\epsilon_0}{1.5\epsilon_0} (12, -6, 9) = \underline{\underline{4\mathbf{a}_\rho - 2\mathbf{a}_\phi + 3\mathbf{a}_z \text{ nC/m}^2}}$$

$$\rho_{v2} = \nabla \bullet \mathbf{P}_2 = 0$$

$$(c) \quad w_{E1} = \frac{1}{2} \mathbf{D}_1 \bullet \mathbf{E}_1 = \frac{1}{2} \frac{\mathbf{D}_1 \bullet \mathbf{D}_1}{\epsilon_0 \epsilon_{r1}} = \frac{1}{2} \frac{(12^2 + 14^2 + 21^2) \times 10^{-18}}{3.5 \times \frac{10^{-9}}{36\pi}} = \underline{\underline{12.62 \mu\text{J/m}^2}}$$

$$w_{E2} = \frac{1}{2} \frac{\mathbf{D}_2 \bullet \mathbf{D}_2}{\epsilon_0 \epsilon_{r2}} = \frac{1}{2} \frac{(12^2 + 6^2 + 9^2) \times 10^{-18}}{1.5 \times \frac{10^{-9}}{36\pi}} = \underline{\underline{9.839 \mu\text{J/m}^2}}$$

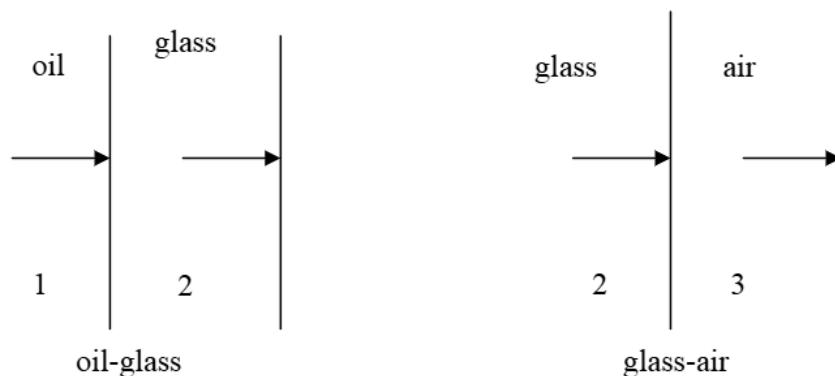
Prob. 5.37

$$Q = \int \mathbf{D} \cdot d\mathbf{S} = \epsilon_1 E_r \frac{4\pi r^2}{2} + \epsilon_2 E_r \frac{4\pi r^2}{2} = 2\pi r^2 (\epsilon_1 + \epsilon_2) E_r$$

$$E_r = \begin{cases} \frac{Q}{2\pi(\epsilon_1 + \epsilon_2)r^2}, & r > a \\ 0, & r < a \end{cases}$$

Prob. 5.38

- (a) The two interfaces are shown below



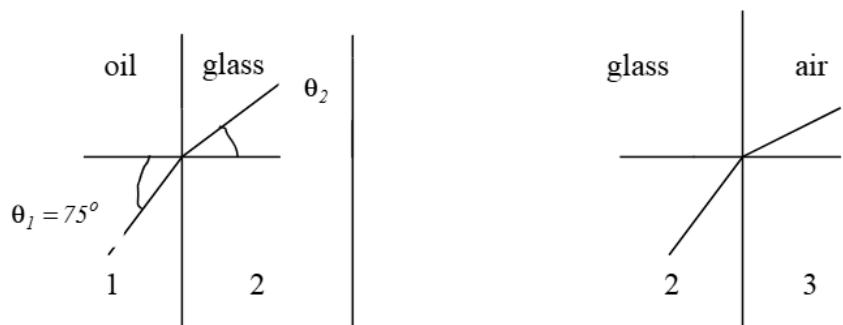
$$E_{1n} = 2000, \quad E_{1t} = 0 = E_{2t} = E_{3t}$$

$$D_{1n} = D_{2n} = D_{3n} \quad \longrightarrow \quad \epsilon_1 E_{1n} = \epsilon_2 E_{2n} = \epsilon_3 E_{3n}$$

$$E_{2n} = \frac{\epsilon_1}{\epsilon_2} E_{1n} = \frac{3.0}{8.5} (2000) = \underline{\underline{705.9 \text{ V/m}, \theta_2 = 0^\circ}}$$

$$E_{3n} = \frac{\epsilon_1}{\epsilon_3} E_{1n} = \frac{3.0}{1.0} (2000) = \underline{\underline{6000 \text{ V/m}, \theta_3 = 0^\circ}}$$

(b)



$$E_{1n} = 2000 \cos 75^\circ = 517.63, \quad E_{1t} = 2000 \sin 75^\circ = E_{2t} = E_{3t} = 1931.85$$

$$E_{2n} = \frac{\epsilon_1}{\epsilon_2} E_{1n} = \frac{3}{8.5} (517.63) = 182.7, \quad E_{3n} = \frac{\epsilon_1}{\epsilon_3} E_{1n} = \frac{3}{1} (517.63) = 1552.89$$

$$E_2 = \sqrt{E_{2n}^2 + E_{2t}^2} = 1940.5, \quad \theta_2 = \tan^{-1} \frac{E_{2t}}{E_{2n}} = \underline{\underline{84.6^\circ}}$$

$$E_3 = \sqrt{E_{3n}^2 + E_{3t}^2} = 2478.6, \quad \theta_3 = \tan^{-1} \frac{E_{3t}}{E_{3n}} = \underline{\underline{51.2^\circ}}$$

Prob. 5.39

$$\begin{aligned} \rho_s &= D_n = \epsilon_0 E = \frac{10^{-9}}{36\pi} \sqrt{30^2 + 40^2 + 20^2} \times 10^{-3} = \frac{\sqrt{2900}}{36\pi} \text{ pC/m}^2 \\ &= \underline{\underline{0.476 \text{ pC/m}^2}} \end{aligned}$$

$$\textbf{Prob. 5.40 (a)} \quad \rho_s = D_n = \epsilon_0 E_n = \frac{10^{-9}}{36\pi} \sqrt{15^2 + 8^2} = \underline{\underline{0.1503 \text{ nC/m}^2}}$$

$$\text{(b)} \quad D_n = \rho_s = -20 \text{ nC}$$

$$D = D_n a_n = (-20 \text{ nC})(-a_y) = \underline{\underline{20a_y \text{ nC/m}^2}}$$

Prob. 5.41

At the interface between ϵ_0 and $2\epsilon_0$,

$$E_{1n} = E_o \cos 30^\circ, \quad E_{1t} = E_o \sin 30^\circ$$

$$E_{2t} = E_{1t} = 0.5E_o$$

$$D_{2n} = D_{1n} \quad \longrightarrow \quad \epsilon_1 E_{1n} = \epsilon_2 E_{2n}$$

$$E_{2n} = \frac{\epsilon_1}{\epsilon_2} E_{1n} = \frac{\epsilon_0}{2\epsilon_0} (0.866E_o) = 0.433E_o$$

The angle **E** makes with the z-axis is

$$\theta_1 = \tan^{-1} \frac{E_{2t}}{E_{2n}} = \tan^{-1} \frac{0.5}{0.433} = \underline{\underline{49.11^\circ}}$$

At the interface between $2\epsilon_0$ and $3\epsilon_0$,

$$E_{3t} = E_{2t} = 0.5E_o$$

$$D_{3n} = D_{2n} \quad \longrightarrow \quad E_{3n} = \frac{\epsilon_2}{\epsilon_3} E_{2n} = \frac{2\epsilon_0}{3\epsilon_0} (0.433E_o) = 0.2887E_o$$

The angle **E** makes with the z-axis is

$$\theta_2 = \tan^{-1} \frac{E_{3t}}{E_{3n}} = \tan^{-1} \frac{0.5}{0.2887} = \underline{\underline{60^\circ}}$$

At the interface between $3\epsilon_0$ and ϵ_0 ,

$$E_{4t} = E_{3t} = 0.5E_o$$

$$D_{4n} = D_{3n} \quad \longrightarrow \quad E_{4n} = \frac{\epsilon_3}{\epsilon_4} E_{3n} = \frac{3\epsilon_0}{\epsilon_0} (0.2887E_o) = 0.866E_o$$

The angle **E** makes with the z-axis is

$$\theta_3 = \tan^{-1} \frac{E_{4t}}{E_{4n}} = \tan^{-1} \frac{0.5}{0.866} = \underline{\underline{30^\circ}}$$

CHAPTER 6

P. E. 6.1

$$\nabla^2 V = -\frac{\rho}{\epsilon} \longrightarrow \frac{d^2 V}{dx^2} = -\frac{\rho_o x}{\epsilon a}$$

$$V = -\frac{\rho_o x^3}{6\epsilon a} + Ax + B$$

$$\mathbf{E} = -\frac{dV}{dx} \mathbf{a}_x = \left(\frac{\rho_o x^2}{2\epsilon a} - A \right) \mathbf{a}_x$$

If $\mathbf{E} = 0$ at $x = 0$, then

$$0 = 0 - A \longrightarrow A = 0$$

If $V = 0$ at $x = a$, then

$$0 = -\frac{\rho_o a^3}{6\epsilon a} + B \longrightarrow B = \frac{\rho_o a^2}{6\epsilon}$$

Thus

$$\underline{\underline{V = \frac{\rho_o}{6\epsilon a} (a^3 - x^3)}}, \quad \underline{\underline{\mathbf{E} = \frac{\rho_o x^2}{2\epsilon a} \mathbf{a}_x}}$$

P. E. 6.2 $V_1 = A_1 x + B_1, \quad V_2 = A_2 x + B_2$

$$V_1(x = d) = V_o = A_1 d + B_1 \longrightarrow B_1 = V_o - A_1 d$$

$$V_1(x = 0) = 0 = 0 + B_2 \longrightarrow B_2 = 0$$

$$V_1(x = a) = V_2(x = a) \longrightarrow aA_1 + B_1 = A_2 a$$

$$D_{1n} = D_{2n} \longrightarrow \epsilon_1 A_1 = \epsilon_2 A_2 \longrightarrow A_2 = \frac{\epsilon_1}{\epsilon_2} A_1$$

$$A_1 a + V_o - A_1 d = \frac{\epsilon_1}{\epsilon_2} a A_1 \longrightarrow V_o = A_1 \left(-a + d + \frac{\epsilon_1}{\epsilon_2} a \right)$$

or

$$A_1 = \frac{V_o}{d - a + \epsilon_1 a / \epsilon_2}, \quad A_2 = \frac{\epsilon_1}{\epsilon_2} A_1 \frac{\epsilon_1 V_o}{\epsilon_2 d - \epsilon_2 a + \epsilon_1 a}$$

Hence

$$\underline{\underline{\mathbf{E}_1 = -A_1 \mathbf{a}_x = \frac{-V_o \mathbf{a}_x}{d - a + \epsilon_1 a / \epsilon_2}}}, \quad \underline{\underline{\mathbf{E}_2 = -A_2 \mathbf{a}_x = \frac{-V_o \mathbf{a}_x}{a + \epsilon_2 d / \epsilon_1 - \epsilon_2 a / \epsilon_1}}}$$

P. E. 6.3 From Example 6.3,

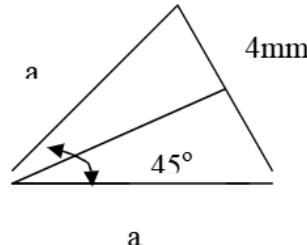
$$\mathbf{E} = -\frac{V_o}{\rho \phi_o} \mathbf{a}_\phi, \quad D = \epsilon_o E$$

$$\rho_s = D_n(\phi = 0) = -\frac{V_o \epsilon}{\rho \phi_o}$$

The charge on the plate $\phi = 0$ is

$$Q = \int \rho_s dS = -\frac{V_o \epsilon}{\phi_o} \int_{z=0}^L \int_{\rho=a}^b \frac{1}{\rho} dz d\rho = -\frac{V_o \epsilon}{\phi_o} L \ln(b/a)$$

$$C = \frac{|Q|}{V_o} = \frac{\epsilon L}{\phi_o} \ln \frac{b}{a}$$



$$a \sin \frac{45^\circ}{2} = 2 \quad \longrightarrow \quad a = \frac{2}{\sin 22.5^\circ} = 5.226 \text{ mm}$$

$$C = \frac{1.5 \times 10^{-9}}{\frac{\pi}{4}} 5 \ln \frac{1000}{5.226} = 444 \text{ pF}$$

$$Q = CV_o = 444 \times 10^{-12} \times 50 \text{ C} = \underline{\underline{22.2 \text{ nC}}}$$

P. E. 6.4 From Example 6.4,

$$V_o = 50, \quad \theta_2 = 45^\circ, \quad \theta_1 = 90^\circ, \quad r = \sqrt{3^2 + 4^2 + 2^2} = \sqrt{29}, \quad \theta = \tan^{-1} \frac{\rho}{z} =$$

$$\tan^{-1} \frac{5}{2} \quad \longrightarrow \quad \theta = 68.2^\circ; \quad \tan 45^\circ = 1$$

$$V = \frac{50 \ln(\tan 34.1^\circ)}{\ln(\tan 22.5^\circ)} = \underline{\underline{22.125 \text{ V}}},$$

$$\mathbf{E} = \frac{50 \mathbf{a}_\theta}{\sqrt{29} \sin 68.2^\circ \ln(\tan 22.5^\circ)} = \underline{\underline{11.35 \mathbf{a}_\theta \text{ V/m}}}$$

P. E. 6.5

$$\begin{aligned} \mathbf{E} &= -\nabla V = -\frac{\partial V}{\partial x} \mathbf{a}_x - \frac{\partial V}{\partial y} \mathbf{a}_y \\ &= -\frac{4V_o}{b} \sum_{n=\text{odd}}^{\infty} \frac{1}{\sinh n\pi a/b} [\cos(n\pi x/b) \sinh(n\pi y/b) \mathbf{a}_x + \sin(n\pi x/b) \cosh(n\pi y/b) \mathbf{a}_y] \end{aligned}$$

(a) At $(x, y) = (a, a/2)$,

$$V = \frac{400}{\pi} (0.3775 - 0.0313 + 0.00394 - 0.000585 + \dots) = \underline{\underline{44.51 \text{ V}}}$$

$$\begin{aligned} \mathbf{E} &= 0 \mathbf{a}_x + (-115.12 + 19.127 - 3.9411 + 0.8192 - 0.1703 + 0.035 - 0.0074 + \dots) \mathbf{a}_y \\ &= \underline{\underline{-99.25 \mathbf{a}_y \text{ V/m}}} \end{aligned}$$

(b) At $(x, y) = (3a/2, a/4)$,

$$V = \frac{400}{\pi} (0.1238 + 0.006226 - 0.00383 + 0.0000264 + \dots) = \underline{\underline{16.50 \text{ V}}}$$

$$\begin{aligned} \mathbf{E} &= (24.757 - 3.7358 - 0.3834 + 0.0369 + 0.00351 - 0.00033 + \dots) \mathbf{a}_x \\ &\quad + (-66.25 - 4.518 + 0.3988 + 0.03722 - 0.00352 - 0.000333 + \dots) \mathbf{a}_y \\ &= \underline{\underline{20.68 \mathbf{a}_x - 70.34 \mathbf{a}_y \text{ V/m}}} \end{aligned}$$

P. E. 6.6

$$V(y=a) = V_o \sin(7\pi x/b) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/b) \sinh(n\pi a/b)$$

By equating coefficients, we notice that $c_n = 0$ for $n \neq 7$. For $n=7$,

$$V_o \sin(7\pi x/b) = c_7 \sin(7\pi x/b) \sinh(7\pi a/b) \longrightarrow c_7 = \frac{V_o}{\sinh(7\pi a/b)}$$

Hence

$$V(x, y) = \frac{V_o}{\sinh(7\pi a/b)} \sin(7\pi x/b) \sinh(7\pi y/b)$$

P. E. 6.7 Let $V(r, \theta, \phi) = R(r)F(\theta)\Phi(\phi)$.

Substituting this in Laplace's equation gives

$$\frac{\Phi F}{r^2} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + \frac{R\Phi}{r^2 \sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{dF}{d\theta} \right) + \frac{RF}{r^2 \sin^2 \theta} \frac{d^2 \Phi}{d\phi^2} = 0$$

Dividing by $RF\Phi / r^2 \sin^2 \theta$ gives

$$\frac{\sin^2 \theta}{R} \frac{d}{dr} (r^2 R') + \frac{\sin \theta}{F} \frac{d}{d\theta} (\sin \theta F') = -\frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} = \lambda^2$$

$$\underline{\underline{\Phi'' + \lambda^2 \Phi = 0}}$$

$$\frac{1}{R} \frac{d}{dr} (r^2 R') + \frac{1}{F \sin \theta} \frac{d}{d\theta} (\sin \theta F') = \lambda^2 / \sin^2 \theta$$

$$\frac{1}{R} \frac{d}{dr} (r^2 R') = \frac{\lambda^2}{\sin^2 \theta} - \frac{1}{F \sin \theta} \frac{d}{d\theta} (\sin \theta F') = \mu^2$$

$$2rR' + r^2 R'' = \mu^2 R$$

or

$$\underline{\underline{R'' + \frac{2}{r} R' - \frac{\mu^2}{r^2} R = 0}}$$

$$\frac{\sin \theta}{F} \frac{d}{d\theta} (\sin \theta F') - \lambda^2 + \mu^2 \sin^2 \theta = 0$$

or

$$\underline{\underline{F'' + \cos \theta F' + (\mu^2 \sin \theta - \lambda^2 \csc \theta) F = 0}}$$

P. E. 6.8 (a) This is similar to Example 6.8(a) except that here $0 < \phi < 2\pi$ instead of $0 < \phi < \pi/2$. Hence

$$I = \frac{2\pi t V_o \sigma}{\ln(b/a)} \quad \text{and} \quad R = \frac{V_o}{I} = \frac{\ln \frac{b}{a}}{2\pi t \sigma}$$

(b) This is similar to Example 6.8(b) except that here $0 < \phi < 2\pi$. Hence

$$I = \frac{V_o \sigma}{t} \int_a^b \int_0^{2\pi} \rho d\rho d\phi = \frac{V_o \sigma \pi (b^2 - a^2)}{t}$$

$$\text{and } R = \frac{V_o}{I} = \frac{t}{\sigma \pi (b^2 - a^2)}$$

P. E. 6.9 From Example 6.9,

$$J_1 = \frac{\sigma_1 V_o}{\rho \ln \frac{b}{a}}, \quad J_2 = \frac{\sigma_2 V_o}{\rho \ln \frac{b}{a}}$$

$$I = \int J \bullet dS = \int_{z=0}^L \left[\int_{\phi=0}^{\pi} J_1 \rho d\phi + \int_{\phi=\pi}^{2\pi} J_2 \rho d\phi \right] dz = \frac{V_o l}{\ln \frac{b}{a}} [\pi \sigma_1 + \pi \sigma_2]$$

$$R = \frac{V_o}{I} = \frac{\ln \frac{b}{a}}{\pi l [\sigma_1 + \sigma_2]}$$

P. E. 6.10 (a) $C = \frac{4\pi\epsilon}{\frac{1}{a} - \frac{1}{b}}$, C_1 and C_2 are in series.

$$C_1 = 4\pi x \frac{10^{-9}}{36\pi} \left(\frac{2.5}{\frac{10^3}{2} - \frac{10^3}{3}} \right) = 5/3 \text{ pF}, \quad C_2 = 4\pi x \frac{10^{-9}}{36\pi} \left(\frac{3.5}{\frac{10^3}{1} - \frac{10^3}{2}} \right) = 7/9 \text{ pF}$$

$$C = \frac{C_1 C_2}{C_1 + C_2} = \frac{(5/3)(7/9)}{(5/3) + (7/9)} = \underline{\underline{0.53 \text{ pF}}}$$

(b) $C = \frac{2\pi\epsilon}{\frac{1}{a} - \frac{1}{b}}$, C_1 and C_2 are in parallel.

$$C_1 = 2\pi \times \frac{10^{-9}}{36\pi} \left(\frac{2.5}{\frac{10^3}{1} - \frac{10^3}{3}} \right) = 5/24 \text{ pF}, \quad C_2 = 2\pi \times \frac{10^{-9}}{36\pi} \left(\frac{3.5}{\frac{10^3}{1} - \frac{10^3}{3}} \right) = 7/24 \text{ pF}$$

$$C = C_1 + C_2 = \underline{\underline{0.5 \text{ pF}}}$$

P. E. 6.11 As in Example 6.8, assuming $V(\rho = a) = 0$, $V(\rho = b) = V_o$,

$$V = V_o \frac{\ln \rho / a}{\ln b / a}, \quad E = -\nabla V = -\frac{V_o}{\rho \ln b / a} a_\rho$$

$$Q = \int \epsilon E \bullet dS = \frac{V_o \epsilon}{\ln b / a} \int_{z=0}^L \int_{\phi=0}^{2\pi} \frac{1}{\rho} dz \rho d\phi = \frac{V_o 2\pi \epsilon L}{\ln b / a}$$

$$C = \frac{Q}{V_o} = \frac{2\pi \epsilon L}{\underline{\ln b / a}}$$

P. E. 6.12

(a) Let C_1 and C_2 be capacitances per unit length of each section and C_T be the total capacitance of 10m length. C_1 and C_2 are in series.

$$C_1 = \frac{2\pi \epsilon_{r1} \epsilon_o}{\ln b / c} = \frac{2\pi x 2.5}{\ln 3 / 2} \frac{10^{-9}}{36\pi} = 342.54 \text{ pF/m},$$

$$C_2 = \frac{2\pi \epsilon_{r2} \epsilon_o}{\ln c / a} = \frac{2\pi x 3.5}{\ln 2} \frac{10^{-9}}{36\pi} = 280.52 \text{ pF/m}$$

$$C = \frac{C_1 C_2}{C_1 + C_2} = \frac{342.54 \times 280.52}{342.54 + 280.52} = 154.22 \text{ pF}$$

$$C_T = Cl = \underline{\underline{1.54}} \text{ nF}$$

(b) C_1 and C_2 are in parallel.

$$C = C_1 + C_2 = \frac{\pi \epsilon_{r1} \epsilon_o}{\ln b / a} + \frac{\pi \epsilon_{r2} \epsilon_o}{\ln b / a} = \frac{\pi (\epsilon_{r1} + \epsilon_{r2}) \epsilon_o}{\ln b / a} = \frac{6\pi}{\ln 3} \frac{10^{-9}}{36\pi} = 151.7 \text{ pF/m}$$

$$C_T = Cl = \underline{\underline{1.52}} \text{ nF}$$

P. E. 6.13 Instead of Eq. (6.31), we now have

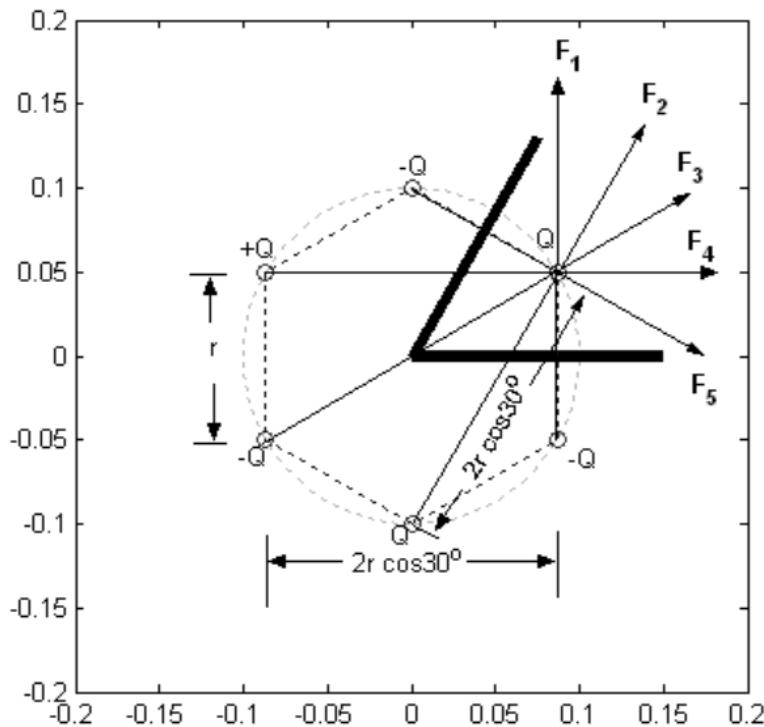
$$V = - \int_b^a \frac{Q dr}{4\pi \epsilon r^2} = - \int_b^a \frac{Q dr}{4\pi \frac{10\epsilon_o}{r} r^2} = - \frac{Q}{40\pi \epsilon_o} \ln b / a$$

$$C = \frac{Q}{|V|} = \frac{40\pi}{\ln 4 / 1.5} \frac{10^{-9}}{36\pi} = \underline{\underline{1.13 \text{ nF}}}$$

P. E. 6.14 Let

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4 + \mathbf{F}_5$$

where \mathbf{F}_i , $i=1,2,\dots,5$ are shown on in the figure below.



$$\begin{aligned}
 \mathbf{F} &= -\frac{Q^2}{4\pi\epsilon_0 r^2} \mathbf{a}_y + \frac{Q^2(\mathbf{a}_x \sin 30^\circ + \mathbf{a}_y \cos 30^\circ)}{4\pi\epsilon_0 (2r \cos 30^\circ)^2} - \frac{Q^2(\mathbf{a}_x \cos 30^\circ + \mathbf{a}_y \sin 30^\circ)}{4\pi\epsilon_0 (2r)^2} + \frac{Q^2 \mathbf{a}_x}{4\pi\epsilon_0 (2r \cos 30^\circ)^2} \\
 &\quad - \frac{Q^2(\mathbf{a}_x \cos 30^\circ - \mathbf{a}_y \sin 30^\circ)}{4\pi\epsilon_0 r^2} \\
 &= \frac{Q^2}{4\pi\epsilon_0 r^2} \left[-\mathbf{a}_y + \frac{1}{3} \left(\frac{\mathbf{a}_x}{2} + \frac{\sqrt{3}\mathbf{a}_y}{2} \right) - \frac{1}{4} \left(\frac{\sqrt{3}\mathbf{a}_x}{2} + \frac{\mathbf{a}_y}{2} \right) + \frac{1}{3} \mathbf{a}_x - \frac{\sqrt{3}\mathbf{a}_x}{2} + \frac{\mathbf{a}_y}{2} \right] \\
 &= 9 \times 10^{-5} \left[\mathbf{a}_x \left(\frac{1}{2} - \frac{5\sqrt{3}}{8} \right) + \mathbf{a}_y \left(\frac{-5}{8} + \frac{\sqrt{3}}{6} \right) \right] = -52.4279 \mathbf{a}_x - 30.27 \mathbf{a}_y \text{ } \mu\text{N}
 \end{aligned}$$

$$|\mathbf{F}| = 60.54 \text{ } \mu\text{N}$$

Note that the force tends to pull Q toward the origin.

P.E. 6.15

- (a) $E_1 = 1.35 \times 10^3$ V/mm and $E_2 = 1 \times 10^3$ V/mm
 (b) $E_1 = 1.72 \times 10^3$ V/mm and $E_2 = 0.59 \times 10^3$ V/mm

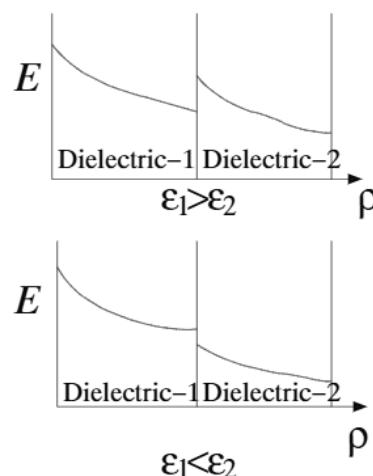


Figure. Electric field intensity distribution in a multi-dielectric coaxial capacitor

It can be noted from this practice exercise that, in a non-uniform field, the dielectric with higher permittivity should be placed in the high field region and the one with lower permittivity in the low field region (this is case-a). The difference in the average stress levels of the two dielectrics reduces. If the placement is done the other way round, the difference increases further (case-b). The electric field intensities in the two cases are plotted as a function of radius in the figure above.

P.E. 6.16

- (a) $V_2 = 66.66$ kV, breakdown voltage $= 6 \times 10 = 60$ kV, breakdown occurs
 (b) $V_2 = 85.71$ kV, breakdown voltage $= 9 \times 10 = 90$ kV, breakdown does not occur

Note: One has to properly size the thickness of dielectric layers with lower permittivity, as in case-b in this practice exercise.

P.E. 6.17

(a) For the parallel-plate capacitor,

$$\mathbf{E} = -\frac{V_o}{d} \mathbf{a}_x$$

From Example 6.11,

$$C = \frac{1}{V_o^2} \int \epsilon |E|^2 dv = \frac{1}{V_o^2} \int \epsilon \frac{V_o^2}{d^2} dv = \frac{\epsilon}{d^2} S d = \frac{\epsilon S}{d}$$

(b) For the cylindrical capacitor,

$$\mathbf{E} = -\frac{V_o}{\rho \ln b/a} \mathbf{a}_\rho$$

From Example 6.8,

$$C = \frac{1}{V_o^2} \iiint \frac{\epsilon V_o^2}{(\rho \ln b/a)^2} \rho d\rho d\phi dz = \frac{2\pi\epsilon L}{(\ln b/a)^2} \int_a^b \frac{d\rho}{\rho} = \frac{2\pi\epsilon L}{\ln b/a}$$

(c) For the spherical capacitor,

$$\mathbf{E} = \frac{V_o}{r^2(1/a-1/b)} \mathbf{a}_r$$

From Example 6.10,

$$C = \frac{1}{V_o^2} \iiint \frac{\epsilon V_o^2}{r^4(1/a-1/b)^2} r^2 \sin\theta d\theta dr d\phi = \frac{\epsilon}{(1/a-1/b)^2} 4\pi \int_a^b \frac{dr}{r^2} = \frac{4\pi\epsilon}{\frac{1}{a}-\frac{1}{b}}$$

Prob. 6.1

(a)

$$-\mathbf{E} = \nabla V = -\frac{20}{r^3} \cos \theta \sin \phi \mathbf{a}_r - \frac{10}{r^3} \sin \theta \sin \phi \mathbf{a}_\theta + \frac{1}{r \sin \theta} \frac{10}{r^2} \cos \theta \cos \phi \mathbf{a}_\phi$$

At P(1, 60°, 30°), r = 1, θ = 60°, φ = 30°

$$\begin{aligned}\mathbf{E} &= \frac{20}{1^3} \cos 60^\circ \sin 30^\circ \mathbf{a}_r + \frac{10}{1^3} \sin 60^\circ \sin 30^\circ \mathbf{a}_\theta - \frac{1}{\sin 60^\circ} \frac{10 \cos 60^\circ \cos 30^\circ}{1^3} \mathbf{a}_\phi \\ &= 5\mathbf{a}_r + 4.33\mathbf{a}_\theta - 5\mathbf{a}_\phi \text{ V/m}\end{aligned}$$

(b)

$$\begin{aligned}\nabla^2 V &= \frac{1}{r^2} \frac{\partial}{\partial r} \left(\frac{-20 \cos \theta \sin \phi}{r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\frac{-10 \sin^2 \theta \sin \phi}{r^2} \right) \\ &\quad - \frac{1}{r^2 \sin^2 \theta} \frac{10 \cos \theta \sin \phi}{r^2} \\ &= \frac{20 \cos \theta \sin \phi}{r^4} - \frac{20 \sin \theta \cos \theta \sin \phi}{r^4 \sin \theta} - \frac{10 \cos \theta \sin \phi}{r^4 \sin^2 \theta} \\ &= -\frac{10 \cos \theta \sin \phi}{r^4 \sin^2 \theta}\end{aligned}$$

$$\nabla^2 V = -\frac{\rho_v}{\epsilon} \longrightarrow \rho_v = -\epsilon \nabla^2 V = \frac{10 \epsilon_0 \cos \theta \sin \phi}{r^4 \sin^2 \theta}$$

At P, r = 1, θ = 60°, φ = 30°

$$\rho_v = 10 \times \frac{10^{-9}}{36\pi} \frac{\cos 60^\circ \sin 30^\circ}{1^4 \sin^2 60^\circ} = 29.47 \text{ pC/m}^3$$

Prob. 6.2

(a)

$$\begin{aligned} \mathbf{E} &= -\nabla V = -\left(\frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial z}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z\right) \\ &= -(15x^2y^2z\mathbf{a}_x + 10x^3yz\mathbf{a}_y + 5x^3y^2\mathbf{a}_z) \end{aligned}$$

At P, x=-3, y=1, z=2,

$$\mathbf{E} = -15(9)(1)(2)\mathbf{a}_x + 10(-27)(1)(2)\mathbf{a}_y - 5(-27)(1)\mathbf{a}_z = \underline{-270\mathbf{a}_x + 540\mathbf{a}_y + 135\mathbf{a}_z \text{ V/m}}$$

$$(b) \quad \rho_v = \nabla \cdot \mathbf{D} \quad \text{or} \quad \rho_v = -\epsilon \nabla^2 V$$

$$\begin{aligned} \nabla^2 V &= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{\partial}{\partial x}(15x^2y^2z) + \frac{\partial}{\partial y}(10x^3yz) + \frac{\partial}{\partial z}(5x^3y^2) \\ &= 30xy^2z + 10x^3z \end{aligned}$$

At P,

$$\rho_v = -\epsilon \nabla^2 V = -2.25 \times \frac{10^{-9}}{36\pi} [30(-3)(1)(2) + 10(-27)(2)] = \underline{14.324 \text{ nC/m}^3}$$

Prob. 6.3

$$(a) \quad \mathbf{E} = -\nabla V = -\frac{\partial V}{\partial \rho}\mathbf{a}_\rho - \frac{1}{\rho} \frac{\partial V}{\partial \phi}\mathbf{a}_\phi = \frac{\sin 3\phi}{\rho^2}\mathbf{a}_\rho - \frac{3 \cos 3\phi}{\rho^2}\mathbf{a}_\phi$$

At A, $\rho=1$, $\phi=20^\circ$, $z=4$,

$$\mathbf{E} = \frac{\sin 60^\circ}{1^2}\mathbf{a}_\rho - \frac{3 \cos 60^\circ}{1^2}\mathbf{a}_\phi = \underline{0.866\mathbf{a}_\rho - 1.5\mathbf{a}_\phi \text{ V/m}}$$

$$\mathbf{P} = \chi_e \epsilon_0 \mathbf{E} = 1.8 \times \frac{10^{-9}}{36\pi} (0.866\mathbf{a}_\rho - 1.5\mathbf{a}_\phi)$$

$$= \underline{13.783\mathbf{a}_\rho - 23.87\mathbf{a}_\phi \text{ pC/m}^2}$$

$$(b) \quad \rho_v = \nabla \cdot \mathbf{D} = \epsilon \nabla \cdot \mathbf{E} = \epsilon \left[\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\frac{\sin 3\phi}{\rho} \right) + \frac{9 \sin 3\phi}{\rho^3} \right]$$

$$\frac{\rho_v}{\epsilon} = \frac{-\sin 3\phi}{\rho^3} + \frac{9 \sin 3\phi}{\rho^3}$$

$$\rho_v = \frac{8\epsilon \sin 3\phi}{\rho^3}$$

$$\text{At A, } \rho_v = 8 \times 2.8 \times \frac{10^{-9}}{36\pi} \frac{\sin 60^\circ}{1^3} = \underline{171.52 \text{ pC/m}^3}$$

Prob. 6.4

$$\nabla^2 V = -\frac{\rho_v}{\epsilon}$$

$$\frac{d^2 V}{dy^2} = -\frac{y}{4\pi} \cdot \frac{10^{-9}}{4\epsilon_0} = -\frac{y}{4\pi} \cdot \frac{10^{-9}}{4 \cdot \frac{10^{-9}}{36\pi}} = -2.25y$$

$$\frac{dV}{dy} = -2.25 \frac{y^2}{2} + B$$

$$V = -0.375y^3 + By + C$$

$$V(1) = 0 = -0.375 + B + C \quad (1)$$

$$V(3) = 50 = -10.125 + 3B + C \quad (2)$$

From (1) and (2), $B=29.875$ and $C=-29.5$

$$V = -0.375y^3 + 29.875y - 29.5$$

$$V(2) = \underline{\underline{27.25 \text{ V}}}$$

Prob. 6.5

$$\nabla^2 V = -\frac{\rho_v}{\epsilon} \longrightarrow \frac{d^2 V}{dz^2} = -\frac{\rho_o z}{\epsilon d}$$

$$\frac{dV}{dz} = -\frac{\rho_o z^2}{2\epsilon d} + A$$

$$V = -\frac{\rho_o z^3}{6\epsilon d} + Az + B$$

$$z = 0, V = 0 \longrightarrow 0 = 0 + B, \quad \text{i.e. } B = 0$$

$$z = d, V = V_o \longrightarrow V_o = -\frac{\rho_o d^2}{6\epsilon} + Ad$$

$$A = \frac{V_o}{d} + \frac{\rho_o d}{6\epsilon}$$

Hence,

$$V = -\frac{\rho_o z^3}{6\epsilon d} + \left(\frac{V_o}{d} + \frac{\rho_o d}{6\epsilon} \right) z$$

Prob. 6.6

$$\nabla^2 V = \frac{d^2 V}{dx^2} = -\frac{\rho_v}{\epsilon} = -\frac{50(1-y^2)x10^{-6}}{\epsilon} = -k(1-y^2)$$

where $k = \frac{50 \times 10^{-6}}{3 \times \frac{10^{-9}}{36\pi}} = 600\pi \times 10^3$

$$\frac{dV}{dy} = -k(y - y^3/3) + A$$

$$V = -k \left(\frac{y^2}{2} - \frac{y^4}{12} \right) + Ay + B = 50\pi \cdot 10^3 y^4 - 300\pi \cdot 10^3 y^2 + Ay + B$$

When $y=2\text{cm}$, $V=30 \times 10^3$,

$$30 \times 10^3 = 50\pi \times 10^3 \times 16 \times 10^{-6} - 300\pi \times 10^3 \times 4 \times 10^{-4} + Ay + B$$

or

$$30,374.5 = 0.02A + B \quad (1)$$

When $y=-2\text{cm}$, $V=30 \times 10^3$,

$$30,374.5 = -0.02A + B \quad (2)$$

From (1) and (2), $A=0$, $B=30,374.5$. Thus,

$$\underline{\underline{V = 157.08y^4 - 942.5y^2 + 30.374 \text{ kV}}}$$

Prob. 6.7

$$\nabla^2 V = -\frac{\rho_v}{\epsilon} = \frac{-\frac{10}{\rho} \times 10^{-12}}{\frac{3.6 \times 10^{-9}}{36\pi}} = -\frac{0.1\pi}{\rho}$$

Let $\alpha = 0.1\pi$.

$$\nabla^2 V = -\frac{\alpha}{\rho} = \frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{dV}{d\rho} \right)$$

$$-\alpha = \frac{d}{d\rho} \left(\rho \frac{dV}{d\rho} \right)$$

$$\rho \frac{dV}{d\rho} = -\alpha\rho + A$$

$$\frac{dV}{d\rho} = -\alpha + \frac{A}{\rho}$$

$$V = -\alpha\rho + A \ln \rho + B$$

$$\text{At } \rho=2, V=0 \longrightarrow 0 = -2\alpha + A \ln 2 + B \quad (1)$$

$$\text{At } \rho=5, V=60 \longrightarrow 60 = -5\alpha + A \ln 5 + B \quad (2)$$

Subtracting (1) from (2),

$$60 = -3\alpha + A \ln 5 / 2 \longrightarrow A = \frac{60 + 3\alpha}{\ln 2.5} = 66.51$$

From (1),

$$B = 2\alpha - A \ln 2 = -45.473$$

$$E = -\frac{dV}{d\rho} a_\rho = (\alpha - \frac{A}{\rho}) a_\rho = (0.3142 - \underline{\underline{\frac{66.51}{\rho}}}) a_\rho$$

Prob. 6.8

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0 + 0 - 2 = -2 \neq 0$$

Does not satisfy Laplace's equation.

Prob. 6.9

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 6xy + 0 + 2c = 0$$

$$\underline{\underline{c = -3xy}}$$

Prob. 6.10

$$V = V(\rho), \quad \nabla^2 V = \frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{dV}{d\rho} \right) = 0$$

$$\rho \frac{dV}{d\rho} = A \quad \longrightarrow \quad \frac{dV}{d\rho} = \frac{A}{\rho}$$

$$V = A \ln \rho + B$$

When $\rho=4$ mm = a, $V=0 \quad \longrightarrow \quad 0=A \ln a + B$ or $B = -A \ln a$

$$\text{When } \rho=12 \text{ mm} = b, V=V_o \quad \longrightarrow \quad V_o = A \ln b - A \ln a \quad \text{or} \quad A = \frac{V_o}{\ln \frac{b}{a}}$$

$$\mathbf{E} = -\nabla V = -\frac{dV}{d\rho} \mathbf{a}_\rho = -\frac{A}{\rho} \mathbf{a}_\rho = -\frac{V_o}{\rho \ln \frac{b}{a}} \mathbf{a}_\rho$$

At $\rho=8$ mm,

$$-6 \mathbf{a}_{\text{kV/m}} = -\frac{V_o}{8 \times 10^{-3} \ln \frac{12}{4}} \mathbf{a}_\rho \quad \longrightarrow \quad V_o = 48 \ln 3 = \underline{\underline{53.73}}$$

Prob. 6.11

$$\nabla^2 V = \frac{d^2 V}{dz^2} = 0 \quad \longrightarrow \quad V = Az + B$$

When $z=0, V=0 \quad \longrightarrow \quad B=0$

When $z=d, V=V_o \quad \longrightarrow \quad V_o = Ad \quad \text{or} \quad A = V_o/d$

Hence,

$$V = \frac{V_o z}{d}$$

$$\mathbf{E} = -\nabla V = -\frac{dV}{dz} \mathbf{a}_z = -\frac{V_o}{d} \mathbf{a}_z$$

$$\mathbf{D} = \epsilon \mathbf{E} = -\epsilon_0 \epsilon_r \frac{V_o}{d} \mathbf{a}_z$$

Since $V_o = 50$ V and $d = 2$ mm,

$$\underline{\underline{V = 25z \text{ kV}, \quad E = -25a_z \text{ kV/m}}}$$

$$\mathbf{D} = -\frac{10^{-9}}{36\pi} (1.5) 25 \times 10^3 \mathbf{a}_z = \underline{\underline{-332a_z \text{ nC/m}^2}}$$

$$\rho_s = D_n = \underline{\underline{\pm 332 \text{ nC/m}^2}}$$

The surface charge density is positive on the plate at $z=d$ and negative on the plate at $z=0$.

Prob. 6.12 From Example 6.8, solving $\nabla^2 V = 0$ when $V = V(\rho)$ leads to

$$V = \frac{V_o \ln \rho / a}{\ln b / a} = V_o \frac{\ln(a / \rho)}{\ln(a / b)}$$

$$\mathbf{E} = -\nabla V = -\frac{V_o}{\rho \ln b / a} \mathbf{a}_\rho = \frac{V_o}{\rho \ln a / b} \mathbf{a}_\rho, \quad \mathbf{D} = \epsilon \mathbf{E} = -\frac{\epsilon_o \epsilon_r V_o}{\rho \ln b / a} \mathbf{a}_\rho$$

$$\rho_s = D_n = \pm \left. \frac{\epsilon_o \epsilon_r V_o}{\rho \ln b / a} \right|_{\rho=a,b}$$

In this case, $V_o = 100$ V, $b = 5$ mm, $a = 15$ mm, $\epsilon_r = 2$. Hence at $\rho = 10$ mm,

$$V = \frac{100 \ln(10 / 15)}{\ln(5 / 15)} = \underline{\underline{36.91 \text{ V}}}$$

$$\mathbf{E} = \frac{100}{10 \times 10^{-3} \ln 3} \mathbf{a}_\rho = \underline{\underline{9.102 \mathbf{a}_\rho \text{ kV/m}}}$$

$$\mathbf{D} = 9.102 \times 10^3 \times \frac{10^{-9}}{36\pi} 2 \mathbf{a}_\rho = \underline{\underline{161 \mathbf{a}_\rho \text{ nC/m}^2}}$$

$$\rho_s (\rho = 5 \text{ mm}) = \frac{10^{-9}}{36\pi} (2) \frac{10^5}{5 \ln 3} = \underline{\underline{322 \text{ nC/m}^2}}$$

$$\rho_s (\rho = 15 \text{ mm}) = -\frac{10^{-9}}{36\pi} (2) \frac{10^5}{15 \ln 3} = \underline{\underline{-107.3 \text{ nC/m}^2}}$$

Prob. 6.13

(a)

$$\begin{aligned} \nabla^2 V &= \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial V}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2} + 0 \\ &= \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(2c_1 \rho^2 - 2c_2 \rho^{-2} \right) \sin 2\phi - \frac{4}{\rho^2} (c_1 \rho^2 + c_2 \rho^{-2}) \sin 2\phi \\ &= (4c_1 + 4c_2 \rho^{-4} - 4c_1 - 4c_2 \rho^{-4}) \sin 2\phi = 0 \end{aligned}$$

(b)

At $P(2, 45^\circ, 1), \rho = 1, \phi = 45^\circ$

$$50 = (c_1 + c_2) \sin 90^\circ = c_1 + c_2 \quad (1)$$

$$\begin{aligned} -\mathbf{E} &= \nabla V = \frac{\partial V}{\partial \rho} \mathbf{a}_\rho + \frac{1}{\rho} \frac{\partial V}{\partial \phi} \mathbf{a}_\phi + 0 \\ &= (2c_1 \rho - 2c_2 \rho^{-3}) \sin 2\phi \mathbf{a}_\rho + (c_1 \rho + c_2 \rho^{-3})(2) \cos 2\phi \mathbf{a}_\phi \end{aligned}$$

At P,

$$\begin{aligned}-\mathbf{E} &= (2c_1 - 2c_2)(1)\mathbf{a}_\rho + \mathbf{0} \\ |\mathbf{E}| = 100 &= 2c_1 - 2c_2 \quad \longrightarrow \quad 50 = c_1 - c_2 \quad (2) \\ \text{From (1) and (2),} \quad \underline{\underline{c_1 = 50, c_2 = 0}}\end{aligned}$$

Prob. 6.14

$$\begin{aligned}\frac{1}{\rho} \frac{d^2V}{d\phi^2} = 0 \quad \longrightarrow \quad \frac{d^2V}{d\phi^2} = 0 \quad \longrightarrow \quad \frac{dV}{d\phi} &= A \\ V = A\phi + B \quad 0 = 0 + B \quad \longrightarrow \quad B &= 0 \\ 50 = A\pi/2 \quad \longrightarrow \quad A &= \frac{100}{\pi} \\ E = -\nabla V = -\frac{1}{\rho} \frac{dV}{d\phi} \mathbf{a}_\phi &= -\frac{A}{\rho} \mathbf{a}_\phi = \underline{\underline{-\frac{100}{\pi\rho} \mathbf{a}_\phi}}\end{aligned}$$

Prob. 6.15

(a)

$$\begin{aligned}\frac{\partial V}{\partial \rho} &= V_o \left(1 + \frac{a^2}{\rho^2}\right) \sin \phi \\ \rho \frac{\partial V}{\partial \rho} &= V_o \left(\rho + \frac{a^2}{\rho}\right) \sin \phi \\ \frac{\partial}{\partial \rho} \left(\rho \frac{\partial V}{\partial \rho} \right) &= V_o \left(1 - \frac{a^2}{\rho^2}\right) \sin \phi \\ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial V}{\partial \rho} \right) &= V_o \left(\frac{1}{\rho} - \frac{a^2}{\rho^3}\right) \sin \phi \\ \frac{\partial^2 V}{\partial \phi^2} &= -V_o \left(\rho - \frac{a^2}{\rho}\right) \sin \phi \\ \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2} &= -V_o \left(\frac{1}{\rho} - \frac{a^2}{\rho^3}\right) \sin \phi \\ \nabla^2 V &= \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial V}{\partial \rho} \right) + \frac{\partial^2 V}{\partial \phi^2} = 0\end{aligned}$$

(b)

If $\rho^2 \gg a^2$, then $a^2/\rho^2 \ll 1$ and $V \approx V_o \rho \sin \phi$

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial \rho} \mathbf{a}_\rho - \frac{1}{\rho} \frac{\partial V}{\partial \phi} \mathbf{a}_\phi = -V_o \sin \phi \mathbf{a}_\rho - V_o \cos \phi \mathbf{a}_\phi$$

Prob. 6.16

$$\nabla^2 V = \frac{d^2 V}{dx^2} = 0 \quad \longrightarrow \quad V = Ax + B$$

At $x = 20 \text{ mm} = 0.02 \text{ m}$, $V = 0$

$$0 = 0.02A + B \quad \longrightarrow \quad (1)$$

$$\mathbf{E} = -\frac{dV}{dx} \mathbf{a}_x \quad \longrightarrow \quad A = 110 \quad \longrightarrow \quad (2)$$

From (1) $B = -0.02A = -2.2$

Then $V = 110x - 2.2$

At $x = 0 \quad \underline{\underline{V = -2.2V}}$

At $x = 50 \text{ mm} = 0.05 \text{ m}$,

$$V = 110 \times 0.05 - 2.2 = \underline{\underline{3.3V}}$$

Prob. 6.17

$$\nabla^2 V = 0 \quad \longrightarrow \quad V = -A/r + B$$

$$\text{At } r=0.5, V=-50 \quad \longrightarrow \quad -50 = -A/0.5 + B$$

Or

$$-50 = -2A + B \quad (1)$$

$$\text{At } r = 1, V = 50 \quad \longrightarrow \quad 50 = -A + B \quad (2)$$

From (1) and (2), $A = 100$, $B = 150$, and

$$\underline{\underline{V = -\frac{100}{r} + 150}}$$

$$\mathbf{E} = -\nabla V = -\frac{A}{r^2} \mathbf{a}_r = -\frac{100}{r^2} \mathbf{a}_r \text{ V/m}$$

Prob. 6.18 From Example 6.4,

$$V = \frac{V_o \ln\left(\frac{\tan \theta / 2}{\tan \theta_1 / 2}\right)}{\ln\left(\frac{\tan \theta_2 / 2}{\tan \theta_1 / 2}\right)}$$

$$V_o = 100, \quad \theta_1 = 30^\circ, \quad \theta_2 = 120^\circ, \quad r = \sqrt{3^2 + 0^2 + 4^2} = 5, \quad \theta = \tan^{-1} \rho / z = \tan^{-1} 3 / 4 = 36.87^\circ$$

$$V = 100 \frac{\ln\left(\frac{\tan 18.435^\circ}{\tan 15^\circ}\right)}{\ln\left(\frac{\tan 60^\circ}{\tan 15^\circ}\right)} = \underline{\underline{11.7 \text{ V}}}$$

$$\mathbf{E} = \frac{-V_o \mathbf{a}_\theta}{r \sin \theta \ln\left(\frac{\tan \theta_2 / 2}{\tan \theta_1 / 2}\right)} = \frac{-100 \mathbf{a}_\theta}{5 \sin 36.87^\circ \ln 6.464} = \underline{\underline{-17.86 \mathbf{a}_\theta \text{ V/m}}}$$

Prob. 6.19

(a)

$$\nabla^2 V = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial V}{\partial \rho} \right) = 0 \quad \longrightarrow \quad V = A \ln \rho + B$$

$$V(\rho = b) = 0 \quad \longrightarrow \quad 0 = A \ln b + B \quad \longrightarrow \quad B = -A \ln b$$

$$V(\rho = a) = V_o \quad \longrightarrow \quad V_o = A \ln a / b \quad \longrightarrow \quad A = -\frac{V_o}{\ln b / a}$$

$$V = -\frac{V_o}{\ln b / a} \ln \rho / b = \frac{V_o \ln b / \rho}{\ln b / a}$$

$$V(\rho = 15 \text{ mm}) = 70 \frac{\ln 2}{\ln 50} = \underline{\underline{12.4 \text{ V}}}$$

(b) As the electron decelerates, potential energy gained = K.E. loss

$$e[70 - 12.4] = \frac{1}{2} m[(10^7)^2 - u^2] \quad \longrightarrow \quad 10^{14} - u^2 = \frac{2e}{m} \times 57.6$$

$$u^2 = 10^{14} - \frac{2 \times 1.6 \times 10^{-19}}{9.1 \times 10^{-31}} \times 57.6 = 10^{12}(100 - 20.25)$$

$$u = 8.93 \times 10^6 \text{ m/s}$$

Prob. 6.20 This is similar to case 1 of Example 6.5.

$$X = c_1x + c_2, \quad Y = c_3y + c_4$$

$$\text{But } X(0) = 0 \longrightarrow 0 = c_2, \quad Y(0) = 0 \longrightarrow 0 = c_4$$

Hence,

$$V(x, y) = XY = a_oxy, \quad a_o = c_1c_3$$

$$\text{Also, } V(xy = 4) = 20 \longrightarrow 20 = 4a_o \longrightarrow a_o = 5$$

Thus,

$$V(x, y) = 5xy \text{ and } \mathbf{E} = -\nabla V = -5ya_x - 5xa_y$$

At (x,y) = (1,2),

$$V = 10 \text{ V}, \quad \mathbf{E} = -10a_x - 5a_y \text{ V/m}$$

Prob. 6.21

(a) As in Example 6.5, $X(x) = A \sin(n\pi x / b)$

For Y,

$$Y(y) = c_1 \cosh(n\pi y / b) + c_2 \sinh(n\pi y / b)$$

$$Y(a) = 0 \longrightarrow 0 = c_1 \cosh(n\pi a / b) + c_2 \sinh(n\pi a / b) \longrightarrow c_1 = -c_2 \tanh(n\pi a / b)$$

$$V = \sum_{n=1}^{\infty} a_n \sin(n\pi x / b) [\sinh(n\pi y / b) - \tanh(n\pi a / b) \cosh(n\pi y / b)]$$

$$V(x, y = 0) = V_o = -\sum_{n=1}^{\infty} a_n \tanh(n\pi a / b) \sin(n\pi x / b)$$

$$-a_n \tanh(n\pi a / b) = \frac{2}{b} \int_0^b V_o \sin(n\pi x / b) dx = \begin{cases} \frac{4V_o}{n\pi}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$$

Hence,

$$\begin{aligned}
 V &= -\frac{4V_o}{\pi} \sum_{n \text{ odd}}^{\infty} \sin(n\pi x / b) \left[\frac{\sinh(n\pi y / b)}{n \tanh(n\pi a / b)} - \frac{\cosh(n\pi y / b)}{n} \right] \\
 &= -\frac{4V_o}{\pi} \sum_{n \text{ odd}}^{\infty} \frac{\sin(n\pi x / b)}{n \sinh(n\pi a / b)} [\sinh(n\pi y / b) \cosh(n\pi a / b) - \cosh(n\pi y / b) \sinh(n\pi a / b)] \\
 &= \underline{\underline{\frac{4V_o}{\pi} \sum_{n \text{ odd}}^{\infty} \frac{\sin(n\pi x / b) \sinh[n\pi(a-y)/b]}{n \sinh(n\pi a / b)}}}
 \end{aligned}$$

Alternatively, for Y

$$Y(y) = c_1 \sinh n\pi(y - c_2) / b$$

$$Y(a) = 0 \quad \longrightarrow \quad 0 = c_1 \sinh[n\pi(a - c_2) / b] \quad \longrightarrow \quad c_2 = a$$

$$V = \sum_{n=1}^{\infty} b_n \sin(n\pi x / b) \sinh[n\pi(y - a) / b]$$

where

$$b_n = \begin{cases} -\frac{4V_o}{n\pi \sinh(n\pi a / b)}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$$

(b) This is the same as Example 6.5 except that we exchange y and x. Hence

$$\begin{aligned}
 V(x, y) &= \frac{4V_o}{\pi} \sum_{n \text{ odd}}^{\infty} \frac{\sin(n\pi y / a) \sinh(n\pi x / a)}{n \sinh(n\pi b / a)} \\
 &= \underline{\underline{\frac{4V_o}{\pi} \sum_{n \text{ odd}}^{\infty} \frac{\sin(n\pi y / a) \sinh[n\pi(b-x)/a]}{n \sinh(n\pi b / a)}}}
 \end{aligned}$$

(c) This is the same as part (a) except that we must exchange x and y. Hence

$$\begin{aligned}
 V(x, y) &= \frac{4V_o}{\pi} \sum_{n \text{ odd}}^{\infty} \frac{\sin(n\pi y / a) \sinh[n\pi(b-x)/a]}{n \sinh(n\pi b / a)} \\
 &= \underline{\underline{\frac{4V_o}{\pi} \sum_{n \text{ odd}}^{\infty} \frac{\sin(n\pi y / a) \sinh[n\pi(b-x)/a]}{n \sinh(n\pi b / a)}}}
 \end{aligned}$$

Prob. 6.22 (a) $X(x)$ is the same as in Example 6.5. Hence

$$V(x, y) = \sum_{n=1}^{\infty} \sin(n\pi x / b) [a_n \sinh(n\pi y / b) + b_n \cosh(n\pi y / b)]$$

At $y=0$, $V = V_1$

$$V_1 = \sum_{n=1}^{\infty} b_n \sin(n\pi x / b) \quad \longrightarrow \quad b_n = \begin{cases} \frac{4V_1}{n\pi}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$$

At $y=a$, $V = V_2$

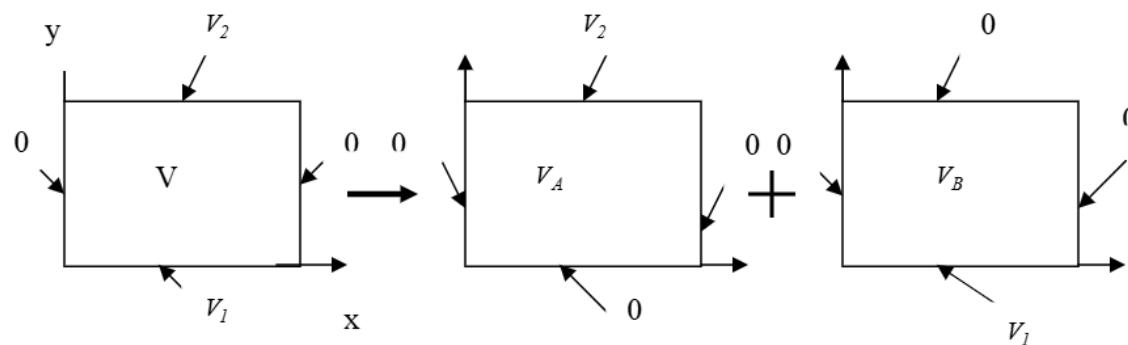
$$V_2 = \sum_{n=1}^{\infty} \sin(n\pi x / b) [a_n \sinh(n\pi a / b) + b_n \cosh(n\pi a / b)]$$

$$a_n \sinh(n\pi a / b) + b_n \cosh(n\pi a / b) = \begin{cases} \frac{4V_2}{n\pi}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$$

or

$$a_n = \begin{cases} \frac{4V_2}{n\pi \sinh(n\pi a / b)} (V_2 - V_1 \cosh(n\pi a / b)), & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$$

Alternatively, we may apply superposition principle.



i.e. $V = V_A + V_B$

V_A is exactly the same as Example 6.5 with $V_o = V_2$, while V_B is exactly the same as Prob. 6.19(a). Hence

$$V = \frac{4}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{\sin(n\pi x/b)}{n \sinh(n\pi a/b)} [V_1 \sinh(n\pi(a-y)/b) + V_2 \sinh(n\pi y/b)]$$

(b)

$$V(x, y) = (a_1 e^{-\alpha x} + a_2 e^{+\alpha x})(a_3 \sin \alpha y + a_4 \cos \alpha y)$$

$$\lim_{x \rightarrow \infty} V(x, y) = 0 \quad \longrightarrow \quad a_2 = 0$$

$$V(x, y=0) = 0 \quad \longrightarrow \quad a_4 = 0$$

$$V(x, y=a) = 0 \quad \longrightarrow \quad \alpha = n\pi/a, \quad n = 1, 2, 3, \dots$$

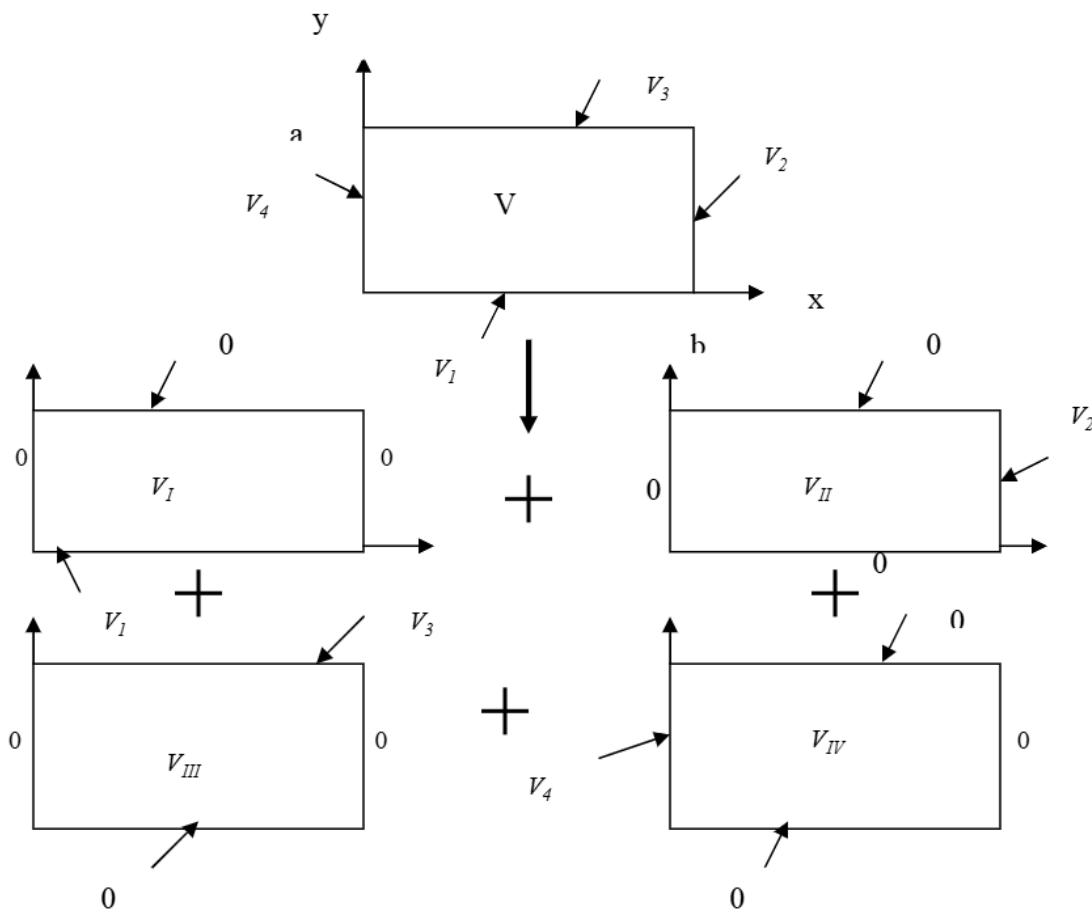
Hence,

$$V(x, y) = \sum_{n=1}^{\infty} a_n e^{-n\pi x/a} \sin(n\pi y/a)$$

$$V(x=0, y) = V_o = \sum_{n=1}^{\infty} a_n \sin(n\pi y/a) \quad \longrightarrow \quad a_n = \begin{cases} \frac{4V_o}{n\pi}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$$

$$V(x, y) = \frac{4V_o}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{\sin(n\pi y/a)}{n} \exp(-n\pi x/a)$$

(c) The problem is easily solved using superposition theorem, as illustrated below.



Therefore,

$$\begin{aligned} V &= V_I + V_{II} + V_{III} + V_{IV} \\ &= \frac{4}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{1}{n} \left\{ \frac{\sin(n\pi x/b)}{\sinh(n\pi a/b)} [V_1 \sinh(n\pi(a-y)/b) + V_3 \sinh(n\pi y/b)] \right. \\ &\quad \left. + \frac{\sin(n\pi x/a)}{\sinh(n\pi b/a)} [V_2 \sinh(n\pi y/a) + V_4 \sinh(n\pi(b-x)/a)] \right\} \end{aligned}$$

where

$$V_I = \frac{4V_1}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{\sin(n\pi x/b) \sinh[n\pi(a-y)/b]}{n \sinh(n\pi a/b)}$$

$$V_{II} = \frac{4V_2}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{\sin(n\pi x/a) \sinh(n\pi y/a)}{n \sinh(n\pi b/a)}$$

$$V_{III} = \frac{4V_3}{\pi} \sum_{n=odd}^{\infty} \frac{\sin(n\pi x/b) \sinh(n\pi y/b)}{n \sinh(n\pi a/b)}$$

$$V_{IV} = \frac{4V_4}{\pi} \sum_{n=odd}^{\infty} \frac{\sin(n\pi y/a) \sinh[n\pi(b-x)/a]}{n \sinh(n\pi b/a)}$$

Prob. 6.23

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial x} \mathbf{a}_x - \frac{\partial V}{\partial y} \mathbf{a}_y$$

$$E_x = \frac{4V_o}{\pi} \sum_{n=odd}^{\infty} \frac{n\pi}{a} \frac{\sin(n\pi y/a)}{n} \exp(-n\pi x/a)$$

$$E_y = -\frac{4V_o}{\pi} \sum_{n=odd}^{\infty} \frac{n\pi}{a} \frac{\cos(n\pi y/a)}{n} \exp(-n\pi x/a)$$

$$\mathbf{E} = \underline{\underline{\frac{4V_o}{a} \sum_{n=odd}^{\infty} \exp(-n\pi x/a) [\sin(n\pi y/a) \mathbf{a}_x - \cos(n\pi y/a) \mathbf{a}_y]}}$$

Prob. 6.24

This is similar to Example 6.5 except that we must exchange x and y. Going through the same arguments, we have

$$V(x, y) = \sum c_n \sinh\left(\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right)$$

Applying the condition at x=a, we get

$$V_o \sin\left(\frac{\pi y}{b}\right) = \sum c_n \sinh\left(\frac{n\pi a}{b}\right) \sin\left(\frac{n\pi y}{b}\right)$$

This yields

$$c_n \sinh\left(\frac{n\pi a}{b}\right) = \begin{cases} V_o, & n=1 \\ 0, & n \neq 1 \end{cases}$$

Hence,

$$V(x, y) = V_o \underline{\underline{\frac{\sinh\left(\frac{\pi x}{b}\right) \sin\left(\frac{\pi y}{b}\right)}{\sinh\left(\frac{\pi a}{b}\right)}}}$$

Prob. 6.25

$$\nabla^2 V = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial V}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2} = 0$$

If we let $V(\rho, \phi) = R(\rho)\Phi(\phi)$,

$$\frac{\Phi}{\rho} \frac{\partial}{\partial \rho} (\rho R') + \frac{1}{\rho^2} R \Phi'' = 0$$

or

$$\frac{\rho}{R} \frac{\partial}{\partial \rho} (\rho R') = -\frac{\Phi''}{\Phi} = \lambda$$

Hence

$$\underline{\underline{\Phi'' + \lambda \Phi = 0}}$$

and

$$\frac{\partial}{\partial \rho} (\rho R') - \frac{\lambda R}{\rho} = 0$$

or

$$\underline{\underline{R'' + \frac{R'}{\rho} - \frac{\lambda R}{\rho^2} = 0}}$$

Prob. 6.26

$$\nabla^2 V = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) = 0$$

If $V(r, \theta) = R(r)F(\theta)$, $r \neq 0$,

$$F \frac{d}{dr} (r^2 R') + \frac{R}{\sin \theta} \frac{d}{d\theta} (\sin \theta F') = 0$$

Dividing through by RF gives

$$\frac{1}{R} \frac{d}{dr} (r^2 R') = -\frac{1}{F \sin \theta} \frac{d}{d\theta} (\sin \theta F') = \lambda$$

Hence,

$$\sin \theta F'' + \cos \theta F' + \lambda F \sin \theta = 0$$

or

$$\underline{F'' + \cot \theta F' + \lambda F = 0}$$

Also,

$$\frac{d}{dr}(r^2 R') - \lambda R = 0$$

or

$$\underline{\underline{R'' + \frac{2R'}{r} - \frac{\lambda}{r^2} R = 0}}$$

Prob. 6.27 If the centers at $\phi = 0$ and $\phi = \pi/2$ are maintained at a potential difference of V_o , from Example 6.3,

$$E_\phi = \frac{2V_o}{\pi\rho}, \quad J = \sigma E$$

Hence,

$$I = \int J \bullet dS = \frac{2V_o \sigma}{\pi} \int_{\rho=a}^b \int_{z=0}^t \frac{1}{\rho} d\rho dz = \frac{2V_o \sigma t}{\pi} \ln(b/a)$$

and

$$R = \frac{V_o}{I} = \frac{\pi}{2\sigma t \ln(b/a)}$$

Prob. 6.28 If $V(r=a) = 0$, $V(r=b) = V_o$, from Example 6.9,

$$E = \frac{V_o}{r^2(1/a - 1/b)}, \quad J = \sigma E$$

Hence,

$$I = \int J \bullet dS = \frac{V_o \sigma}{1/a - 1/b} \int_{\theta=0}^{\alpha} \int_{\phi=0}^{2\pi} \frac{1}{r^2} r^2 \sin \theta d\theta d\phi = \frac{2\pi V_o \sigma}{1/a - 1/b} (-\cos \theta) \Big|_0^\alpha$$

$$R = \frac{V_o}{I} = \frac{\frac{1}{a} - \frac{1}{b}}{2\pi\sigma(1 - \cos \alpha)}$$

Prob. 6.29

This is the same as Problem 6.30 except that $\alpha = \pi$. Hence,

$$R = \frac{1}{2\pi\sigma(1-\cos\pi)} \left(\frac{1}{a} - \frac{1}{b} \right) = \underline{\underline{\frac{1}{4\pi\sigma} \left(\frac{1}{a} - \frac{1}{b} \right)}}$$

Prob. 6.30

For a spherical capacitor, from Eq. (6.38),

$$R = \frac{\frac{1}{a} - \frac{1}{b}}{4\pi\sigma}$$

For the hemisphere, $R' = 2R$ since the sphere consists of two hemispheres in parallel.

As

$$b \longrightarrow \infty,$$

$$R' = \lim_{b \longrightarrow \infty} \frac{2 \left[\frac{1}{a} - \frac{1}{b} \right]}{4\pi\sigma} = \frac{1}{2\pi a\sigma}$$

$$G = 1/R' = 2\pi a\sigma$$

Alternatively, for an isolated sphere, $C = 4\pi\epsilon_0 a$. But

$$RC = \frac{\epsilon}{\sigma} \longrightarrow R = \frac{1}{4\pi a\sigma}$$

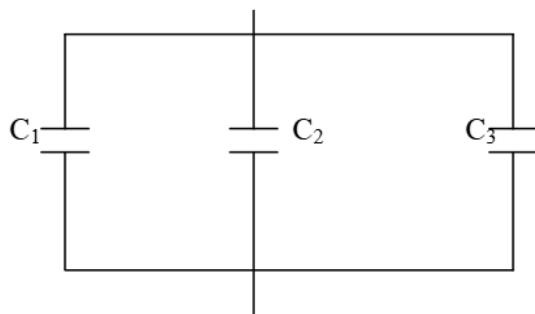
$$R' = 2R = \frac{1}{2\pi a\sigma} \quad \text{or} \quad G = 2\pi a\sigma$$

Prob. 6.31

$$C = \frac{\epsilon S}{d} \longrightarrow S = \frac{Cd}{\epsilon_0 \epsilon_r} = \frac{2 \times 10^{-9} \times 10^{-6}}{4 \times 10^{-9} / 36\pi} \text{ m}^2 = \underline{\underline{0.5655 \text{ cm}^2}}$$

Prob. 6.32

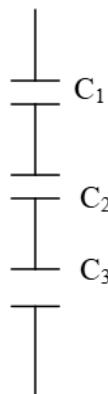
This can be regarded as three capacitors in parallel.



$$\begin{aligned}
 C &= C_1 + C_2 + C_3 = \sum \frac{\epsilon_0 \epsilon_{rk} S_k}{d_k} \\
 &= \frac{\epsilon_0}{2 \times 10^{-3}} [3 \times 15 \times 10^{-2} \times 20 \times 10^{-2} + 5 \times 15 \times 10^{-2} \times 20 \times 10^{-2} + 8 \times 15 \times 10^{-2} \times 20 \times 10^{-2}] \\
 &= \frac{10^{-9}}{36\pi} \times \frac{15 \times 10^{-2} \times 20 \times 10^{-2}}{2 \times 10^{-3}} [3 + 5 + 8] = \underline{\underline{2.122 \text{ nF}}}
 \end{aligned}$$

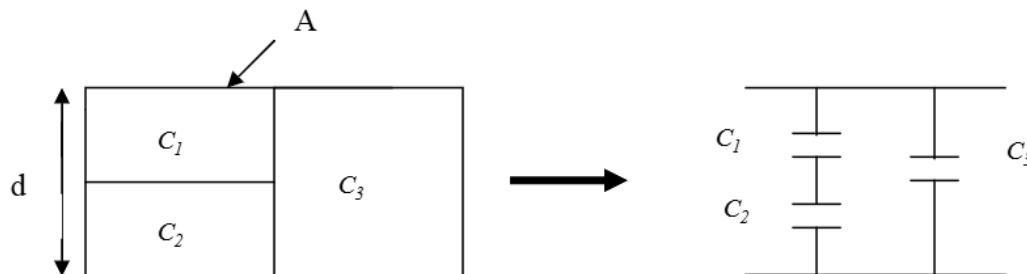
Prob. 6.33

This may be regarded as three capacitors in series.



$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} = \sum_{k=1}^3 \frac{d_k}{\epsilon_k S_k} = \frac{1 \times 10^{-3}}{\frac{10^{-9}}{36\pi} \times 80 \times 10^{-4}} \left[\frac{1}{3} + \frac{1}{5} + \frac{1}{8} \right] = \frac{36\pi}{8} \times 10^9 \times 0.6583$$

$$C_{eq} = \frac{8}{36\pi \times 0.6583} \text{ nF} = \underline{\underline{0.1076 \text{ nF}}}$$

Prob. 6.34

From the figure above,

$$C = \frac{C_1 C_2}{C_1 + C_2} + C_3$$

here

$$C_1 = \frac{\epsilon_0 A / 2}{d / 2} = \frac{\epsilon_0 A}{d}, \quad C_2 = \frac{\epsilon_0 \epsilon_r A}{d}, \quad C_3 = \frac{\epsilon_0 A}{2d}$$

$$C = \frac{\epsilon_0^2 \epsilon_r A^2 / d^2}{\epsilon_0 (\epsilon_r + 1) A / d} + \frac{\epsilon_0 A}{2d} = \frac{\epsilon_0 A}{d} \left(\frac{1}{2} + \frac{\epsilon_r}{\epsilon_r + 1} \right) = \frac{10^{-9}}{36\pi} \frac{10 \times 10^{-4}}{2 \times 10^{-3}} \left(\frac{1}{2} + \frac{6}{7} \right) \equiv 6 \text{ pF}$$

Prob. 6.35

$$C = \frac{\epsilon_0 S}{d} \longrightarrow S = \frac{Cd}{\epsilon_0}$$

$$S = \frac{1 \times 1 \times 10^{-3}}{10^{-9} / 36\pi} = 36\pi \times 10^6$$

$$\underline{\underline{S = 1.131 \times 10^8 \text{ m}^2}}$$

Prob. 6.36

$$Fdx = dW_E \quad \longrightarrow \quad F = \frac{dW_E}{dx}$$

$$W_E = \int \frac{1}{2} \epsilon |E|^2 dv = \frac{1}{2} \epsilon_0 \epsilon_r E^2 x ad + \frac{1}{2} \epsilon_0 E^2 da (1-x)$$

where $E = V_o / d$.

$$\frac{dW_E}{dx} = \frac{1}{2} \epsilon_0 \frac{V_o^2}{d^2} (\epsilon_r - 1) da \quad \longrightarrow \quad F = \frac{\epsilon_0 (\epsilon_r - 1) V_o^2 a}{2d}$$

Alternatively, $W_E = \frac{1}{2} CV_o^2$, where

$$C = C_1 + C_2 = \frac{\epsilon_0 \epsilon_r ax}{d} + \frac{\epsilon_0 \epsilon_r (1-x)}{d}$$

$$\frac{dW_E}{dx} = \frac{1}{2} \epsilon_0 \frac{V_o^2 a}{d} (\epsilon_r - 1)$$

$$F = \underline{\underline{\frac{\epsilon_0 (\epsilon_r - 1) V_o^2 a}{2d}}}$$

Prob.6.37

(a)

$$C = \frac{4\pi \epsilon}{\frac{1}{a} - \frac{1}{b}} = \frac{4\pi \times 2.25 \times \frac{10^{-9}}{36\pi}}{\frac{1}{5 \times 10^{-2}} - \frac{1}{10 \times 10^{-2}}} = \underline{\underline{25 \text{ pF}}}$$

(b) $Q = C V_o = 25 \times 80 \text{ pC}$

$$\rho_s = \frac{Q}{4\pi r^2} = \frac{25 \times 80}{4\pi \times 25 \times 10^{-4}} \text{ pC/m}^2 = \underline{\underline{63.66 \text{ nC/m}^2}}$$

Prob. 6.38

$$C_1 = \frac{\epsilon_0 \epsilon_r S}{d}, \quad C_2 = \frac{\epsilon_0 S}{d}$$

$$\frac{C_1}{C_2} = \epsilon_r \quad \longrightarrow \quad \epsilon_r = \frac{56 \mu\text{F}}{32 \mu\text{F}} = \underline{\underline{1.75}}$$

Prob. 6.39

$$\frac{d^2V}{dz^2} = 0 \longrightarrow V = Az + B$$

At $z=0, V=40 \longrightarrow 40=B$

At $z=2\text{mm}=d, V=0 \longrightarrow 0=Ad+40 \longrightarrow A=-40/d$

$$V = -\frac{40}{d}z + 40 = -\frac{40z}{2 \times 10^{-3}} + 40 = \underline{\underline{-20 \times 10^3 z + 40 \text{ V}}}$$

Prob. 6.40

$$(a) C = \frac{\epsilon S}{d} = \frac{6.8 \times \frac{10^{-9}}{36\pi} \times 0.5}{4 \times 10^{-3}} = \underline{\underline{7.515 \text{ nF}}}$$

$$(b) \rho_s = \frac{Q}{S}, \quad C = \frac{Q}{V} \longrightarrow Q = CV$$

$$\rho_s = \pm \frac{CV}{S} = \pm \frac{7.515 \times 10^{-9} \times 9}{0.5} = \underline{\underline{\pm 135.27 \text{ nC/m}^2}}$$

Prob. 6.41

$$W_E = \frac{Q^2}{2C}$$

$$C = \frac{\epsilon_0 \epsilon_r S}{d}$$

$$W_E = \frac{Q^2 d}{2\epsilon_0 \epsilon_r S}$$

When the plate spacing is doubled

$$W_E = \frac{Q^2(2d)}{2\epsilon_0 \epsilon_r S} = \frac{Q^2 d}{\underline{\underline{\epsilon_0 \epsilon_r S}}}$$

When the plate spacing is halved

$$W_E = \frac{Q^2(d/2)}{2\epsilon_0 \epsilon_r S} = \frac{Q^2 d}{\underline{\underline{4\epsilon_0 \epsilon_r S}}}$$

Prob.6.42

This can be treated as three capacitors in series

$$C_1 = \frac{\epsilon_0 S}{d_1} = \frac{\epsilon_0 (2.5)(0.4)}{1.5 \times 10^{-3}} = 666.67 \epsilon_0$$

$$C_2 = \frac{\epsilon_0 S}{d_2} = \frac{\epsilon_0 (5.6)(0.4)}{1.5 \times 10^{-3}} = 1493.33 \epsilon_0$$

$$C_3 = \frac{\epsilon_0 S}{d_3} = \frac{\epsilon_0 (8.1)(0.4)}{2 \times 10^{-3}} = 1620 \epsilon_0$$

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} = \frac{0.002787}{\epsilon_0}$$

$$C = 358.81 \epsilon_0 = 358.81 \times \frac{10^{-9}}{36\pi}$$

$$C = \underline{\underline{3.1726 \text{ nF}}}$$

Prob. 6.43

(a)

$$C = \frac{\epsilon_0 S}{d} = \frac{10^{-9}}{36\pi} \frac{200 \times 10^{-4}}{3 \times 10^{-3}} = \underline{\underline{59 \text{ pF}}}$$

(b) $\rho_s = D_n = 10^{-6} \text{ nC/m}^2$. But

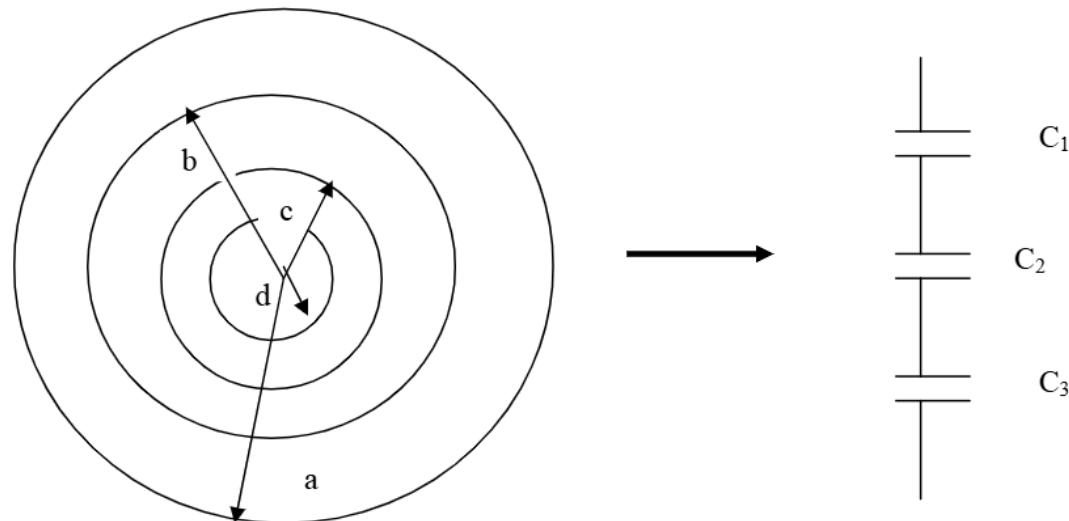
$$D_n = \epsilon E_n = \frac{\epsilon_0 V_0}{d} = \rho_s$$

or

$$V_0 = \frac{\rho_s d}{\epsilon_0} = 10^{-6} \times 3 \times 10^{-3} \times 36\pi \times 10^9 = \underline{\underline{339.3 \text{ V}}}$$

(c)

$$F = \frac{Q^2}{2S\epsilon_0} = \frac{\rho_s^2 S}{2\epsilon_0} = \frac{10^{-12} \times 200 \times 10^{-4} \times 36\pi \times 10^9}{2} = \underline{\underline{1.131 \text{ mN}}}$$

Prob. 6.44

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

where $C_1 = \frac{4\pi\epsilon_3}{\frac{1}{b} - \frac{1}{a}}$, $C_2 = \frac{4\pi\epsilon_2}{\frac{1}{c} - \frac{1}{b}}$, $C_3 = \frac{4\pi\epsilon_1}{\frac{1}{d} - \frac{1}{c}}$,

$$\frac{4\pi}{C} = \frac{1/b - 1/a}{\epsilon_3} + \frac{1/c - 1/b}{\epsilon_2} + \frac{1/d - 1/c}{\epsilon_1}$$

$$C = \frac{4\pi}{\frac{\epsilon_1}{\frac{1}{d} - \frac{1}{c}} + \frac{\epsilon_2}{\frac{1}{c} - \frac{1}{b}} + \frac{\epsilon_3}{\frac{1}{b} - \frac{1}{a}}}$$

Prob. 6.45

We may place a charge Q on the inner conductor. The negative charge $-Q$ is on the outer surface of the shell. Within the shell, $\mathbf{E} = 0$, i.e. between $r=c$ and $r=b$. Otherwise,

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2} \mathbf{a}_r$$

The potential at $r=a$ is

$$\begin{aligned} V_a &= - \int_{-\infty}^a \mathbf{E} \cdot d\mathbf{l} = - \int_{-\infty}^c E_r dr - \int_c^b E_r dr - \int_b^a E_r dr \\ &= - \frac{Q}{4\pi\epsilon_0} \int_{-\infty}^c \frac{dr}{r^2} - 0 - \frac{Q}{4\pi\epsilon_0} \int_b^a \frac{dr}{r^2} = \frac{Q}{4\pi\epsilon_0 c} + \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{a} - \frac{1}{b} \right) \\ C &= \frac{Q}{V_a} = \frac{1}{\frac{1}{4\pi\epsilon_0 c} + \frac{1}{4\pi\epsilon_0} \left(\frac{1}{a} - \frac{1}{b} \right)} \end{aligned}$$

Prob. 6.46

$$C = \frac{2\pi\epsilon L}{\ln(b/a)} = \frac{2\pi \times 2.5 \times \frac{10^{-9}}{36\pi} \times 3 \times 10^3}{\ln(8/5)} = \underline{\underline{0.8665 \mu F}}$$

Prob. 6.47

Let the plate at $\phi=0$ be 0, i.e. $V(0)=0$ and let the plate at $\phi=\pi/4$ be V_o , i.e. $V(\pi/4)=V_o$.

$$\nabla^2 V = \frac{1}{\rho^2} \frac{d^2 V}{d\phi^2} = 0 \quad \longrightarrow \quad \frac{dV}{d\phi} = A \quad \longrightarrow \quad V = A\phi + B$$

$$V(0) = 0 \quad \longrightarrow \quad 0 = 0 + B \quad \longrightarrow \quad B = 0$$

$$V(\pi/4) = V_o \quad \longrightarrow \quad V_o = A\pi/4 \quad \longrightarrow \quad A = \frac{4V_o}{\pi}$$

$$\mathbf{E} = -\nabla V = -\frac{1}{\rho} \frac{dV}{d\phi} \mathbf{a}_\phi = -\frac{A}{\rho} \mathbf{a}_\phi = -\frac{4V_o}{\pi\rho} \mathbf{a}_\phi$$

$$\mathbf{D} = \epsilon \mathbf{E} = -\frac{4\epsilon V_o}{\pi\rho} \mathbf{a}_\phi$$

$$\rho_s = D_n = -\frac{4\epsilon V_o}{\pi\rho}$$

$$Q = \int \rho_s dS = - \int_{\rho=a}^b \int_{z=0}^L \frac{4\epsilon V_o}{\pi\rho} d\rho dz = -\frac{4\epsilon V_o}{\pi} L \ln(b/a)$$

$$C = \frac{|Q|}{V_o} = \frac{4\epsilon L}{\pi} \ln(b/a)$$

Prob. 6.48

$$C = \frac{2\pi\epsilon L}{\ln(b/a)} = \frac{2\pi \times \frac{10^{-9}}{36\pi} \times 4.2 \times 400 \times 10^{-3}}{\ln(3.5/1)} = \underline{\underline{74.5 \text{ pF}}}$$

Prob. 6.49

$$C = \frac{2\pi\epsilon_0 L}{\ln(b/a)} = \frac{2\pi \times \frac{10^{-9}}{36\pi} \times 100 \times 10^{-6}}{\ln(600/20)} = 1.633 \times 10^{-15} \text{ F}$$

$$V = Q/C = \frac{50 \times 10^{-15}}{1.633 \times 10^{-15}} = \underline{\underline{30.62 \text{ V}}}$$

Prob. 6.50

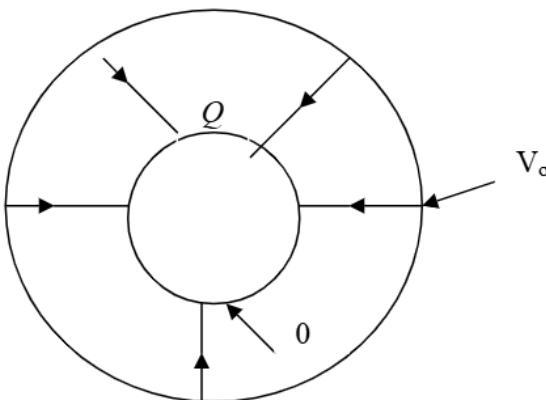
$$C_1 = \frac{2\pi\epsilon_1}{\ln(b/a)}, \quad C_2 = \frac{2\pi\epsilon_2}{\ln(c/b)}$$

Since the capacitance are in series, the total capacitance per unit length is

$$C = \frac{C_1 C_2}{C_1 + C_2} = \frac{2\pi\epsilon_1\epsilon_2}{\underline{\underline{\epsilon_2 \ln(b/a) + \epsilon_1 \ln(c/b)}}}$$

Prob. 6.51

$$\mathbf{E} = \frac{Q}{4\pi\epsilon r^2} \mathbf{a}_r$$



$$W = \frac{1}{2} \int \epsilon |\mathbf{E}|^2 dv = \iiint \frac{Q^2}{32\pi^2\epsilon^2 r^4} \epsilon r^2 \sin\theta d\theta d\phi dr$$

$$= \frac{Q^2}{32\pi^2\epsilon} (2\pi)(2) \int_c^b \frac{dr}{r^2} = \frac{Q^2}{8\pi\epsilon} \left(\frac{1}{c} - \frac{1}{b} \right)$$

$$W = \frac{Q^2(b-c)}{8\pi\epsilon bc}$$

Prob. 6.52

(a) Method 1: $\mathbf{E} = \frac{\rho_s}{\epsilon}(-\mathbf{a}_x)$, where ρ_s is to be determined.

$$V_o = -\int \mathbf{E} \bullet d\mathbf{l} = -\int \frac{-\rho_s}{\epsilon} dx = \rho_s \int_0^d \frac{1}{\epsilon_o} \frac{d}{d+x} dx = \frac{\rho_s}{\epsilon} d \ln(x+d) \Big|_0^d$$

$$V_o = \rho_s d \ln \frac{2d}{d} \longrightarrow \rho_s = \frac{V_o \epsilon_o}{d \ln 2}$$

$$\mathbf{E} = -\frac{\rho_s}{\epsilon} \mathbf{a}_x = -\frac{V_o}{(x+d) \ln 2} \mathbf{a}_x$$

Method 2: We solve Laplace's equation

$$\nabla \bullet (\epsilon \nabla V) = \frac{d}{dx} (\epsilon \frac{dV}{dx}) = 0 \longrightarrow \epsilon \frac{dV}{dx} = A$$

$$\frac{dV}{dx} = \frac{A}{\epsilon} = \frac{Ad}{\epsilon_o(x+d)} = \frac{c_1}{x+d}$$

$$V = c_1 \ln(x+d) + c_2$$

$$V(x=0) = 0 \longrightarrow 0 = c_1 \ln d + c_2 \longrightarrow c_2 = -c_1 \ln d$$

$$V(x=d) = V_o \longrightarrow V_o = c_1 \ln 2d - c_1 \ln d = c_1 \ln 2$$

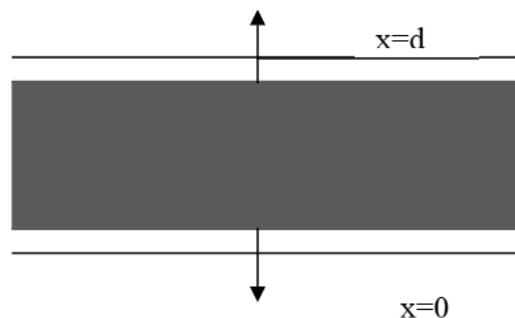
$$c_1 = \frac{V_o}{\ln 2}$$

$$V = c_1 \ln \frac{x+d}{d} = \frac{V_o}{\ln 2} \ln \frac{x+d}{d}$$

$$\mathbf{E} = -\frac{dV}{dx} \mathbf{a}_x = -\frac{V_o}{(x+d) \ln 2} \mathbf{a}_x$$

$$(b) \quad \mathbf{P} = (\epsilon_r - 1)\epsilon_o \mathbf{E} = -\left(\frac{x+d}{d} - 1\right) \frac{\epsilon_o V_o}{(x+d) \ln 2} \mathbf{a}_x = -\frac{\epsilon_o x V_o}{\underline{\underline{d(x+d) \ln 2}}} \mathbf{a}_x$$

(c)



$$\rho_{ps}|_{x=0} = \mathbf{P} \bullet (-\mathbf{a}_x)|_{x=0} = \underline{\underline{0}}$$

$$\rho_{ps}|_{x=d} = \mathbf{P} \bullet \mathbf{a}_x|_{x=d} = -\frac{\epsilon_o V_o}{\underline{\underline{2d \ln 2}}}$$

$$(d) \quad \mathbf{E} = \frac{\rho_s}{\epsilon} \mathbf{a}_x = \frac{Q}{\epsilon S} \mathbf{a}_x = \frac{Q}{\epsilon_o (1 + \frac{x}{d}) S} \mathbf{a}_x$$

$$V = - \int \mathbf{E} \bullet d\mathbf{l} = - \frac{Q}{\epsilon_o S} \int_a^d \frac{dx}{(1 + \frac{x}{d})} = \frac{Q}{\epsilon_o S} d \ln 2$$

$$C = \frac{Q}{V} = \frac{\epsilon_o S}{\underline{\underline{d \ln 2}}}$$

Prob. 6.53

We solve Laplace's equation for an inhomogeneous medium.

$$\nabla \cdot (\epsilon \nabla V) = \frac{d}{dx} \left(\epsilon \frac{dV}{dx} \right) = 0 \quad \longrightarrow \quad \epsilon \frac{dV}{dx} = A$$

$$\frac{dV}{dx} = \frac{A}{\epsilon} = \frac{A}{2\epsilon_0} \left[1 + \left(\frac{x}{d} \right)^2 \right]$$

$$V = \frac{A}{2\epsilon_0} \left(x + \frac{x^3}{3d^2} \right) + B$$

When $x=d$, $V=V_o$,

$$V_o = \frac{A}{2\epsilon_0} \left(d + \frac{d}{3} \right) + B \quad \longrightarrow \quad V_o = \frac{2Ad}{3\epsilon_0} + B \quad (1)$$

When $x = -d$, $V=0$,

$$0 = \frac{A}{2\epsilon_0} \left(-d - \frac{d}{3} \right) + B \quad \longrightarrow \quad 0 = -\frac{2Ad}{3\epsilon_0} + B \quad (2)$$

Adding (1) and (2), $V_o = 2B \quad \longrightarrow \quad B = V_o / 2$

From (2),

$$B = \frac{2Ad}{3\epsilon_0} = \frac{V_o}{2} \quad \longrightarrow \quad A = \frac{3\epsilon_0 V_o}{4d}$$

$$\mathbf{E} = -\nabla V = -\frac{dV}{dx} \mathbf{a}_x = -\frac{A}{\epsilon} \mathbf{a}_x = -\frac{3\epsilon_0 V_o}{4d} \left[1 + \left(\frac{x}{d} \right)^2 \right] \mathbf{a}_x = -\frac{-3V_o}{8d} \left[1 + \left(\frac{x}{d} \right)^2 \right] \mathbf{a}_x$$

$$\rho_s = D \cdot \mathbf{a}_n = \epsilon E \cdot \mathbf{a}_x \Big|_{x=d} = -A = -\frac{3\epsilon_0 V_o}{4d}$$

$$Q = \int_S \rho_s dS = \rho_s S = -\frac{3S\epsilon_0 V_o}{4d}$$

$$C = \frac{|Q|}{V_o} = \underline{\underline{\frac{3\epsilon_0 S}{4d}}}$$

Prob. 6.54

Method 1: Using Gauss's law,

$$Q = \int \mathbf{D} \bullet d\mathbf{S} = 4\pi r^2 D_r \quad \longrightarrow \quad \mathbf{D} = \frac{Q}{4\pi r^2} \mathbf{a}_r, \quad \epsilon = \frac{\epsilon_0 k}{r^2}$$

$$\mathbf{E} = \mathbf{D} / \epsilon = \frac{Q}{4\pi \epsilon_0 k} \mathbf{a}_r$$

$$V = - \int \mathbf{E} \bullet d\mathbf{l} = - \frac{Q}{4\pi \epsilon_0 k} \int_a^b dr = - \frac{Q}{4\pi \epsilon_0 k} (b - a)$$

$$C = \frac{Q}{|V|} = \frac{4\pi \epsilon_0 k}{\underline{b-a}}$$

Method 2: Using the inhomogeneous Laplace's equation,

$$\nabla \bullet (\epsilon \nabla V) = 0 \quad \longrightarrow \quad \frac{1}{r^2} \frac{d}{dr} \left(\frac{\epsilon_0 k}{r^2} r^2 \frac{dV}{dr} \right) = 0$$

$$\epsilon_0 k \frac{dV}{dr} = A' \quad \longrightarrow \quad \frac{dV}{dr} = A \quad \text{or} \quad V = Ar + B$$

$$V(r=a) = 0 \quad \longrightarrow \quad 0 = Aa + B \quad \longrightarrow \quad B = -Aa$$

$$V(r=b) = V_o \quad \longrightarrow \quad V_o = Ab + B = A(b-a) \quad \longrightarrow \quad A = \frac{V_o}{b-a}$$

$$\mathbf{E} = - \frac{dV}{dr} \mathbf{a}_r = -A \mathbf{a}_r = -\frac{V_o}{b-a} \mathbf{a}_r$$

$$\rho_s = D_n = -\frac{V_o}{b-a} \frac{\epsilon_0 k}{r^2} \Big|_{r=a,b}$$

$$Q = \int \rho_s dS = -\frac{V_o \epsilon_0 k}{b-a} \iint \frac{1}{r^2} r^2 \sin \theta d\theta d\phi = -\frac{V_o \epsilon_0 k}{b-a} 4\pi$$

$$C = \frac{|Q|}{V_o} = \frac{4\pi \epsilon_0 k}{\underline{b-a}}$$

Prob. 6.55

$$C = 4\pi \epsilon_0 a = 4\pi \times \frac{10^{-9}}{36\pi} \times 6.37 \times 10^6 = \underline{\underline{0.708 \text{ mF}}}$$

Prob.6.56

$$C = \frac{Q}{V}$$

$$\mathbf{D} = \frac{Q}{2\pi\rho L} \mathbf{a}_\rho$$

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon} = \frac{Q}{2\pi\rho L \epsilon_0 (3)(1+\rho)}$$

$$V = - \int \mathbf{E} \cdot d\mathbf{l} = \frac{Q}{6\pi L \epsilon_0} \int_a^b \frac{d\rho}{\rho(1+\rho)}$$

$$\text{Let } \frac{1}{\rho(1+\rho)} = \frac{A}{\rho} + \frac{B}{1+\rho}$$

Using partial fractions

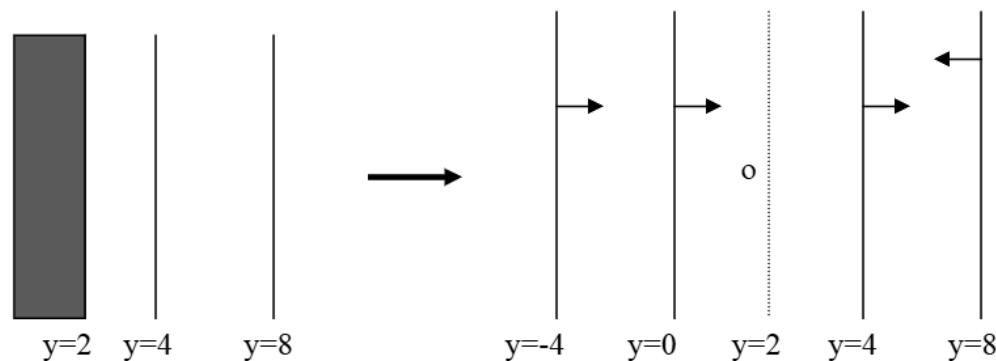
A=1, B=-1

$$\begin{aligned} V &= \frac{Q}{6\pi\epsilon_0 L} \left[\int_a^b \frac{d\rho}{\rho} - \int_a^b \frac{d\rho}{1+\rho} \right] \\ &= \frac{Q}{6\pi\epsilon_0 L} \left[\ln \rho - \ln(1+\rho) \right] \Big|_a^b \\ &= \frac{Q}{6\pi\epsilon_0 L} \left[\ln \frac{b}{1+b} - \ln \frac{a}{1+a} \right] \end{aligned}$$

If a=1 mm, and b=5 mm

$$\begin{aligned} C &= \frac{Q}{|V|} \\ &= \frac{6\pi\epsilon_0}{\ln \frac{b}{1+b} - \ln \frac{a}{1+a}} \\ &= \frac{6\pi \times \frac{10^{-9}}{36\pi}}{\ln \frac{5}{6} - \ln \frac{1}{2}} \\ &= \frac{\frac{1}{6} \times 10^{-9}}{\ln 0.8333 - \ln 0.5} = \frac{1}{6} \times 1.9591 \text{ nF} \end{aligned}$$

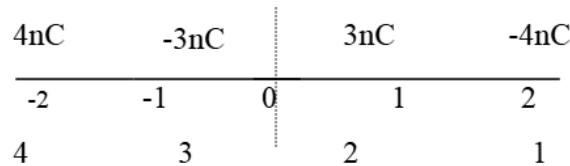
$$C = \underline{\underline{0.326 \text{ nF}}}$$

Prob. 6.57

At $P(0,0,0)$, $\underline{\mathbf{E}} = 0$ since \mathbf{E} does not exist for $y < 2$.

At $Q(-4,6,2)$, $y=6$ and

$$\begin{aligned} \mathbf{E} &= \sum \frac{\rho_s}{2\epsilon_0} \mathbf{a}_n = \frac{10^{-9}}{2 \times 10^{-9} / 36\pi} (-30\mathbf{a}_y + 20\mathbf{a}_y - 20\mathbf{a}_y - 30\mathbf{a}_y) = 18\pi(-60)\mathbf{a}_y \\ &= \underline{\underline{-3.4\mathbf{a}_y \text{ kV/m}}} \end{aligned}$$

Prob. 6.58

(a) $Q_i = -(3nC - 4nC) = \underline{1nC}$

(b) The force of attraction between the charges and the plates is

$$\mathbf{F} = \mathbf{F}_{13} + \mathbf{F}_{14} + \mathbf{F}_{23} + \mathbf{F}_{24}$$

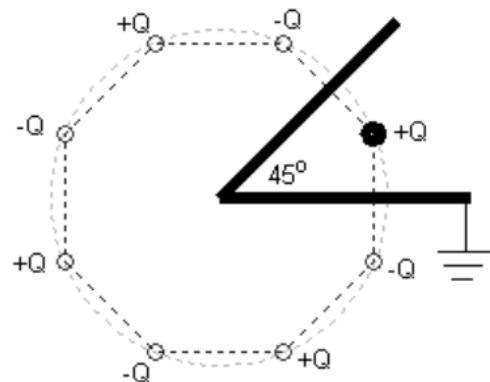
$$|F| = \frac{10^{-18}}{4\pi \times 10^{-9} / 36\pi} \left[\frac{9}{2^2} - \frac{2(12)}{3^2} + \frac{16}{4^2} \right] = \underline{\underline{5.25 \text{ nN}}}$$

Prob. 6.59

We have 7 images as follows: -Q at (-1,1,1), -Q at (1,-1,1), -Q at (1,1,-1),

-Q at (-1,-1,-1), Q at (1,-1,-1), Q at (-1,-1,1), and Q at (-1,1,-1). Hence,

$$\begin{aligned} \mathbf{F} &= \frac{Q^2}{4\pi\epsilon_0} \left[-\frac{2}{2^3} \mathbf{a}_x - \frac{2}{2^3} \mathbf{a}_y - \frac{2}{2^3} \mathbf{a}_z - \frac{(2\mathbf{a}_x + 2\mathbf{a}_y + 2\mathbf{a}_z)}{12^{3/2}} + \frac{(2\mathbf{a}_y + 2\mathbf{a}_z)}{8^{3/2}} \right. \\ &\quad \left. + \frac{(2\mathbf{a}_x + 2\mathbf{a}_y)}{8^{3/2}} + \frac{(2\mathbf{a}_x + 2\mathbf{a}_z)}{8^{3/2}} \right] \\ &= 0.9(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z) \left(-\frac{1}{4} - \frac{1}{12\sqrt{3}} + \frac{1}{4\sqrt{2}} \right) = \underline{\underline{-0.1092(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z) \text{ N}}} \end{aligned}$$

Prob. 6.60

$$N = \left(\frac{360^\circ}{45^\circ} - 1 \right) = 7$$

Prob. 6.61

(a)

$$\begin{aligned} \mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- &= \frac{\rho_L}{2\pi\epsilon_0} \left(\frac{\mathbf{a}_{\rho_1}}{\rho_1} - \frac{\mathbf{a}_{\rho_2}}{\rho_2} \right) = \frac{16 \times 10^{-9}}{2\pi \times \frac{10^{-9}}{36\pi}} \left[\frac{(2, -2, 3) - (3, -2, 4)}{|(2, -2, 3) - (3, -2, 4)|^2} - \frac{(2, -2, 3) - (3, -2, -4)}{|(2, -2, 3) - (3, -2, -4)|^2} \right] \\ &= 18 \times 16 \left[\frac{(-1, 0, -1)}{2} - \frac{(-1, 0, 7)}{50} \right] = \underline{\underline{-138.2\mathbf{a}_x - 184.3\mathbf{a}_y \text{ V/m}}} \end{aligned}$$

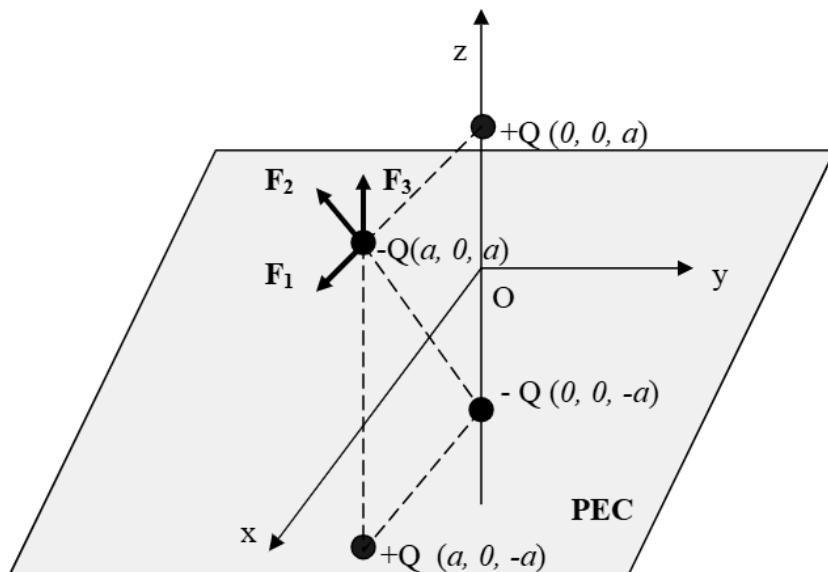
(b) $\rho_s = D_n$

$$\begin{aligned} \mathbf{D} = \mathbf{D}_+ + \mathbf{D}_- &= \frac{\rho_L}{2\pi} \left(\frac{\mathbf{a}_{\rho_1}}{\rho_1} - \frac{\mathbf{a}_{\rho_2}}{\rho_2} \right) = \frac{16 \times 10^{-9}}{2\pi} \left[\frac{(5, -6, 0) - (3, -6, 4)}{|(5, -6, 0) - (3, -6, 4)|^2} - \frac{(5, -6, 0) - (3, -6, -4)}{|(5, -6, 0) - (3, -6, -4)|^2} \right] \\ &= \frac{8}{\pi} \left[\frac{(2, 0, -4)}{20} - \frac{(2, 0, 4)}{20} \right] \text{nC/m}^2 = \underline{\underline{-1.018\mathbf{a}_z \text{ nC/m}^2}} \end{aligned}$$

$$\underline{\underline{\rho_s = -1.018 \text{ nC/m}^2}}$$

Prob. 6.62

The images are shown with proper sign at proper locations. Figure does not show the actual direction of forces but they are expressed as follows:



$$\mathbf{F}_1 = \frac{Q^2}{4\pi\epsilon_0} \left[\frac{-\mathbf{a}_x}{a^2} \right]$$

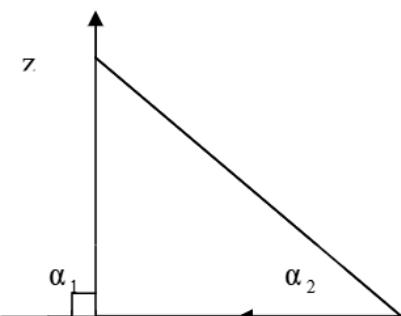
$$\mathbf{F}_2 = \frac{Q^2}{4\pi\epsilon_0} \left[\frac{a\mathbf{a}_x + 2a\mathbf{a}_z}{\left(\sqrt{a^2 + 4a^2}\right)^3} \right]$$

$$\mathbf{F}_3 = \frac{Q^2}{4\pi\epsilon_0} \left[\frac{-\mathbf{a}_z}{4a^2} \right]$$

$$\mathbf{F}_{\text{total}} = \frac{Q^2}{4\pi\epsilon_0 a^2} \left[\left(\frac{1}{5\sqrt{5}} - 1 \right) \mathbf{a}_x + \left(\frac{2}{5\sqrt{5}} - \frac{1}{4} \right) \mathbf{a}_z \right]$$

$$= \underline{\underline{\frac{Q^2}{4\pi\epsilon_0 a^2} \left[-0.91\mathbf{a}_x - 0.071\mathbf{a}_y \right] \text{ N}}}$$

CHAPTER 7

P.E. 7.1


$$\rho = 5, \cos \alpha_1 = 0, \cos \alpha_2 = \sqrt{\frac{2}{27}}$$

$$\mathbf{a}_\phi = \mathbf{a}_l \times \mathbf{a}_\rho = \left(\frac{-\mathbf{a}_x - \mathbf{a}_y}{\sqrt{2}} \right) \times \mathbf{a}_z = \frac{-\mathbf{a}_x + \mathbf{a}_y}{\sqrt{2}}$$

$$\mathbf{H}_3 = \frac{10}{4\pi(5)} \left(\sqrt{\frac{2}{27}} - 0 \right) \left(\frac{-\mathbf{a}_x + \mathbf{a}_y}{\sqrt{2}} \right) = \underline{\underline{-30.63\mathbf{a}_x + 30.63\mathbf{a}_y}} \text{ mA/m}$$

P.E. 7.2

$$(a) \mathbf{H} = \frac{2}{4\pi(2)} \left(1 + \frac{3}{\sqrt{13}} \right) \mathbf{a}_z = \underline{\underline{0.1458\mathbf{a}_z}} \text{ A/m}$$

$$(b) \rho = \sqrt{3^2 + 4^2} = 5, \alpha_2 = 0, \cos \alpha_1 = -\frac{12}{13},$$

$$\mathbf{a}_\phi = -\mathbf{a}_y \times \left(\frac{3\mathbf{a}_x - 4\mathbf{a}_z}{5} \right) = \frac{4\mathbf{a}_x + 3\mathbf{a}_z}{5}$$

$$\begin{aligned} \mathbf{H} &= \frac{2}{4\pi(5)} \left(1 + \frac{12}{13} \right) \left(\frac{4\mathbf{a}_x + 3\mathbf{a}_z}{5} \right) = \frac{1}{26\pi} (4\mathbf{a}_x + 3\mathbf{a}_z) \\ &= \underline{\underline{48.97\mathbf{a}_x + 36.73\mathbf{a}_z}} \text{ mA/m} \end{aligned}$$

P.E. 7.3

(a) From Example 7.3,

$$\mathbf{H} = \frac{Ia^2}{2(a^2 + z^2)^{3/2}} \mathbf{a}_z$$

At (0,0,-1cm), z = 2cm,

$$\mathbf{H} = \frac{50 \times 10^{-3} \times 25 \times 10^{-4}}{2(5^2 + 2^2)^{3/2} \times 10^{-6}} \mathbf{a}_z = \underline{\underline{400.2\mathbf{a}_z}} \text{ mA/m}$$

(b) At (0,0,10cm), z = 9cm,

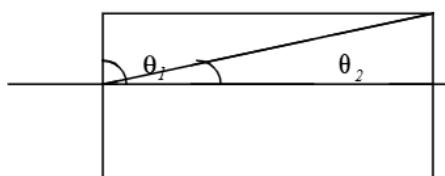
$$\mathbf{H} = \frac{50 \times 10^{-3} \times 25 \times 10^{-4}}{2(5^2 + 9^2)^{3/2} \times 10^{-6}} \mathbf{a}_z = \underline{\underline{57.3 \mathbf{a}_z \text{ mA/m}}}$$

P.E. 7.4

$$\begin{aligned}\mathbf{H} &= \frac{NI}{2L} (\cos \theta_2 - \cos \theta_1) \mathbf{a}_z = \frac{2 \times 10^3 \times 50 \times 10^{-3} (\cos \theta_2 - \cos \theta_1) \mathbf{a}_z}{2 \times 0.75} \\ &= \frac{100}{1.5} (\cos \theta_2 - \cos \theta_1) \mathbf{a}_z\end{aligned}$$

(a) At $(0,0,0)$, $\theta = 90^\circ$, $\cos \theta_2 = \frac{0.75}{\sqrt{0.75^2 + 0.05^2}} = 0.9978$

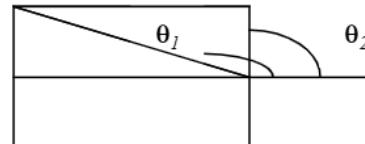
$$\mathbf{H} = \frac{100}{1.5} (0.9978 - 0) \mathbf{a}_z$$



$$= \underline{\underline{66.52 \mathbf{a}_z \text{ A/m}}}$$

(b) At $(0,0,0.75)$, $\theta_2 = 90^\circ$, $\cos \theta_1 = -0.9978$

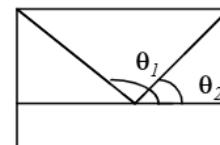
$$\begin{aligned}\mathbf{H} &= \frac{100}{1.5} (0 + 0.9978) \mathbf{a}_z \\ &= \underline{\underline{66.52 \mathbf{a}_z \text{ A/m}}}\end{aligned}$$



(c) At $(0,0,0.5)$, $\cos \theta_1 = \frac{-0.5}{\sqrt{0.5^2 + 0.05^2}} = -0.995$

$$\cos \theta_1 = \frac{0.25}{\sqrt{0.25^2 + 0.05^2}} = 0.9806$$

$$\begin{aligned}\mathbf{H} &= \frac{100}{1.5} (0.9806 + 0.995) \mathbf{a}_z \\ &= \underline{\underline{131.7 \mathbf{a}_z \text{ A/m}}}\end{aligned}$$

**P.E. 7.5**

$$\mathbf{H} = \frac{1}{2} \mathbf{K} \times \mathbf{a}_n$$

(a) $\mathbf{H}(0,0,0) = \frac{1}{2} 50 \mathbf{a}_z \times (-\mathbf{a}_y) = \underline{\underline{25 \mathbf{a}_x \text{ mA/m}}}$

(b) $\mathbf{H}(1,5,-3) = \frac{1}{2} 50 \mathbf{a}_z \times \mathbf{a}_y = \underline{\underline{-25 \mathbf{a}_x \text{ mA/m}}}$

P.E. 7.6

$$|\mathbf{H}| = \begin{cases} \frac{NI}{2\pi\rho}, & \rho - a < \rho < \rho + a, \quad 9 < \rho < 11 \\ 0, & \text{otherwise} \end{cases}$$

(a) At $(3, -4, 0)$, $\rho = \sqrt{3^2 + 4^2} = 5\text{cm} < 9\text{cm}$

$$|\mathbf{H}| = 0$$

(b) At $(6, 9, 0)$, $\rho = \sqrt{6^2 + 9^2} = \sqrt{117} < 11$

$$|\mathbf{H}| = \frac{10^3 \times 100 \times 10^{-3}}{2\pi\sqrt{117} \times 10^2} = \underline{\underline{147.1 \text{ A/m}}}$$

P.E. 7.7

$$(a) \quad \mathbf{B} = \nabla \times \mathbf{A} = (-4xz - 0)\mathbf{a}_x + (0 + 4yz)\mathbf{a}_y + (y^2 - x^2)\mathbf{a}_z$$

$$\underline{\underline{\mathbf{B}(-1, 2, 5) = 20\mathbf{a}_x + 40\mathbf{a}_y + 3\mathbf{a}_z \text{ Wb/m}^2}}$$

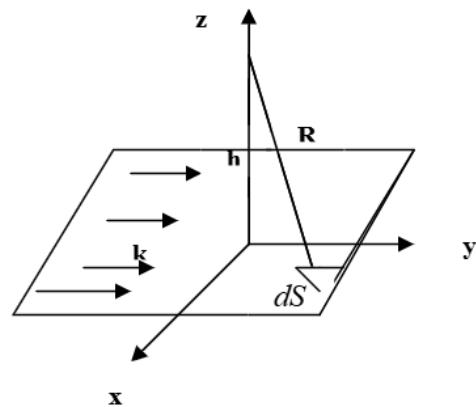
$$(b) \quad \psi = \int \mathbf{B} \cdot d\mathbf{S} = \int_{y=-1}^4 \int_{x=0}^1 (y^2 - x^2) dx dy = \int_{-1}^4 y^2 dy - 5 \int_0^1 x^2 dx$$

$$= \frac{1}{3}(64 + 1) - \frac{5}{3} = \underline{\underline{20 \text{ Wb}}}$$

Alternatively,

$$\begin{aligned} \psi &= \int \mathbf{A} \cdot d\mathbf{l} = \int_0^1 x^2(-1) dx + \int_{-1}^4 y^2(1) dy + \int_1^0 x^2(4) dx + 0 \\ &= -\frac{5}{3} + \frac{65}{3} = \underline{\underline{20 \text{ Wb}}} \end{aligned}$$

P.E. 7.8



$$\mathbf{H} = \int \frac{k dS \times \mathbf{R}}{4\pi R^3},$$

$$\begin{aligned} dS &= dx dy, \mathbf{k} = k_y \mathbf{a}_y, \\ \mathbf{R} &= (-x, -y, h), \end{aligned}$$

$$\begin{aligned}\mathbf{k} \times \mathbf{R} &= (ha_x + xa_z)k_y, \\ \mathbf{H} &= \int \frac{k_y(ha_x + xa_z)dxdy}{4\pi(x^2 + y^2 + h^2)^{3/2}} \\ &= \frac{k_y ha_x}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dxdy}{(x^2 + y^2 + h^2)^{3/2}} + \frac{k_y a_z}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{x dxdy}{(x^2 + y^2 + h^2)^{3/2}}\end{aligned}$$

The integrand in the last term is zero because it is an odd function of x.

$$\begin{aligned}\mathbf{H} &= \frac{k_y ha_x}{4\pi} \int_{\phi=0}^{2\pi} \int_{\rho=0}^{\infty} \frac{\rho d\phi d\rho}{(\rho^2 + h^2)^{3/2}} = \frac{k_y h 2\pi a_x}{4\pi} \int_0^{\infty} (\rho^2 + h^2)^{-3/2} \frac{d(\rho^2)}{2} \\ &= \frac{k_y h}{2} a_x \left(\frac{-1}{(\rho^2 + h^2)^{1/2}} \right) \Big|_0^{\infty} = \frac{k_y}{2} a_x\end{aligned}$$

Similarly, for point (0,0,-h), $\mathbf{H} = -\frac{1}{2} k_y a_x$

Hence,

$$\mathbf{H} = \begin{cases} \frac{1}{2} k_y a_x, & z > 0 \\ \frac{1}{2} k_y a_x, & z < 0 \end{cases}$$

P.E. 7.9

$$\mathbf{H} = \frac{I}{2\pi\rho} \mathbf{a}_\phi$$

$$\text{But } \mathbf{H} = -\nabla V_m \quad (\mathbf{J} = 0)$$

$$\frac{I}{2\pi\rho} \mathbf{a}_\phi = -\frac{1}{\rho} \frac{\partial V_m}{\partial \phi} \mathbf{a}_\phi \rightarrow V_m = -\frac{I}{2\pi} \phi + C$$

$$\text{At } (10, 60^\circ, 7), \phi = 60^\circ = \frac{\pi}{3}, V_m = 0 \rightarrow 0 = -\frac{I}{2\pi} \cdot \frac{\pi}{3} + C$$

$$\text{or } C = \frac{I}{6}$$

$$V_m = -\frac{I}{2\pi} \phi + \frac{I}{6}$$

$$\text{At } (4, 30^\circ, -2), \phi = 30^\circ = \frac{\pi}{6},$$

$$V_m = -\frac{I}{2\pi} \cdot \frac{\pi}{6} + \frac{I}{6} = \frac{I}{12} = \frac{12}{12}$$

$$\underline{\underline{V_m = 1 \text{ A}}}$$

P.E. 7.10

$$(a) \quad \mathbf{B} = \nabla \times \mathbf{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^2y + yz & xy^2 - xz^3 & -6xy + 2z^2y^2 \end{vmatrix}$$

$$\underline{\mathbf{B} = (-6xz + 4x^2y + 3xz^2)\mathbf{a}_x + (y + 6yz - 4xy^2)\mathbf{a}_y + (y^2 - z^3 - 2x^2 - z)\mathbf{a}_z \text{ Wb/m}^2}$$

(b)

$$\begin{aligned} \psi &= \int_{z=0}^2 \int_{y=0}^2 (-6xz + 4x^2y + 3xz^2) dy dz \Big|_{x=1} \\ &= \int \int_0^2 (-6xz) dy dz + 4 \int \int_0^2 x^2y dy dz + 3 \int \int_0^2 xz^2 dy dz \\ &= -6 \int_0^2 dz \int_0^2 dy + 4 \int_0^2 dz \int_0^2 y dy + 3 \int_0^2 dy \int_0^2 z^2 dz \\ &= -6(2)(2) + 4(2) \left(\frac{y^2}{2} \Big|_0^2 \right) + 3(2) \left(\frac{z^3}{3} \Big|_0^2 \right) = -24 + 16 + 16 \end{aligned}$$

$$\underline{\psi = 8 \text{ Wb}}$$

$$(c) \quad \nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 4xy + 2xy - 6xy = 0$$

$$\nabla \cdot \mathbf{B} = -6z + 8xy + 3z^3 + 6z - 8xy + 1 - 3z^3 - 1 = 0$$

As a matter of mathematical necessity,

$$\nabla \bullet \mathbf{B} = \nabla \bullet (\nabla \times \mathbf{A}) = 0$$

Prob. 7.1

(a) See text

(b) Let $\mathbf{H} = \mathbf{H}_y + \mathbf{H}_z$

$$\text{For } \mathbf{H}_z = \frac{I}{2\pi\rho} \mathbf{a}_\phi \quad \rho = \sqrt{(-3)^2 + 4^2} = 5$$

$$a_\phi = -a_z \times \frac{(-3a_x + 4a_y)}{5} = \frac{(3a_y - 4a_x)}{5}$$

$$H_z = \frac{20}{2\pi(25)} (4a_x + 3a_y) = 0.5093a_x + 0.382a_y$$

$$\text{For } H_y = \frac{I}{2\pi\rho} a_\phi, \quad \rho = \sqrt{(-3)^2 + 5^2} = \sqrt{34}$$

$$a_\phi = a_y \times \frac{(-3a_x + 5a_z)}{\sqrt{34}} = \frac{3a_z + 5a_x}{\sqrt{34}}$$

$$\mathbf{H}_y = \frac{10}{2\pi(34)} (5\mathbf{a}_x + 3\mathbf{a}_z) = 0.234\mathbf{a}_x + 0.1404\mathbf{a}_z$$

$$\begin{aligned}\mathbf{H} &= \mathbf{H}_y + \mathbf{H}_z \\ &= 0.7433\mathbf{a}_x + 0.382\mathbf{a}_y + 0.1404\mathbf{a}_z \text{ A/m}\end{aligned}$$

Prob. 7.2

$$d\mathbf{H} = \frac{Idl \times \mathbf{R}}{4\pi R^3}$$

(a) At (1,0,0), $\mathbf{R}=(1,0,0) - (0,0,0) = (1,0,0)$

$$d\mathbf{H} = \frac{4\mathbf{a}_x \times \mathbf{a}_x}{4\pi(1)^3} = \underline{\underline{\underline{\theta}}}$$

(b) At (0, 1,0), $\mathbf{R} = \mathbf{a}_y$

$$d\mathbf{H} = \frac{4\mathbf{a}_x \times \mathbf{a}_y}{4\pi(1)^3} = \underline{\underline{\underline{0.3183\mathbf{a}_z \text{ A/m}}}}$$

(c) At (0,0,1), $\mathbf{R} = \mathbf{a}_z$

$$d\mathbf{H} = \frac{4\mathbf{a}_x \times \mathbf{a}_z}{4\pi(1)^3} = \underline{\underline{\underline{-0.3183\mathbf{a}_y \text{ A/m}}}}$$

(d) At (1,1,1), $\mathbf{R}=(1,1,1)$

$$d\mathbf{H} = \frac{4\mathbf{a}_x \times (\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)}{4\pi(3)^{3/2}} = \underline{\underline{\underline{61.26(-\mathbf{a}_y + \mathbf{a}_z) \text{ mA/m}}}}$$

Prob. 7.3

$$\text{Let } \mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2$$

where \mathbf{H}_1 and \mathbf{H}_2 are respectively due to the lines located at (0,0) and (0,5).

$$\mathbf{H}_1 = \frac{I}{2\pi\rho} \mathbf{a}_\phi, \quad \rho = 5, \quad \mathbf{a}_\phi = \mathbf{a}_\ell \times \mathbf{a}_\rho = \mathbf{a}_z \times \mathbf{a}_x = \mathbf{a}_y$$

$$\mathbf{H}_1 = \frac{10}{2\pi(5)} \mathbf{a}_y = \frac{\mathbf{a}_y}{\pi}$$

$$\mathbf{H}_2 = \frac{I}{2\pi\rho} \mathbf{a}_\phi, \quad \rho = 5\sqrt{2}, \quad \mathbf{a}_\phi = \mathbf{a}_\ell \times \mathbf{a}_\rho, \mathbf{a}_\ell = -\mathbf{a}_z$$

$$\mathbf{a}_\rho = \frac{5\mathbf{a}_x - 5\mathbf{a}_y}{5\sqrt{2}} = \frac{\mathbf{a}_x - \mathbf{a}_y}{\sqrt{2}}$$

$$\mathbf{a}_\phi = -\mathbf{a}_z \times \left(\frac{\mathbf{a}_x - \mathbf{a}_y}{\sqrt{2}} \right) = \frac{-\mathbf{a}_x - \mathbf{a}_y}{\sqrt{2}}$$

$$\mathbf{H}_2 = \frac{10}{2\pi 5\sqrt{2}} \left(\frac{-\mathbf{a}_x - \mathbf{a}_y}{\sqrt{2}} \right) = \frac{1}{2\pi} (-\mathbf{a}_x - \mathbf{a}_y)$$

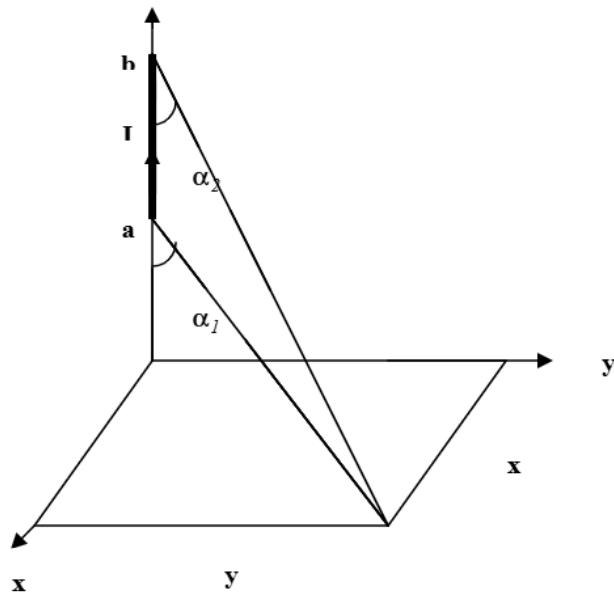
$$\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2 = \frac{\mathbf{a}_y}{\pi} + \frac{1}{2\pi} (-\mathbf{a}_x - \mathbf{a}_y) = \underline{\underline{-0.1592\mathbf{a}_x + 0.1592\mathbf{a}_y}}$$

Prob. 7.4

$$\mathbf{H} = \frac{I}{2\pi\rho} \mathbf{a}_\phi, \quad \rho = 5, I = 12$$

$$\mathbf{a}_\phi = \mathbf{a}_\ell \times \mathbf{a}_\rho = -\mathbf{a}_z \times \left(\frac{3\mathbf{a}_x + 4\mathbf{a}_y}{5} \right) = \frac{4}{5}\mathbf{a}_x - \frac{3}{5}\mathbf{a}_y$$

$$\mathbf{H} = \frac{12}{2\pi(5)} \left(\frac{4}{5}\mathbf{a}_x - \frac{3}{5}\mathbf{a}_y \right) = \underline{\underline{0.3056\mathbf{a}_x - 0.2292\mathbf{a}_y}}$$

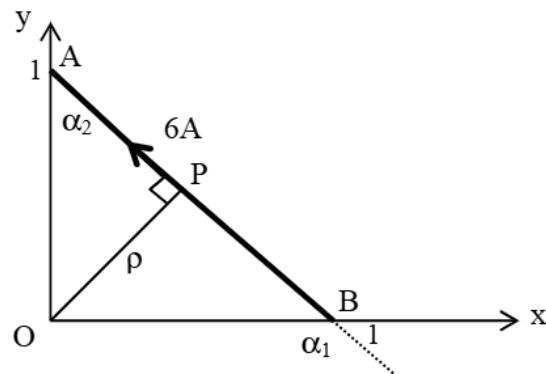
Prob. 7.5

$$\mathbf{H} = \frac{I}{4\pi\rho} (\cos \alpha_2 - \cos \alpha_1) \mathbf{a}_\phi$$

$$\rho = \sqrt{x^2 + y^2}, \cos \alpha_1 = \frac{a}{\sqrt{a^2 + \rho^2}}, \cos \alpha_2 = \frac{b}{\sqrt{b^2 + \rho^2}}$$

$\mathbf{a}_\phi = \mathbf{a}_r \times \mathbf{a}_\rho = \mathbf{a}_z \times \mathbf{a}_\rho = \mathbf{a}_\phi$. Hence,

$$\mathbf{H} = \frac{I}{4\pi\sqrt{x^2 + y^2}} \left[\frac{b}{\sqrt{x^2 + y^2 + b^2}} - \frac{a}{\sqrt{x^2 + y^2 + a^2}} \right] \mathbf{a}_\phi$$

Prob. 7.6

$$\mathbf{H} = \frac{I}{4\pi\rho} (\cos \alpha_2 - \cos \alpha_1) \mathbf{a}_\phi$$

$$\alpha_1 = 135^\circ, \quad \alpha_2 = 45^\circ, \quad \rho = \frac{1}{2}\sqrt{2} = \frac{\sqrt{2}}{2}$$

$$\mathbf{a}_\phi = \mathbf{a}_l \times \mathbf{a}_\rho = \left(\frac{-\mathbf{a}_x + \mathbf{a}_y}{\sqrt{2}} \right) \times \left(\frac{-\mathbf{a}_x - \mathbf{a}_y}{\sqrt{2}} \right) = \frac{1}{2} \begin{vmatrix} -1 & 1 & 0 \\ -1 & -1 & 0 \end{vmatrix} = \mathbf{a}_z$$

$$\mathbf{H} = \frac{6}{4\pi \frac{\sqrt{2}}{2}} (\cos 45^\circ - \cos 135^\circ) \mathbf{a}_z = \frac{3}{\pi} \mathbf{a}_z$$

$$\mathbf{H}(0, 0, 0) = \underline{\underline{0.954 \mathbf{a}_z \text{ A/m}}}$$

Prob. 7.7

(a) At (5,0,0), $\rho=5$, $\mathbf{a}_\phi = \mathbf{a}_y$, $\cos \alpha_1 = 0$, $\cos \alpha_2 = \frac{10}{\sqrt{125}}$

$$\mathbf{H} = \frac{2}{4\pi(5)} \left(\frac{10}{\sqrt{125}} \right) \mathbf{a}_y = \underline{\underline{28.471 \mathbf{a}_y \text{ mA/m}}}$$

(b) At (5,5,0), $\rho=5\sqrt{2}$, $\cos \alpha_1 = 0$, $\cos \alpha_2 = \frac{10}{\sqrt{150}}$

$$\mathbf{a}_\phi = \frac{-\mathbf{a}_x + \mathbf{a}_y}{\sqrt{2}}$$

$$\mathbf{H} = \frac{2}{4\pi(5\sqrt{2})} \left(\frac{10}{\sqrt{150}} \right) \left(\frac{-\mathbf{a}_x + \mathbf{a}_y}{\sqrt{2}} \right) = \underline{\underline{13(-\mathbf{a}_x + \mathbf{a}_y) \text{ mA/m}}}$$

(c) At $(5, 15, 0)$, $\rho = \sqrt{250} = 5\sqrt{10}$, $\cos \alpha_1 = 0$, $\cos \alpha_2 = \frac{10}{\sqrt{350}}$

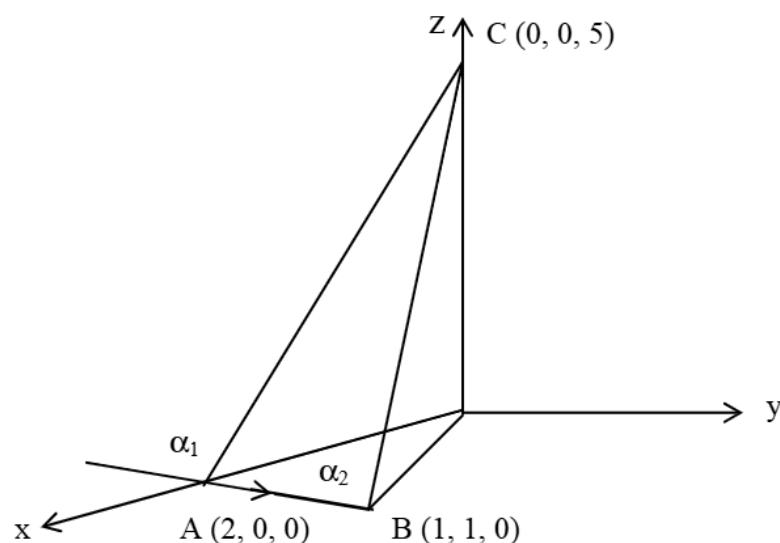
$$\mathbf{a}_\phi = \frac{5\mathbf{a}_y - 15\mathbf{a}_x}{5\sqrt{10}}$$

$$\mathbf{H} = \frac{2}{4\pi(5\sqrt{10})} \left(\frac{10}{\sqrt{350}} \right) \left(\frac{-15\mathbf{a}_x + 5\mathbf{a}_y}{5\sqrt{10}} \right) = \underline{\underline{-5.1\mathbf{a}_x + 1.7\mathbf{a}_y \text{ mA/m}}}$$

d) At $(5, -15, 0)$, by symmetry,

$$\mathbf{H} = \underline{\underline{5.1\mathbf{a}_x + 1.7\mathbf{a}_y \text{ mA/m}}}$$

Prob. 7.8



(a) Consider the figure above.

$$\overline{AB} = (1, 1, 0) - (2, 0, 0) = (-1, 1, 0)$$

$$\overline{AC} = (0, 0, 5) - (2, 0, 0) = (-2, 0, 5)$$

$\overline{AB} \cdot \overline{AC} = 2$, i.e. \overline{AB} and \overline{AC} are not perpendicular.

$$\cos(180^\circ - \alpha_1) = \frac{\overline{AB} \cdot \overline{AC}}{|\overline{AB}| |\overline{AC}|} = \frac{2}{\sqrt{2} \sqrt{29}} \rightarrow \cos \alpha_1 = -\sqrt{\frac{2}{29}}$$

$$\overline{BC} = (0, 0, 5) - (1, 1, 0) = (-1, -1, 5)$$

$$\overline{BA} = (1, -1, 0)$$

$$\cos \alpha_2 = \frac{\overline{BC} \cdot \overline{BA}}{|\overline{BC}| |\overline{BA}|} = \frac{-1+1}{|\overline{BC}| |\overline{BA}|} = 0$$

i.e. $\overline{BC} = \rho = (-1, -1, 5)$, $\rho = \sqrt{27}$

$$\mathbf{a}_\phi = \mathbf{a}_l \times \mathbf{a}_\rho = \frac{(-1, 1, 0)}{\sqrt{2}} \times \frac{(-1, -1, 5)}{\sqrt{27}} = \frac{(5, 5, 2)}{\sqrt{54}}$$

$$\mathbf{H}_2 = \frac{10}{4\pi\sqrt{27}} \left(0 + \sqrt{\frac{2}{29}} \right) \frac{(5, 5, 2)}{\sqrt{2}\sqrt{27}} = \frac{5}{2\pi\sqrt{29}} \cdot \frac{(5, 5, 2)}{27} \text{ A/m}$$

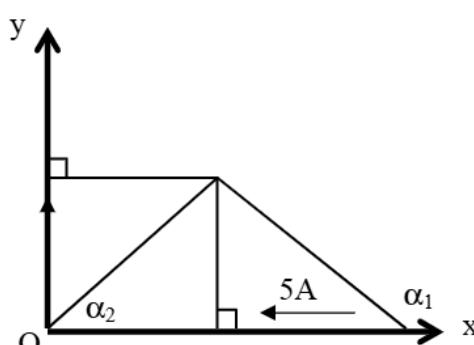
$$= \underline{\underline{27.37 \mathbf{a}_x + 27.37 \mathbf{a}_y + 10.95 \mathbf{a}_z \text{ mA/m}}}$$

$$(b) \quad \mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2 + \mathbf{H}_3 = (0, -59.1, 0) + \underline{\underline{(27.37, 27.37, 10.95)}} \\ + \underline{\underline{(-30.63, 30.63, 0)}} \\ = \underline{\underline{-3.26 \mathbf{a}_x - 1.1 \mathbf{a}_y + 10.95 \mathbf{a}_z \text{ mA/m}}}$$

Prob. 7.9

(a) Let $\mathbf{H} = \mathbf{H}_x + \mathbf{H}_y = 2\mathbf{H}_x$

$$\mathbf{H}_x = \frac{I}{4\pi\rho} (\cos \alpha_2 - \cos \alpha_1) \mathbf{a}_\phi$$



where $\mathbf{a}_\phi = -\mathbf{a}_x \times \mathbf{a}_y = -\mathbf{a}_z$, $\alpha_1 = 180^\circ$, $\alpha_2 = 45^\circ$

$$\begin{aligned}\mathbf{H}_x &= \frac{5}{4\pi(2)} (\cos 45^\circ - \cos 180^\circ) (-\mathbf{a}_z) \\ &= \underline{\underline{-0.6792 \mathbf{a}_z \text{ A/m}}}\end{aligned}$$

(b) $\mathbf{H} = \mathbf{H}_x + \mathbf{H}_y$

$$\begin{aligned}\text{where } \mathbf{H}_x &= \frac{5}{4\pi(2)} (1 - 0) \mathbf{a}_\phi, \quad \mathbf{a}_\phi = -\mathbf{a}_x \times -\mathbf{a}_y = \mathbf{a}_z \\ &= 198.9 \mathbf{a}_z \text{ mA/m}\end{aligned}$$

$$\mathbf{H}_y = 0 \text{ since } \alpha_1 = \alpha_2 = 0$$

$$\mathbf{H} = \underline{\underline{0.1989 \mathbf{a}_z \text{ A/m}}}$$

(c) $\mathbf{H} = \mathbf{H}_x + \mathbf{H}_y$

$$\text{where } \mathbf{H}_x = \frac{5}{4\pi(2)} (1 - 0) (-\mathbf{a}_x \times \mathbf{a}_z) = 198.9 \mathbf{a}_y \text{ mA/m}$$

$$\mathbf{H}_y = \frac{5}{4\pi(2)} (1 - 0) (\mathbf{a}_y \times \mathbf{a}_z) = 198.9 \mathbf{a}_x \text{ mA/m}$$

$$\mathbf{H} = \underline{\underline{0.1989 \mathbf{a}_x + 0.1989 \mathbf{a}_y \text{ A/m.}}}$$

Prob. 7.10

(3)

Let $\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2 + \mathbf{H}_3 + \mathbf{H}_4$



(2)

where \bar{H}_n is the contribution by side n.

(a) $\mathbf{H} = 2\mathbf{H}_1 + \mathbf{H}_2 + \mathbf{H}_4$ since $\mathbf{H}_1 = \mathbf{H}_3$

$$\mathbf{H}_1 = \frac{I}{4\pi\rho} (\cos \alpha_2 - \cos \alpha_1) \mathbf{a}_\phi = \frac{10}{4\pi(2)} \left(\frac{6}{\sqrt{40}} + \frac{1}{\sqrt{2}} \right) \mathbf{a}_z \quad (1)$$

$$\mathbf{H}_2 = \frac{10}{4\pi(6)} \left(2 \times \frac{2}{\sqrt{40}} \right) \mathbf{a}_z, \quad \mathbf{H}_4 = \frac{10}{4\pi(2)} \left(2 \cdot \frac{1}{\sqrt{2}} \right) \mathbf{a}_z$$

$$\mathbf{H} = \left[\frac{5}{2\pi} \left(\frac{3}{\sqrt{10}} + \frac{1}{\sqrt{2}} \right) + \frac{5}{6\pi\sqrt{10}} + \frac{5}{2\pi\sqrt{2}} \right] \mathbf{a}_z = \underline{\underline{1.964 \mathbf{a}_z \text{ A/m}}}$$

(b) At $(4, 2, 0)$, $\mathbf{H} = 2(\mathbf{H}_1 + \mathbf{H}_4)$

$$\mathbf{H}_1 = \frac{10}{4\pi(2)} \frac{8}{\sqrt{20}} \mathbf{a}_z, \quad \mathbf{H}_4 = \frac{10}{4\pi(4)} \frac{4}{\sqrt{20}} \mathbf{a}_z$$

$$\mathbf{H} = \frac{2\sqrt{5}}{\pi} \left(1 + \frac{1}{4} \right) \mathbf{a}_z = \underline{\underline{1.78 \mathbf{a}_z \text{ A/m}}}$$

(c) At $(4, 8, 0)$, $\mathbf{H} = \mathbf{H}_1 + 2\mathbf{H}_2 + \mathbf{H}_3$

$$\begin{aligned}\mathbf{H}_1 &= \frac{10}{4\pi(8)} \left(2 \cdot \frac{4}{4\sqrt{5}} \right) \mathbf{a}_z, \quad \mathbf{H}_2 = \frac{10}{4\pi(4)} \left(\frac{8}{4\sqrt{5}} - \frac{1}{\sqrt{2}} \right) \mathbf{a}_z \\ \mathbf{H}_3 &= \frac{10}{4\pi(4)} \left(\frac{2}{\sqrt{2}} \right) (-\mathbf{a}_z) \\ \mathbf{H} &= \frac{5}{8\pi} (\mathbf{a}_z) \left(\frac{1}{\sqrt{5}} + \frac{4}{\sqrt{5}} - \frac{4}{\sqrt{2}} \right) = \underline{\underline{-0.1178\mathbf{a}_z \text{ A/m}}}\end{aligned}$$

(d) At $(0, 0, 2)$,

$$\begin{aligned}\mathbf{H}_1 &= \frac{10}{4\pi(2)} \left(\frac{8}{\sqrt{68}} - 0 \right) (\mathbf{a}_x \times \mathbf{a}_z) = -\frac{10}{\pi\sqrt{68}} \mathbf{a}_y \\ \mathbf{H}_2 &= \frac{10}{4\pi\sqrt{68}} \left(\frac{4}{\sqrt{84}} - 0 \right) \mathbf{a}_y \times \left(\frac{2\mathbf{a}_z - 8\mathbf{a}_x}{\sqrt{68}} \right) = \frac{5(\mathbf{a}_x + 4\mathbf{a}_z)}{17\pi\sqrt{84}} \\ \mathbf{H}_3 &= \frac{10}{4\pi\sqrt{20}} \left(-\frac{8}{\sqrt{84}} - 0 \right) \mathbf{a}_x \times \left(\frac{2\mathbf{a}_x - 4\mathbf{a}_y}{\sqrt{20}} \right) = \frac{\mathbf{a}_y + 2\mathbf{a}_z}{\pi\sqrt{21}} \\ \mathbf{H}_4 &= \frac{10}{4\pi\sqrt{2}} \left(0 + \frac{4}{\sqrt{20}} \right) (-\mathbf{a}_y \times \mathbf{a}_z) = \frac{-5\mathbf{a}_x}{\pi\sqrt{20}} \\ \mathbf{H} &= \left(\frac{5}{34\pi\sqrt{21}} - \frac{5}{\pi\sqrt{20}} \right) \mathbf{a}_x + \left(\frac{1}{\pi\sqrt{21}} - \frac{10}{\pi\sqrt{68}} \right) \mathbf{a}_y + \left(\frac{20}{34\pi\sqrt{21}} + \frac{2}{\pi\sqrt{21}} \right) \mathbf{a}_z \\ &= \underline{\underline{-0.3457 \bar{a}_x - 0.3165 \bar{a}_y + 0.1798 \bar{a}_z \text{ A/m}}}\end{aligned}$$

Prob. 7.11

For the side of the loop along y-axis,

$$\mathbf{H}_1 = \frac{I}{4\pi\rho} (\cos \alpha_2 - \cos \alpha_1) \mathbf{a}_\phi$$

where $\mathbf{a}_\phi = -\mathbf{a}_x$, $\rho = 2 \tan 30^\circ = \frac{2}{\sqrt{3}}$, $\alpha_2 = 30^\circ$, $\alpha_1 = 150^\circ$

$$\mathbf{H}_1 = \frac{5}{4\pi} \frac{\sqrt{3}}{2} (\cos 30^\circ - \cos 150^\circ) (-\mathbf{a}_x) = -\frac{15}{8\pi} \mathbf{a}_x$$

$$\mathbf{H} = 3\mathbf{H}_1 = -1.79 \mathbf{a}_x \text{ A/m}$$

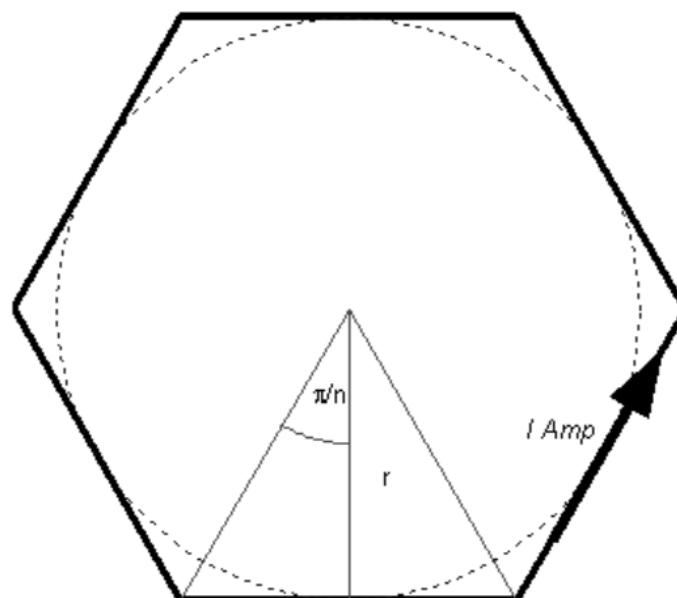
Prob. 7.12

$$\mathbf{H} = 4\mathbf{H}_1 = 4 \frac{I}{4\pi\rho} (\cos \alpha_2 - \cos \alpha_1) \mathbf{a}_\phi$$

$$\rho = a = 2\text{cm}, I = mA, \alpha_2 = 45^\circ, \alpha_1 = 90^\circ + 45^\circ = 135^\circ$$

$$\mathbf{a}_\phi = \mathbf{a}_\ell \times \mathbf{a}_\rho = \mathbf{a}_y \times (-\mathbf{a}_x) = \mathbf{a}_z$$

$$\mathbf{H} = \frac{I}{\pi a} \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right) \mathbf{a}_z = \frac{\sqrt{2}I}{\pi a} \mathbf{a}_z = \frac{\sqrt{2} \times 5 \times 10^{-3}}{\pi \times 2 \times 10^{-2}} \mathbf{a}_z = \underline{\underline{0.1125 \mathbf{a}_z}}$$

Prob. 7.13

- (a) Consider one side of the polygon as shown. The angle subtended by the Side At the center of the circle

$$\frac{360^\circ}{n} = \frac{2\pi}{n}$$

The field due to this side is

$$H_1 = \frac{I}{4\pi\rho} (\cos \alpha_2 - \cos \alpha_1)$$

$$\text{where } \rho = r, \quad \cos \alpha_2 = \cos(90 - \frac{\pi}{n}) = \sin \frac{\pi}{n}$$

$$\cos \alpha_1 = -\sin \frac{\pi}{n}$$

$$H_1 = \frac{I}{4\pi r} 2 \sin \frac{\pi}{n}$$

$$H = nH_1 = \frac{nI}{2\pi r} \sin \frac{\pi}{n}$$

$$(b) \text{ For } n=3, H = \frac{3I}{2\pi r} \sin \frac{\pi}{3}$$

$$r \cot 30^\circ = 2 \rightarrow r = \frac{2}{\sqrt{3}}$$

$$H = \frac{3 \times 5}{2\pi \cancel{2/\sqrt{3}}} \cdot \frac{\sqrt{3}}{2} = \frac{45}{8\pi} = 1.79 \text{ A/m.}$$

$$\text{For } n=4, H = \frac{4I}{2\pi r} \sin \frac{\pi}{4} = \frac{4 \times 5}{2\pi(2)} \cdot \frac{1}{\sqrt{2}} \\ = 1.128 \text{ A/m.}$$

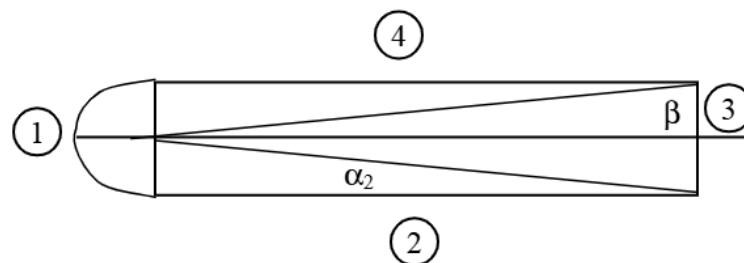
$$(c) \text{ As } n \rightarrow \infty,$$

$$H = \lim_{n \rightarrow \infty} \frac{nI}{2\pi r} \sin \frac{\pi}{n} = \frac{nI}{2\pi r} \cdot \frac{\pi}{n} = \frac{I}{2r}$$

From Example 7.3, when $h=0$,

$$H = \frac{I}{2r}$$

which agrees.

Prob. 7.14

Let $\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2 + \mathbf{H}_3 + \mathbf{H}_4$

$$\mathbf{H}_1 = \frac{I}{4a} \mathbf{a}_z = \frac{10}{4 \times 4 \times 10^{-2}} \mathbf{a}_z = 62.5 \mathbf{a}_z$$

$$\begin{aligned}\mathbf{H}_2 = \mathbf{H}_4 &= \frac{I}{4\pi \times 4 \times 10^{-2}} (\cos \alpha_2 - \cos 90^\circ) \mathbf{a}_z, \quad \alpha_2 = \tan^{-1} \frac{4}{100} = 2.29^\circ \\ &= 19.88 \mathbf{a}_z\end{aligned}$$

$$\begin{aligned}\mathbf{H}_3 &= \frac{I}{4\pi(1)} 2 \cos \beta \mathbf{a}_z, \quad \beta = \tan^{-1} \frac{100}{4} = 87.7^\circ \\ &= \frac{10}{4\pi} 2 \cos 87.7^\circ \mathbf{a}_z = 0.06361 \mathbf{a}_z\end{aligned}$$

$$\begin{aligned}\mathbf{H} &= (62.5 + 2 \times 19.88 + 0.06361) \mathbf{a}_z \\ &= 102.32 \mathbf{a}_z \text{ A/m.}\end{aligned}$$

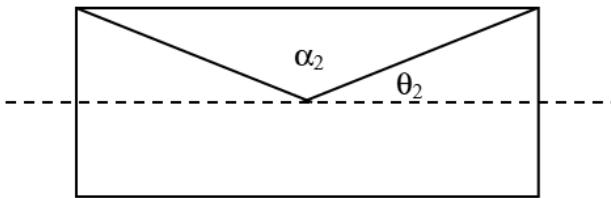
Prob. 7.15

From Example 7.3, \mathbf{H} due to circular loop is

$$\mathbf{H}_i = \frac{\frac{I\rho^2}{2(\rho^2+z^2)}}{\mathbf{a}_z}$$

$$\begin{aligned}(a) \quad \mathbf{H}(0, 0, 0) &= \frac{5 \times 2^2}{2(2^2+0^2)^{3/2}} \mathbf{a}_z + \frac{5 \times 2^2}{2(2^2+4^2)^{3/2}} \mathbf{a}_z \\ &= \underline{\underline{1.36 \mathbf{a}_z \text{ A/m}}}\end{aligned}$$

$$\begin{aligned}(b) \quad \mathbf{H}(0, 0, 2) &= 2 \frac{5 \times 2^2}{2(2^2+2^2)^{3/2}} \mathbf{a}_z \\ &= \underline{\underline{0.884 \mathbf{a}_z \text{ A/m}}}\end{aligned}$$

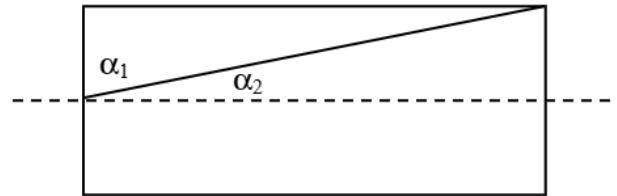
Prob. 7.16

$$|\mathbf{H}| = \frac{nI}{2}(\cos \theta_2 - \cos \theta_1)$$

$$\cos \theta_2 = -\cos \theta_1 = \frac{\ell/2}{\left(a^2 + \ell^2/4\right)^{1/2}}$$

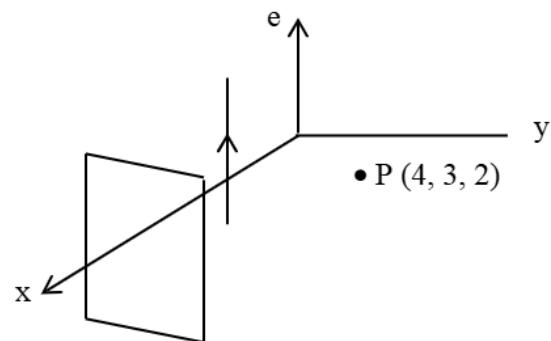
$$|\mathbf{H}| = \frac{nI\ell}{2\left(a^2 + \ell^2/4\right)^{1/2}} = \frac{0.5 \times 150 \times 2 \times 10^{-2}}{2 \times 10^{-3} \times \sqrt{4^2 + 10^2}} = \underline{\underline{69.63 \text{ A/m}}}$$

(b)



$$\alpha_1 = 90^\circ, \tan \theta_2 = \frac{a}{b} = \frac{4}{20} = 0.2 \rightarrow \theta_2 = 11.31^\circ$$

$$|\mathbf{H}| = \frac{nI}{2} \cos \theta_2 = \frac{150 \times 0.5}{2} \cos 11.31^\circ = \underline{\underline{36.77 \text{ A/m}}}$$

Prob. 7.17

$$\text{Let } \mathbf{H} = \mathbf{H}_l + \mathbf{H}_p$$

$$\mathbf{H}_l = \frac{1}{2\pi\rho} \mathbf{a}_\phi$$

$$\boldsymbol{\rho} = (4, 3, 2) - (1, -2, 2) = (3, 5, 0), \quad \rho = |\boldsymbol{\rho}| = \sqrt{34}$$

$$\mathbf{a}_\rho = \frac{3\mathbf{a}_x + 5\mathbf{a}_y}{\sqrt{34}}, \quad \mathbf{a}_1 = \mathbf{a}_z$$

$$\mathbf{a}_\phi = \mathbf{a}_1 \times \mathbf{a}_\rho = \mathbf{a}_z \times \left(\frac{3\mathbf{a}_x + 5\mathbf{a}_y}{\sqrt{34}} \right) = \frac{3\mathbf{a}_y - 5\mathbf{a}_x}{\sqrt{34}}$$

$$\mathbf{H}_1 = \frac{20\pi}{2\pi} \left(\frac{-5\mathbf{a}_x + 3\mathbf{a}_y}{34} \right) \times 10^{-3} = (-1.47\mathbf{a}_y + 0.88\mathbf{a}_y) \text{ mA/m}$$

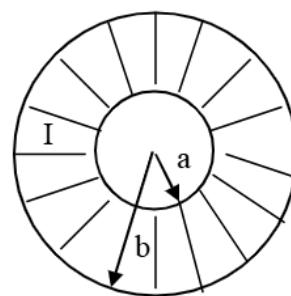
$$\mathbf{H}_p = \frac{1}{2} \mathbf{K} \times \mathbf{a}_n = \frac{1}{2} (100 \times 10^{-3}) \mathbf{a}_z \times (-\mathbf{a}_x) = -0.05\mathbf{a}_y \text{ A/m}$$

$$\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_p = \underline{\underline{-1.47\mathbf{a}_x - 49.12\mathbf{a}_y \text{ mA/m}}}$$

Prob. 7.18

(a) See text

(b)



$$\text{For } \rho < a, \quad \oint \mathbf{H} \cdot d\mathbf{l} = I_{\text{enc}} = 0 \rightarrow H = 0$$

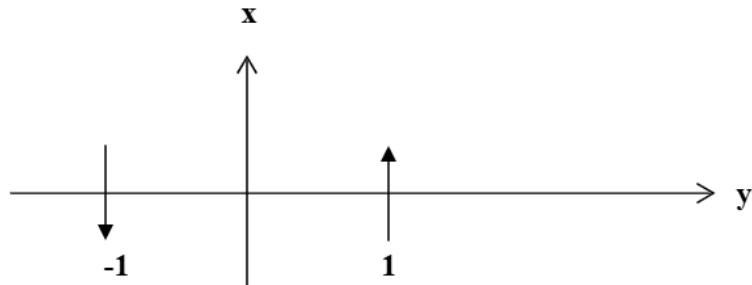
$$\text{For } a < \rho < b, \quad H_\phi \cdot 2\pi\rho = \frac{I\pi(\rho^2 - a^2)}{\pi(b^2 - a^2)}$$

$$H_\phi = \frac{I}{2\pi\rho} \left(\frac{\rho^2 - a^2}{b^2 - a^2} \right)$$

$$\text{For } \rho > b, \quad H_\phi \cdot 2\pi\rho = I \rightarrow H_\phi = \frac{I}{2\pi\rho}$$

Thus,

$$H_\phi = \begin{cases} 0, & \rho < a \\ \frac{I}{2\pi\rho} \left(\frac{\rho^2 - a^2}{b^2 - a^2} \right), & a < \rho < b \\ \frac{I}{2\pi\rho}, & \rho > b \end{cases}$$

Prob. 7.19

$$\begin{aligned}
 \mathbf{H} &= \sum \frac{1}{2} \mathbf{K} \times \mathbf{a}_n \\
 &= \frac{1}{2} (20\mathbf{a}_x) \times (-\mathbf{a}_y) + \frac{1}{2} (-20\mathbf{a}_x) \times \mathbf{a}_y \\
 &= 10(-\mathbf{a}_z) - 10(\mathbf{a}_z) \\
 &= \underline{\underline{-20\mathbf{a}_z \text{ A/m}}}
 \end{aligned}$$

Prob. 7.20

$$\mathbf{H}_P = \frac{1}{2} \mathbf{k} \times \mathbf{a}_n = \frac{1}{2} 10\mathbf{a}_x \times \mathbf{a}_z = -5\mathbf{a}_y$$

$$\mathbf{H}_L = \frac{I}{2\pi\rho} \mathbf{a}_\phi = \frac{I}{2\pi(3)} (\mathbf{a}_x \times -\mathbf{a}_z) = \frac{I}{6\pi} \mathbf{a}_y$$

$$\mathbf{H}_P + \mathbf{H}_L = -5\mathbf{a}_y + \frac{I}{6\pi} \mathbf{a}_y = 0 \quad \longrightarrow \quad I = 30\pi = \underline{\underline{94.25 \text{ A}}}$$

Prob. 7.21

(a) Applying Ampere's law,

$$\begin{aligned}
 H_\phi \cdot 2\pi\rho &= I \cdot \frac{\pi\rho^2}{\pi a^2} \quad \rightarrow \quad H_\phi = I \cdot \frac{\rho^2}{2\pi a^2} \\
 \text{i.e.} \quad \mathbf{H} &= \frac{I\rho}{2\pi a^2} \mathbf{a}_\phi
 \end{aligned}$$

(b) From Eq. (7.29),

$$H_\phi = \begin{cases} \frac{I\rho}{2\pi a^2}, & \rho < a \\ \frac{I}{2\pi\rho}, & \rho > a \end{cases}$$

At $(0, 1 \text{ cm}, 0)$,

$$H_\phi = \frac{3 \times 1 \times 10^{-2}}{2\pi \times 4 \times 10^{-4}} = \frac{300}{8\pi}$$

$$H = \underline{\underline{11.94 a_\phi \text{ A/m}}}$$

At $(0, 4 \text{ cm}, 0)$,

$$H_\phi = \frac{3}{2\pi \times 4 \times 10^{-2}} = \frac{300}{8\pi}$$

$$H = \underline{\underline{11.94 a_\phi \text{ A/m}}}$$

Prob. 7.22

For $0 < \rho < a$

$$\oint_L \mathbf{H} \cdot d\mathbf{l} = I_{enc} = \int \mathbf{J} \cdot d\mathbf{S}$$

$$H_\phi 2\pi\rho = \int_{\phi=0}^{2\pi} \int_{\rho=0}^{\rho} \frac{J_o}{\rho} \rho d\phi d\rho$$

$$= J_o 2\pi\rho$$

$$H_\phi = J_o$$

For $\rho > a$

$$\int \mathbf{H} \cdot d\mathbf{l} = \int \mathbf{J} \cdot d\mathbf{S} = \int_{\phi=0}^{2\pi} \int_{\rho=a}^{\rho} \frac{J_o}{\rho} \rho d\phi d\rho$$

$$H_\rho 2\pi\rho = J_o 2\pi a$$

$$H_\phi = \frac{J_o a}{\rho}$$

$$\text{Hence } H_\phi = \begin{cases} J_o, & 0 < \rho < a \\ \frac{J_o a}{\rho}, & \rho > a \end{cases}$$

Prob. 7.23

$$(a) \mathbf{J} = \nabla \times \mathbf{H} = \frac{1}{\rho} \frac{d}{d\rho} (\rho H_\phi) \mathbf{a}_z = \frac{1}{\rho} \frac{d}{d\rho} (k_o \frac{\rho^2}{a}) \mathbf{a}_z = \underline{\underline{\frac{2k_o}{a} \mathbf{a}_z}}$$

(b) For $\rho > a$,

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{enc} = \int \mathbf{J} \cdot d\mathbf{S} = \int_{\rho=0}^a \int_{\phi=0}^{2\pi} \frac{2k_o}{a} \rho d\rho d\phi = \frac{2k_o}{a} (2\pi) \frac{\rho^2}{2} \Big|_0^a$$

$$H_\phi 2\pi\rho = 2\pi k_o a \quad \longrightarrow \quad H_\phi = \frac{k_o a}{\rho}$$

$$\mathbf{H} = k_o \left(\frac{a}{\rho} \right) \mathbf{a}_\phi, \quad \underline{\underline{\rho > a}}$$

Prob. 7.24

$$\mathbf{J} = \nabla \times \mathbf{H} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & 0 \end{vmatrix} = (2x - 2y) \mathbf{a}_z$$

At (1, -4, 7), x = 1, y = -4, z = 7,

$$\mathbf{J} = [2(1) - 2(-4)] \mathbf{a}_z = \underline{\underline{10 \mathbf{a}_z \text{ A/m}^2}}$$

Prob. 7.25

(a)

$$\mathbf{J} = \nabla \times \mathbf{H} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho H_\phi) \mathbf{a}_z = \frac{1}{\rho} \frac{\partial}{\partial \rho} (10^3 \rho^3) \mathbf{a}_z = \underline{\underline{3\rho \times 10^3 \mathbf{a}_z \text{ A/m}^2}}$$

(b)

Method 1:

$$I = \iint_S \mathbf{J} \cdot d\mathbf{S} = \iint 3\rho \rho d\phi d\rho 10^3 = 3 \times 10^3 \int_0^2 \rho^2 d\rho \int_0^{2\pi} d\phi = 3 \times 10^3 (2\pi) \frac{\rho^3}{3} \Big|_0^2 = 16\pi \times 10^3 A = \underline{\underline{50.265 \text{ kA}}}$$

Method 2:

$$I = \oint_L \mathbf{H} \cdot d\mathbf{l} = 10^3 \int_0^{2\pi} \rho^2 \rho d\phi = 10^3 (8)(2\pi) = \underline{\underline{50.265 \text{ kA}}}$$

Prob. 7.26

Let $\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2$

where \mathbf{H}_1 and \mathbf{H}_2 are due to the wires centered at $x = 0$ and $x = 10\text{cm}$ respectively.

(a) For \mathbf{H}_1 , $\rho = 50\text{ cm}$, $\mathbf{a}_\phi = \mathbf{a}_l \times \mathbf{a}_\rho = \mathbf{a}_z \times \mathbf{a}_x = \mathbf{a}_y$

$$\mathbf{H}_1 = \frac{5}{2\pi(5 \times 10^{-2})} \mathbf{a}_y = \frac{50}{\pi} \mathbf{a}_y$$

For \mathbf{H}_2 , $\rho = 5\text{ cm}$, $\mathbf{a}_\phi = -\mathbf{a}_z \times -\mathbf{a}_x = \mathbf{a}_y$, $\mathbf{H}_2 = \mathbf{H}_1$

$$\begin{aligned}\mathbf{H} &= 2\mathbf{H}_1 = \frac{100}{\pi} \mathbf{a}_y \\ &= \underline{\underline{31.83 \mathbf{a}_y \text{ A/m}}}\end{aligned}$$

(b) For \mathbf{H}_1 , $\mathbf{a}_\phi = \mathbf{a}_z \times \left(\frac{2\mathbf{a}_x + \mathbf{a}_y}{\sqrt{5}} \right) = \frac{2\mathbf{a}_y - \mathbf{a}_x}{\sqrt{5}}$

$$\mathbf{H}_1 = \frac{5}{2\pi 5 \sqrt{5} \times 10^{-2}} \left(\frac{-\mathbf{a}_x + 2\mathbf{a}_y}{\sqrt{5}} \right) = -3.183 \mathbf{a}_x + 6.366 \mathbf{a}_y$$

For \mathbf{H}_2 , $\mathbf{a}_\rho = -\mathbf{a}_z \times \mathbf{a}_y = \mathbf{a}_x$

$$\mathbf{H}_2 = \frac{5}{2\pi(5)} \mathbf{a}_x = 15.915 \mathbf{a}_x$$

$$\begin{aligned}\mathbf{H} &= \mathbf{H}_1 + \mathbf{H}_2 \\ &= \underline{\underline{12.3 \mathbf{a}_x + 6.366 \mathbf{a}_y \text{ A/m}}}\end{aligned}$$

Prob. 7.27

(a) $\mathbf{B} = \frac{\mu_0 I}{2\pi\rho} \mathbf{a}_\phi$

At $(-3, 4, 5)$, $\rho = 5$.

$$B = \frac{4\pi \times 10^{-7} \times 2}{2\pi(5)} \mathbf{a}_\phi = \underline{\underline{80 \mathbf{a}_\phi \text{ nW/m}^2}}$$

$$\begin{aligned}\text{(b)} \quad \Psi &= \int \mathbf{B} \bullet dS = \frac{\mu_0 I}{2\pi} \iint \frac{d\rho dz}{\rho} = \frac{4\pi \times 10^{-7} \times 2}{2\pi} \ln \rho \Big|_2^6 z \Big|_0^4 \\ &= 16 \times 10^{-7} \ln 3 = \underline{\underline{1.756 \mu\text{Wb}}}\end{aligned}$$

Prob. 7.28

(a) $I = \int \mathbf{J} \cdot d\mathbf{S}$

$$\begin{aligned}
 &= \int_{\phi=0}^{2\pi} \int_{\rho=0}^a J_o \left(1 - \frac{\rho^2}{a^2}\right) \rho d\rho d\phi = J_o \int_0^{2\pi} d\phi \int_0^a \left(\rho - \frac{\rho^3}{a^2}\right) d\rho \\
 &= 2\pi J_o \left(\frac{\rho^2}{2} - \frac{\rho^4}{4a^2} \right) \Big|_0^a = \frac{2\pi}{2} J_o \left(a^2 - \frac{a^2}{2} \right) \\
 &= \underline{\underline{\frac{1}{2}\pi a^2 J_o}}
 \end{aligned}$$

(b) $\oint \mathbf{H} \cdot d\mathbf{l} = I_{enc} = \int \mathbf{J} \cdot d\mathbf{S}$

For $\rho < a$,

$$\begin{aligned}
 H_\phi 2\pi\rho &= \int \mathbf{J} \cdot d\mathbf{S} \\
 &= 2\pi J_o \left(\frac{\rho^2}{2} - \frac{\rho^4}{4a^2} \right) \\
 H_\rho \cdot 2\pi\rho &= 2\pi J_o \frac{\rho^2}{4} \left(2 - \frac{\rho^2}{a^2} \right) \\
 H_\rho &= \frac{J_o \rho}{4} \left(2 - \frac{\rho^2}{a^2} \right)
 \end{aligned}$$

For $\rho > a$,

$$\oint \mathbf{H} \cdot d\mathbf{l} = \int J_o dS = I$$

$$H_\phi \cdot 2\pi\rho = \frac{1}{2}\pi a^2 J_o$$

$$H_\phi = \frac{a^2 J_o}{4\rho}$$

Hence $H_\phi = \begin{cases} \frac{J_o \rho}{4} \left(2 - \frac{\rho^2}{a^2} \right), & \rho < a \\ \frac{a J_o}{4\rho}, & \rho > a \end{cases}$

Prob. 7.29

$$\begin{aligned}\mathbf{B} &= \frac{\mu_0 I}{2\pi\rho} \mathbf{a}_\phi \\ \Psi &= \int \mathbf{B} \cdot d\mathbf{S} = \int_{\rho=d}^{d+a} \int_{z=0}^b \frac{\mu_0 I}{2\pi\rho} d\rho dz \\ &= \frac{\mu_0 I b}{2\pi} \ln \frac{d+a}{d}\end{aligned}$$

Prob. 7.30

For a whole circular loop of radius a , Example 7.3 gives

$$\mathbf{H} = \frac{Ia^2 \mathbf{a}_z}{2[a^2 + h^2]^{3/2}}$$

Let $h \rightarrow 0$

$$\mathbf{H} = \frac{I}{2a} \mathbf{a}_z$$

For a semicircular loop, H is halved

$$\mathbf{H} = \frac{I}{4a} \mathbf{a}_z$$

$$\mathbf{B} = \mu_0 \mathbf{H} = \frac{\mu_0 I}{4a} \mathbf{a}_z$$

Prob. 7.31

$$(a) \nabla \bullet \mathbf{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$$

showing that \mathbf{B} satisfies Maxwell's equation.

$$(b) d\mathbf{S} = dy dz \mathbf{a}_x$$

$$\Psi = \int \mathbf{B} \bullet d\mathbf{S} = \int_{z=1}^4 \int_{y=0}^1 y^2 dy dz = \frac{y^3}{3} \Big|_0^1 (z) \Big|_1^4 = 1 \text{ Wb}$$

$$(c) \nabla \times \mathbf{H} = \mathbf{J} \longrightarrow \mathbf{J} = \nabla \times \frac{\mathbf{B}}{\mu_0}$$

$$\nabla \times \mathbf{B} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & z^2 & x^2 \end{vmatrix} = -2z\mathbf{a}_x - 2x\mathbf{a}_y - 2y\mathbf{a}_z$$

$$\mathbf{J} = -\frac{2}{\mu_0} (z\mathbf{a}_x + x\mathbf{a}_y + y\mathbf{a}_z) \text{ A/m}^2$$

Prob. 7.32

On the slant side of the ring, $z = \frac{h}{6} (\rho - a)$

where \mathbf{H}_1 and \mathbf{H}_2 are due to the wires centered at $x = 0$ and $x = 10\text{cm}$ respectively.

$$\begin{aligned} \psi &= \int \mathbf{B} \cdot d\mathbf{S} = \int \frac{\mu_0 I}{2\pi\rho} d\rho dz \\ &= \frac{\mu_0 I}{2\pi} \int_{\rho=a}^{a+b} \int_{z=0}^{\frac{h}{6}(\rho-a)} \frac{dz d\rho}{\rho} = \frac{\mu_0 I h}{2\pi b} \int_{\rho=a}^{a+b} \left(1 - \frac{a}{\rho}\right) d\rho \\ &= \frac{\mu_0 I h}{2\pi b} \left(b - a \ln \frac{a+b}{a}\right) \text{ as required.} \end{aligned}$$

If $a = 30\text{ cm}$, $b = 10\text{ cm}$, $h = 5\text{ cm}$, $I = 10\text{ A}$,

$$\begin{aligned} \psi &= \frac{4\pi \times 10^{-7} \times 10 \times 0.05}{2\pi (10 \times 10^{-2})} \left(0.1 - 0.3 \ln \frac{4}{3}\right) \\ &= \underline{\underline{1.37 \times 10^{-8} \text{ Wb}}} \end{aligned}$$

Prob. 7.33

$$\begin{aligned} \psi &= \int \mathbf{B} \cdot d\mathbf{S} = \mu_0 \int_{z=0}^{0.2} \int_{\phi=0}^{50^\circ} \frac{10^6}{\rho} \sin 2\phi \rho d\phi dz \\ \psi &= 4\pi \times 10^{-7} \times 10^6 (0.2) \left(-\frac{\cos 2\phi}{2}\right) \Big|_0^{50^\circ} \\ &= 0.04\pi (1 - \cos 100^\circ) \\ &= \underline{\underline{0.1475 \text{ Wb}}} \end{aligned}$$

Prob. 7.34

$$\begin{aligned}\psi &= \int_S \mathbf{B} \cdot d\mathbf{S} = \int_{\phi=0}^{\pi/4} \int_{\rho=1}^2 \frac{20}{\rho} \sin^2 \phi \rho d\rho d\phi = 20 \int_1^2 d\rho \int_0^{\pi/4} \sin^2 \phi d\phi \\ &= 20(1) \int_0^{\pi/4} \frac{1}{2}(1 - \cos 2\phi) d\phi = 10(\phi - \frac{1}{2} \sin 2\phi) \Big|_0^{\pi/4} \\ &= 10(\frac{\pi}{4} - \frac{1}{2}) = \underline{\underline{2.854 \text{ Wb}}}\end{aligned}$$

Prob. 7.35

$$\begin{aligned}\psi &= \int_S \mathbf{B} \cdot d\mathbf{S}, \quad d\mathbf{S} = r^2 \sin \theta d\theta d\phi \mathbf{a}_r \\ \psi &= \iint \frac{2}{r^3} \cos \theta r^2 \sin \theta d\theta d\phi \Big|_{r=1} = 2 \int_0^{2\pi} d\phi \int_0^{\pi/3} \cos \theta \sin \theta d\theta \\ &= 2(2\pi) \int_0^{\pi/3} \sin \theta d(\sin \theta) = 4\pi \frac{\sin^2 \theta}{2} \Big|_0^{\pi/3} = 2\pi \sin^2(\pi/3) \\ &= \underline{\underline{4.7123 \text{ Wb}}}\end{aligned}$$

Prob. 7.36

$$\mathbf{B} = \mu_o \mathbf{H} = \frac{\mu_o}{4\pi} \int_v \frac{\mathbf{J} \times \mathbf{R}}{R^3} dv$$

Since current is the flow of charge, we can express this in terms of a charge moving with velocity \mathbf{u} . $\mathbf{J} dv = dq\mathbf{u}$.

$$\mathbf{B} = \frac{\mu_o}{4\pi} \left[\frac{q\mathbf{u} \times \mathbf{R}}{R^3} \right]$$

In our case, \mathbf{u} and \mathbf{R} are perpendicular. Hence,

$$\begin{aligned}B &= \frac{\mu_o}{4\pi} \frac{qu}{R^2} = \frac{4\pi \times 10^{-7}}{4\pi} \times \frac{1.6 \times 10^{-19} \times 2.2 \times 10^6}{(5.3 \times 10^{-11})^2} = \frac{1.6 \times 10^{-20}}{(5.3)^2 \times 10^{-22}} \\ &= \underline{\underline{12.53 \text{ Wb/m}^2}}\end{aligned}$$

Prob. 7.37

(a) $\nabla \cdot \mathbf{A} = -ya \sin ax \neq 0$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y \cos ax & 0 & y + e^x \end{vmatrix}$$

$$= \mathbf{a}_x + e^{-x} \mathbf{a}_y - \cos ax \mathbf{a}_z \neq \mathbf{0}$$

 \mathbf{A} is neither electrostatic nor magnetostatic field

(b) $\nabla \cdot \mathbf{B} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho B_\rho) = \frac{1}{\rho} \frac{\partial}{\partial \rho} (20) = 0$

$\nabla \times \mathbf{B} = \mathbf{0}$

 \mathbf{B} can be E-field in a charge-free region.

(c) $\nabla \cdot \mathbf{C} = \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (r^2 \sin \theta) = 0$

$\nabla \times \mathbf{C} = \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (r^2 \sin^2 \theta) \mathbf{a}_r - \frac{1}{r} \frac{\partial}{\partial r} (r^3 \sin \theta) \mathbf{a}_\theta \neq \mathbf{0}$

 \mathbf{C} is possibly \mathbf{H} field.**Prob. 7.38**

(a) $\nabla \cdot \mathbf{D} = 0$

$$\nabla \times \mathbf{D} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 z & 2(x+1)yz & -(x+1)z^2 \end{vmatrix}$$

$$= 2(x+1)y \mathbf{a}_x + \dots \neq \mathbf{0}$$

 \mathbf{D} is possibly a magnetostatic field.

(b) $\nabla \cdot \mathbf{E} = \frac{1}{\rho} \frac{\partial}{\partial \rho} ((z+1) \cos \phi) + \frac{\partial}{\partial z} \left(\frac{\sin \phi}{\rho} \right) = 0$

$\nabla \times \mathbf{E} = \frac{1}{\rho^2} \cos \theta \mathbf{a}_\rho + \dots \neq \mathbf{0}$

 \mathbf{E} could be a magnetostatic field.

(c) $\nabla \cdot \mathbf{F} = \frac{1}{r^2} \frac{\partial}{\partial r} (2 \cos \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left(\frac{\sin \theta}{r^2} \right) \neq 0$

$\nabla \times \mathbf{F} = \frac{1}{r} \left[\frac{\partial}{\partial r} (r^{-1} \sin \theta) + \frac{2 \sin \theta}{r^2} \right] \mathbf{a}_\theta \neq \mathbf{0}$

 \mathbf{F} can be neither electrostatic nor magnetostatic field.

Prob. 7.39

$$\mathbf{A} = \int \frac{\mu_0 I dl}{4\pi r} = \frac{\mu_0 IL \mathbf{a}_z}{4\pi r}$$

This requires no integration since $L \ll r$.

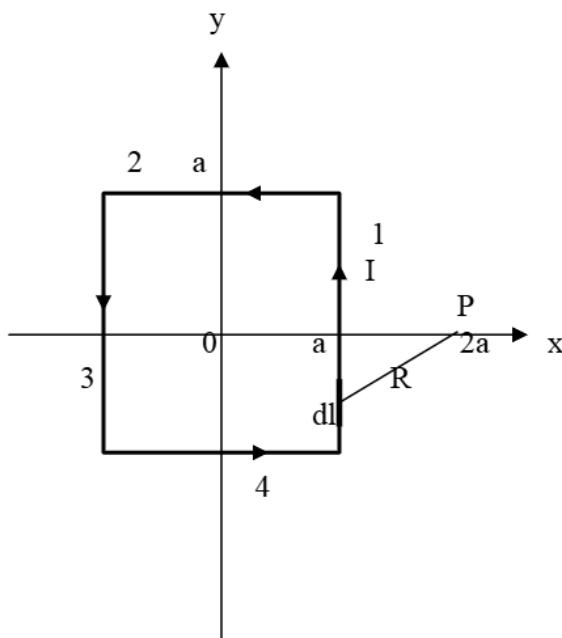
$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{1}{\rho} \frac{\partial A_z}{\partial \phi} \mathbf{a}_\rho - \frac{\partial A_z}{\partial \rho} \mathbf{a}_\phi$$

$$\text{But } r = \sqrt{\rho^2 + z^2}$$

$$\mathbf{A} = \frac{\mu_0 IL \mathbf{a}_z}{4\pi(\rho^2 + z^2)^{1/2}}$$

$$\frac{\partial A_z}{\partial \rho} = \frac{\mu_0 IL}{4\pi} \frac{\partial}{\partial \rho} (\rho^2 + z^2)^{1/2} = \frac{\mu_0 IL}{4\pi} \left(-\frac{1}{2}\right) (\rho^2 + z^2)^{-3/2} (2\rho)$$

$$\mathbf{B} = \underline{\underline{\frac{\mu_0 IL \rho \mathbf{a}_\phi}{4\pi(\rho^2 + z^2)^{3/2}}}} = \frac{\mu_0 IL \rho \mathbf{a}_\phi}{4\pi r^3}$$

Prob. 7.40

Divide the loop into four segments as shown above. Due to segment 1,

$$\begin{aligned} \mathbf{A}_1 &= \int \frac{\mu_o I d\mathbf{l}}{4\pi R}, \quad d\mathbf{l} = dy \mathbf{a}_y, R = \sqrt{y^2 + a^2} \\ \mathbf{A}_1 &= \frac{\mu_o I}{4\pi} \mathbf{a}_y \int_{y=-a}^a \frac{dy}{\sqrt{y^2 + a^2}} = \frac{\mu_o I}{4\pi} \mathbf{a}_y \left(\ln(y + \sqrt{y^2 + a^2}) \right) \Big|_{-a}^a \\ &= \frac{\mu_o I}{4\pi} \mathbf{a}_y \ln \left(\frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right) = \frac{\mu_o I}{2\pi} \ln(\sqrt{2} + 1) \mathbf{a}_y \end{aligned}$$

By symmetry, the contributions due to sides 2 and 4 cancel. For side 3,

$$\begin{aligned} \mathbf{A}_3 &= \int \frac{\mu_o I d\mathbf{l}}{4\pi R}, \quad d\mathbf{l} = dy (-\mathbf{a}_y), R = \sqrt{y^2 + (-3a)^2} \\ \mathbf{A}_3 &= \frac{\mu_o I}{4\pi} (-\mathbf{a}_y) \left(\ln(y + \sqrt{y^2 + 9a^2}) \right) \Big|_{-a}^a = \frac{\mu_o I}{4\pi} (-\mathbf{a}_y) \ln \left(\frac{\sqrt{10} + 1}{\sqrt{10} - 1} \right) \\ &= \frac{\mu_o I}{2\pi} \ln \left(\frac{\sqrt{10} + 1}{3} \right) (-\mathbf{a}_y) \\ \mathbf{A} &= \mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 + \mathbf{A}_4 = \frac{\mu_o I}{2\pi} \ln(\sqrt{2} + 1) \mathbf{a}_y - \frac{\mu_o I}{2\pi} \ln \left(\frac{\sqrt{10} + 1}{3} \right) \mathbf{a}_y \\ &= \underline{\underline{\frac{\mu_o I}{2\pi} \ln \left(\frac{3(\sqrt{2} + 1)}{\sqrt{10} + 1} \right) \mathbf{a}_y}} \end{aligned}$$

Prob. 7.41

$$\begin{aligned} \mathbf{B} &= \nabla \times \mathbf{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & A_z(x, y) \end{vmatrix} = \frac{\partial A_z}{\partial y} \mathbf{a}_x - \frac{\partial A_z}{\partial x} \mathbf{a}_y \\ &= \underline{\underline{-\frac{\pi}{2} \sin \frac{\pi x}{2} \sin \frac{\pi y}{2} \mathbf{a}_x - \frac{\pi}{2} \cos \frac{\pi x}{2} \cos \frac{\pi y}{2} \mathbf{a}_y}} \end{aligned}$$

Prob. 7.42

$$\begin{aligned} \mathbf{B} &= \mu_o \mathbf{H} = \nabla \times \mathbf{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & A_z(x, y) \end{vmatrix} = \frac{\partial A_z}{\partial y} \mathbf{a}_x - \frac{\partial A_z}{\partial x} \mathbf{a}_z = -2\mu_o k y \mathbf{a}_x + 2\mu_o k x \mathbf{a}_y \\ \mathbf{H} &= \underline{\underline{-2k y \mathbf{a}_x + 2k x \mathbf{a}_y}} \end{aligned}$$

Prob. 7.43

$$\mathbf{B}_1 = \nabla \times \mathbf{A}_1 = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & (\sin x + x \sin y) & 0 \end{vmatrix} = (\cos x + \sin y) \mathbf{a}_z$$

$$\mathbf{B}_2 = \nabla \times \mathbf{A}_2 = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos y & \sin x & 0 \end{vmatrix} = (\cos x + \sin y) \mathbf{a}_z$$

$$\mathbf{B}_1 = \mathbf{B}_2 = \mathbf{B}$$

Hence, \mathbf{A}_1 and \mathbf{A}_2 give the same \mathbf{B} .

$$\nabla \cdot \mathbf{B} = 0$$

showing that \mathbf{B} is solenoidal.

Prob. 7.44

$$\begin{aligned} \mathbf{B} &= \nabla \times \mathbf{A} = \frac{1}{\rho} \frac{\partial A_z}{\partial \phi} \mathbf{a}_\rho - \frac{\partial A_z}{\partial \rho} \mathbf{a}_\phi \\ &= \frac{15}{\rho} e^{-\rho} \cos \phi \mathbf{a}_\rho + 15 e^{-\rho} \sin \phi \mathbf{a}_\phi \\ \mathbf{B} \left(3, \frac{\pi}{4}, -10 \right) &= 5 e^{-3} \frac{1}{\sqrt{2}} \mathbf{a}_\rho + 15 e^{-3} \frac{1}{\sqrt{2}} \mathbf{a}_\phi \\ \mathbf{H} &= \frac{\mathbf{B}}{\mu_0} = \frac{10^7}{4\pi} \frac{15}{\sqrt{2}} e^{-3} \left(\frac{1}{3} \mathbf{a}_\rho + \mathbf{a}_\phi \right) \\ \mathbf{H} &= (14 \mathbf{a}_\rho + 42 \mathbf{a}_\phi) \cdot 10^4 \text{ A/m} \\ \psi &= \int \mathbf{B} \cdot d\mathbf{S} = \iint \frac{15}{\rho} e^{-\rho} \cos \phi \rho d\phi dz \\ &= 15 z \Big|_0^{10} (\sin \phi) \Big|_0^{\pi/2} e^{-5} = 150 e^{-5} \quad \Rightarrow \quad \psi = \underline{\underline{1.011 \text{ Wb}}} \end{aligned}$$

Prob. 7.45

$$\begin{aligned}
 \mathbf{B} = \nabla \times \mathbf{A} &= \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (A_\phi \sin \theta) - \frac{\partial A_\theta}{\partial \phi} \right] \mathbf{a}_r + \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r} (r A_\phi) \right] \mathbf{a}_\theta \\
 &\quad + \frac{1}{r} \left[\frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right] \mathbf{a}_\phi \\
 &= \frac{1}{r \sin \theta} \frac{10}{r} 2 \sin \theta \cos \theta \mathbf{a}_r - \frac{1}{r} \frac{\partial}{\partial r} (10) \sin \theta \mathbf{a}_\theta + 0 \mathbf{a}_\phi \\
 \mathbf{B} &= \frac{20}{r^2} \cos \theta \mathbf{a}_r
 \end{aligned}$$

At $(4, 60^\circ, 30^\circ)$, $r = 4$, $\theta = 60^\circ$

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} = \frac{1}{4\pi \times 10^{-7}} \left[\frac{20}{4^2} \cos 60^\circ \mathbf{a}_r \right] = \underline{\underline{4.974 \times 10^5 \mathbf{a}_r \text{ A/m}}}$$

Prob. 7.46

Applying Ampere's law gives

$$H_\phi \cdot 2\pi\rho = J_o \cdot \pi\rho^2$$

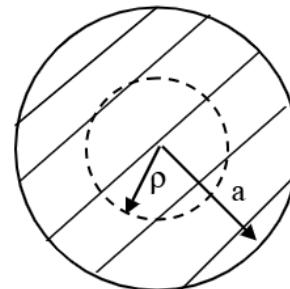
$$H_\phi = \frac{J_o}{2} \rho$$

$$B_\phi = \mu_0 H_\phi = \mu_0 \frac{J_o \rho}{2}$$

$$\text{But } \mathbf{B} = \nabla \times \mathbf{A} = -\frac{\partial A_z}{\partial \rho} \bar{\mathbf{a}}_\phi + \dots$$

$$-\frac{\partial A_z}{\partial \rho} = \frac{1}{2} \mu J_o \rho \longrightarrow A_z = -\mu_0 \frac{J_o \rho^2}{4}$$

$$\text{or } \mathbf{A} = \underline{\underline{-\frac{1}{4} \mu_0 J_o \rho^2 \mathbf{a}_z}}$$



Prob. 7.47

$$\mathbf{B} = \mu_0 \mathbf{H} = \nabla \times \mathbf{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 10 \sin \pi y & 0 & 4 + \cos \pi x \end{vmatrix} = \pi \sin \pi x \mathbf{a}_y - 10 \pi \cos \pi y \mathbf{a}_z$$

$$\underline{\underline{\mathbf{H}}} = \frac{\pi}{\mu_0} \left(\sin \pi x \mathbf{a}_y - 10 \cos \pi y \mathbf{a}_z \right)$$

$$\mathbf{J} = \nabla \times \mathbf{H} = \frac{\pi}{\mu_0} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & \sin \pi x & -10 \cos \pi y \end{vmatrix} = \frac{\pi}{\mu_0} \left(10 \pi \sin \pi y \mathbf{a}_x + \pi \cos \pi x \mathbf{a}_z \right)$$

$$\underline{\underline{\mathbf{J}}} = \frac{\pi^2}{\mu_0} (10 \sin \pi y \mathbf{a}_x + \cos \pi x \mathbf{a}_z)$$

Prob. 7.48

$$\begin{aligned} \mathbf{B} = \nabla \times \mathbf{A} &= \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (A_\phi \sin \theta) \mathbf{a}_r - \frac{1}{r} \frac{\partial}{\partial r} (r A_\phi) \mathbf{a}_\theta \\ &= \frac{1}{r \sin \theta} \frac{A_\phi}{r^2} (2 \sin \theta \cos \theta) \mathbf{a}_r - \frac{A_\phi}{r} \sin \theta (-r^{-2}) \mathbf{a}_\theta \\ &= \frac{A_\phi}{r^3} (2 \cos \theta \mathbf{a}_r + \sin \theta \mathbf{a}_\theta) \end{aligned}$$

Prob. 7.49

$$(a) \quad \mathbf{J} = \nabla \times \mathbf{H} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & x^2 z & -y^2 z \end{vmatrix} = (-2yz - x^2) \mathbf{a}_x + (2xz - 2xy) \mathbf{a}_z$$

At (2, -1, 3), x=2, y=-1, z=3.

$$\underline{\underline{\mathbf{J}}} = 2 \mathbf{a}_x + 16 \mathbf{a}_z \text{ A/m}^2$$

$$(b) \quad -\frac{\partial \rho_v}{\partial t} = \nabla \bullet \mathbf{J} = 0 - 2x + 2x = 0$$

At (2, -1, 3),

$$\frac{\partial \rho_v}{\partial t} = 0 \text{ C/m}^3 \text{s}$$

Prob. 7.50

(a) $\mathbf{B} = \nabla \times \mathbf{A}$

$$\begin{aligned} &= \left[\frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right] \mathbf{a}_\rho + \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right] \mathbf{a}_\phi + \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} (\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi} \right] \mathbf{a}_z \\ &= -\frac{\partial A_z}{\partial \rho} \mathbf{a}_\phi = 20\rho \mathbf{a}_\phi \text{ } \mu\text{Wb/m}^2 \end{aligned}$$

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} = \frac{-20\rho}{\mu_0} \mathbf{a}_\phi \text{ } \mu\text{A/m}$$

$$\begin{aligned} \mathbf{J} = \nabla \times \mathbf{H} &= \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\phi) \mathbf{a}_z \\ &= \frac{1}{\mu_0 \rho} (-40\rho) \mathbf{a}_z = \underline{\underline{\frac{-40}{\mu_0} \mathbf{a}_z \text{ } \mu\text{A/m}^2}} \end{aligned}$$

$$\begin{aligned} (b) \quad I &= \int \mathbf{J} \cdot d\mathbf{S} = \frac{-40}{\mu_0} \int_{\rho=0}^2 \int_{\phi=0}^{2\pi} \rho d\phi d\rho, \quad d\mathbf{S} = \rho d\phi d\rho \mathbf{a}_z \\ &= \frac{-40}{\mu_0} \int_0^2 \rho d\rho \int_0^{2\pi} d\phi = \frac{-40}{\mu_0} \frac{\rho^2}{2} \Big|_0^2 (2\pi) \\ &= \frac{-80\pi \times 2 \times 10^{-6}}{4\pi \times 10^{-7}} = \underline{\underline{-400 \text{ A}}} \end{aligned}$$

Prob. 7.51

$$\mathbf{H} = -\nabla V_m \rightarrow V_m = - \int \mathbf{H} \cdot d\mathbf{l} = -\text{mmf}$$

$$\text{From Example 7.3, } \mathbf{H} = \frac{Ia^2}{2(z^2 + a^2)^{3/2}} \mathbf{a}_z$$

$$V_m = -\frac{Ia^2}{2} \int (z^2 + a^2)^{-3/2} dz = \frac{-Iz}{2(z^2 + a^2)^{1/2}} + c$$

As $z \rightarrow \infty$, $V_m = 0$, i.e.

$$0 = -\frac{I}{2} + c \rightarrow c = \frac{I}{2}$$

Hence,

$$V_m = \frac{I}{2} \left[1 - \frac{z}{\sqrt{z^2 + a^2}} \right]$$

Prob. 7.52

$$\begin{aligned}\nabla' \frac{1}{R} &= \left(\frac{\partial}{\partial x'} \mathbf{a}_x + \frac{\partial}{\partial y'} \mathbf{a}_y + \frac{\partial}{\partial z'} \mathbf{a}_z \right) \left[(x-x')^2 + (y-y')^2 + (z-z')^2 \right]^{-\frac{1}{2}} \\ &= \left(-\frac{1}{2} \right) (-2) (x-x') \mathbf{a}_x \left[(x-x')^2 + (y-y')^2 + (z-z')^2 \right]^{-\frac{3}{2}} + \mathbf{a}_y \text{ and } \mathbf{a}_z \text{ terms} \\ &= \frac{\mathbf{R}}{R^3}\end{aligned}$$

$$R = |\mathbf{r} - \mathbf{r}'| = \left[(x-x')^2 + (y-y')^2 + (z-z')^2 \right]^{\frac{1}{2}}$$

$$\begin{aligned}\nabla \frac{1}{R} &= \left(\frac{\partial}{\partial x} \mathbf{a}_x + \frac{\partial}{\partial y} \mathbf{a}_y + \frac{\partial}{\partial z} \mathbf{a}_z \right) \left[(x-x')^2 + (y-y')^2 + (z-z')^2 \right]^{-\frac{1}{2}} \\ &= -\frac{1}{2} 2 (x-x') \mathbf{a}_x \left[(x-x')^2 + (y-y')^2 + (z-z')^2 \right]^{-\frac{3}{2}} + \mathbf{a}_y \text{ and } \mathbf{a}_z \text{ terms} \\ &= - \left[(x-x') \mathbf{a}_x + (y-y') \mathbf{a}_y + (z-z') \mathbf{a}_z \right] \bigg/ R^3 = -\frac{\bar{\mathbf{R}}}{R^3}\end{aligned}$$

CHAPTER 3

P.E. 3.1

$$(a) \quad \mathbf{F} = m \frac{\partial \mathbf{u}}{\partial t} = Q\mathbf{E} = \underline{\underline{6a_z N}}$$

$$(b) \quad \frac{\partial \mathbf{u}}{\partial t} = 6\mathbf{a}_z = \frac{\partial}{\partial t}(u_x, u_y, u_z) \Rightarrow$$

$$\frac{\partial u_x}{\partial t} = 0 \rightarrow u_x = A$$

$$\frac{\partial u_y}{\partial t} = 0 \rightarrow u_y = B$$

$$\frac{\partial u_z}{\partial t} = 6 \rightarrow u_z = 6t + C$$

Since $\mathbf{u}(t=0) = 0$, $A = B = C = 0$

$$u_x = 0 = u_y, \quad u_z = 6t$$

$$u_x = \frac{\partial x}{\partial t} = 0 \rightarrow x = A$$

$$u_y = \frac{\partial y}{\partial t} = 0 \rightarrow y = B$$

$$u_z = \frac{\partial z}{\partial t} = 6t \rightarrow z = 3t^2 + C_1$$

At $t = 0$, $(x, y, z) = (0, 0, 0)$ $\rightarrow A_1 = 0 = B_1 = C_1$

Hence, $(x, y, z) = (0, 0, 3t^2)$,

$\mathbf{u} = 6t\mathbf{a}_z$ at any time. At $P(0, 0, 12)$, $z = 12 = 3t^2 \rightarrow t = 2s$

$$\underline{\underline{t = 2s}}$$

$$(c) \quad \mathbf{u} = 6t\mathbf{a}_z = 12\mathbf{a}_z \text{ m/s.}$$

$$\mathbf{a} = \frac{\partial \mathbf{u}}{\partial t} = 6\mathbf{a}_z \underline{\underline{\frac{m}{s^2}}}$$

$$(d) \quad K.E. = \frac{1}{2}m|\mathbf{u}|^2 = \frac{1}{2}(1)(144) = \underline{\underline{72J}}$$

P.E. 8.2

$$(a) \quad m\mathbf{a} = e\mathbf{u} \times \mathbf{B} = (eB_o u_y, -eB_o u_x, 0)$$

$$\frac{d^2x}{dt^2} = \frac{eB_o}{m} \frac{dy}{dt} = \omega \frac{dy}{dt} \quad (1)$$

$$\frac{d^2y}{dt^2} = -\frac{eBo}{m} \frac{dx}{dt} = -\omega \frac{dx}{dt} \quad (2)$$

$$\frac{d^2z}{dt^2} = 0; \Rightarrow \frac{dz}{dt} = C_1 \quad (3)$$

From (1) and (2),

$$\frac{d^3x}{dt^3} = \omega \frac{d^2y}{dt^2} = -\omega^2 \frac{dx}{dt}$$

$$(D^2 + \omega^2 D)x = 0 \rightarrow Dx = (0, \pm j\omega)x$$

$$x = c_2 + c_3 \cos \omega t + c_4 \sin \omega t$$

$$\frac{dy}{dt} = \frac{1}{\omega} \frac{d^2x}{dt^2} = -c_3 \omega \cos \omega t - c_4 \omega \sin \omega t$$

At $t = 0$, $\mathbf{u} = (\alpha, 0, \beta)$. Hence,

$$c_1 = \beta, c_3 = 0, c_4 = \frac{\alpha}{\omega}$$

$$\underline{\underline{\frac{dx}{dt} = \alpha \cos \omega t, \frac{dy}{dt} = -\alpha \sin \omega t, \frac{dz}{dt} = \beta}}$$

(b) Solving these yields

$$x = \frac{\alpha}{\omega} \sin \omega t, y = \frac{\alpha}{\omega} \cos \omega t, z = \beta t$$

The starting point of the particle is $(0, \frac{\alpha}{\omega}, 0)$

$$(c) \quad x^2 + y^2 = \frac{\alpha^2}{\omega^2}, z = \beta t$$

showing that the particles move along a helix of radius α/ω placed along the z-axis.

P.E. 8.3

(a) From Example 8.3, $QuB = QE$ regardless of the sign of the charge.

$$E = uB = 8 \times 10^6 \times 0.5 \times 10^{-3} = 4 \text{ kV/m}$$

(b) Yes, since $QuB = QE$ holds for any Q and m .

P.E. 8.4

By Newton's 3rd law, $\mathbf{F}_{12} = \mathbf{F}_{21}$, the force on the infinitely long wire is:

$$\begin{aligned}\mathbf{F}_l = -\mathbf{F} &= \frac{\mu_o I_1 I_2 b}{2\pi} \left(\frac{1}{\rho_o} - \frac{1}{\rho_o + a} \right) \mathbf{a}_\rho \\ &= \frac{4\pi \times 10^{-7} \times 50 \times 3}{2\pi} \left(\frac{1}{2} - \frac{1}{3} \right) \mathbf{a}_\rho = \underline{\underline{5\mathbf{a}_\rho \text{ } \mu N}}\end{aligned}$$

P.E. 8.5

$$\begin{aligned}\mathbf{m} = IS\mathbf{a}_n &= 10 \times 10^{-4} \times 50 \frac{(2, 6, -3)}{7} \\ &= 7.143 \times 10^{-3} (2, 6, -3) \\ &= \underline{\underline{(1.429\mathbf{a}_x + 4.286\mathbf{a}_y - 2.143\mathbf{a}_z) \times 10^{-2} \text{ A-m}^2}}\end{aligned}$$

P.E. 8.6

$$\begin{aligned}(a) \quad \mathbf{T} = \mathbf{m} \times \mathbf{B} &= \frac{10 \times 10^{-4} \times 50}{7 \times 10} \begin{vmatrix} 2 & 6 & -3 \\ 6 & 4 & 5 \end{vmatrix} \\ &= \underline{\underline{0.03\mathbf{a}_x - 0.02\mathbf{a}_y - 0.02\mathbf{a}_z \text{ N-m}}}\end{aligned}$$

$$(b) \quad |\mathbf{T}| = ISB \sin \theta \rightarrow |\mathbf{T}|_{\max} = ISB$$

$$|\mathbf{T}|_{\max} = \frac{50 \times 10^{-3}}{10} |6\mathbf{a}_x + 4\mathbf{a}_y + 5\mathbf{a}_z| = \underline{\underline{0.04387 \text{ Nm}}}$$

P.E. 8.7

$$(a) \quad \mu_r = \frac{\mu}{\mu_o} = 4.6, \chi_m = \mu_r - 1 = \underline{\underline{3.6}}$$

$$(b) \quad \mathbf{H} = \frac{\mathbf{B}}{\mu} = \frac{10 \times 10^{-3} e^{-y}}{4\pi \times 10^{-7} \times 4.6} \mathbf{a}_z \text{ A/m} = \underline{\underline{1730e^{-y}\mathbf{a}_z \text{ A/m}}}$$

$$(c) \quad \mathbf{M} = \chi_m \mathbf{H} = \underline{\underline{6228e^{-y}\mathbf{a}_z \text{ A/m}}}$$

P.E. 8.8

$$\begin{aligned}\mathbf{a}_n &= \frac{3\mathbf{a}_x + 4\mathbf{a}_y}{5} = \frac{6\mathbf{a}_x + 8\mathbf{a}_y}{10} \\ \mathbf{B}_{1n} &= (\mathbf{B}_1 \bullet \mathbf{a}_n) \mathbf{a}_n = \frac{(6+32)(6\mathbf{a}_x + 8\mathbf{a}_y)}{1000}\end{aligned}$$

$$= 0.228\mathbf{a}_x + 0.304\mathbf{a}_y = \mathbf{B}_{2n}$$

$$\mathbf{B}_{1t} = \mathbf{B}_1 - \mathbf{B}_{1n} = -0.128\mathbf{a}_x + 0.096\mathbf{a}_y + 0.2\mathbf{a}_z$$

$$\mathbf{B}_{2t} = \frac{\mu_2}{\mu_1} \mathbf{B}_{1t} = 10\mathbf{B}_{1t} = -1.28\mathbf{a}_x + 0.96\mathbf{a}_y + 2\mathbf{a}_z$$

$$\mathbf{B}_2 = \mathbf{B}_{2n} + \mathbf{B}_{2t} = \underline{\underline{-1.052\mathbf{a}_x + 1.264\mathbf{a}_y + 2\mathbf{a}_z}} \text{ Wb/m}^2$$

P.E. 8.9

(a) $\mathbf{B}_{1n} = \mathbf{B}_{2n} \rightarrow \mu_1 \mathbf{H}_{1n} = \underline{\underline{\mu_2 \mathbf{H}_{2n}}}$

$$\text{or } \mu_1 \mathbf{H}_1 \bullet \mathbf{a}_{n21} = \mu_2 \mathbf{H}_2 \bullet \mathbf{a}_{n21}$$

$$\mu_o \frac{(60+2-36)}{7} = 2\mu_o \frac{(6H_{2x}-10-12)}{7}$$

$$35 = 6H_{2x}$$

$$\underline{\underline{H_{2x} = 5.833 \text{ A/m}}}$$

(b) $\mathbf{K} = (\mathbf{H}_1 - \mathbf{H}_2) \times \mathbf{a}_{n12} = \mathbf{a}_{n21} \times (\mathbf{H}_1 - \mathbf{H}_2)$

$$= \mathbf{a}_{n21} \times [(10, 1, 12) - (35/6, -5, 4)]$$

$$= \frac{1}{7} \begin{vmatrix} 6 & 2 & -3 \\ 25/6 & 6 & 8 \end{vmatrix}$$

$$\underline{\underline{\mathbf{K} = 4.86\mathbf{a}_x - 8.64\mathbf{a}_y + 3.95\mathbf{a}_z \text{ A/m}}}$$

(c) Since $\mathbf{B} = \mu \mathbf{H}$, \mathbf{B}_1 and \mathbf{H}_1 are parallel, i.e. they make the same angle with the normal to the interface.

$$\cos \theta_1 = \frac{\mathbf{H}_1 \bullet \mathbf{a}_{n21}}{|\mathbf{H}_1|} = \frac{26}{7\sqrt{100+1+144}} = 0.2373$$

$$\underline{\underline{\theta_1 = 76.27^\circ}}$$

$$\cos \theta_2 = \frac{\mathbf{H}_2 \bullet \mathbf{a}_{n21}}{|\mathbf{H}_2|} = \frac{13}{7\sqrt{(5.833)^2 + 25 + 16}} = 0.2144$$

$$\underline{\underline{\theta_2 = 77.62^\circ}}$$

P.E. 8.10

(a) $L' = \mu_o \mu_r n^2 S = 4\pi \times 10^{-7} \times 1000 \times 16 \times 10^6 \times 4 \times 10^{-4}$
 $= \underline{\underline{8.042 \text{ H/m}}}$

$$(b) \quad W_m' = \frac{1}{2} L' I^2 = \frac{1}{2} (8.042)(0.5^2) = \underline{\underline{1.005}} \text{ J/m}$$

P.E. 8.11 From Example 8.11,

$$L_{in} = \frac{\mu_o l}{8\pi}$$

$$\begin{aligned} L_{ext} &= \frac{2w_m}{I^2} = \frac{1}{I^2} \iiint \frac{\mu I^2}{4\pi^2 \rho^2} \rho d\rho d\phi dz \\ &= \frac{1}{4\pi^2} \int_0^l dz \int_0^{2\pi} d\phi \int_a^b \frac{2\mu_o}{(1+\rho)\rho} d\rho \\ &= \frac{2\mu_o}{4\pi^2} \bullet 2\pi l \int_a^b \left[\frac{1}{\rho} - \frac{1}{(1+\rho)} \right] d\rho \\ &= \frac{\mu_o l}{\pi} \left[\ln \frac{b}{a} - \ln \frac{1+b}{1+a} \right] \\ L &= L_{in} + L_{ext} = \frac{\mu_o l}{8\pi} + \frac{\mu_o l}{\pi} \left[\ln \frac{b}{a} - \ln \frac{1+b}{1+a} \right] \end{aligned}$$

P.E. 8.12

$$(a) \quad L'_{in} = \frac{\mu_o}{8\pi} = \frac{4\pi \times 10^{-7}}{8\pi} = \underline{\underline{0.05 \text{ } \mu\text{H}/\text{m}}}$$

$$L'_{ext} = L' - L'_{in} = 1.2 - 0.05 = \underline{\underline{1.15 \text{ } \mu\text{H}/\text{m}}}$$

$$\begin{aligned} (b) \quad L' &= \frac{\mu_o}{2\pi} \left[\frac{1}{4} + \ln \frac{d-a}{a} \right] \\ \ln \frac{d-a}{a} &= \frac{2\pi L'}{\mu_o} - 0.25 = \frac{2\pi \times 1.2 \times 10^{-6}}{4\pi \times 10^{-7}} - 0.25 \\ &= 6 - 0.25 = 5.75 \end{aligned}$$

$$\frac{d-a}{a} = e^{5.75} = 314.19$$

$$d-a = 314.19a = 314.19 \times \frac{2.588 \times 10^{-3}}{2} = 406.6mm$$

$$d = 407.9mm = \underline{\underline{40.79cm}}$$

P.E. 8.13

This is similar to Example 8.13. In this case, however, $h=0$ so that

$$\begin{aligned} A_1 &= \frac{\mu_o I_1 a^2 b}{4b^3} \mathbf{a}_\phi \\ \phi_{12} &= \frac{\mu_o I_1 a^2}{4b^2} \bullet 2\pi b = \frac{\mu_o \pi I_1 a^2}{2b} \\ m_{12} &= \frac{\phi_{12}}{I_1} = \frac{\mu_o \pi a^2}{2b} = \frac{4\pi \times 10^{-7} \times \pi \times 4}{2 \times 3} \\ &= \underline{2.632 \mu\text{H}} \end{aligned}$$

P.E. 8.14

$$\begin{aligned} L_{\text{in}} &= \frac{\mu_o}{8\pi} l = \frac{\mu_o 2\pi \rho_o}{8\pi} = \frac{4\pi \times 10^{-7} \times 10 \times 10^{-2}}{4} \\ &= \underline{31.42 \text{ nH}} \end{aligned}$$

P.E. 8.15

(a) From Example 7.6,

$$\begin{aligned} B_{\text{ave}} &= \frac{\mu_o NI}{l} = \frac{\mu_o NI}{2\pi \rho_o} \\ \phi &= B_{\text{ave}} \bullet S = \frac{\mu_o NI}{2\pi \rho_o} \bullet \pi a^2 \\ \text{or } I &= \frac{2\rho_o \phi}{\mu a^2 N} = \frac{2 \times 10 \times 10^{-2} \times 0.5 \times 10^{-3}}{4\pi \times 10^{-7} \times 10^{-4} \times 10^3} \\ &= \underline{795.77 \text{ A}} \end{aligned}$$

Alternatively, using circuit approach

$$\begin{aligned} R &= \frac{l}{\mu S} = \frac{2\pi \rho_o}{\mu_o S} = \frac{2\pi \rho_o}{\mu_o \pi a^2} \\ \Im &= NI = \frac{\phi \Re}{N} = \frac{2\rho_o \phi}{\mu a^2 N}, \quad \text{as obtained before.} \end{aligned}$$

$$\Re = \frac{2\rho_o}{\mu a^2} = \frac{2 \times 10 \times 10^{-2}}{4\pi \times 10^{-7} \times 10^{-4}} = 1.591 \times 10^9$$

$$\Im = \phi \Re = 0.5 \times 10^{-3} \times 1.591 \times 10^9 = 7.9577 \times 10^5$$

$$I = \frac{\Im}{N} = 795.77 \text{ A} \quad \text{as obtained before.}$$

(b) If $\mu = 500\mu_0$,

$$I = \frac{795.77}{500} = \underline{\underline{1.592}} \text{ A}$$

P.E. 8.16

$$\mathfrak{I} = \frac{B^2 a S}{2\mu_0} = \frac{(1.5)^2 \times 10 \times 10^{-4}}{2 \times 4\pi \times 10^{-7}} = \frac{22500}{8\pi} = \underline{\underline{895.25}} \text{ N}$$

P.E. 8.17

We may approximate the longer solenoid as infinite so that $B_1 = \frac{\mu_0 N_1 I_1}{l_1}$. The flux linking the second solenoid is:

$$\psi_2 = N_2 B_1 S_1 = \frac{\mu_0 N_1 I_1}{l_1} \bullet \pi r_1^2 \bullet N_2$$

$$M = \frac{\psi_2}{I_1} = \frac{\mu_0 N_1 N_2}{l_1} \bullet \pi r_1^2$$

Here we assume air-core solenoids.

P.E. 8.18

$$R = \frac{\ell}{\mu S}$$

$$\ell = 2\pi\rho_o = 2\pi \times \frac{1}{2} (5+6) 10^{-2} = 11\pi \times 10^{-2}$$

$$S = 1.5 \times 10^{-2} (6-5) 10^{-2} = 1.5 \times 10^{-4}$$

$$F = NI = \psi R = \psi \frac{\ell}{\mu S} \longrightarrow \mu = \frac{\psi \ell}{NIS}$$

$$\mu = \frac{12 \times 10^{-3} (11\pi \times 10^{-2})}{500(2)1.5 \times 10^{-4}} = \underline{\underline{27.65 \times 10^{-3} \text{ H/m}}}$$

$$B = \frac{\psi}{S} = \frac{12 \times 10^{-3}}{1.5 \times 10^{-4}} = \underline{\underline{80 \text{ Wb}}}$$

Prob. 8.1

$$F = m\omega^2 r = 9.11 \times 10^{-31} \times (2 \times 10^{16})^2 (0.4 \times 10^{-10}) = \underline{\underline{14.576 \text{ nN}}}$$

Prob. 8.2

(a)

$$\begin{aligned} \mathbf{F} = Q(\mathbf{u} \times \mathbf{B}) &= 10^{-3} \begin{vmatrix} 10 & -2 & 6 \\ 0 & 0 & 25 \end{vmatrix} = 10^{-3} (-50\mathbf{a}_x - 250\mathbf{a}_y) \\ &= \underline{\underline{-0.05\mathbf{a}_x - 0.25\mathbf{a}_y \text{ N}}} \end{aligned}$$

(b) Constant velocity implies that acceleration $\mathbf{a} = \mathbf{0}$.

$$\mathbf{F} = m\mathbf{a} = \mathbf{0} = Q(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

$$\mathbf{E} = -\mathbf{u} \times \mathbf{B} = \underline{\underline{50\mathbf{a}_x + 250\mathbf{a}_y \text{ V/m}}}$$

Prob. 8.3

At P, x = 2, y = 5, z = -3

$$\mathbf{E} = 2(2)(5)(-3)\mathbf{a}_x + (2)^2(-3)\mathbf{a}_y + (2)^2(5)\mathbf{a}_z = -60\mathbf{a}_x - 12\mathbf{a}_y + 20\mathbf{a}_z$$

$$\mathbf{B} = (5)^2\mathbf{a}_x + (-3)^2\mathbf{a}_y + 2^2\mathbf{a}_z = 25\mathbf{a}_x + 9\mathbf{a}_y + 4\mathbf{a}_z$$

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

$$\mathbf{u} \times \mathbf{B} = \begin{vmatrix} 1.4 & 3.2 & -1 \\ 25 & 9 & 4 \end{vmatrix} = 21.8\mathbf{a}_x - 30.6\mathbf{a}_y - 67.4\mathbf{a}_z$$

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = (-60, -12, 20) + (21.8, -30.6, -67.4) = (-38.2, -42.6, -47.4)$$

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{u} \times \mathbf{B}) = 4(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \text{ mN}$$

$$= \underline{\underline{-152.8\mathbf{a}_x - 170.4\mathbf{a}_y - 189.6\mathbf{a}_z \text{ mN}}}$$

Prob. 8.4

$$\mathbf{F} = q\mathbf{E} = m\mathbf{a} = m \frac{d\mathbf{u}}{dt}$$

$$\frac{d\mathbf{u}}{dt} = \frac{q\mathbf{E}}{m} = \frac{10 \times 10^{-3}}{2} (30, 0, 0) \times 10^3$$

$$\frac{d}{dt}(u_x, u_y, u_z) = (150, 0, 0)$$

Equating components gives

$$\frac{du_x}{dt} = 150 \quad \longrightarrow \quad u_x = 150t + c_1$$

$$\frac{du_y}{dt} = 0 \quad \longrightarrow \quad u_y = c_2$$

$$\frac{du_z}{dt} = 0 \quad \longrightarrow \quad u_z = c_3$$

At $t = 0$, $\mathbf{u} = (2, 5, 0) \times 10^3$.

$$2000 = 0 + c_1 \quad \longrightarrow \quad c_1 = 2000$$

$$5000 = c_2$$

$$0 = c_3$$

Hence, $\mathbf{u} = (150t + 2000, 5000, 0)$

At $t = 4$ s,

$$\mathbf{u} = \underline{(2600, 5000, 0) \text{ m/s}}$$

$$u_x = \frac{dx}{dt} = 150t + 2000 \quad \longrightarrow \quad x = 75t^2 + 2000t + c_4$$

$$u_y = \frac{dy}{dt} = 5000 \quad \longrightarrow \quad y = 5000t + c_5$$

$$u_z = \frac{dz}{dt} = 0 \quad \longrightarrow \quad z = +c_6$$

$$\text{At } t=0, (x, y, z) = (0, 0, 0) \quad \longrightarrow \quad c_4 = 0 = c_5 = c_6$$

Hence,

$$(x, y, z) = (75t^2 + 2000t, 5000t, 0)$$

$$\text{At } t = 4 \text{s}, \quad x = 9200, \quad y = 20000, \quad z = 0.$$

$$\text{i.e. } (x, y, z) = \underline{(9200, 20000, 0)}$$

Prob. 8.5

$$ma = Q\mathbf{u} \times \mathbf{B}$$

$$10^{-3}\mathbf{a} = -2 \times 10^{-3} \begin{vmatrix} u_x & u_y & u_z \\ 0 & 6 & 0 \end{vmatrix}$$

$$\frac{d}{dt}(u_x, u_y, u_z) = (12u_z, 0, -12u_x)$$

$$\text{i.e. } \frac{du_x}{dt} = 12u_z \quad (1)$$

$$\frac{du_y}{dt} = 0 \rightarrow u_y = A_1 \quad (2)$$

$$\frac{du_z}{dt} = -12u_x \quad (3)$$

From (1) and (3),

$$\ddot{u}_x = 12\dot{u}_z = -144u_x$$

or

$$\ddot{u}_x + 144u_x = 0 \rightarrow u_x = c_1 \cos 12t + c_2 \sin 12t$$

From (1), $u_z = -c_1 \sin 12t + c_2 \cos 12t$

At $t=0$,

$$u_x = 5, u_y = 0, u_z = 0 \rightarrow A_1 = 0 = c_2, c_1 = 5$$

Hence,

$$\mathbf{u} = (5 \cos 12t, 0, -5 \sin 12t)$$

$$\mathbf{u}(t=10s) = (5 \cos 120, 0, -5 \sin 120) = \underline{\underline{4.071\mathbf{a}_x - 2.903\mathbf{a}_z}} \text{ m/s}$$

$$u_x = \frac{dx}{dt} = 5 \cos 12t \rightarrow x = \frac{5}{12} \sin 12t + B_1$$

$$u_y = \frac{dy}{dt} = 0 \rightarrow y = B_2$$

$$u_z = \frac{dz}{dt} = -5 \sin 12t \rightarrow z = \frac{5}{12} \cos 12t + B_3$$

$$\text{At } t=0, (x, y, z) = (0, 1, 2) \rightarrow B_1 = 0, B_2 = 1, B_3 = \frac{19}{12}$$

$$(x, y, z) = \left(\frac{5}{12} \sin 12t, 1, \frac{5}{12} \cos 12t + \frac{19}{12} \right) \quad (4)$$

At t=10s,

$$(x, y, z) = \left(\frac{5}{12} \sin 120, 1, \frac{5}{12} \cos 120 + \frac{19}{12} \right) = (0.2419, 1, 1.923)$$

By eliminating t from (4),

$x^2 + (z - \frac{19}{12})^2 = (\frac{5}{12})^2$, $y = 1$ which is a circle in the $y=1$ plane with center at $(0, 1, 19/12)$. The particle gyrates.

Prob. 8.6

(a) $ma = -e(\mathbf{u} \times \mathbf{B})$

$$-\frac{m}{e} \frac{d}{dt} (u_x, u_y, u_z) = \begin{vmatrix} u_x & u_y & u_z \\ 0 & 0 & B_o \end{vmatrix} = u_y B_o \vec{a}_x - B_o u_x \vec{a}_y$$

$$\frac{du_z}{dt} = 0 \rightarrow u_z = c = 0$$

$$\frac{du_x}{dt} = -u_y \frac{B_o e}{m} = -u_y w, \text{ where } w = \frac{B_o e}{m}$$

$$\frac{du_y}{dt} = u_x w$$

Hence,

$$\ddot{u}_x = -w \dot{u}_y = -w^2 u_x$$

$$\text{or } \ddot{u}_x + w^2 u_x = 0 \rightarrow u_x = A \cos wt + B \sin wt$$

$$u_y = -\frac{\dot{u}_x}{w} = A \sin wt - B \cos wt$$

$$\text{At } t=0, u_x = u_o, u_y = 0 \rightarrow A = u_o, B = 0$$

Hence,

$$u_x = u_o \cos wt = \frac{dx}{dt} \rightarrow x = \frac{u_o}{w} \sin wt + c_1$$

$$u_y = u_o \sin wt = \frac{dy}{dt} \rightarrow y = -\frac{u_o}{w} \cos wt + c_2$$

$$\text{At } t=0, x = 0 = y \rightarrow c_1 = 0, c_2 = \frac{u_o}{w}. \text{ Hence,}$$

$$x = \frac{u_o}{w} \sin wt, y = \frac{u_o}{w} (1 - \cos wt)$$

$$\frac{u_o^2}{w^2} (\cos^2 wt + \sin^2 wt) = \left(\frac{u_o}{w} \right)^2 = x^2 + (y - \frac{u_o}{w})^2$$

showing that the electron would move in a circle centered at $(0, \frac{u_o}{w})$. But since the field does not exist throughout the circular region, the electron passes through a semi-circle and leaves the field horizontally.

(b) $d = \text{twice the radius of the semi-circle}$

$$= \frac{2u_o}{w} = \frac{2u_o m}{B_o e}$$

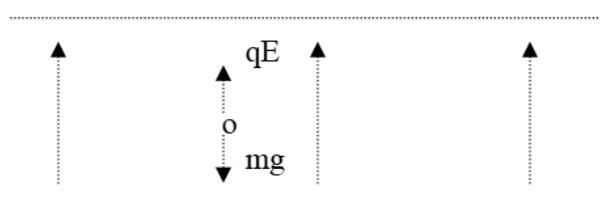
Prob. 8.7

$$\mathbf{F} = \int I dl \times \mathbf{B} = \int_0^{0.2} 2dy (-\mathbf{a}_y) \times (4\mathbf{a}_x - 8\mathbf{a}_z)$$

$$(-\mathbf{a}_y) \times (4\mathbf{a}_x - 8\mathbf{a}_z) = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 0 & -1 & 0 \\ 4 & 0 & -8 \end{vmatrix} = 8\mathbf{a}_x + 4\mathbf{a}_z$$

$$\mathbf{F} = 2(8\mathbf{a}_x + 4\mathbf{a}_z)(0.2) = \underline{\underline{3.2\mathbf{a}_x + 1.6\mathbf{a}_z}} \text{ N}$$

Prob. 8.8



$$mg = qE \quad \longrightarrow \quad q = \frac{mg}{E} = \frac{0.4 \times 10^{-3} \times 9.81}{1.5 \times 10^5} = \underline{\underline{26.67 \text{ nC}}}$$

Prob. 8.9

$$\mathcal{J} = IL \times \mathbf{B} \rightarrow \mathcal{J} = \frac{\mathbf{F}}{L} = I_1 \mathbf{a}_l \times \mathbf{B}_2 = \frac{\mu_o I_1 I_2 \mathbf{a}_l \times \mathbf{a}_\phi}{2\pi\rho}$$

$$(a) \quad \mathbf{F}_{21} = \frac{\mathbf{a}_z \times (-\mathbf{a}_y) 4\pi \times 10^{-7} (-100) (200)}{2\pi} = \underline{\underline{4\mathbf{a}_x \text{ mN/m}}} \text{ (repulsive)}$$

$$(b) \quad \mathbf{F}_{12} = -\mathbf{F}_{21} = -4\mathbf{a}_x \text{ mN/m} \text{ (repulsive)}$$

$$(c) \quad \mathbf{a}_l \times \mathbf{a}_\phi = \mathbf{a}_z \times \left(-\frac{4}{5}\mathbf{a}_x + \frac{3}{5}\mathbf{a}_y \right) = -\frac{3}{5}\mathbf{a}_x - \frac{4}{5}\mathbf{a}_y, \rho = 5$$

$$\mathbf{F}_{31} = \frac{4\pi \times 10^{-7}(-3 \times 10^4)}{2\pi(5)} \left(-\frac{3}{5}\mathbf{a}_x - \frac{4}{5}\mathbf{a}_y \right)$$

$$= 0.72\mathbf{a}_x + 0.96\mathbf{a}_y \text{ mN/m (attractive)}$$

(d) $\mathbf{F}_3 = \mathbf{F}_{31} + \mathbf{F}_{32}$

$$\mathbf{F}_{32} = \frac{4\pi \times 10^{-7} \times 6 \times 10^4}{2\pi(3)} (\mathbf{a}_z \times \mathbf{a}_y) = -4\mathbf{a}_x \text{ mN/m (attractive)}$$

$$\mathbf{F}_3 = -3.28\mathbf{a}_x + 0.96\mathbf{a}_y \text{ mN/m}$$

(attractive due to L₂ and repulsive due to L₁)

Prob. 8.10

$$F = \frac{\mu_o I_1 I_2}{2\pi\rho} = \frac{4\pi \times 10^{-7}(10)10}{2\pi(20 \times 10^{-2})} = 100 \text{ } \mu\text{N}$$

Prob. 8.11

$$W = - \int \mathbf{F} \bullet d\mathbf{l}, \quad \mathbf{F} = \int L d\mathbf{l} \times \mathbf{B} = 3(2\mathbf{a}_z) \times \cos \phi / 3 \mathbf{a}_\phi$$

$$F = 6 \cos \phi / 3 \bar{a}_\phi \text{ N}$$

$$W = - \int_0^{2\pi} 6 \cos \phi / 3 \rho_o d\phi = -6\rho_o \times 3 \sin \phi / 3 \Big|_0^{2\pi} \text{ J}$$

$$= -1.8 \sin \frac{2\pi}{3} = -1.559 \text{ J}$$

Prob. 8.12

(a) $\mathbf{F}_1 = \int_{\rho=2}^6 \frac{\mu_o I_1 I_2}{2\pi\rho} d\rho \mathbf{a}_\rho \times \mathbf{a}_\phi = \frac{4\pi \times 10^{-7}}{2\pi} (2)(5) \ln 6 / 2 \mathbf{a}_z$

$$= 2 \ln 3 \mathbf{a}_z \text{ } \mu\text{N} = 2.197 \mathbf{a}_z \text{ } \mu\text{N}$$

(b) $\mathbf{F}_2 = \int I_2 d\mathbf{l}_2 \times \mathbf{B}_1$

$$= \frac{\mu_o I_1 I_2}{2\pi} \int \frac{1}{\rho} [d\rho \mathbf{a}_\rho + dz \mathbf{a}_z] \times \mathbf{a}_\phi$$

$$= \frac{\mu_o I_1 I_2}{2\pi} \int \frac{1}{\rho} [d\rho \mathbf{a}_z - dz \mathbf{a}_\rho]$$

But $\rho = z+2$, $dz = d\rho$

$$\mathbf{F}_2 = \frac{4\pi \times 10^{-7}}{2\pi} (5)(2) \int_{\rho=4}^2 \frac{1}{\rho} [d\rho \mathbf{a}_z - dz \mathbf{a}_\rho]$$

$$2 \ln \frac{4}{4} (\mathbf{a}_z - \mathbf{a}_\rho) \mu N = 1.386 \mathbf{a}_\rho - 1.386 \mathbf{a}_z \mu N$$

$$\mathbf{F}_3 = \frac{\mu_0 I_1 I_2}{2\pi} \int \frac{1}{\rho} [d\rho \mathbf{a}_z - dz \mathbf{a}_\rho]$$

But $z = -\rho + 6$, $dz = -d\rho$

$$\mathbf{F}_3 = \frac{4\pi \times 10^{-7}}{2\pi} (5)(2) \int_{\rho=6}^4 \frac{1}{\rho} [d\rho \mathbf{a}_z - dz \mathbf{a}_\rho]$$

$$2 \ln \frac{4}{6} (\mathbf{a}_z + \mathbf{a}_\rho) \mu N = -0.8109 \mathbf{a}_\rho - 0.8109 \mathbf{a}_z \mu N$$

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$$

$$= \mathbf{a}_\rho (\ln 4 + \ln 4 - \ln 9) + \mathbf{a}_z (\ln 9 - \ln 4 + \ln 4 - \ln 9)$$

$$= \underline{\underline{0.575 \mathbf{a}_\rho \mu N}}$$

Prob. 8.13

From Prob. 8.7,

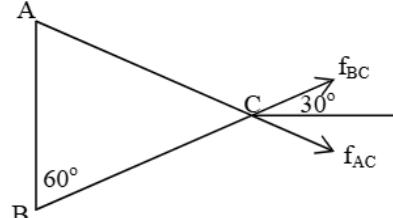
$$\mathbf{f} = \frac{\mu_0 I_1 I_2}{2\pi\rho} \mathbf{a}_\rho$$

$$\mathbf{f} = \mathbf{f}_{AC} + \mathbf{f}_{BC}$$

$$|\mathbf{f}_{AC}| = |\mathbf{f}_{BC}| = \frac{4\pi \times 10^{-7} \times 75 \times 150}{2\pi \times 2} = 1.125 \times 10^{-3}$$

$$\mathbf{f} = 2 \times 1.125 \cos 30^\circ \mathbf{a}_x \text{ mN/m}$$

$$= \underline{\underline{1.949 \mathbf{a}_x \text{ mN/m}}}$$



Prob. 8.14

The field due to the current sheet is

$$\mathbf{B} = \frac{\mu}{2} \mathbf{K} \times \mathbf{a}_n = \frac{\mu_0}{2} 10 \mathbf{a}_x \times (-\mathbf{a}_z) = 5\mu_0 \mathbf{a}_y$$

$$\mathbf{F} = I_2 \int dl_2 \times \mathbf{B} = 2.5 \int_0^L dx \mathbf{a}_x \times (5\mu_0 \mathbf{a}_y) = 2.5 L \times 5\mu_0 (\mathbf{a}_z)$$

$$\frac{\mathbf{F}}{L} = 12.5 \times 4\pi \times 10^{-7} (\mathbf{a}_z) = \underline{\underline{15.71 \mathbf{a}_z \mu N/m}}$$

Prob. 8.15

$$\mathbf{F} = \int Idl \times \mathbf{B} = IL \times \mathbf{B} = 5(2\mathbf{a}_z) \times 40\mathbf{a}_x 10^{-3} = 0.4 \mathbf{a}_y \text{ N}$$

Prob. 8.16

$$\begin{aligned}\mathbf{T} = \mathbf{m} \times \mathbf{B} &= [0.4(0.6)(3)\mathbf{a}_x] \times (0.5\mathbf{a}_x + 0.8\mathbf{a}_y) = 0.72(0.8)\mathbf{a}_z \\ &= \underline{\underline{0.576\mathbf{a}_z \text{ Nm}}}\end{aligned}$$

Prob. 8.17

$$\mathbf{F} = \int I d\ell \times \mathbf{B} \quad \longrightarrow \quad F = IB\ell = 520 \times 0.4 \times 10^{-3} \times 30 \times 10^{-3}$$

$$F = \underline{\underline{6.24 \text{ mN}}}$$

Prob. 8.18

$$\begin{aligned}m = IS &\longrightarrow I = \frac{m}{S} = \frac{m}{\pi r^2} \\ I &= \frac{8 \times 10^{22}}{\pi (6370 \times 10^3)^2} = 6.275 \times 10^8 = \underline{\underline{627.5 \text{ MA}}}\end{aligned}$$

Prob. 8.19

$$\text{Let } \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$$

$$\begin{aligned}\mathbf{F}_1 &= \int I d\ell \times \mathbf{B} = \int_5^0 2dx \mathbf{a}_x \times 30\mathbf{a}_z \text{ mN} \\ &= -60\mathbf{a}_y x \Big|_5^0 = 300\mathbf{a}_y \text{ mN}\end{aligned}$$

$$\begin{aligned}\mathbf{F}_2 &= \int_0^5 2dy \mathbf{a}_y \times 30\mathbf{a}_z \text{ mN} \\ &= 60\mathbf{a}_x y \Big|_0^5 = 300\mathbf{a}_x \text{ mN}\end{aligned}$$

$$\begin{aligned}\mathbf{F}_3 &= \int_0^5 2(dx \mathbf{a}_x + dz \mathbf{a}_z) \times 30\mathbf{a}_z \text{ mN} \\ &= 60(-\mathbf{a}_y)x \Big|_0^5 = -300\mathbf{a}_y \text{ mN}\end{aligned}$$

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = 300\mathbf{a}_y + 300\mathbf{a}_x - 300\mathbf{a}_y \text{ mN} = \underline{\underline{300\mathbf{a}_x \text{ mN}}}$$

$$\mathbf{T} = \mathbf{m} \times \mathbf{B} = ISa_n \times \mathbf{B} = 2(\frac{1}{2})(5)(5)\mathbf{a}_y \times 30\mathbf{a}_z 10^{-3} = \underline{\underline{0.75\mathbf{a}_x \text{ N.m}}}$$

Prob. 8.20

For each turn, $\mathbf{T} = \mathbf{m} \times \mathbf{B}$, $\mathbf{m} = IS\mathbf{a}_n$

For N turns,

$$T = NISB = 50 \times 4 \times 12 \times 10^{-4} \times 100 \times 10^{-3} = \underline{\underline{24 \text{ mNm}}}$$

Prob. 8.21

$$f(x, y, z) = x + 2y - 5z - 12 = 0 \quad \longrightarrow \quad \nabla f = \mathbf{a}_x + 2\mathbf{a}_y - 5\mathbf{a}_z$$

$$\mathbf{a}_n = \frac{\nabla f}{|\nabla f|} = \frac{\mathbf{a}_x + 2\mathbf{a}_y - 5\mathbf{a}_z}{\sqrt{30}}$$

$$\mathbf{m} = NIS\mathbf{a}_n = 2 \times 60 \times 8 \times 10^{-4} \frac{(\mathbf{a}_x + 2\mathbf{a}_y - 5\mathbf{a}_z)}{\sqrt{30}} = \underline{\underline{17.53\mathbf{a}_x + 35.05\mathbf{a}_y - 87.64\mathbf{a}_z \text{ mAm}}}$$

Prob. 8.22

$$M = \chi_m H = \chi_m \frac{B}{\mu_o \mu_r} = \frac{\chi_m B}{\mu_o (1 + \chi_m)}$$

Prob. 8.23

$$(a) \quad \mathbf{M} = \chi_m \mathbf{H} = \chi_m \frac{\mathbf{B}}{\mu_o \mu}$$

$$M = \frac{4999}{5000} \times \frac{1.5}{4\pi \times 10^{-7}} = \underline{\underline{1.193 \times 10^6 \text{ A/m}}}$$

$$(b) \quad \mathbf{M} = \frac{\sum_{k=1}^N \mathbf{m}_k}{\Delta v}$$

If we assume that all \mathbf{m}_k align with the applied \mathbf{B} field,

$$M = \frac{Nm_k}{\Delta v} \rightarrow m_k = \frac{Nm_k}{N/\Delta v} = \frac{1.193 \times 10^6}{8.5 \times 10^{28}}$$

$$m_k = \underline{\underline{1.404 \times 10^{-23} \text{ A} \cdot \text{m}^2}}$$

Prob. 8.24

$$\mu_r = \chi_m + 1 = 6.5 + 1 = \underline{\underline{7.5}}$$

$$M = \chi_m H \quad \longrightarrow \quad H = \frac{M}{\chi_m} = \frac{24y^2}{6.5} a_z$$

At $y = 2\text{cm}$,

$$H = \frac{24 \times 4 \times 10^{-4}}{6.5} a_z = \underline{\underline{1.477 a_z \text{ mA/m}}}$$

$$J = \nabla \times H = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \frac{24y^2}{6.5} \end{vmatrix} = \frac{48y}{6.5} a_x$$

At $y=2\text{cm}$,

$$J = \frac{48 \times 2 \times 10^{-2}}{6.5} a_x = \underline{\underline{0.1477 a_x \text{ A/m}^2}}$$

Prob. 8.25

$$(a) \quad \chi_m = \mu_r - 1 = \underline{\underline{3.5}}$$

$$(b) \quad H = \frac{B}{\mu} = \frac{4y a_z \times 10^{-3}}{4\pi \times 10^{-7} \times 4.5} = \underline{\underline{707.3y a_z \text{ A/m}}}$$

$$(c) \quad M = \chi_m H = \underline{\underline{2.476y a_z \text{ kA/m}}}$$

$$(d) \quad J_b = \nabla \times M = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & M_z(y) \end{vmatrix} = \frac{dM_z}{dy} a_x \\ = \underline{\underline{2.476 a_x \text{ kA/m}^2}}$$

Prob. 8.26

When $H = 250$,

$$B = \frac{2H}{100+H} = \frac{2(250)}{100+250} = 1.4286 \text{ mWb/m}^2$$

But $B = \mu_0 \mu_r H$

$$\mu_r = \frac{B}{\mu_0 H} = \frac{1.4286 \times 10^{-3}}{4\pi \times 10^{-7} \times 250} = \underline{\underline{4.54}}$$

Prob. 8.27

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{\text{enc}}$$

$$H_\phi \cdot 2\pi\rho = \frac{\pi\rho^2}{\pi a^2} \cdot I \rightarrow H_\phi = \frac{I\rho}{2\pi a^2}$$

$$\mathbf{M} = \chi_m \mathbf{H} = \underline{\underline{(\mu_r - 1) \frac{I\rho}{2\pi a^2} \mathbf{a}_\phi}}$$

$$\mathbf{J}_b = \nabla \times \mathbf{M} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho M_\phi) \mathbf{a}_z = \underline{\underline{(\mu_r - 1) \frac{I}{\pi a^2} \mathbf{a}_z}}$$

Prob. 8.28

(a) From $H_{1t} - H_{2t} = K$ and $M = \chi_m H$, we obtain:

$$\underline{\underline{\frac{M_{1t}}{\chi_{m1}} - \frac{M_{2t}}{\chi_{m2}} = K}}$$

Also from $B_{1n} - B_{2n} = 0$ and $B = \mu H = (\mu/\chi_m)M$, we get:

$$\underline{\underline{\frac{\mu_1 M_{1n}}{\chi_{m1}} = \frac{\mu_2 M_{2n}}{\chi_{m2}}}}$$

(b) From $B_1 \cos \theta_1 = B_{1n} = B_{2n} = B_2 \cos \theta_2$ (1)

$$\text{and } \frac{B_1 \sin \theta_1}{\mu_1} = H_{1t} = K + H_{2t} = K + \frac{B_2 \sin \theta_2}{\mu_2} \quad (2)$$

Dividing (2) by (1) gives

$$\frac{\tan \theta_1}{\mu_1} = \frac{k}{B_2 \cos \theta_2} + \frac{\tan \theta_2}{\mu_2} = \frac{\tan \theta_2}{\mu_2} \left(1 + \frac{k \mu_2}{B_2 \sin \theta_2} \right)$$

$$\text{i.e. } \underline{\underline{\frac{\tan \theta_1}{\tan \theta_2} = \frac{\mu_1}{\mu_2} \left(1 + \frac{k \mu_2}{B_2 \sin \theta_2} \right)}}$$

Prob. 8.29

$$\mathbf{B}_{2n} = \mathbf{B}_{1n} = 1.8\mathbf{a}_z$$

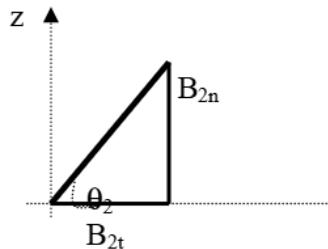
$$\mathbf{H}_{2t} = \mathbf{H}_{1t} \quad \dots \rightarrow \quad \frac{\mathbf{B}_{2t}}{\mu_2} = \frac{\mathbf{B}_{1t}}{\mu_1}$$

$$\mathbf{B}_{2t} = \frac{\mu_2}{\mu_1} \mathbf{B}_{1t} = \frac{4\mu_o}{2.5\mu_o} (6\mathbf{a}_x - 4.2\mathbf{a}_y) = 9.6\mathbf{a}_x - 6.72\mathbf{a}_y$$

$$\mathbf{B}_2 = \mathbf{B}_{2n} + \mathbf{B}_{2t} = 9.6\mathbf{a}_x - 6.72\mathbf{a}_y + 1.8\mathbf{a}_z \text{ mWb/m}^2$$

$$\mathbf{H}_2 = \frac{\mathbf{B}_2}{\mu_2} = \frac{10^{-3}(9.6, -6.72, 1.8)}{4 \times 4\pi \times 10^{-7}}$$

$$= 1,909.86\mathbf{a}_x - 1,336.9\mathbf{a}_y + 358.1\mathbf{a}_z \text{ A/m}$$



$$\tan \theta_2 = \frac{B_{2n}}{B_{2t}} = \frac{1.8}{\sqrt{9.6^2 + 6.72^2}} = 0.1536$$

$$\theta_2 = \underline{\underline{8.73^\circ}}$$

Prob. 8.30

$$(a) \quad \mathbf{B}_{1n} = \mathbf{B}_{2n} = 15\mathbf{a}_\phi$$

$$\mathbf{H}_{1t} = \mathbf{H}_{2t} \rightarrow \frac{\mathbf{B}_{1t}}{\mu_1} = \frac{\mathbf{B}_{2t}}{\mu_2}$$

$$\mathbf{B}_{1t} = \frac{\mu_1}{\mu_2} \mathbf{B}_{2t} = \frac{2}{5} (10\mathbf{a}_\rho - 20\mathbf{a}_z) = 4\mathbf{a}_\rho - 8\mathbf{a}_z$$

Hence,

$$\mathbf{B}_1 = \underline{\underline{4\mathbf{a}_\rho + 15\mathbf{a}_\phi - 8\mathbf{a}_z \text{ mWb/m}^2}}$$

$$(b) \quad W_{m1} = \frac{1}{2} \mathbf{B}_1 \cdot \mathbf{H}_1 = \frac{\mathbf{B}_1^2}{2\mu_1} = \frac{(4^2 + 15^2 + 8^2) \times 10^{-6}}{2 \times 2 \times 4\pi \times 10^{-7}}$$

$$W_{m1} = \underline{\underline{60.68 \text{ J/m}^3}}$$

$$W_{m2} = \frac{\mathbf{B}_2^2}{2\mu_2} = \frac{(10^2 + 15^2 + 20^2) \times 10^{-6}}{2 \times 5 \times 4\pi \times 10^{-7}} = \underline{\underline{57.7 \text{ J/m}^3}}$$

Prob. 8.31

$$\mathbf{B}_{2n} = \mathbf{B}_{1n} = 40\mathbf{a}_x$$

$$\mathbf{B}_{2n} = \mu_2 \mathbf{H}_{2n} \quad \longrightarrow \quad \mathbf{H}_{2n} = \frac{40\mathbf{a}_x}{\mu_2} = \frac{40\mathbf{a}_x}{50\mu_0}$$

$$\mathbf{H}_{2t} = \mathbf{H}_{1t} \quad \longrightarrow \quad \frac{\mathbf{B}_{2t}}{\mu_2} = \frac{\mathbf{B}_{1t}}{\mu_1}$$

$$\mathbf{B}_{2t} = \frac{\mu_2}{\mu_1} \mathbf{B}_{1t}$$

$$\mathbf{H}_{2t} = \frac{\mathbf{B}_{2t}}{\mu_2} = \frac{\mathbf{B}_{1t}}{\mu_1} = \frac{(-30\mathbf{a}_x + 10\mathbf{a}_y)}{\mu_0}$$

$$\mathbf{H}_2 = \mathbf{H}_{2n} + \mathbf{H}_{2t} = \frac{1}{\mu_0} \left(\frac{40}{50}, -30, 10 \right) \cdot 10^{-3} = \frac{10^{-3}}{4\pi \times 10^{-7}} (0.8, -30, 10)$$

$$\mathbf{H}_2 = \underline{\underline{0.6366\mathbf{a}_x - 23.87\mathbf{a}_y + 7.957\mathbf{a}_z \text{ kA/m}}}$$

Prob. 8.32

$$\mathbf{H}_{2t} = \mathbf{H}_{1t} = \alpha\mathbf{a}_x + \delta\mathbf{a}_z$$

$$\mathbf{B}_{2n} = \mathbf{B}_{1n} \quad \longrightarrow \quad \mu_2 \mathbf{H}_{2n} = \mu_1 \mathbf{H}_{1n}$$

$$\mathbf{H}_{2n} = \frac{\mu_1}{\mu_2} \mathbf{H}_{1n} = \frac{\mu_{r1}}{\mu_{r2}} \beta \mathbf{a}_y$$

$$\mathbf{H} = \alpha\mathbf{a}_x + \frac{\mu_{r1}}{\mu_{r2}} \beta \mathbf{a}_y + \delta\mathbf{a}_z$$

Prob. 8.33

$$\mathbf{B}_{2n} = \mathbf{B}_{1n} = 0.6\mathbf{a}_y$$

$$\mathbf{H}_{2t} = \mathbf{H}_{1t} \quad \longrightarrow \quad \frac{\mathbf{B}_{2t}}{\mu_2} = \frac{\mathbf{B}_{1t}}{\mu_1}$$

$$\mathbf{B}_{1t} = \frac{\mu_1}{\mu_2} \mathbf{B}_{2t} = \frac{\mu_0}{12\mu_0} (1.4\mathbf{a}_x - 2\mathbf{a}_z) = 0.1167\mathbf{a}_x - 0.1667\mathbf{a}_z$$

$$\mathbf{B}_1 = \mathbf{B}_{1n} + \mathbf{B}_{1t} = \underline{\underline{0.1167\mathbf{a}_x + 0.6\mathbf{a}_y - 0.1667\mathbf{a}_z \text{ Wb/m}^2}}$$

$$\mathbf{H}_1 = \frac{\mathbf{B}_1}{\mu_1} = \frac{10^{-3}(0.1167, 0.6, -0.1667)}{4\pi \times 10^{-7}}$$

$$= \underline{\underline{(0.0929\mathbf{a}_x + 0.4775\mathbf{a}_y - 0.1327\mathbf{a}_z) \cdot 10^5 \text{ A/m}}}$$

Prob. 8.34

$$f(x, y, z) = x - y + 2z$$

$$\nabla f = \mathbf{a}_x - \mathbf{a}_y + 2\mathbf{a}_z$$

$$\mathbf{a}_n = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{6}}(\mathbf{a}_x - \mathbf{a}_y + 2\mathbf{a}_z)$$

(a)

$$\begin{aligned}\mathbf{H}_{1n} &= (\mathbf{H}_1 \cdot \mathbf{a}_n) \mathbf{a}_n = (40 - 20 - 60) \frac{(\mathbf{a}_x - \mathbf{a}_y + 2\mathbf{a}_z)}{6} \\ &= \underline{-6.667\mathbf{a}_x + 6.667\mathbf{a}_y - 13.333\mathbf{a}_z \text{ A/m}}\end{aligned}$$

(b)

$$\mathbf{H}_2 = \mathbf{H}_{2n} + \mathbf{H}_{2t}$$

$$\text{But } \mathbf{B}_{2n} = \mathbf{B}_{1n} \longrightarrow \mu_2 \mathbf{H}_{2n} = \mu_1 \mathbf{H}_{1n}$$

$$\begin{aligned}\mathbf{B}_2 &= \mu_2 \mathbf{H}_2 = \mu_2 \mathbf{H}_{2n} + \mu_2 \mathbf{H}_{2t} = \mu_1 \mathbf{H}_{1n} + \mu_2 \mathbf{H}_{2t} = \mu_o (2\mathbf{H}_{1n} + 5\mathbf{H}_{2t}) \\ &= 4\pi \times 10^{-7} [(-13.333, 13.333, -26.667) + (233.333, 66.666, -83.333)] \\ &= 4\pi \times 10^{-7} (220, 80, -110) \\ &= \underline{276.5\mathbf{a}_x + 100.5\mathbf{a}_y - 138.2\mathbf{a}_z \text{ }\mu\text{Wb/m}^2}\end{aligned}$$

Prob. 8.35

$$\mathbf{a}_n = \mathbf{a}_\rho$$

$$\mathbf{B}_{2n} = \mathbf{B}_{1n} = 22\mu_o \mathbf{a}_\rho$$

$$\mathbf{H}_{2t} = \mathbf{H}_{1t} \longrightarrow \frac{\mathbf{B}_{2t}}{\mu_2} = \frac{\mathbf{B}_{1t}}{\mu_1}$$

$$\mathbf{B}_{2t} = \frac{\mu_2}{\mu_1} \mathbf{B}_{1t} = \frac{\mu_o}{800\mu_o} (45\mu_o \mathbf{a}_\phi) = 0.05625\mu_o \mathbf{a}_\phi$$

$$\mathbf{B}_2 = \underline{\mu_o (22\mathbf{a}_\rho + 0.05625\mathbf{a}_\phi) \text{ Wb/m}^2}$$

Prob. 8.36

$r = a$ is the interface between the two media.

$$\mathbf{B}_{2n} = \mathbf{B}_{1n} \longrightarrow B_{o1}(1+1.6) \cos \theta \mathbf{a}_r = B_{o2} \cos \theta \mathbf{a}_r$$

$$2.6B_{o1} = B_{o2} \quad (1)$$

$$\begin{aligned}\mathbf{H}_{2t} &= \mathbf{H}_{1t} \longrightarrow \frac{\mathbf{B}_{2t}}{\mu_2} = \frac{\mathbf{B}_{1t}}{\mu_1} \\ \mu_2 \mathbf{B}_{1t} &= \mu_1 \mathbf{B}_{2t} \\ \mu_2 B_{o1}(-0.2) \sin \theta \mathbf{a}_\theta &= \mu_o B_{o2}(-\sin \theta) \mathbf{a}_\theta \\ \mu_2 &= \frac{\mu_o B_{o2}}{0.2 B_{o1}} \quad (2)\end{aligned}$$

Substituting (1) into (2) gives

$$\mu_2 = \frac{\mu_o}{0.2} (2.6) = \underline{\underline{\underline{13\mu_o}}}$$

Prob. 8.37

$$(a) \quad \mathbf{H} = \frac{1}{2} \mathbf{K} \times \mathbf{a}_n = \frac{1}{2} (30 - 40) \mathbf{a}_x \times (-\mathbf{a}_z) = \underline{\underline{\underline{-5\mathbf{a}_y}}} \text{ A/m}$$

$$\mathbf{B} = \mu_o \mathbf{H} = 4\pi \times 10^{-7} (-5\mathbf{a}_y) = \underline{\underline{\underline{-6.28\mathbf{a}_y\mu}}} \text{ Wb/m}^2$$

$$(b) \quad \mathbf{H} = \frac{1}{2} (-30 - 40) \mathbf{a}_y = \underline{\underline{\underline{-35\mathbf{a}_y}}} \text{ A/m}$$

$$\mathbf{B} = \mu_o \mu_r \mathbf{H} = 4\pi \times 10^{-7} (2.5) (-35\mathbf{a}_y) = \underline{\underline{\underline{-110\mathbf{a}_y\mu}}} \text{ Wb/m}^2$$

$$(c) \quad \mathbf{H} = \frac{1}{2} (-30 + 40) \mathbf{a}_y = \underline{\underline{\underline{5\mathbf{a}_y}}}$$

$$\mathbf{B} = \mu_o \mathbf{H} = \underline{\underline{\underline{6.283\mathbf{a}_y\mu}}} \text{ Wb/m}^2$$

Prob. 8.38

$$\mathbf{H}_{1n} = -3\mathbf{a}_z, \quad \mathbf{H}_{1t} = 10\mathbf{a}_x + 15\mathbf{a}_y$$

$$\mathbf{H}_{2t} = \mathbf{H}_{1t} = 10\mathbf{a}_x + 15\mathbf{a}_y$$

$$\mathbf{H}_{2n} = \frac{\mu_1}{\mu_2} \mathbf{H}_{1n} = \frac{1}{200} (-3\mathbf{a}_z) = -0.015\mathbf{a}_z$$

$$\mathbf{H}_2 = 10\mathbf{a}_x + 15\mathbf{a}_y - 0.015\mathbf{a}_z$$

$$\mathbf{B}_2 = \mu_2 \mathbf{H}_2 = 200 \times 4\pi \times 10^{-7} (10, 15, -0.015)$$

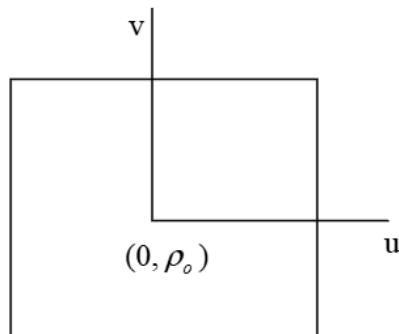
$$\mathbf{B}_2 = \underline{\underline{\underline{2.51\mathbf{a}_x + 3.77\mathbf{a}_y - 0.0037\mathbf{a}_z}}} \text{ mWb/m}^2$$

$$\tan \alpha = \frac{B_{2n}}{B_{2t}}$$

$$\text{or } \alpha = \tan^{-1} \frac{0.0037}{\sqrt{2.51^2 + 3.77^2}} = \underline{\underline{\underline{0.047^\circ}}}$$

Prob. 8.39

- (a) The square cross-section of the toroid is shown below. Let (u, v) be the local coordinates and ρ_o = mean radius. Using Ampere's law around a circle passing through P, we get



$$H(2\pi)(\rho_o + v) = NI \quad \longrightarrow \quad H = \frac{NI}{2\pi(\rho_o + v)}$$

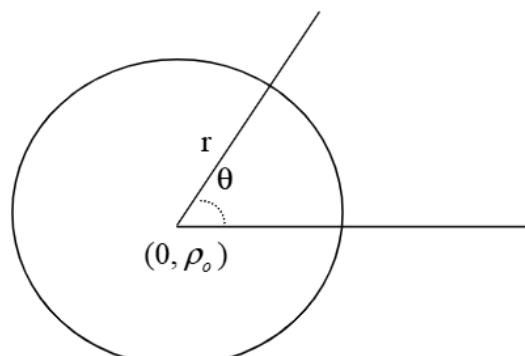
The flux per turn is

$$\Psi = \int_{u=-a/2}^{a/2} \int_{v=-a/2}^{a/2} B du dv = \frac{\mu_o N I a}{2\pi} \ln \left(\frac{\rho_o + a/2}{\rho_o - a/2} \right)$$

$$L = \frac{N\Psi}{I} = \frac{\mu_o N^2 a}{2\pi} \ln \left(\frac{2\rho_o + a}{2\rho_o - a} \right)$$

- (b) The circular cross-section of the toroid is shown below. Let (r, θ) be the local coordinates. Consider a point $P(r \cos \theta, \rho_o + r \sin \theta)$ and apply Ampere's law around a circle that passes through P.

$$H(2\pi)(\rho_o + r \sin \theta) = NI \quad \longrightarrow \quad H = \frac{NI}{2\pi(\rho_o + r \sin \theta)} \approx \frac{NI}{2\pi\rho_o} \left(1 - \frac{r \sin \theta}{\rho_o} \right)$$



$$\text{Flux per turn } \Psi = \int_{r=0}^a \int_{\theta=0}^{2\pi} \frac{\mu NI}{2\kappa_o \rho_o} \left(1 - \frac{r \sin \theta}{\rho_o}\right) r dr d\theta = \frac{\mu NI}{2\kappa_o \rho_o} \frac{a^2}{2} (2\pi)$$

$$L = \frac{N\Psi}{I} = \frac{\mu N^2 a^2}{2\rho_o}$$

Or from Example 8.10,

$$L = L'l = \frac{\mu_o N^2 l S}{l^2} = \frac{\mu_o N^2 \pi a^2}{2\pi \rho_o} = \frac{\mu_o N^2 a^2}{2\rho_o}$$

Prob. 8.40

$$\rho_o = \frac{1}{2}(3+5) = 4\text{ cm}$$

$$a = 2 \text{ cm}$$

$$L = \frac{\mu_o N^2 a}{2\pi} \ln \left[\frac{2\rho_o + a}{2\rho_o - a} \right]$$

$$N^2 = \frac{2\pi L}{\mu_o a \ln \left[\frac{2\rho_o + a}{2\rho_o - a} \right]} = \frac{2\pi(45 \times 10^{-6})}{4\pi \times 10^{-7} (2 \times 10^{-2}) \ln \left(\frac{8+2}{8-2} \right)} = 22,023.17$$

$$\underline{\underline{N = 148.4 \text{ or } 148}}$$

Prob. 8.41

$$L = \frac{\mu_o \ell}{8\pi} \longrightarrow \frac{L}{\ell} = \frac{4\pi \times 10^{-7}}{8\pi} = \underline{\underline{50 \text{ nH/m}}}$$

Prob. 8.42

$$L_{in} = \frac{\mu_o \ell}{8\pi}, \quad L_{ext} = \frac{\mu_o \ell}{2\pi} \ln(b/a)$$

$$\text{If } L_{in} = 2L_{ext} \longrightarrow \frac{\mu_o \ell}{8\pi} = \frac{\mu_o \ell}{\pi} \ln(b/a)$$

$$\ln(b/a) = \frac{1}{8} \quad \frac{b}{a} = e^{1/8} = 1.1331$$

$$b = 1.1331a = \underline{\underline{7.365 \text{ mm}}}$$

Prob. 8.43

$$L' = \frac{\mu}{2\pi} \left[\frac{1}{4} + \ln \frac{b}{a} \right] = \frac{4\pi \times 10^{-7}}{2\pi} [0.25 + \ln(6/2.5)] = \underline{\underline{225 \text{ nH}}}$$

Prob. 8.44

$$\psi_{12} = \int \mathbf{B}_1 \bullet d\mathbf{S} = \int_{\rho=\rho_o}^{\rho_o+a} \int_{z=0}^b \frac{\mu_o I}{2\pi\rho} dz d\rho = \frac{\mu_o I b}{2\pi} \ln \frac{a+\rho_o}{\rho_o}$$

$$M_{12} = \frac{N\psi_{12}}{I} = \frac{N\mu_o b}{2\pi} \ln \frac{a+\rho_o}{\rho_o}$$

For N = 1,

$$M_{12} = \frac{\psi_{12}}{I_1} = \frac{\mu_o b}{2\pi} \ln \frac{a+\rho_o}{\rho_o}$$

$$= \frac{4\pi \times 10^{-7}}{2\pi} (1) \ln 2 = \underline{\underline{0.1386 \mu \text{ H}}}$$

Prob. 8.45

$$\mathbf{H} = \frac{I}{2\pi\rho} \mathbf{a}_\rho$$

$$w_m = \frac{1}{2} \mu |\mathbf{H}|^2 = \frac{1}{2} \mu \frac{I^2}{4\pi^2 \rho^2}$$

$$W = \int w_m dV = \iiint \frac{1}{2} \mu \frac{I^2}{4\pi^2 \rho^2} \rho d\phi d\rho dz = \frac{1}{4\pi} \mu I^2 L \ln(b/a)$$

$$= \frac{1}{4\pi} \times 4 \times 4\pi \times 10^{-7} (625 \times 10^{-6}) 3 \ln(18/12) = \underline{\underline{304.1 \text{ pJ}}}$$

Alternatively,

$$W = \frac{1}{2} LI^2 = \frac{1}{2} \frac{\mu L}{2\pi} \ln \frac{b}{a} \times I^2 = \frac{\mu I^2 L}{4\pi} \ln \frac{b}{a}$$

Prob. 8.46

$$\mu_r = \chi_m + 1 = 20$$

$$w_m = \frac{1}{2} \mathbf{B}_1 \cdot \mathbf{H}_1 = \frac{1}{2} \mu \mathbf{H} \cdot \mathbf{H}$$

$$= \frac{1}{2} \mu (25x^4y^2z^2 + 100x^2y^4z^2 + 225x^2y^2z^4)$$

$$W_m = \int w_m dV$$

$$= \frac{1}{2} \mu \left[25 \int_0^1 x^4 dx \int_0^2 y^2 dy \int_{-1}^2 z^2 dz + 100 \int_0^1 x^2 dx \int_0^2 y^4 dy \int_{-1}^2 z^2 dz \right.$$

$$\left. + 225 \int_0^1 x^2 dx \int_0^2 y^2 dy \int_{-1}^2 z dz \right]$$

$$= \frac{25\mu}{2} \left[\frac{x^5}{5} \Big|_0^1 \frac{y^3}{3} \Big|_0^2 \frac{z^3}{3} \Big|_{-1}^2 + 4 \frac{x^3}{3} \Big|_0^1 \frac{y^5}{5} \Big|_0^2 \frac{z^3}{3} \Big|_{-1}^2 \right.$$

$$\left. + 9 \frac{x^3}{3} \Big|_0^1 \frac{y^3}{3} \Big|_0^2 \frac{z^5}{5} \Big|_{-1}^2 \right]$$

$$= \frac{25\mu}{2} \left(\frac{1}{5} \cdot \frac{8}{3} \cdot \frac{9}{3} + \frac{4}{3} \cdot \frac{32}{3} \cdot \frac{9}{3} + \frac{9}{3} \cdot \frac{8}{3} \cdot \frac{33}{5} \right)$$

$$= \frac{25}{2} \times 4\pi \times 10^{-7} \times 20 \times \frac{3600}{45}$$

$$W_m = \underline{\underline{25.13 \text{ mJ}}}$$

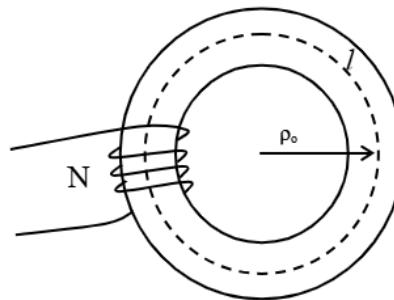
Prob. 8.47

$$W = \frac{1}{2} \int_v \mu H^2 dv = \frac{1}{2} \iiint 4.5 \times 4\pi \times 10^{-7} [200^2 + 500^2] 10^{-6} dx dy dz \\ = 2\pi(4.5)10^{-7}(29 \times 10^4)10^{-6}(2)(2)(2)10^{-6} = \underline{\underline{6.56 \text{ pJ}}}$$

Prob. 8.48

$$NI = HI = \frac{Bl}{\mu}$$

$$N = \frac{Bl}{\mu_0 \mu_r I} = \frac{1.5 \times 0.6\pi}{4\pi \times 10^{-7} \times 600 \times 12} \\ = \underline{\underline{313 \text{ turns}}}$$

**Prob. 8.49**

$$F = NI = 400 \times 0.5 = 200 \text{ A.t}$$

$$R_a = \frac{100}{4\pi} \text{ MAt/Wb}, \quad R_1 = R_2 = \frac{6}{4\pi} \text{ MAt/Wb}, \quad R_3 = \frac{1.8}{4\pi} \text{ MAt/Wb}$$

$$F_a = \frac{R_a F}{R_a + R_3 + R_1 // R_2} = \underline{\underline{190.8 \text{ A.t}}}$$

$$H_a = \frac{F_a}{l_a} = \frac{190.8}{1 \times 10^{-2}} = \underline{\underline{19080 \text{ A/m}}}$$

Prob. 8.50

$$\text{Total } F = NI = 2000 \times 10 = 20,000 \text{ A.t}$$

$$R_c = \frac{l_c}{\mu_0 \mu_r S} = \frac{(24 + 20 - 0.6) \times 10^{-2}}{4\pi \times 10^{-7} \times 1500 \times 2 \times 10^{-4}} = \underline{\underline{0.115 \times 10^7 \text{ A.t/m}}}$$

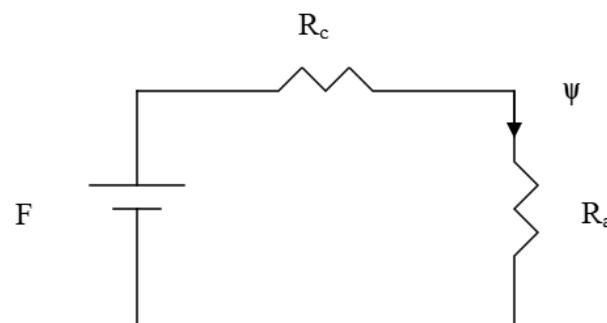
$$R_a = \frac{l_a}{\mu_0 \mu_r S} = \frac{0.6 \times 10^{-2}}{4\pi \times 10^{-7} (1) \times 2 \times 10^{-4}} = \underline{\underline{2.387 \times 10^7 \text{ A.t/m}}}$$

$$R = R_a + R_c = 2.502 \times 10^7 \text{ A.t/m}$$

$$\psi = \frac{\Im}{R} = \psi_a = \psi_c = \frac{20,000}{2.502 \times 10^7} = \underline{\underline{8 \times 10^{-4} \text{ Wb/m}^2}}$$

$$\Im_a = \frac{R_a}{R_a + R_c} \Im = \frac{2.387 \times 20,000}{2.502} = \underline{\underline{19,081 \text{ A.t}}}$$

$$\Im_c = \frac{R_c}{R_a + R_c} \Im = \frac{0.115 \times 20,000}{2.502} = \underline{\underline{919 \text{ A.t}}}$$

Prob. 8.51

$$F = NI = 500 \times 0.2 = 100 \text{ A.t}$$

$$R_c = \frac{l_c}{\mu S} = \frac{42 \times 10^{-2}}{4\pi \times 10^{-7} \times 10^3 \times 4 \times 10^{-4}} = \frac{42 \times 10^6}{16\pi}$$

$$R_a = \frac{l_a}{\mu_o S} = \frac{10^{-3}}{4\pi \times 10^{-7} \times 4 \times 10^{-4}} = \frac{10^8}{16\pi}$$

$$R_a + R_c = \frac{1.42 \times 10^8}{16\pi}$$

$$\psi = \frac{F}{R_a + R_c} = \frac{16\pi \times 100}{1.42 \times 10^8} = \frac{16\pi}{1.42} \text{ } \mu\text{Wb}$$

$$B_a = \frac{\psi}{S} = \frac{16\pi \times 10^{-6}}{1.42 \times 4 \times 10^{-4}} = \underline{\underline{88.5 \text{ mWb/m}^2}}$$

Prob. 8.52

$$F = \frac{B^2 S}{2\mu_0} = \frac{\psi^2}{2\mu_0 S} = \frac{4 \times 10^{-6}}{2 \times 4\pi \times 10^{-7} \times 0.3 \times 10^{-4}} = \underline{\underline{53.05}} \text{ kN}$$

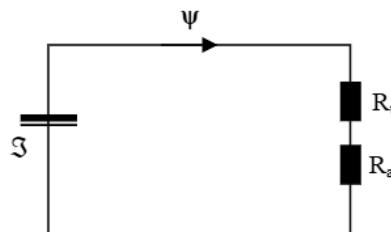
Prob. 8.53

(a) $F = NI = 200 \times 10^{-3} \times 750 = 150 \text{ A.t.}$

$$R_a = \frac{l_a}{\mu_0 S} = \frac{10^{-3}}{25 \times 10^{-6} \mu_0} = 3.183 \times 10^7$$

$$R_t = \frac{l_t}{\mu_0 \mu_r S} = \frac{2\pi \times 0.1}{\mu_0 \times 300 \times 25 \times 10^{-6}} = 6.7 \times 10^7$$

$$\psi = \frac{\mathfrak{I}}{R_a + R_t} = \frac{150}{10^7 (3.183 + 20/3)} = 15.23 \times 10^{-7}$$



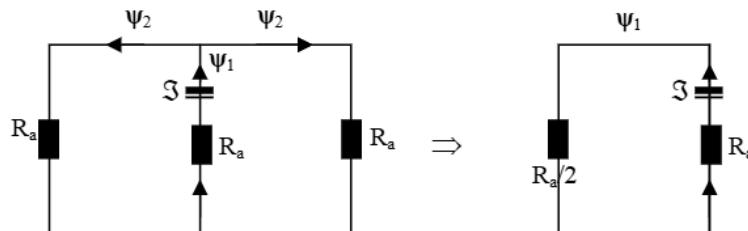
$$F = \frac{B^2 S}{2\mu_0} = \frac{\psi^2}{2\mu_0 S} = \frac{2.32 \times 10^{-12}}{2 \times 4\pi \times 10^{-7} \times 25 \times 10^{-6}}$$

$$= \underline{\underline{37}} \text{ mN}$$

(b) If $\mu_t \rightarrow \infty$, $R_t = 0$, $\psi = \frac{\mathfrak{I}}{R_a} = \frac{150}{3.183 \times 10^7}$

$$F_2 = I_2 dl_2 \bullet B_1 = I_2 dl_2 \frac{\psi_1}{S} = \frac{2 \times 10^{-3} \times 5 \times 10^{-3} \times 150}{3.183 \times 10^7 \times 25 \times 10^{-6}}$$

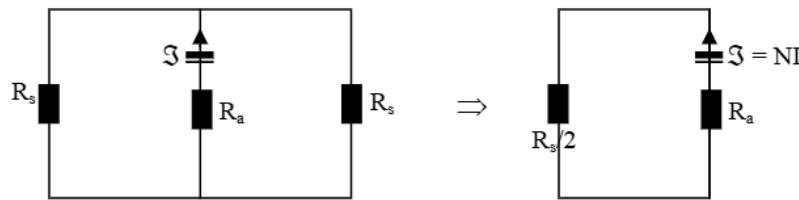
$$F_2 = \underline{\underline{1.885}} \mu\text{N}$$

Prob. 8.54

$$\psi_1 = 2\psi_2, \psi_1 = \frac{\mathfrak{I}}{\frac{3}{2}R_a} = \frac{2\mathfrak{I}}{3R_a} \rightarrow \psi_2 = \frac{\mathfrak{I}}{3R_a}$$

$$\mathfrak{I} = 2 \left(\frac{\psi_2^2}{2\mu_0 S} \right) + \frac{\psi_1}{2\mu_0 S} = \frac{3\psi_1^2}{4\mu_0 S} = \frac{\mathfrak{I}^2}{3R_a^2 \mu_0 S}$$

$$\begin{aligned}
 &= \frac{\mu_0 S \mathfrak{I}^2}{3l_a^2} = \frac{4\pi \times 10^{-7} \times 200 \times 10^{-4} \times 9 \times 10^6}{3 \times 10^{-6}} \\
 &= 24\pi \times 10^3 = mg \rightarrow m = \frac{24\pi \times 10^3}{9.8} = \underline{\underline{7694 \text{ kg}}}
 \end{aligned}$$

Prob. 8.55

Since $\mu \rightarrow \infty$ for the core (see Figure), $R_c = 0$.

$$\begin{aligned}
 \mathfrak{I} &= NI = \psi \left(R_a + \frac{R_s}{2} \right) = \frac{\psi(a/2 + x)}{\mu_0 S} \\
 &= \frac{\psi(2x + a)}{2\mu_0 S} \\
 \mathfrak{I} &= \frac{B^2 S}{2\mu_0} = \psi^2 \frac{1}{2\mu_0 S} = \frac{1}{2\mu_0 S} \bullet \frac{N^2 I^2 4\mu_0^2 S^2}{(a+2x)^2} \\
 &= \frac{2N^2 I^2 \mu_0 S}{(a+2x)^2}
 \end{aligned}$$

$F = -Fa_x$ since the force is attractive, i.e.

$$F = \frac{-2N^2 I^2 \mu_0 S a_x}{(a+2x)^2}$$

CHAPTER 9

P.E. 9.1

(a) $V_{emf} = \int (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} = uBl = 8(0.5)(0.1) = \underline{\underline{0.4}} \text{ V}$

(b) $I = \frac{V_{emf}}{R} = \frac{0.4}{20} = \underline{\underline{20}} \text{ mA}$

(c) $\mathbf{F}_m = Il \times \mathbf{B} = 0.02(-0.1\mathbf{a}_y \times 0.5\mathbf{a}_z) = \underline{\underline{-\mathbf{a}_x}} \text{ mN}$

(d) $P = FU = I^2 R = 8 \text{ mW}$

or $P = \frac{V_{emf}^2}{R} = \frac{(0.4)^2}{20} = \underline{\underline{8}} \text{ mW}$

P.E. 9.2

(a) $V_{emf} = \int (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}$

where $\mathbf{B} = B_o \mathbf{a}_y = B_o (\sin \phi \mathbf{a}_\rho + \cos \phi \mathbf{a}_\phi)$, $B_o = 0.05 \text{ Wb/m}^2$

$$(\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} = -\rho \omega B_o \sin \phi dz = -0.2\pi \sin(\omega t + \pi/2) dz$$

$$V_{emf} = \int_0^{0.03} (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} = -6\pi \cos(100\pi t) \text{ mV}$$

At $t = 1 \text{ ms}$,

$$V_{emf} = -6\pi \cos 0.1\pi = \underline{\underline{-17.93}} \text{ mV}$$

$$i = \frac{V_{emf}}{R} = -60\pi \cos(100\pi t) \text{ mA}$$

At $t = 3 \text{ ms}$, $i = -60\pi \cos 0.3\pi = \underline{\underline{-110.8}} \text{ mA}$

(b) Method 1:

$$\Psi = \int \mathbf{B} \cdot d\mathbf{S} = \int B_o t (\cos \phi \mathbf{a}_\rho - \sin \phi \mathbf{a}_\phi) \cdot d\rho dz \mathbf{a}_\phi = - \int_0^{0.03} \int_0^{z_o} B_o t \sin \phi d\rho dz = -B_o \rho_o z_o t \sin \phi$$

where $B_o = 0.02$, $\rho_o = 0.04$, $z_o = 0.03$

$$\phi = \omega t + \pi/2$$

$$\Psi = -B_o \rho_o z_o t \cos \omega t$$

$$V_{emf} = -\frac{\partial \Psi}{\partial t} = B_o \rho_o z_o \cos \omega t - B_o \rho_o z_o t \omega \sin \omega t$$

$$\begin{aligned}
 &= (0.02)(0.04)(0.03)[\cos \omega t - \omega t \sin \omega t] \\
 &= 24[\cos \omega t - \omega t \sin \omega t] \mu V
 \end{aligned}$$

Method 2:

$$\begin{aligned}
 V_{emf} &= - \int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} + \int (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} \\
 \mathbf{B} &= B_o t \mathbf{a}_x = B_o t (\cos \phi \mathbf{a}_\rho - \sin \phi \mathbf{a}_\phi), \phi = \omega t + \pi/2
 \end{aligned}$$

$$\frac{\partial \mathbf{B}}{\partial t} = B_o (\cos \phi \mathbf{a}_\rho - \sin \phi \mathbf{a}_\phi)$$

Note that only explicit dependence of \mathbf{B} on time is accounted for, i.e. we make ϕ = constant because it is transformer (stationary) emf. Thus,

$$\begin{aligned}
 V_{emf} &= -B_o \int_0^{\rho_o} \int_0^{z_o} (\cos \phi \mathbf{a}_\rho - \sin \phi \mathbf{a}_\phi) \cdot d\rho dz \mathbf{a}_\phi + \int_{z_o}^0 -\rho_o \omega B_o t \cos \phi dz \\
 &= B_o \rho_o z_o (\sin \phi - \omega t \cos \phi), \phi = \omega t + \pi/2
 \end{aligned}$$

$= B_o \rho_o z_o (\cos \omega t - \omega t \sin \omega t)$ as obtained earlier.

At $t = 1\text{ms}$,

$$\begin{aligned}
 V_{emf} &= 24[\cos 18^\circ - 100\pi \times 10^{-3} \sin 18^\circ] \mu V \\
 &= \underline{20.5 \mu V}
 \end{aligned}$$

At $t = 3\text{ms}$,

$$\begin{aligned}
 i &= 240[\cos 54^\circ - 0.03\pi \sin 54^\circ] \text{mA} \\
 &= \underline{-41.93 \text{ mA}}
 \end{aligned}$$

P.E. 9.3

$$\begin{aligned}
 V_1 &= -N_1 \frac{d\psi}{dt}, V_2 = -N_2 \frac{d\psi}{dt} \\
 \frac{V_2}{V_1} &= \frac{N_2}{N_1} \rightarrow V_2 = \frac{N_2}{N_1} V_1 = \frac{300 \times 120}{500} = \underline{\underline{72V}}
 \end{aligned}$$

P.E. 9.4

$$\text{(a)} \quad \mathbf{J}_d = \frac{\partial \mathbf{D}}{\partial t} = \underline{\underline{-20\omega \epsilon_o \sin(\omega t - 50x) \mathbf{a}_y A/m^2}}$$

$$(b) \quad \nabla \times \mathbf{H} = \mathbf{J}_d \rightarrow -\frac{\partial H_z}{\partial x} \mathbf{a}_y = -20\omega\epsilon_0 \sin(\omega t - 50x) \mathbf{a}_y$$

or $\mathbf{H} = \frac{20\omega\epsilon_0}{50} \cos(\omega t - 50x) \mathbf{a}_z$

$$= \underline{\underline{0.4\omega\epsilon_0 \cos(\omega t - 50x) \mathbf{a}_z}} \text{ A/m}$$

$$(c) \quad \nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} \rightarrow \frac{\partial E_y}{\partial x} \mathbf{a}_z = 0.4\mu_0\omega^2\epsilon_0 \sin(\omega t - 50x) \mathbf{a}_z$$

$$1000 = 0.4\mu_0\epsilon_0\omega^2 = 0.4 \frac{\omega^2}{c^2}$$

or $\omega = \underline{\underline{1.5 \times 10^{10} \text{ rad/s}}}$

P.E. 9.5

$$(a) \quad j^3 \left(\frac{1+j}{2-j} \right)^2 = -j \left[\frac{\sqrt{2}\angle 45^\circ}{\sqrt{5}\angle -26.56^\circ} \right]^2 = -j \left(\frac{2}{5} \angle 143.13^\circ \right)$$

$$= \underline{\underline{0.24 + j0.32}}$$

$$(b) \quad 6\angle 30^\circ + j5 - 3 + e^{j45^\circ} = 5.196 + j3 + j5 - 3 + 0.7071(1+j)$$

$$= \underline{\underline{2.903 + j8.707}}$$

P.E. 9.6

$$\mathbf{P} = 2 \sin(10t + x - \pi/4) \mathbf{a}_y = 2 \cos(10t + x - \pi/4 - \pi/2) \mathbf{a}_y, w = 10$$

$$= R_e \left(2e^{j(x-\pi/4)} \mathbf{a}_y e^{jwt} \right) = R_e \left(\underline{\underline{\mathbf{P}_s e^{jwt}}} \right)$$

$$\text{i.e. } \mathbf{P}_s = \underline{\underline{2e^{j(x-\pi/4)} \mathbf{a}_y}}$$

$$\mathbf{Q} = R_e \left(\underline{\underline{\mathbf{Q}_s e^{jwt}}} \right) = R_e \left(e^{j(x+wt)} (\mathbf{a}_x - \mathbf{a}_z) \right) \sin \pi y$$

$$= \underline{\underline{\sin \pi y \cos(wt+x) (\mathbf{a}_x - \mathbf{a}_z)}}$$

P.E. 9.7

$$-\mu \frac{\partial \mathbf{H}}{\partial t} = \nabla \times \mathbf{E} = \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (E_\phi \sin \theta) \mathbf{a}_r - \frac{1}{r} \frac{\partial}{\partial r} (r E_\phi) \mathbf{a}_\theta$$

$$\begin{aligned}
 &= \frac{2\cos\theta}{r^2} \cos(\omega t - \beta r) \mathbf{a}_r - \frac{\beta}{r} \sin\theta \sin(\omega t - \beta r) \mathbf{a}_\theta \\
 \mathbf{H} &= -\frac{2\cos\theta}{\mu_0\omega r^2} \sin(\omega t - \beta r) \mathbf{a}_r - \frac{\beta}{\mu_0\omega r} \sin\theta \cos(\omega t - \beta r) \mathbf{a}_\theta \\
 \beta &= \frac{\omega}{c} = \frac{6 \times 10^7}{3 \times 10^8} = \underline{\underline{0.2}} \text{ rad/m} \\
 \mathbf{H} &= \underline{\underline{\frac{1}{12\pi r^2} \cos\theta \sin(6 \times 10^7 - 0.2r) \mathbf{a}_r - \frac{1}{120\pi r} \sin\theta \cos(6 \times 10^7 - 0.2r) \mathbf{a}_\theta}}
 \end{aligned}$$

P.E. 9.8

$$\begin{aligned}
 \omega &= \frac{3}{\sqrt{\mu_0\epsilon_0}} = \frac{3c}{\sqrt{\mu_0\epsilon_0}} = \frac{9 \times 10^8}{\sqrt{10}} = \underline{\underline{2.846 \times 10^8}} \text{ rad/s} \\
 \mathbf{E} &= \frac{1}{\epsilon_0} \int \nabla \times \mathbf{H} dt = -\frac{6}{\omega\epsilon_0} \cos(\omega t - 3y) \mathbf{a}_x \\
 &= \frac{-6}{\frac{9 \times 10^8}{\sqrt{10}} \cdot \frac{10^{-9}}{36\pi}} \cos(\omega t - 3y) \mathbf{a}_x \\
 &= \underline{\underline{-476.86 \cos(2.846 \times 10^8 t - 3y) \mathbf{a}_x \text{ V/m}}}
 \end{aligned}$$

P.E. 9.9

$$V = -\frac{\partial \psi}{\partial t} = -\frac{\partial}{\partial t} \int \mathbf{B} \bullet d\mathbf{S} = -\frac{\partial \mathbf{B}}{\partial t} \bullet \mathbf{S}$$

$$= 3770 \sin 377t \times \pi (0.2)^2 \times 10^{-3}$$

$$= \underline{\underline{0.4738 \sin 377t \text{ V}}}$$

P.E. 9.10

$$V = \int (\mathbf{A} \times \mathbf{B}) \bullet d\mathbf{l}$$

$$u = \rho \omega \mathbf{a}_\phi, \quad B = B_o \mathbf{a}_z$$

$$\begin{aligned}
 V &= \int_{\rho=0}^r \rho \omega B_o d\rho = \frac{1}{2} \omega B_o \rho^2 \Big|_0^r = \frac{\omega B_o \ell^2}{2} \\
 &= \frac{30}{2} \times 60 \times 10^{-3} (8 \times 10^{-2})^2 = \underline{\underline{5.76 \text{ mV}}}
 \end{aligned}$$

Prob. 9.1

Measuring the induced emf in the clockwise direction,

$$\begin{aligned}
 V_{\text{emf}} &= \oint (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} \\
 &= \int_0^{1.2} (5\mathbf{a}_x \times 0.2\mathbf{a}_z) \cdot d\mathbf{y} \mathbf{a}_y + \int_{1.2}^0 (15\mathbf{a}_x \times 0.2\mathbf{a}_z) \cdot d\mathbf{y} \mathbf{a}_x \\
 &= - \int_0^{1.2} (1) dy - \int_{1.2}^0 (3) dy \\
 &= -1.2 + 1.2 \times 3 = -1.2 + 3.6 \\
 &= 2.4 \text{ V}
 \end{aligned}$$

Prob. 9.2

$$\psi = \mathbf{B} \cdot \mathbf{S} = (0.2)^2 \pi \cdot 40 \times 10^{-3} \sin 10^4 t$$

$$V = -\frac{\partial \psi}{\partial t} = -16\pi \cos 10^4 t$$

$$\begin{aligned}
 i &= \frac{V}{R} = \frac{16\pi}{4} \cos 10^4 t \\
 &= -12.57 \cos 10^4 t \text{ A}
 \end{aligned}$$

Prob. 9.3

$$\begin{aligned}
 V_{\text{emf}} &= -\frac{\partial \psi}{\partial t} = -\frac{\partial}{\partial t} \iint \mathbf{B} \cdot d\mathbf{S} = -\iint \frac{\partial \mathbf{B}}{\partial t} \cdot dx dy \mathbf{a}_z \\
 &= \int_{y=0}^{0.1} \int_{x=0}^{0.8} 30\pi \times 40 \sin(30\pi t - 3y) dx dy \text{ mV} \\
 &= 1200\pi \int_0^{0.8} dx \int_0^{0.1} \sin(30\pi t - 3y) dy \\
 &= 1200\pi (0.8) \left(-\frac{1}{-3} \cos(30\pi t - 3y) \Big|_0^{0.1} \right) \\
 &= 320\pi [\cos(30\pi t - 0.3) - \cos(30\pi t)] \text{ mV}
 \end{aligned}$$

$$\begin{aligned}
 I &= \frac{V_{\text{emf}}}{R} = \frac{V_{\text{emf}}}{10+4} = \frac{320\pi}{14} [-2 \sin(30\pi t - 0.15) \sin(-0.15)] \\
 &= 143.62 \sin(30\pi t - 0.15) \sin(0.15)
 \end{aligned}$$

$$I = 21.46 \sin(30\pi t - 0.15) \text{ mA}$$

Prob. 9.4

$$\begin{aligned} V_{\text{emf}} &= -\frac{\partial}{\partial t} \int \mathbf{B} \bullet d\mathbf{S} = -\int \frac{\partial \mathbf{B}}{\partial t} \bullet d\mathbf{S} \\ &= -\iint (-4\omega) \sin \omega t \sqrt{x^2 + y^2} dx dy = 4\omega \sin \omega t \iint \sqrt{x^2 + y^2} dx dy \end{aligned}$$

We change variables from Cartesian to cylindrical coordinates.

$$\begin{aligned} V_{\text{emf}} &= 4\omega \sin \omega t \int_{\phi=0}^{2\pi} \int_{\rho=0}^3 \rho \cdot \rho d\rho d\phi = 4\omega \sin \omega t (2\pi) \frac{\rho^3}{3} \Big|_0^3 \\ &= 72\pi\omega \sin \omega t = \underline{\underline{226.2\omega \sin \omega t \text{ V}}} \end{aligned}$$

Prob. 9.5

$$\mathbf{B} = \frac{\mu_o I}{2\pi y} (-a_x)$$

$$\begin{aligned} \psi &= \int \mathbf{B} \bullet d\mathbf{S} = \frac{\mu_o I}{2\pi} \int_{z=0}^a \int_{y=\rho}^{\rho+a} \frac{dz dy}{y} = \frac{\mu_o I a}{2\pi} \ln \frac{\rho+a}{\rho} \\ V_{\text{emf}} &= -\frac{\partial \psi}{\partial t} = -\frac{\partial \psi}{\partial \rho} \bullet \frac{\partial \rho}{\partial t} = -\frac{\mu_o I a}{2\pi} u_o \frac{d}{d\rho} [\ln(\rho+a) - \ln \rho] \\ &= -\frac{\mu_o I a}{2\pi} u_o \left[\frac{1}{\rho+a} - \frac{1}{\rho} \right] = \underline{\underline{\frac{\mu_o a^2 I u_o}{2\pi \rho (\rho+a)}}} \end{aligned}$$

where $\rho = \rho_o + u_o t$

Prob. 9.6

$$\begin{aligned} V_{\text{emf}} &= \int_{\rho}^{\rho+a} 3a_z \times \frac{\mu_o I}{2\pi \rho} a_\phi \bullet d\rho a_\rho = -\frac{3\mu_o I}{2\pi} \ln \frac{\rho+a}{\rho} \\ &= -\frac{4\pi \times 10^{-7}}{2\pi} \times 15 \times 3 \ln \frac{60}{20} = -9.888 \mu V \end{aligned}$$

Thus the induced emf = 9.888 μV, point A at higher potential.

Prob. 9.7

$$\begin{aligned} V_{\text{emf}} &= -N \frac{\partial \psi}{\partial t} = -N \frac{\partial}{\partial t} \int \mathbf{B} \bullet d\mathbf{S} = -NB \frac{dS}{dt} \\ &= -NB\ell \frac{d}{dt}(\rho\phi) = -NB\ell \rho \frac{d\phi}{dt} = -NB\ell \rho \omega \\ &= -50(0.2)(30 \times 10^{-4})(60) = \underline{\underline{-1.8 \text{ V}}} \end{aligned}$$

Prob. 9.8Method 1:

We assume that the sliding rode is on $-\ell < z < \ell$

$$\ell = x/\sqrt{3} = 5t/\sqrt{3}$$

$$V_{\text{emf}} = \int (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} = \int 5\mathbf{a}_x \times 0.6\mathbf{a}_z \cdot dy \mathbf{a}_y = -3x \int_{-\ell}^{\ell} dy = -3x \times 25t^2 = \underline{\underline{-86.6025t^2}}$$

Method 2:

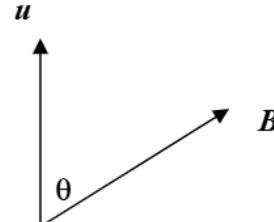
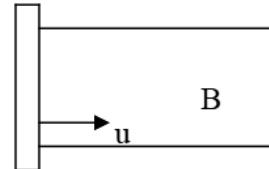
The flux linkage is given by

$$\psi = \int_{x=0}^{5t} \int_{y=-x/\sqrt{3}}^{x/\sqrt{3}} 0.6xdxdy = 0.6 \times \frac{2}{\sqrt{3}} \times 125t^3 / 3 = 28,8675t^3$$

$$V_{\text{emf}} = -\frac{d\psi}{dt} = \underline{\underline{-86.602t^2}}$$

Prob. 9.9

$$V_{\text{emf}} = uBl = 410 \times 0.4 \times 10^{-6} \times 36 = \underline{\underline{5.904 \text{ mV}}}$$

Prob. 9.10

$$\begin{aligned} V_{\text{emf}} &= \int (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} = uBl \cos \theta \\ &= \left(\frac{120 \times 10^3}{3600} \text{ m/s} \right) (4.3 \times 10^{-5}) (1.6) \cos 65^\circ \\ &= 2.293 \cos 65^\circ = \underline{\underline{0.97 \text{ mV}}} \end{aligned}$$

Prob. 9.11

$$d\psi = 0.64 - 0.45 = 0.19, dt = 0.02$$

$$V_{emf} = N \frac{d\psi}{dt} = 10 \left(\frac{0.19}{0.02} \right) = 95V$$

$$I = \frac{V_{emf}}{R} = \left(\frac{95}{15} \right) = \underline{\underline{6.33 \text{ A}}}$$

Using Lenz's law, the direction of the induced current is counterclockwise.

Prob. 9.12

$$V = \int (\mathbf{u} \times \mathbf{B}) \bullet d\mathbf{l}, \text{ where } \mathbf{u} = \rho \omega \mathbf{a}_\phi, \quad \mathbf{B} = B_o \mathbf{a}_z$$

$$V = \int_{\rho_1}^{\rho_2} \rho \omega B_o d\rho = \frac{\omega B_o}{2} (\rho_2^2 - \rho_1^2)$$

$$V = \frac{60 \times 15}{2} \bullet 10^{-3} (100 - 4) \bullet 10^{-4} = \underline{\underline{4.32 \text{ mV}}}$$

Prob. 9.13

$$J_{ds} = j\omega D_s \rightarrow |J_{ds}|_{\max} = \sigma \epsilon E_s = \sigma \epsilon \frac{V_s}{d}$$

$$= \frac{10^{-9}}{36\pi} \times \frac{2\pi \times 20 \times 10^6 \times 50}{0.2 \times 10^{-3}}$$

$$= \underline{\underline{277.8 \text{ A/m}^2}}$$

$$I_{ds} = J_{ds} \bullet S = \frac{1000}{3.6} \times 2.8 \times 10^{-4} = \underline{\underline{77.78 \text{ mA}}}$$

Prob. 9.14

$$J_c = \sigma E, \quad J_d = \frac{\partial D}{\partial t} = \epsilon \frac{\partial E}{\partial t}$$

$$|J_c| = \sigma |E|, \quad |J_d| = \epsilon \omega |E|$$

$$\text{If } I_c = I_d, \text{ then } |J_c| = |J_d| \longrightarrow \sigma = \epsilon \omega$$

$$\omega = 2\pi f = \frac{\sigma}{\epsilon}$$

$$f = \frac{\sigma}{2\pi\epsilon} = \frac{4}{2\pi \times 9 \times \frac{10^{-9}}{36\pi}} = \underline{\underline{8 \text{ GHz}}}$$

Prob. 9.15. $\frac{J_c}{J_d} = \frac{\sigma E}{\omega \epsilon E} = \frac{\sigma}{\omega \epsilon}$

(a) $\frac{\sigma}{\omega \epsilon} = \frac{2 \times 10^{-3}}{2\pi \times 10^9 \times 81 \times \frac{10^{-9}}{36\pi}} = \underline{\underline{0.444 \times 10^{-3}}}$

(b) $\frac{\sigma}{\omega \epsilon} = \frac{25}{2\pi \times 10^9 \times 81 \times \frac{10^{-9}}{36\pi}} = \underline{\underline{5.555}}$

(c) $\frac{\sigma}{\omega \epsilon} = \frac{2 \times 10^{-4}}{2\pi \times 10^9 \times 5 \times \frac{10^{-9}}{36\pi}} = \underline{\underline{7.2 \times 10^{-4}}}$

Prob. 9.16

$$\frac{J_d}{J} = \frac{\omega \epsilon E}{\sigma E} = \frac{\omega \epsilon}{\sigma} = 1 \quad \longrightarrow \quad \omega = \frac{\sigma}{\epsilon} = \frac{10^{-4}}{3 \times \frac{10^{-9}}{36\pi}} = 12\pi \times 10^5$$

$$2\pi f = 12\pi \times 10^5 \quad \longrightarrow \quad f = \underline{\underline{600 \text{ kHz}}}$$

Prob. 9.17

$$J_c = \sigma E = 0.4 \cos(2\pi \times 10^3 t)$$

$$E = \frac{0.4}{\sigma} \cos(2\pi \times 10^3 t)$$

$$\begin{aligned} J_d &= \epsilon \frac{\partial E}{\partial t} = -\frac{0.4\epsilon}{\sigma} (2\pi \times 10^3) \sin(2\pi \times 10^3 t) \\ &= -\frac{0.4 \times 4.5 \times \frac{10^{-9}}{36\pi}}{10^{-4}} (2\pi \times 10^3) \sin(2\pi \times 10^3 t) \\ &= \underline{\underline{-100 \sin(2\pi \times 10^3 t) \text{ A/m}^2}} \end{aligned}$$

Prob. 9.18

(a) $\nabla \bullet \mathbf{E}_s = \rho_s / \epsilon, \nabla \bullet \mathbf{H}_s = 0$

$$\underline{\underline{\nabla \times \mathbf{E}_s = j\omega \mu \mathbf{H}_s, \quad \nabla \times \mathbf{H}_s = (\sigma - j\omega \epsilon) \mathbf{E}_s}}$$

$$(b) \quad \nabla \bullet \mathbf{D} = \rho_v \rightarrow \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho_v \quad (1)$$

$$\nabla \bullet \mathbf{B} = 0 \rightarrow \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \quad (2)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \rightarrow \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t} \quad (3)$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t} \quad (4)$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\partial B_z}{\partial t} \quad (5)$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \rightarrow \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = J_x + \frac{\partial D_x}{\partial t} \quad (6)$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y + \frac{\partial D_y}{\partial t} \quad (7)$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z + \frac{\partial D_z}{\partial t} \quad (8)$$

Prob. 9.19

$$\text{If } \mathbf{J} = 0 = \rho_v, \text{ then } \nabla \bullet \mathbf{B} = 0 \quad (1)$$

$$\nabla \bullet \mathbf{D} = \rho_v \quad (2)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (3)$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad (4)$$

Since $\nabla \bullet \nabla \times \mathbf{A} = 0$ for any vector field \mathbf{A} ,

$$\nabla \bullet \nabla \times \mathbf{E} = -\frac{\partial}{\partial t} \nabla \bullet \mathbf{B} = 0$$

$$\nabla \bullet \nabla \times \mathbf{H} = -\frac{\partial}{\partial t} \nabla \bullet \mathbf{D} = 0$$

showing that (1) and (2) are incorporated in (3) and (4). Thus Maxwell's equations can be reduced to (3) and (4), i.e.

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$$

Prob. 9.20

$$\nabla \cdot \mathbf{E} = 0 \longrightarrow (1)$$

$$\nabla \cdot \mathbf{H} = 0 \longrightarrow (2)$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \longrightarrow (3)$$

$$\nabla \times \mathbf{E} = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y(x, t) & 0 \end{bmatrix}$$

$$= \frac{\partial E_y}{\partial x} \mathbf{a}_z = -E_o \sin x \cos t \mathbf{a}_z$$

$$H = -\frac{1}{\mu} \int \nabla \times \mathbf{E} dt = \frac{E_o}{\mu_o} \sin x \sin t \mathbf{a}_z$$

$$\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t} \longrightarrow (4)$$

$$\nabla \times \mathbf{H} = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & H_z(x, t) \end{bmatrix}$$

$$= -\frac{\partial H_z}{\partial x} \mathbf{a}_y = -\frac{E_o}{\mu_o} \cos x \sin t \mathbf{a}_y$$

$$\mathbf{E} = \frac{1}{\epsilon} \int \nabla \times \mathbf{H} dt = \frac{E_o}{\mu_o \epsilon} \cos x \cos t \mathbf{a}_y$$

which is off the given \mathbf{E} by a factor. Thus, Maxwell's equations (1) to (3) are satisfied, but (4) is not. The only way (4) is satisfied is for $\mu_o \epsilon = 1$ which is not true.

Prob. 9.21

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \nabla \times \mathbf{E} = -\frac{\partial}{\partial t} \nabla \times \mathbf{B} = -\mu \frac{\partial}{\partial t} \nabla \times \mathbf{H} = -\mu \frac{\partial \mathbf{J}}{\partial t} - \mu \epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

But

$$\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \bullet \mathbf{E}) - \nabla^2 \mathbf{E}$$

$$\nabla(\nabla \bullet \mathbf{E}) - \nabla^2 \mathbf{E} = -\mu \frac{\partial \mathbf{J}}{\partial t} - \mu \epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}, \quad \mathbf{J} = \sigma \mathbf{E}$$

In a source-free region, $\nabla \bullet \mathbf{E} = \rho_v / \epsilon = 0$. Thus,

$$\underline{\underline{\nabla^2 E = \mu \sigma \frac{\partial E}{\partial t} + \mu \epsilon \frac{\partial^2 E}{\partial t^2}}}$$

Prob. 9.22

$$\nabla \bullet \mathbf{J} = (0 + 0 + 3z^2) \sin 10^4 t = -\frac{\partial \rho_v}{\partial t}$$

$$\rho_v = - \int \nabla \bullet \mathbf{J} dt = - \int 3z^2 \sin 10^4 t dt = \frac{3z^2}{10^4} \cos 10^4 t + C_o$$

If $\rho_v|_{z=0} = 0$, then $C_o = 0$ and

$$\underline{\underline{\rho_v = 0.3z^2 \cos 10^4 t \text{ mC/m}^3}}$$

Prob. 9.23

$$\mathbf{J}_d = \frac{\partial \mathbf{D}}{\partial t} = \epsilon_o \frac{\partial \mathbf{E}}{\partial t} = \frac{50 \epsilon_o}{\rho} (-10^8) \sin(10^8 t - kz) \mathbf{a}_\rho = -\frac{4.421 \times 10^{-2}}{\rho} \sin(10^8 t - kz) \mathbf{a}_\rho \text{ A/m}^2$$

$$\nabla \times \mathbf{E} = -\mu_o \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \times \mathbf{E} = \frac{\partial E_\rho}{\partial z} \mathbf{a}_\phi = \frac{50k}{\rho} \sin(10^8 t - kz) \mathbf{a}_\phi$$

$$\mathbf{H} = -\frac{1}{\mu_o} \int \nabla \times \mathbf{E} dt = \frac{1}{4\pi \times 10^{-7}} \frac{50k}{10^8 \rho} \cos(10^8 t - kz) \mathbf{a}_\phi$$

$$\mathbf{H} = \frac{2.5k}{2\pi\rho} \cos(10^8 t - kz) \mathbf{a}_\phi \text{ A/m}$$

$$\nabla \times \mathbf{H} = -\frac{\partial H_\phi}{\partial z} \mathbf{a}_\rho = -\frac{2.5k^2}{2\pi\rho} \sin(10^8 t - kz) \mathbf{a}_\rho$$

$$\nabla \times \mathbf{H} = \mathbf{J}_d \quad \longrightarrow \quad -\frac{4.421 \times 10^{-2}}{\rho} \sin(10^8 t - kz) \mathbf{a}_\rho = \frac{-2.5k^2}{2\pi\rho} \sin(10^8 t - kz) \mathbf{a}_\rho$$

$$k^2 = \frac{2\pi}{2.5} \times 4.421 \times 10^{-2} \quad \longrightarrow \quad \underline{\underline{k = 0.333}}$$

Prob. 9.24

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} = 0 + \epsilon \frac{\partial \mathbf{E}}{\partial t} \quad \longrightarrow \quad \mathbf{E} = \frac{1}{\epsilon} \int \nabla \times \mathbf{H} dt$$

$$\nabla \times \mathbf{H} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & 10 \cos(\omega t + \beta x) \end{vmatrix} = 10\beta \sin(\omega t + \beta x) \mathbf{a}_y$$

$$\mathbf{E} = \frac{1}{\epsilon} \int 10\beta \sin(\omega t + \beta x) dt \mathbf{a}_y = \frac{-10\beta}{\omega\epsilon} \cos(\omega t + \beta x) \mathbf{a}_y$$

$$\text{But } \nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \quad \longrightarrow \quad \mathbf{H} = -\frac{1}{\mu} \int \nabla \times \mathbf{E} dt$$

$$\nabla \times \mathbf{E} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & \frac{-10\beta}{\omega\epsilon} \cos(\omega t + \beta x) & 0 \end{vmatrix} = \frac{10\beta^2}{\omega\epsilon} \sin(\omega t + \beta x) \mathbf{a}_z$$

$$\mathbf{H} = -\frac{1}{\mu} \int \frac{10\beta^2}{\omega\epsilon} \sin(\omega t + \beta x) dt \mathbf{a}_z = \frac{10\beta^2}{\omega^2 \mu \epsilon} \cos(\omega t + \beta x) \mathbf{a}_z$$

Comparing this with the given \mathbf{H} ,

$$10 = \frac{10\beta^2}{\omega^2 \mu \epsilon} \quad \longrightarrow \quad \beta = \omega \sqrt{\mu \epsilon} = 2\pi \times 10^9 \sqrt{4\pi \times 10^{-7} \times \frac{10^{-9}}{36\pi} \times 81}$$

$$\beta = 60\pi = \underline{\underline{188.5 \text{ rad/m}}}$$

$$\mathbf{E} = \frac{-10\beta}{\omega\epsilon} \cos(\omega t + \beta x) \mathbf{a}_y = \underline{\underline{-148 \cos(\omega t + \beta x) \mathbf{a}_y \text{ V/m}}}$$

Prob. 9.25

$$\begin{aligned}\mathbf{D} &= \underline{\epsilon_o E} = \underline{\underline{\epsilon_o E_o \cos(\omega t - \beta z) \mathbf{a}_x}} \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \quad \longrightarrow \quad \mathbf{B} = -\int \nabla \times \mathbf{E} dt \\ \nabla \times \mathbf{E} &= \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x \cos(\omega t - \beta z) & 0 & 0 \end{vmatrix} = -\beta E_o \sin(\omega t - \beta z) \mathbf{a}_y \\ \mathbf{B} &= \underline{\underline{\frac{\beta E_o}{\mu_o} \cos(\omega t - \beta z) \mathbf{a}_y}} \\ \mathbf{H} &= \underline{\underline{\frac{\mathbf{B}}{\mu_o}}} = \underline{\underline{\frac{\beta E_o}{\mu_o \omega} \cos(\omega t - \beta z) \mathbf{a}_y}}\end{aligned}$$

Prob. 9.26

$$\begin{aligned}(a) \quad \mathbf{J}_d &= \frac{\partial \mathbf{D}}{\partial t} \quad \longrightarrow \quad \mathbf{D} = \int \mathbf{J}_d dt \\ \mathbf{D} &= \frac{-60 \times 10^{-3}}{10^9} \cos(10^9 t - \beta z) \mathbf{a}_x = \underline{\underline{-60 \times 10^{-12} \cos(10^9 t - \beta z) \mathbf{a}_x \text{ C/m}^2}} \\ \nabla \times \mathbf{E} &= \mu \frac{\partial \mathbf{H}}{\partial t} \quad \longrightarrow \quad \nabla \times \frac{\mathbf{D}}{\epsilon} = -\mu \frac{\partial \mathbf{H}}{\partial t} \\ \nabla \times \frac{\mathbf{D}}{\epsilon} &= \frac{1}{\epsilon} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ D_x & 0 & 0 \end{vmatrix} = \frac{1}{\epsilon} (-60)(-1) \times 10^{-12} \sin(10^9 t - \beta z) \mathbf{a}_x \\ &= \frac{60\beta}{\epsilon} \times 10^{-12} \sin(10^9 t - \beta z) \mathbf{a}_y \\ \mathbf{H} &= -\frac{1}{\mu} \int \nabla \times \frac{\mathbf{D}}{\epsilon} dt = -\frac{1}{\mu} (-1) \frac{60\beta}{\epsilon} \times \frac{10^{-12}}{10^9} \cos(10^9 t - \beta z) \mathbf{a}_y \\ &= \underline{\underline{\frac{60\beta}{\mu\epsilon} \times 10^{-21} \cos(10^9 t - \beta z) \mathbf{a}_y \text{ A/m}}}\end{aligned}$$

$$(b) \quad \nabla \times \mathbf{H} = \mathbf{J} + \mathbf{J}_\phi = 0 + \mathbf{J}_d$$

$$\mathbf{J}_d = \nabla \times \mathbf{H} = \begin{vmatrix} \frac{\partial}{\partial_x} & \frac{\partial}{\partial_y} & \frac{\partial}{\partial_z} \\ 0 & H_y & 0 \end{vmatrix} = \frac{(-\beta)(-1)60\beta}{\mu\epsilon} \times (10^{-21}) \sin(10^9 t - \beta z) \mathbf{a}_x$$

Equating this with the given \mathbf{J}_d

$$60 \times 10^{-3} = \frac{60\beta^2}{\mu\epsilon} \times 10^{-21}$$

$$\beta^2 = \mu\epsilon \cdot 10^{18} = 2 \times 4\pi \times 10^{-7} \times 10 \times \frac{10^{-9}}{36\pi} = \frac{2000}{9}$$

$$\beta = \underline{14.907 \text{ rad/m}}$$

Prob. 9.27

$$\nabla \times \mathbf{E} = -\mu_o \frac{\partial \mathbf{H}}{\partial t} \quad \longrightarrow \quad \mathbf{H} = -\frac{1}{\mu_o} \int \nabla \times \mathbf{E} dt$$

$$\begin{aligned} \nabla \times \mathbf{E} &= \frac{1}{r} \frac{\partial}{\partial r} (r E_\theta) \mathbf{a}_\phi = \frac{1}{r} \frac{\partial}{\partial r} [10 \sin \theta \cos(\omega t - \beta r)] \mathbf{a}_\phi \\ &= \frac{10\beta}{r} \sin \theta \cos(\omega t - \beta r) \mathbf{a}_\phi \end{aligned}$$

$$\begin{aligned} \mathbf{H} &= -\frac{10\beta}{\mu r} \sin \theta \int \sin(\omega t - \beta r) dt \mathbf{a}_\phi \\ &= \frac{10\beta}{\omega \mu_o r} \sin \theta \cos(\omega t - \beta r) \mathbf{a}_\phi \end{aligned}$$

Prob. 9.28

$$(a) \quad \nabla \bullet \mathbf{A} = 0$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & E_z(x, t) \end{vmatrix} = -\frac{\partial E_z(x, t)}{\partial x} \mathbf{a}_y \neq 0$$

Yes, A is a possible EM field.

(b) $\nabla \bullet B = 0$

$$\nabla \times \mathbf{B} = \frac{1}{\rho} \frac{\partial}{\partial \rho} [10 \cos(\omega t - 2\rho)] \mathbf{a}_z \neq 0$$

Yes, B is a possible EM field.

(c) $\nabla \bullet C = \frac{1}{\rho} \frac{\partial}{\partial \rho} (3\rho^3 \cot \phi \sin \omega t) - \frac{\sin \phi \sin \omega t}{\rho^2} \neq 0$

$$\nabla \times \mathbf{C} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\cos \phi \sin \omega t) \mathbf{a}_z - 3\rho^2 \frac{\partial}{\partial \phi} (\cot \phi \sin \omega t) \mathbf{a}_z \neq 0$$

No, C cannot be an EM field.

(d) $\nabla \bullet D = \frac{1}{r^2 \sin \theta} \sin(\omega t - 5r) \frac{\partial}{\partial \theta} (\sin^2 \theta) \neq 0$

$$\nabla \times \mathbf{D} = -\frac{\partial D_\theta}{\partial \phi} \mathbf{a}_r + \frac{1}{r} \frac{\partial}{\partial r} (r D_\theta) \mathbf{a}_\phi = \frac{1}{r} \sin \theta (-5) \sin(\omega t - 5r) \mathbf{a}_\phi \neq 0$$

No, D cannot be an EM field.

Prob. 9.29

From Maxwell's equations,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (1)$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad (2)$$

Dotting both sides of (2) with \vec{E} gives:

$$\mathbf{E} \bullet (\nabla \times \mathbf{H}) = \mathbf{E} \bullet \mathbf{J} + \mathbf{E} \bullet \frac{\partial \mathbf{D}}{\partial t} \quad (3)$$

But for any arbitrary vectors \vec{A} and \vec{B} ,

$$\nabla \bullet (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \bullet (\nabla \times \mathbf{A}) - \mathbf{A} \bullet (\nabla \times \mathbf{B})$$

Applying this on the left-hand side of (3) by letting $\mathbf{A} \equiv \mathbf{H}$ and $\mathbf{B} \equiv \mathbf{E}$, we get

$$\mathbf{H} \bullet (\nabla \times \mathbf{E}) + \nabla \bullet (\mathbf{H} \times \mathbf{E}) = \mathbf{E} \bullet \mathbf{J} + \frac{1}{2} \frac{\partial}{\partial t} (\mathbf{D} \bullet \mathbf{E}) \quad (4)$$

From (1),

$$\mathbf{H} \bullet (\nabla \times \mathbf{E}) = \mathbf{H} \bullet \left(-\frac{\partial \mathbf{B}}{\partial t} \right) = \frac{1}{2} \frac{\partial}{\partial t} (\mathbf{B} \bullet \mathbf{H})$$

Substituting this in (4) gives:

$$-\frac{1}{2} \frac{\partial}{\partial t} (\mathbf{B} \bullet \mathbf{H}) - \nabla \bullet (\mathbf{E} \times \mathbf{H}) = \mathbf{J} \bullet \mathbf{E} + \frac{1}{2} \frac{\partial}{\partial t} (\mathbf{D} \bullet \mathbf{E})$$

Rearranging terms and then taking the volume integral of both sides:

$$\int_v \nabla \bullet (\mathbf{E} \times \mathbf{H}) dv = -\frac{\partial}{\partial t} \frac{1}{2} \int_v (\mathbf{E} \bullet \mathbf{D} + \mathbf{H} \bullet \mathbf{B}) dv - \int_v \mathbf{J} \bullet \mathbf{E} dv$$

Using the divergence theorem, we get

$$\oint_s (\mathbf{E} \times \mathbf{H}) \bullet d\mathbf{S} = -\frac{\partial W}{\partial t} - \int_v \mathbf{J} \bullet \mathbf{E} dv$$

or $\frac{\partial W}{\partial t} = -\oint_s (\mathbf{E} \times \mathbf{H}) \bullet d\mathbf{S} - \int_v \mathbf{E} \bullet \mathbf{J} dv$ as required.

Prob. 9.30

$$\begin{aligned} -\frac{\partial \mathbf{B}}{\partial t} &= \nabla \times \mathbf{E} = \beta E_o \sin(\omega t + \beta y - \beta z) (\mathbf{a}_y + \mathbf{a}_z) \\ -\mu \frac{\partial \mathbf{H}}{\partial t} &= \nabla \times \mathbf{E} \quad \longrightarrow \quad \mathbf{H} = -\frac{1}{\mu} \int \nabla \times \mathbf{E} dt \\ \mathbf{H} &= \frac{\beta E_o}{\mu \omega} \cos(\omega t + \beta y - \beta z) (\mathbf{a}_y + \mathbf{a}_z) \text{ A/m} \end{aligned}$$

Prob. 9.31

Using Maxwell's equations,

$$\nabla \times \mathbf{H} = \sigma \mathbf{E} + \epsilon \frac{\partial \mathbf{E}}{\partial t} \quad (\sigma = 0) \quad \longrightarrow \quad \mathbf{E} = \frac{1}{\epsilon} \int \nabla \times \mathbf{H} dt$$

But

$$\begin{aligned} \nabla \times \mathbf{H} &= -\frac{1}{r \sin \theta} \frac{\partial H_\theta}{\partial \phi} \mathbf{a}_r + \frac{1}{r} \frac{\partial}{\partial r} (r H_\theta) \mathbf{a}_\phi = \frac{12 \sin \theta}{r} \beta \sin(2\pi \times 10^8 t - \beta r) \mathbf{a}_\phi \\ \mathbf{E} &= \frac{12 \sin \theta}{\epsilon_0} \beta \int \sin(2\pi \times 10^8 t - \beta r) dt \mathbf{a}_\phi \\ &= -\frac{12 \sin \theta}{\omega \epsilon_0 r} \beta \cos(\omega t - \beta r) \mathbf{a}_\phi, \quad \omega = 2\pi \times 10^8 \end{aligned}$$

Prob. 9.32

With the given \mathbf{A} , we need to prove that

$$\nabla^2 \mathbf{A} = \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2}$$

$$\nabla^2 \mathbf{A} = \mu\epsilon(j\omega)(j\omega)\mathbf{A} = -\omega^2 \mu\epsilon \mathbf{A}$$

Let $\beta^2 = \omega^2 \mu\epsilon$, then $\nabla^2 \mathbf{A} = -\beta^2 \mathbf{A}$ is to be proved. We recognize that

$$\mathbf{A} = \frac{\mu_o}{4\pi r} e^{j\omega t} e^{-j\beta r} \mathbf{a}_z$$

$$\text{Assume } \varphi = \frac{e^{-j\beta r}}{r}, \quad \mathbf{A} = \frac{\mu_o}{4\pi} e^{j\omega t} \varphi \mathbf{a}_z$$

$$\begin{aligned} \nabla^2 \varphi &= \frac{1}{r^2 \sin \theta} \left[\frac{\partial}{\partial r} (r^2 \sin \theta \frac{\partial \varphi}{\partial r}) \right] = \frac{1}{r^2} \left[\frac{\partial}{\partial r} (r^2) \left(\frac{-j\beta}{r} - \frac{1}{r^2} \right) e^{-j\beta r} \right] \\ &= \frac{1}{r^2} (-\beta^2 r + j\beta - j\beta) e^{-j\beta r} = -\beta^2 \frac{e^{-j\beta r}}{r} = -\beta^2 \varphi \end{aligned}$$

$$\text{Therefore, } \nabla^2 \mathbf{A} = -\beta^2 \mathbf{A}$$

We can find V using Lorentz gauge.

$$\begin{aligned} V &= \frac{-1}{\mu_o \epsilon_o} \int \nabla \bullet \mathbf{A} dt = \frac{-1}{j\omega \mu_o \epsilon_o} \nabla \bullet \mathbf{A} \\ &= \frac{-1}{j\omega \mu_o \epsilon_o} \frac{\partial}{\partial r} \left(\frac{\mu_o}{4\pi r} e^{-j\beta r} e^{j\omega t} \right) = \frac{-1}{j\omega \epsilon_o (4\pi)} \left(\frac{-j\beta}{r} - \frac{1}{r^2} \right) e^{-j\beta r} e^{j\omega t} \cos \theta \\ V &= \underline{\underline{\frac{\cos \theta}{j4\pi\omega\epsilon_o r} \left(j\beta + \frac{1}{r} \right) e^{j(\omega t - \beta r)}}} \end{aligned}$$

Prob. 9.33

Take the curl of both sides of the equation.

$$\nabla \times \mathbf{E} = -\nabla \times \nabla V - \frac{\partial}{\partial t} \nabla \times \mathbf{A}$$

But $\nabla \times \nabla V = \mathbf{0}$ and $\mathbf{B} = \nabla \times \mathbf{A}$. Hence,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

which is Faraday's law.

Prob. 9.34

$$(a) \quad \nabla \cdot \mathbf{A} = \frac{\partial A_z}{\partial z} = \frac{x}{c}, \quad \frac{\partial V}{\partial t} = -xc, \quad -\mu_o \epsilon_o \frac{\partial V}{\partial t} = \frac{x}{c^2} c = \frac{x}{c}$$

$$\text{Hence, } \nabla \cdot \mathbf{A} = -\mu_o \epsilon_o \frac{\partial V}{\partial t}$$

$$(b) \quad \mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = -\left(\frac{\partial V}{\partial x} \mathbf{a}_x + \frac{\partial V}{\partial z} \mathbf{a}_z\right) + x \mathbf{a}_z = -(z \mathbf{a}_x + x \mathbf{a}_z) + x \mathbf{a}_z$$

$$\underline{\underline{\mathbf{E} = -z \mathbf{a}_x}}$$

Prob. 9.35

$$\nabla \cdot \mathbf{A} = 0 = -\mu \epsilon \frac{\partial V}{\partial t} \quad \longrightarrow \quad \underline{\underline{V = \text{constant}}}$$

$$(a) \quad \mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = \underline{\underline{0}} - A_o \omega \cos(\omega t - \beta z) \mathbf{a}_x$$

$$= \underline{\underline{-A_o \omega \cos(\omega t - \beta z) \mathbf{a}_x}}$$

(b) Using Maxwell's equations, we can show that

$$\underline{\underline{\beta = \omega \sqrt{\mu_o \epsilon_o}}}$$

Prob. 9.36

(a)

$$\begin{aligned} z &= 4 \angle 30^\circ - 10 \angle 50^\circ = 3.464 + 2j - 6.427 - j7.66 = -2.963 - j5.66 \\ &= 6.389 \angle -117.64^\circ \\ z^{1/2} &= \underline{\underline{2.5277 \angle -58.82^\circ}} \end{aligned}$$

(b)

$$\begin{aligned} \frac{1+j2}{6-j8-7 \angle 15^\circ} &= \frac{2.236 \angle 63.43^\circ}{6-j8-7.761-j1.812} = \frac{2.236 \angle 63.43^\circ}{9.841 \angle 265.57^\circ} \\ &= \underline{\underline{0.2272 \angle -202.1^\circ}} \end{aligned}$$

$$(c) \quad z = \frac{(5 \angle 53.13^\circ)^2}{12-j7-6-j10} = \frac{25 \angle 106.26^\circ}{18.028 \angle -70.56^\circ}$$

$$= \underline{\underline{1.387 \angle 176.8^\circ}}$$

(d)

$$\frac{1.897 \angle -100^\circ}{(5.76 \angle 90^\circ)(9.434 \angle -122^\circ)} = \underline{\underline{0.0349 \angle -68^\circ}}$$

Prob. 9.37

$$(a) \quad \mathbf{A} = 5 \cos(2t + \pi/3 - \pi/2) \mathbf{a}_x + 3 \cos(2t + 30^\circ) \mathbf{a}_y = \text{Re}(\mathbf{A}_s e^{j\omega t}), \omega = 2$$

$$\underline{\underline{\mathbf{A}_s = 5e^{-j30^\circ} \mathbf{a}_x + 3e^{j30^\circ} \mathbf{a}_y}}$$

$$(b) \quad \underline{\underline{\underline{B}}} = \frac{100}{\rho} \cos(\omega t - 2\pi z - 90^\circ) \underline{\underline{\underline{a}}}_\rho$$

$$\underline{\underline{\underline{B}}}_z = \frac{100}{\rho} e^{-j(2\pi z + 90^\circ)} \underline{\underline{\underline{a}}}_\rho$$

$$(c) \quad \underline{\underline{\underline{C}}} = \frac{\cos \theta}{r} \cos(\omega t - 3r - 90^\circ) \underline{\underline{\underline{a}}}_\theta$$

$$\underline{\underline{\underline{C}}}_s = \frac{\cos \theta}{r} e^{-j(3r + 90^\circ)} \underline{\underline{\underline{a}}}_z$$

$$(d) \quad \underline{\underline{\underline{D}}}_s = 10 \cos(k_1 x) e^{-jk_2 z} \underline{\underline{\underline{a}}}_y$$

Prob. 9.39

We can use Maxwell's equations or borrow ideas from chapter 10.

$$\eta = \sqrt{\frac{\mu}{\epsilon}} = \eta_o \sqrt{\frac{1}{\epsilon_r}} = \frac{120\pi}{9}$$

$$H_o = \frac{E_o}{\eta} = \frac{10 \times 9}{120\pi} = \underline{\underline{\underline{0.2387}}}$$

$$\beta = \omega \sqrt{\mu \epsilon} = \frac{\omega}{c} \sqrt{\epsilon_r} = \frac{2\pi \times 10^9}{3 \times 10^8} \sqrt{81} = 60\pi = \underline{\underline{\underline{188.5 \text{ rad/m}}}}$$

Prob. 9.40

(a)

$$\begin{aligned} \underline{\underline{\underline{H}}} &= \operatorname{Re} \left[40 e^{j(10^9 t - \beta z)} \underline{\underline{\underline{a}}}_x \right], \quad \omega = 10^9 \\ &= \operatorname{Re} \left[40 e^{-j\beta z} \underline{\underline{\underline{a}}}_x e^{j\omega t} \right] = \operatorname{Re} \left[\underline{\underline{\underline{H}}}_s e^{j\omega t} \right] \end{aligned}$$

$$\underline{\underline{\underline{H}}}_s = \underline{\underline{\underline{40 e^{-j\beta z} a_x}}}$$

(b)

$$\begin{aligned} \underline{\underline{\underline{J}}}_d &= \nabla \times \underline{\underline{\underline{H}}} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 40 \cos(10^9 t - \beta z) & 0 & 0 \end{vmatrix} \\ &= \underline{\underline{\underline{40 \beta \sin(10^9 t - \beta z) a_y}} \text{ A/m}^2} \end{aligned}$$

Prob. 9.41

$$(j\omega)^2 Y + 4j\omega Y + Y = 2 \angle 0^\circ, \quad \omega = 3$$

$$Y(-\omega^2 + 4j\omega + 1) = 2$$

$$Y = \frac{2}{-\omega^2 + 4j\omega + 1} = \frac{2}{-9 + j12 + 1} = \frac{2}{-8 + j12} = -0.0769 - j0.1154 \\ = 0.1387 \angle -123.7^\circ$$

$$y(t) = \text{Re}(Ye^{j\omega t}) = \underline{\underline{0.1387 \cos(3t - 123.7^\circ)}}$$

CHAPTER 10

P. E. 10.1 (a)

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{2 \times 10^8} = 31.42 \text{ ns},$$

$$\lambda = uT = 3 \times 10^8 \times 31.42 \times 10^{-9} = 9.425 \text{ m}$$

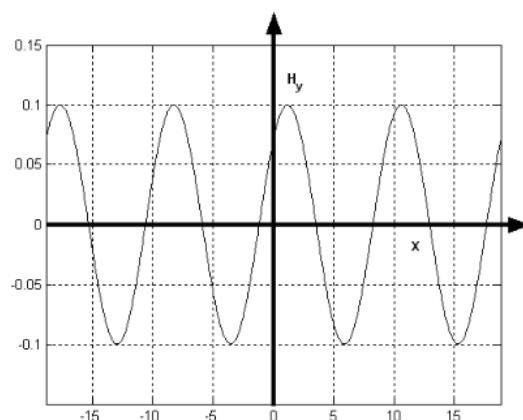
$$k = \beta = 2\pi / \lambda = 0.6667 \text{ rad/m}$$

$$(b) t_1 = T/8 = 3.927 \text{ ns}$$

(c)

$$\mathbf{H}(t=t_1) = 0.1 \cos(2 \times 10^8 \frac{\pi}{8 \times 10^8} - 2x/3) \mathbf{a}_y = 0.1 \cos(2x/3 - \pi/4) \mathbf{a}_y$$

as sketched below.

**P. E. 10.2** Let $x_o = \sqrt{1 + (\sigma / \omega\epsilon)^2}$, then

$$\alpha = \omega \sqrt{\frac{\mu_o \epsilon_o}{2}} \mu_r \epsilon_r (x_o - 1) = \frac{\omega}{c} \sqrt{\frac{16}{2}} \sqrt{x_o - 1}$$

$$\text{or } \sqrt{x_o - 1} = \frac{\alpha c}{\omega \sqrt{8}} = \frac{1/3 \times 3 \times 10^8}{10^8 \sqrt{8}} = \frac{1}{\sqrt{8}} \longrightarrow x_o = 9/8$$

$$x_o^2 = \frac{81}{64} = 1 + (\sigma / \omega\epsilon)^2 \longrightarrow \frac{\sigma}{\omega\epsilon} = 0.5154$$

$$\tan 2\theta_\eta = 0.5154 \longrightarrow \theta_\eta = 13.63^\circ$$

$$\frac{\beta}{\alpha} = \sqrt{\frac{x_o + 1}{x_o - 1}} = \sqrt{17}$$

$$(a) \quad \beta = \alpha \sqrt{17} = \frac{\sqrt{17}}{3} = 1.374 \text{ rad/m}$$

$$(b) \quad \frac{\sigma}{\omega \epsilon} = 0.5154$$

$$(c) \quad |\eta| = \frac{\sqrt{\mu/\epsilon}}{\sqrt{x_o}} = \frac{120\pi\sqrt{2/8}}{\sqrt{9/8}} = 177.72$$

$$\eta = 177.72 \angle 13.63^\circ \Omega$$

$$(d) \quad u = \frac{\omega}{\beta} = \frac{10^8}{1.374} = 7.278 \times 10^7 \text{ m/s}$$

$$(e) \quad \mathbf{a}_H = \mathbf{a}_E \times \mathbf{a}_H \longrightarrow \mathbf{a}_z \times \mathbf{a}_x = \mathbf{a}_H \longrightarrow \mathbf{a}_H = \mathbf{a}_y$$

$$\mathbf{H} = \frac{0.5}{177.5} e^{-z/3} \sin(10^8 t - \beta z - 13.63^\circ) \mathbf{a}_y = \underline{\underline{2.817 e^{-z/3} \sin(10^8 t - \beta z - 13.63^\circ) \mathbf{a}_y \text{ mA/m}}}$$

P. E. 10.3 (a) Along -z direction

$$(b) \quad \lambda = \frac{2\pi}{\beta} = 2\pi / 2 = 3.142 \text{ m}$$

$$f = \frac{\omega}{2\pi} = \frac{10^8}{2\pi} = 15.92 \text{ MHz}$$

$$\beta = \omega \sqrt{\mu \epsilon} = \omega \sqrt{\mu_o \epsilon_o} \sqrt{\mu_r \epsilon_r} = \frac{\omega}{c} \sqrt{(1) \epsilon_r}$$

$$\text{or } \sqrt{\epsilon_r} = \beta c / \omega = \frac{3 \times 10^8 \times 2}{10^8} = 6 \longrightarrow \epsilon_r = 36$$

$$(c) \quad \theta_\eta = 0, |\eta| = \sqrt{\mu/\epsilon} = \sqrt{\mu_o/\epsilon_o} \sqrt{1/\epsilon_r} = \frac{120\pi}{6} = 20\pi$$

$$\mathbf{a}_k = \mathbf{a}_E \times \mathbf{a}_H \longrightarrow -\mathbf{a}_z = \mathbf{a}_y \times \mathbf{a}_H \longrightarrow \mathbf{a}_H = \mathbf{a}_x$$

$$\mathbf{H} = \frac{50}{20\pi} \sin(\omega t + \beta z) \mathbf{a}_x = \underline{\underline{795.8 \sin(10^8 t + 2z) \mathbf{a}_x}} \text{ mA/m}$$

P. E. 10.4 (a)

$$\frac{\sigma}{\omega\epsilon} = \frac{10^{-2}}{10^9 \pi \times 4 \times \frac{10^{-9}}{36\pi}} = 0.09$$

$$\alpha \equiv \omega \sqrt{\frac{\mu\epsilon}{2} \left[1 + \frac{1}{2} \left(\frac{\sigma}{\omega\epsilon} \right)^2 - 1 \right]} = \frac{\omega}{2c} \sqrt{\mu_r \epsilon_r} \frac{\sigma}{\omega\epsilon} = \frac{10^9 \pi}{2 \times 3 \times 10^8} (2)(0.09) = 0.9425 \text{ Np/m}$$

$$\beta \equiv \omega \sqrt{\frac{\mu\epsilon}{2} \left[1 + \frac{1}{2} \left(\frac{\sigma}{\omega\epsilon} \right)^2 + 1 \right]} = \frac{10^9 \pi}{3 \times 10^8} \sqrt{2[2 + 0.5(0.09)^2]} = 20.965 \text{ rad/m}$$

$$\mathbf{E} = 30e^{-0.9425y} \cos(10^9 \pi t - 20.96y + \pi/4) \mathbf{a}_z$$

At t = 2ns, y = 1m,

$$\mathbf{E} = 30e^{-0.9425} \cos(2\pi - 20.96 + \pi/4) \mathbf{a}_z = \underline{\underline{2.844 \mathbf{a}_z}} \text{ V/m}$$

$$(b) \quad \beta y = 10^\circ = \frac{10\pi}{180} \text{ rad}$$

or

$$y = \frac{\pi}{18} \frac{1}{\beta} = \frac{\pi}{18 \times 20.965} = \underline{\underline{8.325 \text{ mm}}}$$

$$(c) \quad 30(0.6) = 30 e^{-\alpha y}$$

$$y = \frac{1}{\alpha} \ln(1/0.6) = \frac{1}{0.9425} \ln \frac{1}{0.6} = \underline{\underline{542 \text{ mm}}}$$

(d)

$$|\eta| \equiv \frac{\sqrt{\mu/\epsilon}}{[1 + \frac{1}{4}(0.09)^2]} = \frac{60\pi}{1.002} = 188.11 \Omega$$

$$2\theta_\eta = \tan^{-1} 0.09 \quad \longrightarrow \quad \theta_\eta = 2.571^\circ$$

$$\mathbf{a}_H = \mathbf{a}_k \times \mathbf{a}_E = \mathbf{a}_y \times \mathbf{a}_z = \mathbf{a}_x$$

$$\mathbf{H} = \frac{30}{188.11} e^{-0.9425y} \cos(10^9 \pi t - 20.96y + \pi/4 - 2.571^\circ) \mathbf{a}_x$$

At y = 2m, t = 2ns,

$$\mathbf{H} = (0.1595)(0.1518) \cos(-34.8963 \text{ rad}) \mathbf{a}_x = \underline{\underline{-22.83 \mathbf{a}_x}} \text{ mA/m}$$

P. E. 10.5

$$I_s = \int_0^w \int_0^\infty J_{xs} dy dz = J_{xs}(0) \int_0^w dy \int_0^\infty e^{-z(1+j)/\delta} dz = \frac{J_{xs}(0) w \delta}{1+j}$$

$$|I_s| = \underline{\underline{\frac{J_{xs}(0) w \delta}{\sqrt{2}}}}$$

P. E. 10.6 (a)

$$\frac{R_{ac}}{R_{dc}} = \frac{a}{2\delta} = \frac{a}{2} \sqrt{\pi f \mu \sigma} = \frac{1.3 \times 10^{-3}}{2} \sqrt{\pi \times 10^7 \times 4\pi \times 10^{-7} \times 3.5 \times 10^7} = \underline{\underline{24.16}}$$

(b)

$$\frac{R_{ac}}{R_{dc}} = \frac{1.3 \times 10^{-3}}{2} \sqrt{\pi \times 2 \times 10^9 \times 4\pi \times 10^{-7} \times 3.5 \times 10^7} = \underline{\underline{341.7}}$$

P. E. 10.7

$$\begin{aligned} E &= \operatorname{Re}[E_s e^{j\omega t}] = \operatorname{Re} [E_o e^{j\omega t} e^{-j\beta z} \mathbf{a}_x + E_o e^{-j\pi/2} e^{j\omega t} e^{-j\beta z} \mathbf{a}_y] \\ &= E_o \cos(\omega t - \beta z) \mathbf{a}_x + E_o \cos(\omega t - \beta z - \pi/2) \mathbf{a}_y \\ &= E_o \cos(\omega t - \beta z) \mathbf{a}_x + E_o \sin(\omega t - \beta z) \mathbf{a}_y \end{aligned}$$

At z = 0, E_x = E_o cos ωt, E_y = E_o sin ωt

$$\cos^2 \omega t + \sin^2 \omega t = 1 \quad \longrightarrow \quad \left(\frac{E_x}{E_o} \right)^2 + \left(\frac{E_y}{E_o} \right)^2 = 1$$

which describes a circle. Hence the polarization is circular.

P. E. 10.8

$$\mathcal{P}_{\text{ave}} = \frac{1}{2} \eta H_o^2 \mathbf{a}_x$$

(a) Let $f(x, z) = x + y - 1 = 0$

$$\mathbf{a}_n = \frac{\nabla f}{|\nabla f|} = \frac{\mathbf{a}_x + \mathbf{a}_y}{\sqrt{2}}, \quad d\mathbf{S} = dS \mathbf{a}_n$$

$$P_t = \int \mathbf{P} \cdot d\mathbf{S} = \mathbf{P} \cdot S \mathbf{a}_n = \frac{1}{2} \eta H_o^2 \mathbf{a}_x \cdot \frac{\mathbf{a}_x + \mathbf{a}_y}{\sqrt{2}}$$

$$= \frac{1}{2\sqrt{2}} (120\pi)(0.2)^2 (0.1)^2 = \underline{53.31 \text{ mW}}$$

$$(b) \quad d\mathbf{S} = dy dz \mathbf{a}_x, \quad P_t = \int \mathcal{P} \cdot d\mathbf{S} = \frac{1}{2} \eta H_o^2 S$$

$$P_t = \frac{1}{2} (120\pi)(0.2)^2 \pi (0.05)^2 = \underline{59.22 \text{ mW}}$$

$$\mathbf{P. E. 10.9} \quad \eta_1 = \eta_o = 120\pi, \eta_2 = \sqrt{\frac{\mu}{\epsilon}} = \frac{\eta_o}{2}$$

$$\tau = \frac{2\eta_2}{\eta_2 + \eta_1} = 2/3, \Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = -1/3$$

$$E_{ro} = \Gamma E_{io} = -\frac{10}{3}$$

$$\underline{\underline{E_{rs} = -\frac{10}{3} e^{j\beta_1 z} \mathbf{a}_x \text{ V/m}}}$$

where $\beta_1 = \omega/c = 100\pi/3$.

$$E_{to} = \tau E_{io} = \frac{20}{3}$$

$$\underline{\underline{E_{ts} = \frac{20}{3} e^{-j\beta_2 z} \mathbf{a}_x \text{ V/m}}}$$

where $\beta_2 = \omega\sqrt{\epsilon_r}/c = 2\beta_1 = 200\pi/3$.

P. E. 10.10

$$\alpha_1 = 0, \quad \beta_1 = \frac{\omega}{c} \sqrt{\mu_r \epsilon_r} = \frac{2\omega}{c} = 5 \longrightarrow \omega = 5c / 2 = 7.5 \times 10^8$$

$$\frac{\sigma_2}{\omega \epsilon_2} = \frac{0.1}{7.5 \times 10^8 \times 4 \times \frac{10^{-9}}{36\pi}} = 1.2\pi$$

$$\alpha_2 = \frac{\omega}{c} \sqrt{\frac{4}{2} \left[\sqrt{1+1.44\pi^2} - 1 \right]} = 6.021$$

$$\beta_2 = \frac{\omega}{c} \sqrt{\frac{4}{2} \left[\sqrt{1+1.44\pi^2} + 1 \right]} = 7.826$$

$$|\eta_2| = \frac{60\pi}{\sqrt[4]{1+1.44\pi^2}} = 95.445, \eta_1 = 120\pi\sqrt{\epsilon_{r1}} = 754$$

$$\tan 2\theta_{\eta_2} = 1.2\pi \longrightarrow \theta_{\eta_2} = 37.57^\circ$$

$$\eta_2 = 95.445 \angle 37.57^\circ$$

(a)

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{95.445 \angle 37.57^\circ - 754}{95.445 \angle 37.57^\circ + 754} = 0.8186 \angle 171.08^\circ$$

$$\tau = 1 + \Gamma = 0.2295 \angle 33.56^\circ$$

$$s = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.8186}{1 - 0.8186} = 10.025$$

$$(b) \quad \mathbf{E}_i = 50 \sin(\omega t - 5x) \mathbf{a}_y = \text{Im}(\mathbf{E}_{is} e^{j\omega t}), \text{ where } \mathbf{E}_{is} = 50 e^{-j5x} \mathbf{a}_y.$$

$$E_{ro} = \Gamma E_{io} = 0.8186 e^{j171.08^\circ} (50) = 40.93 e^{j171.08^\circ}$$

$$\mathbf{E}_{rs} = 40.93 e^{j5x + j171.08^\circ} \mathbf{a}_y$$

$$\mathbf{E}_r = \text{Im}(\mathbf{E}_{rs} e^{j\omega t}) = 40.93 \sin(\omega t + 5x + 171.1^\circ) \mathbf{a}_y \text{ V/m}$$

$$\mathbf{a}_H = \mathbf{a}_k \times \mathbf{a}_E = -\mathbf{a}_x \times \mathbf{a}_y = -\mathbf{a}_z$$

$$\underline{\underline{\underline{H_r = -\frac{40.93}{754} \sin(\omega t + 5x + 171.1^\circ) \mathbf{a}_z}}}$$

(c)

$$E_{to} = \tau E_{io} = 0.229 e^{j33.56^\circ} (50) = 11.475 e^{j33.56^\circ}$$

$$\underline{\underline{\underline{E_{ts} = 11.475 e^{-j\beta_2 x + j33.56^\circ} e^{-\alpha_2 x} \mathbf{a}_y}}}$$

$$\underline{\underline{\underline{E_t = \text{Im}(E_{ts} e^{j\omega t}) = 11.475 e^{-6.021x} \sin(\omega t - 7.826x + 33.56^\circ) \mathbf{a}_y}}}$$

$$\mathbf{a}_H = \mathbf{a}_k \times \mathbf{a}_E = \mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z$$

$$\underline{\underline{\underline{H_t = \frac{11.495}{95.445} e^{-6.021x} \sin(\omega t - 7.826x + 33.56^\circ - 37.57^\circ) \mathbf{a}_z}}}$$

$$\underline{\underline{\underline{= 0.1202 e^{-6.021x} \sin(\omega t - 7.826x - 4.01^\circ) \mathbf{a}_z}}}$$

(d)

$$\underline{\underline{\underline{\mathcal{P}_{\text{ave}} = \frac{E_{io}^2}{2\eta_1} \mathbf{a}_x + \frac{E_{ro}^2}{2\eta_1} (-\mathbf{a}_x) = \frac{1}{2(240\pi)} [50^2 \mathbf{a}_x - 40.93^2 \mathbf{a}_x] = 0.5469 \mathbf{a}_x}}}$$

$$\underline{\underline{\underline{P_{\text{ave}} = \frac{E_{to}^2}{2|\eta_2|} e^{-2\alpha_2 x} \cos \theta_{\eta_2} \mathbf{a}_x = \frac{(11.475)^2}{2(95.445)} \cos 37.57^\circ e^{-2(6.021)x} \mathbf{a}_x = 0.5469 e^{-12.04x} \mathbf{a}_x}}}$$

P. E. 10.11 (a)

$$k = -2\mathbf{a}_y + 4\mathbf{a}_z \longrightarrow k = \sqrt{2^2 + 4^2} = \sqrt{20}$$

$$\omega = kc = 3 \times 10^8 \sqrt{20} = 1.342 \times 10^9 \text{ rad/s},$$

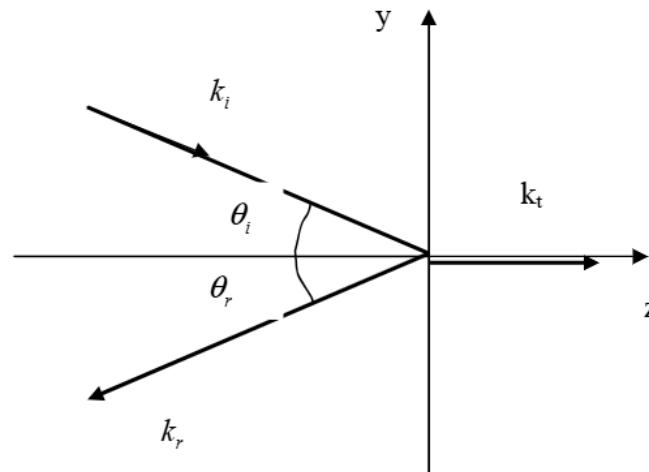
$$\lambda = 2\pi / k = \underline{\underline{\underline{1.405 \text{m}}}}$$

$$(b) \underline{\underline{\underline{H = \frac{\mathbf{a}_k \times \mathbf{E}}{\eta_o} = \frac{(-2\mathbf{a}_y + 4\mathbf{a}_z)}{\sqrt{20}(120\pi)} \times (10\mathbf{a}_y + 5\mathbf{a}_z) \cos(\omega t - \mathbf{k} \cdot \mathbf{r})}}}$$

$$\underline{\underline{\underline{= -29.66 \cos(1.342 \times 10^9 t + 2y - 4z) \mathbf{a}_x}}}$$

$$(c) \quad \mathcal{P}_{\text{ave}} = \frac{|E_o|^2}{2\eta_o} \mathbf{a}_k = \frac{125}{2(120\pi)} \frac{(-2\mathbf{a}_y + 4\mathbf{a}_z)}{\sqrt{20}} = \underline{-74.15\mathbf{a}_y + 148.9\mathbf{a}_z} \quad \text{mW/m}^2$$

P. E. 10.12 (a)



$$\tan \theta_i = \frac{k_y}{k_{iz}} = \frac{2}{4} \longrightarrow \underline{\underline{\theta_i = 26.56^\circ = \theta_r}}$$

$$\sin \theta_t = \sqrt{\frac{\mu_1 \epsilon_1}{\mu_2 \epsilon_2}} \sin \theta_i = \frac{1}{2} \sin 26.56^\circ \longrightarrow \underline{\underline{\theta_t = 12.92^\circ}}$$

(b) $\eta_1 = \eta_o, \eta_2 = \eta_o / 2$ \mathbf{E} is parallel to the plane of incidence. Since $\mu_1 = \mu_2 = \mu_o$,

we may use the result of Prob. 10.42, i.e.

$$\Gamma_{\text{II}} = \frac{\tan(\theta_t - \theta_i)}{\tan(\theta_t + \theta_i)} = \frac{\tan(-13.64^\circ)}{\tan(39.48^\circ)} = \underline{\underline{-0.2946}}$$

$$\tau_{\text{II}} = \frac{2 \cos 26.56^\circ \sin 12.92^\circ}{\sin 39.48^\circ \cos(-13.64^\circ)} = \underline{\underline{0.6474}}$$

(c) $\mathbf{k}_r = -\beta_1 \sin \theta_r \mathbf{a}_y - \beta_1 \cos \theta_r \mathbf{a}_z$. Once k_r is known, E_r is chosen such that

$\mathbf{k}_r \cdot \mathbf{E}_r = 0$ or $\nabla \cdot \mathbf{E}_r = 0$. Let

$$\mathbf{E}_r = \pm E_{or} (-\cos \theta_r \mathbf{a}_y + \sin \theta_r \mathbf{a}_z) \cos(\omega t + \beta_1 \sin \theta_r y + \beta_1 \cos \theta_r z)$$

Only the positive sign will satisfy the boundary conditions. It is evident that

$$\mathbf{E}_i = E_{oi}(\cos \theta_i \mathbf{a}_y + \sin \theta_i \mathbf{a}_z) \cos(\omega t + 2y - 4z)$$

Since $\theta_r = \theta_i$,

$$E_{or} \cos \theta_r = \Gamma_{//} E_{oi} \cos \theta_i = 10 \Gamma_{//} = -2.946$$

$$E_{or} \sin \theta_r = \Gamma_{//} E_{oi} \sin \theta_i = 5 \Gamma_{//} = -1.473$$

$$\beta_1 \sin \theta_r = 2, \beta_1 \cos \theta_r = 4$$

i.e.

$$E_r = -(2.946 \mathbf{a}_y - 1.473 \mathbf{a}_z) \cos(\omega t + 2y + 4z)$$

$$\mathbf{E}_1 = \mathbf{E}_i + \mathbf{E}_r = \underline{\underline{(10\mathbf{a}_y + 5\mathbf{a}_z) \cos(\omega t + 2y - 4z)}} + \underline{\underline{(-2.946\mathbf{a}_y + 1.473\mathbf{a}_z) \cos(\omega t + 2y + 4z)}}$$

V/m

$$(d) \mathbf{k}_t = -\beta_2 \sin \theta_t \mathbf{a}_y + \beta_2 \cos \theta_t \mathbf{a}_z. \quad \text{Since } \mathbf{k}_r \bullet \mathbf{E}_r = 0, \text{ let}$$

$$\mathbf{E}_t = E_{ot}(\cos \theta_t \mathbf{a}_y + \sin \theta_t \mathbf{a}_z) \cos(\omega t + \beta_2 y \sin \theta_t - \beta_2 z \cos \theta_t)$$

$$\beta_2 = \omega \sqrt{\mu_2 \epsilon_2} = \beta_1 \sqrt{\epsilon_{r2}} = 2\sqrt{20}$$

$$\sin \theta_t = \frac{1}{2} \sin \theta_i = \frac{1}{2\sqrt{5}}, \quad \cos \theta_t = \frac{\sqrt{9}}{\sqrt{20}}$$

$$\beta_2 \cos \theta_t = 2\sqrt{20} \sqrt{\frac{19}{20}} = 8.718$$

$$E_{ot} \cos \theta_t = \tau_{//} E_{oi} \cos \theta_t = 0.6474 \sqrt{125} \sqrt{\frac{19}{20}} = 7.055$$

$$E_{ot} \sin \theta_t = \tau_{//} E_{oi} \sin \theta_t = 0.6474 \sqrt{125} \sqrt{\frac{1}{20}} = 1.6185$$

Hence

$$\mathbf{E}_2 = \mathbf{E}_t = \underline{\underline{(7.055\mathbf{a}_y + 1.6185\mathbf{a}_z) \cos(\omega t + 2y - 8.718z)}} \text{ V/m}$$

$$(d) \tan \theta_{B//} = \sqrt{\frac{\epsilon_2}{\epsilon_1}} = 2 \longrightarrow \underline{\underline{\theta_{B//} = 63.43^\circ}}$$

P.E. 10.13

$$S_i = \frac{1+0.4}{1-0.4} = \frac{1.4}{0.6} = \underline{\underline{2.333}}$$

$$S_o = \frac{1+0.2}{1-0.2} = \frac{1.2}{0.8} = \underline{\underline{1.5}}$$

Prob. 10.1 (a) Wave propagates along +a_x.

(b)

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{2\pi \times 10^6} = \underline{\underline{1\mu s}}$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{6} = \underline{\underline{1.047\text{m}}}$$

$$u = \frac{\omega}{\beta} = \frac{2\pi \times 10^6}{6} = \underline{\underline{1.047 \times 10^6\text{m/s}}}$$

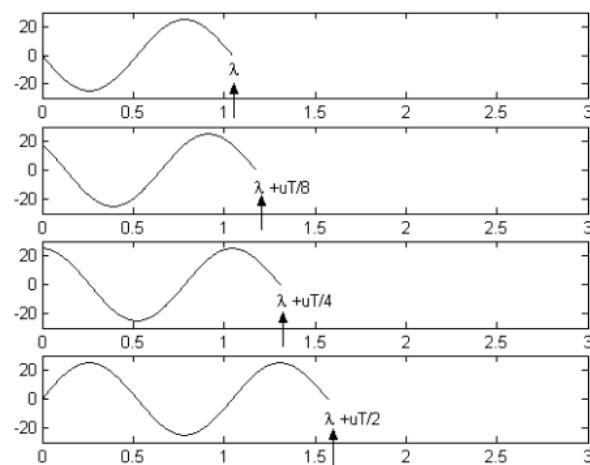
(c) At $t=0$, $E_z = 25 \sin(-6x) = -25 \sin 6x$

$$\text{At } t=T/8, E_z = 25 \sin\left(\frac{2\pi}{T} \frac{T}{8} - 6x\right) = 25 \sin\left(\frac{\pi}{4} - 6x\right)$$

$$\text{At } t=T/4, E_z = 25 \sin\left(\frac{2\pi}{T} \frac{T}{4} - 6x\right) = 25 \sin(-6x + 90^\circ) = 25 \cos 6x$$

$$\text{At } t=T/2, E_z = 25 \sin\left(\frac{2\pi}{T} \frac{T}{2} - 6x\right) = 25 \sin(-6x + \pi) = 25 \sin 6x$$

These are sketched below.

**Prob. 10.2**

(a)

$$\frac{\partial E}{\partial x} = -\sin(x + \omega t) - \sin(x - \omega t)$$

$$\frac{\partial^2 E}{\partial x^2} = -\cos(x + \omega t) - \cos(x - \omega t) = -E$$

$$\frac{\partial E}{\partial t} = -\omega \sin(x + \omega t) - \omega \sin(x - \omega t)$$

$$\frac{\partial^2 E}{\partial t^2} = -\omega^2 \cos(x + \omega t) - \omega^2 \cos(x - \omega t) = -\omega^2 E$$

$$\frac{\partial^2 E}{\partial t^2} - u^2 \frac{\partial^2 E}{\partial x^2} = -\omega^2 E + u^2 E = 0$$

if $\omega^2 = u^2$ and hence, eq. (10.1) is satisfied.(b) $u = \omega$ **Prob. 10.3**(a) $\omega = 10^8 \text{ rad/s}$

$$(b) \beta = \frac{\omega}{c} = \frac{10^8}{3 \times 10^8} = \underline{\underline{0.333 \text{ rad/m}}}$$

$$(c) \lambda = \frac{2\pi}{\beta} = 6\pi = \underline{\underline{18.85 \text{ m}}}$$

(d) Along $-\mathbf{a}_y$
At $y=1, t=10\text{ms}$,

$$(e) H = 0.5 \cos(10^8 t \times 10 \times 10^{-9} + \frac{1}{3} \times 3) = 0.5 \cos(1 + 1) \\ = \underline{\underline{-0.1665 \text{ A/m}}}$$

Prob. 10.4

$$(a) \lambda = \frac{c}{f} = \frac{3 \times 10^8}{60} = \underline{\underline{5 \times 10^6 \text{ m}}}$$

$$(b) \lambda = \frac{3 \times 10^8}{2 \times 10^6} = \underline{\underline{150 \text{ m}}}$$

$$(c) \lambda = \frac{3 \times 10^8}{120 \times 10^6} = \underline{\underline{2.5 \text{ m}}}$$

$$(d) \lambda = \frac{3 \times 10^8}{2.4 \times 10^9} = \underline{\underline{0.125 \text{ m}}}$$

Prob. 10.5 If

$$\gamma^2 = j\omega\mu(\sigma + j\omega\epsilon) = -\omega^2\mu\epsilon + j\omega\mu\sigma \quad \text{and } \gamma = \alpha + j\beta, \text{ then}$$

$$|\gamma^2| = \sqrt{(\alpha^2 - \beta^2)^2 + 4\alpha^2\beta^2} = \sqrt{(\alpha^2 + \beta^2)^2} = \alpha^2 + \beta^2$$

i.e.

$$\alpha^2 + \beta^2 = \omega\mu\sqrt{(\sigma^2 + \omega^2\epsilon^2)} \quad (1)$$

$$\text{Re}(\gamma^2) = \alpha^2 - \beta^2 = -\omega^2\mu\epsilon$$

$$\beta^2 - \alpha^2 = \omega^2\mu\epsilon \quad (2)$$

Subtracting and adding (1) and (2) lead respectively to

$$\alpha = \omega \sqrt{\frac{\mu\epsilon}{2} \left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon} \right)^2} - 1 \right]}$$

$$\beta = \omega \sqrt{\frac{\mu\epsilon}{2} \left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon} \right)^2} + 1 \right]}$$

(b) From eq. (10.25), $\mathbf{E}_s(z) = E_o e^{-\gamma z} \mathbf{a}_x$.

$$\nabla \times \mathbf{E} = -j\omega\mu \mathbf{H}_s \quad \longrightarrow \quad \mathbf{H}_s = \frac{j}{\omega\mu} \nabla \times \mathbf{E}_s = \frac{j}{\omega\mu} (-\gamma E_o e^{-\gamma z} \mathbf{a}_y)$$

$$\text{But } \mathbf{H}_s(z) = H_o e^{-\gamma z} \mathbf{a}_y, \text{ hence } H_o = \frac{E_o}{\eta} = -\frac{j\gamma}{\omega\mu} E_o$$

$$\eta = \frac{j\omega\mu}{\gamma}$$

(c) From (b),

$$\eta = \frac{j\omega\mu}{\sqrt{j\omega\mu(\sigma + j\omega\epsilon)}} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}} = \frac{\sqrt{\mu/\epsilon}}{\sqrt{1 - j\frac{\sigma}{\omega\epsilon}}}$$

$$|\eta| = \sqrt{\frac{\mu/\epsilon}{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2}}, \tan 2\theta_\eta = \left(\frac{\omega\epsilon}{\sigma}\right)^{-1} = \frac{\sigma}{\omega\epsilon}$$

Prob. 10.6 (a)

$$\frac{\sigma}{\omega\epsilon} = \frac{8 \times 10^{-2}}{2\pi \times 50 \times 10^6 \times 3.6 \times \frac{10^{-9}}{36\pi}} = 8$$

$$\alpha = \omega \sqrt{\frac{\mu\epsilon}{2} \left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1 \right]} = \frac{2\pi \times 50 \times 10^6}{3 \times 10^8} \sqrt{\frac{2.1 \times 3.6}{2} [\sqrt{65} - 1]} = 5.41$$

$$\beta = \omega \sqrt{\frac{\mu\epsilon}{2} \left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1 \right]} = 6.129$$

$$\gamma = \underline{\alpha + j\beta} = \underline{\underline{5.41 + j6.129}} / \text{m}$$

$$(b) \lambda = \frac{2\pi}{\beta} = \frac{2\pi}{6.129} = \underline{\underline{1.025}} \text{ m}$$

$$(c) u = \frac{\omega}{\beta} = \frac{2\pi \times 50 \times 10^6}{6.129} = \underline{\underline{5.125 \times 10^7}} \text{ m/s}$$

$$(d) |\eta| = \frac{\sqrt{\frac{\mu}{\epsilon}}}{\sqrt[4]{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2}} = \frac{120\pi\sqrt{\frac{2.1}{3.6}}}{\sqrt[4]{65}} = 101.4$$

$$\tan 2\theta_\eta = \frac{\sigma}{\omega\epsilon} = 8 \longrightarrow \theta_\eta = 41.44^\circ$$

$$\eta = \underline{\underline{101.41 \angle 41.44^\circ \Omega}}$$

$$(e) H_s = \mathbf{a}_k \times \frac{\mathbf{E}_s}{\eta} = \mathbf{a}_x \times \frac{6}{\eta} e^{-\gamma z} \mathbf{a}_z = -\frac{6}{\eta} e^{-\gamma z} \mathbf{a}_y = \underline{\underline{-59.16 e^{-j41.44^\circ} e^{-\gamma z} \mathbf{a}_y}} \text{ mA/m}$$

Prob. 10.7

$$(a) \tan \theta = \frac{\sigma}{\omega\epsilon} = \frac{10^{-2}}{2\pi \times 12 \times 10^6 \times 10 \times \frac{10^{-9}}{36\pi}} = \underline{\underline{1.5}}$$

$$(b) \tan \theta = \frac{10^{-4}}{2\pi \times 12 \times 10^6 \times 4 \times \frac{10^{-9}}{36\pi}} = \underline{\underline{3.75 \times 10^{-2}}}$$

$$(c) \tan \theta = \frac{4}{2\pi \times 12 \times 10^6 \times 81 \times \frac{10^{-9}}{36\pi}} = \underline{\underline{74.07}}$$

Prob. 10.8

(a)

$$\alpha = \omega \sqrt{\frac{\mu\epsilon}{2} \left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon} \right)^2} - 1 \right]} = \frac{2\pi \times 15 \times 10^9}{3 \times 10^8} \sqrt{\frac{1 \times 9.6}{2} \left[\sqrt{1 + 9 \times 10^{-8}} - 1 \right]} \\ = 100\pi \sqrt{4.8 \left(\frac{1}{2} \times 9 \times 10^{-8} \right)} = 0.146$$

$$\delta = \frac{1}{\alpha} = \underline{\underline{6.85 \text{ m}}}$$

$$(b) A = \alpha \ell = 0.146 \times 5 \times 10^{-3} = \underline{\underline{0.73 \times 10^{-3} \text{ Np}}}$$

Prob. 10.9

The phase difference is θ_η

$$\tan 2\theta_\eta = \frac{\sigma}{\omega\epsilon} = \frac{8 \times 10^{-3}}{2\pi \times 20 \times 10^6 \times 4 \times \frac{10^{-9}}{36\pi}} = 1.8$$

$$2\theta_\eta = 60.9^\circ \quad \longrightarrow \quad \theta_\eta = \underline{\underline{30.47^\circ}}$$

Prob. 10.10

For silver, the loss tangent is

$$\frac{\sigma}{\omega\epsilon} = \frac{6.1 \times 10^7}{2\pi \times 10^8 \times \frac{10^{-9}}{36\pi}} = 6.1 \times 18 \times 10^8 \gg 1$$

Hence, silver is a good conductor

For rubber,

$$\frac{\sigma}{\omega\epsilon} = \frac{10^{-15}}{2\pi \times 10^8 \times 3.1 \times \frac{10^{-9}}{36\pi}} = \frac{18}{3.1} \times 10^{-14} \ll 1$$

Hence, rubber is a poor conductor or a good insulator.

Prob. 10.11

$$\frac{\sigma}{\omega\epsilon} = \frac{4}{2\pi \times 10^5 \times 80 \times 10^{-9} / 36\pi} = 9,000 \gg 1$$

$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\frac{2\pi \times 10^5}{2} \times 4\pi \times 10^{-7} \times 4} = 0.4\pi$$

$$(a) \quad u = \omega / \beta = \frac{2\pi \times 10^5}{0.4\pi} = \underline{\underline{5 \times 10^5}} \text{ m/s}$$

$$(b) \quad \lambda = 2\pi / \beta = \frac{2\pi}{0.4\pi} = \underline{\underline{5}} \text{ m}$$

$$(c) \quad \delta = 1 / \alpha = \frac{1}{0.4\pi} = \underline{\underline{0.796}} \text{ m}$$

$$(d) \quad \eta = |\eta| \angle \theta_\eta, \theta_\eta = 45^\circ$$

$$|\eta| = \frac{\sqrt{\frac{\mu}{\epsilon}}}{\sqrt[4]{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2}} \equiv \sqrt{\frac{\mu \omega\epsilon}{\epsilon \sigma}} = \sqrt{\frac{4\pi \times 10^{-7} \times 2\pi \times 10^5}{4}} = 0.4443$$

$$\eta = \underline{0.4443 \angle 45^\circ \Omega}$$

Prob. 10.12

$$\frac{\sigma}{\omega\epsilon} = \frac{1}{2\pi \times 10^9 \times 4 \times \frac{10^{-9}}{36\pi}} = 4.5$$

$$\begin{aligned} \alpha &= \omega \sqrt{\frac{\mu\epsilon}{2} \left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1 \right]} \\ &= 2\pi \times 10^9 \sqrt{\frac{4\pi}{2} \times 10^{-7} \times 4 \times 9 \times \frac{10^{-9}}{36\pi} \left[\sqrt{1 + 4.5^2} - 1 \right]} \\ &= \underline{20\pi \sqrt{2[\sqrt{21.25} - 1]}} = 168.8 \text{ Np/m} \end{aligned}$$

$$\beta = \omega \sqrt{\frac{\mu\epsilon}{2} \left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1 \right]} = 20\pi \sqrt{2[\sqrt{21.25} + 1]} = \underline{210.5 \text{ rad/m}}$$

$$\tan 2\theta_\eta = \frac{\sigma}{\omega\epsilon} = 4.5 \quad \longrightarrow \quad \theta_\eta = 38.73^\circ$$

$$|\eta| = \frac{\sqrt{\mu/\epsilon}}{\sqrt[4]{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2}} = \frac{120\pi\sqrt{9/4}}{\sqrt[4]{1 + 4.5^2}} = 263.38$$

$$\eta = \underline{263.38 \angle 38.73^\circ \Omega}$$

$$u = \frac{\omega}{\beta} = \frac{2\pi \times 10^9}{210.5} = \underline{2.985 \times 10^7 \text{ m/s}}$$

Prob. 10.13

This is a lossy medium in which $\mu = \mu_0$.

$$\text{Let } x = \left(\frac{\sigma}{\omega\epsilon} \right)^2$$

$$\eta = \frac{j\omega\mu}{\gamma} = \frac{j2\pi \times 10^9 \times 4\pi}{100 + j200} = 35.31 \angle 26.57^\circ$$

$$E_o = 0.05 \times 35.31 = 1.765$$

$$\mathbf{a}_E = \mathbf{a}_H \times \mathbf{a}_k = -\mathbf{a}_z$$

Thus, we obtain

$$\underline{\underline{\mathbf{E} = -1.765 \cos(2\pi \times 10^9 t - 200x + 26.57^\circ) \mathbf{a}_z \text{ V/m}}}$$

$$\sqrt{\epsilon_r(1/3)} = \frac{\alpha c}{\omega} = \frac{100 \times 3 \times 10^8}{2\pi \times 10^9} = \frac{15}{\pi}$$

$$\epsilon_r \frac{1}{3} = 4.776 \quad \longrightarrow \quad \epsilon_r = 14.32$$

$$\tan 2\theta_\eta = \frac{\sigma}{\omega\epsilon} = \frac{4}{3} \quad \longrightarrow \quad \theta_\eta = 26.57^\circ$$

$$|\eta| = \frac{\sqrt{\mu/\epsilon}}{\sqrt[4]{1+x}} = \frac{\sqrt{14.32}}{\sqrt[4]{5/3}} = 77.175$$

$$E_o = |\eta| H_o = 77.175 \times 50 \times 10^{-3} = 3.858$$

$$\mathbf{a}_E = -(\mathbf{a}_k \times \mathbf{a}_H) = -(\mathbf{a}_x \times \mathbf{a}_y) = -\mathbf{a}_z$$

$$\underline{\underline{\mathbf{E} = -3.858 e^{-100x} \cos(2\pi \times 10^9 t - 200x + 26.57^\circ) \mathbf{a}_z \text{ V/m}}}$$

Prob. 10.14 (a)

$$T = 1/f = 2\pi/\omega = \frac{2\pi}{\pi \times 10^8} = \underline{\underline{20 \text{ ns}}}$$

$$(b) \text{ Let } x = \sqrt{1 + \left(\frac{\sigma}{\omega\epsilon} \right)^2}$$

$$\frac{\alpha}{\beta} = \left(\frac{x-1}{x+1} \right)^{1/2}$$

$$\text{But } \alpha = \frac{\omega}{c} \sqrt{\frac{\mu_r \epsilon_r}{2}} \sqrt{x-1}$$

$$\sqrt{x-1} = \frac{\alpha c}{\omega \sqrt{\frac{\mu_r \epsilon_r}{2}}} = \frac{0.1 \times 3 \times 10^8}{\pi \times 10^8 \sqrt{2}} = 0.06752 \longrightarrow x = 1.0046$$

$$\beta = \left(\frac{x+1}{x-1} \right)^{1/2} \quad \alpha = \left(\frac{2.0046}{0.0046} \right)^{1/2} \quad 0.1 = 2.088$$

$$\lambda = 2\pi / \beta = \frac{2\pi}{2.088} = 3 \text{ m}$$

$$(c) \quad |\eta| = \frac{\sqrt{\mu/\epsilon}}{\sqrt{x}} = \frac{377}{2\sqrt{1.0046}} = 188.1$$

$$x = \sqrt{1 + \left(\frac{\sigma}{\omega \epsilon} \right)^2} = 1.0046$$

$$\frac{\sigma}{\omega \epsilon} = 0.096 = \tan 2\theta_\eta \longrightarrow \theta_\eta = 2.74^\circ$$

$$\eta = 188.1 \angle 2.74^\circ \quad \Omega$$

$$E_o = \eta H_o = 12 \times 188.1 = 2257.2$$

$$\mathbf{a}_E \times \mathbf{a}_H = \mathbf{a}_k \longrightarrow \mathbf{a}_E \times \mathbf{a}_x = \mathbf{a}_y \longrightarrow \mathbf{a}_E = \mathbf{a}_z$$

$$\underline{\underline{\mathbf{E} = 2.257 e^{-0.1y} \sin(\pi \times 10^8 t - 2.088 y + 2.74^\circ) \mathbf{a}_z}} \quad \text{kV/m}$$

(d) The phase difference is 2.74°.

Prob. 10.15

$$(a) \quad \beta = 6.5 = \omega \sqrt{\mu_o \epsilon_o} = \frac{\omega}{c}$$

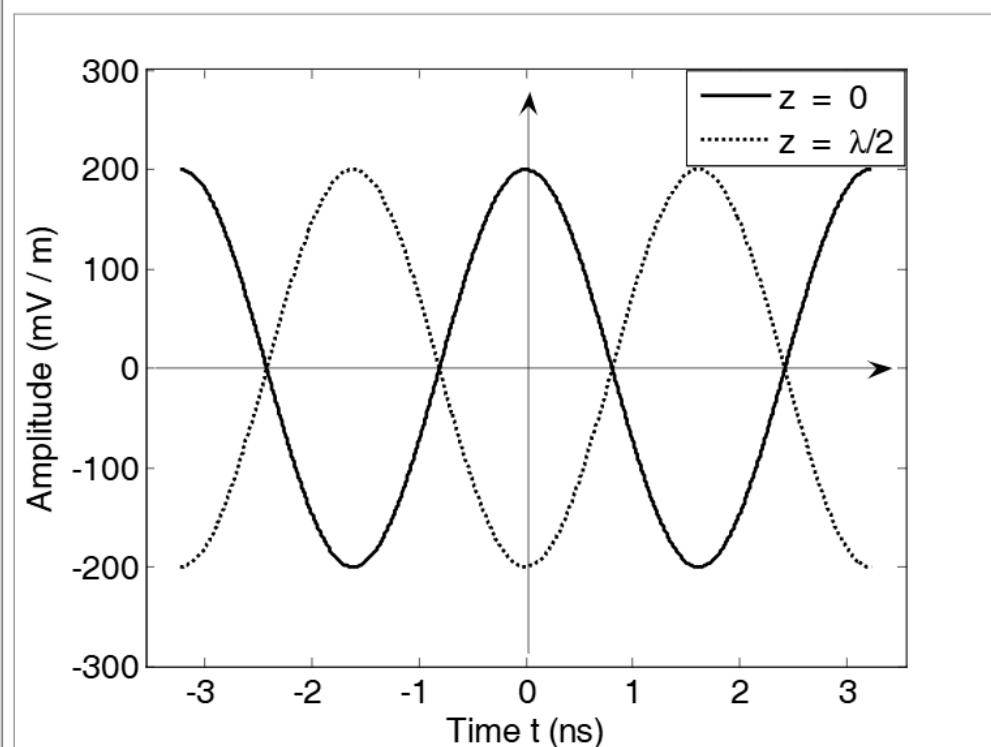
$$\omega = \beta c = 6.5 \times 3 \times 10^8 = 1.95 \times 10^9 \text{ rad/s}$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{6.5} = 0.9666 \text{ m}$$

$$(b) \quad \text{For } z=0, \quad E_z = 0.2 \cos \omega t$$

$$\text{For } z=\lambda/2, \quad E_z = 0.2 \cos(\omega t - \frac{2\pi}{\lambda} \frac{\lambda}{2}) = -0.2 \cos \omega t$$

The two waves are sketched below.



$$(c) \quad \mathbf{H} = H_o \cos(\omega t - 6.5z) \mathbf{a}_H$$

$$H_o = \frac{E_o}{\eta_o} = \frac{0.2}{377} = 5.305 \times 10^{-4}$$

$$\mathbf{a}_E \times \mathbf{a}_H = \mathbf{a}_k \quad \longrightarrow \quad \mathbf{a}_x \times \mathbf{a}_H = \mathbf{a}_z \quad \longrightarrow \quad \mathbf{a}_H = \mathbf{a}_y$$

$$\mathbf{H} = 0.5305 \cos(\omega t - 6.5z) \mathbf{a}_y \text{ mA/m}$$

Prob. 10.16

$$u = \frac{c}{\sqrt{\epsilon_r \mu_r}} = \frac{3 \times 10^8}{\sqrt{3 \times 4}} = \underline{\underline{8.66 \times 10^7 \text{ m/s}}}$$

$$\lambda = \frac{u}{f} = \frac{8.66 \times 10^7}{60 \times 10^6} = \underline{\underline{1.443 \text{ m}}}$$

$$\eta = \eta_o \sqrt{\frac{\mu_r}{\epsilon_r}} = 377 \sqrt{\frac{4}{3}} = \underline{\underline{435.32 \Omega}}$$

Prob. 10.17 (a) Along x direction.

$$(b) \beta = 6, \omega = 2 \times 10^8,$$

$$\beta = \omega \sqrt{\mu \epsilon} = \frac{\omega}{c} \sqrt{\mu_r \epsilon_r}$$

$$\sqrt{\epsilon_r} = \beta c / \omega = \frac{6 \times 3 \times 10^8}{2 \times 10^8} = 9 \longrightarrow \epsilon_r = 81$$

$$\epsilon = \epsilon_o \epsilon_r = \frac{10^{-9}}{36\pi} \times 81 = \underline{\underline{7.162 \times 10^{-10} \text{ F/m}}}$$

$$(c) \eta = \sqrt{\mu / \epsilon} = \sqrt{\mu_o / \epsilon_o} \sqrt{\mu_r / \epsilon_r} = \frac{120\pi}{9} = 41.89 \Omega$$

$$E_o = H_o \eta = 25 \times 10^{-3} \times 41.88 = 1.047$$

$$\mathbf{a}_E \times \mathbf{a}_H = \mathbf{a}_k \longrightarrow \mathbf{a}_E \times \mathbf{a}_y = -\mathbf{a}_x \longrightarrow \mathbf{a}_E = \mathbf{a}_z$$

$$E = \underline{\underline{1.047 \sin(2 \times 10^8 t + 6x) \mathbf{a}_z \text{ V/m}}}$$

$$\text{Prob. 10.18 (a)} \frac{\sigma}{\omega \epsilon} = \frac{10^{-6}}{2\pi \times 10^7 \times 5 \times \frac{10^{-9}}{36\pi}} = 3.6 \times 10^{-4} \ll 1$$

Thus, the material is lossless at this frequency.

$$(b) \beta = \omega \sqrt{\mu \epsilon} = \frac{2\pi \times 10^7}{3 \times 10^8} \sqrt{5 \times 750} = \underline{\underline{12.83 \text{ rad/m}}}$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{12.83} = \underline{\underline{0.49}} \text{ m}$$

(c) Phase difference = $\beta l = \underline{\underline{25.66}} \text{ rad}$

$$(d) \eta = \sqrt{\mu/\epsilon} = 120\pi \sqrt{\frac{\mu_r}{\epsilon_r}} = 120\pi \sqrt{\frac{750}{5}} = \underline{\underline{4.62 \text{ k}\Omega}}$$

Prob. 10.19

(a) $E = \text{Re}[E_s e^{j\omega t}] = (5a_x + 12a_y)e^{-0.2z} \cos(\omega t - 3.4z)$

At $z = 4\text{m}$, $t = T/8$, $\omega t = \frac{2\pi T}{T/8} = \frac{\pi}{4}$

$$E = (5a_x + 12a_y)e^{-0.8} \cos(\pi/4 - 13.6)$$

$$|E| = 13e^{-0.8} |\cos(\pi/4 - 13.6)| = \underline{\underline{5.662 \text{ V/m}}}$$

(b) loss = $\alpha \Delta z = 0.2(3) = 0.6 \text{ Np}$. Since $1 \text{ Np} = 8.686 \text{ dB}$,

$$\text{loss} = 0.6 \times 8.686 = \underline{\underline{5.212 \text{ dB}}}$$

(c) Let $x = \sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2}$

$$\frac{\alpha}{\beta} = \left(\frac{x-1}{x+1}\right)^{1/2} = 0.2/3.4 = \frac{1}{17}$$

$$\frac{x-1}{x+1} = 1/289 \quad \longrightarrow \quad x = 1.00694$$

$$\alpha = \omega \sqrt{\mu\epsilon/2} \sqrt{x-1} = \frac{\omega}{c} \sqrt{\epsilon_r/2} \sqrt{x-1}$$

$$\sqrt{\frac{\epsilon_r}{2}} = \frac{\alpha c}{\omega \sqrt{x-1}} = \frac{0.2 \times 3 \times 10^8}{10^8 \sqrt{0.00694}} = 7.2 \quad \longrightarrow \quad \epsilon_r = 103.68$$

$$|\eta| = \frac{\sqrt{\mu_0} \cdot \frac{1}{\sqrt{\epsilon_r}}}{\sqrt{x}} = \frac{120\pi}{\sqrt{103.68 \times 1.00694}} = 36.896$$

$$\tan 2\theta_\eta = \frac{\sigma}{\omega\epsilon} = \sqrt{x^2 - 1} = 0.118 \quad \longrightarrow \quad \theta_\eta = 3.365^\circ$$

$$\eta = \underline{36.896 \angle 3.365^\circ \Omega}$$

Prob. 10.20

This is a lossless material.

$$\eta = \sqrt{\frac{\mu}{\epsilon}} = 377 \sqrt{\frac{\mu_r}{\epsilon_r}} = 105 \quad (1)$$

$$u = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{\mu\epsilon}} = \frac{c}{\sqrt{\mu_r\epsilon_r}} = 7.6 \times 10^7 \quad (2)$$

From (1),

$$\sqrt{\frac{\mu_r}{\epsilon_r}} = \frac{105}{377} = 0.2785 \quad (1)a$$

From (2),

$$\frac{1}{\sqrt{\mu_r\epsilon_r}} = \frac{7.6 \times 10^7}{3 \times 10^8} = 0.2533 \quad (2)a$$

Multiplying (1)a by (2)a,

$$\frac{1}{\epsilon_r} = 0.2785 \times 0.2533 = 0.07054 \quad \longrightarrow \quad \epsilon_r = \underline{\underline{14.175}}$$

Dividing (1)a by (2)a,

$$\mu_r = \frac{0.2785}{0.2533} = \underline{\underline{1.0995}}$$

Prob. 10.21

$$(a) \quad \nabla \times \mathbf{E} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x(z,t) & E_y(z,t) & 0 \end{vmatrix} = -\frac{\partial E_y}{\partial z} \mathbf{a}_x + \frac{\partial E_x}{\partial z} \mathbf{a}_y$$

$$= -6\beta \cos(\omega t - \beta z) \mathbf{a}_x + 8\beta \sin(\omega t - \beta z) \mathbf{a}_y$$

$$\text{But } \nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \quad \longrightarrow \quad \mathbf{H} = -\frac{1}{\mu} \int \nabla \times \mathbf{E} dt$$

$$\mathbf{H} = \underline{\underline{\frac{6\beta}{\mu\omega} \sin(\omega t - \beta z) \mathbf{a}_x + \frac{8\beta}{\mu\omega} \cos(\omega t - \beta z) \mathbf{a}_y}}$$

$$(b) \beta = \omega\sqrt{\mu\epsilon} = \frac{2\pi f}{c}\sqrt{4.5} = \frac{2\pi \times 40 \times 10^6}{3 \times 10^8} \sqrt{4.5} = \underline{\underline{1.777 \text{ rad/m}}}$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{1.777} = \underline{\underline{3.536 \text{ m}}}$$

$$\eta = \sqrt{\frac{\mu}{\epsilon}} = \frac{120\pi}{\sqrt{4.5}} = \underline{\underline{177.72 \Omega}}$$

$$u = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{\sqrt{4.5}} = \frac{3 \times 10^8}{\sqrt{4.5}} = \underline{\underline{1.4142 \times 10^8 \text{ m/s}}}$$

Prob. 10.22

$$0.4E_o = E_o e^{-\alpha z} \longrightarrow \frac{1}{0.4} = e^{2\alpha}$$

$$\text{Or } \alpha = \frac{1}{2} \ln \frac{1}{0.4} = 0.4581 \longrightarrow \delta = 1/\alpha = \underline{\underline{2.183 \text{ m}}}$$

$$\lambda = 2\pi / \beta = 2\pi / 1.6$$

$$u = f\lambda = 10^7 \times \frac{2\pi}{1.6} = \underline{\underline{3.927 \times 10^7 \text{ m/s}}}$$

Prob. 10.23

(a)

$$\beta = \omega\sqrt{\mu\epsilon} = \frac{\omega}{c} = \frac{10^8\pi}{3 \times 10^8} = \frac{\pi}{3} = \underline{\underline{1.0472 \text{ rad/m}}}$$

(b)

$$E = 0 \longrightarrow \sin(10^8\pi t_o - \beta x_o) = 0 = \sin(n\pi), n = 1, 2, 3, \dots$$

$$10^8\pi t_o - \beta x_o = \pi$$

$$10^8\pi \times 5 \times 10^{-3} - \frac{\pi}{3} x_o = \pi \longrightarrow x_o \simeq \underline{\underline{5 \times 10^5 \text{ m}}}$$

(c)

$$\mathbf{H} = H_o \sin(10^8\pi t - \beta x) \mathbf{a}_H$$

$$H_o = \frac{E_o}{\eta} = \frac{50 \times 10^{-3}}{120\pi} = 132.63 \mu\text{A/m}$$

$$\mathbf{a}_H = \mathbf{a}_k \times \mathbf{a}_E = \mathbf{a}_x \times \mathbf{a}_z = -\mathbf{a}_y$$

$$\mathbf{H} = -132.63 \sin(10^8\pi t - 1.0472x) \mathbf{a}_y \mu\text{A/m}$$

Prob. 10.24

$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} \longrightarrow \sigma = \frac{2\alpha^2}{\omega\mu} = \frac{2 \times 12^2}{2\pi \times 10^6 \times 4\pi \times 10^{-7}} = \underline{\underline{36.48}}$$

$$\eta = |\eta| \angle \theta_\eta = \sqrt{\frac{\omega\mu}{\sigma}} \angle 45^\circ$$

$$|\eta| = \sqrt{\frac{\omega\mu}{\sigma}} = \sqrt{\frac{2\pi \times 10^6 \times 4\pi \times 10^{-7}}{36.48}} = 0.4652$$

$$E_o = |\eta| H_o = 0.4652 \times 20 \times 10^{-3} = 9.305 \times 10^{-3}$$

$$\mathbf{a}_E = \mathbf{a}_H \times \mathbf{a}_k = \mathbf{a}_y \times (-\mathbf{a}_z) = -\mathbf{a}_x$$

$$\begin{aligned} \mathbf{E} &= E_o e^{-\alpha z} \sin(\omega t + \beta z) \mathbf{a}_x \\ &= -9.305 e^{-12z} \sin(2\pi \times 10^6 t + 12z + 45^\circ) \mathbf{a}_x \text{ mV/m} \end{aligned}$$

Prob. 10.25 For a good conductor, $\frac{\sigma}{\omega\epsilon} \gg 1$, say $\frac{\sigma}{\omega\epsilon} > 100$

$$(a) \quad \frac{\sigma}{\omega\epsilon} = \frac{10^{-2}}{2\pi \times 8 \times 10^6 \times 15 \times \frac{10^{-9}}{36\pi}} = 1.5 \longrightarrow \text{lossy}$$

No, not conducting.

$$(b) \quad \frac{\sigma}{\omega\epsilon} = \frac{0.025}{2\pi \times 8 \times 10^6 \times 16 \times \frac{10^{-9}}{36\pi}} = 3.515 \longrightarrow \text{lossy}$$

No, not conducting.

$$(c) \quad \frac{\sigma}{\omega\epsilon} = \frac{25}{2\pi \times 8 \times 10^6 \times 81 \times \frac{10^{-9}}{36\pi}} = 694.4 \longrightarrow \text{conducting}$$

Yes, conducting.

Prob. 10.26

$$\alpha = \omega \sqrt{\frac{\mu\epsilon}{2} \left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon} \right)^2} - 1 \right]} = \frac{2\pi f}{c} \sqrt{\frac{\mu_r \epsilon_r}{2} \left[\sqrt{1.0049} - 1 \right]} = \frac{2\pi \times 6 \times 10^6}{3 \times 10^8} \sqrt{\frac{4}{2} \times 2.447 \times 10^{-3}}$$

$$\alpha = 8.791 \times 10^{-3}$$

$$\delta = 1/\alpha = \underline{\underline{113.75 \text{ m}}}$$

$$\beta = \omega \sqrt{\frac{\mu\epsilon}{2} \left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon} \right)^2} + 1 \right]} = \frac{4\pi}{100} \sqrt{\frac{4}{2} \left[\sqrt{1.0049} + 1 \right]} = 0.2515$$

$$u = \omega / \beta = \frac{2\pi \times 6 \times 10^6}{0.2515} = \underline{\underline{1.5 \times 10^8 \text{ m/s}}}$$

Prob. 10.27 (a)

$$R_{dc} = \frac{l}{\sigma S} = \frac{l}{\sigma \pi a^2} = \frac{600}{5.8 \times 10^7 \times \pi \times (1.2)^2 \times 10^{-6}} = \underline{\underline{2.287 \Omega}}$$

(b) $R_{ac} = \frac{l}{\sigma 2\pi a \delta}$. At 100 MHz, $\delta = 6.6 \times 10^{-3} \text{ mm} = 6.6 \times 10^{-6} \text{ m}$ mm for copper (see Table 10.2).

$$R_{ac} = \frac{600}{5.8 \times 10^7 \times 2\pi \times (1.2 \times 10^{-3}) \times 6.6 \times 10^{-6}} = \underline{\underline{207.61 \Omega}}$$

$$(c) \quad \frac{R_{ac}}{R_{dc}} = \frac{a}{2\delta} = 1 \quad \longrightarrow \quad \delta = a/2 = \frac{66.1 \times 10^{-3}}{\sqrt{f}}$$

$$\sqrt{f} = \frac{66.1 \times 2 \times 10^{-3}}{a} = \frac{66.1 \times 2}{1.2} \quad \longrightarrow \quad f = \underline{\underline{12.137 \text{ kHz}}}$$

Prob. 10.28

$$(a) \tan \theta = \frac{\sigma}{\omega\epsilon} = \frac{3.5 \times 10^7}{2\pi \times 150 \times 10^6 \times \frac{10^{-9}}{36\pi}} = \frac{3.5 \times 18 \times 10^9}{15} \gg 1$$

$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f \mu\sigma} = \sqrt{150\pi \times 10^6 \times 4\pi \times 10^{-7} \times 3.5 \times 10^7} = 143,965.86$$

$$\gamma = \alpha + j\beta = \underline{\underline{1.44(1+j) \times 10^5 \text{ /m}}}$$

$$(b) \quad \delta = 1/\alpha = \underline{\underline{6.946 \times 10^{-6} \text{ m}}}$$

$$(c) \quad u = \frac{\omega}{\beta} = \frac{150 \times 2\pi \times 10^6}{1.44 \times 10^5} = \underline{\underline{6547 \text{ m/s}}}$$

Prob. 10.29

$$\frac{\sigma}{\omega\epsilon} = \frac{4}{2\pi \times 2 \times 10^9 \times 24 \times \frac{10^{-9}}{36\pi}} = 1.5$$

$$\alpha = \omega \sqrt{\frac{\mu\epsilon}{2} \left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon} \right)^2} - 1 \right]} = 2\pi \times 2 \times 10^9 \sqrt{\frac{4\pi \times 10^{-7} \times 24 \times \frac{10^{-9}}{36\pi}}{2} \left[\sqrt{1 + (1.5)^2} - 1 \right]}$$

$$= 130.01 \text{ Np/m}$$

$$10^{-5} E_o = E_o e^{-\alpha d}$$

Taking the log of both sides gives

$$-5\ln 10 = -\alpha d \quad \longrightarrow \quad d = \frac{5\ln 10}{\alpha} = \frac{5\ln 10}{130.01} = \underline{\underline{0.0886 \text{ m}}}$$

Prob. 10.30

$$\alpha = \beta = 1/\delta$$

$$\lambda = 2\pi/\beta = 2\pi\delta = 6.283\delta \quad \longrightarrow \quad \delta = 0.1591\lambda$$

showing that δ is shorter than λ .

Prob. 10.31

$$t = 5\delta = \frac{5}{\sqrt{\pi f \mu \sigma}} = \frac{5}{\sqrt{\pi \times 12 \times 10^9 \times 4\pi \times 10^{-7} \times 6.1 \times 10^7}} = \underline{\underline{2.94 \times 10^{-6} \text{ m}}}$$

Prob. 10.32

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} \quad \longrightarrow \quad f = \frac{1}{\delta^2 \pi \mu \sigma}$$

$$f = \frac{1}{4 \times 10^{-6} \times \pi \times 4\pi \times 10^{-7} \times 6.1 \times 10^7} = \underline{\underline{1.038 \text{ kHz}}}$$

Prob. 10.33

- (a) Linearly polarized along \mathbf{a}_z
 (b) $\omega = 2\pi f = 2\pi \times 10^7 \longrightarrow f = 10^7 = \underline{\underline{10 \text{ MHz}}}$

$$\beta = \omega \sqrt{\mu \epsilon} = \omega \sqrt{\mu_o \epsilon_o} \sqrt{\epsilon_r} = \frac{\omega}{c} \sqrt{\epsilon_r}$$

$$(c) \sqrt{\epsilon_r} = \frac{\beta c}{\omega} = \frac{3 \times 3 \times 10^8}{2\pi \times 10^7} = 14.32 \longrightarrow \epsilon_r = \underline{\underline{205.18}}$$

$$\text{Let } \mathbf{H} = H_o \sin(\omega t - 3y) \mathbf{a}_H$$

$$H_o = \frac{E_o}{\eta}, \quad \eta = \sqrt{\frac{\mu}{\epsilon}} = \frac{\eta_o}{\sqrt{\epsilon_r}} = \frac{120\pi}{14.32} = 26.33$$

$$(d) H_o = \frac{12}{26.33} = 0.456$$

$$\mathbf{a}_H = \mathbf{a}_k \times \mathbf{a}_E = \mathbf{a}_y \times \mathbf{a}_z = \mathbf{a}_x$$

$$\mathbf{H} = 0.456 \sin(2\pi \times 10^7 t - 3y) \mathbf{a}_x \text{ A/m}$$

Prob. 10.34

$$\mathbf{E} = (2\mathbf{a}_y - 5\mathbf{a}_z) \sin(\omega t - \beta x)$$

The ratio E_y / E_z remains the same as t changes. Hence the wave is linearly polarized

Prob. 10.35

(a)

$$E_x = E_o \cos(\omega t + \beta y), \quad E_y = E_o \sin(\omega t + \beta y)$$

$$E_x(0, t) = E_o \cos \omega t \longrightarrow \cos \omega t = \frac{E_x(0, t)}{E_o}$$

$$E_y(0, t) = E_o \sin \omega t \longrightarrow \sin \omega t = \frac{E_y(0, t)}{E_o}$$

$$\cos^2 \omega t + \sin^2 \omega t = 1 \longrightarrow \left(\frac{E_x}{E_o} \right)^2 + \left(\frac{E_y}{E_o} \right)^2 = 1$$

Hence, we have circular polarization.

(b)

$$E_x = E_o \cos(\omega t - \beta y), \quad E_y = -3E_o \sin(\omega t - \beta y)$$

In the $y=0$ plane,

$$E_x(0,t) = E_o \cos \omega t \quad \longrightarrow \quad \cos \omega t = \frac{E_x(0,t)}{E_o}$$

$$E_y(0,t) = E_o \sin \omega t \quad \longrightarrow \quad \sin \omega t = \frac{-E_y(0,t)}{3E_o}$$

$$\cos^2 \omega t + \sin^2 \omega t = 1 \quad \longrightarrow \quad \left(\frac{E_x}{E_o} \right)^2 + \frac{1}{9} \left(\frac{E_y}{E_o} \right)^2 = 1$$

Hence, we have elliptical polarization.**Prob. 10.36**

(a) We can write

$$\mathbf{E} = \operatorname{Re}(E_s e^{j\omega t}) = (40\mathbf{a}_x + 60\mathbf{a}_y) \cos(\omega t - 10z)$$

Since E_x / E_y does not change with time, the wave is linearly polarized.(b) This is elliptically polarized.**Prob. 10.37**

(a)

When $\phi = 0$,

$$\mathbf{E}(y,t) = (E_{o1}\mathbf{a}_x + E_{o2}\mathbf{a}_z) \cos(\omega t - \beta y)$$

The two components are in phase and the wave is linearly polarized.

(b)

When $\phi = \pi/2$,

$$E_z = E_{o2} \cos(\omega t - \beta y + \pi/2) = -E_{o2} \sin(\omega t - \beta y)$$

We can combine E_x and E_z to show that the wave is elliptically polarized.

(c)

When $\phi = \pi$,

$$\begin{aligned} \mathbf{E}(y,t) &= E_{o1} \cos(\omega t - \beta y) \mathbf{a}_x + E_{o2} \cos(\omega t - \beta y + \pi) \mathbf{a}_z \\ &= (E_{o1} \mathbf{a}_x - E_{o2} \mathbf{a}_y) \cos(\omega t - \beta y) \end{aligned}$$

Thus, the wave is linearly polarized.

Prob. 10.38

We can write \mathbf{E}_s as

$$\mathbf{E}_s = \mathbf{E}_1(z) + \mathbf{E}_2(z)$$

where

$$\mathbf{E}_1(z) = \frac{1}{2} E_o (\mathbf{a}_x - j\mathbf{a}_y) e^{-j\beta z}$$

$$\mathbf{E}_2(z) = \frac{1}{2} E_o (\mathbf{a}_x + j\mathbf{a}_y) e^{-j\beta z}$$

We recognize that \mathbf{E}_1 and \mathbf{E}_2 are circularly polarized waves. The problem is therefore proved.

Prob. 10.39

(a) The wave is elliptically polarized.

(b)

$$\text{Let } \mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2,$$

$$\text{where } \mathbf{E}_1 = 40 \cos(\omega t - \beta z) \mathbf{a}_x, \quad \mathbf{E}_2 = 60 \sin(\omega t - \beta z) \mathbf{a}_y$$

$$\mathbf{H}_1 = H_{o1} \cos(\omega t - \beta z) \mathbf{a}_{H1}$$

$$H_{o1} = \frac{40}{\eta_o} = \frac{40}{120\pi} = 0.106$$

$$\mathbf{a}_{H1} = \mathbf{a}_k \times \mathbf{a}_E = \mathbf{a}_z \times \mathbf{a}_x = \mathbf{a}_y$$

$$\mathbf{H}_1 = 0.106 \cos(\omega t - \beta z) \mathbf{a}_y$$

$$\mathbf{H}_2 = H_{o2} \sin(\omega t - \beta z) \mathbf{a}_{H2}$$

$$H_{o2} = \frac{60}{\eta_o} = \frac{60}{120\pi} = 0.1592$$

$$\mathbf{a}_{H2} = \mathbf{a}_k \times \mathbf{a}_E = \mathbf{a}_z \times \mathbf{a}_y = -\mathbf{a}_x$$

$$\mathbf{H}_2 = -0.1592 \sin(\omega t - \beta z) \mathbf{a}_x$$

$$\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2 = -159.2 \sin(\omega t - \beta z) \mathbf{a}_x + 106 \cos(\omega t - \beta z) \mathbf{a}_y \text{ mA/m}$$

Prob. 10.40

$$\text{Let } \mathbf{E}_s = \mathbf{E}_r + j\mathbf{E}_i \quad \text{and} \quad \mathbf{H}_s = \mathbf{H}_r + j\mathbf{H}_i$$

$$\mathbf{E} = \operatorname{Re}(\mathbf{E}_s e^{j\omega t}) = \mathbf{E}_r \cos \omega t - \mathbf{E}_i \sin \omega t$$

Similarly,

$$\mathbf{H} = \mathbf{H}_r \cos \omega t - \mathbf{H}_i \sin \omega t$$

$$\begin{aligned}
 \mathcal{P} &= \mathbf{E} \times \mathbf{H} = \mathbf{E}_r \times \mathbf{H}_r \cos^2 \omega t + \mathbf{E}_i \times \mathbf{H}_i \sin^2 \omega t - \frac{1}{2} (\mathbf{E}_r \times \mathbf{H}_i + \mathbf{E}_i \times \mathbf{H}_r) \sin 2\omega t \\
 \mathcal{P}_{\text{ave}} &= \frac{1}{T} \int_0^T \mathcal{P} dt = \frac{1}{T} \int_0^T \cos^2 \omega t (\mathbf{E}_r \times \mathbf{H}_r) + \frac{1}{T} \int_0^T \sin^2 \omega t (\mathbf{E}_i \times \mathbf{H}_i) - \frac{1}{2T} \int_0^T \sin 2\omega t (\mathbf{E}_r \times \mathbf{H}_i + \mathbf{E}_i \times \mathbf{H}_r) \\
 &= \frac{1}{2} (\mathbf{E}_r \times \mathbf{H}_r + \mathbf{E}_i \times \mathbf{H}_i) = \frac{1}{2} \operatorname{Re}[(\mathbf{E}_r + j\mathbf{E}_i) \times (\mathbf{H}_r - j\mathbf{H}_i)] \\
 \mathcal{P}_{\text{ave}} &= \frac{1}{2} \operatorname{Re}(\mathbf{E}_s \times \mathbf{H}_s^*)
 \end{aligned}$$

as required.

Prob. 10.41

(a)

$$\beta = \omega \sqrt{\mu \epsilon} = \omega \sqrt{\mu_o \epsilon_o} \sqrt{\epsilon_r} = \frac{\omega}{c} \sqrt{\epsilon_r}$$

$$\sqrt{\epsilon_r} = \frac{\beta c}{\omega} = \frac{8 \times 3 \times 10^8}{10^9} = 2.4$$

$$\epsilon_r = \underline{\underline{\epsilon_r}}$$

$$(b) \quad \eta = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu_o}{\epsilon_o}} \frac{1}{\sqrt{\epsilon_r}} = \frac{377}{2.4} = \underline{\underline{\underline{\underline{\eta}}}}$$

$$(c) \quad u = \frac{\omega}{\beta} = \frac{10^9}{8} = \underline{\underline{\underline{\underline{1.25 \times 10^8 \text{ m/s}}}}$$

(d)

$$\text{Let } \mathbf{H} = H_o \cos(10^9 t + 8x) \mathbf{a}_H$$

$$H_o = \frac{E_o}{\eta} = \frac{150}{157.1} = 0.955$$

$$\mathbf{a}_H = \mathbf{a}_k \times \mathbf{a}_E = -\mathbf{a}_x \times \mathbf{a}_z = \mathbf{a}_y$$

$$\mathbf{H} = 0.955 \cos(10^9 t + 8x) \mathbf{a}_y \text{ A/m}$$

(e)

$$\mathcal{P} = \mathbf{E} \times \mathbf{H} = -150(0.955) \cos^2(10^9 t + 8x) \mathbf{a}_x$$

$$= \underline{\underline{\underline{-143.25 \cos^2(10^9 t + 8x) \mathbf{a}_x \text{ W/m}^2}}}$$

Prob. 10.42

$$\begin{aligned}
 \mathbf{P} = \mathbf{E} \times \mathbf{H} &= \begin{vmatrix} E_x & 0 & E_z \\ 0 & H_y & 0 \end{vmatrix} = -E_z H_y \mathbf{a}_x + E_x H_y \mathbf{a}_z \\
 &= -H_o E_{o2} \sin \alpha x \cos \alpha x \sin(\omega t - \beta x) \cos(\omega t - \beta z) \mathbf{a}_x \\
 &\quad + H_o E_{o1} \cos^2 \alpha x \cos^2(\omega t - \beta z) \mathbf{a}_z \\
 &= -\frac{1}{4} H_o E_{o2} \sin 2\alpha x \sin 2(\omega t - \beta x) \mathbf{a}_x \\
 &\quad + H_o E_{o1} \cos^2 \alpha x \cos^2(\omega t - \beta z) \mathbf{a}_z \\
 \mathbf{P}_{ave} &= \frac{1}{T} \int_0^T \mathbf{P} dt = 0 + \frac{1}{2} H_o E_{o1} \cos^2 \alpha x \mathbf{a}_z \\
 &= \underline{\underline{\frac{1}{2} E_{o1} H_o \cos^2 \alpha x \mathbf{a}_z}}
 \end{aligned}$$

Prob. 10.43

(a)

$$\text{Let } \mathbf{H}_s = \frac{H_o}{r} \sin \theta e^{-j3r} \mathbf{a}_H$$

$$H_o = \frac{E_o}{\eta_o} = \frac{10}{120\pi} = \frac{1}{12\pi}$$

$$\mathbf{a}_H = \mathbf{a}_k \times \mathbf{a}_E = \mathbf{a}_r \times \mathbf{a}_\theta = \mathbf{a}_\phi$$

$$\mathbf{H}_s = \underline{\underline{\frac{1}{12\pi r} \sin \theta e^{-j3r} \mathbf{a}_\phi \text{ A/m}}}$$

(b)

$$\mathcal{P}_{ave} = \frac{1}{2} \operatorname{Re}(\mathbf{E}_s \times \mathbf{H}_s) = \frac{10}{2 \times 12\pi r^2} \sin^2 \theta \mathbf{a}_r$$

$$P_{ave} = \int_S \mathcal{P}_{ave} \cdot dS, \quad dS = r^2 \sin \theta d\theta d\phi \mathbf{a}_r$$

$$\begin{aligned}
 P_{ave} &= \frac{10}{24\pi} \int_{\phi=0}^{\pi} \int_{\theta=0}^{\pi/6} r^2 \sin^3 \theta d\theta d\phi \Big|_{r=2} = \frac{5}{8} - \frac{5\sqrt{3}}{32} = 0.007145 \\
 &= \underline{\underline{7.145 \text{ mW}}}
 \end{aligned}$$

Prob. 10.44

$$(a) P_{ave} = \frac{1}{2} \operatorname{Re}(E_s H_s^*) = \frac{1}{2} \operatorname{Re}\left(\frac{|E_s|}{|\eta|}\right) = \frac{8^2}{2|\eta|} e^{-0.2z}$$

$$\alpha = \omega \sqrt{\frac{\mu \epsilon}{2} \left[\sqrt{1 + \left(\frac{\sigma}{\omega \epsilon} \right)^2} - 1 \right]}$$

$$\beta = \omega \sqrt{\frac{\mu \epsilon}{2} \left[\sqrt{1 + \left(\frac{\sigma}{\omega \epsilon} \right)^2} + 1 \right]}$$

Let $x = \sqrt{1 + \left(\frac{\sigma}{\omega \epsilon} \right)^2}$

$$\frac{\alpha}{\beta} = \frac{\sqrt{x-1}}{\sqrt{x+1}} = 0.1 / 0.3 = 1 / 3$$

$$\frac{x-1}{x+1} = \frac{1}{9} \quad \longrightarrow \quad x = 5 / 4$$

$$\frac{5}{4} = \sqrt{1 + \left(\frac{\sigma}{\omega \epsilon} \right)^2} \quad \longrightarrow \quad \frac{\sigma}{\omega \epsilon} = 3 / 4$$

$$|\eta| = \frac{\sqrt{\frac{\mu}{\epsilon}}}{\sqrt[4]{1 + \left(\frac{\sigma}{\omega \epsilon} \right)^2}} = \frac{120\pi / \sqrt{81}}{\sqrt[4]{\frac{5}{4}}} = 37.4657$$

$$P_{ave} = \frac{64}{2(37.4657)} e^{-0.2z} = \underline{\underline{0.8541 e^{-0.2z} \text{ W/m}^2}}$$

$$(b) 20dB = 10 \log \frac{P_1}{P_2} \quad \longrightarrow \quad \frac{P_1}{P_2} = 100$$

$$\frac{P_2}{P_1} = e^{-0.2z} = \frac{1}{100} \quad \longrightarrow \quad e^{0.2z} = 100$$

$$z = 5 \log 100 = \underline{\underline{23 \text{ m}}}$$

Prob. 10.45

$$(a) u = \omega / \beta \quad \longrightarrow \quad \omega = u\beta = \frac{\beta c}{\sqrt{4.5}} = \frac{2 \times 3 \times 10^8}{\sqrt{4.5}} = \underline{\underline{2.828 \times 10^8 \text{ rad/s}}}$$

$$\eta = \frac{120\pi}{\sqrt{4.5}} = 177.7 \Omega$$

$$\underline{\underline{H}} = \underline{\underline{a}_k} \times \frac{\underline{\underline{E}}}{\eta} = \frac{\underline{\underline{a}_z}}{\eta} \times \frac{40}{\rho} \sin(\omega t - 2z) \underline{\underline{a}_\rho} = \frac{0.225}{\rho} \sin(\omega t - 2z) \underline{\underline{a}_\phi} \text{ A/m}$$

$$(b) \quad \underline{\underline{P}} = \underline{\underline{E}} \times \underline{\underline{H}} = \frac{9}{\rho^2} \sin^2(\omega t - 2z) \underline{\underline{a}_z} \text{ W/m}^2$$

$$(c) \quad \mathcal{P}_{ave} = \frac{4.5}{\rho^2} \underline{\underline{a}_z}, \quad dS = \rho d\phi d\rho \underline{\underline{a}_z}$$

$$P_{ave} = \int \underline{\underline{P}}_{ave} \bullet dS = 4.5 \int_{2mm}^{3mm} \frac{d\rho}{\rho} \int_0^{2\pi} d\phi = 4.5 \ln(3/2)(2\pi) = \underline{\underline{11.46 \text{ W}}}$$

Prob. 10.46

$$\underline{\underline{\mathcal{P}}} = \underline{\underline{E}} \times \underline{\underline{H}} = \frac{E_o^2}{\eta r^2} \sin^2 \theta \sin^2 \omega(t - r/c) \underline{\underline{a}_r}$$

$$\underline{\underline{P}}_{ave} = \frac{1}{T} \int_0^T \underline{\underline{\mathcal{P}}} dt = \frac{E_o^2}{2\eta r^2} \sin^2 \theta \underline{\underline{a}_r}$$

Prob. 10.47

$$\beta = \frac{\omega}{c} \longrightarrow \omega = \beta c = 40(3 \times 10^8) = \underline{\underline{12 \times 10^9 \text{ rad/s}}}$$

$$E = \frac{1}{\epsilon} \int \nabla \times \underline{\underline{H}} dt$$

$$\begin{aligned} \nabla \times \underline{\underline{H}} &= \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 10 \sin(\omega t - 40x) & -20 \sin(\omega t - 40x) \\ -800 \cos(\omega t - 40x) \underline{\underline{a}_y} & -400 \cos(\omega t - 40x) \underline{\underline{a}_z} \end{vmatrix} \\ &= -800 \cos(\omega t - 40x) \underline{\underline{a}_y} - 400 \cos(\omega t - 40x) \underline{\underline{a}_z} \end{aligned}$$

$$\begin{aligned} E &= \frac{1}{\epsilon} \int \nabla \times \underline{\underline{H}} dt = -\frac{800}{\omega \epsilon} \sin(\omega t - 40x) \underline{\underline{a}_y} - \frac{400}{\omega \epsilon} \sin(\omega t - 40x) \underline{\underline{a}_z} \\ &= -\frac{800}{12 \times 10^9 \times \frac{10^{-9}}{36\pi}} \sin(\omega t - 40x) \underline{\underline{a}_y} - \frac{400}{12 \times 10^9 \times \frac{10^{-9}}{36\pi}} \sin(\omega t - 40x) \underline{\underline{a}_z} \\ &= -7.539 \sin(\omega t - 40x) \underline{\underline{a}_y} - 3.77 \sin(\omega t - 40x) \underline{\underline{a}_z} \text{ kV/m} \end{aligned}$$

$$\begin{aligned}\mathbf{P} = \mathbf{E} \times \mathbf{H} &= \begin{vmatrix} 0 & E_y & E_z \\ 0 & H_y & H_z \end{vmatrix} = (E_y H_z - E_z H_y) \mathbf{a}_x \\ &= [20(7.537) \sin^2(\omega t - 40x) + 37.7 \sin^2(\omega t - 40x)] \mathbf{a}_x 10^3 \\ \mathbf{P}_{\text{ave}} &= \frac{1}{2} [20(7.537) + 37.7] \mathbf{a}_x 10^3 = \underline{\underline{94.23 \mathbf{a}_x \text{ kW/m}^2}}\end{aligned}$$

Prob. 10.48

$$\begin{aligned}P &= \frac{E_o^2}{2\eta_o} \quad \longrightarrow \quad E_o^2 = 2\eta_o P = 2(120\pi)10 \times 10^{-3} = 7.539 \\ E_o &= \underline{\underline{2.746 \text{ V/m}}}\end{aligned}$$

Prob. 10.49

Let $T = \omega t - \beta z$.

$$\begin{aligned}-\frac{\partial \mathbf{B}}{\partial t} &= \nabla \times \mathbf{E} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos T & \sin T & 0 \end{vmatrix} \\ -\mu \frac{\partial \mathbf{H}}{\partial t} &= \beta \cos T \mathbf{a}_x + \beta \sin T \mathbf{a}_y \\ \mathbf{H} &= -\frac{\beta}{\mu} \int [\cos T \mathbf{a}_x + \sin T \mathbf{a}_y] dt = -\frac{\beta}{\mu\omega} \sin T \mathbf{a}_x + \frac{\beta}{\mu\omega} \cos T \mathbf{a}_y \\ \mathcal{P} &= \mathbf{E} \times \mathbf{H} = \begin{vmatrix} \cos T & \sin T & 0 \\ -\frac{\beta}{\mu\omega} \sin T & \frac{\beta}{\mu\omega} \cos T & 0 \end{vmatrix} = \frac{\beta}{\mu\omega} (\cos^2 T + \sin^2 T) \mathbf{a}_z \\ &= \frac{\beta}{\mu\omega} \mathbf{a}_z = \sqrt{\frac{\epsilon}{\mu}} \mathbf{a}_z\end{aligned}$$

which is constant everywhere.

Prob. 10.50

$$\begin{aligned}\mathcal{P} &= \frac{E_o^2}{2\eta_o} \\ P &= \mathcal{P}S = \frac{E_o^2 S}{2\eta_o} = \frac{(2.4 \times 10^3)^2 \times 450 \times 10^{-4}}{2 \times 377} = \underline{\underline{343.8 \text{ W}}}\end{aligned}$$

Prob. 10.51

$$\mathcal{P} = \mathbf{E} \times \mathbf{H} = \frac{V_o I_o}{2\pi\rho^2 \ln(b/a)} \sin^2(\omega t - \beta z) \mathbf{a}_z$$

$$(a) \quad \mathcal{P}_{ave} = \frac{1}{T} \int_0^T \mathcal{P} dt = \frac{V_o I_o}{2\pi\rho^2 \ln(b/a)} \frac{1}{T} \int_0^T \sin^2(\omega t - \beta z) dt \mathbf{a}_z = \frac{V_o I_o}{2\pi\rho^2 \ln(b/a)} \frac{1}{2} \mathbf{a}_z$$

$$= \underline{\underline{\frac{V_o I_o}{4\pi\rho^2 \ln(b/a)} \mathbf{a}_z}}$$

(b)

$$P_{ave} = \iint_S \mathcal{P}_{ave} \cdot d\mathbf{S}, \quad d\mathbf{S} = \rho d\rho d\phi \mathbf{a}_z$$

$$= \frac{V_o I_o}{4\pi \ln(b/a)} \int_{\phi=0}^{2\pi} \int_{\rho=a}^b \frac{1}{\rho^2} \rho d\rho d\phi = \frac{V_o I_o}{4\pi \ln(b/a)} (2\pi) \ln(b/a)$$

$$= \underline{\underline{\frac{1}{2} V_o I_o}}$$

Prob. 10.52

$$(a) \quad P_{i,ave} = \frac{E_{io}^2}{2\eta_1}, \quad P_{r,ave} = \frac{E_{ro}^2}{2\eta_1}, \quad P_{t,ave} = \frac{E_{to}^2}{2\eta_2}$$

$$R = \frac{P_{r,ave}}{P_{i,ave}} = \frac{E_{ro}^2}{E_{io}^2} = \Gamma^2 = \left(\frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \right)^2$$

$$R = \left(\frac{\sqrt{\frac{\mu_o}{\epsilon_2}} - \sqrt{\frac{\mu_o}{\epsilon_1}}}{\sqrt{\frac{\mu_o}{\epsilon_2}} + \sqrt{\frac{\mu_o}{\epsilon_1}}} \right)^2 = \left(\frac{\sqrt{\mu_o \epsilon_1} - \sqrt{\mu_o \epsilon_2}}{\sqrt{\mu_o \epsilon_1} + \sqrt{\mu_o \epsilon_2}} \right)^2$$

$$\text{Since } n_1 = c\sqrt{\mu_1 \epsilon_1} = c\sqrt{\mu_o \epsilon_1}, \quad n_2 = c\sqrt{\mu_o \epsilon_2},$$

$$R = \left(\frac{n_1 - n_2}{n_1 + n_2} \right)^2$$

$$T = \frac{P_{t,ave}}{P_{i,ave}} = \frac{\eta_1}{\eta_2} \frac{E_{to}^2}{E_{io}^2} = \frac{\eta_1}{\eta_2} \tau^2 = \frac{\eta_1}{\eta_2} (1 + \Gamma)^2 = \frac{4n_1 n_2}{(n_1 + n_2)^2}$$

(b) If $P_{r,\text{ave}} = P_{t,\text{ave}} \longrightarrow RP_{i,\text{ave}} = TP_{i,\text{ave}} \longrightarrow R = T$

$$\text{i.e. } (n_1 - n_2)^2 = 4n_1 n_2 \longrightarrow n_1^2 - 6n_1 n_2 + n_2^2 = 0$$

$$\text{or } \left(\frac{n_1}{n_2}\right)^2 - 6\left(\frac{n_1}{n_2}\right) + 1 = 0, \text{ so}$$

$$\frac{n_1}{n_2} = 3 \pm \sqrt{8} = 5.828 \quad \underline{\text{or}} \quad 0.1716$$

(Note that these values are mutual reciprocals, reflecting the inherent symmetry of the problem.)

Prob. 10.53 (a) $\eta_1 = \eta_o, \quad \eta_2 = \sqrt{\frac{\mu}{\epsilon}} = \eta_o / 2$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\eta_o / 2 - \eta_o}{3\eta_o / 2} = \underline{\underline{-1/3}}, \quad \tau = \frac{2\eta_2}{\eta_2 + \eta_1} = \frac{\eta_o}{3\eta_o / 2} = \underline{\underline{2/3}}$$

$$s = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 1/3}{1 - 1/3} = \underline{\underline{2}}$$

$$(b) \quad E_{or} = \Gamma E_{oi} = -\frac{1}{3} \times (30) = -10$$

$$\mathbf{E}_r = -10 \cos(\omega t + z) \mathbf{a}_x \text{ V/m}$$

Let $\mathbf{H}_r = H_{or} \cos(\omega t + z) \mathbf{a}_H$

$$\mathbf{a}_E \times \mathbf{a}_H = \mathbf{a}_k \longrightarrow -\mathbf{a}_x \times \mathbf{a}_H = -\mathbf{a}_z \longrightarrow \mathbf{a}_H = \mathbf{a}_y$$

$$\mathbf{H}_r = \frac{10}{120\pi} \cos(\omega t + z) \mathbf{a}_y = \underline{\underline{26.53 \cos(\omega t + z) \mathbf{a}_y \text{ mA/m}}}$$

Prob. 10.54

$$\beta_1 = \frac{\omega}{c} = \frac{2\pi \times 10^8}{3 \times 10^8} = \frac{2\pi}{3}$$

$$\eta_1 = \eta_o, \quad \eta_2 = 0$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{0 - \eta_o}{0 + \eta_o} = -1, \quad \tau = 1 + \Gamma = 0$$

$$\mathbf{E}_t = \mathbf{0}$$

$$\underline{\underline{\mathbf{E}_r = -50 \sin(2\pi \times 10^8 t + \beta_1 x) \mathbf{a}_z \text{ V/m}}}$$

Prob. 10.55 (a) Medium 1 is free space. Given that $\beta = 1$,

$$\beta = 1 = \frac{\omega}{c} \longrightarrow \omega = c = \underline{\underline{3 \times 10^8 \text{ rad/s}}}$$

$$(b) \eta_1 = \eta_o, \quad \eta_2 = \eta_o \sqrt{\frac{\mu_r}{\epsilon_r}} = \eta_o \sqrt{\frac{3}{12}} = \eta_o / 2$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = -1/3, \quad \tau = 1 + \Gamma = 2/3$$

$$s = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 1/3}{1 - 1/3} = 2$$

(c) Let $\mathbf{H}_r = H_{or} \cos(\omega t + z) \mathbf{a}_H$, where

$$\mathbf{E}_r = -\frac{1}{3}(30) \cos(\omega t + z) \mathbf{a}_y = -10 \cos(\omega t + z) \mathbf{a}_y, \quad H_{or} = \frac{10}{\eta_o} = \frac{10}{120\pi}$$

$$\mathbf{a}_E \times \mathbf{a}_H = \mathbf{a}_k \longrightarrow -\mathbf{a}_y \times \mathbf{a}_H = -\mathbf{a}_z \longrightarrow \mathbf{a}_H = -\mathbf{a}_x$$

$$\mathbf{H}_r = -\frac{10}{120\pi} \cos(3 \times 10^8 t + z) \mathbf{a}_x \text{ A/m} = \underline{\underline{-26.53 \cos(3 \times 10^8 t + z) \mathbf{a}_x \text{ mA/m}}}$$

$$(d) \quad \mathbf{P}_t = \frac{E_{ot}^2}{2\eta_2} \mathbf{a}_z, \quad E_{ot} = \tau E_{oi} = \frac{2}{3}(30) = 20, \quad \eta_2 = 60\pi$$

$$\mathbf{P}_t = \frac{20^2}{120\pi} (\mathbf{a}_z) = \underline{\underline{1.061 \mathbf{a}_z \text{ W/m}^2}}$$

Prob. 10.56 $\eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}} = \eta_o / 2, \quad \eta_2 = \eta_o$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = 1/3, \quad \tau = 1 + \Gamma = 4/3$$

$$E_{or} = \Gamma E_{io} = (1/3)(5) = 5/3, \quad E_{ot} = \tau E_{io} = 20/3$$

$$\beta = \frac{\omega}{c} \sqrt{\mu_r \epsilon_r} = \frac{10^8}{3 \times 10^8} \sqrt{4} = 2/3$$

(a) $\mathbf{E}_r = \frac{5}{3} \cos(10^8 t - 2y/3) \mathbf{a}_z$

$$\mathbf{E}_1 = \mathbf{E}_i + \mathbf{E}_r = 5 \cos(10^8 t + \frac{2}{3}y) \mathbf{a}_z + \frac{5}{3} \cos(10^8 t - \frac{2}{3}y) \mathbf{a}_z \text{ V/m}$$

(b) $\mathcal{P}_{ave1} = \frac{E_{io}^2}{2\eta_1} (-\mathbf{a}_y) + \frac{E_{ro}^2}{2\eta_1} (+\mathbf{a}_y) = \frac{25}{2(60\pi)} \left(1 - \frac{1}{9}\right) (-\mathbf{a}_y) = \underline{\underline{-0.0589 \mathbf{a}_y \text{ W/m}^2}}$

(c) $\mathcal{P}_{ave2} = \frac{E_{to}^2}{2\eta_2} (-\mathbf{a}_y) = \frac{400}{9(2)(120\pi)} (-\mathbf{a}_y) = \underline{\underline{-0.0589 \mathbf{a}_y \text{ W/m}^2}}$

Prob. 10.57

(a) $\eta_1 = \eta_o$

$$\mathbf{E}_i = E_{io} \sin(\omega t - 5x) \mathbf{a}_E$$

$$E_{io} = H_{io} \eta_o = 120\pi \times 4 = 480\pi$$

$$\mathbf{a}_E \times \mathbf{a}_H = \mathbf{a}_k \longrightarrow \mathbf{a}_E \times \mathbf{a}_y = \mathbf{a}_x \longrightarrow \mathbf{a}_E = -\mathbf{a}_z$$

$$\mathbf{E}_i = -480\pi \sin(\omega t - 5x) \mathbf{a}_z$$

$$\eta_2 = \sqrt{\frac{\mu_o}{4\epsilon_o}} = \frac{120\pi}{\sqrt{4}} = 60\pi$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{60\pi - 120\pi}{60\pi + 120\pi} = -1/3, \quad \tau = 1 + \Gamma = 2/3$$

$$E_{ro} = \Gamma E_{io} = (-1/3)(-480\pi) = 160\pi$$

$$\mathbf{E}_r = 160\pi \sin(\omega t + 5x) \mathbf{a}_z$$

$$\mathbf{E}_1 = \mathbf{E}_i + \mathbf{E}_r = \underline{\underline{-1.508 \sin(\omega t - 5x) \mathbf{a}_z}} + \underline{\underline{0.503 \sin(\omega t + 5x) \mathbf{a}_z}} \text{ kV/m}$$

$$(b) \quad E_{to} = \tau E_{io} = (2/3)(480\pi) = 320\pi$$

$$\mathcal{P} = \frac{E_{to}^2}{2\eta_2} \mathbf{a}_x = \underline{\underline{\frac{(320\pi)^2}{2(60\pi)} \mathbf{a}_x}} = 2.68 \mathbf{a}_x \text{ kW/m}^2$$

$$(c) \quad s = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+1/3}{1-1/3} = 2$$

Prob. 10.58

$$(a) \text{ In air, } \beta_1 = 1, \lambda_1 = 2\pi / \beta_1 = 2\pi = \underline{\underline{6.283 \text{ m}}}$$

$$\omega = \beta_1 c = \underline{\underline{3 \times 10^8 \text{ rad/s}}}$$

In the dielectric medium, ω is the same .

$$\omega = \underline{\underline{3 \times 10^8 \text{ rad/s}}}$$

$$\beta_2 = \frac{\omega}{c} \sqrt{\epsilon_{r2}} = \beta_1 \sqrt{\epsilon_{r2}} = \sqrt{3}$$

$$\lambda_2 = \frac{2\pi}{\beta_2} = \frac{2\pi}{\sqrt{3}} = \underline{\underline{3.6276 \text{ m}}}$$

$$(b) \quad H_o = \frac{E_o}{\eta_o} = \frac{10}{120\pi} = 0.0265$$

$$\mathbf{a}_H = \mathbf{a}_k \times \mathbf{a}_E = \mathbf{a}_z \times \mathbf{a}_y = -\mathbf{a}_x$$

$$\mathbf{H}_i = \underline{\underline{-26.5 \cos(\omega t - z) \mathbf{a}_x \text{ mA/m}}}$$

$$(c) \quad \eta_1 = \eta_o, \quad \eta_2 = \eta_o / \sqrt{3}$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{(1/\sqrt{3}) - 1}{(1/\sqrt{3}) + 1} = \underline{\underline{-0.268}}, \quad \tau = 1 + \Gamma = \underline{\underline{0.732}}$$

$$(d) \quad E_{to} = \tau E_{io} = 7.32, \quad E_{ro} = \Gamma E_{io} = -2.68$$

$$\underline{\underline{E_1 = E_i + E_r = 10 \cos(\omega t - z) \mathbf{a}_y - 2.68 \cos(\omega t + z) \mathbf{a}_y \text{ V/m}}}$$

$$\underline{\underline{E_2 = E_t = 7.32 \cos(\omega t - z) \mathbf{a}_y \text{ V/m}}}$$

$$\mathcal{P}_{ave1} = \frac{1}{2\eta_1} (\mathbf{a}_z) [E_{io}^2 - E_{ro}^2] = \frac{1}{2(120\pi)} (\mathbf{a}_z) (10^2 - 2.68^2) = \underline{\underline{0.1231 \mathbf{a}_z \text{ W/m}^2}}$$

$$\mathcal{P}_{ave2} = \frac{E_{to}^2}{2\eta_2} (\mathbf{a}_z) = \frac{\sqrt{3}}{2 \times 120\pi} (7.32)^2 (\mathbf{a}_z) = \underline{\underline{0.1231 \mathbf{a}_z \text{ W/m}^2}}$$

Prob. 10.59

$$\eta_1 = \eta_o = 120\pi$$

For seawater (lossy medium),

$$\eta_2 = \sqrt{\frac{j\omega\mu_o}{\sigma + j\omega\epsilon}} = \sqrt{\frac{j2\pi \times 10^8 \times 4}{4 + j2\pi \times 10^8 \times 81 \times \frac{10^{-9}}{36\pi}}} = 10.44 + j9.333$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = 0.9461 \angle 177.16$$

$$|\Gamma|^2 = 0.8952, \quad 1 - |\Gamma| = 0.1084$$

$$\frac{P_r}{P_i} = 89.51\%, \quad \frac{P_t}{P_i} = 10.84\%,$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{7.924 \angle 43.975 - 377}{7.924 \angle 43.975 + 377} = 0.9702 \angle 178.2^\circ$$

The fraction of the incident power reflected is

$$\frac{P_r}{P_i} = |\Gamma|^2 = 0.9702^2 = \underline{\underline{0.9413}}$$

The transmitted fraction is

$$\frac{P_t}{P_i} = 1 - |\Gamma|^2 = 1 - 0.9702^2 = \underline{\underline{0.0587}}$$

Prob. 10.60

(a)

$$\eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}} = \frac{120\pi}{\sqrt{4}} = 188.5, \quad \eta_2 = \sqrt{\frac{\mu_2}{\epsilon_2}} = \frac{120\pi}{\sqrt{3.2}} = 210.75$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{210.75 - 188.5}{210.75 + 188.5} = 0.0557, \quad \tau = \frac{2\eta_2}{\eta_2 + \eta_1} = \frac{2 \times 210.75}{210.75 + 188.5} = 1.0557$$

$$E_{ro} = \Gamma E_{io} = (0.0557)(12) = 0.6684 \quad E_{to} = \tau E_{io} = 1.0557(12) = 12.668$$

$$\beta_1 = \omega \sqrt{\mu_1 \epsilon_1} = \frac{\omega}{c} \sqrt{4} \quad \longrightarrow \quad \omega = \frac{\beta_1 c}{2} = \frac{40\pi(3 \times 10^8)}{2} = \underline{\underline{6\pi \times 10^9 \text{ rad/s}}}$$

(b)

$$\beta_2 = \omega \sqrt{\mu_2 \epsilon_2} = \frac{\omega}{c} \sqrt{3.2} = \frac{6\pi \times 10^9 \sqrt{3.2}}{3 \times 10^8} = 112.4$$

$$E_r = E_{ro} \cos(\omega t + 40\pi x) \mathbf{a}_z = \underline{\underline{0.6684 \cos(6\pi \times 10^9 t + 40\pi x) \mathbf{a}_z \text{ V/m}}}$$

$$E_t = E_{to} \cos(\omega t - \beta_2 x) \mathbf{a}_z = \underline{\underline{12.668 \cos(6\pi \times 10^9 t - 112.4x) \mathbf{a}_z \text{ V/m}}}$$

Prob. 10.61 (a) $\omega = \beta c = 3 \times 3 \times 10^8 = \underline{\underline{9 \times 10^8 \text{ rad/s}}}$ (b) $\lambda = 2\pi / \beta = 2\pi / 3 = \underline{\underline{2.094 \text{ m}}}$

$$(c) \frac{\sigma}{\omega \epsilon} = \frac{4}{9 \times 10^8 \times 80 \times 10^{-9} / 36\pi} = 2\pi = \underline{\underline{6.288}}$$

$$\tan 2\theta_\eta = \frac{\sigma}{\omega \epsilon} = 6.288 \quad \longrightarrow \quad \theta_\eta = 40.47^\circ$$

$$|\eta_2| = \frac{\sqrt{\mu_2 / \epsilon_2}}{\sqrt[4]{1 + \left(\frac{\sigma_2}{\omega \epsilon_2}\right)^2}} = \frac{377 / \sqrt{80}}{\sqrt[4]{1 + 4\pi^2}} = 16.71$$

$$\eta_2 = \underline{\underline{16.71 \angle 40.47^\circ \Omega}}$$

$$(d) \quad \Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{16.71 \angle 40.47^\circ - 377}{16.71 \angle 40.47^\circ + 377} = 0.935 \angle 179.7^\circ$$

$$E_{or} = \Gamma E_{oi} = 9.35 \angle 179.7^\circ$$

$$E_r = \underline{\underline{9.35 \sin(\omega t - 3z + 179.7) \mathbf{a}_x \text{ V/m}}}$$

$$\alpha_2 = \frac{\omega}{c} \sqrt{\frac{\mu_{r2}\epsilon_{r2}}{2} \left[\sqrt{1 + \left(\frac{\sigma_2}{\omega\epsilon_2} \right)^2} - 1 \right]} = \frac{9 \times 10^8}{3 \times 10^8} \sqrt{\frac{80}{2} \left[\sqrt{1 + 4\pi^2} - 1 \right]} = 43.94 \text{ Np/m}$$

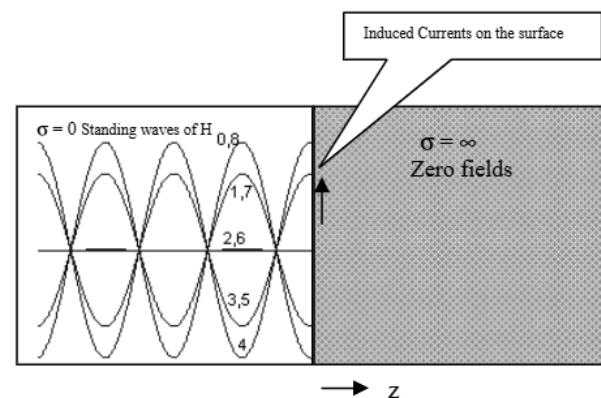
$$\beta_2 = \frac{9 \times 10^8}{3 \times 10^8} \sqrt{\frac{80}{2} \left[\sqrt{1 + 4\pi^2} + 1 \right]} = 51.48 \text{ rad/m}$$

$$\tau = \frac{2\eta_2}{\eta_2 + \eta_1} = \frac{2 \times 16.71 \angle 40.47^\circ}{16.71 \angle 40.47^\circ + 377} = 0.0857 \angle 38.89^\circ$$

$$E_{ot} = \tau E_o = 0.857 \angle 38.89^\circ$$

$$E_t = 0.857 e^{43.94z} \sin(9 \times 10^8 t + 51.48z + 38.89^\circ) \mathbf{a}_x \text{ V/m}$$

Prob. 10.62



Curve 0 is at $t = 0$; curve 1 is at $t = T/8$; curve 2 is at $t = T/4$; curve 3 is at $t = 3T/8$, etc.

Prob. 10.63 Since $\mu_o = \mu_1 = \mu_2$,

$$\sin \theta_{t1} = \sin \theta_i \sqrt{\frac{\epsilon_o}{\epsilon_1}} = \frac{\sin 45^\circ}{\sqrt{4.5}} = 0.3333 \quad \longrightarrow \quad \underline{\underline{\theta_{t1} = 19.47^\circ}}$$

$$\sin \theta_{t2} = \sin \theta_{t1} \sqrt{\frac{\epsilon_1}{\epsilon_2}} = \frac{1}{3} \sqrt{\frac{4.5}{2.25}} = 0.4714 \quad \longrightarrow \quad \underline{\underline{\theta_{t2} = 28.13^\circ}}$$

Prob. 10.64

$$\begin{aligned}\mathbf{E}_s &= \frac{20(e^{jk_x x} - e^{-jk_x x})}{j2} \frac{(e^{jk_y y} + e^{-jk_y y})}{2} \mathbf{a}_z \\ &= -j5 \left[e^{j(k_x x + k_y y)} + e^{j(k_x x - k_y y)} - e^{-j(k_x x - k_y y)} - e^{-j(k_x x + k_y y)} \right] \mathbf{a}_z\end{aligned}$$

which consists of four plane waves.

$$\begin{aligned}\nabla \times \mathbf{E}_s &= -j\omega\mu_o \mathbf{H}_s \quad \longrightarrow \quad \mathbf{H}_s = \frac{j}{\omega\mu_o} \nabla \times \mathbf{E}_s = \frac{j}{\omega\mu_o} \left(\frac{\partial E_z}{\partial y} \mathbf{a}_x - \frac{\partial E_z}{\partial x} \mathbf{a}_y \right) \\ \mathbf{H}_s &= -\frac{j20}{\omega\mu_o} \left[k_y \sin(k_x x) \sin(k_y y) \mathbf{a}_x + k_x \cos(k_x x) \cos(k_y y) \mathbf{a}_y \right]\end{aligned}$$

Prob. 10.65

$$\eta_1 = \eta_o = 377 \Omega$$

For η_2 ,

$$\frac{\sigma_2}{\omega\epsilon_2} = \frac{4}{2\pi \times 1.2 \times 10^9 \times 50 \times \frac{10^{-9}}{36\pi}} = 1.2$$

$$\tan 2\theta_{\eta_2} = \frac{\sigma_2}{\omega\epsilon_2} = 1.2 \quad \longrightarrow \quad \theta_{\eta_2} = 25.1^\circ$$

$$|\eta_2| = \sqrt{\frac{\mu/\epsilon}{1 + \left(\frac{\sigma_2}{\omega\epsilon_2}\right)^2}} = \frac{120\pi\sqrt{1/50}}{\sqrt[4]{1+1.2^2}} = 42.658$$

$$\eta_2 = 42.658 \angle 25.1^\circ$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{42.658 \angle 25.1^\circ - 377}{42.658 \angle 25.1^\circ + 377} = \underline{\underline{0.8146 \angle 174.4^\circ}}$$

Prob. 10.66

(a)

$$P_t = (1 - |\Gamma|^2) P_i$$

$$s = \frac{1 + |\Gamma|}{1 - |\Gamma|} \quad \longrightarrow \quad |\Gamma| = \frac{s-1}{s+1}$$

$$\frac{P_t}{P_i} = 1 - \left(\frac{s-1}{s+1} \right)^2 = \underline{\underline{\frac{4s}{(s+1)^2}}}$$

$$(b) \quad P_i = P_r + P_t \quad \longrightarrow \quad \frac{P_r}{P_i} = 1 - \frac{P_t}{P_i} = \underline{\underline{\left(\frac{s-1}{s+1}\right)^2}}$$

Prob. 10.67

If \mathbf{A} is a uniform vector and $\Phi(r)$ is a scalar,

$$\nabla \times (\Phi \mathbf{A}) = \nabla \Phi \times \mathbf{A} + \Phi (\nabla \times \mathbf{A}) = \nabla \Phi \times \mathbf{A}$$

since $\nabla \times \mathbf{A} = \mathbf{0}$.

$$\begin{aligned} \nabla \times \mathbf{E} &= \left(\frac{\partial}{\partial x} \mathbf{a}_x + \frac{\partial}{\partial y} \mathbf{a}_y + \frac{\partial}{\partial z} \mathbf{a}_z \right) \times \mathbf{E}_o e^{j(k_x x + k_y y + k_z z - \omega t)} = j(k_x \mathbf{a}_x + k_y \mathbf{a}_y + k_z \mathbf{a}_z) e^{j(k \cdot r - \omega t)} \times \mathbf{E}_o \\ &= j \mathbf{k} \times \mathbf{E}_o e^{j(k \cdot r - \omega t)} = j \mathbf{k} \times \mathbf{E} \end{aligned}$$

$$\text{Also, } -\frac{\partial \mathbf{B}}{\partial t} = j \omega \mu \mathbf{H}. \quad \text{Hence } \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \text{ becomes } \mathbf{k} \times \mathbf{E} = \omega \mu \mathbf{H}$$

From this, $\mathbf{a}_k \times \mathbf{a}_E = \mathbf{a}_H$

Prob. 10.68

$$k = |\mathbf{k}| = \sqrt{124^2 + 124^2 + 263^2} = 316.1$$

$$\lambda = \frac{2\pi}{k} = \underline{\underline{19.88 \text{ mm}}}$$

$$k = \omega \sqrt{\mu \epsilon} = \frac{2\pi f}{c} \quad \longrightarrow \quad f = \frac{kc}{2\pi} = \frac{316.1 \times 3 \times 10^8}{2\pi} = \underline{\underline{\underline{15.093 \text{ GHz}}}}$$

$$\mathbf{k} \bullet \mathbf{a}_x = k \cos \theta_x \quad \longrightarrow \quad \cos \theta_x = \frac{124}{316.1} \quad \longrightarrow \quad \theta_x = 66.9^\circ = \theta_y$$

$$\theta_z = \cos^{-1} \frac{263}{361.1} = 33.69^\circ$$

Thus,

$$\underline{\underline{\underline{\theta_x = \theta_y = 66.9^\circ, \theta_z = 33.69^\circ}}}$$

Prob. 10.69

$$\mathbf{k} = -3.4\mathbf{a}_x + 4.2\mathbf{a}_y$$

$$\mathbf{k} \cdot \mathbf{E} = 0 \quad \longrightarrow \quad 0 = -3.4E_o + 4.2$$

$$E_o = \frac{4.2}{3.4} = \underline{\underline{1.235}}$$

$$k = |\mathbf{k}| = \beta = \sqrt{(-3.4)^2 + (4.2)^2} = 5.403$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{5.403} = 1.162$$

$$f = \frac{c}{\lambda} = \frac{3 \times 10^8}{1.162} = \underline{\underline{258 \text{ MHz}}}$$

$$\mathbf{H}_s = \frac{1}{\mu\omega} \mathbf{k} \times \mathbf{E}_s = \frac{1}{\mu k c} \mathbf{k} \times \mathbf{E}_s$$

$$= \frac{1}{4\pi \times 10^{-7} \times 5.403 \times 3 \times 10^8} \begin{vmatrix} -3.4 & 4.2 & 0 \\ E_o & 1 & 3+j4 \end{vmatrix} A_o$$

$$\text{where } A_o = e^{-j3.4x+4.2y}$$

$$\begin{aligned} \mathbf{H}_s &= 4.91A_o \times 10^{-4} [4.2(3+j4)\mathbf{a}_x + 3.4(3+j4)\mathbf{a}_y + (-3.4-4.2E_o)\mathbf{a}_z] \\ &= 0.491 [(12.6+j16.8)\mathbf{a}_x + (10.2+j13.6)\mathbf{a}_y - 8.59\mathbf{a}_z] e^{-j3.4x+4.2y} \text{ mA/m} \end{aligned}$$

Prob.10.70

$$\nabla \cdot \mathbf{E} = (\frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z) \cdot \mathbf{E}_o e^{j(k_x x + k_y y + k_z z - \omega t)} = j(k_x \mathbf{a}_x + k_y \mathbf{a}_y + k_z \mathbf{a}_z) e^{j(k \cdot \mathbf{r} - \omega t)} \cdot \mathbf{E}_o$$

$$= j\mathbf{k} \cdot \mathbf{E}_o e^{j(k \cdot \mathbf{r} - \omega t)} = j\mathbf{k} \cdot \mathbf{E} = 0 \quad \longrightarrow \quad \mathbf{k} \cdot \mathbf{E} = 0$$

Similarly,

$$\nabla \cdot \mathbf{H} = j\mathbf{k} \cdot \mathbf{H} = 0 \quad \longrightarrow \quad \mathbf{k} \cdot \mathbf{H} = 0$$

It has been shown in the previous problem that

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \longrightarrow \quad \mathbf{k} \times \mathbf{E} = \omega \mu \mathbf{H}$$

Similarly,

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} \quad \longrightarrow \quad k_x H = -\epsilon \omega E$$

From $\mathbf{k} \times \mathbf{E} = \omega \mu \mathbf{H}$, $\mathbf{a}_k \times \mathbf{a}_E = \mathbf{a}_H$ and

From $\mathbf{k} \times \mathbf{H} = -\epsilon \omega \mathbf{E}$, $\mathbf{a}_k \times \mathbf{a}_H = -\mathbf{a}_E$

Prob. 10.71

$$\text{If } \mu_o = \mu_1 = \mu_2, \eta_1 = \frac{\eta_o}{\sqrt{\epsilon_{r1}}}, \eta_2 = \frac{\eta_o}{\sqrt{\epsilon_{r2}}}$$

$$\Gamma_{\text{W}} = \frac{\frac{1}{\sqrt{\epsilon_{r2}}} \cos \theta_t - \frac{1}{\sqrt{\epsilon_{r1}}} \cos \theta_i}{\frac{1}{\sqrt{\epsilon_{r2}}} \cos \theta_t + \frac{1}{\sqrt{\epsilon_{r1}}} \cos \theta_i}$$

$$\sqrt{\epsilon_{r1}} \sin \theta_i = \sqrt{\epsilon_{r2}} \sin \theta_t \quad \longrightarrow \quad \frac{\sqrt{\epsilon_{r2}}}{\sqrt{\epsilon_{r1}}} = \frac{\sin \theta_i}{\sin \theta_t}$$

$$\begin{aligned} \Gamma_{\text{W}} &= \frac{\cos \theta_t - \frac{\sin \theta_i}{\sin \theta_t} \cos \theta_i}{\cos \theta_t + \frac{\sin \theta_i}{\sin \theta_t} \cos \theta_i} = \frac{\sin \theta_t \cos \theta_t - \sin \theta_i \cos \theta_i}{\sin \theta_t \cos \theta_t + \sin \theta_i \cos \theta_i} \\ &= \frac{\sin 2\theta_t - \sin 2\theta_i}{\sin 2\theta_t + \sin 2\theta_i} = \frac{\sin(\theta_t - \theta_i) \cos(\theta_t + \theta_i)}{\cos(\theta_t - \theta_i) \sin(\theta_t + \theta_i)} = \frac{\tan(\theta_t - \theta_i)}{\tan(\theta_t + \theta_i)} \end{aligned}$$

Similarly,

$$\begin{aligned} \tau_{\text{W}} &= \frac{\frac{2}{\sqrt{\epsilon_{r2}}} \cos \theta_i}{\frac{1}{\sqrt{\epsilon_{r2}}} \cos \theta_t + \frac{1}{\sqrt{\epsilon_{r1}}} \cos \theta_i} = \frac{2 \cos \theta_i}{\cos \theta_t + \frac{\sin \theta_i}{\sin \theta_t} \cos \theta_i} \\ &= \frac{2 \cos \theta_i \sin \theta_t}{\sin \theta_t \cos \theta_t (\sin^2 \theta_i + \cos^2 \theta_i) + \sin \theta_i \cos \theta_i (\sin^2 \theta_t + \cos^2 \theta_t)} \\ &= \frac{2 \cos \theta_i \sin \theta_t}{(\sin \theta_i \cos \theta_t + \sin \theta_t \cos \theta_i)(\cos \theta_i \cos \theta_t + \sin \theta_i \sin \theta_t)} \\ &= \frac{2 \cos \theta_i \sin \theta_t}{\sin(\theta_i + \theta_t) \cos(\theta_i - \theta_t)} \end{aligned}$$

$$\Gamma_{\perp} = \frac{\frac{1}{\sqrt{\epsilon_{r2}}} \cos \theta_i - \frac{1}{\sqrt{\epsilon_{r1}}} \cos \theta_t}{\frac{1}{\sqrt{\epsilon_{r2}}} \cos \theta_i + \frac{1}{\sqrt{\epsilon_{r1}}} \cos \theta_t} = \frac{\cos \theta_i - \frac{\sin \theta_i}{\sin \theta_t} \cos \theta_t}{\cos \theta_i + \frac{\sin \theta_i}{\sin \theta_t} \cos \theta_t} = \frac{\sin(\theta_t - \theta_i)}{\sin(\theta_t + \theta_i)}$$

$$\tau_{\perp} = \frac{\frac{2}{\sqrt{\epsilon_{r2}}} \cos \theta_i}{\frac{1}{\sqrt{\epsilon_{r2}}} \cos \theta_i + \frac{1}{\sqrt{\epsilon_{r1}}} \cos \theta_t} = \frac{2 \cos \theta_i}{\cos \theta_i + \frac{\sin \theta_i}{\sin \theta_t} \cos \theta_t} = \frac{2 \cos \theta_i \sin \theta_i}{\sin(\theta_t + \theta_i)}$$

Prob. 10.72

$$(a) n_1 = 1, \quad n_2 = c \sqrt{\mu_2 \epsilon_2} = c \sqrt{6.4 \epsilon_o \times \mu_o} = \sqrt{6.4} = 2.5298$$

$$\sin \theta_t = \frac{n_1}{n_2} \sin \theta_i = \frac{1}{2.5298} \sin 12^\circ = 0.082185 \quad \longrightarrow \quad \theta_t = 4.714^\circ$$

$$\eta_1 = 120\pi, \quad \eta_2 = 120\pi \sqrt{\frac{1}{6.4}} = 47.43\pi$$

$$\begin{aligned} \frac{E_{ro}}{E_{io}} &= \Gamma = \frac{\eta_2 \cos \theta_t - \eta_1 \cos \theta_i}{\eta_2 \cos \theta_t + \eta_1 \cos \theta_i} = \frac{47.43\pi \cos 4.714^\circ - 120\pi \cos 12^\circ}{47.43\pi \cos 4.714^\circ + 120\pi \cos 12^\circ} \\ &= \frac{47.27 - 117.38}{47.27 + 117.38} = \underline{\underline{-0.4258}} \end{aligned}$$

$$\frac{E_{io}}{E_{ro}} = \tau = \frac{2\eta_2 \cos \theta_i}{\eta_2 \cos \theta_t + \eta_1 \cos \theta_i} = \frac{2 \times 47.43 \cos 12^\circ}{47.27 + 117.38} = \frac{92.787}{164.65} = \underline{\underline{0.5635}}$$

Prob. 10.73

$$(a) \quad \mathbf{k}_i = 4\mathbf{a}_y + 3\mathbf{a}_z$$

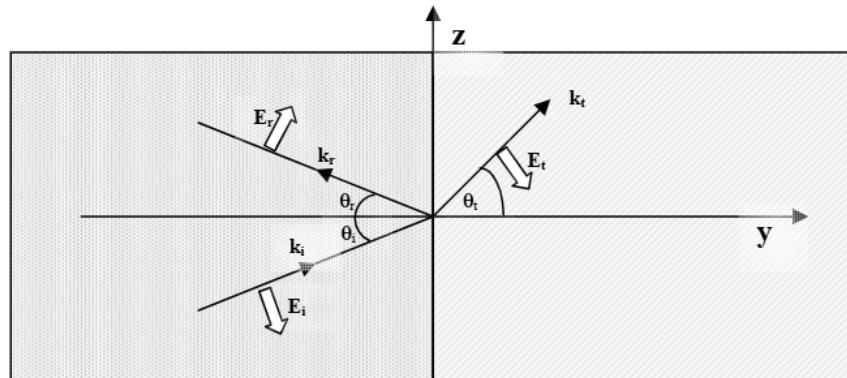
$$\mathbf{k}_i \bullet \mathbf{a}_n = k_i \cos \theta_i \quad \longrightarrow \quad \cos \theta_i = 4/5 \quad \longrightarrow \quad \underline{\underline{\theta_i = 36.87^\circ}}$$

(b)

$$\mathbf{P}_{ave} = \frac{1}{2} \operatorname{Re}(\mathbf{E}_s \times \mathbf{H}_s^*) = \frac{E_o^2}{2\eta} \mathbf{a}_k = \frac{(\sqrt{8^2 + 6^2})^2}{2 \times 120\pi} \frac{(4\mathbf{a}_y + 3\mathbf{a}_z)}{5} = \underline{\underline{106.1\mathbf{a}_y + 79.58\mathbf{a}_z \text{ mW/m}^2}}$$

(c) $\theta_r = \theta_i = 36.87^\circ$. Let

$$\mathbf{E}_r = (E_{ry}\mathbf{a}_x + E_{rz}\mathbf{a}_z)\sin(\omega t - \mathbf{k}_r \bullet \mathbf{r})$$



From the figure, $\mathbf{k}_r = k_{rz}\mathbf{a}_z - k_{ry}\mathbf{a}_y$. But $k_r = k_i = 5$

$$k_{rz} = k_r \sin \theta_r = 5(3/5) = 3, \quad k_{ry} = k_r \cos \theta_r = 5(4/5) = 4,$$

Hence, $\mathbf{k}_r = -4\mathbf{a}_y + 3\mathbf{a}_z$

$$\sin \theta_t = \frac{n_1}{n_2} \sin \theta_i = \frac{c \sqrt{\mu_1 \epsilon_1}}{c \sqrt{\mu_2 \epsilon_2}} \sin \theta_i = \frac{3/5}{\sqrt{4}} = 0.3$$

$$\theta_t = 17.46, \quad \cos \theta_t = 0.9539, \quad \eta_1 = \eta_o = 120\pi, \eta_2 = \eta_o / 2 = 60\pi$$

$$\Gamma_{//} = \frac{E_{ro}}{E_{io}} = \frac{\eta_2 \cos \theta_t - \eta_1 \cos \theta_i}{\eta_2 \cos \theta_t + \eta_1 \cos \theta_i} = \frac{\frac{\eta_o}{2}(0.9539) - \eta_o(0.8)}{\frac{\eta_o}{2}(0.9539) + \eta_o(0.8)} = -0.253$$

$$E_{ro} = \Gamma_{//} E_{io} = -0.253(10) = -2.53$$

$$\text{But } (E_{ry}\mathbf{a}_y + E_{rz}\mathbf{a}_z) = E_{ro}(\sin \theta_r \mathbf{a}_y + \cos \theta_r \mathbf{a}_z) = -2.53\left(\frac{3}{5}\mathbf{a}_y + \frac{4}{5}\mathbf{a}_z\right)$$

$$\underline{\underline{\mathbf{E}_r = -(1.518\mathbf{a}_y + 2.024\mathbf{a}_z)\sin(\omega t + 4y - 3z) \text{ V/m}}}$$

Similarly, let

$$\mathbf{E}_t = (E_{ty}\mathbf{a}_y + E_{tz}\mathbf{a}_z) \sin(\omega t - \mathbf{k}_t \bullet \mathbf{r})$$

$$k_t = \beta_2 = \omega \sqrt{\mu_2 \epsilon_2} = \omega \sqrt{4\mu_o \epsilon_o}$$

$$\text{But } k_i = \beta_1 = \omega \sqrt{\mu_o \epsilon_o}$$

$$\frac{k_t}{k_i} = 2 \quad \longrightarrow \quad k_t = 2k_i = 10$$

$$k_y = k_t \cos \theta_t = 9.539, \quad k_z = k_t \sin \theta_t = 3,$$

$$k_t = 9.539\mathbf{a}_y + 3\mathbf{a}_z$$

Note that $k_{iz} = k_{rz} = k_{tz} = 3$

$$\tau_{\text{\\}} = \frac{E_{to}}{E_{io}} = \frac{2\eta_2 \cos \theta_i}{\eta_2 \cos \theta_t + \eta_1 \cos \theta_i} = \frac{\eta_o(0.8)}{\frac{\eta_o}{2}(0.9539) + \eta_o(0.8)} = 0.6265$$

$$E_{to} = \tau_{\text{\\}} E_{io} = 6.265$$

But

$$(E_{ty}\mathbf{a}_y + E_{tz}\mathbf{a}_z) = E_{to} (\sin \theta_t \mathbf{a}_y - \cos \theta_t \mathbf{a}_z) = 6.256(0.3\mathbf{a}_y - 0.9539\mathbf{a}_z)$$

Hence,

$$\underline{\underline{\mathbf{E}_t = (1.879\mathbf{a}_y - 5.968\mathbf{a}_z) \sin(\omega t - 9.539y - 3z) \text{ V/m}}}$$

Prob. 10.74

$$(a) n = \frac{c}{u} = \sqrt{\mu_r \epsilon_r} = \sqrt{2.1 \times 1} = \underline{\underline{1.45}}$$

$$(b) n = \sqrt{\mu_r \epsilon_r} = \sqrt{1 \times 81} = \underline{\underline{9}}$$

$$(c) n = \sqrt{\epsilon_r} = \sqrt{2.7} = \underline{\underline{1.643}}$$

Prob. 10.75

(a) From air to seawater,

$$\epsilon_{r1} = 1, \quad \epsilon_{r2} = 81$$

$$\tan \theta_B = \sqrt{\frac{\epsilon_2}{\epsilon_1}} = \sqrt{\frac{81}{1}} = 9 \quad \longrightarrow \quad \underline{\underline{\theta_B = 83.66^\circ}}$$

(b) From seawater to air,

$$\epsilon_{r1} = 81, \quad \epsilon_{r2} = 1$$

$$\tan \theta_B = \sqrt{\frac{\epsilon_2}{\epsilon_1}} = \sqrt{\frac{1}{81}} = \frac{1}{9} \quad \longrightarrow \quad \underline{\underline{\theta_B = 6.34^\circ}}$$

Prob. 10.76

(a)

$$\tan \theta_i = \frac{k_{ix}}{k_{iz}} = \frac{1}{\sqrt{8}} \quad \longrightarrow \quad \underline{\underline{\theta_i = \theta_r = 19.47^\circ}}$$

$$\sin \theta_t = \sin \theta_i \sqrt{\frac{\epsilon_{r1}}{\epsilon_{r2}}} = \frac{1}{3}(3) = 1 \quad \longrightarrow \quad \underline{\underline{\theta_t = 90^\circ}}$$

$$(b) \quad \beta_1 = \frac{\omega}{c} \sqrt{\epsilon_{r1}} = \frac{10^9}{3 \times 10^8} \times 3 = 10 = k \sqrt{1+8} = 3k \quad \longrightarrow \quad \underline{\underline{k = 3.333}}$$

$$(c) \quad \lambda = 2\pi / \beta, \quad \lambda_l = 2\pi / \beta_1 = 2\pi / 10 = \underline{\underline{0.6283 \text{ m}}}$$

$$\beta_2 = \omega / c = 10 / 3, \quad \lambda_2 = 2\pi / \beta_2 = 2\pi \times 3 / 10 = \underline{\underline{1.885 \text{ m}}}$$

$$(d) \quad E_i = \eta_1 H_x \times a_k = 40\pi(0.2) \cos(\omega t - \mathbf{k} \cdot \mathbf{r}) a_y \times \frac{(a_x + \sqrt{8}a_z)}{3}$$

$$= (23.6954a_x - 8.3776a_z) \cos(10^9 t - kx - k\sqrt{8}z) \text{ V/m}$$

$$(e) \quad \tau_{//} = \frac{2 \cos \theta_i \sin \theta_t}{\sin(\theta_i + \theta_t) \cos(\theta_t - \theta_i)} = \frac{2 \cos 19.47^\circ \sin 90^\circ}{\sin 19.47^\circ \cos 19.47^\circ} = 6$$

$$\Gamma_{//} = -\frac{\cot 19.47^\circ}{\cot 19.47^\circ} = -1$$

Let $E_t = -E_{io} (\cos \theta_t a_x - \sin \theta_t a_z) \cos(10^9 t - \beta_2 x \sin \theta_t - \beta_2 z \cos \theta_t)$

where

$$\mathbf{E}_t = -E_{io}(\cos \theta_i \mathbf{a}_x - \sin \theta_i \mathbf{a}_z) \cos(10^9 t - \beta_1 x \sin \theta_i - \beta_1 z \cos \theta_i)$$

$$\sin \theta_t = 1, \quad \cos \theta_t = 0, \quad \beta_2 \sin \theta_t = 10/3$$

$$E_{to} \sin \theta_t = \tau_{\text{v}} E_{io} = 6(24\pi)(3)(1) = 1357.2$$

Hence,

$$\mathbf{E}_t = 1357 \cos(10^9 t - 3.333x) \mathbf{a}_z \text{ V/m}$$

$$\text{Since } \Gamma = -1, \quad \theta_r = \theta_i$$

$$\mathbf{E}_r = (213.3 \mathbf{a}_x + 75.4 \mathbf{a}_z) \cos(10^9 t - kx + k\sqrt{8}z) \text{ V/m}$$

$$(f) \quad \tan \theta_{B//} = \sqrt{\frac{\epsilon_2}{\epsilon_1}} = \sqrt{\frac{\epsilon_o}{9\epsilon_o}} = 1/3 \quad \longrightarrow \quad \underline{\underline{\theta_{B//} = 18.43^\circ}}$$

Prob.10.77

Microwave is used:

- (1) For surveying land with a piece of equipment called the *tellurometer*. This radar system can precisely measure the distance between two points.
- (2) For guidance. The guidance of missiles, the launching and homing guidance of space vehicles, and the control of ships are performed with the aid of microwaves.
- (3) In semiconductor devices. A large number of new microwave semiconductor devices have been developed for the purpose of microwave oscillator, amplification, mixing/detection, frequency multiplication, and switching. Without such achievement, the majority of today's microwave systems could not exist.

Prob.10.78

(a) In terms of the S-parameters, the T-parameters are given by

$$T_{11} = 1/S_{21}, \quad T_{12} = -S_{22}/S_{21}, \quad T_{21} = S_{11}/S_{21}, \quad T_{22} = S_{12} - S_{11}S_{22}/S_{21}$$

$$(b) \quad T_{11} = 1/0.4 = 2.5, \quad T_{12} = -0.2/0.4,$$

$$T_{21} = 0.2/0.4, \quad T_{22} = 0.4 - 0.2 \times 0.2/0.4 = 0.3$$

Hence,

$$T = \begin{bmatrix} 2.5 & -0.5 \\ 0.5 & 0.3 \end{bmatrix}$$

Prob. 10.79

Since $Z_L = Z_o$, $\Gamma_L = 0$.

$$\Gamma_i = S_{11} = \underline{0.33 - j0.15}$$

$$\Gamma_g = (Z_g - Z_o) / (Z_g + Z_o) = (2 - 1) / (2 + 1) = 1/3$$

$$\begin{aligned}\Gamma_o &= S_{22} + S_{12}S_{21}\Gamma_g / (1 - S_{11}\Gamma_g) \\ &= 0.44 - j0.62 + 0.56 \times 0.56 \times (1/3) / [1 - (0.11 - j0.05)] \\ &= \underline{0.5571 - j0.6266}\end{aligned}$$

Prob. 10.80 The microwave wavelengths are of the same magnitude as the circuit components. The wavelength in air at a microwave frequency of 300 GHz, for example, is 1 mm. The physical dimension of the lumped element must be in this range to avoid interference. Also, the leads connecting the lumped element probably have much more inductance and capacitance than is needed.

CHAPTER 11

P.E. 11.1 Since Z_o is real and $\alpha \neq 0$, this is a distortionless line.

$$Z_o = \sqrt{\frac{R}{G}} \quad (1)$$

$$\text{or } \frac{L}{R} = \frac{C}{G} \quad (2)$$

$$\alpha = \sqrt{RG} \quad (3)$$

$$\beta = \omega L \sqrt{\frac{G}{R}} = \frac{\omega L}{Z_o} \quad (4)$$

$$(1) \times (3) \rightarrow R = \alpha Z_o = 0.04 \times 80 = \underline{\underline{3.2 \Omega / \text{m}}},$$

$$(3) \div (1) \rightarrow G = \frac{\alpha}{Z_o} = \frac{0.04}{80} = \underline{\underline{5 \times 10^{-4} \text{ S} / \text{m}}}$$

$$L = \frac{\beta Z_o}{\omega} = \frac{1.5 \times 80}{2\pi \times 5 \times 10^8} = \underline{\underline{38.2 \text{ nH/m}}}$$

$$C = \frac{LG}{R} = \frac{12}{\pi} \times 10^{-8} \times \frac{0.04}{80} \times \frac{1}{0.04 \times 80} = \underline{\underline{5.97 \text{ pF/m}}}$$

P.E. 11.2

$$(a) Z_o = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{0.03 + j2\pi \times 0.1 \times 10^{-3}}{0 + j2\pi \times 0.02 \times 10^{-6}}}$$

$$= 70.73 - j1.688 = \underline{\underline{70.75 \angle -1.367^\circ \Omega}}$$

$$(b) \gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \sqrt{(0.03 + j0.2\pi)(j0.4 \times 10^{-4}\pi)}$$

$$= \underline{\underline{2.121 \times 10^{-4} + j8.888 \times 10^{-3} / \text{m}}}$$

$$(c) u = \frac{\omega}{\beta} = \frac{2\pi \times 10^3}{8.888 \times 10^{-3}} = \underline{\underline{7.069 \times 10^5 \text{ m/s}}}$$

P.E. 11.3

(a) $Z_o = Z_l \rightarrow Z_{in} = Z_o = \underline{\underline{30 + j60\Omega}}$

(b) $V_{in} = V_o = \frac{Z_{in}}{Z_{in} + Z_o} V_g = \frac{V_g}{2} = \underline{\underline{7.5\angle 0^\circ \text{ V}_{ms}}}$

$$I_{in} = I_o = \frac{V_g}{Z_g + Z_{in}} = \frac{V_g}{2Z_o} = \frac{15\angle 0^\circ}{2(30 + j60)}$$

$$= 0.2236\angle -63.43^\circ A$$

assuming that $Z_g = 0$.(c) Since $Z_o = Z_r$, $\Gamma = 0 \rightarrow V_o^- = 0, V_o^+ = V_o$ The load voltage is $V_L = V_s(z = l) = V_o^+ e^{-\gamma l}$

$$e^{-\gamma l} = \frac{V_o^+}{V_L} = \frac{7.5\angle 0^\circ}{5\angle -48^\circ} 1.5\angle 48^\circ$$

$$e^{\alpha l} e^{j\beta l} = 1.5\angle 48^\circ$$

$$e^{\alpha l} = 1.5 \rightarrow \alpha = \frac{1}{l} \ln(1.5) = \frac{1}{40} \ln(1.5) = 0.0101$$

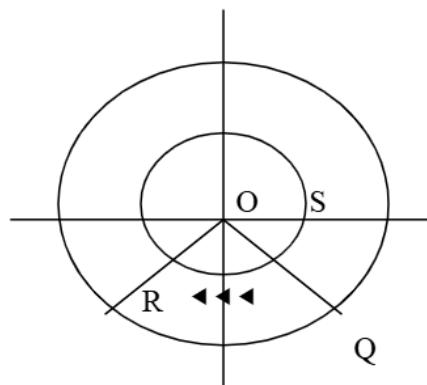
$$e^{j\beta l} = e^{j48^\circ} \rightarrow \beta = \frac{48^\circ}{l 180^\circ} \pi \text{ rad} = 0.02094$$

$$\underline{\underline{\gamma = 0.0101 + j0.02094 \text{ /m}}}$$

assuming that $Z_g = 0$.**P.E. 11.4**(a) Using the Smith chart, locate S at $s = 1.6$. Draw a circle of radius OS. Locate P where $\theta_\Gamma = 300^\circ$. At P,

$$|\Gamma| = \frac{OP}{OQ} = \frac{2.1\text{cm}}{9.2\text{cm}} = 0.228$$

$$\underline{\underline{\Gamma = 0.228\angle 300^\circ}}$$

Also at P, $z_L = 1.15 - j0.48$,

$$Z_L = Z_o z_L = 70(1.15 - j0.48) = \underline{\underline{80.5 - j33.6 \Omega}}$$

$$l = 0.6\lambda \rightarrow 0.6 \times 720^\circ = 432^\circ = \underline{\underline{360^\circ + 73^\circ}}$$

From P, move 432° to R. At R, $z_{in} = 0.68 - j0.25$

$$Z_{in} = Z_o Z_{in} = 70(0.68 - j0.25) = \underline{\underline{47.6 - j17.5 \Omega}}$$

- (b) The minimum voltage (the only one) occurs at $\theta_\Gamma = 180^\circ$; its distance from the load is $\frac{180 - 60}{720}\lambda = \frac{\lambda}{6} = \underline{\underline{0.1667\lambda}}$

Values obtained using formulas are as follows:

$$\Gamma = \frac{s-1}{s+1} \angle 300^\circ = 0.2308 \angle 300^\circ$$

$$Z_L = 80.5755 - j34.018 \Omega$$

$$Z_{in} = 48.655 - j17.63 \Omega$$

These are pretty close.

P.E. 11.5

$$(a) \Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{60 + j60 - 60}{60 + j60 + 60} = \frac{j}{2+j} = \underline{\underline{0.4472 \angle 63.43^\circ}}$$

$$s = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.4472}{1 - 0.4472} = \underline{\underline{2.618}}$$

$$\text{Let } x = \tan(\beta l) = \tan \frac{2\pi l}{\lambda}$$

$$Z_{in} = Z_o \left[\frac{Z_L + jZ_o \tan(\beta l)}{Z_o + jZ_L \tan(\beta l)} \right]$$

$$120 - j60 = 60 \left[\frac{60 + j60 + j60x}{60 + j(60 + j60)x} \right]$$

$$\text{Or } 2 - j = \frac{1 + j(1+x)}{1 - x + jx} \rightarrow 1 - x + j(2x - 2) = 0$$

$$\text{Or } x = 1 = \tan(\beta l)$$

$$\frac{\pi}{4} + n\pi = \frac{2\pi l}{\lambda}$$

i.e. $l = \frac{\lambda}{8}(1+4n), n=0,1,2,3\dots$

(b) $z_L = \frac{Z_L}{Z_o} \frac{60+j60}{60} = 1+j$

Locate the load point P on the Smith chart.

$$|\Gamma| = \frac{OP}{OQ} = \frac{4.1\text{cm}}{9.2\text{cm}} = 0.4457, \theta_\Gamma = 62^\circ$$

$$\Gamma = 0.4457 \angle 62^\circ$$

Locate the point S on the Smith chart. At S, $r = s = 2.6$

$$Z_{in} = \frac{Z_{in}}{Z_o} = \frac{120+j60}{60} = 2-j, \text{ which is located at R on the chart. The angle between}$$

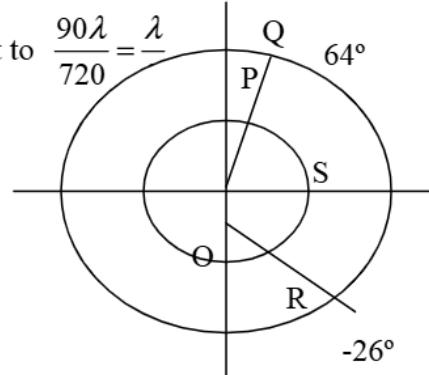
OP and OR is $64^\circ - (-25^\circ) = 90^\circ$ which is equivalent to $\frac{90\lambda}{720} = \frac{\lambda}{8}$

Hence $l = \frac{\lambda}{8} + n\frac{\lambda}{2} = \frac{\lambda}{8}(1+4n), n=0,1,2\dots$

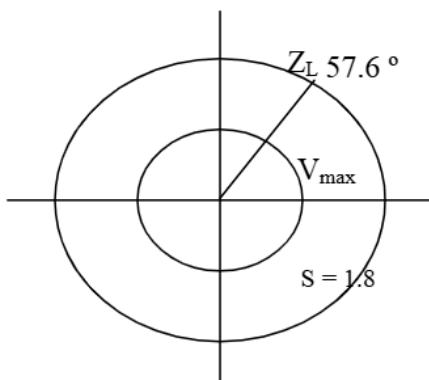
$$(Z_{in})_{max} = sZ_o = 2.618(60) = 157.08\Omega$$

$$(Z_{in})_{min} = Z_o / s = 60 / 2.618 = 22.92 \Omega$$

$$l = \frac{62^\circ}{720^\circ} \lambda = 0.0851\lambda$$



P.E. 11.6



$$\frac{\lambda}{2} = 37.5 - 25 = 12.5\text{cm} \text{ or } \lambda = 25\text{cm}$$

$$l = 37.5 - 35.5 = 2\text{cm} = \frac{2\lambda}{25}$$

$$l = 0.08\lambda \rightarrow 57.6^\circ$$

$$z_L = 1.184 + j0.622$$

$$Z_L = Z_o z_L = 50(1.184 - j0.622)$$

$$= 59.22 + j31.11\Omega$$

P.E. 11.7 See the Smith chart

$$z_L = \frac{100 - j80}{75} = 1.33 - j1.067$$

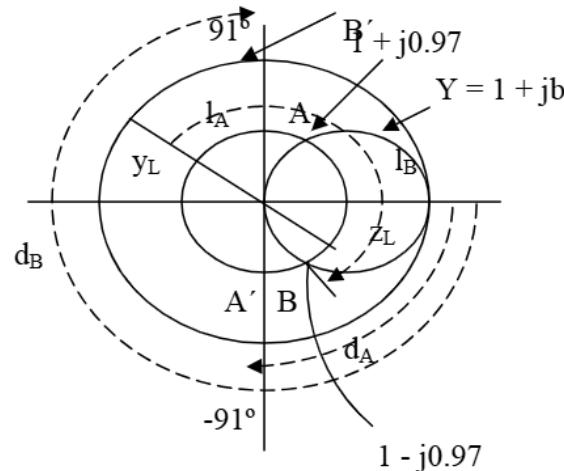
$$l_A = \frac{132^\circ - 65^\circ}{72^\circ} \lambda = \underline{\underline{0.093\lambda}}$$

$$l_B = \frac{132^\circ + 64^\circ}{720^\circ} = \underline{\underline{0.272\lambda}}$$

$$d_A = \frac{91}{720} \lambda = 0.126\lambda$$

$$d_B = 0.5\lambda - d_A = 0.374\lambda$$

$$Y_s = \pm \frac{j0.95}{75} = \underline{\underline{\pm j12.67 \text{ mS}}}$$

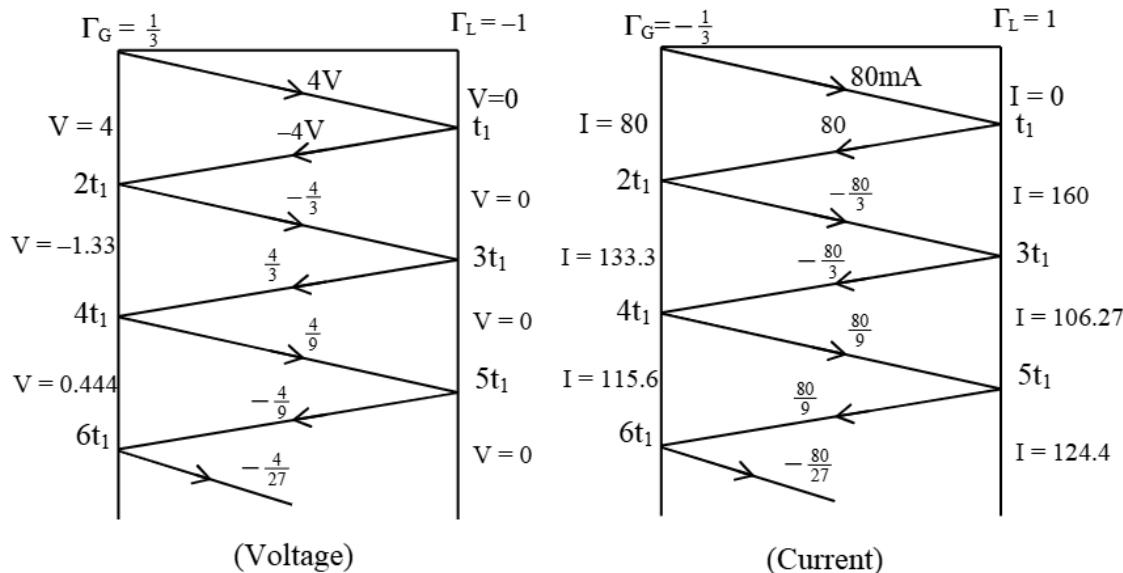


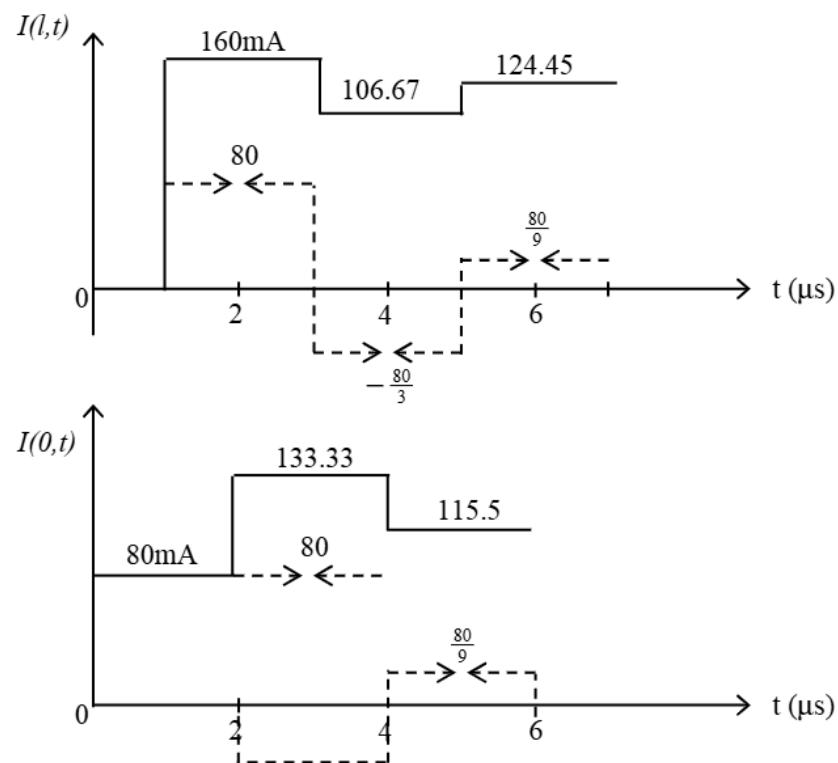
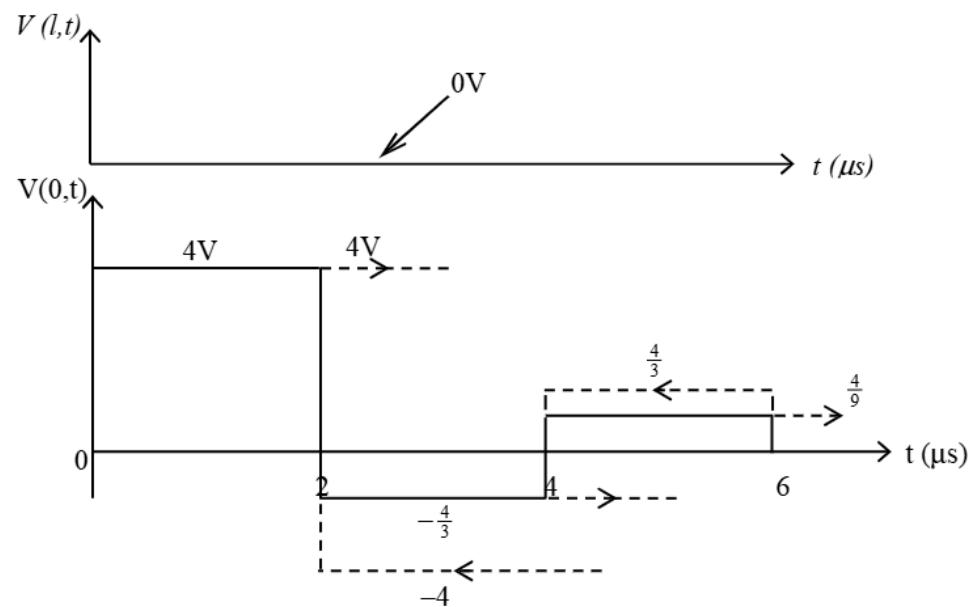
P.E. 11.8

$$(a) \Gamma_G = \frac{1}{3}, \Gamma_L = z_L \xrightarrow{\lim} \frac{Z_L - Z_o}{Z_L + Z_o} = -1$$

$$V_\infty =_{z_L} \xrightarrow{\lim} \frac{Z_L}{Z_L + Z_g} V_g = 0, \quad I_\infty =_{z_L} \xrightarrow{\lim} \frac{V_g}{Z_g + Z_L} = \frac{V_g}{Z_g} = \frac{12}{100} = 120mA$$

Thus the bounce diagrams for current and waves are as shown below.

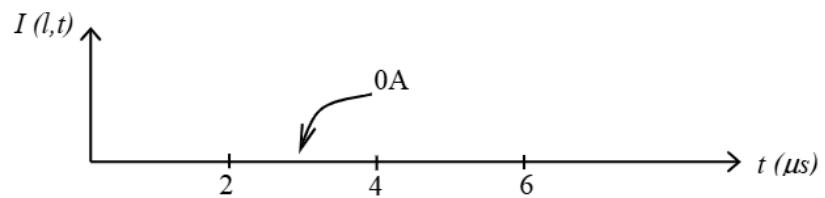
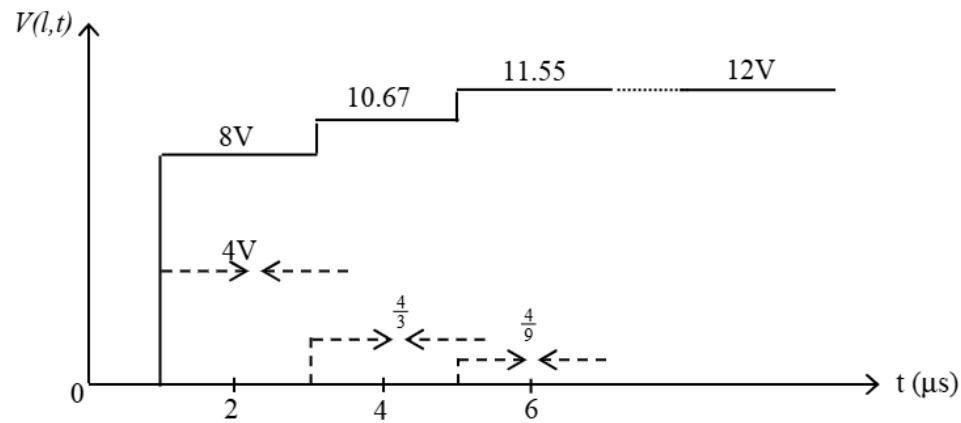
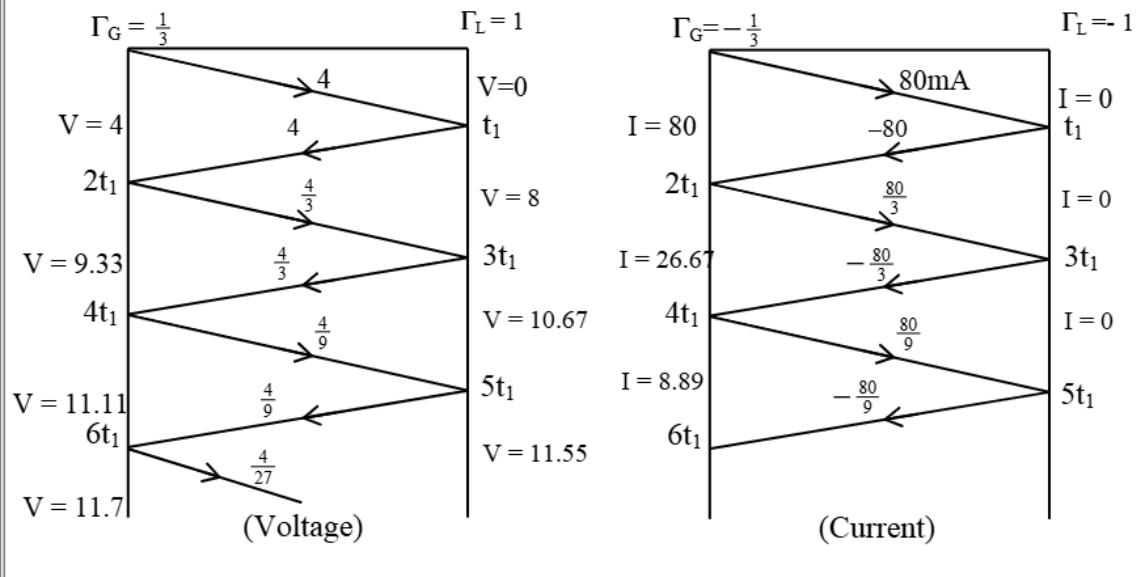


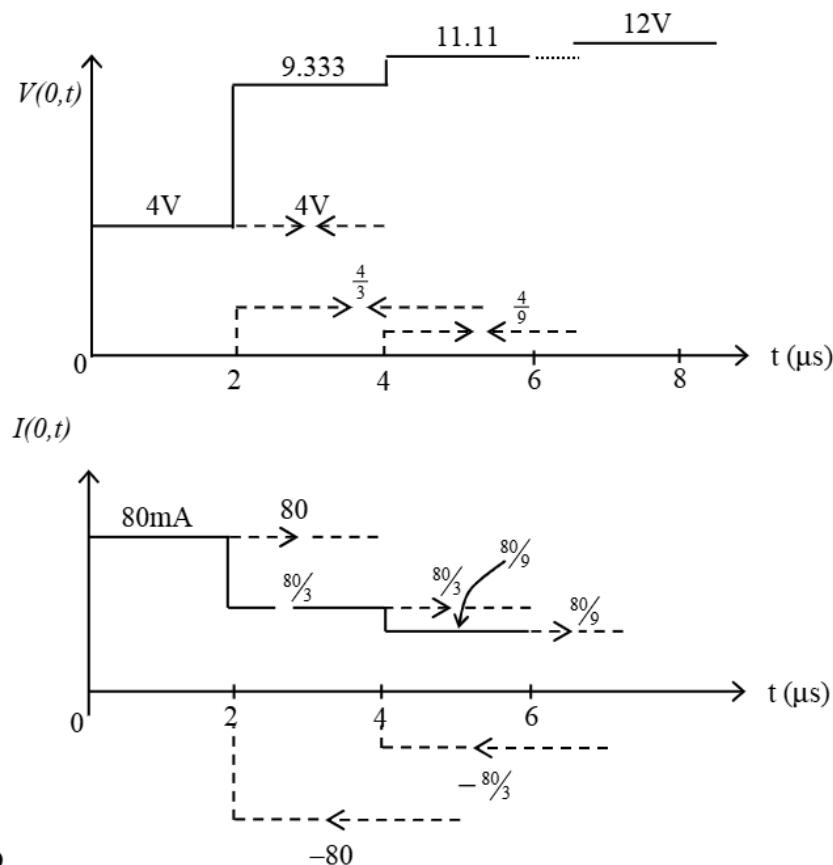


$$(b) \quad \Gamma_G = \frac{1}{3}, \Gamma_L = z_L \xrightarrow{\lim} \infty \quad \frac{Z_L - Z_o}{Z_L + Z_o} = 1$$

$$V_\infty = z_L \xrightarrow{\lim} \infty \quad \frac{Z_L}{Z_L + Z_g} V_g = V_g = 12V, \quad I_\infty = z_L \xrightarrow{\lim} \infty \quad \frac{V_g}{Z_L + Z_g} = 0$$

The bounce diagrams for current and voltage waves are as shown below.

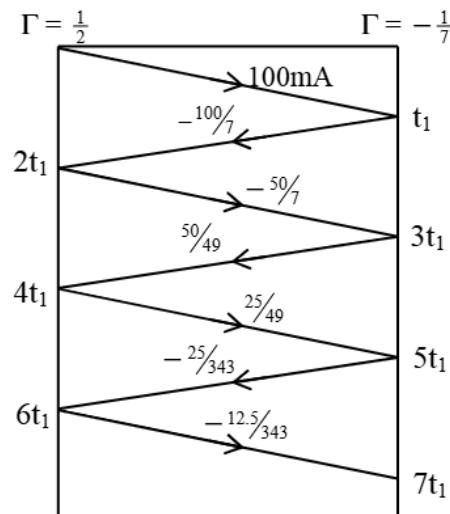


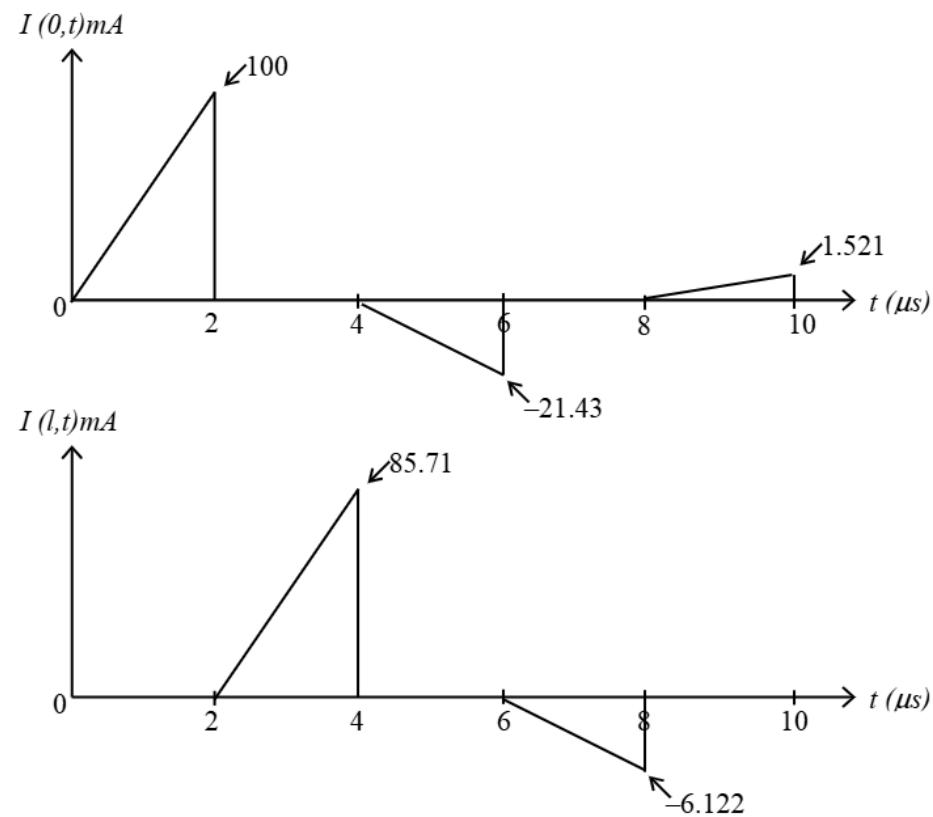
**P.E. 11.9**

$$\Gamma_g = -\frac{1}{2}, \Gamma_L = \frac{1}{7}, t_1 = 2\mu s$$

$$(I_o)_{\max} = \frac{(V_g)_{\max}}{Z_g + Z_o} = \frac{10}{100} = 100mA$$

The bounce diagrams for maximum current are as shown below.



**P.E. 11.10**

$$(a) \text{ For } w/h = 0.8, \quad \epsilon_{\text{eff}} = \frac{4.8}{2} + \frac{2.8}{2} \left[1 + \frac{12}{0.8} \right]^{-\frac{1}{2}} = \underline{\underline{2.75}}$$

$$(b) Z_o = \frac{60}{\sqrt{2.75}} \ln \left(\frac{8}{0.8} + \frac{0.8}{4} \right) = 36.18 \ln 10.2 = \underline{\underline{84.03 \Omega}}$$

$$(c) \lambda = \frac{3 \times 10^8}{10^{10} \sqrt{2.75}} = \underline{\underline{18.09 \text{ mm}}}$$

P.E. 11.11

$$R_s = \sqrt{\frac{\pi f \mu_o}{\sigma_c}} = \sqrt{\frac{\pi \times 20 \times 10^9 \times 4\pi \times 10^{-7}}{5.8 \times 10^7}} \\ = 3.69 \times 10^{-2}$$

$$\alpha_c = 8.685 \frac{R_s}{w Z_o} = \frac{8.686 \times 3.69 \times 10^{-2}}{2.5 \times 10^{-3} \times 50} \\ = \underline{\underline{2.564 \text{ dB/m}}}$$

P.E. 11.12

$$Z_L = (1 + j2)Z_o \quad \longrightarrow \quad z_L = \frac{Z_L}{Z_o} = 1 + j2$$

We locate z_L on the Smith chart.

$$\frac{\lambda}{4} \quad \longrightarrow \quad \frac{720^\circ}{4} = 180^\circ$$

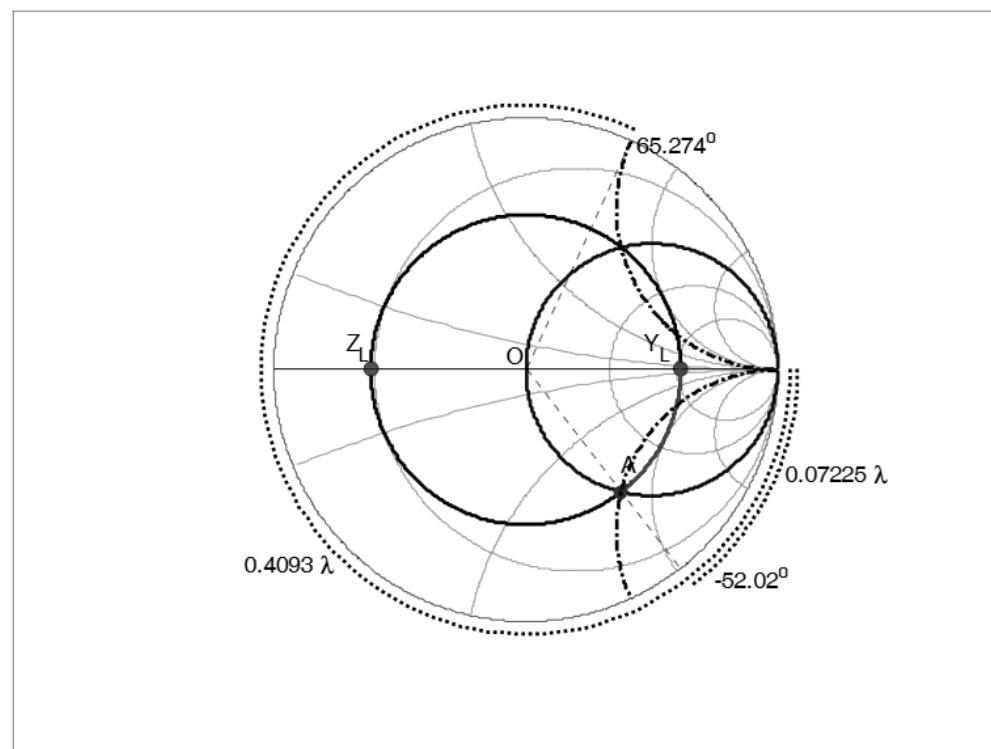
We move 180° toward the generator and locate point Q at which

$$z = 0.2 - j0.4$$

$$Z = zZ_o = \underline{\underline{(0.2 - j0.4)Z_o}}$$

P.E. 11.13

$$\frac{\lambda}{4} \rightarrow \frac{720^\circ}{4} = 180^\circ$$



$$\text{At } A, \quad y_{in} = 1 - j1.561$$

$$y_{stub} = j1.5614$$

$$\text{Position of the stub} = \underline{\underline{0.0723\lambda}}$$

$$\text{Length of the stub} = \underline{\underline{0.4093\lambda}}$$

Prob. 11.1

$$C = \frac{\pi \epsilon l}{\cosh^{-1}(d/2a)} \approx \frac{\pi \epsilon l}{\ln(d/a)}$$

since $(d/2a)^2 = 11.11 \gg 1$.

$$C = \frac{\pi \times \frac{10^{-9}}{36\pi} \times 16 \times 10^{-3}}{\ln(2/0.3)} = \underline{\underline{0.2342 \text{ pF}}}$$

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma_c}} = \frac{1}{\sqrt{\pi \times 10^7 \times 4\pi \times 10^{-7} \times 5.8 \times 10^7}} = 2.09 \times 10^{-5} \text{ m} \ll a$$

$$R_{ac} = \frac{l}{\pi a \delta \sigma_c} = \frac{16 \times 10^{-3}}{\pi \times 0.3 \times 10^{-3} \times 2.09 \times 10^{-5} \times 5.8 \times 10^7} = \underline{\underline{1.4 \times 10^{-2} \Omega}}$$

Prob. 11.2

$$L = \frac{\mu}{2\pi} \ln(b/a), \quad C = \frac{2\pi\epsilon}{\ln(b/a)}$$

$$\frac{L}{C} = \frac{\mu}{2\pi} \frac{\left(\ln \frac{b}{a}\right)^2}{2\pi\epsilon}$$

$$Z_o = \sqrt{\frac{L}{C}} = \frac{\ln(b/a)}{2\pi} \sqrt{\frac{\mu}{\epsilon}} = 50$$

$$50 = \frac{\ln(b/a)}{2\pi} \sqrt{\frac{\mu_o}{\epsilon_o \epsilon_r}}$$

$$\epsilon_r = \frac{\left(\ln \frac{b}{a}\right)^2}{4\pi^2} \frac{\eta_o^2}{50^2} = \frac{\left(\ln \frac{20}{6}\right)^2}{4\pi^2} \frac{377^2}{50^2} = \underline{\underline{2.0874}}$$

Prob. 11.3Method 1:

Assume a charge per unit length Q on the surface of the inner conductor and $-Q$ on the surface of the outer conductor. Using Gauss's law,

$$E_\rho = \frac{Q}{2\pi\epsilon\rho}, \quad a < \rho < b$$

$$V = - \int_a^b E \cdot dl = \frac{Q}{2\pi\epsilon} \ln(b/a)$$

$$J = \sigma E = \frac{\sigma Q}{2\pi\epsilon\rho}$$

$$I = \int_S \mathbf{J} \cdot d\mathbf{S} = \int_{\phi=0}^{2\pi} \frac{\sigma Q}{2\pi\epsilon\rho} (1) \rho d\phi = \frac{\sigma Q}{\epsilon}$$

$$G = \frac{I}{V} = \frac{\frac{\sigma Q}{\epsilon}}{\frac{Q}{2\pi\epsilon} \ln(b/a)} = \frac{2\pi\sigma}{\underline{\underline{\ln(b/a)}}}$$

Method 2:

Consider a section of unit length. Assume that a total current of I flows from inner conductor to outer conductor. At any radius ρ between a and b ,

$$\mathbf{J} = \frac{I}{2\pi\rho} \mathbf{a}_\rho, \quad a < \rho < b$$

$$\mathbf{E} = \frac{\mathbf{J}}{\sigma} = \frac{I}{2\pi\sigma\rho} \mathbf{a}_\rho$$

$$V = - \int_a^b \mathbf{E} \cdot d\mathbf{l} = \frac{I}{2\pi\sigma} \ln(b/a)$$

$$G = \frac{I}{V} = \frac{2\pi\sigma}{\ln(b/a)}$$

Prob. 11.4

$$\delta = \frac{1}{\sqrt{\pi f \mu_c \sigma_c}} = \frac{1}{\sqrt{\pi \times 80 \times 10^6 \times 4\pi \times 10^{-7} \times 5.28 \times 10^7}} = 7.744 \times 10^{-6}$$

$$R = \frac{1}{2\pi\delta\sigma_c} \left[\frac{1}{a} + \frac{1}{b} \right] = \frac{\left[\frac{1}{0.8 \times 10^{-3}} + \frac{1}{2.6 \times 10^{-3}} \right]}{2\pi \times 7.744 \times 10^{-6} \times 5.28 \times 10^7} = \frac{10^3(1.25 + 0.3836)}{2569.09} = \underline{\underline{0.6359 \Omega/m}}$$

$$L = \frac{\mu}{2\pi} \ln \frac{b}{a} = \frac{4\pi \times 10^{-7}}{2\pi} \ln \frac{2.6}{0.8} = \underline{\underline{2.357 \times 10^{-7} \text{ H/m}}}$$

$$G = \frac{2\pi\sigma}{\ln \frac{b}{a}} = \frac{2\pi \times 10^{-5}}{\ln \frac{2.6}{0.8}} = \underline{\underline{5.33 \times 10^{-5} \text{ S/m}}}$$

$$C = \frac{2\pi\epsilon}{\ln \frac{b}{a}} = \frac{2\pi \times 3.5 \times \frac{10^{-9}}{36\pi}}{\ln \frac{2.6}{0.8}} = \underline{\underline{1.65 \times 10^{-10} \text{ F/m}}}$$

Prob. 11.5

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma_c}} = \frac{1}{\sqrt{\pi \times 500 \times 10^6 \times 4\pi \times 10^{-7} \times 7 \times 10^7}} =$$

$$\delta = 2.6902 \times 10^{-6}$$

$$R = \frac{2}{w\delta\sigma_c} = \frac{2}{0.3 \times 2.6902 \times 10^{-6} \times 7 \times 10^7} = \underline{\underline{0.0354 \Omega / m}}$$

$$L = \frac{\mu_o d}{w} = \frac{4\pi \times 10^{-7} \times 1.2 \times 10^{-2}}{0.3} = \underline{\underline{50.26 \text{ nH/m}}}$$

$$C = \frac{\epsilon_o w}{d} = \frac{10^{-9}}{36\pi} \times \frac{0.3}{1.2 \times 10^{-2}} = \underline{\underline{221 \text{ pF/m}}}$$

Since $\sigma = 0$ for air,

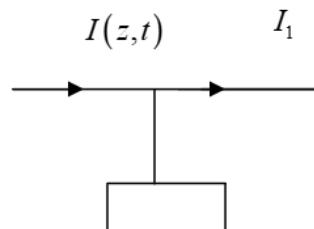
$$G = \frac{\sigma w}{d} = 0$$

Prob. 11.6

(a) Applying Kirchhoff's voltage law to the loop yields

$$V(z + \Delta z, t) = V(z, t) - R\Delta z I_1 - L\Delta z \frac{\partial I_1}{\partial t}$$

$$\text{But } I_1 = I(z, t) - \frac{C}{2} \Delta z \frac{\partial V(z, t)}{\partial t} - \frac{G}{2} \Delta z V(z, t)$$



Hence,

$$V(z + \Delta z, t) = V(z, t) - R\Delta z \left[I(z, t) - \frac{C}{2} \Delta z \frac{\partial V}{\partial t} - \frac{G}{2} \Delta z V \right] - L\Delta z \left[\frac{\partial I}{\partial t} - \frac{C}{2} \Delta z \frac{\partial^2 V}{\partial t^2} - \frac{G}{2} \Delta z \frac{\partial V}{\partial t} \right]$$

Dividing by Δz and taking limits as $\Delta t \rightarrow 0$ give

$$\lim_{\Delta z \rightarrow 0} \frac{V(z + \Delta z, t) - V(z, t)}{\Delta z} = \lim_{\Delta z \rightarrow 0} \left[-RI - L \frac{\partial I}{\partial t} + \frac{RC}{2} \Delta z \frac{\partial V}{\partial t} + \frac{RG}{2} \Delta z V + \frac{LC}{2} \Delta z \frac{\partial^2 V}{\partial t^2} + \frac{LG}{2} \Delta z \frac{\partial V}{\partial t} \right]$$

$$\text{or } -\frac{\partial V}{\partial z} = RI + L \frac{\partial I}{\partial t}$$

Similarly, applying Kirchhoff's law to the node leads to

$$I(z + \Delta z, t) - I(z, t) = -G\Delta z \left(\frac{V(z, t) + V(z + \Delta z)}{2} \right) - C\Delta z \frac{\partial}{\partial t} \left(\frac{V(z, t) + V(z + \Delta z, t)}{2} \right)$$

Let $\Delta z \rightarrow 0$, we get

$$-\frac{\partial I}{\partial z} = GV + C \frac{\partial V}{\partial t}$$

(b) Applying Kirchhoff's voltage law,

$$V(z, t) = R \frac{\Delta I}{2} I(z, t) + L \frac{\Delta I}{2} \frac{\partial I}{\partial t}(z, t) + V(z + \Delta l / 2, t)$$

or

$$-\frac{V(z + \Delta l / 2, t) - V(z, t)}{\Delta l / 2} = RI + L \frac{\partial I}{\partial t}$$

$$\text{As } \Delta l \rightarrow 0, \quad -\frac{\partial V}{\partial z} = RI + L \frac{\partial I}{\partial t}$$

Here, we take $\Delta l = \Delta z$. Applying Kirchhoff's current law,

$$I(z, t) = I(z + \Delta l, t) + G\Delta l V(z + \Delta l / 2, t) + C\Delta l \frac{\partial V(z + \Delta l / 2, t)}{\partial t}$$

or

$$-\frac{I(z + \Delta l, t) - I(z, t)}{\Delta l} = GV(z + \Delta l / 2, t) + C \frac{\partial V(z + \Delta l / 2, t)}{\partial t}$$

$$\text{As } \Delta l \rightarrow 0, \quad -\frac{\partial I(z, t)}{\partial z} = GV(z, t) + C \frac{\partial V(z, t)}{\partial t}$$

Prob. 11.7

(a)

$$\begin{aligned} \gamma &= \sqrt{(R + j\omega L)(G + j\omega C)} = j\omega \sqrt{LC} \sqrt{\left(1 + \frac{R}{j\omega L}\right)\left(1 + \frac{G}{j\omega C}\right)} \\ &= j\omega \sqrt{LC} \sqrt{1 - \frac{RG}{\omega^2 LC} + \frac{R}{j\omega L} + \frac{G}{j\omega C}} \end{aligned}$$

As $R \ll \omega L$ and $G \ll \omega C$, dropping the ω^2 term gives

$$\begin{aligned} \gamma &\approx j\omega \sqrt{LC} \sqrt{1 + \frac{R}{j\omega L} + \frac{G}{j\omega C}} \approx j\omega \sqrt{LC} \left[1 + \frac{R}{2j\omega L} + \frac{G}{2j\omega C} \right] \\ &= \underline{\left(\frac{R}{2} \sqrt{\frac{C}{L}} + \frac{G}{2} \sqrt{\frac{L}{C}} \right) + j\omega \sqrt{LC}} \end{aligned}$$

(b)

$$\begin{aligned} Z_o &= \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{L}{C}} \sqrt{\frac{1 + \frac{R}{j\omega L}}{1 + \frac{G}{j\omega C}}} = \sqrt{\frac{L}{C}} \left(1 + \frac{R}{j\omega L} \right)^{1/2} \left(1 + \frac{G}{j\omega C} \right)^{-1/2} \\ &\approx \sqrt{\frac{L}{C}} \left(1 + \frac{R}{2j\omega L} + \dots \right) \left(1 - \frac{G}{j2\omega C} + \dots \right) = \sqrt{\frac{L}{C}} \left(1 - j \frac{R}{2\omega L} + j \frac{G}{2\omega C} + \dots \right) \\ &\approx \underline{\sqrt{\frac{L}{C}} \left[1 + j \left(\frac{G}{2\omega C} - \frac{R}{2\omega L} \right) \right]} \end{aligned}$$

Prob. 11.8

For a lossless line,

$$u = \frac{1}{\sqrt{LC}}, \quad Z_o = \sqrt{\frac{L}{C}}$$

$$uZ_o = \frac{1}{C}$$

$$u = \frac{1}{Z_o C}$$

Prob. 11.9

$$Z_o = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \quad (1)$$

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)} \quad (2)$$

$$\alpha = 0.04 \text{ dB/m} = \frac{0.04}{8.686} \text{ Np/m} = 0.00461 \text{ Np/m}$$

Multiplying (1) and (2),

$$Z_o(\alpha + j\beta) = R + j\omega L \longrightarrow 50(0.00461 + j2.5) = R + j\omega L$$

$$R = 50 \times 0.00461 = \underline{\underline{0.2305 \Omega/m}}$$

$$L = \frac{50 \times 2.5}{2\pi \times 60 \times 10^6} = \underline{\underline{0.3316 \mu H/m}}$$

Dividing (2) by (1),

$$\frac{\alpha + j\beta}{Z_o} = G + j\omega C$$

$$G = \frac{\alpha}{Z_o} = \frac{0.00461}{50} = \underline{\underline{92.2 \mu S/m}}$$

$$C = \frac{\beta}{\omega Z_o} = \frac{2.5}{2\pi \times 60 \times 10^6 \times 50} = \underline{\underline{0.1326 nF/m}}$$

Prob. 11.10

$$Z_o = \sqrt{\frac{L}{c}} = \sqrt{\frac{\mu d}{w}} \cdot \frac{d}{\epsilon w} = \frac{d}{w} \sqrt{\frac{\mu}{\epsilon}}$$

$$Z_o = \eta_o \frac{d}{w} = 78$$

$$Z_o = \eta_o \frac{d}{w} = 75$$

$$\frac{78}{75} = \frac{w'}{w} \rightarrow w' = 1.04w$$

i.e. the width must be increased by 4%.

Prob. 11.11

$$(a) Z_o = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{6.8 + j2\pi \times 10^3 \times 3.4 \times 10^{-3}}{0.42 \times 10^{-6} + j2\pi \times 10^3 \times 8.4 \times 10^{-9}}} \\ = 10^3 \sqrt{\frac{6.8 + j21.36}{0.42 + j52.78}} = \underline{\underline{644.3 - j97 \Omega}}$$

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = 10^{-3} \sqrt{(6.8 + j21.36)(0.42 - j52.78)} \\ = \underline{\underline{(5.415 + j33.96) \times 10^{-3} / \text{mi}}}$$

$$(b) u = \frac{\omega}{\beta} = \frac{2\pi \times 10^3}{33.96 \times 10^{-3}} = \underline{\underline{1.85 \times 10^5 \text{ mi/s}}}$$

$$(c) \lambda = \frac{2\pi}{\beta} = \frac{2\pi}{33.96 \times 10^{-3}} = \underline{\underline{185.02 \text{ mi}}}$$

Prob. 11.12

Using eq. (11.42a),

$$Z_{in} = -jZ_o \cot \beta \ell$$

$$Z_o = 250, \quad \lambda = \frac{c}{f} = \frac{3 \times 10^8}{400 \times 10^6} = 0.75 \text{ m}$$

$$\beta \ell = \frac{2\pi}{\lambda} \ell = \frac{2\pi \times 0.1}{0.75} = 48^\circ$$

$$Z_{in} = -j(250) \cot 48^\circ = \underline{\underline{-j225.1 \Omega}}$$

Prob. 11.13

Assume that the line is lossless.

$$Z_o = \sqrt{\frac{L}{C}}$$

$$\text{From Table 11.1, } L = \frac{\mu}{2\pi} \ln \frac{b}{a}, \quad C = \frac{2\pi\epsilon}{\ln \frac{b}{a}}$$

$$\frac{L}{C} = \frac{\mu}{\epsilon} \left(\frac{1}{2\pi} \ln \frac{b}{a} \right)^2$$

$$Z_o = \sqrt{\frac{L}{C}} = \frac{1}{2\pi} \ln \frac{b}{a} \times \sqrt{\frac{\mu}{\epsilon}} = \frac{\eta_o}{2\pi\sqrt{\epsilon_r}} \ln \frac{b}{a}$$

$$\ln \frac{b}{a} = 2\pi\sqrt{\epsilon_r} \frac{Z_o}{\eta_o} = 2\pi\sqrt{2.25} \frac{75}{120\pi} = 1.875$$

$$\frac{b}{a} = e^{1.875} \quad \longrightarrow \quad a = b e^{-1.875} = 3e^{-1.875} \text{ mm} = \underline{\underline{0.46 \text{ mm}}}$$

Prob. 11.14

(a) For a lossless line, $R = 0 = G$.

$$\gamma = j\omega\sqrt{LC} \quad \longrightarrow \quad \beta = \omega\sqrt{LC} = \omega\sqrt{\mu_o c_o} = \frac{\omega}{c}$$

$$u = \frac{\omega}{\beta} = c = \frac{1}{\sqrt{LC}}$$

(b) For lossless line, $R = 0 = G$

$$L = \frac{\mu}{\pi} \cosh^{-1} \frac{d}{2a}, \quad C = \frac{\pi\epsilon}{\cosh^{-1} \frac{d}{2a}}$$

$$Z_o = \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu}{\pi} \cdot \frac{1}{\pi\epsilon}} \cosh^{-1} \frac{d}{2a} = \frac{120\pi}{\pi\sqrt{\epsilon_r}} \cosh^{-1} \frac{d}{2a}$$

$$= \frac{120}{\sqrt{\epsilon_r}} \cosh^{-1} \frac{d}{2a}$$

Yes, true for other lossless lines.

Prob. 11.15

$$L = \frac{\mu}{\pi} \cosh^{-1} \frac{d}{2a} = 4 \times 10^{-7} \cosh^{-1} \frac{0.32}{0.12}$$

$$\underline{\underline{L = 0.655 \mu\text{H/m}}}$$

$$C = \frac{\pi \epsilon}{Cosh^{-1} \frac{d}{2a}} = \frac{\pi \times \frac{10^{-9}}{36\pi} \times 3.5}{Cosh^{-1} 2.667}$$

$$\underline{\underline{C = 59.4 \text{ pF/m}}}$$

$$Z_o = \sqrt{\frac{L}{C}} = \sqrt{\frac{0.655 \times 10^{-6}}{59.4 \times 10^{-12}}} = \underline{\underline{105 \Omega}}$$

or

$$Z_o = \frac{120}{\sqrt{3.5}} \cosh^{-1} 2.667 = \underline{\underline{105 \Omega}}$$

Prob. 11.16

For a distortionless cable,

$$\frac{R}{L} = \frac{G}{C} \longrightarrow RC = LG \quad (1)$$

$$Z_o = \sqrt{\frac{L}{C}} = 60 \quad (2)$$

$$u = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} = \frac{\ell}{t_o} = \frac{4}{80 \times 10^{-6}} \quad (3)$$

$$\alpha\ell = 0.24 \text{ dB} = \frac{0.24}{8.686} Np = 0.0276$$

$$\alpha = \sqrt{RG} = 0.00069 \quad (4)$$

From (2) and (3),

$$\frac{1}{C} = \frac{60 \times 4}{80 \times 10^{-6}} \longrightarrow C = \frac{8 \times 10^{-5}}{240} = \underline{\underline{333.3 \text{ nF/m}}}$$

From (2),

$$L = (60)^2 C = 3600 \times 333.3 \times 10^{-9} = \underline{\underline{1.20 \text{ mH/m}}}$$

From (1) and (4),

$$\frac{C}{G} = \frac{LG}{0.00069^2} \longrightarrow G^2 = \frac{0.00069^2}{60^2}$$

$$G = \frac{0.00069}{60} = \underline{\underline{11.51 \mu\text{S}/\text{m}}}$$

From (4),

$$R = \frac{0.00069^2}{G} = 0.00069 \times 60 = \underline{\underline{0.0414 \Omega/\text{m}}}$$

Prob. 11.17

$$(a) \frac{R}{L} = \frac{G}{C} \rightarrow G = \frac{R}{L} C = \frac{20 \times 63 \times 10^{-12}}{0.3 \times 10^{-6}}$$

$$G = 4.2 \times 10^{-3} \text{ S/m}$$

$$\alpha = \sqrt{RG} = \sqrt{20 \times 4.2 \times 10^{-3}} = 0.2898$$

$$\beta = \omega \sqrt{LC} = 2\pi \times 120 \times 10^6 \sqrt{0.3 \times 10^{-6} \times 63 \times 10^{-12}} = 3.278$$

$$\underline{\underline{\gamma = 0.2898 + j3.278 \text{ /m}}}$$

$$u = \frac{\omega}{\beta} = \frac{2\pi \times 120 \times 10^6}{3.278} = \underline{\underline{2.3 \times 10^8 \text{ m/s}}}$$

$$Z_o = \sqrt{\frac{L}{C}} = \sqrt{\frac{0.3 \times 10^{-6}}{63 \times 10^{-12}}} = \underline{\underline{69 \Omega}}$$

(b) Let V_o be its original magnitude

$$V_o e^{-\alpha z} = 0.2 V_o \rightarrow e^{-\alpha z} = 5$$

$$z = \frac{1}{\alpha} \ln 5 = \underline{\underline{5.554 \text{ m}}}$$

$$(c) \beta l = 45^\circ = \frac{\pi}{4} \rightarrow l = \frac{\pi}{4\beta} = \frac{4}{4 \times 3.278} = \underline{\underline{0.3051 \text{ m}}}$$

Prob. 11.18

(a) $\alpha = \underline{\underline{0.0025 \text{ Np/m}}}, \beta = \underline{\underline{2 \text{ rad/m}}}$

$$u = \frac{\omega}{\beta} = \frac{10^8}{2} = \underline{\underline{5 \times 10^7 \text{ m/s}}}$$

(b) $\Gamma = \frac{V_o}{V_o^+} = \frac{60}{120} = \frac{1}{2}$

But $\Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} \rightarrow \frac{1}{2} = \frac{300 - Z_o}{300 + Z_o} \rightarrow Z_o = \underline{\underline{100 \Omega}}$

$$\begin{aligned} I(l') &= \frac{120}{Z_o} e^{0.0025l'} \cos(10^8 + 2l') - \frac{60}{Z_o} e^{-0.0025l'} \cos(10^8 t - 2l') \\ &= 1.2e^{0.0025l'} \cos(10^8 + 2l') - 0.6e^{-0.0025l'} \cos(10^8 t - 2l') A \end{aligned}$$

Prob. 11.19

$\alpha = 10^{-3}, \beta = 0.01$

$\gamma = \alpha + j\beta = 0.001 + j0.01 = \underline{\underline{(1 + j10) \times 10^{-3} / \text{m}}}$

$u = \frac{\omega}{\beta} = \frac{2\pi \times 10^4}{0.01} = \underline{\underline{6.283 \times 10^6 \text{ m/s}}}$

Prob. 11.20

$Z_o = \sqrt{\frac{L}{C}} = \sqrt{\frac{0.6 \times 10^{-6}}{82 \times 10^{-12}}} = \underline{\underline{85.54 \Omega}}$

$RC = LG \quad \longrightarrow \quad G = \frac{RC}{L}$

$\alpha = \sqrt{RG} = \sqrt{R \frac{RC}{L}} = \frac{R}{Z_o} = \frac{10 \times 10^{-3}}{85.54} = 1.169 \times 10^{-4} \text{ Np/m}$

$$\begin{aligned} \beta &= \omega \sqrt{LC} = 2\pi \times 80 \times 10^6 \sqrt{0.6 \times 10^{-6} \times 82 \times 10^{-12}} \\ &= 3.5258 \text{ rad/m} \end{aligned}$$

Prob. 11.21

$$R + j\omega L = 6.5 + j2\pi \times 2 \times 10^6 \times 3.4 \times 10^{-6} = 6.5 + j42.73$$

$$G + j\omega C = 8.4 \times 10^{-3} + j2\pi \times 2 \times 10^6 \times 21.5 \times 10^{-12} = (8.4 + j0.27) \times 10^{-3}$$

$$Z_o = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{6.5 + j42.73}{(8.4 + j0.27) \times 10^{-3}}}$$

$$Z_o = 71.71 \angle 39.75^\circ = \underline{\underline{55.12 + j45.85 \Omega}}$$

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \sqrt{(43.19 \angle 81.34^\circ)(8.4 \times 10^{-3} \angle 1.84^\circ)} \\ = 0.45 + j0.4 \text{ /m}$$

$$\begin{array}{c} \uparrow \quad \uparrow \\ \alpha \quad \beta \\ t = \frac{l}{u}, \text{ but } u = \frac{\omega}{\beta}, \end{array}$$

$$t = \frac{\beta l}{\omega} = \frac{0.39 \times 5.6}{2\pi \times 2 \times 10^6} = \underline{\underline{0.1783 \mu s}}$$

Prob. 11.22

$$Z_{in} = Z_o \left[\frac{Z_L + Z_o \tanh \gamma \ell}{Z_0 + Z_L \tanh \gamma \ell} \right]$$

$$\gamma \ell = \alpha \ell + j\beta \ell = 1.4 \times 0.5 + j2.6 \times 0.5 = 0.7 + j1.3$$

$$\tanh \gamma \ell = 1.4716 + j0.3984$$

$$Z_{in} = (75 + j60) \left[\frac{200 + (75 + j60)(1.4716 + j0.3984)}{(75 + j60) + 200(1.4716 + j0.3984)} \right] \\ = \underline{\underline{57.44 + j48.82 \Omega}}$$

Prob. 11.23

(a) For a lossy line,

$$Z_{in} = Z_o \left[\frac{Z_L + Z_o \tanh \gamma \ell}{Z_0 + Z_L \tanh \gamma \ell} \right]$$

For a short-circuit, $Z_L = 0$.

$$Z_{sc} = Z_{in} \Big|_{Z_L=0} = Z_o \tanh \gamma \ell$$

$$\tanh \gamma \ell = \frac{Z_{sc}}{Z_o} = \frac{30 - j12}{80 + j60} = 0.168 - j0.276$$

$$\gamma \ell = \alpha \ell + j\beta \ell = \tanh^{-1}(0.168 - j0.276) = 0.1571 - j0.2762$$

$$\alpha = \frac{0.1571}{2.1} = \underline{\underline{0.0748 \text{ Np/m}}}$$

$$\beta = \frac{0.2762}{2.1} = \underline{\underline{0.1316 \text{ rad/m}}}$$

(b)

$$\begin{aligned} Z_{in} &= (80 + j60) \left[\frac{(40 + j30) + (80 + j60)(0.168 - j0.276)}{(80 + j60) + (40 + j30)(0.168 - j0.276)} \right] \\ &= \underline{\underline{61.46 + j24.43 \Omega}} \end{aligned}$$

Prob. 11.24

$$\begin{aligned} \text{(a)} \quad T_L &= \frac{V_L}{V_o^+} = \frac{Z_L I_L}{\cancel{\frac{1}{2}(V_L + Z_o I_L)}} = \frac{2Z_L I_L}{Z_L I_L + Z_o I_L} \\ &= \frac{2Z_L}{Z_L + Z_o} \end{aligned}$$

$$1 + \Gamma_L = 1 + \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{2Z_L}{Z_L + Z_o}$$

$$\text{(b) (i)} \quad \tau_L = \frac{2nZ_o}{nZ_o + Z_o} = \frac{2n}{n+1}$$

$$\text{(ii)} \quad \tau_L = \underset{Y_L}{\cancel{\frac{2}{1 + \frac{Z_o}{Z_L}}}} = 2$$

$$\text{(iii)} \quad \tau_L = \underset{Z_L}{\cancel{\frac{2Z_L}{Z_L + Z_o}}} = 0$$

$$\text{(iv)} \quad \tau_L = \frac{2Z_o}{2Z_o} = 1$$

Prob. 11.25

$$\begin{aligned}\gamma &= \sqrt{(R + j\omega L)(G + j\omega C)} = \sqrt{(3.5 + j2\pi \times 400 \times 10^6 \times 2 \times 10^{-6})(0 + j2\pi \times 400 \times 10^6 \times 120 \times 10^{-12})} \\ &= \sqrt{(3.5 + j5026.55)(j0.3016)} = 0.0136 + j38.94 \\ \alpha &= \underline{\underline{0.0136 \text{ Np/m}}}, \quad \beta = \underline{\underline{38.94 \text{ rad/m}}} \\ u &= \frac{\omega}{\beta} = \frac{2\pi \times 400 \times 10^6}{38.94} = \underline{\underline{6.452 \times 10^7 \text{ m/s}}} \\ Z_o &= \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{3.5 + j5026.55}{j0.3016}} = \underline{\underline{129.1 - j0.045 \Omega}}\end{aligned}$$

Prob. 11.26

From eq. (11.33)

$$\begin{aligned}Z_{sc} &= Z_{in} \Big|_{Z_L=0} = Z_o \tanh \gamma l \\ Z_{oc} &= Z_{in} \Big|_{Z_L=\infty} = \frac{Z_o}{\tanh \gamma l} = Z_o \coth(\gamma l)\end{aligned}$$

For lossless line, $\gamma = j\beta, \tan(\gamma l) = \tanh(j\beta l) = j \tan(\beta l)$

$$Z_{sc} = jZ_o \tan(\beta l), Z_{oc} = -jZ_o \cot(\beta l)$$

Prob. 11.27

$$(a) \quad \beta l = \frac{1}{4} \times 100 = 25 \text{ rad} = 1432.4^\circ = 352.4^\circ$$

$$Z_{in} = 60 \left[\frac{j40 + j60 \tan 352.4^\circ}{60 - 40 \tan 352.4^\circ} \right] = \underline{\underline{j29.375 \Omega}}$$

$$\begin{aligned}V(z=0) &= V_o = \frac{Z_{in}}{Z_{in} + Z_g} V_g = \frac{j29.375(10 \angle 0^\circ)}{j29.375 + 50 - j40} \\ &= \frac{293.75 \angle 90^\circ}{51.116 \angle -12^\circ} = \underline{\underline{5.75 \angle 102^\circ}}\end{aligned}$$

$$(b) \quad Z_{in} = Z_L = \underline{\underline{j40 \Omega}}$$

$$V_o^+ = \frac{V_g}{(e^{j\beta l} + \Gamma e^{-j\beta l})} \quad (\Gamma \text{ is from the load})$$

$$V_L = \frac{V_g(1 + \Gamma)}{(e^{j\beta l} + \Gamma e^{-j\beta l})} = \underline{\underline{12.62 \angle 0^\circ \text{ V}}}$$

$$(c) \quad \beta l' = \frac{1}{4} \times 4 = 1 \text{ rad} = 57.3^\circ$$

$$Z_{in} = 60 \left[\frac{j40 + j60 \tan 57.3^\circ}{60 - 40 \tan 57.3^\circ} \right] = \underline{\underline{-j3471.88 \Omega}}$$

$$V = \frac{V_g(e^j + \Gamma e^{-j})}{(e^{j25} + \Gamma e^{-j25})} = \underline{\underline{22.74 \angle 0^\circ \text{ V}}}$$

(d) 3m from the source is the same as 97m from the load., i.e.

$$l' = 100 - 3 = 97 \text{ m}, \quad \beta l' = \frac{1}{4} \times 97 = 24.25 \text{ rad} = 309.42^\circ$$

$$Z_{in} = 60 \left[\frac{j40 + j60 \tan 309.42^\circ}{60 - 40 \tan 309.42^\circ} \right] = \underline{\underline{-j18.2 \Omega}}$$

$$V = \frac{V_g(e^{j97/4} + \Gamma e^{-j97/4})}{(e^{j25} + \Gamma e^{-j25})} = \underline{\underline{6.607 \angle 180^\circ \text{ V}}}$$

Prob. 11.28

$$(a) \quad \Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{120 - 50}{170} = \underline{\underline{0.4118}}$$

$$\text{For resistive load, } s = \frac{Z_L}{Z_o} = \underline{\underline{2.4}}$$

$$(b) \quad Z_{in} = Z_o \frac{Z_L + jZ_o \tan(\beta l)}{Z_o + jZ_L \tan(\beta l)}$$

$$\beta l = \frac{2\pi}{\lambda} \cdot \frac{\lambda}{6} = 60^\circ$$

$$Z_{in} = 50 \left[\frac{120 + j50 \tan(60^\circ)}{50 + j120 \tan(60^\circ)} \right] = \underline{\underline{34.63 \angle -40.65^\circ \Omega}}$$

Prob. 11.30

$$V_1 = V_s(z=0) = V_o^+ + V_o^- \quad (1)$$

$$V_2 = V_s(z=l) = V_o^+ e^{-\gamma l} + V_o^- e^{\gamma l} \quad (2)$$

$$I_1 = I_s(z=0) = \frac{V_o^+}{Z_o} - \frac{V_o^-}{Z_o} \quad (3)$$

$$I_2 = -I_s(z=l) = -\frac{V_o^+}{Z_o} e^{-\gamma l} + \frac{V_o^-}{Z_o} e^{\gamma l} \quad (4)$$

$$(1) + (3) \rightarrow V_o^+ = \frac{1}{2}(V_1 + Z_o I_1)$$

$$(1) - (3) \rightarrow V_o^- = \frac{1}{2}(V_1 - Z_o I_1)$$

Substituting V_o^+ and V_o^- in (2) gives

$$\begin{aligned} V_2 &= \frac{1}{2}(V_1 + Z_o I_1)e^{-\gamma l} + \frac{1}{2}(V_1 - Z_o I_1)e^{\gamma l} \\ &= \frac{1}{2}(e^{\gamma l} + e^{-\gamma l})V_1 + \frac{1}{2}Z_o(e^{-\gamma l} - e^{\gamma l})I_1 \end{aligned}$$

$$V_2 = \cosh \gamma l V_1 - Z_o \sinh \gamma l I_1 \quad (5)$$

Substituting V_o^+ and V_o^- in (4),

$$\begin{aligned} I_2 &= -\frac{1}{2Z_o}(V_1 + Z_o I_1)e^{-\gamma l} + \frac{1}{2Z_o}(V_1 - Z_o I_1)e^{\gamma l} \\ &= \frac{1}{2Z_o}(e^{\gamma l} - e^{-\gamma l})V_1 + \frac{1}{2}(e^{\gamma l} + e^{-\gamma l})I_1 \\ I_2 &= \frac{1}{Z_o} \sinh \gamma l V_1 - \cosh \gamma l I_1 \end{aligned} \quad (6)$$

From (5) and (6)

$$\begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} = \begin{bmatrix} \cosh \gamma l & -Z_o \sinh \gamma l \\ -\frac{1}{Z_o} \sinh \gamma l & \cosh \gamma l \end{bmatrix} \begin{bmatrix} V_1 \\ I_1 \end{bmatrix}$$

But

$$\begin{bmatrix} \cosh \gamma l & -Z_o \sinh \gamma l \\ -\frac{1}{Z_o} \sinh \gamma l & \cosh \gamma l \end{bmatrix}^{-1} = \begin{bmatrix} \cosh \gamma l & Z_o \sinh \gamma l \\ \frac{1}{Z_o} \sinh \gamma l & \cosh \gamma l \end{bmatrix}$$

Thus

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \cosh \gamma l & Z_o \sinh \gamma l \\ \frac{1}{Z_o} \sinh \gamma l & \cosh \gamma l \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

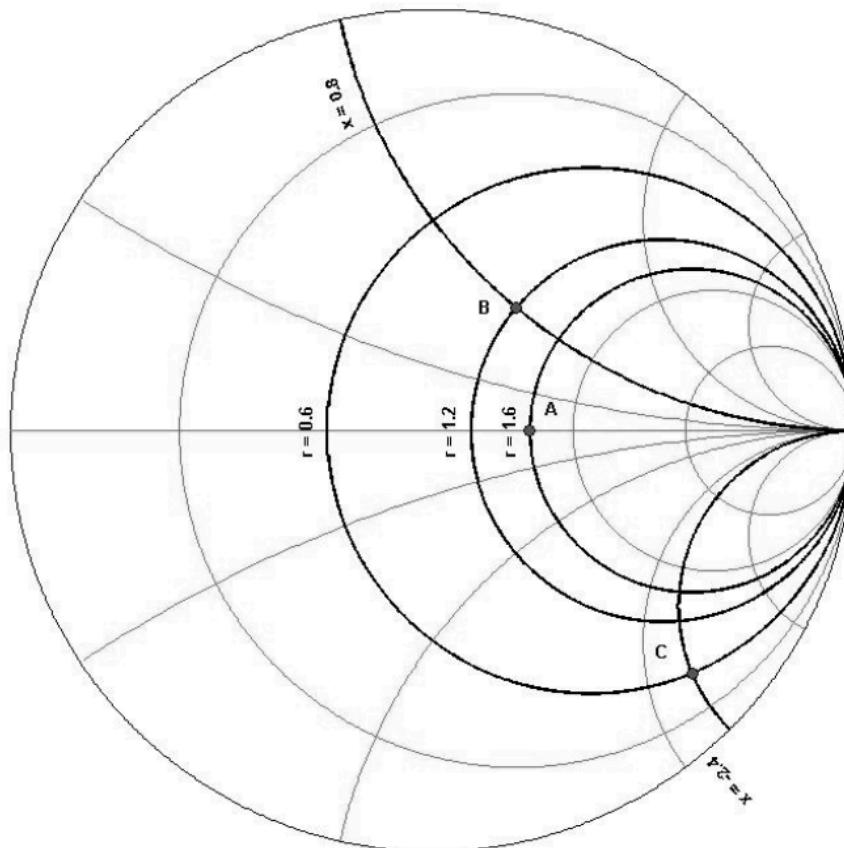
Prob. 11.31

$$(a) \quad z_a = \frac{Z_a}{Z_o} = \frac{80}{50} = 1.6$$

$$(b) \quad z_b = \frac{Z_b}{Z_o} = \frac{60 + j40}{50} = 1.2 + j0.8$$

$$(c) \quad z_c = \frac{Z_c}{Z_o} = \frac{30 - j120}{50} = 0.6 - j2.4$$

The three loads are located on the Smith chart, as A, B, and C as shown next.



Prob. 11.32

$$z_L = \frac{Z_L}{Z_o} = \frac{210}{100} = 2.1 = s$$

$$\text{Or } \Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{110}{310},$$

$$s = \frac{1 + |\Gamma|}{1 - |\Gamma|} = 2.1$$

$$\text{But } s = \frac{V_{\max}}{V_{\min}} \rightarrow V_{\max} = s V_{\min}$$

Since the line is $\frac{\lambda}{4}$ long, $\frac{\lambda}{4} \rightarrow \frac{720^\circ}{4} = 120^\circ$

Hence the sending end will be V_{\min} , while the receiving end at V_{\max}

$$V_{\min} = V_{\max} / s = 80 / 2.1 = 38.09$$

$$V_{\text{sending}} = \underline{\underline{38.09 \angle 90^\circ}}$$

Prob. 11.33

(a) Method 1: At Y,

$$Z_{in} = Z_o \left[\frac{Z_L + jZ_o \tan \beta \ell}{Z_o + jZ_L \tan \beta \ell} \right]$$

$$\beta \ell = \frac{2\pi}{\lambda} \frac{\lambda}{2} = \pi, \quad \tan \pi = 0$$

$$Z_{in} = Z_o \frac{Z_L}{Z_o} = Z_L = 150 \Omega$$

At X, $Z_L' = 150 \Omega$

$$\beta \ell = \frac{2\pi}{\lambda} \frac{\lambda}{4} = \pi/2, \quad \tan \pi/2 = \infty$$

$$Z_{in} = \lim_{\tan \beta \ell \rightarrow \infty} Z_o \left[\frac{jZ_o + \frac{Z_L'}{\tan \beta \ell}}{jZ_L' + \frac{Z_o}{\tan \beta \ell}} \right] = \frac{Z_o^2}{Z_L'} = \frac{(75)^2}{150} = 37.5 \Omega$$

Method 2: Using the Smith chart,

$$Z_L' = \frac{Z_L}{Z_o} = \frac{150}{50} = 3$$

Since $\ell = \lambda/2$, we must move 360° toward the generator. We arrive at the same point. Hence,

$$Z_{in} = Z_L = 150$$

$$Z_L' = \frac{150}{75} = 2$$

$$\ell = \frac{\lambda}{4} \rightarrow 180^\circ$$

We move 180° toward the generator. $Z_{in} = 0.5$

$$Z_{in} = 75(0.6) = \underline{\underline{37.5\Omega}}$$

(b) From the Smith chart,

$$s = 3 \text{ for section XY}$$

$$s = 2 \text{ for section YZ}$$

$$(c) \quad \Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{150 - 50}{150 + 50} = 0.5$$

Prob. 11.34

$$Z_{in} = Z_o \left[\frac{Z_L + jZ_o \tan \beta \ell}{Z_o + jZ_L \tan \beta \ell} \right]$$

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{4 \times 10^8} = 0.75$$

$$\beta \ell = \frac{2\pi}{\lambda} \ell = \frac{2\pi \times 3.2}{0.75} = 26.81$$

$$Z_{in} = 50 \left[\frac{(30 - j50) + j50 \tan(26.81)}{50 + j(30 - j50) \tan(26.81)} \right] = \underline{\underline{26.13 + j44.23 \Omega}}$$

Prob. 11.35

$$z_L = \frac{Z_L}{Z_o} = \frac{40 - j25}{50} = 0.8 - j0.5$$

We locate this at point P on the Smith chart shown below

$$|\Gamma_L| = \frac{OP}{OQ} = \frac{2.4 \text{ cm}}{8 \text{ cm}} = 0.3, \quad \theta_\Gamma = -96^\circ$$

$$\underline{\underline{\Gamma_L = 0.3 \angle -96^\circ}}$$

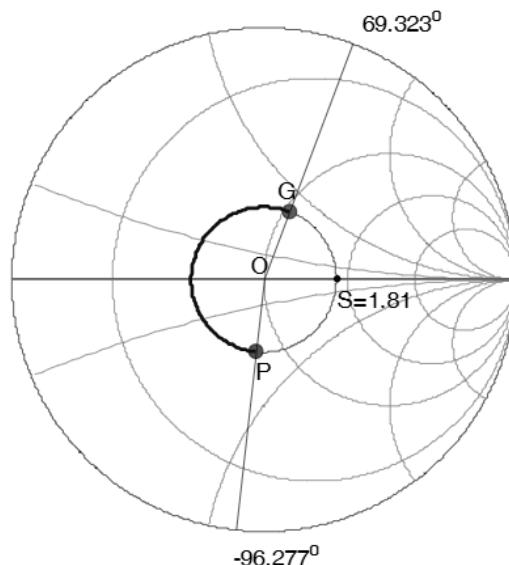
$$\text{At } S, \quad s = r = \underline{\underline{1.81}}$$

$$\ell = 0.27\lambda \quad \longrightarrow \quad 0.27 \times 720^\circ = 194.4^\circ$$

From P, we move 194.4° toward the generator to G. At G,

$$z_{in} = 1.0425 + j0.6133$$

$$Z_{in} = Z_o z_{in} = 50(1.0425 + j0.6133) = \underline{\underline{52.13 - j30.66 \Omega}}$$

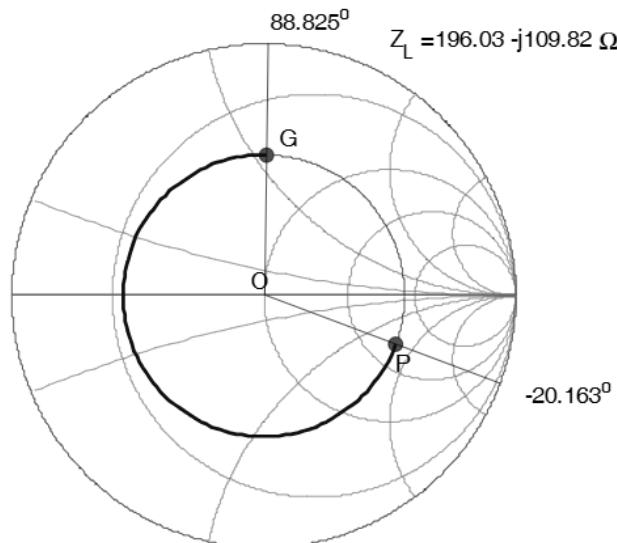


Prob. 11.36

$$u = \frac{c}{\sqrt{\epsilon_r}} = \frac{3 \times 10^8}{\sqrt{3.62}} = 1.5767 \times 10^8$$

$$\lambda = \frac{u}{f} = \frac{1.5767 \times 10^8}{400 \times 10^6} = 39.42 \text{ cm}$$

$$\ell = 132 \text{ cm} \quad \longrightarrow \quad 3.3485\lambda \quad \longrightarrow \quad \theta = 720^\circ \times 0.3485 = 250.92^\circ$$



$$z_{in} = \frac{Z_{in}}{Z_o} = \frac{40 + j65}{75} = 0.5333 + j0.8666$$

We move toward the load from G (corresponding to Z_{in}) to P (corresponding to the load). At P,

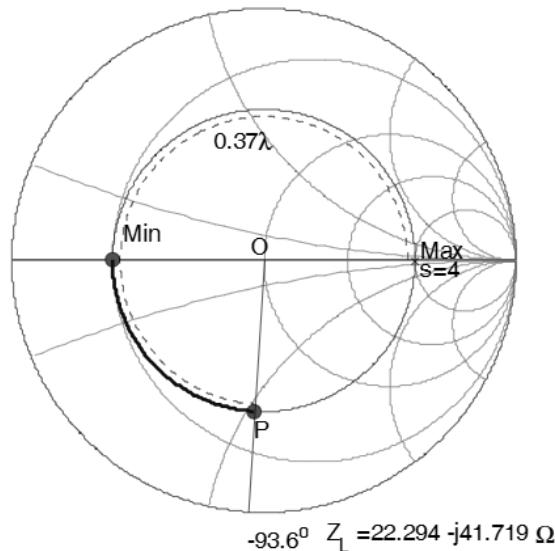
$$z_L = 2.6137 - j1.4643$$

$$Z_L = Z_o z_L = 75(2.6137 - j1.4643) = \underline{\underline{196.03 - j109.82 \Omega}}$$

Prob. 11.37

(a) $0.12\lambda \longrightarrow 0.12 \times 720^\circ = 86.4^\circ$

We draw the $s=4$ circle and locate V_{\min} . We move from that location 86.4° toward the load.



At P, $z_L = 0.45 - j0.83$

$Z = 50(0.45 - j0.83) = \underline{\underline{22.3 - j41.72 \Omega}}$

(b) the load is capacitive.

(c) V_{\min} and V_{\max} are $\lambda/4$ apart. Hence the first maximum occurs at

$0.12\lambda + 0.25\lambda = \underline{\underline{0.37\lambda}}$

Prob. 11.38

$$(a) \ z_L = \frac{Z_L}{Z_o} = \frac{75 + j60}{50} = 1.5 + j1.2$$

$$|\Gamma| = \frac{OP}{OQ} = \frac{3.8\text{cm}}{8\text{cm}} = 0.475, \quad \theta_\Gamma = 42^\circ$$

$$\Gamma = \underline{\underline{0.475 \angle 42^\circ}}$$

(Exact value = $0.4688 \angle 41.76^\circ$)

$$(b) \ s=2.8$$

(Exact value = 2.765)

$$(c) \ 0.2\lambda \rightarrow 0.2 \times 720^\circ = 144^\circ$$

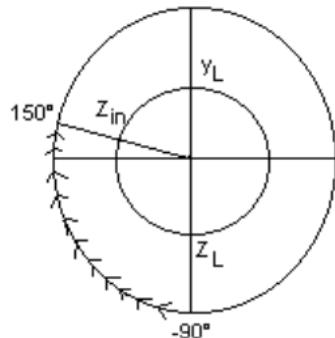
$$z_{in} = 0.55 - j0.65$$

$$Z_{in} = Z_o z_{in} = 50(0.55 + j0.65) = \underline{\underline{27.5 + j32.5 \Omega}}$$

(d) Since $\theta_\Gamma = 42^\circ$, V_{min} occurs at

$$\frac{42}{720}\lambda = \underline{\underline{0.05833\lambda}}$$

(e) same as in (d), i.e.. $\underline{\underline{0.05833\lambda}}$

Prob. 11.39

$$\text{If } \lambda \rightarrow 720^\circ, \text{ then } \frac{\lambda}{6} \rightarrow 120^\circ$$

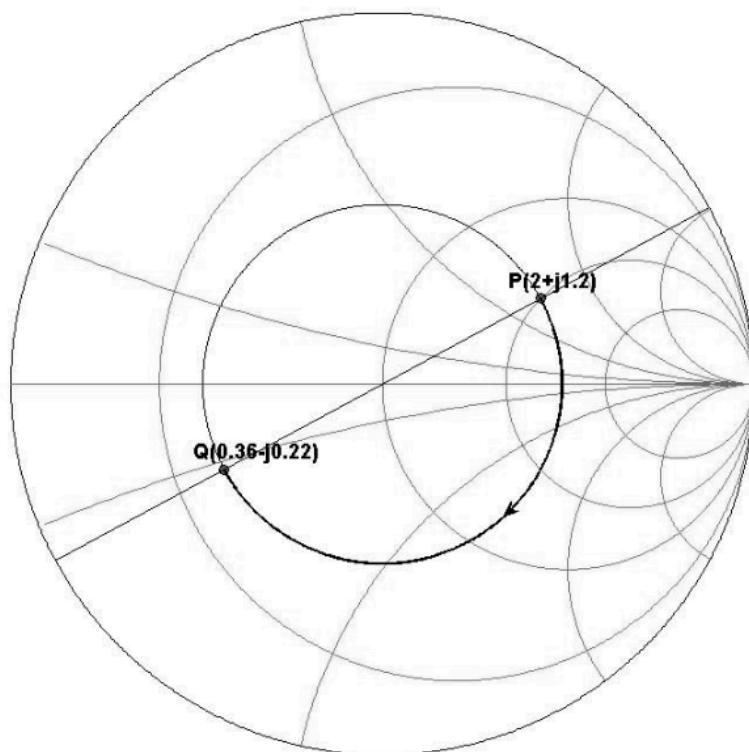
$$z_{in} = \underline{\underline{0.35 + j0.24}}$$

Prob. 11.40

$$z = \frac{Z}{Z_0} = \frac{100 + j60}{50} = 2 + j1.2$$

We locate z on the Smith chart. We move 180° toward the generator to reach point Q. At Q, $y = 0.36 - j0.22$

$$Y = yY_0 = \frac{1}{50}(0.36 - j0.22) = \underline{\underline{7.4 - j4.4 \text{ mS}}}$$



Prob. 11.41

$$\lambda = \frac{u}{f} = \frac{0.5 \times 3 \times 10^8}{160 \times 10^6} = 0.9375 \text{ m}$$

$$z_L = \frac{Z_L}{Z_0} = \frac{50 + j30}{50} = 1 + j0.6$$

We locate z at P on the Smith chart. We draw a circle that passes through P.

We locate point Q as the point where the circle crosses the Γ_r - axis. At Q,

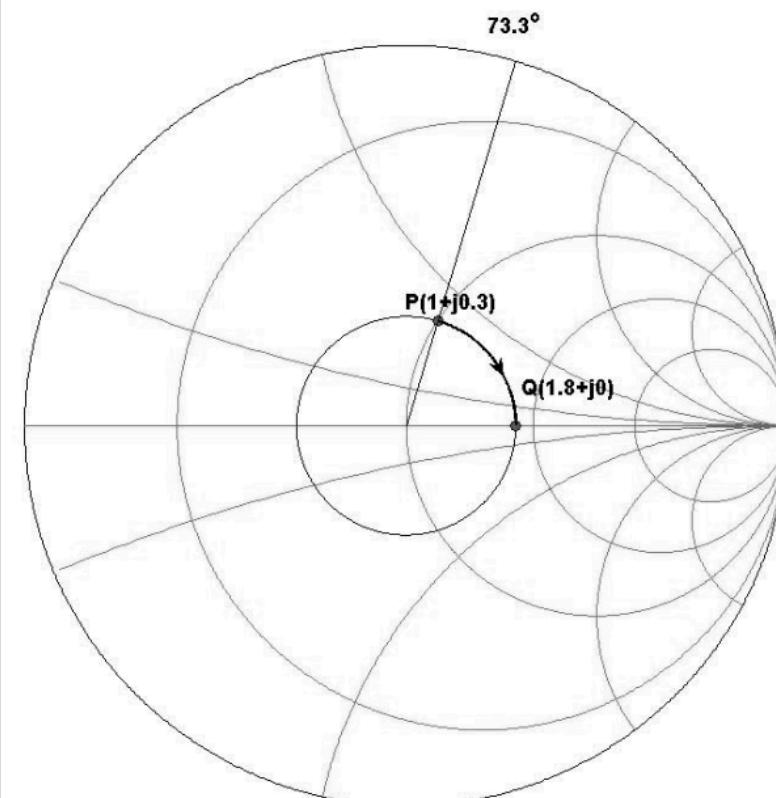
$$z_{in} = 1.8$$

$$Z_{in} = Z_0 z_{in} = 50(1.8) = \underline{\underline{90 \Omega}}$$

The angular distance between P and Q is $\theta_\Gamma = 73.3^\circ$.

$$\text{If } \lambda \rightarrow 720^\circ, \quad 73.3^\circ \rightarrow \frac{\lambda}{720^\circ} 73.3^\circ$$

$$\ell = \frac{73.3^\circ}{720^\circ} \times 0.9375 = \underline{\underline{0.0954 \text{ m}}}$$



Prob. 11.42

$$z_L = \frac{Z_L}{Z_o} = \frac{40 - j30}{50} = 0.8 - j0.6$$

Locate this load at point P on the Smith chart.

$$\frac{\lambda}{4} \rightarrow \frac{720^\circ}{4} = 180^\circ$$

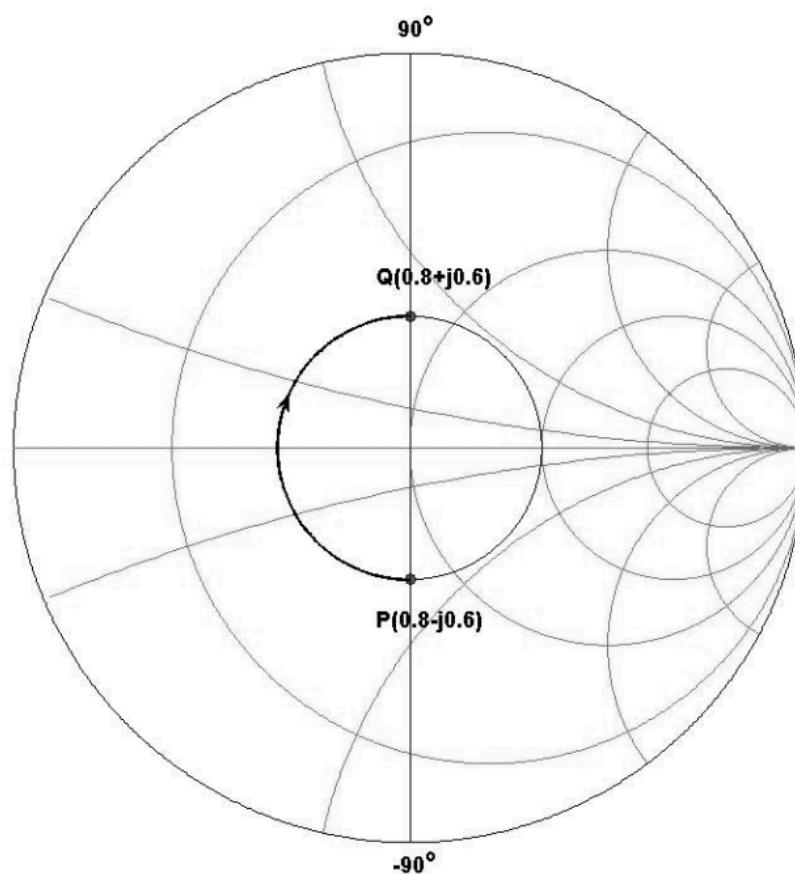
Draw a circle that passes through P and move 180° toward the generator.

At point Q,

$$z_{in} = 0.8 + j0.6$$

$$Z_{in} = Z_o z_{in} = 50(0.8 + j0.6) = 40 + j30$$

$$Y_{in} = \frac{1}{Z_{in}} = 0.016 - j0.012 = \underline{16 - j12 \text{ mS}}$$



Prob. 11.43

$$(a) \Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{0.5 - j - 1}{0.5 - j + 1} = 0.0769 - j0.6154 = \underline{\underline{0.6202 \angle -82.87^\circ}}$$

(b)

$$\Gamma = \frac{z_L - 1}{z_L + 1} \longrightarrow z_L = \frac{1 + \Gamma}{1 - \Gamma} = \frac{1 + 0.4 \angle 25^\circ}{1 - 0.4 \angle 25^\circ}$$

$$z_L = 1.931 + j0.7771$$

$$Z_L = \underline{\underline{(1.931 + j0.7771)Z_o}}$$

Prob. 11.44

$$z_L = \frac{Z_L}{Z_o} = \frac{40 + j25}{50} = 0.8 + j0.5$$

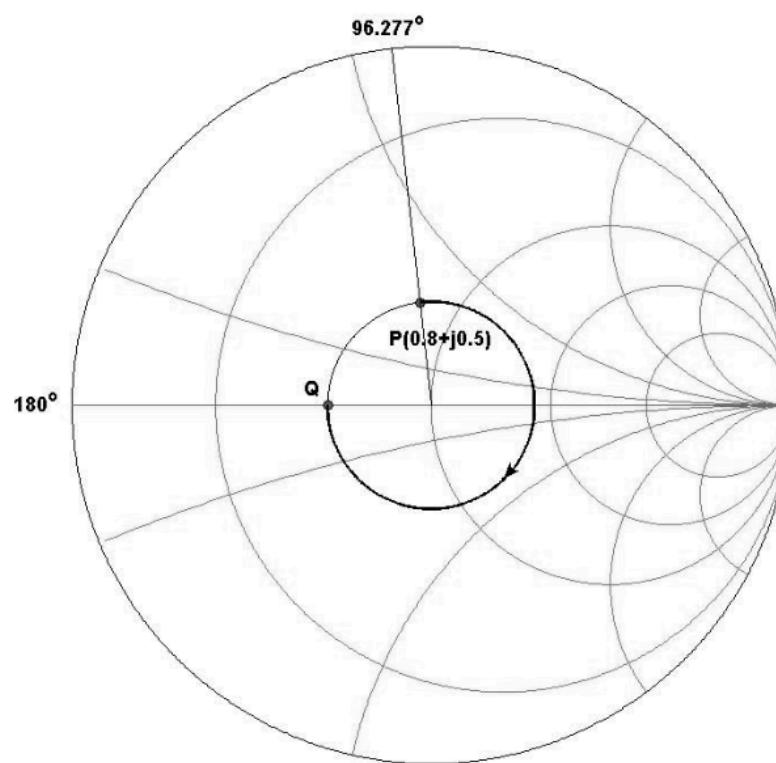
Locate this load at point P on the Smith chart. Draw a circle that passes through P. Locate point Q where the negative Γ_r – axis crosses the circle.

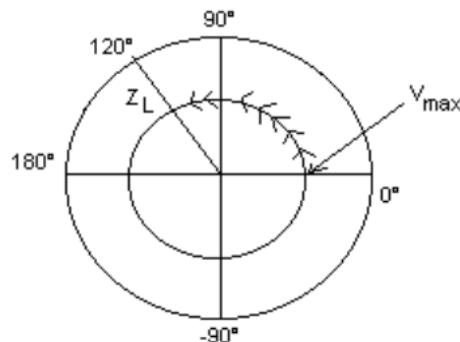
$\theta_p = 97^\circ$. The angular distance between P and Q is

$$\theta = 180 + \theta_p = 277^\circ.$$

$$720^\circ \rightarrow \lambda$$

$$277^\circ \rightarrow \ell = \frac{\lambda}{720^\circ} 277^\circ = \underline{\underline{0.3847\lambda}}$$



Prob. 11.45

$$(a) \frac{\lambda}{2} = 120\text{cm} \rightarrow \lambda = 2.4\text{m}$$

$$u = f\lambda \rightarrow f = \frac{u}{\lambda} = \frac{3 \times 10^8}{2.4} = \underline{\underline{125\text{MHz}}}$$

$$(b) 40\text{cm} = \frac{40\lambda}{240} = \frac{\lambda}{6} \rightarrow \frac{720^\circ}{6} = 120^\circ$$

$$\begin{aligned} Z_L &= Z_o z_L = 150(0.48 + j0.48) \\ &= \underline{\underline{72 + j72 \Omega}} \end{aligned}$$

(Exact value = $73.308 + j70.324 \Omega$)

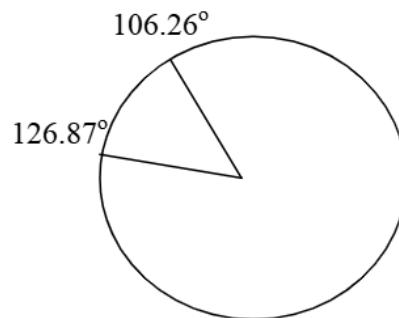
$$(c) |\Gamma| = \frac{s-1}{s+1} = \frac{1.6}{3.9} = 0.444, \\ \Gamma = \underline{\underline{0.444 \angle 120^\circ}}$$

Prob. 11.46

(a)

$$z_L = \frac{Z_L}{Z_o} = \frac{j60}{80} = j0.75, \quad z_{in} = \frac{Z_{in}}{Z_o} = \frac{j40}{80} = j0.5$$

The two loads fall on the $r=0$ circle, the outermost resistance circle. The shortest distance between them is



$$\frac{360^\circ - (126.87^\circ - 106.26^\circ)\lambda}{720^\circ} = \underline{\underline{0.4714\lambda}}$$

$$(b) \underline{\underline{s = \infty}}, \quad \underline{\underline{\Gamma_L = 1 \angle 106.26^\circ}}$$

Prob. 11.47

$$(a) \ Z_{in} = \frac{Z_{in}}{Z_o} = \frac{100 - j120}{80} = 1.25 - j1.5$$

$$\lambda = \frac{u}{f} = \frac{0.8 \times 3 \times 10^8}{12 \times 10^6} = 20 \text{m}$$

$$l_1 = 22 \text{m} = \frac{22\lambda}{20} = 1.1\lambda \rightarrow 720^\circ + 72^\circ$$

$$l_2 = 28 \text{m} = \frac{28\lambda}{20} = 1.4\lambda \rightarrow 720^\circ + 72^\circ + 216^\circ$$

To locate P(the load), we move 2 revolutions plus 72° toward the load. At P,

$$|\Gamma_L| = \frac{OP}{OQ} = \frac{5.1 \text{cm}}{9.2 \text{cm}} = 0.5543$$

$$\theta_\Gamma = 72^\circ - 47^\circ = 25^\circ$$

$$\Gamma_L = \underline{\underline{0.5543 \angle 25^\circ}}$$

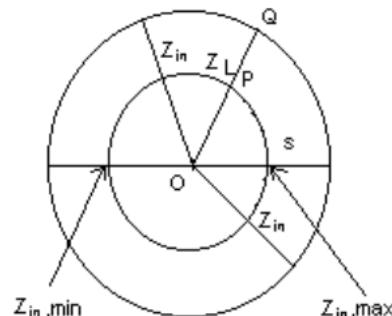
(Exact value = $0.5624 \angle 25.15^\circ$)

$$Z_{in,\max} = sZ_o = 3.7(80) = \underline{\underline{296 \Omega}}$$

(Exact value = 285.59Ω)

$$Z_{in,\min} = \frac{Z_o}{s} = \frac{80}{3.7} = \underline{\underline{21.622 \Omega}}$$

(Exact value = 22.41Ω)



(b) Also, at P, $Z_L = 2.3 + j1.55$

$$Z_L = 80(2.3 + j1.55) = \underline{\underline{184 + j124\Omega}}$$

(Exact value = $183.45 + j128.25 \Omega$)

At S, $s = \underline{\underline{3.7}}$

To Locate Z'_{in} , we move 216° from Z_{in} toward the generator.

At Z'_{in} ,

$$z'_{in} = 0.48 + j0.76$$

$$Z'_{in} = 80(0.48 + j0.76) = \underline{\underline{38.4 + j60.8\Omega}}$$

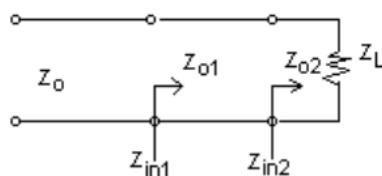
(Exact = $37.56 + j61.304 \Omega$)

(c) Between Z_L and Z_{in} , we move 2 revolutions and 72° . During the movement, we pass through $Z_{in,max}$ 3 times and $Z_{in,min}$ twice.

Thus there are:

$$\underline{\underline{3 Z_{in,max} \text{ and } 2 Z_{in,min}}}$$

Prob. 11.48



$$(a) \text{ From Eq. (11.43), } Z_{in2} = \frac{Z_{o2}^2}{Z_L}$$

$$Z_{in1} = \frac{Z_{o1}^2}{Z_{in2}} = Z_o, \text{ i.e. } Z_{in2} = \frac{Z_{o1}^2}{Z_o} = \frac{Z_{o2}^2}{Z_L}$$

$$Z_{o1} = Z_{o2} \sqrt{\frac{Z_o}{Z_L}} = 30 \sqrt{\frac{50}{75}} = \underline{\underline{24.5\Omega}}$$

$$(b) \text{ Also, } \frac{Z_o}{Z_{o1}} = \left(\frac{Z_{o2}}{Z_L} \right) \rightarrow Z_{o2} = \frac{Z_o Z_L}{Z_{o1}} \quad (1)$$

$$\text{Also, } \frac{Z_{o1}}{Z_{o2}} = \left(\frac{Z_{o2}}{Z_L} \right)^2 \rightarrow (Z_{o2})^3 = Z_{o1} Z_L^2 \quad (2)$$

$$\text{From (1) and (2), } (Z_{o2})^3 = Z_{o1}Z_L^2 = \frac{Z_o^3 Z_L^3}{Z_{o1}^3} \quad (3)$$

$$\text{or } Z_{o1} = \sqrt[4]{Z_o^3 Z_L} = \sqrt[4]{(50)^3 (75)} = \underline{\underline{55.33\Omega}}$$

$$\text{From (3), } Z_{o2} = \sqrt[3]{Z_{o1}Z_L^2} = \sqrt[3]{(55.33)(75)^2} = \underline{\underline{67.74\Omega}}$$

Prob. 11.49

$$l_1 = \frac{\lambda}{4} \rightarrow Z_{in1} = \frac{Z_o^2}{Z_L} \text{ or } y_{in1} = \frac{Z_L}{Z_o^2}$$

$$y_{in1} = \frac{200 + j150}{(100)^2} = \underline{\underline{20 + j15 \text{ mS}}}$$

$$l_2 = \frac{\lambda}{8} \rightarrow Z_{in2} = Z_L \lim_{Z_o \rightarrow \infty} Z_o \left(\frac{Z_L + jZ_o \tan \frac{\pi}{4}}{Z_o + jZ_L \tan \frac{\pi}{4}} \right) = jZ_o$$

$$y_{in2} = \frac{1}{jZ_o} = \frac{1}{j100} = \underline{\underline{-j10 \text{ mS}}}$$

$$l_3 = \frac{7\lambda}{8} \rightarrow Z_{in3} = Z_o \left(\frac{Z_i + jZ_o \tan \frac{7\pi}{4}}{Z_o + jZ_i \tan \frac{7\pi}{4}} \right) = \frac{Z_o(Z_i - jZ_o)}{(Z_o - jZ_i)}$$

But

$$y_i = y_{in1} + y_{in2} = 20 + j5 \text{ mS}$$

$$z_i = \frac{1}{y_i} = \frac{1000}{20 + j5} = 47.06 - j11.76$$

$$y_{in3} = \frac{Z_o - jZ_{in}}{Z_o(Z_{in} - jZ_o)} = \frac{100 - j47.06 - j11.76}{100(47.06 - j11.76 - j100)} \\ = \underline{\underline{6.408 + j5.189 \text{ mS}}}$$

If the shorted section were open,

$$y_{in1} = \underline{\underline{20 + j15 \text{ mS}}}$$

$$y_{in2} = \frac{1}{Z_{in2}} = \frac{j \tan \frac{\pi}{4}}{Z_o} = \frac{j}{100} = \underline{\underline{j10 \text{ mS}}}$$

$$l_3 = \frac{7\lambda}{8} \rightarrow Z_{in3} = Z_o \frac{\left(Z_i + jZ_o \tan \frac{7\pi}{4} \right)}{\left(Z_o + jZ_i \tan \frac{7\pi}{4} \right)} = \frac{Z_o(Z_i - jZ_o)}{(Z_o - jZ_i)}$$

$$y_i = y_{in1} + y_{in2} = 20 + j15 + j10 = 20 + j25 \text{ mS}$$

$$Z_i = \frac{1}{y_i} = \frac{1000}{20 + j25} = 19.51 - j24.39 \Omega$$

$$y_{in3} = \frac{Z_o - jZ_i}{Z_o(Z_i - jZ_o)} = \frac{75.61 - j19.51}{100(19.51 - j124.39)}$$

$$= \underline{\underline{2.461 + j5.691 \text{ mS}}}$$

Prob. 11.50

From the previous problem, $Z_{in} = 148 \Omega$

$$I_{in} = \frac{V_g}{Z_g + Z_{in}} = \frac{120}{80 + 148} = 0.5263 A$$

$$P_{ave} = \frac{1}{2} |I_{in}|^2 R_{in} = \frac{1}{2} (0.5263)^2 (148) = 20.5 W$$

Since the lines are lossless, the average power delivered to either antenna is 10.25 W

Prob. 11.51

$$(a) \quad \beta l = \frac{2\pi}{4} \cdot \frac{\lambda}{4} = \frac{\pi}{2}, \quad \tan \beta l = \infty$$

$$Z_{in} = Z_o \left(\frac{Z_L + jZ_o \tan \beta l}{Z_o + jZ_L \tan \beta l} \right) = Z_o \frac{\left(\frac{Z_L}{\tan \beta l} + jZ_o \right)}{\left(\frac{Z_o}{\tan \beta l} + jZ_L \right)}$$

As $\tan \beta l \rightarrow \infty$,

$$Z_{in} = \frac{Z_o^2}{Z_L} = \frac{(50)^2}{200} = \underline{\underline{12.5 \Omega}}$$

(b) If $Z_L = 0$,

$$Z_{in} = \frac{Z_o^2}{0} = \underline{\underline{\infty}} \quad (\text{open})$$

$$(c) \quad Z_L = 25 / \infty = \frac{25 \times \infty}{25 + \infty} = \frac{25}{1 + \frac{25}{\infty}} = 25\Omega$$

$$Z_{in} = \frac{(50)^2}{12.5} = \underline{\underline{200\Omega}}$$

Prob. 11.52

$$\frac{\lambda}{4} \rightarrow 180^\circ, \quad z_L = \frac{74}{50} = 1.48, \quad \frac{1}{z_L} = 0.6756$$

This acts as the load to the left line. But there are two such loads in parallel due to

the two lines on the right. Thus

$$Z_L' = 50 \frac{\left(\frac{1}{z_L}\right)}{2} = 25(0.6756) = 16.892$$

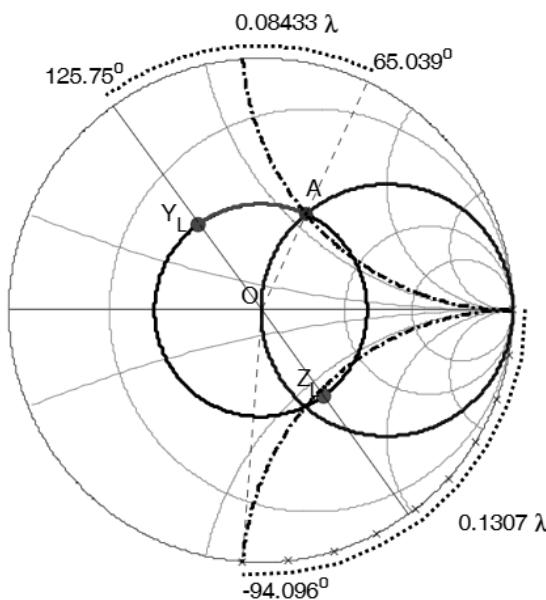
$$z_L' = \frac{16.892}{50} = 0.3378, \quad z_{in} = \frac{1}{z_L'} = 2.96$$

$$Z_{in} = 50(2.96) = \underline{\underline{148\Omega}}$$

Prob. 11.53

$$z_L = \frac{Z_L}{Z_o} = \frac{60 - j50}{50} = 1.2 - j1$$

$$y_L = \frac{1}{z_L}$$



At A, $y = 1 + j0.92$, $y_s = -j0.92$

$$Y_s = Y_o y_s = \frac{-j0.92}{50} = -j18.4 \text{ mS}$$

Stub length = 0.1307λ

Stub position = 0.0843λ

Prob. 11.54

$$d_A = 0.12\lambda \rightarrow 0.12 \times 720^\circ = 86.4^\circ$$

$$l_A = 0.3\lambda \rightarrow 0.3 \times 720^\circ = 216^\circ$$

(a) From the Smith Chart below,

$$z_L = 0.57 + j0.69$$

$$Z_L = 60(0.57 + j0.69)$$

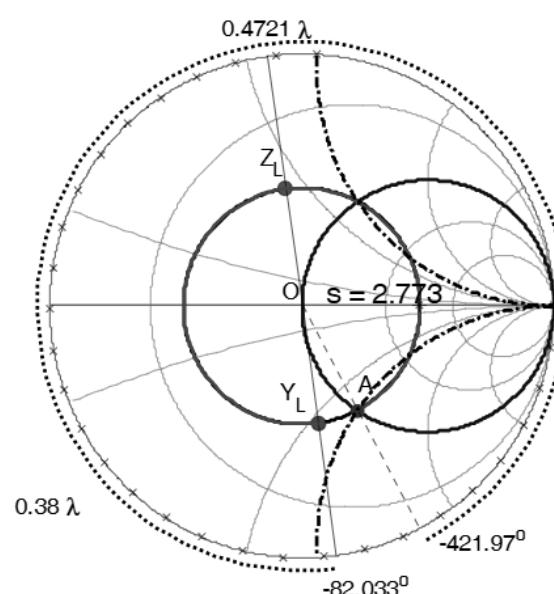
$$= 34.2 + j41.4\Omega$$

$$(b) d_B = \frac{360^\circ - 86.4^\circ}{720^\circ} \lambda = \underline{\underline{0.38\lambda}}$$

$$l_B = \frac{\lambda}{2} - \frac{(-62.4^\circ - -82^\circ)}{720^\circ} \lambda = \underline{\underline{0.473\lambda}}$$

$$(c) \underline{\underline{s = 2.65}}$$

(Exact value = 2.7734)



Prob. 11.55

$$z_L = \frac{Z_L}{Z_o} = \frac{120 + j220}{50} = 2.4 + j4.4$$

We follow Example 11.7. At A, $y_s = -j3$ and at B, $y_s = +j3$. The required stub admittance is

$$Y_s = Y_o y_s = \frac{\pm j3}{50} = \pm j0.06 \text{ S}$$

The distance between the load and the stub is determined as follows. For A, value = 0.2308 λ)

For B,

$$l_B = \frac{180 + 10 + 17}{720} \lambda = \underline{\underline{0.2875\lambda}}$$

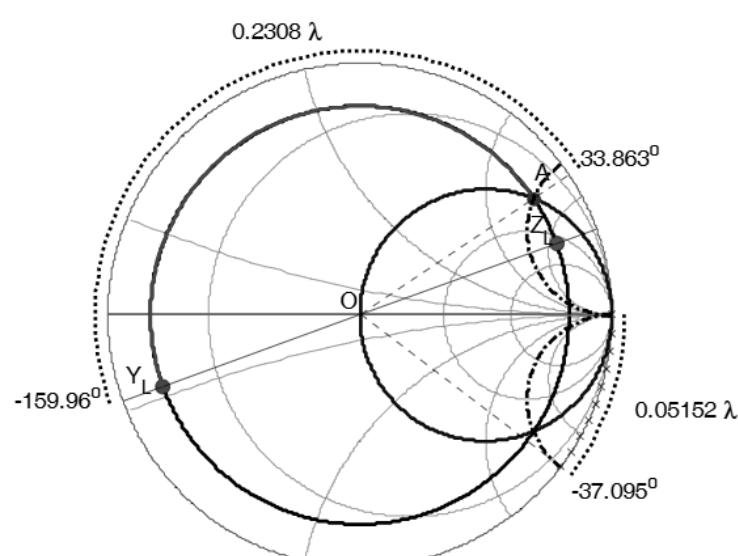
The length of the stub line is determined as follows.

$$d_A = \frac{19}{720} \lambda = \underline{\underline{0.0264\lambda}}$$

(Exact value = 0.0515 λ)

$$d_B = \frac{360 - 19}{720} \lambda = \underline{\underline{0.4736\lambda}}$$

(Exact value = 0.4485 λ)



Prob. 11.56

$$s = \frac{V_{\max}}{V_{\min}} = \frac{4V}{1V} = 4$$

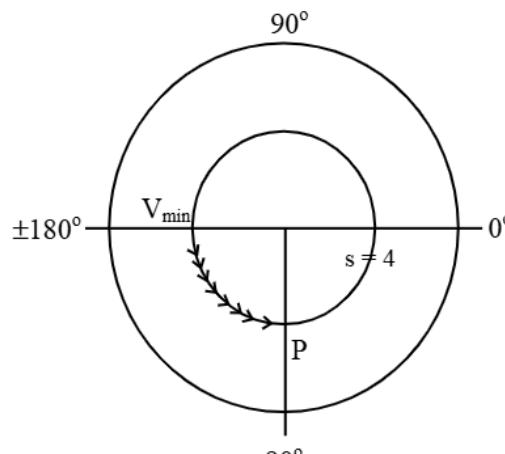
$$|\Gamma| = \frac{s-1}{s+1} = \frac{3}{5} = 0.6$$

$$\frac{\lambda}{2} = 25 \text{ cm} - 5 \text{ cm} = 20 \text{ cm}$$

$$\rightarrow \lambda = 40 \text{ cm}$$

The load is $l=5\text{cm}$ from V_{\min} , i.e.

$$l = \frac{5\lambda}{40} = \frac{\lambda}{8} \rightarrow 90^\circ$$



On the $s = 4$ circle, move 90° from V_{\min} towards the load and obtain $Z_L = 0.46 - j0.88$ at P.

$$Z_L = Z_0 z_L = 60(0.46 - j0.88) = \underline{27.6 - j52.8 \Omega}$$

(Exact value = $28.2353 - j52.9412 \Omega$)

$$\theta_\Gamma = 270^\circ \text{ or } -90^\circ$$

$$\underline{\underline{\Gamma = 0.6 \angle -90^\circ}}$$

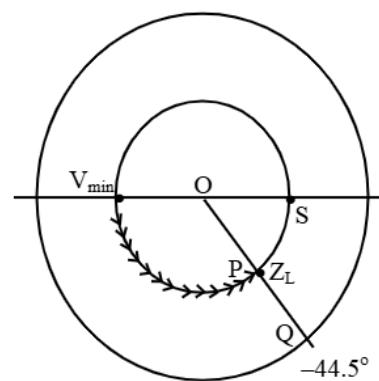
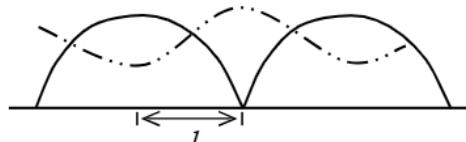
Prob. 11.57

$$s = \frac{V_{\max}}{V_{\min}} = \frac{0.95}{0.45} = 2.11$$

$$\frac{\lambda}{2} = 22.5 - 14 = 8.5 \rightarrow \lambda = 17 \text{ cm}$$

$$f = \frac{c}{\lambda} = \frac{3 \times 10^8}{0.17} = \underline{1.764 \text{ GHz}}$$

$$l = 3.2 \text{ cm} = \frac{3.2}{17} \lambda \rightarrow 135.5^\circ$$



At P, $z_L = 1.4 - j0.8$

$$Z_L = 50(1.4 - j0.8) = \underline{\underline{70 - j40\Omega}}$$

(Exact value = $70.606-j40.496 \Omega$)

$$|\Gamma| = \frac{s-1}{s+1} = \frac{1.11}{3.11} = 0.357, \quad \theta_\Gamma = -44.5^\circ$$

$$\Gamma = \underline{\underline{0.357 \angle -44.5^\circ}}$$

(Exact value = $0.3571 \angle -44.471^\circ$)

Prob. 11.58

$$\Gamma_s = \frac{R_g - R_o}{R_g + R_o} = \frac{0 - 50}{0 + 50} = \underline{\underline{-1}}$$

$$\Gamma_L = \frac{R_L - R_o}{R_L + R_o} = \frac{80 - 50}{80 + 50} = \underline{\underline{0.231}}$$

Prob. 11.59

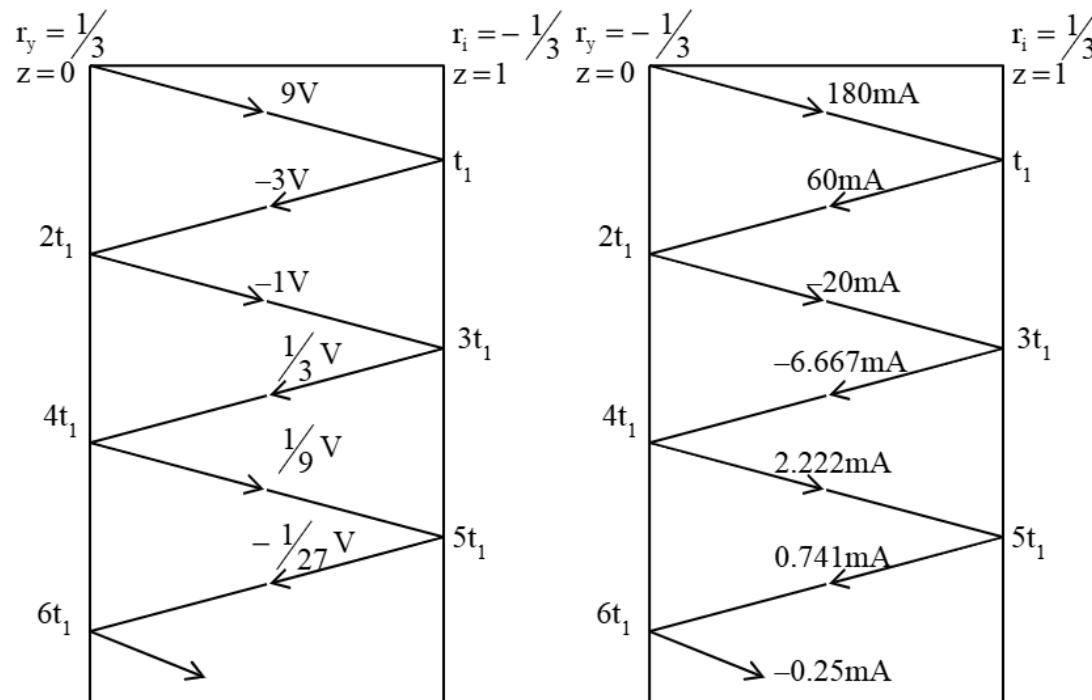
$$\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{0.5Z_o - Z_o}{1.5Z_o} = -\frac{1}{3}$$

$$\Gamma_g = \frac{Z_g - Z_o}{Z_g + Z_o} = \frac{Z_o}{3Z_o} = \frac{1}{3}$$

$$t_1 = \frac{l}{u} = 2\mu s, \quad V_o = \frac{Z_o}{3Z_o}(27) = 9 \text{ V}, \quad I_o = \frac{V_o}{Z_o} = 180 \text{ mA}$$

$$V_\infty = \frac{Z_L}{Z_g + Z_L} V_g = \frac{0.5}{2.5}(27) = 5.4 \text{ V}, \quad I_\infty = \frac{V_\infty}{Z_L} = 216 \text{ mA}$$

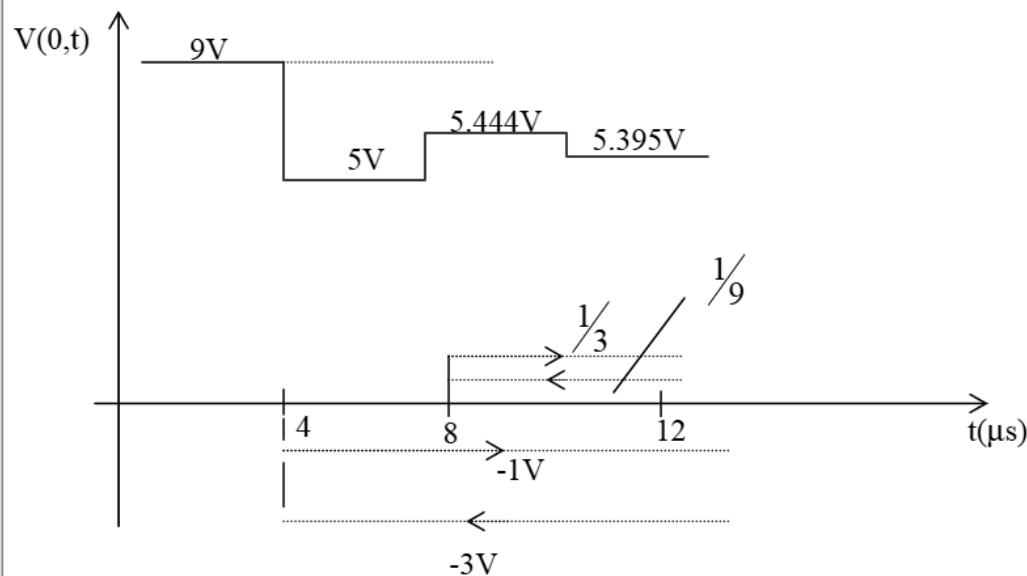
The voltage and current bounce diagrams are shown below

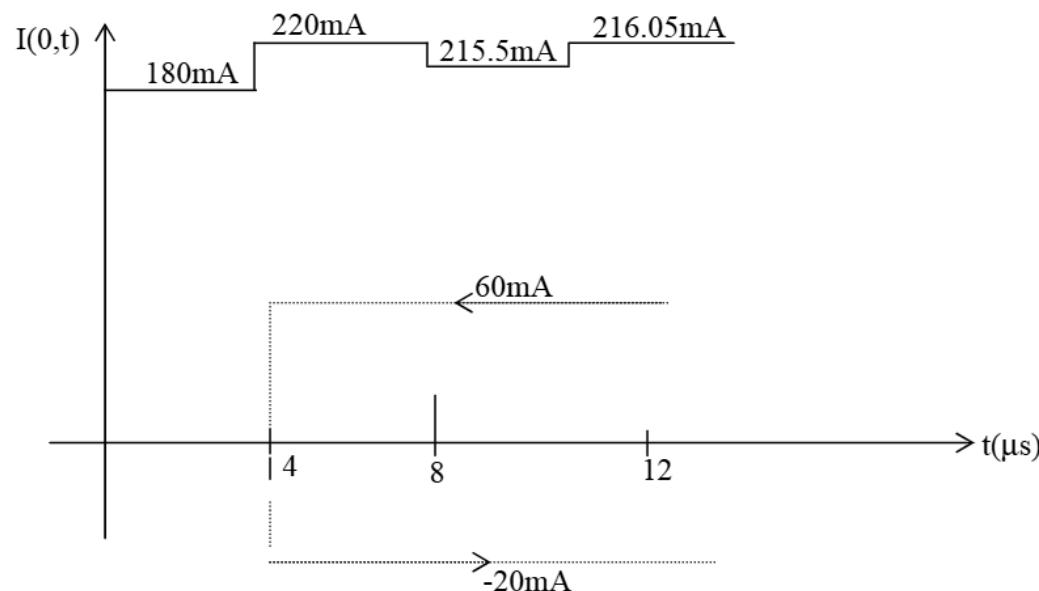


(Voltage bounce diagram)

(Current bounce diagram)

From the bounce diagrams, we obtain $V(0,t)$ and $I(0,t)$ as shown below:



**Prob. 11.60**

Using Thevenin equivalent at $z = 0$ gives

$$R_g = R_s = 4Z_o = 200\Omega$$

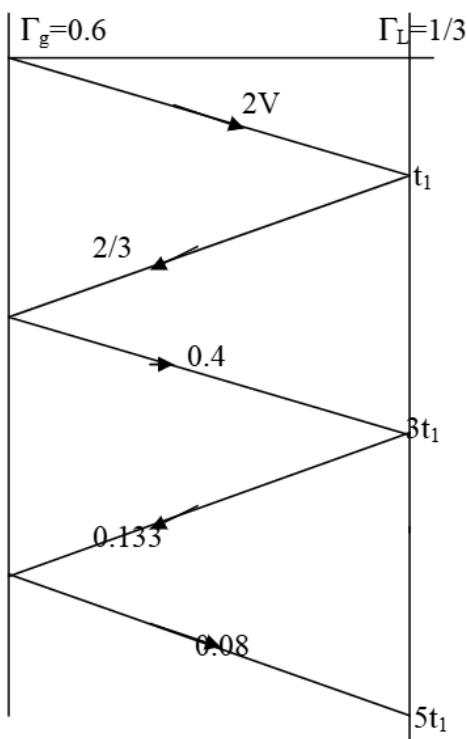
$$V_g = I_s R_s = 10 \times 200 \times 10^{-3} = 2V$$

$$\Gamma_g = \frac{Z_g - Z_o}{Z_g + Z_o} = \frac{4Z_o - Z_o}{4Z_o + Z_o} = \frac{3}{5}$$

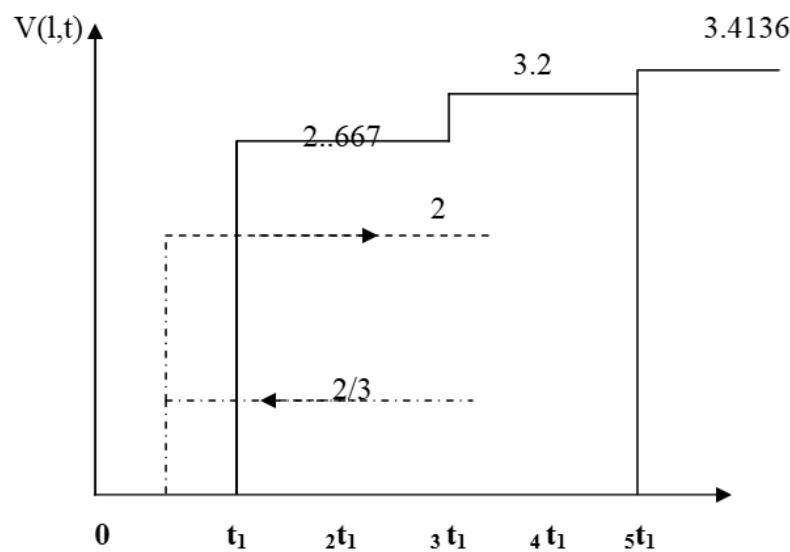
$$\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{2Z_o - Z_o}{2Z_o + Z_o} = \frac{1}{3}$$

$$t_1 = \frac{\ell}{u} = \frac{10}{2 \times 10^8} = 50 \text{ ns}$$

The bounce diagram is shown below.



The load voltage is sketched below.



$$I(\ell, t) = \frac{V(\ell, t)}{Z_L} = \frac{V(\ell, t)}{100}$$

To get $I(l, t)$, we just scale down $V(l, t)$ by 100.

Prob. 11.61

$$\Gamma_g = \frac{Z_g - Z_o}{Z_g + Z_o} = \frac{32 - 75}{32 + 75} = 0.4019$$

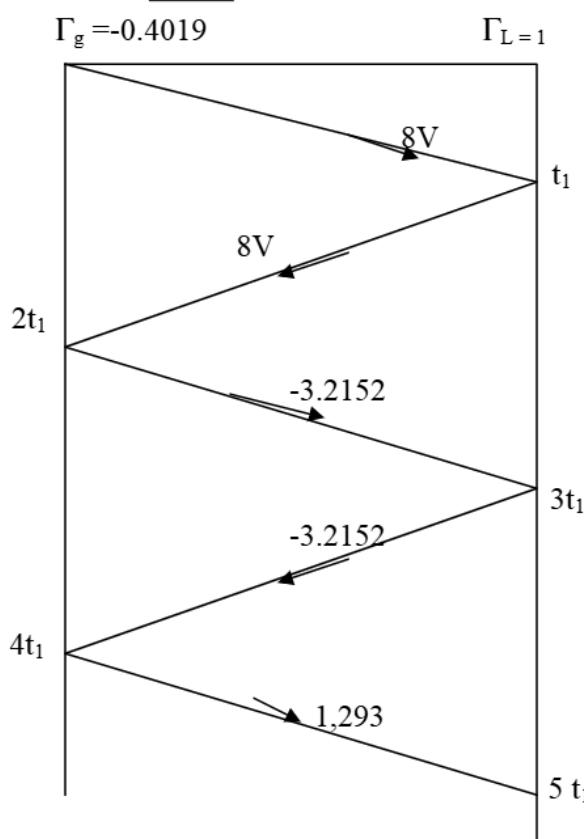
$$\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{2 \times 10^6 - 75}{2 \times 10^6 + 75} \approx 1$$

$$t_1 = \frac{\ell}{u} = \frac{50 \times 10^{-2}}{2 \times 10^8} = 2.5 \text{ ns}$$

The bounce diagram is shown below.

At $t = 20 \text{ ns} = 4t_1$,

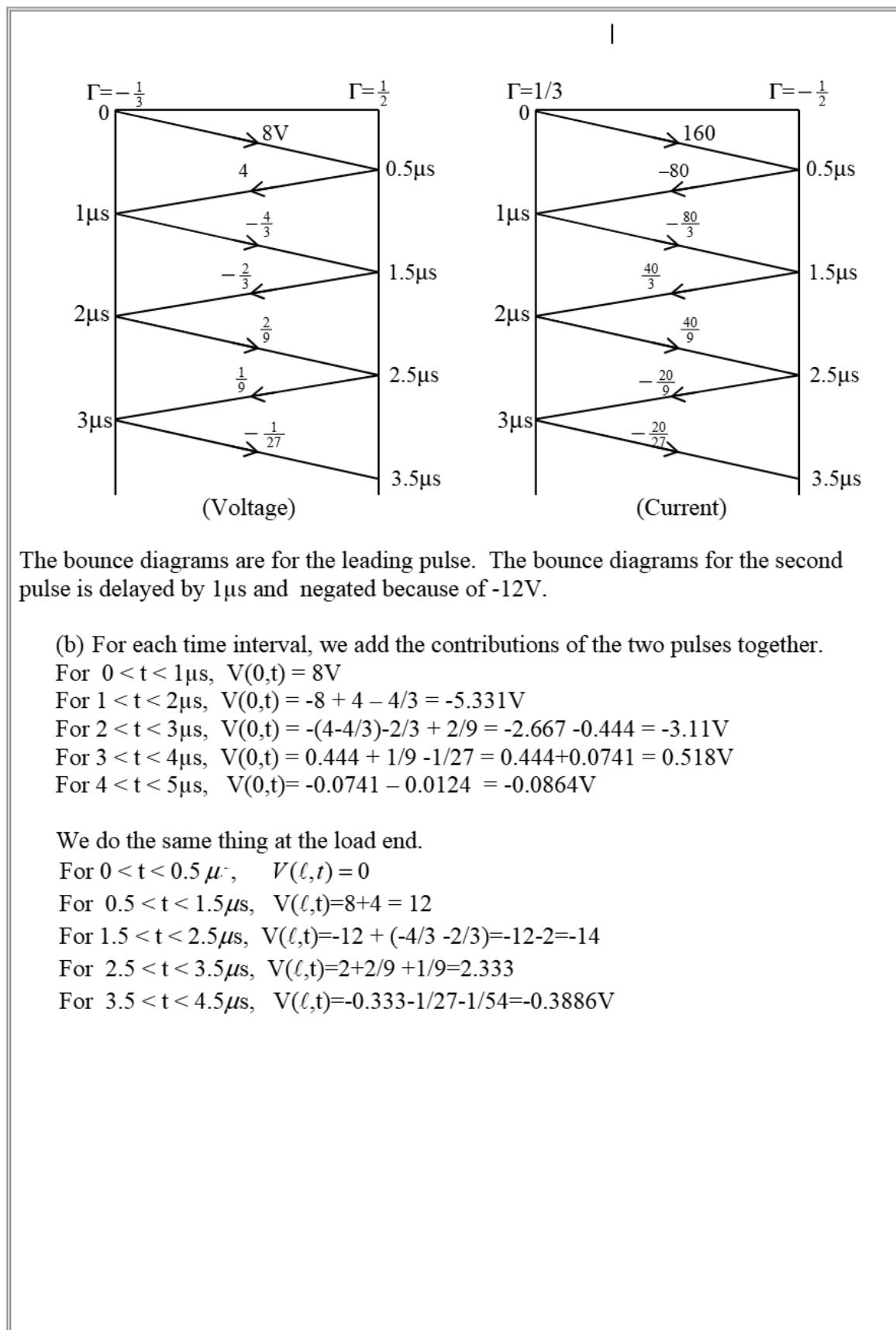
$$V = 8 + 8 - 3.2152 - 3.2152 = \underline{\underline{9.57 \text{ V}}}$$

**Prob. 11.62**

$$(a) \quad t_1 = \frac{l}{u} = \frac{150}{3 \times 10^8} = 0.5 \mu\text{s},$$

$$\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{150 - 50}{150 + 150} = \frac{1}{2}, \quad \Gamma_g = \frac{Z_g - Z_o}{Z_g + Z_o} = \frac{25 - 50}{75} = -\frac{1}{3},$$

$$V_o = \frac{Z_o V_g}{Z_o + Z_g} = \frac{50(12)}{75} = 8V, \quad I_o = \frac{V_g}{Z_g + Z_o} = \frac{12}{75} = 160 \text{ mA}$$



The bounce diagrams are for the leading pulse. The bounce diagrams for the second pulse is delayed by $1\mu s$ and negated because of $-12V$.

(b) For each time interval, we add the contributions of the two pulses together.

$$\text{For } 0 < t < 1\mu s, V(0,t) = 8V$$

$$\text{For } 1 < t < 2\mu s, V(0,t) = -8 + 4 - 4/3 = -5.331V$$

$$\text{For } 2 < t < 3\mu s, V(0,t) = -(4-4/3)-2/3 + 2/9 = -2.667 - 0.444 = -3.11V$$

$$\text{For } 3 < t < 4\mu s, V(0,t) = 0.444 + 1/9 - 1/27 = 0.444 + 0.0741 = 0.518V$$

$$\text{For } 4 < t < 5\mu s, V(0,t) = -0.0741 - 0.0124 = -0.0864V$$

We do the same thing at the load end.

$$\text{For } 0 < t < 0.5\mu s, V(\ell,t) = 0$$

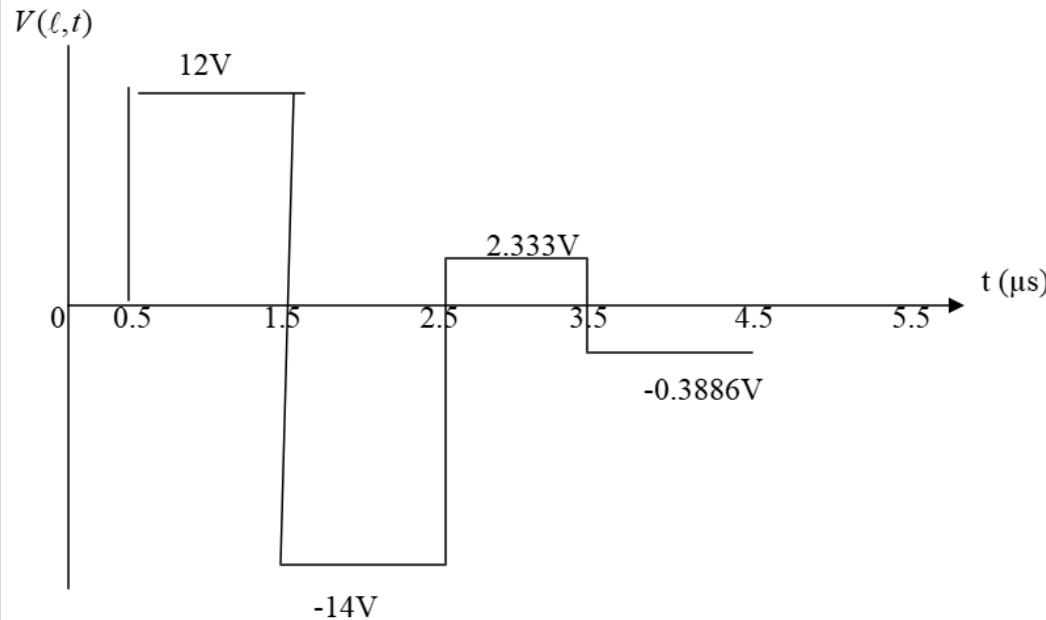
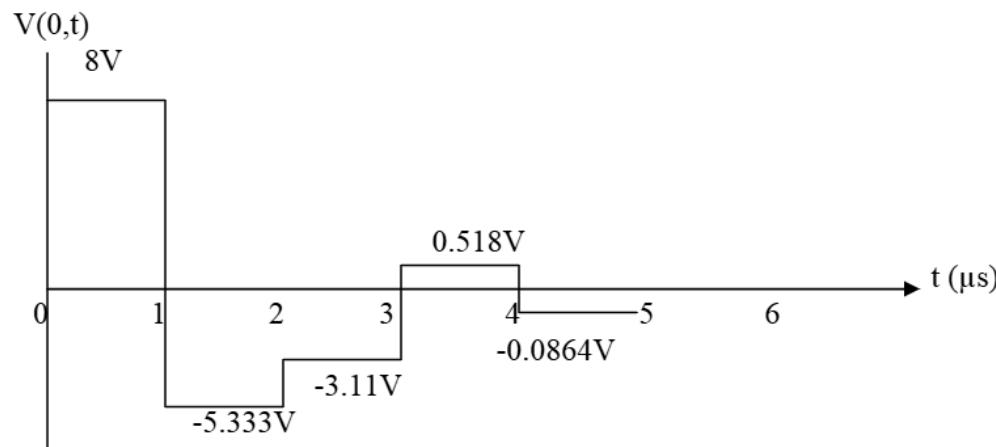
$$\text{For } 0.5 < t < 1.5\mu s, V(\ell,t) = 8 + 4 = 12$$

$$\text{For } 1.5 < t < 2.5\mu s, V(\ell,t) = -12 + (-4/3 - 2/3) = -12 - 2 = -14$$

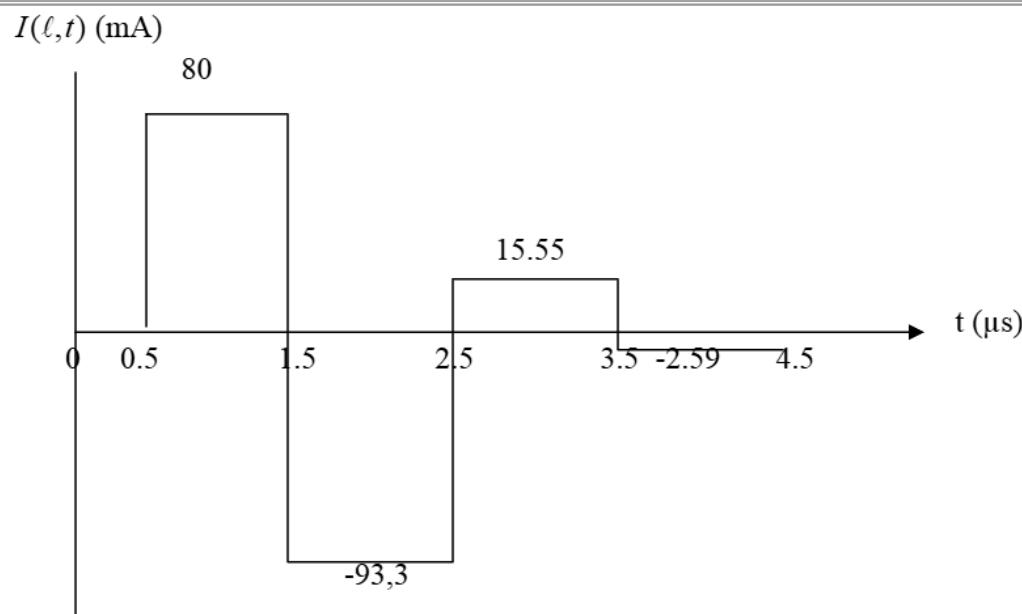
$$\text{For } 2.5 < t < 3.5\mu s, V(\ell,t) = 2 + 2/9 + 1/9 = 2.333$$

$$\text{For } 3.5 < t < 4.5\mu s, V(\ell,t) = -0.333 - 1/27 - 1/54 = -0.3886V$$

The results are shown below.



Since $I(\ell,t) = \frac{V(\ell,t)}{Z_L} = \frac{V(\ell,t)}{150}$, we scale $V(\ell,t)$ by a factor of 1/150 as shown below.

**Prob. 11.63**

$$V_o = 8V = \frac{Z_o}{Z_o + Z_g} V_g = \frac{50}{50 + 60} V_g \quad \longrightarrow \quad V_g = \frac{8 \times 110}{50} = \underline{\underline{17.6 \text{ V}}}$$

$$2t_1 = 4\mu s \quad \longrightarrow \quad t_1 = 2\mu s = \frac{\ell}{u}$$

$$\ell = ut_1 = 3 \times 10^8 \times 2 \times 10^{-6} = \underline{\underline{600 \text{ m}}}$$

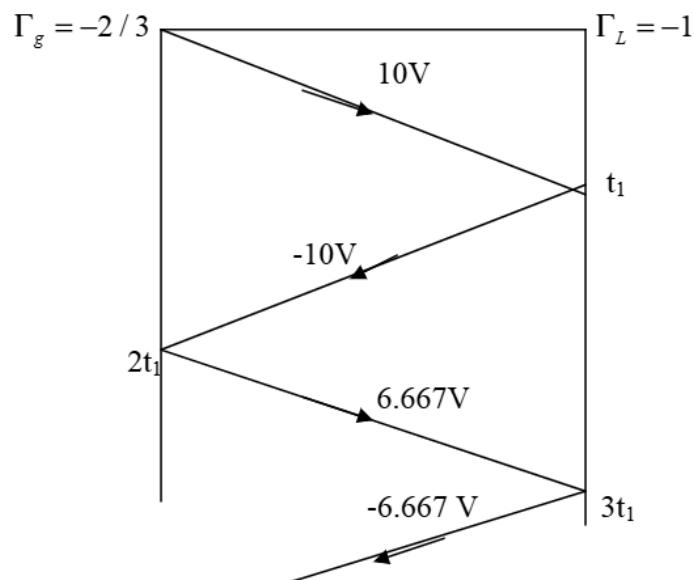
Prob. 11.64

$$t_1 = \frac{20}{2 \times 10^8} = 10^{-7} = 0.1\mu s, \quad V_o = \frac{Z_o}{Z_o + Z_g} V_g = \frac{50}{60} (12) = 10V$$

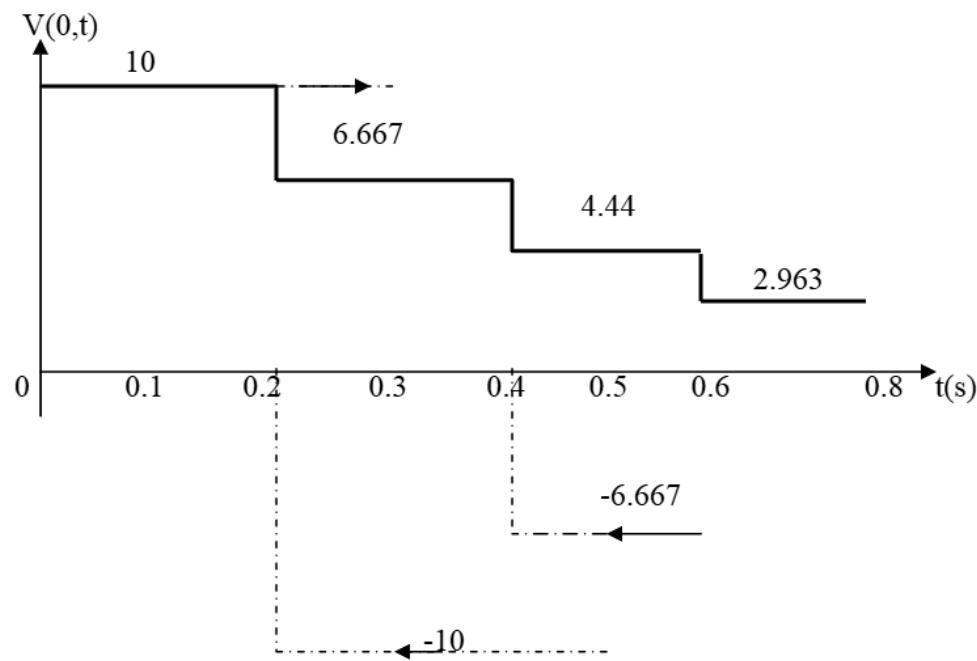
$$\Gamma_g = \frac{Z_g - Z_o}{Z_g + Z_o} = \frac{10 - 50}{10 + 50} = -2/3$$

$$\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{0 - 50}{0 + 50} = -1$$

The voltage bounce diagram is shown below



From the bounce diagram, we obtain $V(0,t)$ as shown. $V(l,t) = 0$ due to the short circuit.



Prob. 11.65

The initial pulse on the line is

$$V_o = \frac{Z_o}{Z_o + Z_g} V_g \quad (1)$$

The reflection coefficient is

$$\Gamma = \frac{Z_g - Z_o}{Z_g + Z_o} \quad (2)$$

The first reflected wave has amplitude ΓV_o , while the second reflected wave has amplitude $\Gamma^2 V_o$, etc. For a very long time, the total voltage is

$$V_T = V_o + \Gamma V_o + \Gamma^2 V_o + \dots = V_o \left(1 + \Gamma + \Gamma^2 + \Gamma^3 + \dots \right) = V_o \left(\frac{1}{1 - \Gamma} \right) \quad (3)$$

Substituting (1) and (2) into (3) gives

$$V_T = \frac{Z_o}{Z_o + Z_g} V_g \left(\frac{1}{1 - \frac{Z_g - Z_o}{Z_g + Z_o}} \right) = Z_o V_g \frac{1}{Z_g + Z_o - Z_g + Z_o} = \underline{\underline{\frac{V_g}{2}}}$$

Prob. 11.66

For $w = 0.4 \text{ mm}$, $\frac{w}{h} = \frac{0.4 \text{ mm}}{2 \text{ m}} = 0.2 \rightarrow \text{narrow strip}$

For $\frac{w}{h} = 0.2$, $\epsilon_{\text{eff}} = 5.851$, $Z_o = 91.53 \Omega$

For $\frac{w}{h} = 0.4$, $\epsilon_{\text{eff}} = 6.072$, $Z_o = 73.24 \Omega$

Hence,

$$\underline{\underline{73.24 \Omega < Z_o < 91.53 \Omega}}$$

Prob. 11.67

(a) Let $x = w/h$. If $x < 1$,

$$50 = \frac{60}{\sqrt{4.6}} \ln \left(\frac{8}{x} + x \right)$$

$$5\sqrt{4.6} - 6\ln \left(\frac{8}{x} + x \right) = 0$$

we solve for x (e.g using Maple) and get $x = 2.027$ or 3.945

which contradicts our assumption that $x < 1$. If $x > 1$,

$$50 = \frac{120\pi}{\sqrt{4.6} [x + 1.393 + 0.667 \ln(x + 1.444)]}$$

We solve this iteratively and obtain:

$$x = 1.8628, w = xh = 14.9024 \text{ mm}$$

For this w and h ,

$$\epsilon_{\text{eff}} = 3.4598$$

$$(b) \quad \beta = \frac{\omega\sqrt{\epsilon_{\text{eff}}}}{c}$$

$$\beta\ell = 45^\circ = \frac{\pi}{4} = \frac{\omega\ell\sqrt{\epsilon_{\text{eff}}}}{c}$$

$$\ell = \frac{\pi c}{4\sqrt{\epsilon_{\text{eff}}} 2\pi f} = \frac{3 \times 10^8}{8 \times \sqrt{3.4598} \times 8 \times 10^9}$$

$$\underline{\underline{\ell = 0.00252 \text{ m}}}$$

Prob. 11.68

$$w = 1.5 \text{ cm}, \quad h = 1 \text{ cm}, \quad \frac{w}{h} = 1.5$$

$$(a) \quad \epsilon_{\text{eff}} = \left(\frac{6+1}{2} \right) + \frac{\epsilon_r - 1}{2\sqrt{1 + 12h/w}} = 1.6 + \frac{0.6}{\sqrt{1 + 12/1.5}} = \underline{\underline{1.8}}$$

$$Z_0 = \frac{377}{\sqrt{1.8} (1.5 + 1.393 + 0.667 \ln(2.944))} = \frac{281}{3.613} = \underline{\underline{77.77 \Omega}}$$

$$(b) \quad \alpha_c = 8.686 \frac{R_s}{w Z_0}$$

$$R_s = \frac{1}{\sigma_c \sigma} = \sqrt{\frac{\mu \pi f}{\sigma_c}} = \sqrt{\frac{19 \times 2.5 \times 10^9 \times 4\pi \times 10^{-3}}{1.1 \times 10^7}}$$

$$= 2.995 \times 10^{-2}$$

$$\alpha_c = \frac{8.686 \times 2.995 \times 10^{-2}}{1.5 \times 10^{-2} \times 77.77} = \underline{\underline{0.223 \text{ dB/m}}}$$

$$u = \frac{c}{\sqrt{\epsilon_{\text{eff}}}} \rightarrow \lambda = \frac{u}{f} = \frac{c}{f \sqrt{\epsilon_{\text{eff}}}} = \frac{3 \times 10^8}{2.5 \times 10^9 \sqrt{1.8}} = 8.944 \times 10^{-2}$$

$$\alpha_d = 27.3 \times \frac{0.8(2.2)}{1.2 \sqrt{1.8}} \frac{2 \times 10^{-2}}{8.944 \times 10^{-2}} = \frac{96.096}{14.3996}$$

$$\alpha_d = \underline{\underline{6.6735 \text{ dB/m}}}$$

$$(c) \quad \alpha = \alpha_c + \alpha_d = 6.8965 \text{ dB/m}$$

$$\alpha \ell = 20 \text{ dB} \rightarrow \ell = \frac{20}{\alpha} = \frac{20}{6.8965} = \underline{\underline{2.9 \text{ m}}}$$

Prob. 11.69

$$w' = 0.5 + \frac{0.1}{3.2} \ln \left(\frac{5 \times 1.2}{0.1} \right) = 0.5 + 0.1279 = 0.6279$$

$$\begin{aligned} Z_o &= \frac{377}{2\pi\sqrt{2}} \ln \left\{ 1 + \frac{4 \times 1.2}{\pi \times 0.6177} \left[\frac{8 \times 1.2}{4 \times 0.6279} + 3.354 \right] \right\} \\ &= 42.43 \ln \{ 1 + 2.49(3.822 + 3.354) \} = 42.43 \ln(20.255) \\ &= \underline{\underline{127.64 \Omega}} \end{aligned}$$

Prob. 11.70

Suppose we guess that $w/h < 2$

$$A = \frac{75}{60} \sqrt{\frac{3.3}{2}} + \frac{1.3}{3.3} \left(0.23 + \frac{0.11}{2.3} \right) = 1.117$$

$$\frac{w}{h} = \frac{8e^A}{e^{2A} - 2} = \frac{24.44}{7.337} = 3.331 \rightarrow w = 3.331h = \underline{\underline{4mm}}$$

If we guess that $w/h > 2$,

$$B = \frac{60\pi^2}{Z_o \sqrt{\epsilon_r}} = \frac{60\pi^2}{75\sqrt{2.3}} = 5.206$$

$$\frac{w}{h} = \frac{2}{\pi} \left[4.266 - \ln 9.412 + \frac{1.3}{4.6} \left(\ln 4.206 + 0.39 - \frac{0.61}{2.3} \right) \right]$$

$$= 1.665 < 2$$

Thus $\frac{w}{h} = 3.331 > 2$

$$\varepsilon_{\text{eff}} = \frac{3.3}{2} + \frac{1.3}{2\sqrt{1 + \frac{12}{3.331}}} = 1.953$$

$$u = \frac{3 \times 10^8}{\sqrt{1.953}} = \underline{\underline{2.1467 \times 10^8 \text{ m/s}}}$$

Prob. 11.71

$$\Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{100 - 150}{250} = -0.2$$

$$RL = -20 \log |\Gamma| = \underline{\underline{13.98 \text{ dB}}}$$

Prob. 11.72

$$\Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{120 - 50}{170} = 0.4118$$

$$RL = -20 \log_{10} |\Gamma| = -20 \log_{10}(0.4118) = \underline{\underline{7.706 \text{ dB}}}$$

CHAPTER 12

P. E. 12.1 (a) For TE₁₀, f_c = 3 GHz,

$$\sqrt{1 - (f_c/f)^2} = \sqrt{1 - (3/15)^2} = \sqrt{0.96}, \quad \beta_o = \omega/u_o = 4\pi f/c$$

$$\beta = \frac{4\pi f}{c} \sqrt{0.96} = \frac{4\pi \times 15 \times 10^9}{3 \times 10^8} \sqrt{0.96} = \underline{\underline{615.6}} \text{ rad/m}$$

$$u = \frac{\omega}{\beta} = \frac{2\pi \times 15 \times 10^9}{615.6} = \underline{\underline{1.531 \times 10^8}} \text{ m/s}$$

$$\eta' = \sqrt{\frac{\mu}{\epsilon}} = 60\pi, \quad \eta_{TE} = \frac{60\pi}{\sqrt{0.96}} = \underline{\underline{192.4\Omega}}$$

(b) For TM₁₁, f_c = 3 $\sqrt{7.25}$ GHz, $\sqrt{1 - (f_c/f)^2} = 0.8426$

$$\beta = \frac{4\pi f}{c} (0.8426) = \frac{4\pi \times 15 \times 10^9 (0.8426)}{3 \times 10^8} = \underline{\underline{529.4}} \text{ rad/m}$$

$$u = \frac{\omega}{\beta} = \frac{2\pi \times 15 \times 10^9}{529.4} = \underline{\underline{1.78 \times 10^8}} \text{ m/s}$$

$$\eta_{TM} = 60\pi(0.8426) = \underline{\underline{158.8\Omega}}$$

P. E. 12.2 (a) Since $E_z \neq 0$, this is a TM mode

$$E_{zs} = E_o \sin(m\pi x/a) \sin(n\pi y/b) e^{-j\beta z}$$

$$E_o = 20, \quad \frac{m\pi}{a} = 40\pi \quad \longrightarrow \quad m=2, \quad \frac{n\pi}{b} = 50\pi \quad \longrightarrow \quad n=1$$

i.e. TM₂₁ mode.

$$(b) \quad f_c = \frac{u'}{2} \sqrt{(m/a)^2 + (n/b)^2} = \frac{3 \times 10^8}{2} \sqrt{40^2 + 50^2} = 1.5\sqrt{41} \text{ GHz}$$

$$\beta = \omega \sqrt{\mu \epsilon} \sqrt{1 - (f_c/f)^2} = \frac{2\pi f}{c} \sqrt{f^2 - f_c^2} = \frac{2\pi \times 10^9}{3 \times 10^8} \sqrt{225 - 92.25} = \underline{\underline{241.3 \text{ rad/m.}}}$$

(c)

$$E_{xs} = \frac{-j\beta}{h^2} (40\pi) 20 \cos 40\pi x \sin 50\pi y e^{-j\beta z}$$

$$E_{ys} = \frac{-j\beta}{h^2} (50\pi) 20 \sin 40\pi x \cos 50\pi y e^{-j\beta z}$$

$$\frac{E_y}{E_x} = \underline{\underline{1.25 \tan 40\pi x \cot 50\pi y}}$$

P. E. 12.3 If TE₁₃ mode is assumed, f_c and β remain the same.

$$f_c = 28.57 \text{ GHz}, \beta = 1718.81 \text{ rad/m}, \gamma = j\beta$$

$$\eta_{TE13} = \frac{377 / 2}{\sqrt{1 - (28.57 / 50)^2}} = 229.69 \Omega$$

For m=1, n=3, the field components are:

$$E_z = 0$$

$$H_z = H_o \cos(\pi x / a) \cos(3\pi y / b) \cos(\omega t - \beta z)$$

$$E_x = -\frac{\omega\mu}{h^2} \left(\frac{3\pi}{b} \right) H_o \cos(\pi x / a) \sin(3\pi y / b) \sin(\omega t - \beta z)$$

$$E_y = \frac{\omega\mu}{h^2} \left(\frac{\pi}{a} \right) H_o \sin(\pi x / a) \cos(3\pi y / b) \sin(\omega t - \beta z)$$

$$H_x = -\frac{\beta}{h^2} \left(\frac{\pi}{a} \right) H_o \sin(\pi x / a) \cos(3\pi y / b) \sin(\omega t - \beta z)$$

$$H_y = -\frac{\beta}{h^2} \left(\frac{3\pi}{a} \right) H_o \cos(\pi x / a) \sin(3\pi y / b) \sin(\omega t - \beta z)$$

$$\text{Given that } H_{ox} = 2 = -\frac{\beta}{h^2} (\pi / a) H_o,$$

$$H_{oy} = -\frac{\beta}{h^2} (3\pi / b) H_o = 6a / b = 6(1.5) / 8 = 11.25$$

$$H_{oz} = H_o = -\frac{2h^2 a}{\beta\pi} = \frac{-2 \times 14.51\pi^2 \times 10^4 \times 1.5 \times 10^{-2}}{1718.81\pi} = -7.96$$

$$E_{oy} = \frac{\omega\mu}{h^2} \left(\frac{\pi}{a} \right) H_o = -\frac{2\omega\mu}{\beta} = 2\eta_{TE} = -459.4$$

$$E_{ox} = -E_{oy} \frac{3a}{b} = 459.4(4.5 / 0.8) = 2584.1$$

$$E_x = 2584.1 \cos(\pi x / a) \sin(3\pi y / b) \sin(\omega t - \beta z) \text{ V/m},$$

$$E_y = -459.4 \sin(\pi x / a) \cos(3\pi y / b) \sin(\omega t - \beta z) \text{ V/m},$$

$$E_z = 0,$$

$$H_y = 11.25 \cos(\pi x / a) \sin(3\pi y / b) \sin(\omega t - \beta z) \text{ A/m},$$

$$H_z = -7.96 \cos(\pi x / a) \cos(3\pi y / b) \cos(\omega t - \beta z) \text{ A/m}$$

P. E. 12.4

$$f_{c11} = \frac{u'}{2} \sqrt{\frac{1}{a^2} + \frac{1}{b^2}} = \frac{3 \times 10^8 \times 10^2}{2} \sqrt{1/8.636^2 + 1/4.318^2} = 3.883 \text{ GHz}$$

$$u_p = \frac{3 \times 10^8}{\sqrt{1 - (3.883/4)^2}} = \underline{\underline{12.5 \times 10^8}} \text{ m/s},$$

$$u_g = \frac{9 \times 10^{16}}{12.5 \times 10^8} = \underline{\underline{7.2 \times 10^7}} \text{ m/s}$$

P. E. 12.5 The dominant mode becomes TE₀₁ mode

$$f_{c01} = \frac{c}{2b} = 3.75 \text{ GHz}, \quad \eta_{TE} = 406.7 \Omega$$

From Example 12.2,

$$E_x = -E_o \sin(3\pi y / b) \sin(\omega t - \beta z), \quad \text{where } E_o = \frac{\omega \mu b}{\pi} H_o.$$

$$\mathcal{P}_{ave} = \int_{x=0}^a \int_{y=0}^b \frac{|E_{xs}|^2}{2\eta} dx dy = \frac{E_o^2 ab}{4\eta}$$

Hence $E_o = 63.77 \text{ V/m}$ as in Example 12.5.

$$H_o = \frac{\pi E_o}{\omega \mu b} = \frac{\pi \times 63.77}{2\pi \times 10^{10} \times 4\pi \times 10^{-7} \times 4 \times 10^{-2}} = \underline{\underline{63.34}} \text{ mA/m}$$

P. E. 12.6 (a) For m=1, n=0, $f_c = u'/(2a)$

$$\frac{\sigma}{\omega\epsilon} = \frac{10^{-15}}{2\pi \times 9 \times 10^9 \times 2.6 \times 10^{-9} / (36\pi)} = \frac{10^{-15}}{1.3} \ll 1$$

Hence,

$$u' \equiv \frac{1}{\sqrt{\mu\epsilon}} = c / \sqrt{2.6}, \quad f_c = \frac{3 \times 10^8}{2 \times 2.4 \times 10^{-2} \sqrt{2.6}} = 2.2149 \text{ GHz}$$

$$\alpha_d = \frac{\sigma\eta'}{2\sqrt{1-(f_c/f)^2}} = \frac{10^{-15} \times 377 / \sqrt{2.6}}{2\sqrt{1-(2.2149/9)^2}} = 1.205 \times 10^{-13} \text{ Np/m}$$

For n = 0, m=1,

$$\begin{aligned} \alpha_c &= \frac{2R_s}{b\eta'\sqrt{1-(f_c/f)^2}} \left[\frac{1}{2} + \frac{b}{a} (f_c/f)^2 \right] \\ &= \frac{2\sqrt{2.6}\sqrt{\pi \times 9 \times 10^9 \times 1.1 \times 10^7 \times 4\pi \times 10^{-7}}}{377 \times 1.5 \times 10^{-2} \times 1.1 \times 10^7 \sqrt{1-(2.2149/9)^2}} [0.5 + (2.4/1.5)(2.2148/9)^2] = 2 \times 10^{-2} \text{ Np/m} \end{aligned}$$

(b) Since $\alpha_c \gg \alpha_d$, $\alpha = \alpha_c + \alpha_d \approx \alpha_c = 2 \times 10^{-2}$

$$\text{loss} = \alpha l = 2 \times 10^{-2} \times 0.4 = 0.8 \times 10^{-2} \text{ Np} = 0.06945 \text{ dB}$$

P. E. 12.7 For TE₁₁, m = 1 = n,

$$H_{zs} = H_o \cos(\pi x/a) \cos(\pi y/b) e^{-\gamma z}$$

$$E_{xs} = \frac{j\omega}{h^2} (\pi/b) H_o \cos(\pi x/a) \sin(\pi y/b) e^{-\gamma z}$$

$$E_{ys} = -\frac{j\omega\mu}{h^2} (\pi/a) H_o \sin(\pi x/a) \cos(\pi y/b) e^{-\gamma z}$$

$$H_{xs} = \frac{j\beta}{h^2} (\pi/a) H_o \sin(\pi x/a) \cos(\pi y/b) e^{-\gamma z}$$

$$H_{ys} = \frac{j\beta}{h^2} (\pi/b) H_o \cos(\pi x/a) \sin(\pi y/b) e^{-\gamma z}$$

$$E_{zs} = 0$$

For the electric field lines,

$$\frac{dy}{dx} = \frac{E_y}{E_x} = (a/b) \tan(\pi x/a) \cot(\pi y/b)$$

For the magnetic field lines

$$\frac{dy}{dx} = \frac{H_y}{H_x} = -(a/b) \cot(\pi x/a) \tan(\pi y/b)$$

Notice that $(\frac{E_y}{E_x})(\frac{H_y}{H_x}) = -1$

showing that the electric and magnetic field lines are mutually orthogonal. The field lines are as shown in Fig. 12.14.

P. E. 12.8

$$u' = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{\sqrt{\epsilon_r}}$$

$$f_{TE101} = \frac{1.5 \times 10^{10}}{\sqrt{3}} \sqrt{1/25 + 0 + 1/100} = \underline{\underline{1.936 \text{ GHz}}}$$

$$Q_{TE101} = \frac{1}{61\delta}, \text{ where}$$

$$\delta = \frac{1}{\sqrt{\pi f_{101} \mu \sigma_c}} = \frac{1}{\sqrt{\pi \times 1.936 \times 10^9 \times 4\pi \times 10^{-7} \times 5.8 \times 10^7}} = 1.5 \times 10^{-6}$$

$$Q_{TE101} = \frac{10^6}{61 \times 1.5} = \underline{\underline{10,929}}$$

P. E. 12.9

(a) By Snell's law, $n_1 \sin \theta_1 = n_2 \sin \theta_2$. Thus
 $\theta_2 = 90^\circ \rightarrow \sin \theta_2 = 1$

$$\sin \theta_1 = n_2/n_1, \quad \theta_1 = \sin^{-1} n_2/n_1 = \sin^{-1} 1.465/1.48 = \underline{\underline{81.83^\circ}}$$

$$(b) NA = \sqrt{n_1^2 - n_2^2} = \sqrt{1.48^2 - 1.465^2} = \underline{\underline{0.21}}$$

P. E. 12.10

$$\alpha l = 10 \log P(0)/P(l) = 0.2 \times 10 = 2$$

$$P(0)/P(l) = 10^{0.2}, \text{ i.e. } P(l) = P(0) 10^{-0.2} = 0.631 P(0)$$

i.e. 63.1 %

Prob. 12.1

$$\begin{aligned} f_c &= \frac{u'}{2} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} = \frac{3 \times 10^8}{2\sqrt{2.25} \times 10^{-2}} \left[\left(\frac{m}{2.28}\right)^2 + \left(\frac{n}{1.01}\right)^2 \right]^{1/2} \\ &= \frac{15}{\sqrt{2.25}} \left[\left(\frac{m}{2.28}\right)^2 + \left(\frac{n}{1.01}\right)^2 \right]^{1/2} \text{ GHz} \end{aligned}$$

Using this formula, we obtain the cutoff frequencies for the given modes as shown below.

Mode	f_c (GHz)
TE ₀₁	9.901
TE ₁₀	4.386
TE ₁₁	10.829
TE ₀₂	19.802
TE ₂₂	21.658
TM ₁₁	10.829
TM ₁₂	20.282
TM ₂₁	13.228

Prob. 12.2

$$f_{cmn} = \frac{u'}{2} \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} = \frac{3 \times 10^8}{2} \times 10^2 \sqrt{\left(\frac{m}{6}\right)^2 + \left(\frac{n}{4}\right)^2} = 15 \sqrt{\left(\frac{m}{6}\right)^2 + \left(\frac{n}{4}\right)^2} \text{ GHz}$$

$$f_{10} = \frac{15}{6} = 2.5 \text{ GHz}$$

$$f_{c01} = \frac{15}{4} = 3.75 \text{ GHz}$$

$$f_{c20} = 15 \times \frac{2}{6} = 5 \text{ GHz}$$

$$f_{c11} = 15 \sqrt{\frac{1}{6^2} + \frac{1}{4^2}} = 15 \times 0.3005 = 4.51 \text{ GHz}$$

Possible modes are TE_{10} , TE_{01} , TE_{20} , TE_{11} and TM_{11} .

Prob. 12.3

$$(a) \text{ For TE}_{10} \text{ mode, } f_c = \frac{u'}{2a} = \frac{3 \times 10^3}{2 \times 6 \times 10^{-2}} = \underline{\underline{2.5 \text{ GHz}}}$$

$$(b) f = 3f_c = 7.5 \text{ GHz}$$

$$\begin{aligned} f_{cmn} &= \frac{u'}{2} \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} = \frac{3 \times 10^2 \times 10^2}{2} \sqrt{\left(\frac{m}{6}\right)^2 + \left(\frac{n}{4}\right)^2} \\ &= 15 \sqrt{\left(\frac{m}{6}\right)^2 + \left(\frac{n}{4}\right)^2} \text{ GHz} \end{aligned}$$

$$f_{c20} = 15 \times \frac{2}{6} = 5 \text{ GHz}$$

$$f_{c01} = 3.75 \text{ GHz}, \quad f_{c02} = 7.5 \text{ GHz}$$

$$f_{c10} = 2.5 \text{ GHz}, \quad f_{c20} = 5.0 \text{ GHz}$$

$$f_{c21} = 6.25 \text{ GHz}, \quad f_{c30} = 7.5 \text{ GHz}$$

$$f_{12} = 15 \sqrt{\left(\frac{1}{6}\right)^2 + \left(\frac{2}{4}\right)^2} = 7.91 \text{ GHz}$$

$$f_{11} = 15 \sqrt{\left(\frac{1}{6}\right)^2 + \left(\frac{1}{4}\right)^2} = 4.507 \text{ GHz}$$

The following modes are transmitted

$\text{TE}_{01}, \text{TE}_{02}, \text{TE}_{10}, \text{TE}_{11}, \text{TE}_{20}, \text{TE}_{21}, \text{TE}_{30}$

$\text{TM}_{11}, \text{TM}_{21}$

i.e. 7 TE modes and 2 TM modes

Prob. 12.4

(a)

For TE_{10} mode,

$$f_c = \frac{u'}{2} \sqrt{\left(\frac{1}{a}\right)^2} = \frac{3 \times 10^8}{2 \times 2.4 \times 10^{-2}} = \underline{\underline{6.25 \text{ GHz}}}$$

For TE_{01} mode,

$$f_c = \frac{u'}{2} \sqrt{\left(\frac{1}{b}\right)^2} = \frac{3 \times 10^8}{2 \times 1.2 \times 10^{-2}} = \underline{\underline{12.5 \text{ GHz}}}$$

For TE_{20} mode,

$$f_c = \frac{u'}{2} \sqrt{\left(\frac{2}{a}\right)^2} = 2 \times 6.25 = \underline{\underline{12.5 \text{ GHz}}}$$

For TE_{02} mode,

$$f_c = \frac{u'}{2} \sqrt{\left(\frac{2}{b}\right)^2} = 2 \times 12.5 = \underline{\underline{25 \text{ GHz}}}$$

(b) Since $f = 12 \text{ GHz}$, TE_{10} mode will propagate.**Prob. 12.5** (a) For TE_{10} mode, $f_c = \frac{u'}{2a}$

$$\text{Or } a = \frac{u'}{2f_c} = \frac{3 \times 10^8}{2 \times 5 \times 10^9} = \underline{\underline{3 \text{ cm}}}$$

$$\text{For } \text{TE}_{01} \text{ mode, } f_c = \frac{u'}{2b}$$

$$\text{Or } b = \frac{u'}{2f_c} = \frac{3 \times 10^8}{2 \times 12 \times 10^9} = \underline{\underline{1.25 \text{ cm}}}$$

(b) Since $a > b$, $1/a < 1/b$, the next higher modes are calculated as shown below.

Mode	f_c (GHz)
TE_{10}	5
$*\text{TE}_{20}$	10
TE_{30}	15
TE_{40}	20
$*\text{TE}_{01}$	12
TE_{02}	24
$*\text{TE}_{11}$	13
TE_{21}	15.62

The next three higher modes are starred ones, i.e. TE₂₀, TE₀₁, TE₁₁

$$(c) u' = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{\sqrt{2.25}} = 2 \times 10^8 \text{ m/s}$$

For TE₁₁ modes,

$$f_c = \frac{3 \times 10^8}{2 \times 10^{-2} \sqrt{2.25}} \sqrt{\frac{1}{3^2} + \frac{1}{1.25^2}} = \underline{\underline{8.67 \text{ GHz}}}$$

Prob. 12.6 For the dominant mode,

$$f_c = \frac{c}{2a} = \frac{3 \times 10^8}{2 \times 8} = 18.75 \text{ MHz}$$

- (a) It will not pass the AM signal, (b) it will pass the FM signal.

Prob. 12.7 $a/b = 3 \longrightarrow a = 3b$

$$f_{c10} = \frac{u'}{2a} \longrightarrow a = \frac{u'}{2f_{c10}} = \frac{3 \times 10^8}{2 \times 18 \times 10^9} \text{ m} = 0.833 \text{ cm}$$

A design could be a = 9mm, b = 3mm.

Prob. 12.8

$$\text{Let } F_{12} = \sqrt{1 - \left(\frac{f_{c12}}{f} \right)^2} = \sqrt{1 - \left(\frac{25}{40} \right)^2} = 0.7806$$

$$\lambda' = \frac{c}{f} = \frac{3 \times 10^8}{40 \times 10^9} = 0.0075 \text{ m} = 7.5 \times 10^{-3} \text{ m}$$

$$\lambda_{12} = \frac{\lambda'}{F_{12}} = \frac{7.5 \times 10^{-3} \text{ m}}{0.7806} = \underline{\underline{9.608 \times 10^{-3} \text{ m}}}$$

$$u_{12} = \frac{u'}{F_{12}} = \frac{3 \times 10^8}{0.7806} = \underline{\underline{3.843 \times 10^8 \text{ m/s}}}$$

$$\beta_{12} = \frac{2\pi}{\lambda_{12}} = \frac{2\pi}{9.608 \times 10^{-3}} = \underline{\underline{653.95 \text{ rad/m}}}$$

$$\eta_{TE12} = \frac{\eta'}{F_{12}} = \frac{120\pi}{0.7806} = \underline{\underline{482.95 \Omega}}$$

Prob. 12.9

$$u = \frac{\omega}{\beta} = \frac{u'}{\sqrt{1 - (f_c/f)^2}} = \frac{3 \times 10^8}{\sqrt{1 - (6.5/7.2)^2}} = 6.975 \times 10^8 \text{ m/s}$$

$$u_g = \frac{9 \times 10^{16}}{u} = 1.2903 \times 10^8 \text{ m/s}$$

$$t = \frac{2l}{u_g} = \frac{300}{1.2903 \times 10^8} = 2.325 \mu\text{s}$$

Prob. 12.10

$$f = 1.12 f_{c10} = 1.12 \frac{c}{2a} \longrightarrow a = 1.12 \frac{c}{2f} = \frac{1.12 \times 3 \times 10^8}{2 \times 4 \times 10^9} = 4.2 \text{ cm}$$

The next higher-order mode is $f_{c01} = c/2b$.

$$f = 0.85 f_{c01} = 0.85 \frac{c}{2b} \longrightarrow b = \frac{0.85c}{2f} = \frac{0.85 \times 3 \times 10^8}{2 \times 4 \times 10^9} = 3.187 \text{ cm}$$

Prob. 12.11

$$\begin{aligned} (a) f_{c10} &= \frac{u'}{2a} = \frac{c}{2a\sqrt{\epsilon_r}} \\ &= \frac{3 \times 10^8}{2 \times 1.067 \times 10^{-2} \sqrt{6.8}} \\ &= \frac{30}{2 \times 1.067 \sqrt{6.8}} \text{ GHz} \\ &= 5.391 \text{ GHz} \end{aligned}$$

(b)

$$\begin{aligned} F &= \sqrt{1 - \left(\frac{f_c}{f}\right)^2} \\ &= \sqrt{1 - \left(\frac{5.391}{6}\right)^2} \\ &= 0.439 \end{aligned}$$

$$u_p = \frac{u'}{F} = \frac{c}{F\sqrt{\epsilon_r}} = \frac{3 \times 10^8}{0.439 \times \sqrt{6.8}} = \underline{\underline{2.62 \times 10^8 \text{ m/s}}}$$

$$(c) \lambda = \frac{\lambda'}{F} = \frac{u'/f}{F} = \frac{c}{fF\sqrt{\epsilon_r}} = \frac{3 \times 10^8}{0.439 \times 6 \times 10^9 \times \sqrt{6.8}}$$

$$= \frac{10^{-1}}{2 \times 0.439 \sqrt{6.8}} = 0.04368 \text{ m} = \underline{\underline{4.368 \text{ cm}}}$$

Prob. 12.12

In evanescent mode,

$$k^2 = \omega^2 \mu \epsilon < \left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2$$

$$\beta = 0, \quad \gamma = \alpha = \sqrt{\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 - k^2} = \sqrt{4\pi^2 \mu \epsilon f_c^2 - \omega^2 \mu \epsilon}$$

$$\alpha = \sqrt{\mu \epsilon} \sqrt{4\pi^2 f_c^2 - 4\pi^2 f^2} = 2\pi \sqrt{\mu \epsilon} f_c \sqrt{1 - \left(\frac{f}{f_c} \right)^2}$$

Prob. 12.13 $E_z \neq 0$. This must be TM₂₃ mode (m=2, n=3). Since a=2b,

$$f_c = \frac{c}{4b} \sqrt{m^2 + 4n^2} = \frac{3 \times 10^8}{4 \times 3 \times 10^{-2}} \sqrt{4 + 36} = 15.81 \text{ GHz}, \quad f = \frac{\omega}{2\pi} = \frac{10^{12}}{2\pi} = 159.2 \text{ GHz}$$

$$\eta_{TM} = 377 \sqrt{1 - (15.81/159.2)^2} = \underline{\underline{375.1 \Omega}}$$

$$\mathcal{P}_{ave} = \frac{|E_{xs}|^2 + |E_{ys}|^2}{2\eta_{TM}} \mathbf{a}_z$$

$$= \frac{\beta^2 E_o^2}{2h^4 \eta_{TM}} \left[(2\pi/a)^2 \cos^2(2\pi x/a) \sin^2(3\pi y/b) + (3\pi/b)^2 \sin^2(2\pi x/a) \cos^2(3\pi y/b) \right] \mathbf{a}_z$$

$$P_{ave} = \int_{x=0}^a \int_{y=0}^b \mathcal{P}_{ave} \cdot dx dy \mathbf{a}_z$$

$$= \frac{\beta^2 E_o^2}{2h^4 \eta_{TM}} \frac{ab}{4} \left[\frac{4\pi^2}{a^2} + \frac{9\pi^2}{b^2} \right] = \frac{\beta^2 E_o^2 ab}{8h^2 \eta_{TM}}$$

But

$$\beta = \frac{\omega}{c} \sqrt{1 - (f_c/f)^2} = \frac{10^{12}}{3 \times 10^8} \sqrt{1 - (15.81/159.2)^2} = 3.317 \times 10^3$$

$$h^2 = \frac{4\pi^2}{a^2} + \frac{9\pi^2}{b^2} = \frac{10\pi^2}{b^2} = 1.097 \times 10^5$$

$$P_{ave} = \frac{(3.317)^2 \times 10^6 \times 5^2 \times 18 \times 10^{-4}}{8 \times (1.098 \times 10^5) \times 375.1} = \underline{\underline{1.5 \text{ mW}}}$$

Prob. 12.14 (a) Since $m=2$ and $n=1$, we have TE₂₁ mode

$$(b) \beta = \beta' \sqrt{1 - (f_c/f)^2} = \omega \sqrt{\mu_0 \epsilon_0} \sqrt{1 - (\omega_c/\omega)^2}$$

$$\beta c = \sqrt{\omega^2 - \omega_c^2} \quad \longrightarrow \quad \omega_c^2 = \sqrt{\omega^2 - \beta^2 c^2}$$

$$f_c = \frac{\omega_c}{2\pi} = \sqrt{f^2 - \frac{\beta^2 c^2}{4\pi^2}} = \sqrt{36 \times 10^{18} - \frac{144 \times 9 \times 10^{16}}{4\pi^2}} = \underline{\underline{5.973 \text{ GHz}}}$$

$$(c) \eta_{TE} = \frac{\eta}{\sqrt{1 - (f_c/f)^2}} = \frac{377}{\sqrt{1 - (5.973/6)^2}} = \underline{\underline{3978 \Omega}}$$

(d) For TE mode,

$$E_y = \frac{\omega \mu}{h^2} (m\pi/a) H_o \sin(m\pi x/a) \cos(n\pi y/b) \sin(\omega t - \beta z)$$

$$H_x = \frac{-\beta}{h^2} (m\pi/a) H_o \sin(m\pi x/a) \cos(n\pi y/b) \sin(\omega t - \beta z)$$

$$\beta = 12, m = 2, n = 1$$

$$E_{oy} = \frac{\omega \mu}{h^2} (m\pi/a) H_o, \quad H_{ox} = \frac{\beta}{h^2} (m\pi/a) H_o$$

$$\eta_{TE} = \frac{E_{oy}}{H_{ox}} = \frac{\omega \mu}{\beta} = \frac{2\pi \times 6 \times 10^9 \times 4\pi \times 10^{-7}}{12} = 4\pi^2 \times 100$$

$$H_{ox} = \frac{E_{oy}}{\eta_{TE}} = \frac{5}{4\pi^2 \times 100} = 1.267 \text{ mA/m}$$

$$\underline{\underline{H_x = -1.267 \sin(m\pi x/a) \cos(n\pi y/b) \sin(\omega t - \beta z) \text{ mA/m}}}$$

Prob. 12.15 (a) Since $m=2, n=3$, the mode is TE₂₃.

$$(b) \quad \beta = \beta' \sqrt{1 - (f_c / f)^2} = \frac{2\pi f}{c} \sqrt{1 - (f_c / f)^2}$$

But

$$f_c = \frac{u'}{2} \sqrt{(m/a)^2 + (n/b)^2} = \frac{3 \times 10^8}{2 \times 10^{-2}} \sqrt{(2/2.86)^2 + (3/1.016)^2} = 46.19 \text{ GHz}, \quad f = 50 \text{ GHz}$$

$$\beta = \frac{2\pi \times 50 \times 10^9}{3 \times 10^8} \sqrt{1 - (46.19/50)^2} = 400.68 \text{ rad/m}$$

$$\gamma = j\beta = j\underline{\underline{400.7}} \text{ /m}$$

$$(c) \quad \eta = \frac{\eta'}{\sqrt{1 - (f_c/f)^2}} = \frac{377}{\sqrt{1 - (46.19/50)^2}} = \underline{\underline{\underline{985.3\Omega}}}$$

Prob. 12.16 In free space,

$$\eta_1 = \frac{\eta_o}{\sqrt{1 - (f_c/f)^2}}, \quad f_c = \frac{c}{2a} = \frac{3 \times 10^8}{2 \times 5 \times 10^{-2}} = 3 \text{ GHz}$$

$$\eta_1 = \frac{377}{\sqrt{1 - (3/8)^2}} = 406.7\Omega$$

$$\eta_2 = \frac{\eta'_1}{\sqrt{1 - (f_c/f)^2}}, \quad \eta' = \frac{120\pi}{\sqrt{2.25}} = 80\pi, \quad f_c = \frac{u'}{2a}, \quad u' = \frac{c}{\sqrt{\epsilon_r}}$$

$$f_c = \frac{3 \times 10^8}{2 \times 5 \times 10^{-2} \sqrt{2.25}} = 2 \text{ GHz}, \quad \eta_2 = \frac{80\pi}{\sqrt{1 - (2/8)^2}} = 259.57\Omega$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = -0.2208$$

$$s = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \underline{\underline{\underline{1.5667}}}$$

Prob. 12.17 Substituting $E_z = R\Phi Z$ into the wave equation,

$$\frac{\Phi Z}{\rho} \frac{d}{d\rho} (\rho R') + \frac{RZ}{\rho^2} \Phi'' + R\Phi Z'' + k^2 R\Phi Z = 0$$

Dividing by $R\Phi Z$,

$$\frac{1}{R\rho} \frac{d}{d\rho} (\rho R') + \frac{\Phi''}{\Phi\rho^2} + k^2 = -\frac{Z''}{Z} = -k_z^2$$

i.e. $\underline{\underline{Z'' - k_z^2 Z = 0}}$

$$\frac{1}{R\rho} \frac{d}{d\rho} (\rho R') + \frac{\Phi''}{\Phi\rho^2} + (k^2 + k_z^2) = 0$$

$$\frac{\rho}{R} \frac{d}{d\rho} (\rho R') + (k^2 + k_z^2) \rho^2 = -\frac{\Phi''}{\Phi} = k_\phi^2$$

or

$$\underline{\underline{\Phi'' + k_\phi^2 \Phi = 0}}$$

$$\rho \frac{d}{d\rho} (\rho R') + (k_\rho^2 \rho^2 - k_\phi^2) R = 0, \text{ where } k_\rho^2 = k^2 + k_z^2. \text{ Hence}$$

$$\underline{\underline{\rho^2 R'' + \rho R' + (k_\rho^2 \rho^2 - k_\phi^2) R = 0}}$$

Prob. 12.18

(a)

For TE₁₀ mode,

$$f_c = \frac{u'}{2a} = \frac{3 \times 10^8}{2 \times 7.2 \times 10^{-2}} = 2.083 \text{ GHz}$$

$$\text{Let } F = \sqrt{1 - \left(\frac{f_c}{f}\right)^2} = \sqrt{1 - \left(\frac{2.083}{6.2}\right)^2} = 0.942$$

$$\beta = \omega \sqrt{\mu \epsilon} F = \frac{\omega F}{c} = \frac{2\pi \times 6.2 \times 10^9 \times 0.942}{3 \times 10^8} = \underline{\underline{122.32 \text{ rad/m}}}$$

$$u_p = \frac{\omega}{\beta} = \frac{c}{F} = \frac{3 \times 10^8}{0.942} = \underline{\underline{3.185 \times 10^8 \text{ m/s}}}$$

$$u_g = u' F = 3 \times 10^8 (0.942) = \underline{\underline{2.826 \times 10^8 \text{ m/s}}}$$

$$\eta_{TE} = \frac{\eta'}{F} = \frac{377}{0.942} = \underline{\underline{400.21 \Omega}}$$

(b)

$$u' = \frac{1}{\sqrt{\mu \epsilon}} = \frac{c}{\sqrt{\epsilon_r}} = \frac{3 \times 10^8}{\sqrt{2.25}} = 2 \times 10^8$$

$$f_c = \frac{u'}{2a} = \frac{2 \times 10^8}{2 \times 7.2 \times 10^{-2}} = 1.389 \text{ GHz}$$

$$\text{Let } F = \sqrt{1 - \left(\frac{f_c}{f}\right)^2} = \sqrt{1 - \left(\frac{1.389}{6.2}\right)^2} = 0.9746$$

$$\beta = \omega \sqrt{\mu \epsilon} F = \frac{\omega F \sqrt{\epsilon_r}}{c} = \frac{2\pi \times 6.2 \times 10^9 \times 0.9746 \times 1.5}{3 \times 10^8} = \underline{\underline{189.83 \text{ rad/m}}}$$

$$u_p = \frac{\omega}{\beta} = \frac{2\pi \times 6.2 \times 10^9}{189.83} = \underline{\underline{2.052 \times 10^8 \text{ m/s}}}$$

$$u_g = u' F = 2 \times 10^8 (0.9746) = \underline{\underline{1.949 \times 10^8 \text{ m/s}}}$$

$$\eta_{TE} = \frac{\eta'}{F} = \frac{377}{1.5 \times 0.9746} = \underline{\underline{257.88 \Omega}}$$

Prob. 12.19

$$f_{c10} = \frac{u'}{2a} = \frac{\frac{1}{\sqrt{\mu\epsilon}}}{2a} = \frac{c}{2a\sqrt{\epsilon_r}} = \frac{3 \times 10^8}{2 \times 7.214 \times 10^{-2} \sqrt{2.5}} = 1.315 \text{ GHz}$$

$$\text{Let } F = \sqrt{1 - \left(\frac{f_c}{f}\right)^2} = \sqrt{1 - \left(\frac{1.315}{4}\right)^2} = 0.9444$$

$$\beta = \omega \sqrt{\mu\epsilon} F = \frac{\omega \sqrt{\epsilon_r} F}{c}$$

$$u_p = \frac{\omega}{\beta} = \frac{c}{F\sqrt{\epsilon_r}} = \frac{3 \times 10^8}{0.9444 \times \sqrt{2.5}} = \underline{\underline{2.009 \times 10^8 \text{ m/s}}}$$

$$u_g = u'F = \frac{cF}{\sqrt{\epsilon_r}} = \frac{3 \times 10^8 \times 0.9444}{\sqrt{2.5}} = \underline{\underline{1.792 \times 10^8 \text{ m/s}}}$$

Prob. 12.20

$$f_c = \frac{u'}{2a}$$

$$u_g = u' \sqrt{1 - \left(\frac{f_c}{f}\right)^2} \quad \longrightarrow \quad \left(\frac{f_c}{f}\right)^2 = 1 - \left(\frac{u_g}{u'}\right)^2 = 1 - \left(\frac{1.8 \times 10^8}{3 \times 10^8}\right)^2 = 0.208$$

$$f_c = \sqrt{0.208} f = 2.0523 \text{ GHz}$$

$$a = \frac{u'}{2f_c} = \frac{3 \times 10^8}{2\sqrt{2.2} \times 2.053 \times 10^9} = \underline{\underline{4.927 \text{ cm}}}$$

Prob. 12.21

$$\text{Let } F = \sqrt{1 - (f_c/f)^2} = \sqrt{1 - (16/24)^2} = 0.7453$$

$$u' = \frac{1}{\sqrt{\mu\epsilon}} = \frac{3 \times 10^8}{\sqrt{2.25}} = 2 \times 10^8, \quad u_p = \frac{u'}{F}, \quad u_g = u'F = 2 \times 10^8 \times 0.7453 = \underline{\underline{1.491 \times 10^8 \text{ m/s}}}$$

$$\eta_{TE} = \eta'/F = \frac{377}{1.5 \times 0.7453} = \underline{\underline{337.2 \Omega}}$$

Prob. 12.22

$$f_c = \frac{3 \times 10^8}{2} \sqrt{(m/0.025)^2 + (n/0.01)^2} = 15 \sqrt{n^2 + (m/2.5)^2} \text{ GHz}$$

$f_{c10} = 6 \text{ GHz}$, $f_{c20} = 12 \text{ GHz}$, $f_{c01} = 15 \text{ GHz}$.

Since $f_{c20}, f_{c10} > 11 \text{ GHz}$, only the dominant TE_{10} mode is propagated.

$$(a) \frac{u_p}{u} = \frac{1}{\sqrt{1 - (f_c/f)^2}} = \frac{1}{\sqrt{1 - (6/11)^2}} = \underline{\underline{1.193}}$$

$$(b) \frac{u_g}{u} = \sqrt{1 - (6/11)^2} = \underline{\underline{0.8381}}$$

Prob. 12.23

For the TE_{10} mode,

$$H_{zs} = H_o \cos\left(\frac{\pi x}{a}\right) e^{-j\beta z}$$

$$H_{xs} = \frac{j\beta a}{\pi} H_o \sin\left(\frac{\pi x}{a}\right) e^{-j\beta z}$$

$$E_{ys} = -\frac{j\omega\mu a}{\pi} H_o \sin\left(\frac{\pi x}{a}\right) e^{-j\beta z}$$

$$E_{xs} = 0 = E_{zs} = H_{ys}$$

$$\mathbf{E}_s \times \mathbf{H}_s^* = \begin{vmatrix} 0 & E_{ys} & 0 \\ H_{xs}^* & 0 & H_{zs}^* \end{vmatrix} = E_{ys} H_{zs}^* \mathbf{a}_x - E_{ys} H_{xs}^* \mathbf{a}_z$$

$$= -\frac{j\omega\mu a}{\pi} H_o^2 \cos\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi x}{a}\right) \mathbf{a}_x + \frac{\omega\mu\beta a^2}{\pi^2} H_o^2 \sin^2\left(\frac{\pi x}{a}\right) \mathbf{a}_z$$

$$\mathcal{P}_{ave} = \frac{1}{2} \operatorname{Re} [\mathbf{E}_s \times \mathbf{H}_s^*] = \underline{\underline{\frac{\omega\mu\beta a^2}{2\pi^2} H_o^2 \sin^2\left(\frac{\pi x}{a}\right) \mathbf{a}_z}}$$

Prob. 12.24

$$\mathbf{P}_{ave} = \frac{|E_{xs}|^2 + |E_{ys}|^2}{2\eta} \mathbf{a}_z = \underline{\underline{\frac{\omega^2\mu^2\pi^2}{2\eta b^2 h^4} H_o^2 \sin^2 \pi y/b \mathbf{a}_z}}$$

where $\eta = \eta_{TE10}$.

$$P_{ave} = \int \mathbf{P}_{ave} \cdot d\mathbf{S} = \frac{\omega^2\mu^2\pi^2}{2\eta b^2 h^4} H_o^2 \int_{x=0}^a \int_{y=0}^b \sin^2 \pi y/b dx dy$$

$$P_{ave} = \underline{\underline{\frac{\omega^2\mu^2\pi^2}{2\eta b^2 h^4} H_o^2 ab / 2}}$$

But $h^2 = (m\pi/a)^2 + (n\pi/b)^2 = \frac{\pi^2}{b^2}$,

$$\underline{P_{ave} = \frac{\omega^2 \mu^2 ab^3 H_o^2}{4\pi^2 \eta}}$$

Prob. 12.25

$$R_s = \sqrt{\frac{\pi \mu f}{\sigma_c}} = \sqrt{\frac{\pi \times 12 \times 10^9 \times 4\pi \times 10^{-7}}{5.8 \times 10^7}} = 2.858 \times 10^{-2}$$

$$f_{c10} = \frac{u'}{2a} = \frac{3 \times 10^8}{2\sqrt{2.6} \times 2 \times 10^{-2}} = 4.651 \text{ GHz}$$

$$f_{c11} = \frac{u'}{2} \left[\frac{1}{a^2} + \frac{1}{b^2} \right]^{1/2} = 10.4 \text{ GHz}$$

$$\eta' = \sqrt{\frac{\mu}{\epsilon}} = \frac{377}{\sqrt{2.6}} = 233.81 \Omega$$

(a) For TE₁₀ mode, eq.(12.57) gives

$$\alpha_d + j\beta_d = \sqrt{-\omega^2 \mu \epsilon + k_x^2 + k_y^2 + j\omega \mu \sigma_d}$$

$$= \sqrt{-\omega^2 / u^2 + \frac{\pi^2}{a^2} + j\omega \mu \sigma_d}$$

$$= \sqrt{-\left(\frac{2\pi \times 12 \times 10^9}{3 \times 10^8}\right)^2 (2.6) + \frac{\pi^2}{(2 \times 10^{-2})^2} + j2\pi \times 12 \times 10^9 \times 4\pi \times 10^{-7} \times 10^{-4}}$$

$$= 0.012682 + j373.57$$

$$\underline{\alpha_d = 0.012682 \text{ Np/m}}$$

$$\alpha_c = \frac{2R_s}{b\eta' \sqrt{1 - (f_c/f)^2}} \left[\frac{1}{2} + \frac{b}{a} \left(\frac{f_c}{f} \right)^2 \right]$$

$$= \frac{2 \times 2.858 \times 10^{-2}}{10^{-2} (233.81) \sqrt{1 - (4.651/12)^2}} \left[\frac{1}{2} + \frac{1}{2} \left(\frac{4.651}{12} \right)^2 \right] = \underline{0.0153 \text{ Np/m}}$$

(b) For TE₁₁ mode,

$$\begin{aligned}\alpha_d + j\beta_d &= \sqrt{-\omega^2 / u^2 + 1 / a^2 + 1 / b^2 + j\omega\mu\sigma_d} \\ &= \sqrt{-139556.21 + \frac{\pi^2}{(10^{-2})^2} + j9.4748} = 0.02344 + j202.14\end{aligned}$$

$$\underline{\underline{\alpha_d = 0.02344 \text{ Np/m}}}$$

$$\alpha_c = \frac{2R_s}{b\eta' \sqrt{1 - (f_c/f)^2}} \left[\frac{(b/a)^3 + 1}{(b/a)^2 + 1} \right] = \frac{2 \times 2.858 \times 10^{-2}}{10^{-2} (233.81) \sqrt{1 - (10.4/12)^2}} \left[\frac{(1/8) + 1}{(1/4) + 1} \right]$$

$$\underline{\underline{\alpha_c = 0.0441 \text{ Np/m}}}$$

Prob. 12.26 $\epsilon_c = \epsilon' - j\epsilon'' = \epsilon - j\frac{\sigma}{\omega}$

Comparing this with

$$\epsilon_c = 16\epsilon_o(1 - j10^{-4}) = 16\epsilon_o - j16\epsilon_o \times 10^{-4}$$

$$\epsilon = 16\epsilon_o, \quad \frac{\sigma}{\omega} = 16\epsilon_o \times 10^{-4}$$

For TM₂₁ mode,

$$f_c = \frac{u'}{2} \left[\frac{m^2}{a^2} + \frac{n^2}{b^2} \right]^{1/2} = 2.0963 \text{ GHz}, \quad f = 1.1f_c = 2.3059 \text{ GHz}$$

$$\sigma = 16\epsilon_o \omega \times 10^{-4} = 16 \times 2\pi \times 2.3059 \times 10^9 \times \frac{10^{-9}}{36\pi} \times 10^{-4} = 2.0525 \times 10^{-4}$$

$$\eta' = \sqrt{\frac{\mu}{\epsilon}} = 30\pi$$

$$\alpha_d = \frac{\sigma\eta'}{2\sqrt{1 - (f_c/f)^2}} = \frac{4.1 \times 10^{-4} \times 30\pi}{2\sqrt{1 - 1/1.12}} = \underline{\underline{\underline{\underline{\alpha_d = 0.0231 \text{ Np/m}}}}}$$

$$E_o e^{-\alpha_d z} = 0.8 E_o \quad \longrightarrow \quad z = \frac{1}{\alpha_d} \ln(1/0.8) = \underline{\underline{\underline{\underline{9.66 \text{ m}}}}}$$

Prob. 12.27For TM₂₁ mode,

$$\alpha_c = \frac{2R_s}{b\eta' \sqrt{1 - (f_c/f)^2}}$$

$$R_s = \frac{1}{\sigma_c \delta} = \sqrt{\frac{\pi f \mu}{\sigma_c}} = \sqrt{\frac{\pi \times 2.3059 \times 10^9 \times 4\pi \times 10^{-7}}{1.5 \times 10^7}} = 0.0246$$

$$\alpha_c = \frac{2 \times 0.0246}{4\pi \times 10^{-2} \times 30\pi \times 0.4166} = 0.0314 \text{ Np/m}$$

$$E_o e^{-(\alpha_c + \alpha_d)z} = 0.7 E_o \quad \longrightarrow \quad z = \frac{1}{\alpha_c + \alpha_d} \ln(1/0.7) = \underline{\underline{6.5445 \text{ m}}}$$

Prob. 12.28

$$f_{c10} = \frac{u'}{2a} = \frac{3 \times 10^8}{2 \times 6 \times 10^{-2}} = 2.5 \text{ GHz}$$

$$\eta_{TE} = \frac{\eta'}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}} = \frac{377}{\sqrt{1 - \left(\frac{2.5}{4}\right)^2}} = 483$$

From Example 12.5,

$$P_{ave} = \frac{E_o^2 ab}{2\eta} = \frac{(2.2)^2 \times 10^6 \times 6 \times 3 \times 10^{-4}}{2 \times 483} = \underline{\underline{9.0196 \text{ mW}}}$$

Prob. 12.29 For TE₁₀ mode,

$$f_c = \frac{u'}{2a} = \frac{3 \times 10^8}{2\sqrt{2.11} \times 4.8 \times 10^{-2}} = 2.151 \text{ GHz}$$

$$(a) \text{ loss tangent } = \frac{\sigma}{\omega\epsilon} = d$$

$$\sigma = d\omega\epsilon = 3 \times 10^{-4} \times 2\pi \times 4 \times 10^9 \times 2.11 \times \frac{10^{-9}}{36\pi} = 1.4086 \times 10^{-4}$$

$$\eta' = \frac{120\pi}{\sqrt{2.11}} = 259.53$$

$$\alpha_d = \frac{\sigma\eta'}{2\sqrt{1-(f_c/f)^2}} = \frac{1.4067 \times 10^{-4} \times 259.53}{2\sqrt{1-(2.151/4)^2}} = \underline{\underline{2.165 \times 10^{-2} \text{ Np/m}}}$$

$$(b) R_s = \sqrt{\frac{\mu f \pi}{\sigma_c}} = \sqrt{\frac{\pi \times 4 \times 10^9 \times 4\pi \times 10^{-7}}{4.1 \times 10^7}} = 1.9625 \times 10^{-2}$$

$$\begin{aligned} \alpha_c &= \frac{2R_s}{b\eta' \sqrt{1-(f_c/f)^2}} \left[\frac{1}{2} + \frac{b}{a} \left(\frac{f_c}{f} \right)^2 \right] = \frac{3.925 \times 10^{-2} (0.5 + 0.5 \times 0.2892)}{2.4 \times 10^{-2} \times 259.53 \times 0.8431} \\ &= \underline{\underline{4.818 \times 10^{-3} \text{ Np/m}}} \end{aligned}$$

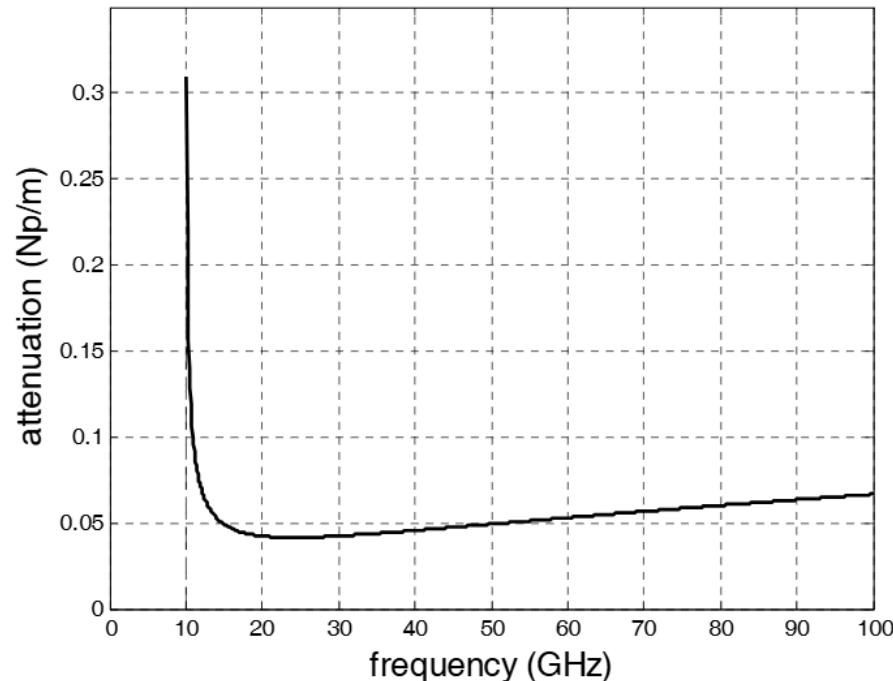
Prob. 12.30

$$\begin{aligned} \alpha_c &= \frac{2R_s}{b\eta' \sqrt{1-\left(\frac{f_c}{f}\right)^2}} \left[\frac{1}{2} + \frac{b}{a} \left(\frac{f_c}{f} \right)^2 \right] = \frac{2\sqrt{\frac{\pi f \mu}{\sigma_c}}}{b\eta' \sqrt{1-\left(\frac{f_c}{f}\right)^2}} \left[\frac{1}{2} + \frac{1}{2} \left(\frac{f_c}{f} \right)^2 \right] \\ &= \frac{2\sqrt{4\pi \times 10^{-7} \times \pi} \sqrt{f} \times \frac{1}{2}}{0.5 \times 10^{-2} \times (120\pi / \sqrt{2.25}) \sqrt{5.8 \times 10^7} \sqrt{1-\left(\frac{f_c}{f}\right)^2}} \left[1 + \left(\frac{f_c}{f} \right)^2 \right] \\ &= \frac{10^{-5} \sqrt{f}}{30\sqrt{(5.8/2.25)} \sqrt{1-\left(\frac{f_c}{f}\right)^2}} \left[1 + \left(\frac{f_c}{f} \right)^2 \right] \end{aligned}$$

The MATLAB code is shown below

```
k=10^(-5)/(30*sqrt(5.8/2.25));
fc=10^10;
for n=1:1000
    f(n)=fc*(n/100+1);
    fn=f(n);
    num=sqrt(fn)*(1 +(fc/fn)^2);
    den=sqrt(1- (fc/fn)^2);
    alpha(n) =k*num/den;
end
plot(f/10^9,alpha)
xlabel('frequency (GHz)')
ylabel('attenuation')
grid
```

The plot of attenuation versus frequency is shown below.



Prob. 12.31 (a) For TE₁₀ mode,

$$f_c = \frac{u'}{2a}, \quad u' = \frac{c}{\sqrt{2.11}}$$

$$f_c = \frac{3 \times 10^8}{\sqrt{2.11}(2 \times 2.25 \times 10^{-2})} = \underline{\underline{4.589 \text{ GHz}}}$$

$$(b) \quad \alpha_{cTE10} = \frac{2R_s}{b\eta' \sqrt{1 - (f_c/f)^2}} \left[\frac{1}{2} + \frac{b}{a} (f_c/f)^2 \right]$$

$$R_s = \sqrt{\frac{\pi f \mu}{\sigma_c}} = \sqrt{\frac{\pi \times 5 \times 10^9 \times 4\pi \times 10^{-7}}{1.37 \times 10^7}} = 3.796 \times 10^{-2}$$

$$\eta' = \frac{377}{\sqrt{2.11}} = 259.54$$

$$\alpha_c = \frac{2 \times 3.796 \times 10^{-2} [0.5 + \frac{1.5}{2.25} (4.589 / 5)^2]}{1.5 \times 10^{-4} (259.54) \sqrt{1 - (4.589 / 5)^2}} = \underline{\underline{0.05217 \text{ Np/m}}}$$

Prob. 12.32

$$(a) f_{c10} = \frac{u'}{2a} = \frac{3 \times 10^8}{2 \times 3.8 \times 10^{-2}} = 3.947 \text{ GHz}$$

$$u_g = u' \sqrt{1 - \left(\frac{f_c}{f} \right)^2} = 3 \times 10^8 \sqrt{1 - (0.3947)^2} = \underline{\underline{2.756 \times 10^8 \text{ m/s}}}$$

$$(b) \alpha = \alpha_d + \alpha_c$$

$\alpha_d = 0$ since the guide is air-filled.

$$R_s = \sqrt{\frac{\pi f \mu}{\sigma_c}} = \sqrt{\frac{\pi \times 10^{10} \times 4\pi \times 10^{-7}}{5.8 \times 10^7}} = 2.609 \times 10^{-2} \Omega$$

$$\alpha_c = \frac{2R_s}{b\eta' \sqrt{1 - \left(\frac{f_c}{f} \right)^2}} \left[0.5 + \frac{b}{a} \left(\frac{f_c}{f} \right)^2 \right]$$

$$= \frac{2 \times 2.609 \times 10^{-2}}{1.6 \times 10^{-2} (377) \sqrt{1 - (0.3947)^2}} \left[0.5 + \frac{1.6}{3.8} (0.3947)^2 \right] = \frac{5.218 \times 0.5656}{554.23}$$

$$= \underline{\underline{5.325 \times 10^{-3} \text{ Np/m}}}$$

$$\alpha_c (dB) = 8.686 \times 5.325 \times 10^{-3} = 0.04626 \text{ dB/m}$$

Prob. 12.33

$$f_{c10} = \frac{u'}{2a} = \frac{c}{2a\sqrt{\epsilon_r \mu_r}} = \frac{3 \times 10^8}{2 \times 2.5 \times 10^{-2} \sqrt{2.26}} = 3.991 \text{ GHz}$$

$$\beta' = \frac{\omega}{u'} = \frac{2\pi f \sqrt{\epsilon_r}}{c}$$

$$F = \sqrt{1 - \left(\frac{f_c}{f} \right)^2} = \sqrt{1 - \left(\frac{3.991}{7.5} \right)^2} = 0.8467$$

$$\beta = \beta' F = \frac{2\pi \times 7.5 \times 10^9 \sqrt{2.26}}{3 \times 10^8} 0.8467 = \underline{\underline{199.94 \text{ rad/m}}}$$

$$\alpha_d = \frac{\sigma\eta'}{2F} = \frac{\sigma\eta_o}{2F\sqrt{\epsilon_r}} = \frac{10^{-4}(377)}{2 \times 0.8467\sqrt{2.26}} = \underline{\underline{1.481 \times 10^{-2} \text{ Np/m}}}$$

$$\alpha_c = \frac{2R_s}{b\eta' \sqrt{1 - \left(\frac{f_c}{f}\right)^2}} \left[0.5 + \frac{b}{a} \left(\frac{f_c}{f} \right)^2 \right]$$

$$R_s = \sqrt{\frac{\pi f \mu}{\sigma_c}} = \sqrt{\frac{\pi \times 7.5 \times 10^9 \times 4\pi \times 10^{-7}}{1.1 \times 10^7}} = 0.0519$$

$$\alpha_c = \frac{2 \times 0.0519 \left[0.5 + \frac{1.5}{2.5} \left(\frac{3.991}{7.5} \right)^2 \right]}{1.5 \times 10^{-2} \times \frac{377}{\sqrt{2.26}} \times 0.8467} = \frac{0.1038 \times 0.6698}{3.1848}$$

$$= \underline{\underline{0.02183 \text{ Np/m}}}$$

$$u_p = \frac{u'}{F} = \frac{c}{F\sqrt{\epsilon_r}} = \frac{3 \times 10^8}{0.8467\sqrt{2.26}} = \underline{\underline{2.357 \times 10^8 \text{ m/s}}}$$

$$u_g = u'F = \frac{3 \times 10^8 \times 0.8467}{\sqrt{2.26}} = \underline{\underline{1.689 \times 10^8 \text{ m/s}}}$$

$$\lambda_c = \frac{u'}{f_c} = \frac{c}{f_c\sqrt{\epsilon_r}} = \frac{3 \times 10^8}{3.991 \times 10^9 \sqrt{2.26}} = 0.05 \text{ m} = \underline{\underline{5 \text{ cm}}} (= 2a, \text{ as expected})$$

Prob. 12.34

The cutoff frequency of the dominant mode is

$$f_{c10} = \frac{u}{2a} = \frac{3 \times 10^8}{4.576 \times 10^{-2}} = 6.56 \text{ GHz}$$

The surface resistance is

$$R_s = \sqrt{\frac{\pi f \mu}{\sigma_c}} = \sqrt{\frac{\pi \times 8.4 \times 10^9 \times 4\pi \times 10^{-7}}{5.8 \times 10^7}} = 23.91 \times 10^{-3}$$

For TE₁₀ mode,

$$\alpha_c = \frac{2R_s}{b\eta' \sqrt{1 - \left(\frac{f_c}{f}\right)^2}} \left[0.5 + \frac{b}{a} \left(\frac{f_c}{f} \right)^2 \right]$$

$$\frac{f_c}{f} = \frac{6.56}{8.4} = 0.781, \quad \eta' = \eta_o = 377$$

$$\begin{aligned}\alpha_c &= \frac{2 \times 23.91 \times 10^{-3}}{1.016 \times 10^{-2} \times 377 \sqrt{1 - 0.781^2}} \left[0.5 + \frac{1.016}{2.286} (0.781)^2 \right] \\ &= \frac{47.82 \times 10^{-3} (0.5 + 0.2711)}{3.83 \times 0.6245} = 15.42 \times 10^{-3} \text{ Np/m} \\ &= 15.42 \times 10^{-3} \times 8.686 \text{ dB/m} = \underline{\underline{0.1339 \text{ dB/m}}}\end{aligned}$$

Prob. 12.35 For TE₁₀ mode,

$$\alpha_c = \frac{2R_s}{b\eta' \sqrt{1 - (f_c/f)^2}} \left[\frac{1}{2} + \frac{b}{a} \left(\frac{f_c}{f} \right)^2 \right]$$

$$\text{But } a = b, \quad R_s = \frac{1}{\sigma_c \delta} = \sqrt{\frac{\pi f \mu}{\sigma_c}}$$

$$\alpha_c = \frac{2 \sqrt{\frac{\pi f \mu}{\sigma_c}}}{a\eta' \sqrt{1 - (f_c/f)^2}} \left[\frac{1}{2} + \left(\frac{f_c}{f} \right)^2 \right] = \frac{k \sqrt{f} \left[\frac{1}{2} + \left(\frac{f_c}{f} \right)^2 \right]}{\sqrt{1 - (f_c/f)^2}}$$

where k is a constant.

$$\frac{d\alpha_c}{df} = \frac{k [1 - (\frac{f_c}{f})^2]^{1/2} [\frac{1}{4} f^{-1/2} - \frac{3}{2} f_c^2 f^{-5/2}] - \frac{k}{2} [\frac{1}{2} f^{1/2} + f_c^2 f^{-3/2}] (2 f_c^2 f^{-3}) [1 - (\frac{f_c}{f})^2]^{-1/2}}{1 - (f_c/f)^2}$$

For minimum value, $\frac{d\alpha_c}{df} = 0$. This leads to $f = 2.962 f_c$.

Prob. 12.36 For the TE mode to z,

$$E_{zs} = 0, H_{zs} = H_o \cos(m\pi x/a) \cos(n\pi y/b) \sin(p\pi z/c)$$

$$E_{ys} = -\frac{\gamma}{h^2} \frac{\partial E_{zs}}{\partial y} + \frac{j\omega\mu}{h^2} \frac{\partial H_{zs}}{\partial x} = -\frac{j\omega\mu}{h^2} (m\pi/a) H_o \sin(m\pi x/a) \cos(n\pi y/b) \sin(p\pi z/c)$$

as required.

$$E_{xs} = -\frac{\gamma}{h^2} \frac{\partial E_{zs}}{\partial x} - \frac{j\omega\mu}{h^2} \frac{\partial H_{zs}}{\partial y} = \frac{j\omega\mu}{h^2} (n\pi/b) H_o \cos(m\pi x/a) \sin(n\pi y/b) \sin(p\pi z/c)$$

From Maxwell's equation,

$$-j\omega\mu H_s = \nabla \times E_s = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_{xs} & E_{ys} & 0 \end{vmatrix}$$

$$H_{xs} = \frac{1}{j\omega\mu} \frac{\partial E_{ys}}{\partial z} = -\frac{1}{h^2} (m\pi/a)(p\pi/c) H_o \sin(m\pi x/a) \cos(n\pi y/b) \cos(p\pi z/c)$$

Prob. 12.37 Maxwell's equation can be written as

$$H_{xs} = \frac{j\omega\epsilon}{h^2} \frac{\partial E_{zs}}{\partial y} - \frac{\gamma}{h^2} \frac{\partial H_{zs}}{\partial x}$$

For a rectangular cavity,

$$h^2 = k_x^2 + k_y^2 = (m\pi/a)^2 + (n\pi/b)^2$$

For TM mode, $H_{zs} = 0$ and

$$E_{zs} = E_o \sin(m\pi x/a) \sin(n\pi y/b) \cos(p\pi z/c)$$

Thus

$$H_{xs} = \frac{j\omega\epsilon}{h^2} \frac{\partial E_{zs}}{\partial y} = \frac{j\omega\epsilon}{h^2} (n\pi/b) E_o \sin(m\pi x/a) \cos(n\pi y/b) \cos(p\pi z/c)$$

as required.

$$\begin{aligned} H_{xs} &= -\frac{j\omega\epsilon}{h^2} \frac{\partial E_{zs}}{\partial x} - \frac{\gamma}{h^2} \frac{\partial H_{zs}}{\partial y} \\ &= -\frac{j\omega\epsilon}{h^2} (m\pi/a) E_o \cos(m\pi x/a) \sin(n\pi y/b) \cos(p\pi z/c) \end{aligned}$$

From Maxwell's equation,

$$j\omega\epsilon E_s = \nabla \times H_s = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_{xs} & H_{ys} & 0 \end{vmatrix}$$

$$E_{ys} = \frac{1}{j\omega\epsilon} \frac{\partial H_{xs}}{\partial z} = \frac{1}{h^2} (n\pi/b)(p\pi/c) E_o \sin(m\pi x/a) \cos(n\pi y/b) \cos(p\pi z/c)$$

Prob. 12.38

$$f_r = \frac{u'}{2} \sqrt{(m/a)^2 + (n/b)^2 + (p/c)^2}$$

where for TM mode to z, m = 1, 2, 3, ..., n=1, 2, 3,, p = 0, 1, 2,

and for TE mode to z, m = 0,1, 2, 3, ..., n=0,1, 2, 3,, p = 1, 2, 3, ..., (m + n) ≠ 0.

(a) If a < b < c, 1/a > 1/b > 1/c,

The lowest TM mode is TM₁₁₀ with $f_r = \frac{u'}{2} \sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$

The lowest TE mode is TE₀₁₁ with $f_r = \frac{u'}{2} \sqrt{\frac{1}{b^2} + \frac{1}{c^2}} < \frac{u'}{2} \sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$

Hence the dominant mode is TE₀₁₁.

(b) If a > b > c, 1/a < 1/b < 1/c,

The lowest TM mode is TM₁₁₀ with $f_r = \frac{u'}{2} \sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$

The lowest TE mode is TE₁₀₁ with $f_r = \frac{u'}{2} \sqrt{\frac{1}{a^2} + \frac{1}{c^2}} > \frac{u'}{2} \sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$

Hence the dominant mode is TM₁₁₀.

(c) If a = c > b, 1/a = 1/c < 1/b,

The lowest TM mode is TM₁₁₀ with $f_r = \frac{u'}{2} \sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$

The lowest TE mode is TE₁₀₁ with $f_r = \frac{u'}{2} \sqrt{\frac{1}{a^2} + \frac{1}{c^2}} < \frac{u'}{2} \sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$

Hence the dominant mode is TE₁₀₁.

Prob. 12.39

(a) Since $a > b < c$, the dominant mode is TE_{101} .

$$f_r = \frac{c}{2} \sqrt{\frac{1}{a^2} + \frac{1}{c^2}} = \frac{3 \times 10^8}{2} \sqrt{\frac{1}{(2 \times 10^{-2})^2} + \frac{1}{(3 \times 10^{-2})^2}} = \underline{\underline{9.014 \text{ GHz}}}$$

$$(b) \quad \delta = \frac{1}{\sqrt{\pi f_{r101} \mu_0 \sigma}} = \frac{1}{\sqrt{\pi \times 9.014 \times 10^9 \times 4\pi \times 10^{-7} \times 5.8 \times 10^7}} \\ = \frac{1}{\sqrt{4\pi^2 \times 5.8 \times 9.014 \times 10^9}}$$

$$Q = \frac{(a^2 + c^2)abc}{\delta [2b(a^3 + c^3) + ac(a^2 + c^2)]} = \frac{(4+9)6 \times 10^{-2}}{\delta [2(8+27) + 6(4+9)]} = \frac{0.78}{148\delta} \\ = \frac{0.78}{148} \sqrt{4\pi^2 \times 5.8 \times 9.014 \times 10^9} = \underline{\underline{7571.5}}$$

Prob. 12.40

$$u' = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{\sqrt{2.5}} = \frac{3 \times 10^8}{\sqrt{2.5}} = 1.897 \times 10^8$$

$$f_c = \frac{u'}{2} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{c}\right)^2} = \frac{1.897 \times 10^8 \times 10^2}{2} \sqrt{\left(\frac{m}{1}\right)^2 + \left(\frac{n}{2}\right)^2 + \left(\frac{p}{3}\right)^2} \\ = 9.485 \sqrt{m^2 + 0.25n^2 + 0.111p^2} \text{ GHz}$$

$$f_{r101} = 9.485 \sqrt{1+0+0.111} = 10 \text{ GHz}$$

$$f_{r011} = 9.485 \sqrt{0+0.25+0.111} = 5.701 \text{ GHz}$$

$$f_{r012} = 9.485 \sqrt{0+0.25+0.444} = 7.906 \text{ GHz}$$

$$f_{r013} = 9.485 \sqrt{0+0.25+0.999} = 10.61 \text{ GHz}$$

$$f_{r021} = 9.485 \sqrt{0+1+0.111} = 10 \text{ GHz}$$

Thus, the first five resonant frequencies are:

5.701 GHz (TE_{011})

7.906 GHz (TE_{012})

10 GHz (TE_{101} and TE_{021})

10.61 GHz (TE_{013} or TM_{110})

11.07 GHz (TE_{111} or TM_{111})

Prob. 12.41

$$Q = \frac{(a^2 + c^2)abc}{\delta [2b(a^3 + c^3) + ac(a^2 + c^2)]}$$

When $a = b = c$,

$$Q = \frac{2a^2a^3}{\delta [2a \times 2a^3 + a^2 \times 2a^2]} = \frac{2a^5}{6\delta a^4} = \underline{\underline{\frac{a}{3\delta}}}$$

Prob. 12.42

(a) Since $a > b < c$, the dominant mode is TE_{101}

$$f_{r101} = \frac{u'}{2} \sqrt{\frac{1}{a^2} + 0 + \frac{1}{c^2}} = \frac{3 \times 10^8 \times 10^2}{2} \sqrt{\frac{1}{2^2} + \frac{1}{1^2}} = \underline{\underline{16.77 \text{ GHz}}}$$

$$\begin{aligned} \text{(b)} \quad Q_{\text{TE}101} &= \frac{(a^2 + c^2)abc}{\delta [2b(a^3 + c^3) + ac(a^2 + c^2)]} \\ &= \frac{(400 + 100)20 \times 8 \times 10 \times 10^{-3}}{\delta [16(8000 + 1000) + 200(400 + 100)]} = \frac{3.279 \times 10^{-3}}{\delta} \end{aligned}$$

$$\text{But } \delta = \frac{1}{\sqrt{\pi f_{r101} \mu_o \sigma}} = \frac{1}{\sqrt{\pi 16.77 \times 10^9 \times 4\pi \times 10^{-7} \times 6.1 \times 10^7}} = \frac{10^{-4}}{200.961}$$

$$Q_{\text{TE}101} = 3.279 \times 10^{-3} \frac{200.961}{10^{-4}} = \underline{\underline{6589.51}}$$

Prob. 12.43

$$f_r = \frac{c}{2a} \sqrt{m^2 + n^2 + p^2}$$

The lowest possible modes are TE_{101} , TE_{011} , and TM_{110} . Hence

$$f_r = \frac{c}{2a} \sqrt{2} \longrightarrow a = \frac{c}{f_r \sqrt{2}} = \frac{3 \times 10^8}{\sqrt{2} \times 3 \times 10^9} = 7.071 \text{ cm}$$

$$\underline{\underline{a = b = c = 7.071 \text{ cm}}}$$

Prob. 12.44(a) $a = b = c$

$$f_r = \frac{u'}{2a} \sqrt{m^2 + n^2 + p^2}$$

For the dominant mode TE_{101} ,

$$f_r = \frac{u'}{2a} \sqrt{1+1} = \frac{c}{2a} \sqrt{2}$$

$$a = \frac{c\sqrt{2}}{2f_r} = \frac{3 \times 10^8 \sqrt{2}}{2 \times 5.6 \times 10^9} = 0.03788 \text{ m}$$

$$\underline{\underline{a = b = c = 3.788 \text{ cm}}}$$

(b)

$$\text{For } \epsilon_r = 2.05, \quad u' = \frac{c}{\sqrt{\epsilon_r}}$$

$$a = \frac{c\sqrt{2}}{2f_r\sqrt{\epsilon_r}} = \frac{0.03788}{\sqrt{2.05}} = 0.02646$$

$$\underline{\underline{a = b = c = 2.646 \text{ cm}}}$$

Prob. 12.45

(a)

This is a TM mode to z. From Maxwell's equations,

$$\nabla \times \mathbf{E}_s = -j\omega\mu\mathbf{H}_s$$

$$\mathbf{H}_s = -\frac{1}{j\omega\mu} \nabla \times \mathbf{E}_s = \frac{j}{\omega\mu} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & E_{zs}(x,y) \end{vmatrix} = \frac{j}{\omega\mu} \left(\frac{\partial E_{zs}}{\partial y} \mathbf{a}_x - \frac{\partial E_{zs}}{\partial x} \mathbf{a}_y \right)$$

But

$$E_{zs} = 200 \sin 30\pi x \sin 30\pi y, \quad \frac{1}{\omega\mu} = \frac{1}{6 \times 10^9 \times 4\pi \times 10^{-7}} = \frac{10^{-2}}{24\pi}$$

$$\mathbf{H}_s = \frac{j10^{-2}}{24\pi} \times 200 \times 30\pi \{ \sin 30\pi x \cos 30\pi y \mathbf{a}_x - \cos 30\pi x \sin 30\pi y \mathbf{a}_y \}$$

$$\mathbf{H} = \text{Re} (\mathbf{H}_s e^{j\omega t})$$

$$\mathbf{H} = 2.5 \{ -\sin 30\pi x \cos 30\pi y \mathbf{a}_x + \cos 30\pi x \sin 30\pi y \mathbf{a}_y \} \sin 6 \times 10^9 \pi t \text{ A/m}$$

$$(b) \quad \mathbf{E} = E_z \mathbf{a}_z, \quad \mathbf{H} = H_x \mathbf{a}_x + H_y \mathbf{a}_y$$

$$\mathbf{E} \cdot \mathbf{H} = 0$$

Prob. 12.46

$$(a) \quad a = b = c \quad \longrightarrow \quad f_{r101} = \frac{3 \times 10^8}{a\sqrt{2}} = 12 \times 10^9$$

$$a = \frac{3 \times 10^8}{\sqrt{2} \times 12 \times 10^9} = \underline{\underline{1.77 \text{ cm}}}$$

$$(b) \quad Q_{TE101} = \frac{a}{3\delta} = \frac{a\sqrt{\pi f_{r101} \mu \sigma}}{3}$$

$$= \frac{1.77 \times 10^{-2} \sqrt{\pi \times 12 \times 10^9 \times 4\pi \times 10^{-7} \times 5.8 \times 10^7}}{3} = \underline{\underline{9767.61}}$$

Prob. 12.47

$$f_r = \frac{u'}{2} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{c}\right)^2}$$

$$f_{r101} = \frac{3 \times 10^8}{2} \sqrt{\frac{1}{(10.2)^2} + \frac{1}{(3.6)^2}} = 44.186 \text{ MHz}$$

$$f_{r011} = 150 \sqrt{\frac{1}{(8.7)^2} + \frac{1}{(3.6)^2}} \text{ MHz} = 45.093 \text{ MHz}$$

$$f_{r111} = 150 \sqrt{\frac{1}{(10.2)^2} + \frac{1}{(8.7)^2} + \frac{1}{(3.6)^2}} \text{ MHz} = 47.43 \text{ MHz}$$

$$f_{r110} = 150 \sqrt{\frac{1}{(10.2)^2} + \frac{1}{(8.7)^2}} \text{ MHz} = 22.66 \text{ MHz}$$

$$f_{r102} = 150 \sqrt{\frac{1}{(10.2)^2} + \frac{4}{(3.6)^2}} \text{ MHz} = 84.62 \text{ MHz}$$

$$f_{r201} = 150 \sqrt{\frac{4}{(10.2)^2} + \frac{1}{(3.6)^2}} \text{ MHz} = 51 \text{ MHz}$$

Thus, the resonant frequencies below 50 MHz are

$f_{r110}, f_{r101}, f_{r011}, \text{ and } f_{r111}$

Prob. 12.48

$$n = c/u_m = \frac{3 \times 10^8}{2.1 \times 10^8} = \underline{\underline{1.4286}}$$

Prob. 12.49 When an optical fiber is used as the transmission medium, cable radiation is eliminated. Thus, optical fibers offer total EMI isolation because they neither emit nor pick up EM waves.

Prob. 12.50

$$(a) \text{NA} = \sqrt{n_1^2 - n_2^2} = \sqrt{1.62^2 - 1.604^2} = \underline{0.2271}$$

$$(b) \text{NA} = \sin \theta_a = 0.2271 \text{ or } \theta_a = \sin^{-1} 0.2271 = \underline{13.13^\circ}$$

$$(c) V = \frac{\pi d}{\lambda} \text{NA} = \frac{\pi \times 50 \times 10^{-6} \times 0.2271}{1300 \times 10^{-9}} = 27.441$$

$$N = V^2/2 \quad \underline{6 \text{ modes}}$$

Prob. 12.51

$$V = \frac{\pi d}{\lambda} \sqrt{n_1^2 - n_2^3} = \frac{\pi \times 2 \times 5 \times 10^{-6}}{1300 \times 10^{-9}} \sqrt{1.48^2 - 1.46^2} = 5.86$$

$$N = \frac{V^2}{2} = 17.17 \text{ or } \underline{\underline{17 \text{ modes}}}$$

Prob. 12.52

$$(a) \text{NA} = \sin \theta_a = \sqrt{n_1^2 - n_2^2} = \sqrt{1.53^2 - 1.45^2} = 0.4883 \\ \theta_a = \sin^{-1} 0.4883 = \underline{29.23^\circ}$$

$$(b) P(l)/P(0) = 10^{-\alpha l / 10} = 10^{-0.4 \times 5 / 10} = 0.631$$

$$\text{i.e. } \underline{63.1 \%}$$

Prob. 12.53

$$P(l) = P(0) 10^{-\alpha l / 10} = 10 \times 10^{-0.5 \times 0.85 / 10} = \underline{\underline{9.0678 \text{ mW}}}$$

Prob. 12.54 As shown in Eq. (10.35), $\log_{10} P_1/P_2 = 0.434 \ln P_1/P_2$,

$$1 \text{ Np} = 20 \log_{10} e = 8.686 \text{ dB} \text{ or } 1 \text{ Np/km} = 8.686 \text{ dB/km},$$

or $1 \text{ Np/m} = 8686 \text{ dB/km}$. Thus,

$$\alpha_{12} = \underline{\underline{8686 \alpha_{10}}}$$

Prob. 12.55

$$\alpha\ell = 10 \log_{10} \frac{P_{in}}{P_{out}} = 10 \log_{10} \frac{1.2 \times 10^{-3}}{1 \times 10^{-6}} = 30.792$$

$$\alpha = 0.4 \text{ dB/km} = \frac{0.4}{8.686} \text{ Np/km}$$

$$\ell = \frac{30.792}{\alpha} = \frac{30.792 \text{ dB}}{0.4 \text{ dB/km}} = \underline{\underline{76.98 \text{ km}}}$$

CHAPTER 13

P.E. 13.1

(a) For this case, r is at near field.

$$H_{\phi s} = \frac{I_o dl \sin \theta}{4\pi} \left(\frac{j\beta}{r} + \frac{1}{r^2} \right) e^{-j\beta r}, \quad \beta r = \frac{2\pi}{\lambda} \cdot \frac{\lambda}{5} = 72^\circ$$

$$\lambda = \frac{2\pi c}{\omega} = \frac{2\pi \times 3 \times 10^8}{10^8} = 6\pi, \quad \beta = \frac{2\pi}{\lambda} = \frac{1}{3}$$

$$H_{\phi s} = \frac{(0.25) \frac{6\pi}{100} \sin 30^\circ}{4\pi} \left(\frac{j1/3}{6\pi/5} + \frac{1}{(6\pi/5)^2} \right) e^{-j72^\circ} = 0.2119 \angle -20.511^\circ \text{ mA/m}$$

$$\underline{\underline{\boldsymbol{H}}} = \text{Im} (H_{\phi s} e^{j\omega t} \boldsymbol{a}_\phi) \quad \text{Im is used since } I = I_o \sin \omega t$$

$$= 0.2119 \sin(10^8 - 20.5^\circ) \boldsymbol{a}_\phi \text{ mA/m}$$

(b) For this case, r is at far field. $\beta = \frac{2\pi}{\lambda} \times 200\lambda = 0^\circ$

$$H_{\phi s} = \frac{j(0.25) \left(\frac{2\pi}{\lambda} \right) \frac{\lambda}{100} \sin 60^\circ e^{-j0^\circ}}{4\pi (6\pi \times 200)} = 0.2871 e^{j90^\circ} \mu Am$$

$$\underline{\underline{\boldsymbol{H}}} = \text{Im} (H_{\phi s} \boldsymbol{a}_\phi e^{j\omega t}) = 0.2871 \sin(10^8 + 90^\circ) \boldsymbol{a}_\phi \mu Am.$$

P. E. 13.2

$$(a) l = \frac{\lambda}{4} = \underline{\underline{1.5m}},$$

$$(b) I_o = \underline{\underline{83.3mA}}$$

$$(c) R_{rad} = 36.56 \Omega, P_{rad} = \frac{1}{2} (0.0833)^2 36.56$$

$$= \underline{\underline{126.8 \text{ mW}}}.$$

$$(d) Z_L = 36.5 + j21.25,$$

$$\Gamma = \frac{36.5 + j21.25 - 75}{36.5 + j21.25 + 75} = 0.3874 \angle 140.3^\circ$$

$$s = \frac{1 + 0.3874}{1 - 0.3874} = \underline{\underline{2.265}}$$

P.E. 13.3

$$D = \frac{4\pi U_{\max}}{P_{rad}}$$

(a) For the Hertzian monopole

$$U(\theta, \phi) = \sin^2 \theta, \quad 0 < \theta < \pi/2, \quad 0 < \phi < 2\pi, \quad U_{\max} = 1$$

$$P_{rad} = \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} \sin^2 \theta d\theta d\phi = \frac{4\pi}{3}$$

$$D = \frac{4\pi(1)}{4\pi/3} = \underline{\underline{3}}$$

(b) For the $\frac{\lambda}{4}$ monopole,

$$U(\theta, \phi) = \frac{\cos^2(\frac{\pi}{2} \cos \theta)}{\sin^2 \theta}, \quad U_{\max} = 1$$

$$P_{rad} = \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} \frac{\cos^2(\frac{\pi}{2} \cos \theta)}{\sin^2 \theta} \sin \theta d\theta d\phi = 2\pi(0.609)$$

$$D = \frac{4\pi(1)}{2\pi(0.609)} = \underline{\underline{3.28}}$$

P. E. 13.4

(a) $P_{rad} = \eta_r P_{in} = 0.95(0.4)$

$$D = \frac{4\pi U_{\max}}{P_{rad}} = \frac{4\pi(0.5)}{0.4 \times 0.95} = \underline{\underline{16.53}}$$

$$(b) D = \frac{4\pi(0.5)}{0.3} = \underline{\underline{20.94}}$$

P. E. 13.5

$$P_{rad} = \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} \sin \theta \sin \theta d\theta d\phi = \frac{\pi^2}{2}, \quad U_{max} = 1$$

$$D = \frac{4\pi(1)}{\pi^2/2} = \underline{\underline{2.546}}$$

P. E. 13.6

(a) $f(\theta) = |\cos \theta| \cos \left[\frac{1}{2} (\beta d \cos \theta + \alpha) \right]$

where $\alpha = \pi, \beta d = \frac{2\pi}{\lambda} \cdot \frac{\lambda}{2} = \pi$

$$f(\theta) = |\cos \theta| \cos \left[\frac{1}{2} (\pi \cos \theta + \pi) \right]$$

↓ ↓
unit pattern group pattern

For the group pattern, we have nulls at

$$\frac{\pi}{2} (\cos \theta + 1) = \pm \frac{\pi}{2} \longrightarrow \theta = \pm \frac{\pi}{2}$$

and maxima at

$$\frac{\pi}{2} (\cos \theta + 1) = 0, \pi \longrightarrow \cos \theta = -1, 1 \longrightarrow \theta = 0, \pi$$

Thus the group pattern and the resultant patterns are as shown in Fig.13.15(a)

(b) $f(\theta) = |\cos \theta| \cos \left[\frac{1}{2} (\beta d \cos \theta + \alpha) \right]$

where $\alpha = -\frac{\pi}{2}, \beta d = \pi / 2$

$$f(\theta) = |\cos \theta| \cos \left[\frac{1}{2} \left(\frac{\pi}{2} \cos \theta - \frac{\pi}{2} \right) \right]$$

↓ ↓
unit pattern group pattern

For the group pattern, the nulls are at

$$\frac{\pi}{4}(\cos \theta - 1) = -\frac{\pi}{2} \quad \rightarrow \quad \theta = 180^\circ$$

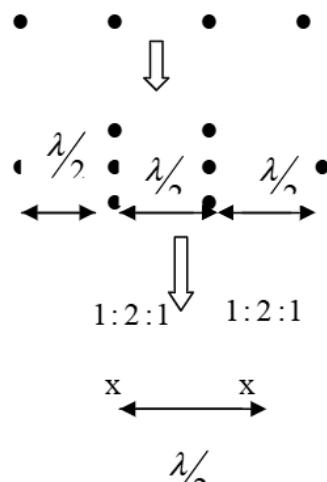
and maxima at

$$\cos \theta - 1 = 0 \quad \rightarrow \quad \theta = 0$$

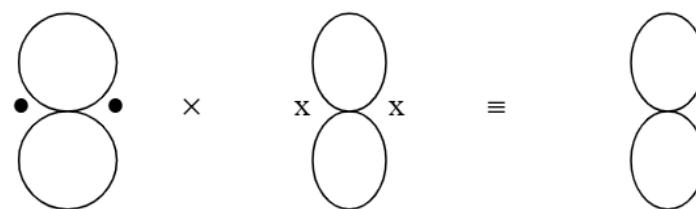
Thus the group pattern and the resultant patterns are as shown in Fig.13.15(b)

P. E. 13.7

(a)



Thus, we take a pair at a time and multiply the patterns as shown below.



(b) The group pattern is the normalized array factor, i.e.

$$(AF)_n = \frac{1}{\sum} \left| 1 + Ne^{j\psi} + \frac{N(N-1)}{2!} e^{j2\psi} + \frac{N(N-1)(N-2)}{3!} e^{j3\psi} + \dots + e^{j(N-1)\psi} \right|$$

$$\text{where } \sum = \sum_{i=1}^{N-1} \binom{N}{i} = 1 + N + \frac{N-1}{2!} + \frac{N(N-1)(N-2)}{3!} + \dots$$

$$= (1+1)^{N-1} = 2^{N-1}$$

$$(AF)_n = \frac{1}{2^{N-1}} \left| 1 + e^{j\psi} \right|^{N-1} = \frac{1}{2^{N-1}} \left| e^{\frac{j\psi}{2}} \left| e^{-\frac{j\psi}{2}} + e^{\frac{j\psi}{2}} \right|^{N-1} \right|$$

$$= \frac{1}{2^{N-1}} \left| 2 \cos \frac{\psi}{2} \right|^{N-1} = \underline{\underline{\left| \cos \frac{\psi}{2} \right|^{N-1}}}$$

P. E. 13.8

$$A_e = \frac{\lambda^2}{4\pi} G_d, \quad \lambda = \frac{c}{f} = \frac{3 \times 10^8}{10^8} = 3 \text{ m}$$

For the Hertzian dipole,

$$\begin{aligned} G_d &= 1.5 \sin^2 \theta \\ A_e &= \frac{\lambda^2}{4\pi} (1.5 \sin^2 \theta) \\ A_{e,\max} &= \frac{1.5 \lambda^2}{4\pi} = \frac{1.5 \times 9}{4\pi} = \underline{\underline{1.074 \text{ m}^2}} \end{aligned}$$

By definition,

$$\begin{aligned} P_r = A_e P_{ave} \longrightarrow P_{ave} &= \frac{P_r}{A_e} = \frac{3 \times 10^{-6}}{1.074} \\ &= \underline{\underline{2.793 \mu\text{W/m}^2}} \end{aligned}$$

P. E. 13.9

$$\begin{aligned} \text{(a)} \quad G_d &= \frac{4\pi r^2 P_{ave}}{P_{rad}} = \frac{4\pi r^2 \frac{1}{2} \frac{E^2}{\eta}}{P_{rad}} = \frac{2\pi r^2 E^2}{\eta P_{rad}} \\ &= \frac{2\pi \times 400 \times 10^6 \times 144 \times 10^{-6}}{120\pi \times 100 \times 10^3} = 0.0096 \end{aligned}$$

$$G = 10 \log_{10} G_d = \underline{\underline{-20.18 \text{ dB}}}$$

$$\text{(b)} \quad G = \eta_r G_d = 0.98 \times 0.0096 = \underline{\underline{9.408 \times 10^{-3}}}$$

P. E. 13.10

$$r = \left[\frac{\lambda^2 G_d^2 \sigma}{(4\pi)^3} \frac{P_{rad}}{P_r} \right]^{1/4}$$

$$\text{where } \lambda = \frac{c}{f} = \frac{3 \times 10^8}{6 \times 10^9} = 0.05 \text{ m}$$

$$A_e = 0.7\pi a^2 = 0.7\pi(1.8)^2 = 7.125 m^2$$

$$G_d = \frac{4\pi A_e}{\lambda^2} = \frac{4\pi(7.125)}{25 \times 10^{-4}} = 3.581 \times 10^4$$

$$r = \left[\frac{25 \times 10^{-4} \times (3.581)^2 \times 10^8 \times 5 \times 60 \times 10^3}{(4\pi)^3 \times 0.26 \times 10^{-3}} \right]^{1/4}$$

$$= 1168.4 \text{m} = \underline{\underline{0.631 \text{nm}}}$$

At $r = \frac{r_{\max}}{2} = 584.2 \text{m}$,

$$P = \frac{G_d P_{rad}}{4\pi r^2} = \frac{3.581 \times 10^4 \times 60 \times 10^3}{4\pi(584.2)^2} = \underline{\underline{501 \text{W/m}^2}}$$

Prob. 13.1

Using vector transformation,

$$A_{rs} = A_{xs} \sin \theta \cos \phi, \quad A_{\theta s} = A_{xs} \cos \theta \cos \phi, \quad A_{\phi s} = -A_{xs} \sin \phi$$

$$A_s = \frac{50 e^{-j\beta r}}{r} (\sin \theta \cos \phi \mathbf{a}_r + \cos \theta \cos \phi \mathbf{a}_\theta - \sin \phi \mathbf{a}_\phi)$$

$$\begin{aligned} \frac{\nabla \times \mathbf{A}_s}{\mu} &= \mathbf{H}_s = \frac{100 \cos \theta \sin \phi}{\mu r^2 \sin \theta} e^{-j\beta r} \mathbf{a}_r - \frac{50}{\mu r^2} (1 - j\beta r) \sin \phi e^{-j\beta r} \mathbf{a}_\theta \\ &\quad - \frac{50}{\mu r^2} \cos \theta \cos \phi (1 + j\beta r) e^{-j\beta r} \mathbf{a}_\phi \end{aligned}$$

At far field, only $\frac{1}{r}$ term remains. Hence

$$\mathbf{H}_s = \frac{j50}{\mu r} \beta e^{-j\beta r} (\sin \phi \mathbf{a}_\theta - \cos \theta \cos \phi \mathbf{a}_\phi)$$

$$\mathbf{E}_s = -\eta \mathbf{a}_r \times \mathbf{H}_s = \frac{-j50 \beta \eta e^{-j\beta r}}{\mu r} (\sin \phi \mathbf{a}_\phi + \cos \theta \cos \phi \mathbf{a}_\theta)$$

$$\mathbf{E} = \operatorname{Re}[\mathbf{E}_s e^{j\omega t}] = \frac{50\eta\beta}{\mu r} \sin(\omega t - \beta r) \underline{\underline{\sin \phi \mathbf{a}_\phi + \cos \theta \cos \phi \mathbf{a}_\theta}} \text{ V/m}$$

$$\mathbf{H} = \operatorname{Re}[\mathbf{H}_s e^{j\omega t}] = \frac{-50}{\mu r} \beta \sin(\omega t - \beta r) \underline{\underline{\sin \phi \mathbf{a}_\theta - \cos \theta \cos \phi \mathbf{a}_\phi}} \text{ A/m}$$

Prob. 13.2

$$(a) \quad \lambda = \frac{c}{f} = \frac{3 \times 10^8}{400 \times 10^6} = 0.75 \text{ m}$$

$$r_{\min} = \frac{2d^2}{\lambda} = \frac{2(0.02\lambda)^2}{\lambda} \ll r$$

i.e. r is in the far field.

$$H_{\phi s} = \frac{jI_o \beta dl}{4\pi r} \sin \theta e^{-j\beta r}$$

$$|H_{\phi s}| = \frac{I_o \beta dl}{4\pi r} \sin \theta = \frac{3 \times \frac{2\pi}{\lambda} \times 0.02\lambda \times \sin 90^\circ}{4\pi(60)} = 5 \times 10^{-4} = 0.5 \text{ mA/m}$$

$$|E_{\theta s}| = \eta_o |H_{\phi s}| = 0.1885 \text{ V/m}$$

$$(b) \quad |H_{\phi s}| = 0.5 \text{ mA/m}$$

$$(c) \quad R_{\text{rad}} = 80\pi^2 \left(\frac{dl}{\lambda} \right)^2 = 80\pi^2 (0.02)^2 = 0.3158 \Omega$$

$$(d) \quad P_{\text{rad}} = \frac{1}{2} |I_o|^2 R_{\text{rad}} = \frac{1}{2} (9)(0.3158) = 1.421 \text{ W}$$

Prob. 13.4

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{50 \times 10^6} = 6 \text{ m}$$

$$r_{\min} = \frac{2d^2}{\lambda} = \frac{2(5 \times 10^{-3})^2}{6} = 8.333 \times 10^{-6} \text{ m} \ll r = 15 \text{ cm}$$

i.e. r is in the far field.

$$H_{\phi s} = \frac{jI_o \beta dl}{4\pi r} \sin \theta e^{-j\beta r}$$

$$|H_{\phi s}| = \frac{I_o \beta dl}{4\pi r} \sin \theta = \frac{\frac{2\pi}{6} \times 5 \times 10^{-3} \times \sin 30^\circ}{4\pi \times 15 \times 10^{-2}} = \frac{5}{18} \times 10^{-2} = \underline{\underline{2.778 \text{ mA/m}}}$$

$$|E_{\theta s}| = \eta_o |H_{\phi s}| = 377 \times 2.778 \times 10^{-3} = \underline{\underline{1.047 \text{ V/m}}}$$

Prob. 13.5

$$(a) A_{zs} = \frac{e^{-j\beta r}}{4\pi r} \int_{-\frac{l}{2}}^{\frac{l}{2}} I_o \left(1 - \frac{2|z|}{l}\right) e^{j\beta z \cos \theta} dz$$

$$= \frac{e^{-j\beta r}}{4\pi r} I_o \left[\int_{-\frac{l}{2}}^{\frac{l}{2}} \left(1 - \frac{2|z|}{l}\right) \cos(\beta z \cos \theta) dz + j \int_{-\frac{l}{2}}^{\frac{l}{2}} \left(1 - \frac{2|z|}{l}\right) \sin(\beta z \cos \theta) dz \right]$$

$$= \frac{e^{-j\beta r}}{4\pi r} 2I_o \int_0^{\frac{l}{2}} \left(1 - \frac{2z}{l}\right) \cos(\beta z \cos \theta) dz$$

$$= \frac{I_o e^{-j\beta r}}{2\pi r \beta^2 \cos^2 \theta} \cdot \frac{2}{l} \left[1 - \cos\left(\frac{\beta l}{2} \cos \theta\right) \right]$$

$$E_s = -j\omega \mu A_s \longrightarrow E_{\theta s} = j\omega \mu \sin \theta A_{zs} = j\beta \eta \sin \theta A_{zs}$$

$$E_{\theta s} = \frac{j\eta I_o e^{-j\beta r}}{\pi r l} \frac{\sin \theta \left[1 - \cos\left(\frac{\beta l}{2} \cos \theta\right) \right]}{\beta \cos^2 \theta}$$

$$\text{If } \frac{\beta l}{2} \ll 1, \cos\left(\frac{\beta l}{2} \cos \theta\right) = 1 - \frac{(\frac{\beta l}{2} \cos \theta)^2}{2!}.$$

Hence

$$E_{\theta s} = \frac{j\eta I_o}{8\pi r} \beta l e^{-j\beta r} \sin \theta, \quad H_{\phi s} = E_{\theta s} / \eta$$

$$P_{ave} = \frac{|E_{\theta s}|^2}{2\eta}, \quad P_{rad} = \int P_{ave} dS$$

$$P_{rad} = \int_0^{2\pi} \int_0^{\pi} \frac{n}{2} \left(\frac{I_o \beta l}{8\pi} \right)^2 \frac{1}{r^2} \sin^2 \theta r^2 \sin \theta d\theta d\phi$$

$$= 10\pi^2 I_o^2 \left(\frac{l}{\lambda} \right)^2 = \frac{1}{2} I_o^2 R_{rad}$$

$$\text{or } R_{rad} = 20\pi^2 \left(\frac{l}{\lambda} \right)^2$$

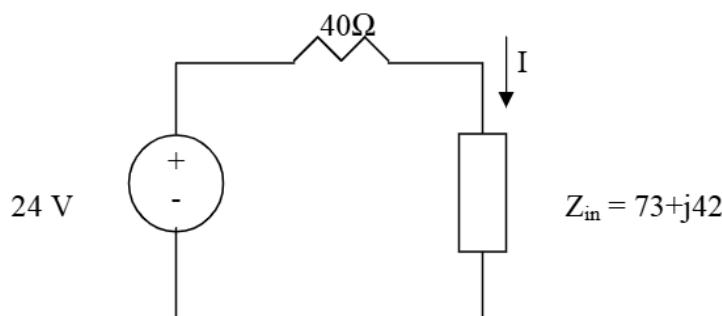
$$(b) \quad 0.5 = 20\pi^2 \left(\frac{l}{\lambda}\right)^2 \rightarrow l = 0.05\lambda$$

Prob. 13.6

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{3 \times 10^6} = 100 \text{ m}$$

$\ell = 4 \text{ m} \ll \lambda$. Hence the antenna is electrically short.

$$P_{rad} = \frac{1}{2} I_o^2 R_{rad} = 40 \left(\frac{\pi I_o d l}{\lambda} \right)^2 = 40 \left(\frac{\pi \times 3 \times 4}{100} \right)^2 = 5.685 \text{ W}$$

Prob. 13.8

$$I = \frac{V}{R_s + Z_{in}} = \frac{24}{40 + 73 + j42} = 0.1866 - j0.0694$$

$$P_{rad} = \frac{1}{2} |I|^2 R_{rad}, \quad R_{rad} = 73$$

$$P_{rad} = \frac{1}{2} (0.1991)^2 \times 73 = 1.447 \text{ W}$$

Prob. 13.9

Change the limits in Eq. (13.16) to $\pm l/2$ i.e.

$$\begin{aligned} A_z &= \frac{\mu I_o e^{j\beta z \cos \theta}}{4\pi r} \left. (j\beta \cos \theta \cos \beta t + \beta \sin \beta t) \right|_{-\frac{l}{2}}^{\frac{l}{2}} \\ &= \frac{\mu I_o e^{j\beta r}}{2\pi r} \frac{1}{\beta \sin^2 \theta} \left[\sin \frac{\beta l}{2} \cos \left(\frac{\beta l}{2} \cos \theta \right) - \cos \theta \cos \frac{\beta l}{2} \sin \left(\frac{\beta l}{2} \cos \theta \right) \right] \end{aligned}$$

But $\mathbf{B} = \mu \mathbf{H} = \nabla \times \mathbf{A}$

$$H_{\phi s} = \frac{1}{\mu r} \left[\frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right],$$

where $A_\theta = -A_z \sin \theta, A_r = A_z \cos \theta$

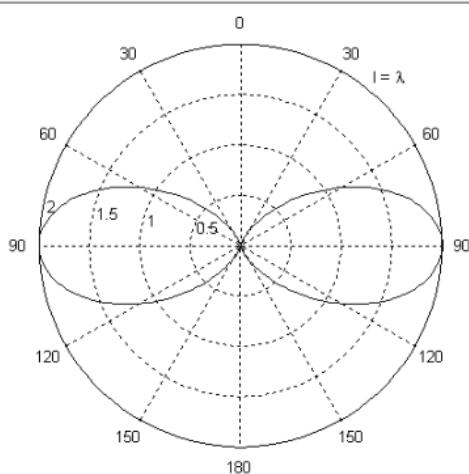
$$H_{\phi s} = \frac{I_o}{2\pi r} \frac{e^{-j\beta r}}{\beta} \left(\frac{j\beta}{\sin \theta} \right) \left[\sin \frac{\beta l}{2} \cos \left(\frac{\beta l}{2} \cos \theta \right) - \cos \theta \cos \frac{\beta l}{2} \sin \left(\frac{\beta l}{2} \cos \theta \right) \right] + \frac{I_o}{2\pi r^2} e^{-j\beta r} (\dots)$$

For far field, only the $\frac{1}{r}$ -term remains. Hence

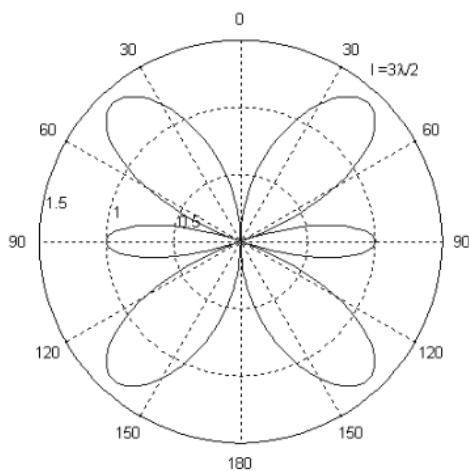
$$H_{\phi s} = \frac{j I_o}{2\pi r} e^{-j\beta r} \frac{\left[\sin \frac{\beta l}{2} \cos \left(\frac{\beta l}{2} \cos \theta \right) - \cos \theta \cos \frac{\beta l}{2} \sin \left(\frac{\beta l}{2} \cos \theta \right) \right]}{\sin \theta}$$

$$(b) f(\theta) = \frac{\cos \left(\frac{\beta l}{2} \cos \theta \right) - \cos \frac{\beta l}{2}}{\sin \theta}$$

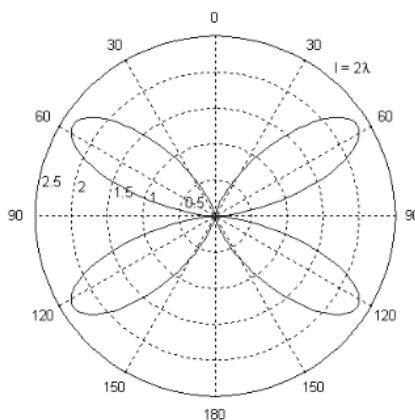
$$\text{For } l = \lambda, f(\theta) = \frac{\cos(\pi \cos \theta) + 1}{\sin \theta}$$



$$\text{For } l = \frac{3\lambda}{2}, f(\theta) = \frac{\cos\left(\frac{3\pi}{2}\cos\theta\right)}{\sin\theta}$$



$$\text{For } l = 2\lambda, f(\theta) = \frac{\cos\theta\sin(2\pi\cos\theta)}{\sin\theta}$$

**Prob. 13.10**

(a)

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{450 \times 10^6} = 0.6667 \text{ m}$$

$$\ell = \frac{\lambda}{2} = \underline{\underline{0.333 \text{ m}}}$$

(b)

$$\frac{\sigma}{\omega\epsilon} = \frac{4}{2\pi \times 450 \times 10^6 \times 81 \times \frac{10^{-9}}{36\pi}} = 1.975$$

$$\begin{aligned} \beta &= \omega \sqrt{\frac{\mu\epsilon}{2} \left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon} \right)^2} + 1 \right]} = \frac{2\pi \times 460 \times 10^6}{c} \sqrt{\frac{81}{2} \left[\sqrt{1 + (1.975)^2} + 1 \right]} \\ &= \frac{2\pi \times 460 \times 10^6}{3 \times 10^8} \times 11.4086 = 109.91 \end{aligned}$$

$$\lambda = \frac{2\pi}{\beta} = 0.0572$$

$$\ell = \frac{\lambda}{2} = \underline{\underline{28.58 \text{ mm}}}$$

Prob. 13.11

(a)

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{1.150 \times 10^6} = 260.8 \text{ m}$$

$$\ell = \frac{\lambda}{4} = \underline{\underline{65.22 \text{ m}}}$$

(b)

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{90 \times 10^6} = 3.333 \text{ m}$$

$$\ell = \frac{\lambda}{4} = \underline{\underline{0.8333 \text{ m}}}$$

(c)

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{80 \times 10^6} = 3.75 \text{ m}$$

$$\ell = \frac{\lambda}{4} = \underline{\underline{0.9375 \text{ m}}}$$

(d)

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{600 \times 10^6} = 0.5 \text{ m}$$

$$\ell = \frac{\lambda}{4} = \underline{\underline{0.125 \text{ m}}}$$

Prob. 13.13

$$(a) \quad \lambda = \frac{c}{f} = \frac{3 \times 10^8}{10 \times 10^6} = 30 \text{ m}$$

$$E_{\max} = \frac{\eta \pi I_o S}{r \lambda^2} \longrightarrow I_o = \frac{E_{\max} r \lambda^2}{\eta \pi S}$$

$$I_o = \frac{50 \times 10^{-3} \times 3 \times 30^2}{120 \pi^2 \pi (0.2)^2 100} = \underline{\underline{9.071 \text{ mA}}} \quad (\text{S} = \text{N} \pi r^2)$$

$$(b) \quad R_{rad} = \frac{320 \pi^4 S^2}{\lambda^4} = \frac{320 \pi^4 \pi^2 (0.2)^4 \times 10^4}{30^4} = 6.077 \Omega$$

$$P_{rad} = \frac{1}{2} I_o^2 R_{rad} = \frac{1}{2} (9.071)^2 \times 10^{-6} \times 6.077$$

$$= \underline{\underline{0.25 \text{ mW}}}$$

Prob. 13.14

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{80 \times 10^6} = 3.75 \text{ m}$$

$$S = N\pi\rho_o^2$$

$$R_{rad} = \frac{320\pi^4 S^2}{\lambda^4} = \frac{320\pi^4 N^2 \pi^2 \rho_o^4}{\lambda^4} \quad \longrightarrow \quad N^2 = \frac{\lambda^4 R_{rad}}{320\pi^6 (1.2 \times 10^{-2})^4}$$

$$N^2 = \frac{(3.75)^4 \times 8}{320\pi^6 (1.2 \times 10^{-2})^4} = 248006 \quad \longrightarrow \quad N \simeq \underline{\underline{498}}$$

Prob. 13.15

$$(a) \quad R_{rad} = \frac{320\pi^4 S^2}{\lambda^4}$$

$$S = \pi\rho_o^2 = \pi(0.4)^2 = 0.5027 \text{ m}^2$$

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{6 \times 10^6} = 50 \text{ m}$$

$$R_{rad} = \frac{320\pi^4 (0.5027)^2}{(50)^4} = \underline{\underline{1.26 \text{ m}\Omega}}$$

$$(b) \quad P_{rad} = \frac{1}{2} I_o^2 R_{rad} = \frac{1}{2} (50)^2 \times 1.26 \times 10^{-3} = \underline{\underline{1.575 \text{ W}}}$$

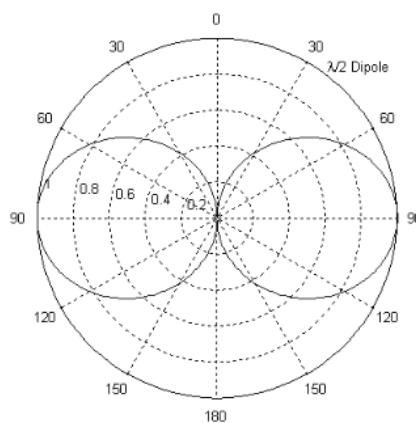
$$(c) \quad R_\ell = \frac{a}{2\delta} R_{dc} = \frac{a}{2\delta} \frac{\ell}{\sigma S} = \frac{a\ell}{2\sigma\pi a^2} \sqrt{\pi f \mu \sigma} = \frac{2\pi R}{2\pi a} \sqrt{\frac{\mu f \pi}{\sigma}}$$

$$R_\ell = \frac{R}{a} \sqrt{\frac{\mu f \pi}{\sigma}} = \frac{0.4}{4 \times 10^{-3}} \sqrt{\frac{4\pi \times 10^{-7} \times 6 \times 10^6 \times \pi}{5.8 \times 10^7}} = 63.91 \text{ m}\Omega$$

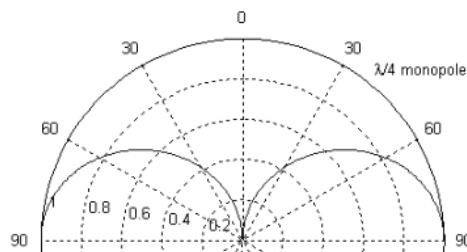
$$\eta = \frac{R_{rad}}{R_{rad} + R_\ell} = \frac{1.26}{1.26 + 63.91} \times 100\% = \underline{\underline{1.933\%}}$$

Prob. 13.16

$$(a) f(\theta) = \left| \frac{\cos\left(\frac{\pi}{2} \cos \theta\right)}{\sin \theta} \right|$$



(b) The same as for $\frac{\lambda}{2}$ dipole except that the fields are zero for $\theta > \frac{\pi}{2}$ as shown.

**Prob. 13.17**

Let P_{rad1} and P_{rad2} be the old and new radiated powers respectively.

Let P_{ohm1} and P_{ohm2} be the old and new ohmic powers respectively.

$$\eta_{r1} = 20\% = \frac{P_{rad1}}{P_{rad1} + P_{ohm1}} = \frac{1}{5} \quad \longrightarrow \quad 4P_{rad1} = P_{ohm1} \quad (1)$$

$$\text{But } P_{ohm1} = \frac{1}{2} I^2 R_s \Delta z$$

$$P_{ohm2} = \frac{1}{2} I^2 R_s 2 \Delta z = 2P_{ohm1} \quad (2)$$

$$P_{rad1} = \frac{1}{2} I_o^2 R_{rad} = \frac{1}{2} I_o^2 \times 80\pi^2 \left(\frac{\Delta z}{\lambda} \right)^2$$

$$P_{rad2} = \frac{1}{2} I_o^2 R_{rad} = \frac{1}{2} I_o^2 \times 80\pi^2 \left(\frac{2\Delta z}{\lambda} \right)^2 = 4P_{rad1} \quad (3)$$

From (1) to (3),

$$\eta_{r2} = \frac{P_{rad2}}{P_{rad2} + P_{ohm2}} = \frac{4P_{rad1}}{4P_{rad1} + 2P_{ohm1}} = \frac{P_{ohm1}}{3P_{ohm1}} = \underline{\underline{33.3\%}}$$

Prob. 13.18

(a) Let $\mathbf{H}_s = \frac{\cos 2\theta}{\eta_0 r} e^{-j\beta r} \mathbf{a}_H$

$$\mathbf{a}_E \times \mathbf{a}_H = \mathbf{a}_k \quad \longrightarrow \quad \mathbf{a}_\theta \times \mathbf{a}_H = \mathbf{a}_r \quad \longrightarrow \quad \mathbf{a}_H = \mathbf{a}_\phi$$

$$\mathbf{H}_s = \frac{\cos 2\theta}{120\pi r} e^{-j\beta r} \mathbf{a}_\phi$$

(b) $\mathbf{P}_{ave} = \frac{|E_s|^2}{2\eta} \mathbf{a}_r = \frac{\cos^2(2\theta)}{2\eta r^2} \mathbf{a}_r$

$$P_{rad} = \frac{1}{2\eta} \iint \frac{\cos^2 2\theta}{r^2} r^2 \sin \theta d\theta d\phi = \frac{1}{240\pi} (2\pi) \int_0^\pi \cos^2 2\theta \sin \theta d\theta$$

But $\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 2\cos^2 \theta - 1$

$$\begin{aligned} P_{rad} &= -\frac{1}{120} \int_0^\pi (2\cos^2 \theta - 1)^2 d(\cos \theta) \\ &= -\frac{1}{120} \int_0^\pi (4\cos^4 \theta - 4\cos^2 \theta + 1) d(\cos \theta) \\ &= -\frac{1}{120} \left(\frac{4\cos^5 \theta}{5} - \frac{4\cos^3 \theta}{3} + \cos \theta \right) \Big|_0^\pi \\ &= -\frac{1}{120} \left[-\frac{4}{5} + \frac{4}{3} - 1 - \frac{4}{5} + \frac{4}{3} - 1 \right] = \frac{1}{120} \left(\frac{14}{15} \right) \\ &= \underline{\underline{7.778 \text{ mW}}} \end{aligned}$$

(c)

$$\begin{aligned} P_{rad} &= -\frac{1}{120} \int_{60^\circ}^{120^\circ} (2\cos^2 \theta - 1)^2 d(\cos \theta) \\ &= -\frac{1}{120} \left(\frac{4\cos^5 \theta}{5} - \frac{4\cos^3 \theta}{3} + \cos \theta \right) \Big|_{60^\circ}^{120^\circ} \\ &= -\frac{1}{120} \left[\frac{4}{5} \left(-\frac{1}{32} \right) - \frac{4}{3} \left(-\frac{1}{8} \right) - \frac{1}{2} - \frac{4}{5} \left(\frac{1}{32} \right) + \frac{4}{3} \left(\frac{1}{8} \right) - \frac{1}{2} \right] = \frac{1}{60} \left[\frac{1}{40} + \frac{1}{2} - \frac{1}{6} \right] \\ &= 5.972 \text{ mW} \end{aligned}$$

which is $\frac{5.972}{7.778} = 0.7678 \text{ or } \underline{\underline{76.78\%}}$

Prob. 13.20

$$(a) P_{rad} = \int \mathbf{P}_{rad} \cdot d\mathbf{S} = P_{ave} \cdot 2\pi r^2 \quad (\text{hemisphere})$$

$$P_{ave} = \frac{P_{rad}}{2\pi r^2} = \frac{200 \times 10^3}{2\pi(2500 \times 10^6)} = 12.73 \mu W / m^2$$

$$\underline{\underline{\mathbf{P}_{ave}}} = 12.73 \mu W / m^2 .$$

$$(b) P_{ave} = \frac{(E_{max})^2}{2\eta}$$

$$E_{max} = \sqrt{2\eta P_{ave}} = \sqrt{240\pi \times 12.73 \times 10^{-6}}$$

$$= 0.098 \text{ V/m}$$

Prob. 13.21

$$G_d = \frac{U}{U_{ave}} = \frac{4\pi r^2 P_{ave}}{\int P_{ave} \cdot dS} = \frac{8\pi \sin \theta \cos \phi}{\int P_{ave} \cdot dS}$$

$$\begin{aligned} \text{But } \int P_{ave} \cdot dS &= \int_{\theta=0}^{\pi} \int_{\phi=0}^{\pi/2} 2 \sin \theta \cos \phi \sin \theta d\theta d\phi \\ &= 2 \int_0^{\pi/2} \cos \phi d\phi \int_0^{\pi} \sin^2 \theta d\theta = 2 \sin \phi \Big|_0^{\pi/2} \left(\frac{\pi}{2} \right) = \pi \end{aligned}$$

$$G_d = \frac{8 \sin \theta \cos \phi}{\int P_{ave} \cdot dS}$$

$$D = G_{d,max} = \underline{\underline{8}}$$

Prob. 13.22

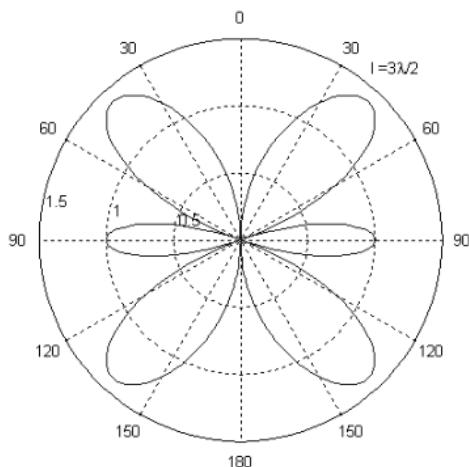
From Prob. 13.11, set $\ell = \frac{3\lambda}{2}$, $\beta\ell = \frac{2\pi}{\lambda} \frac{3\lambda}{2} = 3\pi$

$$\begin{aligned} H_{\phi s} &= \frac{jI_o e^{-\beta r}}{2\pi r} \left[\cos\left(\frac{2\pi}{\lambda} \frac{1}{2} \frac{3\lambda}{2} \cos\theta\right) - \cos\left(\frac{2\pi}{\lambda} \frac{1}{2} \frac{3\lambda}{2}\right) \right] \\ &= \frac{jI_o e^{-\beta r}}{2\pi r} \left[\cos\left(\frac{3\pi}{2} \cos\theta\right) - \cos\left(\frac{3\pi}{2}\right) \right] = \frac{jI_o e^{-\beta r}}{2\pi r} \frac{\cos(1.5\pi \cos\theta)}{\sin\theta} \end{aligned}$$

Hence, the normalized radiated field pattern is

$$f(\theta) = \frac{\cos(1.5\pi \cos\theta)}{\sin\theta}$$

which is plotted below.



Prob. 13.23

The MATLAB code is shown below

```
N=20;
del= 2*pi/N;
sum=0;
for k=1:N
    theta = del*k;
    term = (1 - cos(theta))/theta;
    sum = sum + term;
end
int = del*sum
```

When the program is run, it gives the value of 2.4335. The accuracy may be increased by increasing N.

Prob. 13.24

$$(a) E_{\theta s} = \frac{j\eta I_o \beta dl}{4\pi r} \sin \theta e^{-j\beta r}$$

$$R_{rad} = 80\pi^2 \left(\frac{dl}{\lambda} \right)^2$$

$$G_d = \frac{4\pi r^2 P_{ave}}{P_{rad}} = \frac{4\pi r^2 \cdot \frac{1}{2\eta} |E_{\theta s}|^2}{\frac{1}{2} I_o^2 R_{rad}}$$

$$= \frac{4\pi r^2}{I_o^2} \cdot \frac{1}{80\pi^2} \left(\frac{\lambda}{dl} \right)^2 \cdot \frac{1}{\eta} \frac{\eta^2 I_o^2 \beta^2 (dl)^2 \sin^2 \theta}{16\pi^2 r^2}$$

$$G_d = \underline{\underline{1.5 \sin^2 \theta}}$$

$$(b) D = G_{d,\max} = \underline{\underline{1.5}}$$

$$(c) A_e = \frac{\lambda^2}{4\pi} G_d = \frac{1.5 \lambda^2 \sin^2 \theta}{4\pi}$$

$$(d) R_{rad} = 80\pi^2 \left(\frac{1}{16} \right)^2 = \underline{\underline{3.084 \Omega}}$$

Prob. 13.25

$$(a) \quad E_{\phi s} = \frac{120\pi^2 I_o}{r} \frac{S}{\lambda^2} \sin \theta e^{-j\beta r}$$

$$R_{rad} = \frac{320\pi^4 S^2}{\lambda^4}$$

$$G_d = \frac{4\pi U(\theta, \phi)}{P_{rad}} = \frac{4\pi r^2 P_{ave}}{\frac{1}{2} I_o^2 R_{rad}} = \frac{8\pi r^2}{I_o^2} \cdot \frac{1}{2\eta} \frac{|E_{\phi s}|^2}{R_{rad}}$$

$$= \frac{8\pi r^2}{I_o^2} \cdot \frac{1}{2\eta} \cdot 14400\pi^4 \frac{I_o^2}{r^2} \frac{S^2}{\lambda^4} \sin^2 \theta \frac{\lambda^2}{320\pi^4 S^2}$$

$$G_d = \underline{\underline{1.5 \sin^2 \theta}}$$

$$(b) \quad D = \underline{\underline{1.5}}$$

$$(c) \quad A_e = \frac{\lambda^2 G_d}{4\pi} = \frac{\lambda^2}{4\pi} \underline{\underline{1.5 \sin^2 \theta}}$$

$$(d) \quad S = \pi a^2 = \frac{\pi d^2}{4} = \frac{320\pi^6}{(576)^2}$$

$$R_{rad} = \underline{\underline{0.927 \Omega}}$$

Prob. 13.26

(a)

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{1.2 \times 10^6} = 250$$

$$\ell = \frac{\lambda}{4} = \underline{\underline{62.5 \text{ m}}}$$

(b) From eq. (13.30), $R_{rad} = \underline{\underline{36.5 \Omega}}$

(c)

For $\lambda/4$ -monopole,

$$f(\theta) = \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}, \quad 0 < \theta < \pi/2$$

$$G_d(\theta, \phi) = \frac{4\pi f^2(\theta)}{\int f^2(\theta) d\Omega} = \frac{4\pi \cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\int_0^{2\pi} \int_0^{\pi/2} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} d\theta d\phi}$$

$$= \frac{4\pi \cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta} \frac{1}{2\pi(0.6094)} = \frac{3.282 \cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$$

$$D = G_{d,\max} = \underline{\underline{3.282}}$$

Prob. 13.27

(a) $U_{\max} = 1$

$$U_{ave} = \frac{P_{rad}}{4\pi} = \frac{\int U d\Omega}{4\pi}$$

$$= \frac{1}{4\pi} \int \int \sin^2 2\theta \sin\theta d\theta d\phi$$

$$= \frac{1}{4\pi} (2\pi) \int_0^\pi (2\sin\theta \cos\theta)^2 d(-\cos\theta)$$

$$= 2 \int_0^\pi (\cos^4\theta - \cos^2\theta) d(\cos\theta)$$

$$= 2 \left[\frac{\cos^5\theta}{5} - \frac{\cos^3\theta}{3} \right]_0^\pi$$

$$= 2 \left[-\frac{2}{5} + \frac{2}{3} \right] = \frac{8}{15}$$

$$U_{ave} = \underline{\underline{0.5333}}$$

$$D = \frac{U_{\max}}{U_{\text{ave}}} = \underline{\underline{1.875}}$$

(b) $U_{\max} = 4$

$$U_{\text{ave}} = \frac{1}{4\pi} \int U d\Omega = \frac{4}{4\pi} \int \int \frac{\sin \theta}{\sin^2 \theta} d\theta d\phi$$

$$= \frac{1}{\pi} \int_0^\pi d\phi \int_{\pi/3}^{\pi/2} \csc \theta d\theta = \frac{\pi}{\pi} \ln \sqrt{3}$$

$$U_{\text{ave}} = \underline{0.5493}$$

$$D = \frac{U_{\max}}{U_{\text{ave}}} = \frac{16}{3 \ln \sqrt{3}} = \underline{\underline{9.7092}}$$

(c) $U_{\max} = 2$

$$\begin{aligned} U_{\text{ave}} &= \frac{1}{4\pi} \int U d\Omega = \frac{1}{4\pi} \int \int 2 \sin^2 \theta \sin^2 \phi \sin \theta d\theta d\phi \\ &= \frac{1}{2\pi} \int_0^\pi \sin^2 \phi d\phi \int_0^\pi (1 - \cos^2 \theta) d(-\cos \theta) \\ &= \frac{1}{2\pi} \cdot \frac{\pi}{2} \left(\frac{\cos^3 \theta}{3} - \cos \theta \right) \Big|_0^\pi = \frac{1}{4} \left[-\frac{2}{3} + 2 \right] = \frac{1}{3} \end{aligned}$$

$$U_{\text{ave}} = \underline{0.333}$$

$$D = \frac{U_{\max}}{U_{\text{ave}}} = \underline{\underline{6}}$$

Prob. 13.28

$$(a) \quad G_d(\theta, \phi) = \frac{U(\theta, \phi)}{U_{ave}}$$

$$U_{ave} = \frac{1}{4\pi} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} 10 \sin \theta \sin^2 \phi \times \sin \theta d\theta d\phi$$

$$= \frac{10}{4\pi} \int_0^{2\pi} \sin^2 \phi d\phi \int_0^{\pi} \sin^2 \theta d\theta$$

$$= \frac{10}{4\pi} \frac{1}{2} \left(\theta - \frac{\sin 2\theta}{2} \right) \Big|_0^{2\pi} \frac{1}{2} \left(\phi - \frac{\sin 2\phi}{2} \right) \Big|_0^{\pi}$$

$$= \frac{10}{16\pi} (2\pi - 0)(\pi - 0) = \frac{5\pi}{4}$$

$$G_d(\theta, \phi) = \frac{40 \sin \theta \sin^2 \phi}{5\pi} = \underline{\underline{2.546 \sin \theta \sin^2 \phi}}$$

$$D = G_{d,\max} = \underline{\underline{2.546}}$$

$$(b) \quad U_{ave} = \frac{1}{4\pi} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} 2 \sin^2 \theta \sin^3 \phi \times \sin \theta d\theta d\phi$$

$$= \frac{2}{4\pi} \int_0^{2\pi} \sin^3 \phi d\phi \int_0^{\pi} \sin^3 \theta d\theta = \frac{2}{4\pi} \left[\int_0^{\pi} (1 - \cos^2 \phi) d(-\cos \phi) \right]^2$$

$$= \frac{1}{2\pi} \left[\left(\frac{\cos^3 \phi}{3} - \cos \phi \right) \Big|_0^{\pi} \right] = \frac{1}{2\pi} \left(\frac{4}{3} \right)^2 = \frac{16}{18\pi}$$

$$G_d(\theta, \phi) = \frac{18\pi}{16} 2 \sin^2 \theta \sin^3 \phi = \underline{\underline{2.25\pi \sin^2 \theta \sin^3 \phi}}$$

$$D = G_{d,\max} = \underline{\underline{7.069}}$$

$$\begin{aligned}
 (c) \quad U_{ave} &= \frac{1}{4\pi} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} 5(1 + \sin^2 \theta \sin^2 \phi) \times \sin \theta d\theta d\phi \\
 &= \frac{5}{4\pi} \left[\int_0^{\pi} \sin \theta d\theta \int_0^{2\pi} d\phi + \int_0^{\pi} \sin^3 \theta d\theta \int_0^{2\pi} \sin^2 d\phi \right] \\
 &= \frac{5}{4\pi} \left[2\pi(-\cos \theta) \Big|_0^{\pi} + \frac{4}{3} \left(\frac{\phi}{2} - \frac{\sin 2\phi}{4} \right) \Big|_0^{2\pi} \right] \\
 &= \frac{5}{4\pi} \left[4\pi + \frac{4}{3}\pi \right] = \underline{\underline{\frac{20}{3}}} \\
 G_d(\theta, \phi) &= \frac{3}{20} 5(1 + \sin^2 \theta \sin^2 \phi) = \underline{\underline{0.75(1 + \sin^2 \theta \sin^2 \phi)}} \\
 D &= G_{d,\max} = \underline{\underline{1.5}}
 \end{aligned}$$

Prob. 13.29

$$U_{\max} = 4$$

$$\begin{aligned}
 U_{ave} &= \frac{1}{4\pi} \int U d\Omega = \frac{1}{4\pi} \iint 4 \sin^2 \theta \sin \frac{\phi}{2} \sin \theta d\theta d\phi \\
 &= \frac{1}{\pi} \int_0^{\pi} \sin^3 \theta d\theta \int_0^{\pi} \sin \frac{\phi}{2} d\phi = \frac{1}{\pi} \int_0^{\pi} (1 - \cos^2 \theta) d(-\cos \theta) (-2 \cos \frac{\phi}{2}) \Big|_0^{\pi} \\
 &= \frac{1}{\pi} \left(\frac{4}{3} \right) (2) = \underline{\underline{\frac{8}{3\pi}}} \\
 D &= \frac{U_{\max}}{U_{ave}} = 4 \times \frac{3\pi}{8} = \underline{\underline{4.712}}
 \end{aligned}$$

Prob. 13.30

$$\begin{aligned}
 \mathbf{P}_{ave} &= \frac{|E_r|^2}{2\eta} \mathbf{a}_r = \frac{I_o^2 \sin^2 \theta}{2\eta r^2} \mathbf{a}_r \\
 P_{rad} &= \frac{I_o^2}{2\eta} \iint \frac{\sin^2 \theta}{r^2} r^2 \sin \theta d\theta d\phi = \frac{I_o^2}{240\pi} (2\pi) \int_0^{2\pi} (1 - \cos^2 \theta) d(-\cos \theta) \\
 &= \frac{I_o^2}{120} \left(\frac{\cos^3 \theta}{3} - \cos \theta \right) \Big|_0^{\pi} = \frac{I_o^2}{120} (-1/3 + 1 - 1/3 + 1) = \underline{\underline{\frac{I_o^2}{90}}}
 \end{aligned}$$

$$I_o^2 = 90 P_{ave} = 90 \times 50 \times 10^{-3} \longrightarrow I_o = \underline{\underline{2.121 \text{ A}}}$$

Prob. 13.31

$$\begin{aligned}
 P_{rad} &= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{|\mathbf{E}|^2}{2\eta_o} r^2 \sin \theta d\theta d\phi = \frac{100}{r^2} \frac{r^2}{2\eta_o} (2\pi) \int_0^{\pi} \sin^5 \theta d\theta \\
 &= \frac{2\pi}{2 \times 377} 100 \left(-\frac{5}{8} \cos \theta + \frac{5}{48} \cos 3\theta - \frac{1}{80} \cos 5\theta \right) \Big|_0^\pi \\
 &= \frac{100\pi}{377} \left(\frac{5}{8} + \frac{5}{8} - \frac{5}{48} - \frac{5}{48} + \frac{1}{80} + \frac{1}{80} \right) = \underline{\underline{0.889 \text{ W}}}
 \end{aligned}$$

Prob. 13.32

This is similar to Fig. 13.10 except that the elements are z-directed.

$$\mathbf{E}_s = \mathbf{E}_{s1} + \mathbf{E}_{s2} = \frac{j\eta\beta I_o dl}{4\pi} \left[\sin \theta_1 \frac{e^{-j\beta r_1}}{r_1} \mathbf{a}_{\theta 1} + \sin \theta_2 \frac{e^{-j\beta r_2}}{r_2} \mathbf{a}_{\theta 2} \right]$$

$$\text{where } r_1 \equiv r - \frac{d}{2} \cos \theta, \quad r_2 \equiv r + \frac{d}{2} \cos \theta, \quad \theta_1 \equiv \theta_2 \equiv \theta, \quad \mathbf{a}_{\theta 1} \equiv \mathbf{a}_{\theta 2} = \mathbf{a}_\theta$$

$$\mathbf{E}_s = \frac{j\eta\beta I_o dl}{4\pi} \sin \theta \mathbf{a}_\theta \left[e^{j\beta d \cos \theta / 2} + e^{-j\beta d \cos \theta / 2} \right]$$

$$\mathbf{E}_s = \frac{j\eta\beta I_o dl}{2\pi} \sin \theta \cos \left(\frac{1}{2} \beta d \cos \theta \right) \mathbf{a}_\theta$$

Prob. 13.33

Equation (13.33) applies except that $\cos \theta$ must be replaced by $\sin \theta$. Hence, the radiation pattern is

$$f(\theta) = \sin \theta 2 \cos \left[\frac{1}{2} (\beta d \cos \theta + \alpha) \right]$$

$$\alpha = 0, \quad d = 2(\lambda / 4) = \lambda / 2, \quad \beta d = \frac{2\pi}{\lambda} \frac{\lambda}{2} = \pi$$

$$f(\theta) = 2 \sin \theta \cos \left(\frac{1}{2} \pi \cos \theta \right)$$

Prob. 13.34

$$\text{(a) } \text{AF} = 2 \cos \left[\frac{1}{2} (\beta d \cos \theta + \alpha) \right], \quad \alpha = 0, \quad \beta d = \frac{2\pi}{\lambda} \lambda = 2\pi$$

$$\text{AF} = \underline{\underline{2 \cos(\pi \cos \theta)}}$$

(b) Nulls occur when

$$\cos(\pi \cos \theta) = 0 \longrightarrow \pi \cos \theta = \pm\pi / 2, \pm 3\pi / 2, \dots$$

or

$$\theta = \underline{\underline{60^\circ, 120^\circ}}$$

(c) Maxima and minima occur when

$$\frac{df}{d\theta} = 0 \longrightarrow \sin(\pi \cos \theta)\pi \sin \theta = 0$$

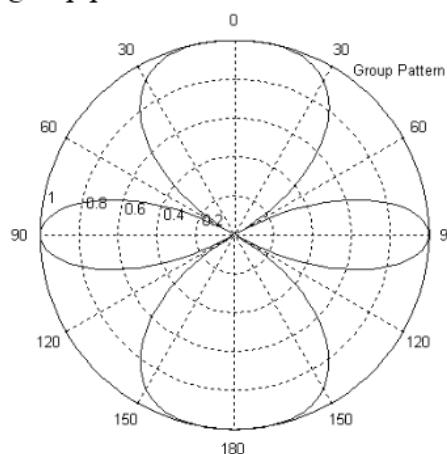
i.e. $\sin \theta = 0 \longrightarrow \theta = 0^\circ, 180^\circ$

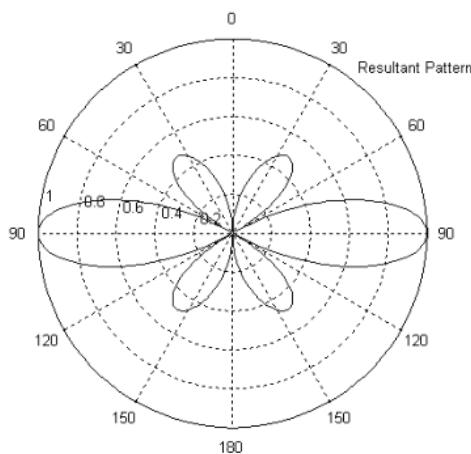
$\cos \theta = 0 \longrightarrow \theta = 90^\circ$

or

$$\theta = \underline{\underline{0^\circ, 90^\circ, 180^\circ}}$$

(d) The group pattern is sketched below.



**Prob. 13.35**

$$f(\theta) = \cos\left[\frac{1}{2}(\beta d \cos \theta + \alpha)\right]$$

(a) $\alpha = \pi/2, \beta d = \frac{2\pi}{\lambda}, \lambda = 2\pi$

$$f(\theta) = \cos\left(\pi \cos \theta + \frac{\pi}{4}\right)$$

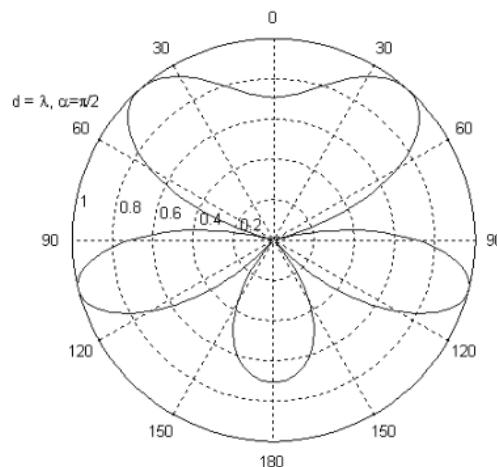
Nulls occur at $\pi \cos \theta + \frac{\pi}{4} = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \dots$ or $\theta = 75.5^\circ, 138.6^\circ$

Maxima occur at $\frac{\partial f}{\partial \theta} = 0 \rightarrow \sin \theta = 0 \rightarrow \theta = 0^\circ, 180^\circ$

Or $\sin\left(\pi \cos \theta + \frac{\pi}{4}\right) = 0 \rightarrow \theta = 41.4^\circ, 104.5^\circ$

With $f_{\max} = 0.711$.

Hence the group pattern is sketched below.



$\theta = 41.0138^\circ, \text{abs}(f) = 1,$
$\theta = 104.6083^\circ, \text{abs}(f) = 1,$
$\theta = 180^\circ, \quad \text{abs}(f) = 1/\sqrt{2} = 0.707$

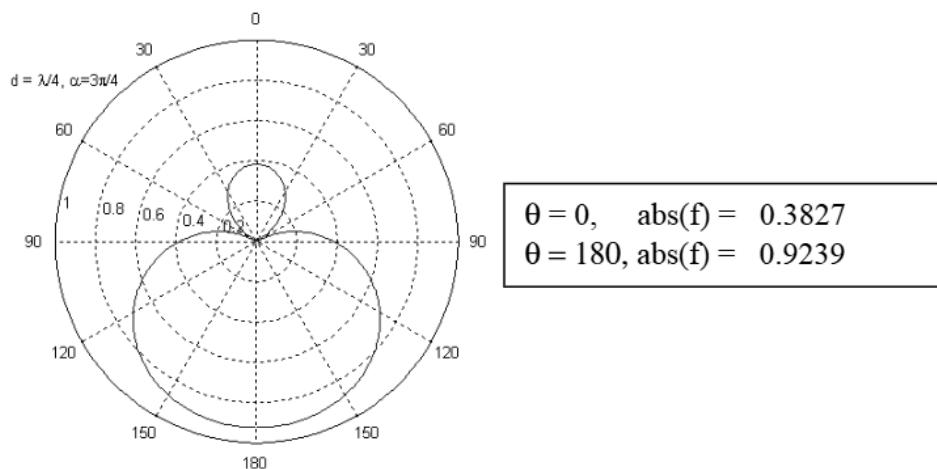
$$(b) \quad \alpha = \frac{3\pi}{4}, \beta d = \frac{2\pi}{\lambda} \cdot \frac{\lambda}{4} = \frac{\pi}{2}$$

$$f(\theta) = \left| \cos\left(\frac{\pi}{4} \cos \theta + \frac{3\pi}{8}\right) \right|$$

Nulls occur at $\frac{\pi}{4} \cos \theta + \frac{3\pi}{8} = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \dots \rightarrow \theta = 60^\circ$

Minima and maxima occur at $\sin \theta \cos\left(\frac{\pi}{4} \cos \theta + \frac{3\pi}{8}\right) = 0$

i.e. $\theta = 0^\circ, 180^\circ \rightarrow f(\theta) = 0.383, 0.924$



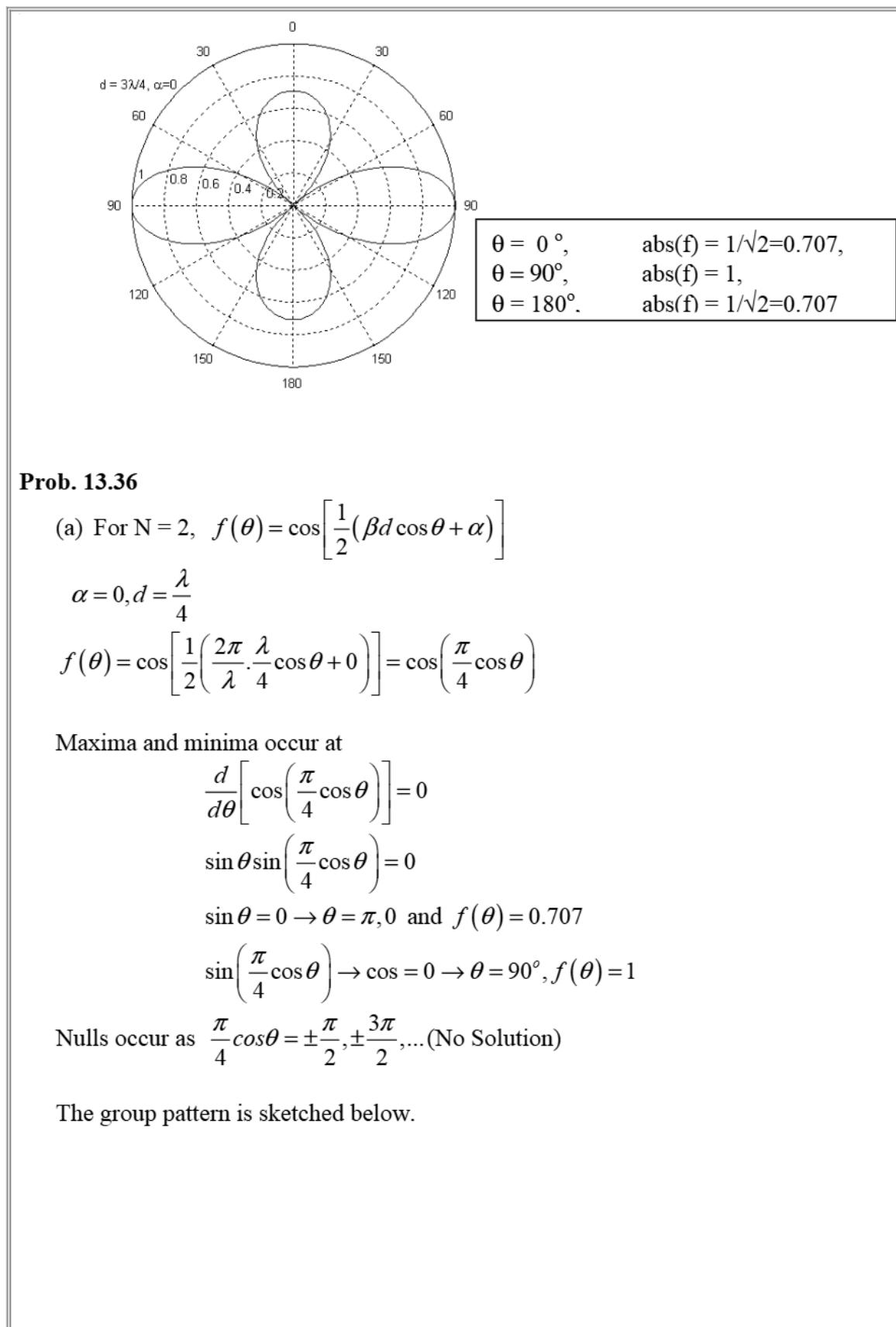
$$(c) \quad \alpha = 0, \beta d = \frac{2\pi}{\lambda} \cdot \frac{3\lambda}{4} = \frac{3\pi}{2}$$

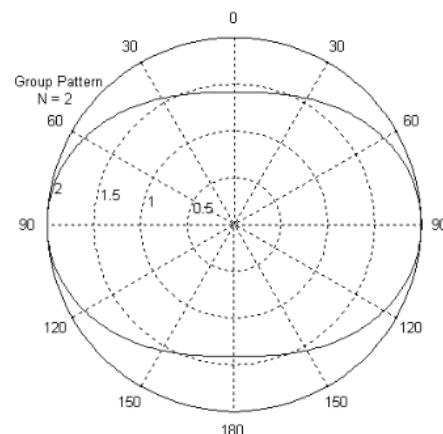
$$f(\theta) = \left| \cos\left(\frac{3\pi}{4} \cos \theta\right) \right|$$

It has nulls at $\frac{3\pi}{4} \cos \theta = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \dots \rightarrow \theta = 48.2^\circ, 131.8^\circ$

It has maxima and minima at $\frac{df}{d\theta} = 0 \rightarrow \sin \theta \sin\left(\frac{3\pi}{4} \cos \theta\right) = 0$

i.e. $\theta = 0^\circ, 180^\circ \rightarrow f(\theta) = 0.71, 1, \quad \theta = \pm 90^\circ, \rightarrow f(\theta) = 1$





(b) For $N = 4$,

$$AF = \frac{\sin 2(\beta d \cos \theta + 0)}{\sin \frac{1}{2}(\beta d \cos \theta + 0)}$$

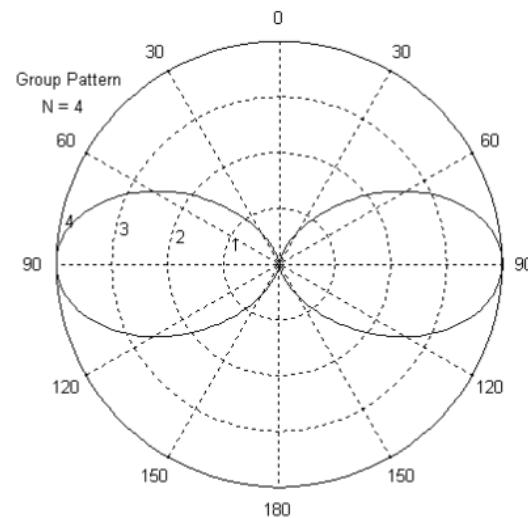
$$\text{Now, } \frac{\sin 4\theta}{\sin \theta} = \frac{2 \sin 2\theta \cos 2\theta}{\sin \theta} = 4 \cos 2\theta \cos \theta$$

$$AF = 4 \cos(\beta d \cos \theta) \cos\left(\frac{1}{2} \beta d \cos \theta\right)$$

$$f(\theta) = \cos\left(\frac{2\pi}{\lambda} \cdot \frac{\lambda}{4} \cos \theta\right) \cos\left(\frac{1}{2} \frac{2\pi}{\lambda} \frac{\lambda}{4} \cos \theta\right) \cos \theta$$

$$= \cos\left(\frac{\pi}{2} \cos \theta\right) \cos\left(\frac{\pi}{4} \cos \theta\right)$$

The plot is shown below.

**Prob. 13.37**

The MATLAB code is shown below.

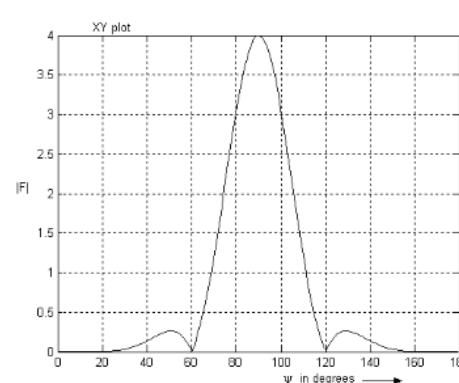
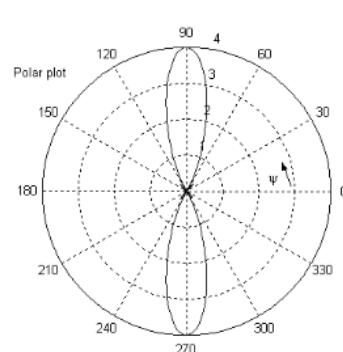
for n=1:180

```

phi=n*pi/180;
p(n)=n;
sn=sin(2*pi*cos(phi));
cn=cos(0.5*pi*cos(phi));
sd=sin(0.5*pi*cos(phi));
fun=sn*cn/cn/sd;
f(n)= abs(fun);
end
```

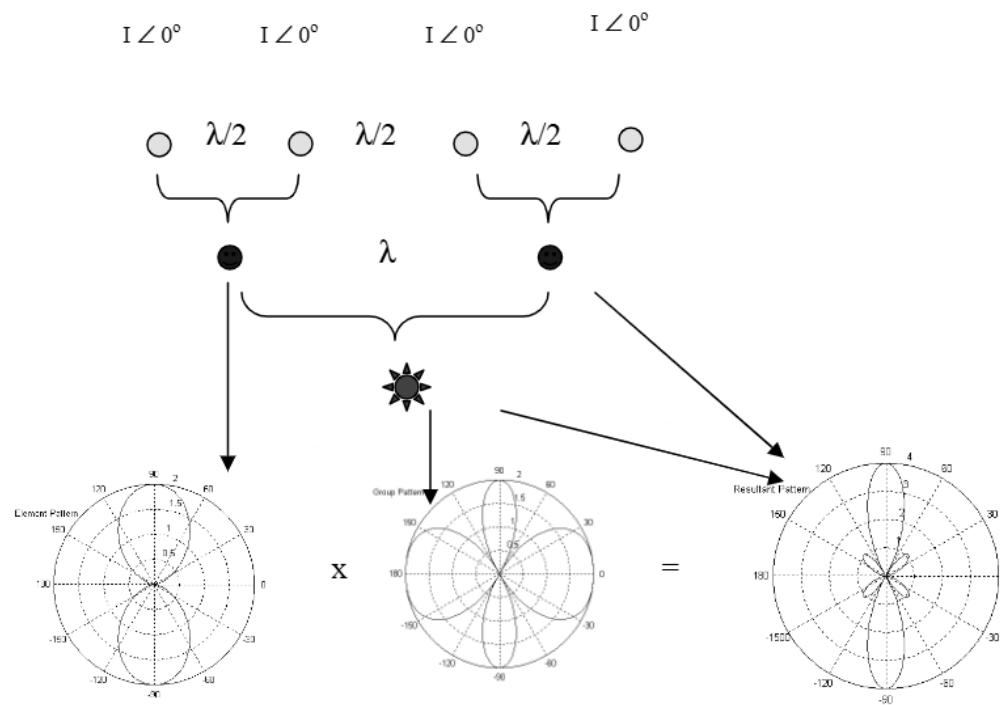
polar(p,f)

The polar plot and the xy plot are shown below.



Prob. 13.38

(a) The resultant pattern is obtained as follows.



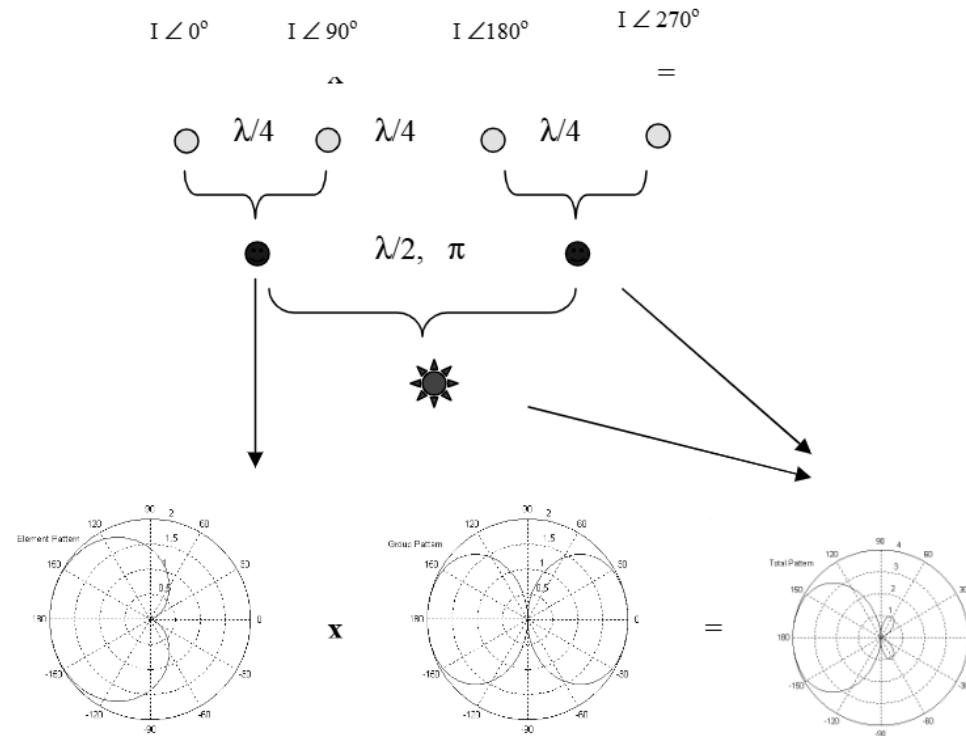
(b) The array is replaced by

$$+ \quad + \\ \angle 0^\circ \quad \frac{\lambda}{4} \quad \angle \pi/2$$

where + stands for

$$\angle 0^\circ \quad \angle \pi$$

Thus the resultant pattern is obtained as shown.



Prob. 13.39

$$G_d(dB) = 20dB = 10\log_{10} G_d \longrightarrow G_d = 10^2 = 100$$

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{10 \times 10^9} = 3 \times 10^{-2}$$

$$A_e = \frac{\lambda^2}{4\pi} G_d = \frac{9 \times 10^{-4}}{4\pi} 100 = \underline{\underline{7.162 \times 10^{-2} \text{ m}^2}}$$

Prob. 13.40

$$A_e = \frac{P_r}{P_{ave}} = \frac{P_r}{|E_r|^2} = \frac{2\eta P_r}{|E_r|^2}$$

$$= \frac{2 \times 120\pi \times 2 \times 10^{-6}}{25 \times 10^2 \times 10^{-6}} = \frac{48\pi}{250} = \underline{\underline{0.6031}}$$

Prob. 13.41

Friis equation states that

$$\frac{P_r}{P_t} = G_r G_t \left(\frac{\lambda}{4\pi r} \right)^2$$

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{200 \times 10^6} = 1.5 \text{ m}, \quad r = 238,857 \times 1.609 \times 10^3 = 3.843 \times 10^8$$

$$G_t(dB) = 15dB = 10 \log_{10} G_t \quad \longrightarrow \quad G_t = 10^{15/10} = 31.623$$

$$G_r = \left(\frac{4\pi r}{\lambda} \right)^2 \frac{P_r}{P_t} = \left(\frac{4\pi \times 3.843 \times 10^8}{1.5} \right)^2 \frac{4 \times 10^{-9}}{120 \times 10^{-3}} = 34.55 \times 10^{10}$$

$$G_r(dB) = 10 \log_{10} G_r = 10 \log_{10} 34.55 \times 10^{10} = \underline{\underline{115.384 \text{ dB}}}$$

Prob. 13.42

Using Frii's equation,

$$P_r = G_r G_t \left[\frac{\lambda}{4\pi r} \right]^2 P_t$$

$$P_t = \left[\frac{4\pi r}{\lambda} \right]^2 \frac{P_r}{G_r G_t}$$

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{3 \times 10^9} = 0.1, \quad r = 42 \text{ km}$$

$$G_t(dB) = 10 \log_{10} G_t = 25 \quad \longrightarrow \quad G_t = 10^{2.5} = 316.23$$

$$G_r(dB) = 10 \log_{10} G_r = 20 \quad \longrightarrow \quad G_t = 10^2 = 100$$

$$P_t = \left(\frac{4\pi \times 42 \times 10^3}{0.1} \right)^2 \frac{3 \times 10^{-6}}{31623} = \underline{\underline{2.642 \text{ kW}}}$$

Prob. 13.43

$$G_d = 10^4, G_{dt} = 10^{3.2} = 1585$$

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{15 \times 10^9} = 0.02 \text{ m} = \frac{1}{50}$$

$$P_r = G_{dr} G_{dt} \left(\frac{\lambda}{4\pi r} \right)^2 P_t = 10^4 (1585) \left(\frac{0.02}{4\pi \times 2.456741 \times 10^7} \right)^2 320 \\ = 2.129 \times 10^{-11} \text{ W} = \underline{\underline{21.29 \text{ pW}}}$$

Prob. 13.44

Using Friis's equation,

$$P_r = G^2 \left[\frac{\lambda}{4\pi r} \right]^2 P_t$$

$$P_t = \left[\frac{4\pi r}{\lambda} \right]^2 \frac{P_r}{G^2}$$

$$G(dB) = 25 \longrightarrow G = 10^{2.5} = 316.23$$

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{600 \times 10^6} = 0.5$$

$$P_t = \left[\frac{4\pi \times 450}{0.5} \right]^2 \frac{4 \times 10^{-3}}{(316.23)^2} = \underline{\underline{5.1163 \text{ W}}}$$

Prob. 13.45

$$30dB = \log \frac{P_t}{P_r} \rightarrow \frac{P_t}{P_r} = 10^3 = 1000$$

$$\text{But } P_r = (G_d)^2 \left(\frac{3}{50 \times 4\pi \times 12} \right)^2 P_t = P_t \left(\frac{G_d}{800\pi} \right)^2$$

$$\left(\frac{G_d}{800\pi} \right)^2 = \frac{P_r}{P_t} = \frac{1}{1000} = \left(\frac{1}{10\sqrt{10}} \right)^2$$

$$\text{or } G_d = \frac{800\pi}{10\sqrt{10}} = 79.476$$

$$G_d = 10 \log 79.476 = \underline{\underline{19 \text{ dB}}}$$

Prob. 13.46

$$(a) P_i = \frac{|E|^2}{2\eta_o} = \frac{P_{rad}G_d}{4\pi r^2} \rightarrow |E_i| = \sqrt{\frac{240\pi P_{rad}G_d}{4\pi r^2}}$$

$$|E_i| = \frac{1}{r} \sqrt{60P_{rad}G_d} = \frac{1}{120 \times 10^3} \sqrt{60 \times 200 \times 10^3 \times 3500}$$

$$= \underline{\underline{1.708 \text{ V/m}}}$$

$$(b) |E_s| = \sqrt{\frac{|E_i|^2 \sigma}{4\pi r^2}} = \sqrt{\frac{1.708^2 \times 8}{4\pi \times 14400 \times 10^6}} = \underline{\underline{11.36 \mu\text{V/m}}}$$

$$(c) P_c = P_i \sigma = \frac{1.708^2}{240\pi} (8) = \underline{\underline{30.95 \text{ mW}}}$$

$$(d) P_i = \frac{|E|^2}{2\eta_o} = \frac{(11.36)^2 \times 10^{-12}}{240\pi} = 1.712 \times 10^{-13} \text{ W/m}^2$$

$$\lambda = \frac{3 \times 10^8}{15 \times 10^8} = 0.2 \text{ m}, A_{2r} = \frac{\lambda^2 G}{4\pi} = \frac{0.04 \times 3500}{4\pi}$$

$$P_r = P_a A_{er} = 1.712 \times 10^{-13} \times 11.14 = 1.907 \times 10^{-12}$$

$$\text{or } P_r = \frac{(\lambda G_d)^2 \sigma P_{rad}}{(4\pi)^3 r^4} = \frac{(0.2 \times 3500)^2 \times 8 \times 2 \times 10^5}{(4\pi)^3 \times 12^4 \times 10^{16}}$$

$$= \underline{\underline{1.91 \times 10^{-12} \text{ W}}}$$

Prob. 13.47

$$P_r = \frac{(\lambda G_d)^2 \sigma P_{rad}}{(4\pi)^3 r^4}$$

$$G_d(\text{dB}) = 30 \text{ dB} = 10 \log_{10} G_d \longrightarrow G_d = 10^3$$

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{4 \times 10^9} = 0.075 \text{ m}$$

$$P_r = \frac{(0.075 \times 10^3)^2 \times 12 \times 80 \times 10^3}{(4\pi)^3 (10 \times 10^3)^4} = \underline{\underline{272.1 \text{ pW}}}$$

Prob. 13.48

$$P_{rad} = \frac{4\pi}{G_{dt} G_{dr}} \left(\frac{4\pi r_1 r_2}{\lambda} \right)^2 \frac{P_r}{\sigma}$$

But $G_{dt} = 36 dB = 10^{3.6} = 3981.1$

$$G_{dr} = 20 dB = 10^2 = 100$$

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{5 \times 10^9} = 0.06$$

$$r_1 = 3 km, r_2 = 5 km$$

$$P_{rad} = \frac{4\pi}{3981.1 \times 100} \left(\frac{4\pi \times 15 \times 10^6}{6 \times 10^{-2}} \right)^2 \frac{8 \times 10^{-12}}{2.4}$$

$$= \underline{\underline{1.038 \text{ kW}}}$$

Prob. 13.49

$$P_r = \frac{(\lambda G_d)^2 \sigma P_{rad}}{(4\pi)^3 r^4} \rightarrow P_{rad} = \frac{(4\pi)^3 r^4 P_r}{(\lambda G_d)^2 \sigma}$$

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{6 \times 10^9} = \frac{1}{20} \ll r = 250 m$$

$$40 = \log_{10} G_d \rightarrow G_d = 10^4$$

$$P_{rad} = \frac{(4\pi)^3 (0.25 \times 10^3)^4 \times 2 \times 10^{-6}}{\left(\frac{1}{20} \times 10^4 \right)^2 \times 0.8} = \underline{\underline{77.52 \text{ W}}}$$

Prob. 13.50

(a)

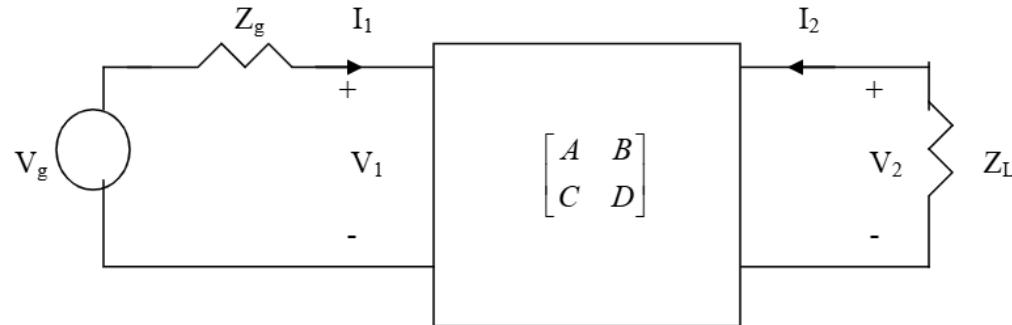
$$F = \frac{\pi f L}{R} = \frac{\pi \times 300 \times 10^6 \times 50 \times 10^{-9}}{20} = 2.356$$

$$IL = 10 \log_{10}(1 + F^2) = 10 \log_{10}(1 + 2.356^2) = \underline{\underline{8.164 \text{ dB}}}$$

(b)

$$F = \pi fRC = \pi \times 300 \times 10^6 \times 10 \times 10^3 \times 60 \times 10^{-12} = 180\pi = 565.5$$

$$IL = 10 \log_{10}(1 + F^2) = 10 \log_{10}(1 + 565.5^2) = \underline{\underline{55.05 \text{ dB}}}$$

Prob. 13.51

By definition,

$$V_1 = AV_2 - BI_2 \quad (1)$$

$$I_1 = CV_2 - DI_2 \quad (2)$$

Let V_2 and \bar{V}_2 be respectively the load voltages when the filter circuit is present and when it is absent.

$$V_2 = -I_2 Z_L = \frac{I_1 Z_L}{C Z_L + D}$$

$$= \frac{V_g Z_L}{\left(Z_g + \frac{V_1}{I_1} \right) (C Z_L + D)} = \frac{V_g Z_L}{\left(Z_g + \frac{AV_2 - BI_2}{CV_2 - DI_2} \right) (C Z_L + D)}$$

$$= \frac{V_g Z_L}{\left(Z_g + \frac{AZ_L + B}{C Z_L + D} \right) (C Z_L + D)}$$

$$= \frac{V_g Z_L}{(Z_g (C Z_L + D) + AZ_L + B)}$$

$$\bar{V}_2 = \frac{V_g Z_L}{(Z_g + Z_L)}$$

Ratio and modulus give

$$\left| \frac{\bar{V}_2}{V_2} \right| = \frac{(Z_g(CZ_L + D) + AZ_L + B)}{Z_g + Z_L}$$

Insertion loss =

$$IL = 20 \log_{10} \left| \frac{\bar{V}_2}{V_2} \right| = 20 \log_{10} \left| \frac{(Z_g(CZ_L + D) + AZ_L + B)}{Z_g + Z_L} \right|$$

which is the required result

Prob. 13.52

$$\begin{aligned} SE &= 20 \log_{10} \frac{E_i}{E_o} = 20 \log_{10} \frac{6}{20 \times 10^{-6}} = 20 \log_{10}(3 \times 10^5) \\ &= \underline{\underline{109.54 \text{ dB}}} \end{aligned}$$

CHAPTER 14

P. E. 14.1 The program in Fig. 14.3 was used to obtain the plot in Fig. 14.5.

P. E. 14.2 For the exact solution,

$$\xrightarrow{(D^2 + 1)y = 0} y = A \cos x + B \sin x$$

$$y(0) = 0 \rightarrow A = 0$$

$$y(1) = 1 \rightarrow 1 = B \sin 1 \text{ or } B = 1/\sin 1$$

$$\text{Thus, } y = \sin x / \sin 1$$

For the finite difference solution,

$$y'' + y = 0 \rightarrow \frac{y(x + \Delta) - 2y(x) + y(x - \Delta)}{\Delta^2} + y = 0$$

or

$$y(x) = \frac{y(x + \Delta) + y(x - \Delta)}{2 - \Delta^2}, y(0) = 0, y(1) = 1, \Delta = 1/4$$

With the MATLAB program shown below, we obtain the exact result y_e and FD result y .

```

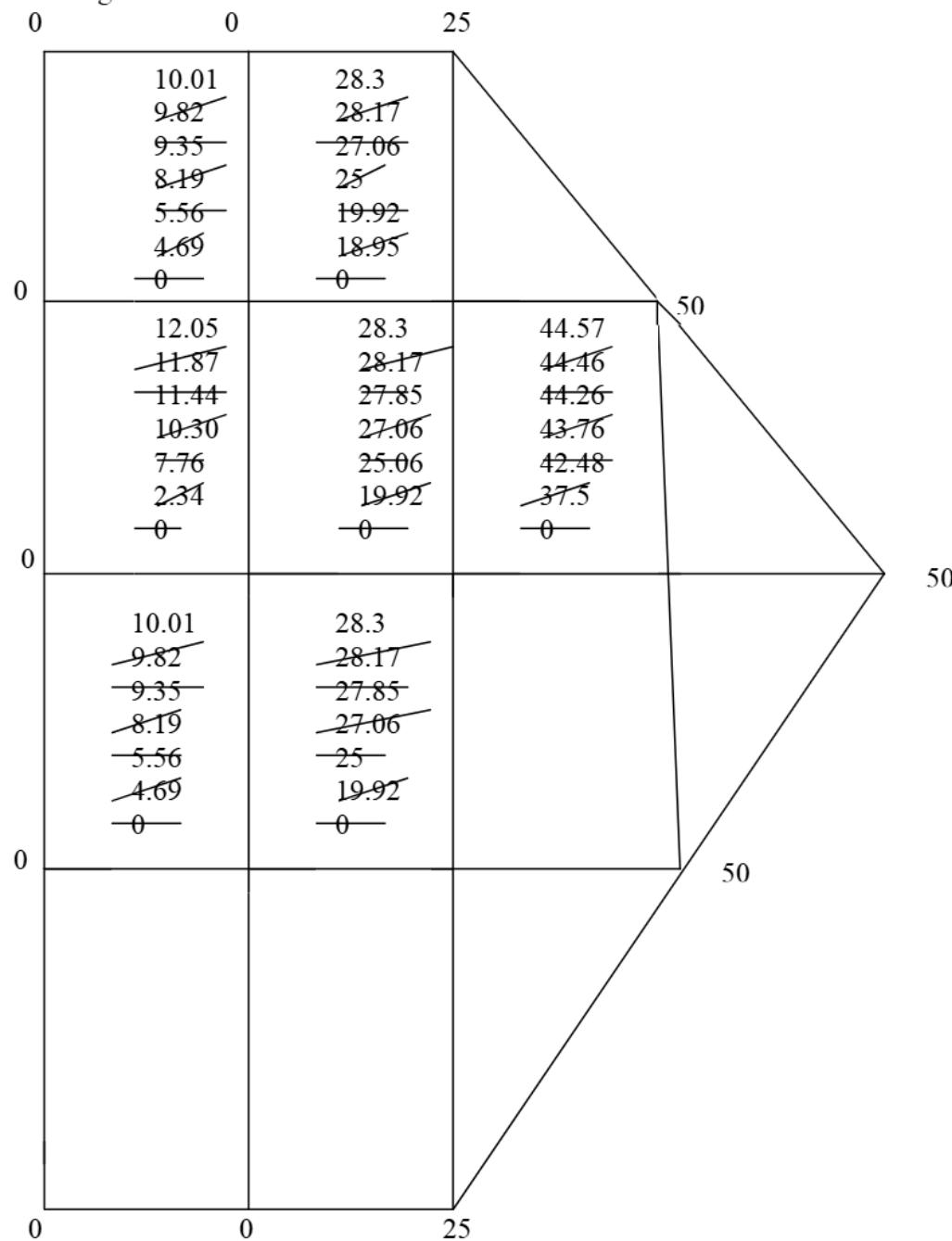
y(1)=0.0;
y(5)=1.0;
del=0.25;
for n=1:20
    for k=2:4
        y(k)=(y(k+1)+y(k-1))/(2-del*del)
        x=(k-1)*del;
        ye=sin(x)/sin(1.0)
    end
end

```

The results are listed below.

y(x)	N=5	N=10	N=15	N=20	Exact $y_e(x)$
y(0.25)	0.2498	0.2924	0.2942	0.2943	0.2940
y(0.5)	0.5242	0.5682	0.5701	0.5702	0.5697
y(0.75)	0.7867	0.8094	0.8104	0.8104	0.8101

P. E. 14.3 By applying eq. (14.16) to each node as shown below, we obtain the following results after 5 iterations.



P. E. 14.4 (a) Using the program in Fig. 14.16 with $nx = 4+1=5$ and $ny = 8+1=9$, we obtain the potential at center as

$$V(3,5) = \underline{23.796} \text{ V}$$

(b) Using the same program with $nx = 12+1=13$ and $ny = 24+1=25$, the potential at the center is

$$V(7,13) = \underline{23.883} \text{ V}$$

P. E. 14.5 $V = 0.79$ (the solution improves further and gets closer to the expected value due to a finer grid).

Code: It is the same as in Example 14.5 except the following lines:

```
grid_size=0.05;
nt=90; %% Top plate at 90th grid row from the upper boundary
nB=110; %% Bottom plate at 110th grid row from the upper boundary
ns1=90; %% Both plates start from 90th grid column
ns2=110; %% Both plates ends at 110th grid column
```

$$v = V((nt \times lx) + 100)$$

P. E. 14.6 The code remains essentially the same as in Example 14.6 except for boundary conditions which need to be explicitly defined as explained at the end of Section 14.4 above Practice Exercise 14.6.

P. E. 14.7 By combining the ideas in Figs. 14.23 and 14.27, and dividing each wire into N segments, the results listed in Table 14.2 is obtained.

P.E. 14.8 To determine V and \mathbf{E} at $(-1, 4, 5)$, we use the program in Fig. 14.25.

$$V = \int_0^L \frac{\rho_L dl}{4\pi\epsilon_o R}, \text{ where } R = \sqrt{26 + (4 - y')^2}$$

$$V = \frac{\Delta}{4\pi\epsilon} \sum_{k=1}^N \frac{\rho_k}{\sqrt{26 + (y - y_k)^2}}$$

$$E = \int_0^L \frac{\rho_L dl R}{4\pi\epsilon_o R^3}$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}' = (-1, 4-y', 5)$, $R = |\mathbf{R}|$

$$E_x \equiv \frac{\Delta}{4\pi\epsilon} \sum_{k=1}^N \frac{(-1)\rho_k}{[26 + (4 - y_k)^2]^{3/2}}$$

$$E_y \equiv \frac{\Delta}{4\pi\epsilon} \sum_{k=1}^N \frac{(4 - y_k)\rho_k}{[26 + (4 - y_k)^2]^{3/2}}$$

$$E_z = -5E_x$$

For $N = 20$, $V_o = 1V$, $L = 1m$, $a = 1mm$, the program in Fig. 14.23 is modified. The result is:

$$V = 12.47 \text{ mV}, E = -0.3266 \mathbf{a}_x + 1.1353 \mathbf{a}_y + 1.6331 \mathbf{a}_z \text{ mV/m}$$

P. E. 14.9 Consider Poisson's equation as the partial differential equation and Dirac delta function as the forcing function.

$$\nabla^2 G = -g$$

Green's function is the impulse response of the given differential equation. So forcing function f in the above expression is Dirac-delta function.

$$\nabla^2 G = -g$$

By considering axial symmetry the variation in the direction of ϕ and Z is zero.

$$\nabla^2 G = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial G}{\partial \rho} \right)$$

Substituting and integrating over the surface of ρ and ϕ

$$\iint \nabla^2 G \rho d\rho d\phi = \iint -\delta \rho d\rho d\phi$$

$$\iint \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial G}{\partial \rho} \rho d\rho d\phi = \iint \partial \left(\rho \frac{\partial G}{\partial \rho} \right) d\phi = -1$$

$$\int \partial \left(\rho \frac{\partial G}{\partial \rho} \right) \int_0^{2\pi} d\phi = \rho \frac{\partial G}{\partial \rho} 2\pi = -1$$

$$\int \partial G = \int -\frac{1}{2\pi} \frac{\partial \rho}{\rho}$$

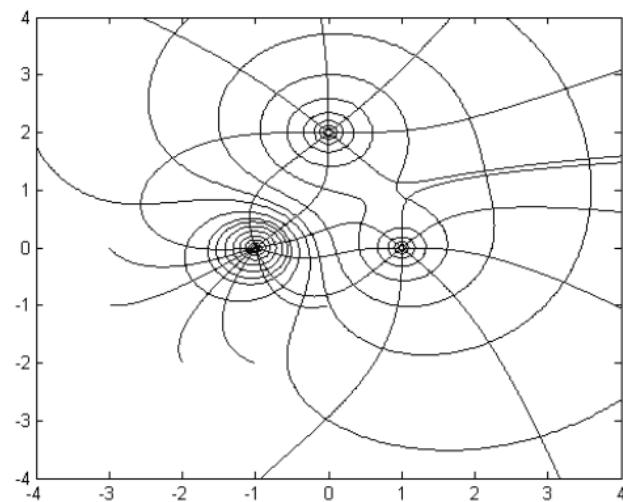
$$G = -\frac{1}{2\pi} \ln(\rho)$$

ρ is distance between the source and observation points.

Prob. 14.1 (a) Using the Matlab code in Fig. 14.3, we input the data as:

```
>> plotit([-1 2 1], [-1 0; 0 2; 1 0], 1, 1, 0.01, 0.01, 8, 2, 5)
```

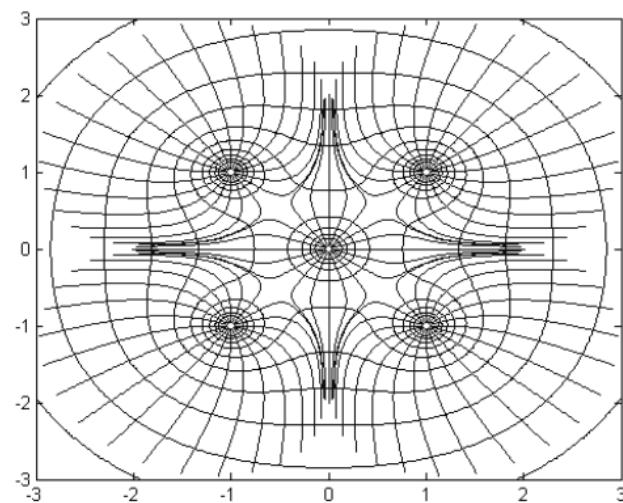
and the plot is shown below.



(b) Using the MATLAB code in Fig. 14.3, we input the required data as:

```
>> plotit([1 1 1 1 1], [-1 -1; -1 1; 1 -1; 1 1; 0 0], 1, 1, 0.02, 0.01, 6, 2, 5)
```

and obtain the plot shown below.



Prob. 14.2

$$\nabla^2 V = \frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial V}{\partial \rho} + \frac{\partial^2 V}{\partial z^2} = 0$$

The equivalent finite difference expression is

$$\begin{aligned} & \frac{V(\rho_o + \Delta\rho, z_o) - 2V(\rho_o, z_o) + V(\rho_o - \Delta\rho, z_o)}{(\Delta\rho)^2} + \frac{1}{\rho_o} \frac{V(\rho_o + \Delta\rho, z_o) - V(\rho_o - \Delta\rho, z_o)}{2\Delta\rho} \\ & + \frac{V(\rho_o, z_o + \Delta z) - 2V(\rho_o, z_o) + V(\rho_o, z_o - \Delta z)}{(\Delta z)^2} = 0 \end{aligned}$$

If $\Delta z = \Delta\rho = h$, rearranging terms gives

$$\begin{aligned} V(\rho_o, z_o) &= \frac{1}{4}V(\rho_o, z_o + h) + \frac{1}{4}V(\rho_o, z_o - h) + \left(1 + \frac{h}{2\rho_o}\right)V(\rho_o + h, z_o) \\ &+ \left(1 - \frac{h}{2\rho_o}\right)V(\rho_o - h, z_o) \end{aligned}$$

as expected.

Prob. 14.3 (a)

$$\frac{dV}{dx} = \frac{V(x + \Delta x) - V(x - \Delta x)}{2\Delta x}$$

For $\Delta x = 0.05$ and at $x = 0.15$,

$$\frac{dV}{dx} = \frac{2.0134 - 1.00}{0.05 \times 2} = \underline{\underline{10.117}}$$

$$\frac{d^2V}{dx^2} = \frac{V(x + \Delta x) - 2V(x) + V(x - \Delta x)}{(\Delta x)^2} = \frac{2.0134 + 1.0017 - 2 \times 1.5056}{(0.05)^2} = \underline{\underline{1.56}}$$

(b) $V = 10 \sinh x$, $dV/dx = 10 \cosh x$. At $x = 0.15$, $dV/dx = \underline{\underline{10.113}}$

which is close to the numerical estimate.

$$d^2V/dx^2 = 10 \sinh x. \text{ At } x = 0.15, d^2V/dx^2 = \underline{\underline{1.5056}}$$

which is slightly lower than the numerical value.

Prob.14.4

Exact solution:

$$(D^2 + 4)y = 0 \longrightarrow y(x) = A\cos 2x + B\sin 2x$$

$$y(0) = 0 \longrightarrow 0 = A$$

$$y(1) = 10 \longrightarrow 10 = B\sin 2 \longrightarrow B = \frac{10}{\sin 2}$$

$$y(x) = 10 \frac{\sin 2x}{\sin 2}$$

$$y(0.25) = 10 \frac{\sin 0.5}{\sin 2} = \underline{\underline{5.272}}$$

Finite difference solution:

$$\frac{y(x + \Delta) - 2y(x) + y(x - \Delta)}{\Delta^2} + 4y(x) = 0$$

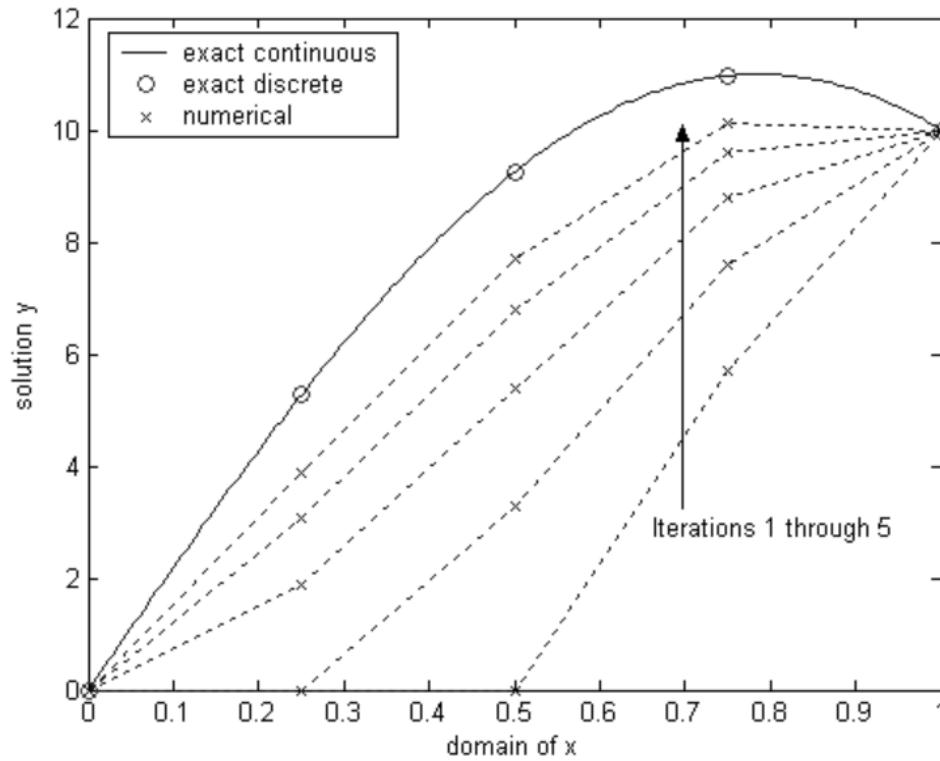
$$y(x + \Delta) + y(x - \Delta) = 2y(x) - 4\Delta^2 y(x) = (2 - 4\Delta^2)y(x)$$

or

$$y(x) = \frac{y(x + \Delta) + y(x - \Delta)}{(2 - 4\Delta^2)}, \quad \Delta = 0.25$$

Using this scheme, we obtain the result shown below. The number of iterations is not enough to get accurate result. The numerical results are compared with the exact solution as shown in the figure below.

Iteration	0	0.25	0.5	0.75	1.0
0	0	0	0	0	10
1	0	0	0	5.7143	10
2	0	0	3.2653	7.5802	10
3	0	1.8659	5.398	8.7987	10
4	0	3.0844	6.7904	9.5945	10
5	0	3.8802	7.7904	10.1142	10

**Prob. 14.5**

$$\nabla^2 V = \frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial V}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2} = 0, \quad (1)$$

$$\frac{\partial^2 V}{\partial \rho^2} = \frac{V_{m+1}^{n+1} - 2V_m^n + V_{m-1}^n}{(\Delta \rho)^2}, \quad (2)$$

$$\frac{\partial^2 V}{\partial \phi^2} = \frac{V_m^{n+1} - 2V_m^n + V_m^{n-1}}{(\Delta \phi)^2}, \quad (3)$$

$$\frac{\partial V}{\partial \rho} \Big|_{m,n} = \frac{V_{m+1}^n - V_{m-1}^n}{2\Delta \rho}. \quad (4)$$

Substituting (2) to (4) into (1) gives

$$\nabla^2 V = \frac{V_{m+1}^n - V_{m-1}^n}{m\Delta \rho(2\Delta \rho)} + \frac{V_{m+1}^{n+1} - 2V_m^n + V_{m-1}^n}{(\Delta \rho)^2} + \frac{V_m^{n+1} - 2V_m^n + V_m^{n-1}}{(m\Delta \rho\Delta \phi)^2}$$

$$= \frac{1}{(\Delta\rho)^2} \left[\left(1 - \frac{1}{2m}\right) V_{m-1}^{n-1} - 2V_m^n + \left(1 + \frac{1}{2m}\right) V_{m+1}^{n-1} + \frac{1}{(m\Delta\phi)^2} (V_m^{n+1} - 2V_m^n + V_m^{n-1}) \right]$$

as required.

Prob. 14.6

Iteration →	0	1	2	3	4	5
V ₁	0.0000	25.0000	35.6250	38.9063	39.7266	39.9316
V ₂	0.0000	26.2500	32.8125	34.4531	34.8633	34.9658
V ₃	0.0000	16.2500	22.8125	24.4531	24.8633	24.9658
V ₄	0.0000	15.6250	18.9063	19.7266	19.9316	19.9829

Prob. 14.7

$$V_o = \frac{V_1 + V_2 + V_3 + V_4}{4} = \frac{10 - 40 + 50 + 80}{4} = \underline{\underline{25V}}$$

Prob. 14.8

$$V_1 = \frac{1}{4} [20 + 20 + 40 + V_4] = 20 + \frac{V_4}{4}$$

$$V_2 = \frac{1}{4} [-10 + 20 + 0 + V_3] = 2.5 + \frac{V_3}{4}$$

$$V_3 = \frac{1}{4} [0 + 20 + V_2 + V_4] = 5 + \frac{1}{4}(V_2 + V_4)$$

$$V_4 = \frac{1}{4} [0 + 40 + V_1 + V_3] = 10 + \frac{1}{4}(V_1 + V_3)$$

Using these relationships, we obtain the data in the table below.

Iteration	1st	2nd	3rd	4th	5th
V ₁	20	24.1	24.64	24.72	24.73
V ₂	2.5	3.91	5.02	5.22	5.26
V ₃	5.625	10.08	10.89	11.03	11.05
V ₄	16.406	18.54	18.88	18.94	18.95

Prob. 14.9

(a) We follow Example 6.5 with $a=b$.

$$V = V_1 + V_2 = \frac{4V_o}{\pi} \sum_{n=odd} \frac{\sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right)}{n \sinh(n\pi)} + \frac{4V_o}{\pi} \sum_{n=odd} \frac{\sin\left(\frac{n\pi y}{a}\right) \sinh\left(\frac{n\pi x}{a}\right)}{n \sinh(n\pi)}$$

(b) At the center of the region, finite difference gives

$$V(a/2, a/2) = \frac{1}{4}(0 + 0 + V_o + V_o) = \frac{V_o}{2} = \underline{\underline{25 \text{ V}}}$$

Prob. 14.10

$$k = \frac{h^2 \rho_s}{\epsilon} = 10^{-4} \times \frac{50 \times 10^{-9}}{\frac{10^{-9}}{36\pi}} = 0.18\pi = 0.5655$$

At node 1,

$$V_1 = \frac{1}{4}[0 + V_2 + V_3 + k] \longrightarrow 4V_1 - V_2 - V_3 = k \quad (1)$$

At node 2,

$$V_2 = \frac{1}{4}[0 + V_1 + V_4 + k] \longrightarrow 4V_2 - V_1 - V_4 = k \quad (2)$$

At node 3,

$$V_3 = \frac{1}{4}[0 + 2V_1 + V_4 + k] \longrightarrow 4V_3 - 2V_1 - V_4 = k \quad (3)$$

At node 4,

$$V_4 = \frac{1}{4}[0 + 2V_2 + V_3 + k] \longrightarrow 4V_4 - 2V_2 - V_3 = k \quad (4)$$

Putting (1) to (4) in matrix form,

$$\begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -2 & 0 & 4 & -1 \\ 0 & -2 & -1 & 4 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} 0.5655 \\ 0.5655 \\ 0.5655 \\ 0.5655 \end{bmatrix}$$

Using a calculator or MATLAB, we obtain

$$\underline{\underline{V_1 = V_2 = 0.3231 \text{ V}, \quad V_3 = V_4 = 0.4039 \text{ V}}}$$

Prob. 14.11

(a)

$$\left[\begin{array}{cccccc} -4 & 1 & 0 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 \\ 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 1 & 0 & 1 & -4 \end{array} \right] \left[\begin{array}{c} V_a \\ V_b \\ V_c \\ V_d \\ V_e \\ V_f \end{array} \right] = \left[\begin{array}{c} -200 \\ -100 \\ -100 \\ -100 \\ 0 \\ 0 \end{array} \right]$$

(b)

$$\left[\begin{array}{ccccccc} -4 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -4 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{array} \right] \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \\ V_8 \end{bmatrix} = \begin{bmatrix} -30 \\ -15 \\ -30 \\ -7.5 \\ 0 \\ -7.5 \\ 0 \\ 0 \end{bmatrix}$$

Prob. 14.12 (a) Matrix [A] remains the same. To each term of matrix [B], we add

$$-h^2 \rho_v / \varepsilon.$$

(b) Let $\Delta x = \Delta y = h = 0.25$ so that $n_x = 5 = n_y$.

$$\frac{\rho_v}{\varepsilon} = \frac{x(y-1)10^{-9}}{10^{-9}/36\pi} = 36\pi x(y-1)$$

Modify the program in Fig. 14.16 as follows.

```

H=0.25;
for I=1:nx-1
    for J=1: ny-1
        X = H*I;
        Y=H*J;
        RO = 36.0*pi*X*(Y-1);
        V(I,J) = 0.25*(V(I+1,J) + V(I-1,J) + V(I,J+1) + V(I,J-1) + H*H*RO);
    end
end

```

This is the major change. However, in addition to this, we must set

```
v1 = 0.0;
v2 = 10.0;
v3 = 20.0;
v4 = -10.0;
nx = 5;
ny = 5;
```

The results are:

$$\begin{aligned} V_a &= 4.6095 & V_b &= 9.9440 & V_c &= 11.6577 \\ V_d &= -1.5061 & V_e &= 3.5090 & V_f &= 6.6867 \\ \underline{V_g = -3.2592} & & \underline{V_h = 0.2366} & & \underline{V_i = 3.3472} \end{aligned}$$

Prob. 14.13

$$V_1 = \frac{1}{4}(0 + 0 + V_2 + V_4) = \frac{1}{4}(V_2 + V_4)$$

$$V_2 = \frac{1}{4}(0 + 50 + V_1 + V_3) = \frac{1}{4}(50 + V_1 + V_3)$$

$$V_3 = \frac{1}{4}(0 + 100 + 50 + V_2) = \frac{1}{4}(150 + V_2)$$

$$V_4 = \frac{1}{4}(0 + 50 + V_1 + V_5) = \frac{1}{4}(50 + V_1 + V_5)$$

$$V_5 = \frac{1}{4}(0 + 0 + V_4 + V_6) = \frac{1}{4}(V_4 + V_6)$$

$$V_6 = \frac{1}{4}(0 + 50 + V_5 + V_7) = \frac{1}{4}(50 + V_5 + V_7)$$

$$V_7 = \frac{1}{4}(0 + 100 + V_6 + 50) = \frac{1}{4}(150 + V_6)$$

Initially set all free potentials equal to zero. Apply the seven formulas above iteratively and obtain the results shown below.

n	1	2	3	4	5
V ₁	0	6.25	9.77	10.63	10.97
V ₂	12.5	24.22	25.83	26.15	26.25
V ₃	40.625	43.55	43.96	44.04	44.06
V ₄	12.5	14.84	16.70	17.73	17.97
V ₅	3.12	7.03	10.29	10.93	11.05
V ₆	13.281	24.46	25.98	26.23	26.28
V ₇	40.82	43.62	43.99	44.06	44.07

Prob. 14.14

$$\frac{1}{c^2} \frac{\Phi_{m,n}^{j+1} + \Phi_{m,n}^{j-1} - 2\Phi_{m,n}^j}{(\Delta t)^2} = \frac{\Phi_{m+1,n}^j + \Phi_{m-1,n}^j - 2\Phi_{m,n}^j}{(\Delta x)^2}$$

$$+ \frac{\Phi_{m,n+1}^j + \Phi_{m,n-1}^j - 2\Phi_{m,n}^j}{(\Delta z)^2}$$

If $h = \Delta x = \Delta z$, then after rearranging we obtain

$$\Phi_{m,n}^{j+1} = 2\Phi_{m,n}^j - \Phi_{m,n}^{j-1} + \alpha(\Phi_{m+1,n}^j + \Phi_{m-1,n}^j - 2\Phi_{m,n}^j)$$

$$+ \alpha(\Phi_{m,n+1}^j + \Phi_{m,n-1}^j - 2\Phi_{m,n}^j)$$

where $\alpha = (c\Delta t / h)^2$.

Prob. 14.15

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 V}{\partial t^2} \quad \longrightarrow \quad \frac{V(x + \Delta x, t) - 2V(x, t) + V(x - \Delta x, t)}{(\Delta x)^2} =$$

$$\frac{V(x, t + \Delta t) - 2V(x, t) + V(x, t - \Delta t)}{(\Delta t)^2}$$

$$V(x, t + \Delta t) = \left(\frac{\Delta t}{\Delta x} \right)^2 [V(x + \Delta x, t) - 2V(x, t) + V(x - \Delta x, t)] + 2V(x, t) - V(x, t - \Delta t)$$

or

$$V(i, j + 1) = \alpha[V(i + 1, j) + v(i - 1, j)] + 2(1 - \alpha)V(i, j) - V(i, j - 1)$$

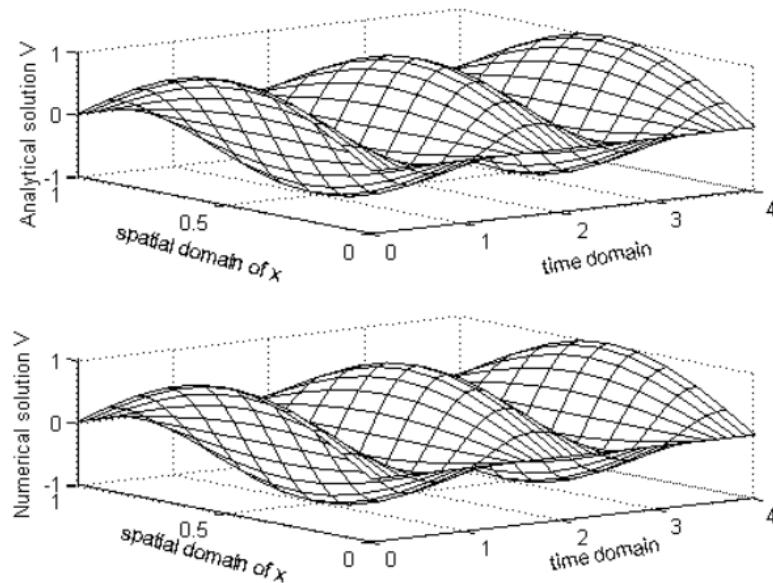
where $\alpha = \left(\frac{\Delta t}{\Delta x} \right)^2$. Applying the finite difference formula derived above, the following

programs was developed.

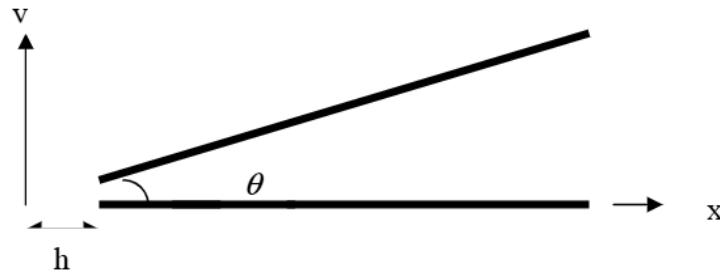
```
xd=0:.1:1;td=0:.1:4;
[t,x]=meshgrid(td,xd);
Va=sin(pi*x).*cos(pi*t);%Analytical result
subplot(211);mesh(td,xd,Va);colormap([0 0 0])
% Numerical result
N=length(xd);M=length(td);
v(:,1)=sin(pi*xd');
v(2:N-1,2)=(v(1:N-2,1)+v(3:N,1))/2;
for k=2:M-1
    v(2:N-1,k+1)=-v(2:N-1,k-1)+v(1:N-2,k)+v(3:N,k);
```

```
end
subplot(212);mesh(td,xd,v);colormap([0 0 0])
```

The results of the finite difference algorithm agree perfectly with the exact solution as shown below.



Prob. 14.16



To find C, take the following steps:

- (1) Divide each line into N equal segments. Number the segments in the lower conductor as 1, 2, ..., N and segments in the upper conductor as N+1, N+2, ..., 2N,
- (2) Determine the coordinate (x_k, y_k) for the center of each segment.

For the lower conductor, $y_k = 0, k=1, \dots, N, x_k = h + \Delta(k-1/2), k = 1, 2, \dots, N$

For the upper conductor, $y_k = [h + \Delta(k-1/2)] \sin \theta$, $k=N+1, N+2, \dots, 2N$,
 $x_k = [h + \Delta(k-1/2)] \cos \theta$, $k = N+1, N+2, \dots, 2N$

where h is determined from the gap g as

$$h = \frac{g}{2 \sin \theta / 2}$$

(3) Calculate the matrices $[V]$ and $[A]$ with the following elements

$$V_k = \begin{cases} V_o, k = 1, \dots, N \\ -V_o, k = N+1, \dots, 2N \end{cases}$$

$$A_{ij} = \begin{cases} \frac{\Delta}{4\pi\epsilon R_{ij}}, i \neq j \\ 2\ln\Delta/a, i = j \end{cases}$$

where $R_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$

(4) Invert matrix $[A]$ and find $[\rho] = [A]^{-1} [V]$.

(5) Find the charge Q on one conductor

$$Q = \sum \rho_k \Delta = \Delta \sum_{k=1}^N \rho_k$$

(6) Find $C = |Q|/2V_o$

Taking $N=10$, $V_o=1.0$, a program was developed to obtain the following result.

θ	C (in pF)
10	8.5483
20	9.0677
30	8.893
40	8.606
50	13.004
60	8.5505
70	9.3711
80	8.7762
90	8.665
100	8.665
110	10.179
120	8.544
130	9.892
140	8.7449
150	9.5106
160	8.5488
170	11.32
180	8.6278

Prob. 14.17 Combining the ideas in the programs in Figs. 14.23 and 14.27, we develop a MATLAB code which gives

$$N = 20 \longrightarrow C = 19.4 \text{ pF/m}$$

$$N = 40 \longrightarrow C = 13.55 \text{ pF/m}$$

$$N = 100 \longrightarrow C = 12.77 \text{ pF/m}$$

For the exact value, $d/2a = 50/10 = 5$

$$C = \frac{\pi \epsilon}{\cosh^{-1} \frac{d}{2a}} = \frac{\pi \times 10^{-9} / 36\pi}{\cosh^{-1} 5} = 12.12 \text{ pF/m}$$

Prob. 14.18 We may modify the program in Fig. 14.27 and obtain the result in the table below. $Z_o \equiv \underline{\underline{100 \Omega}}$.

N	Z_o , in Ω
10	97.2351
20	97.8277
30	98.0515
40	98.1739
50	98.2524

Prob. 14.19

We make use of the formulas in Problem 14.18.

$$V_i = \sum_{j=1}^{2N} A_j \rho_i$$

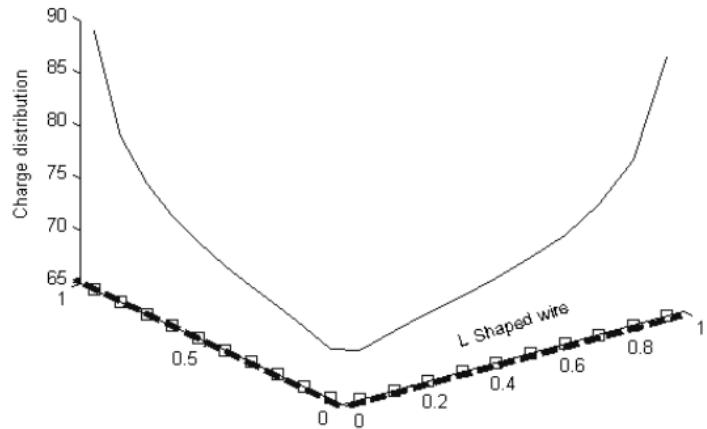
where N is the number of divisions on each arm of the conductor.

The MATLAB codes is as follows:

```

aa=0.001;
L=2.0;
N=10; %no.of divisions on each arm
NT=N*2;
delta=L/(NT);
x=zeros(NT,1);
y=zeros(NT,1);
%Second calculate the elements of the coefficient matrix
for i=1:N-1
    y(i)=0;
    x(i)=delta*(i-0.5)
end
for i=N+1:NT
    x(i)=0;
    y(i)=delta*(i-N-0.5);
end
for i=1:NT
    for j=1:NT
        if (i ~= j)
            R=sqrt((x(i)-x(j))^2 + (y(i)-y(j))^2)
            A(i,j)=-delta*R;
        else
            A(i,j)=-delta*(log(delta)-1.5);
        end
    end
end
%Determine the matrix of constant vector B and find rho
B=2*pi*eo*vo*ones(NT,1);
rho=inv(A)*B;
```

The result is presented below.



Segment	x	y	ρ in pC/m
1	0.9500		89.6711
2	0.8500	0	80.7171
3	0.7500	0	77.3794
4	0.6500	0	75.4209
5	0.5500	0	74.0605
6	0.4500	0	73.0192
7	0.3500	0	72.1641
8	0.2500	0	71.4150
9	0.1500	0	70.6816
10	0.0500	0	69.6949
11	0	0	69.6949
12	0	0.0500	70.6816
13	0	0.1500	71.4150
14	0	0.2500	72.1641
15	0	0.3500	73.0192
16	0	0.4500	74.0605
17	0	0.5500	75.4209
18	0	0.6500	77.3794
19	0	0.7500	80.7171
20	0	0.8500	89.6711

Prob. 14.20(a) Exact solution yields

$$C = 2\pi\epsilon / \ln(\Delta / a) = 8.02607 \times 10^{-11} \text{ F/m and } Z_o = 41.559\Omega$$

where $a = 1\text{cm}$ and $\Delta = 2\text{cm}$. The numerical solution is shown below.

N	C (pF/m)	$Z_o(\Omega)$
10	82.386	40.486
20	80.966	41.197
40	80.438	41.467
100	80.025	41.562

(b) For this case, the numerical solution is shown below.

N	C (pF/m)	$Z_o(\Omega)$
10	109.51	30.458
20	108.71	30.681
40	108.27	30.807
100	107.93	30.905

Prob. 14.21 We modify the MATLAB code in Fig. 14.27 (for Example 14.7) by changing the input data and matrices [A] and [B]. We let

$$x_i = h + \Delta(i-1/2), \quad i = 1, 2, \dots, N, \quad \Delta = L/N$$

$$y_i = h/2, \quad j = 1, 2, \dots, N, \quad z_k = t/2, \quad k = 1, 2, \dots, N$$

and calculate

$$R_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}$$

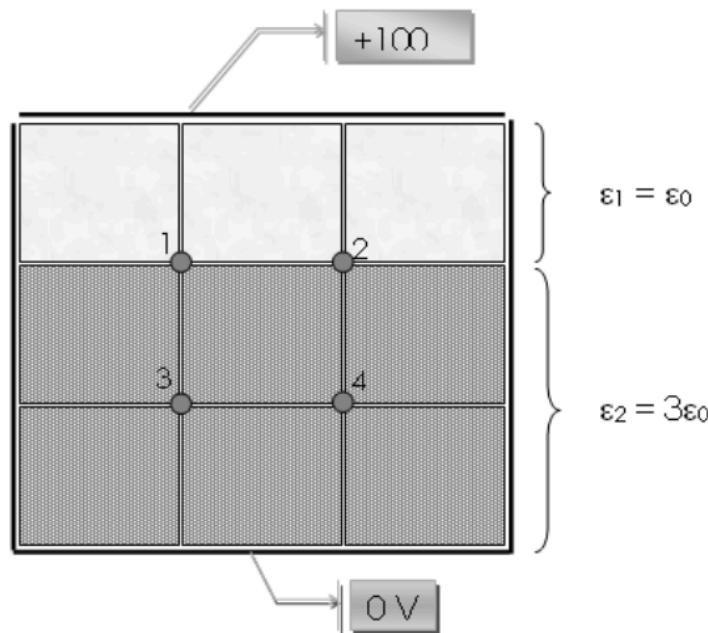
We obtain matrices [A] and [B]. Inverting [A] gives

$$[q] = [A]^{-1} [B], \quad [\rho_v] = [q]/(ht\Delta), \quad C = \frac{\sum_{i=1}^N q_i}{10}$$

The computed values of $[\rho_v]$ and C are shown below.

Prob. 14.22

The MATLAB code is similar to the one in Fig.14.34. When the program is run, it gives $Z_o = \underline{\underline{40.587 \Omega}}$.

Prob. 14.23

On the interface,

$$\frac{\epsilon_1}{2(\epsilon_1 + \epsilon_2)} = \frac{1}{8}, \quad \frac{\epsilon_2}{2(\epsilon_1 + \epsilon_2)} = \frac{3}{8}$$

$$V_1 = \frac{V_2}{4} + \frac{3V_3}{8} + 12.5$$

$$V_2 = 12.5 + \frac{3V_4}{8} + \frac{V_1}{4}$$

$$V_3 = \frac{1}{4}(V_1 + V_4)$$

$$V_4 = \frac{1}{4}(V_2 + V_3)$$

Applying this iteratively, we obtain the results shown in the table below.

No. of iterations	0	1	2	3	4	5...	100
V ₁	0	12.5	17.57	19.25	19.77	19.93	20
V ₂	0	15.62	18.65	19.58	19.87	19.96	20
V ₃	0	3.125	5.566	6.33	6.56	6.6634	6.667
V ₄	0	4.688	6.055	6.477	6.608	6.649	6.667

Prob. 14.24

$$V_1 = \frac{1}{4}(0 + 0 + 100 + V_2) = 25 + \frac{V_2}{4}$$

$$V_2 = \frac{1}{4}(0 + 100 + V_1 + V_3) = 25 + \frac{V_1 + V_3}{4}$$

$$V_3 = \frac{1}{4}(0 + 0 + 100 + V_2) = 25 + \frac{V_2}{4}$$

$$V_4 = \frac{1}{4}(0 + 0 + 100 + V_5) = 25 + \frac{V_5}{4}$$

$$V_5 = \frac{1}{4}(0 + 0 + 100 + V_4 + V_6) = 25 + \frac{(V_4 + V_6)}{4}$$

$$V_6 = \frac{1}{4}(0 + 0 + 100 + V_5) = 25 + \frac{V_5}{4}$$

We initially set $V_1 = V_2 = V_3 = V_4 = V_5 = V_6 = 0$ and then apply above formulas iteratively. The solutions are presented in the table below.

iteration	1st	2nd	3rd	4th	5th
V ₁	25	32.81	35.35	35.67	35.71
V ₂	31.25	41.41	42.68	42.83	42.85
V ₃	32.81	35.35	35.67	35.71	35.71
V ₄	25	23.81	35.35	35.67	35.71
V ₅	31.25	41.41	42.68	42.83	42.85
V ₆	32.81	35.35	35.67	35.71	35.71

$$\underline{V_1 = V_4 = 35.71 \text{ V}, \quad V_2 = V_5 = 42.85 \text{ V}, \quad V_3 = V_6 = 35.71 \text{ V}}$$

Alternatively, if we take advantage of the symmetry, $V_1 = V_3 = V_4 = V_6$ and $V_2 = V_5$. We need to find solve two equations, namely,

$$V_1 = 25 + V_2 / 4$$

$$V_2 = 25 + V_1 / 2$$

Solving these gives

$$V_1 = 35.714$$

$$V_2 = 42.857$$

Other node voltages follow.

Prob. 14.25

The finite difference solution is obtained by following the same steps as in Example 14.10. We obtain $\underline{\underline{Z_o}} = 43 \Omega$

Prob.14.26

$$V_1 = \frac{1}{4}(V_2 + 100 + 100 + 100) = \frac{1}{4}V_2 + 75$$

$$V_2 = \frac{1}{4}(V_1 + V_4 + 2V_3)$$

$$V_3 = \frac{1}{4}(V_2 + V_5 + 200) = \frac{1}{4}(V_2 + V_5) + 50$$

$$V_4 = \frac{1}{4}(V_2 + V_7 + 2V_5)$$

$$V_5 = \frac{1}{4}(V_3 + V_4 + V_6 + V_8)$$

$$V_6 = \frac{1}{4}(V_5 + V_9 + 200) = \frac{1}{4}(V_5 + V_9) + 50$$

$$V_7 = \frac{1}{4}(V_4 + 2V_8 + 0) = \frac{1}{4}(V_4 + 2V_8)$$

$$V_8 = \frac{1}{4}(V_5 + V_7 + V_9)$$

$$V_9 = \frac{1}{4}(V_6 + V_8 + 100 + 0) = \frac{1}{4}(V_6 + V_8) + 25$$

Using these equations, we apply iterative method and obtain the results shown below.

	1 st	2 nd	3 rd	4 th	5 th
V ₁	75	79.687	87.11	89.91	92.01
V ₂	18.75	48.437	59.64	68.06	74.31
V ₃	54.69	65.82	73.87	79.38	82.89
V ₄	4.687	19.824	34.57	46.47	53.72
V ₅	14.687	35.14	49.45	57.24	61.78
V ₆	53.71	68.82	74.2	77.01	78.6
V ₇	1.172	6.958	18.92	26.08	30.194
V ₈	4.003	20.557	28.93	33.53	36.153
V ₉	39.43	47.34	50.78	52.63	53.69

Prob. 14.27

Applying the difference method,

$$V_1 = \frac{V_3}{4} + \frac{V_2}{2} + 25$$

$$V_2 = \frac{1}{4}(V_1 + V_4) + 50$$

$$V_3 = \frac{1}{4}(V_1 + 2V_4)$$

$$V_4 = \frac{1}{4}(V_2 + V_3 + V_5)$$

$$V_5 = \frac{V_4}{4} + 50$$

Applying these equations iteratively, we obtain the results below.

Iterations	0	1	2	3	4	5...	100
V ₁	0	25.0	54.68	64.16	70.97	73.79	74.68
V ₂	0	56.25	67.58	74.96	78.23	79.54	80.41
V ₃	0	6.25	17.58	33.91	38.72	40.63	41.89
V ₄	0	15.63	35.74	41.96	44.86	45.31	45.95
V ₅	0	53.91	58.74	60.49	61.09	61.37	51.49

CHAPTER 15

P. E. 15.1 From Table 15.1, the functional for the two-dimensional diffusion equation is

$$F = \frac{1}{2} \int_v \left(\left(\left(\frac{\partial A}{\partial x} \right)^2 + \left(\frac{\partial A}{\partial y} \right)^2 \right) + j\omega\mu\sigma A^2 \right) dx dy$$

The function of the functional is $f = \frac{1}{2} \left(\left(\left(\frac{\partial A}{\partial x} \right)^2 + \left(\frac{\partial A}{\partial y} \right)^2 \right) + j\omega\mu\sigma A^2 \right) = \frac{1}{2} (A'_x)^2 + A'_y)^2 + j\omega\mu\sigma A^2)$

Putting this in the Euler-Lagrange equation,

$$\frac{\partial f}{\partial A} - \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial A'_x} \right) - \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial A'_y} \right) = 0$$

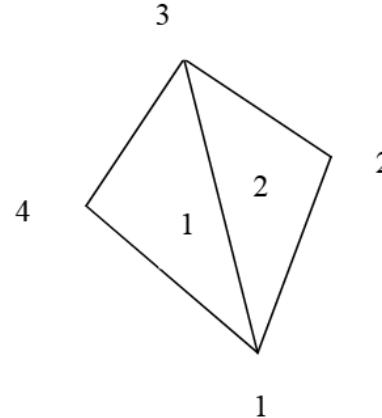
$$j\omega\mu\sigma A - (A''_x) - (A''_y) = 0$$

$$(A''_x) + (A''_y) - j\omega\mu\sigma A = 0$$

Hence, minimizing the above functional leads to the solution of the diffusion equation.

P. E. 15.2

(a)



For element 1, local 1-2-3 corresponds with global 1-3-4 so that $A_1 = 0.35$,

$$P_1 = 0.8, P_2 = 0.6, P_3 = -1.4, Q_1 = -0.5, Q_2 = 0.5, Q_3 = 0$$

$$\mathcal{N}^{(1)} = \begin{bmatrix} 0.6357 & 0.1643 & -0.8 \\ 0.1643 & 0.4357 & -0.6 \\ -0.8 & -0.6 & 1.4 \end{bmatrix}$$

For element 2, local 1-2-3 corresponds with global 1-2-3 so that $A_2 = 0.7$,

$$P_1 = 0.1, P_2 = 1.4, P_3 = -1.5, Q_1 = -1, Q_2 = 0, Q_3 = 1$$

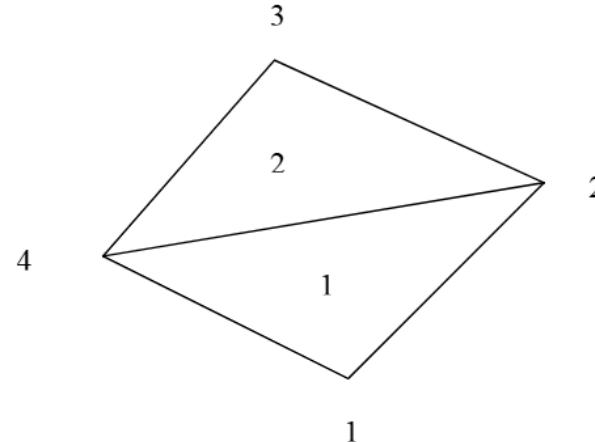
$$C^{(2)} = \begin{bmatrix} 0.3607 & 0.05 & -0.4107 \\ 0.05 & 0.7 & -0.75 \\ -0.4107 & -0.75 & 1.1607 \end{bmatrix}$$

The global coefficient matrix is given by

$$C = \begin{bmatrix} C_{11}^{(1)} + C_{11}^{(2)} & C_{12}^{(2)} & C_{12}^{(1)} + C_{13}^{(2)} & C_{13}^{(1)} \\ C_{21}^{(2)} & C_{22}^{(2)} & C_{23}^{(2)} & 0 \\ C_{21}^{(1)} + C_{31}^{(2)} & C_{32}^{(2)} & C_{22}^{(1)} + C_{33}^{(2)} & C_{23}^{(1)} \\ C_{31}^{(1)} & 0 & C_{32}^{(2)} & C_{33}^{(1)} \end{bmatrix}$$

$$= \begin{bmatrix} 0.9964 & 0.05 & -0.2464 & -0.8 \\ 0.05 & 0.7 & -0.75 & 0 \\ -0.2464 & -0.75 & 1.596 & -0.6 \\ -0.8 & 0 & -0.75 & 1.4 \end{bmatrix}$$

(b)



For element 1, local 1-2-3 corresponds with global 1-2-4 .

$$P_1 = 0.9000; P_2 = 0.6000; P_3 = -1.5000$$

$$Q_1 = -1.5000; Q_2 = 0.5000; Q_3 = 1;$$

$$A_1 = 0.6750;$$

$$C^{(1)} = \begin{bmatrix} 1.1333 & -0.0778 & -1.0556 \\ -0.0778 & 0.2259 & -0.1481 \\ -1.0556 & -0.1481 & 1.2037 \end{bmatrix}$$

For element 2, local numbering 1-2-3 corresponds with global numbering 2-3-4.

$$P_1 = 0.8000; P_2 = -0.9000; P_3 = 0.1000;$$

$$Q_1 = -0.5000; Q_2 = 1.5000; Q_3 = -1;$$

$$A_2 = 0.3750;$$

$$C^{(2)} = \begin{bmatrix} 0.5933 & -0.9800 & 0.3867 \\ -0.9800 & 2.0400 & -1.0600 \\ 0.3867 & -1.0600 & 0.6733 \end{bmatrix}$$

The global coefficient matrix is

$$C = \begin{bmatrix} C_{11}^{(1)} & C_{12}^{(1)} & 0 & C_{13}^{(1)} \\ C_{21}^{(1)} & C_{22}^{(1)} + C_{11}^{(2)} & C_{12}^{(2)} & C_{23}^{(1)} + C_{13}^{(2)} \\ 0 & C_{12}^{(2)} & C_{22}^{(2)} & C_{23}^{(2)} \\ C_{31}^{(1)} & C_{32}^{(1)} + C_{31}^{(2)} & C_{32}^{(2)} & C_{33}^{(1)} + C_{33}^{(2)} \end{bmatrix}$$

$$C = \begin{bmatrix} 1.1333 & -0.0778 & 0 & -1.0556 \\ -0.0778 & 0.8193 & -0.9800 & 0.2385 \\ 0 & -0.9800 & 2.0400 & -1.0600 \\ -1.0556 & 0.2385 & -1.0600 & 1.8770 \end{bmatrix}$$

P.E. 15.3 We use the MATLAB program in Fig. 15.8. The input data for the region in Fig. 15.9 is as follows:

```
NE = 32; ND = 26; NP = 18;
NL = [ 1 2 4
       2 5 4
       2 3 5
       3 6 5
       4 5 9
       5 10 9
       5 6 10 ]
```

```

6 11 10
7 8 12
8 13 12
8 9 13
9 14 13
9 10 14
10 15 14
10 11 15
11 16 15
12 13 17
13 18 17
13 14 18
14 19 18
14 15 19
15 20 19
15 16 20
16 21 20
17 18 22
18 23 22
18 19 23
19 24 23
19 20 24
20 25 24
20 21 25
21 26 25];
X = [ 1.0 1.5 2.0 1.0 1.5 2.0 0.0 0.5 1.0 1.5 2.0 0.0 0.5 1.0 1.5 2.0 0.0 0.5 1.0 1.5
      2 0.0 0.5 1.0 1.5 2.0];
Y = [ 0.0 0.0 0.0 0.5 0.5 0.5 1.0 1.0 1.0 1.0 1.0 1.5 1.5 1.5 1.5 1.5 1.5 2.0 2.0 2.0
      2.0 2.0 2.5 2.5 2.5 2.5 2.5 ];
NDP = [ 1 2 3 6 11 16 21 26 25 24 23 22 17 12 7 8 9 4];
VAL = [0.0 0.0 15.0 30.0 30.0 30.0 25.0 20.0 20.0 20.0 10.0 0.0 0.0 0.0
      0.0 0.0 0.0];

```

With this data, the finite element (FEM) solution is compared with the finite difference (FD) solution as shown in the table below.

Node #	X	Y	FEM	FD
5	1.5	0.5	11.265	11.25
10	1.5	1.0	15.06	15.02
13	0.5	1.5	4.958	4.705
14	1.0	1.5	9.788	9.545
15	1.0	1.5	18.97	18.84
18	0.5	2.0	10.04	9.659
19	1.0	2.0	15.32	15.85
20	1.5	2.0	21.05	20.87

P. E. 15.4

Using Eq. (15.22):

Energy stored =

$$\frac{1}{2} \times 10^6 \times [4 \ 5 \ 6] \begin{bmatrix} 0.56 & -0.46 & -0.1 \\ -0.46 & 0.82 & -0.37 \\ -0.1 & -0.37 & 0.47 \end{bmatrix} \times 10^6 \times \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \times 8.85 \times 10^{-12} = 4.3 \text{ J/m}$$

P. E. 15.5

$$\begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e3} \end{bmatrix} = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$V_e = [1 \ x \ y] \frac{1}{(x_2 - x_1)(x_3 - x_1)(x_3 - x_2)} \begin{bmatrix} (x_3 - x_2)x_3x_2 & (x_1 - x_3)x_3x_1 & (x_2 - x_1)x_2x_1 \\ -(x_3 - x_2)(x_3 + x_2) & -(x_1 - x_3)(x_1 + x_3) & -(x_2 - x_1)(x_2 + x_1) \\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1) \end{bmatrix} \begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e3} \end{bmatrix}$$

$$V_e = \sum_{i=1}^3 \alpha_i(x, y) V_{ei}$$

$$\alpha_1 = \frac{1}{(x_2 - x_1)(x_3 - x_1)(x_3 - x_2)} [(x_3 - x_2)x_3x_2 - (x_3 - x_2)(x_3 + x_2)x \\ + (x_3 - x_2)x^2] = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)}$$

$$\alpha_2 = \frac{1}{(x_2 - x_1)(x_3 - x_1)(x_3 - x_2)} [(x_1 - x_3)x_3x_1 - (x_1 - x_3)(x_1 + x_3)x \\ + (x_1 - x_3)x^2] = \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)}$$

$$\alpha_3 = \frac{1}{(x_2 - x_1)(x_3 - x_1)(x_3 - x_2)} [(x_2 - x_1)x_2x_1 - (x_2 - x_1)(x_2 + x_1)x \\ + (x_2 - x_1)x^2] = \frac{(x - x_2)(x - x_1)}{(x_3 - x_2)(x_3 - x_1)}$$

Therefore, for an i^{th} node of an element having i, j and k nodes, the shape function is

$$\alpha_i = \frac{(x - x_j)(x - x_k)}{(x_i - x_j)(x_i - x_k)}$$

P. E. 15.6

$$\mathbf{E} = -\nabla V$$

Using eq. (15.38),

$$V_e = \alpha_1 V_{e1} + \alpha_2 V_{e2} + \alpha_3 V_{e3}$$

Using eq. (15.11), we obtain:

$$\begin{aligned} V_e &= \frac{(x_2y_3 - x_3y_2) + P_1x + Q_1y}{2\Delta} V_{e1} + \frac{(x_3y_1 - x_1y_3) + P_2x + Q_2y}{2\Delta} V_{e2} \\ &\quad + \frac{(x_1y_2 - x_2y_1) + P_3x + Q_3y}{2\Delta} V_{e3} \\ \mathbf{E}^e &= -\left(\frac{P_1V_{e1} + P_2V_{e2} + P_3V_{e3}}{2\Delta}\right) \mathbf{a}_x - \left(\frac{Q_1V_{e1} + Q_2V_{e2} + Q_3V_{e3}}{2\Delta}\right) \mathbf{a}_y \\ \mathbf{E}^e &= -\left(\frac{(0.5 \times -1) + (0.25 \times 1) + 0}{2 \times 0.5 \times 10^{-6}}\right) \mathbf{a}_x - \left(\frac{(0 + (0.25 \times -1) + (0.16 \times 1)) \times 10^{-6}}{2 \times 0.5 \times 10^{-6}}\right) \mathbf{a}_y \\ \mathbf{E}^e &= 0.25\mathbf{a}_x + 0.9\mathbf{a}_y \end{aligned}$$

P. E. 15.7

Area of element 1 = $0.5 \times 10^{-6} \text{ m}^2$

Area of element 2 = $0.5 \times 10^{-6} \text{ m}^2$

Using equation (15.53),

$$T^{(1)} = T^{(2)} = \frac{\Delta}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \frac{0.5 \times 10^{-6}}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Assembling of elemental matrices $T^{(e)}$ into global matrix $[T]$ is similar to assembling the element coefficient matrix given in Section 15.3c.

Therefore global matrix $[T]$ is

$$\begin{aligned} [T] &= \begin{bmatrix} T_{11}^{(1)} & T_{12}^{(1)} & T_{13}^{(1)} & 0 \\ T_{12}^{(1)} & T_{22}^{(1)} + T_{11}^{(2)} & T_{23}^{(1)} + T_{13}^{(2)} & T_{12}^{(2)} \\ T_{13}^{(1)} & T_{23}^{(1)} + T_{13}^{(2)} & T_{33}^{(1)} + T_{33}^{(2)} & T_{23}^{(2)} \\ 0 & T_{12}^{(2)} & T_{23}^{(2)} & T_{22}^{(2)} \end{bmatrix} \\ [T] &= \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 4 & 2 & 1 \\ 1 & 2 & 4 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix} \frac{0.5 \times 10^{-6}}{12} \end{aligned}$$

P. E. 15.8

For element 1 the local node numbers are assigned as 1-2-3 for the global node numbers 1-2-4.

From eq. (15.11b) shape function of node 2 in element 1 is

$$\alpha_2 = \frac{1}{2\Delta_1} [(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y]$$

From eq. (15.12), the area of element 1:

$$\Delta_1 = \frac{1}{2} [(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)] = \frac{1}{2} [|(0.6)(0.9) - (0.4)(-0.4)|] \\ \Delta_1 = 0.35 \text{ unit}^2$$

$$\alpha_2 = \frac{1}{2 \times 0.35} [(1.2)(1.8) - (0.8)(2.7)] + 0.9x - 0.4y \\ \alpha_2 = \frac{1}{0.7} [0.9x - 0.4y] \quad (A)$$

Similarly for element 2 the local nodes are assigned as 1-2-3 for the global node numbers 2-3-4

From eq. (15.12), the area of element 2 is 0.525 unit²

$$\alpha_2 = \frac{1}{2 \times 0.525} [(2.7)(2.1) - (1.2)(2.1)] - 0.6x - 0.9y \\ \alpha_2 = \frac{1}{1.05} [3.15 - 0.6x - 0.9y] \quad (B)$$

Line equation of segment 4-2:

$$y - 1.4 = \frac{1.3}{-0.2} (x - 1.4) \\ y = -0.65x + 10.5 \quad (C)$$

The expression of the two shape functions of node 2 can be rewritten by eliminating y from eq. (A) and eq. (B) using eq. (C).

The expression of the shape function in element 1:

$$\alpha_2 = \frac{1}{0.7} [0.9x - 0.4(-0.65x + 10.5)] = \frac{1}{0.7} [3.5x - 4.2] = 5x - 6 \\ \alpha_2 = 5x - 6$$

The expression of the shape function in element 2:

$$\alpha_2 = \frac{1}{0.7} [3.15 - 0.6x - 0.9(-0.65x + 10.5)] = \frac{1}{0.7} [5.25x - 6.3] = 5x - 6 \\ \alpha_2 = 5x - 6$$

Thus, we can conclude that the expression of the shape function is same on the segment 4-2 for both elements.

P. E. 15.9

```
% The geometry is the same as that in Example 14.4
% The geometry is drawn in the pde toolbox and p e t matrices are imported into
% the MATLAB code using procedure described in Section 15.7
clc;
clear all;
load meshdata;%% loading p, e, t matrices
[A3 n_elements]=size(t);% A3=4
[A2 n_edges]=size(e);% A2=7
[A1 n_nodes]=size(p);% A1=2
C=zeros(n_nodes,n_nodes);% Initialization of the global coefficient matrix
K=zeros(n_nodes,n_nodes);
V1=40;% Potential of the top plate
V2=100;% Potential of the plate on the right hand side
V3=0;% Potential of the plate on the left hand side
V4=10;% Potential of the bottom plate
V=zeros(n_nodes,1);% Initialization of matrix of unknown potentials
B=zeros(n_nodes,1);% Initialization of the right hand side matrix
%contributed by boundary conditions
%Formation of coefficient matrices for all elements
for element=1:n_elements
    nodes=t(1:3,element);% 'nodes' is a 3 by 1 matrix containing global
    % node numbers of element under consideration
    Xc=p(1,nodes); % X coordinates of nodes
    Yc=p(2,nodes); % Y coordinates of nodes
    P=zeros(3,1);
    Q=zeros(3,1);
    P(1)=Yc(2)-Yc(3);
    P(2)=Yc(3)-Yc(1);
    P(3)=Yc(1)-Yc(2);
    Q(1)=Xc(3)-Xc(2);
    Q(2)=Xc(1)-Xc(3);
    Q(3)=Xc(2)-Xc(1);
    % eq. (15.21c)
    delta= 0.5*abs((P(2)*Q(3))-(P(3)*Q(2)));%Absolute value is taken since
    % the three nodes may not have been numbered in the anticlockwise direction

    for i=1:3
        for j=1:3
            % eq. (15.21b)
            c(element,i,j)=((P(i)*P(j))+(Q(i)*Q(j)))/(4*delta);
        end
    end
end
```

```
for element=1:n_elements
    nodes=t(1:3,element);
    % Formation of the global coefficient matrix
    for i=1:3
        for j=1:3
            C(nodes(i),nodes(j))=c(element,i,j)+C(nodes(i),nodes(j));
        end
    end
    end
    % Imposing boundary conditions
    K=C;
    for edge=1:n_edges
        if e(5,edge)==1
            node1=e(1,edge);
            node2=e(2,edge);
            B(node1)=V1;
            B(node2)=V1;
            K(node1,:)=zeros(1,n_nodes);
            K(node1,node1)=1;
            K(node2,:)=zeros(1,n_nodes);
            K(node2,node2)=1;
        end
        if e(5,edge)==2
            node1=e(1,edge);
            node2=e(2,edge);
            B(node1)=V2;
            B(node2)=V2;
            K(node1,:)=zeros(1,n_nodes);
            K(node1,node1)=1;
            K(node2,:)=zeros(1,n_nodes);
            K(node2,node2)=1;
        end
        if e(5,edge)==3
            node1=e(1,edge);
            node2=e(2,edge);
            B(node1)=V4;
            B(node2)=V4;
            K(node1,:)=zeros(1,n_nodes);
            K(node1,node1)=1;
            K(node2,:)=zeros(1,n_nodes);
            K(node2,node2)=1;
        end
    end
```

```
if e(5,edge)==4
    node1=e(1,edge);
    node2=e(2,edge);
    B(node1)=V3;
    B(node2)=V3;
    K(node1,:)=zeros(1,n_nodes);
    K(node1,node1)=1;
    K(node2,:)=zeros(1,n_nodes);
    K(node2,node2)=1;
end

end
% When the goometry of Fig. 14.14 is drawn using the "rectangle"
% option in the pdetoolbox, the four corner nodes get numbered as 1,2,3 and 4
% These nodes need to be made floating to represent gaps between the
% different plates.
for i=1:n_nodes
    if (i==1||i==2||i==3||i==4)
        K(i,:)=C(i,:);
        B(i)=0;
    end
end
%solving for unknown potentials
V=K\B;
figure(1);
pdeplot(p,e,t,'xydata',V,'mesh','off','colormap','jet','colorbar','on',
'contour','on','levels',20);
[Ex,Ey]=pdegrad(p,t,V);% Calculation of E field if required
E=sqrt(Ex.^2+Ey.^2);
figure(2)
pdeplot(p,e,t,'xydata',E,'colorbar','off')
disp('potential at the point (0.5, 0.5)')
x=0.5;
y=0.5;
for i=1:n_nodes
    xc=p(1,i);
    yc=p(2,i);
    dist(i)=((x-xc)^2+(y-yc)^2)^0.5;
end
[d,node]=min(dist);
V(node,1)
```

P. E. 15.10

The code remains essentially the same as in Example 15.10 except for boundary conditions which need to be explicitly defined as explained in the code for TM modes and at the end of Section 14.4 above Practice Exercise 14.6.

Prob. 15.1 From the given figure, we obtain

$$\alpha_1 = \frac{A_1}{A} = \frac{1}{2A} \begin{vmatrix} 1 & x & y \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \frac{1}{2A} [(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y]$$

as expected. The same applies for α_2 and α_3 .

Prob. 15.2

(a) $P_1 = 1.5, P_2 = 0.5, P_3 = -2, Q_1 = -1, Q_2 = 1.5, Q_3 = -0.5$

$$A = \frac{1}{2}(P_2Q_3 - P_3Q_2) = 1.375$$

$$C_{ij} = \frac{1}{4A} [P_i P_j + Q_i Q_j]$$

$$C = \begin{bmatrix} 0.5909 & -0.1364 & -0.4545 \\ -0.1364 & 0.4545 & -0.3182 \\ -0.4545 & -0.3182 & 0.7727 \end{bmatrix}$$

(b)

$$P_1 = -4, P_2 = 4, P_3 = 0, Q_1 = 0, Q_2 = -3, Q_3 = 3$$

$$A = \frac{1}{2}(P_2Q_3 - P_3Q_2) = 6$$

$$C = \begin{bmatrix} 0.6667 & -0.6667 & 0 \\ -0.6667 & 1.042 & -0.375 \\ 0 & -0.375 & 0.375 \end{bmatrix}$$

Prob. 15.3 (a)

$$2A = \begin{vmatrix} 1 & 1/2 & 1/2 \\ 1 & 3 & 1/2 \\ 1 & 2 & 2 \end{vmatrix} = 15/4$$

$$\alpha_1 = \frac{4}{15}[(6-1) + (-1\frac{1}{2})x + (-1)y] = \frac{4}{15}(5 - 1.5x - y)$$

$$\alpha_2 = \frac{4}{15}[(1-1) + \frac{3}{2}x - \frac{3}{2}y] = \frac{4}{15}(1.5x - 1.5y)$$

$$\alpha_3 = \frac{4}{15}[(1/4 - 3/2) + 0x + \frac{5}{2}y] = \frac{4}{15}(-1.25 + 2.5y)$$

$$V = \alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3$$

Substituting $V=80$, $V_1 = 100$, $V_2 = 50$, $V_3 = 30$, α_1 , α_2 , and α_3 leads to

$$20 = 7.5x + 10y + 3.75$$

Along side 12, $y=1/2$ so that

$$20 = 15x/2 + 5 + 15/4 \quad \longrightarrow \quad x=3/2, \text{ i.e. } (1.5, 0.5)$$

Along side 13, $x=y$

$$20 = 15x/2 + 10x + 15/4 \quad \longrightarrow \quad x=13/4, \text{ i.e. } (13/14, 13/14)$$

Along side 23, $y = -3x/2 + 5$

$$20 = 15x/2 - 15 + 50 + 15/4 \quad \longrightarrow \quad x=-5/2 \text{ (not possible)}$$

Hence intersection occurs at

(1.5, 0.5) along 12 and (0.9286, 0.9286) along 13

(b) At (2,1),

$$\alpha_1 = \frac{4}{15}, \quad \alpha_2 = \frac{6}{15}, \quad \alpha_3 = \frac{5}{15}$$

$$V(2,1) = \alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 = (400 + 300 + 150)/15 = \underline{\underline{56.67 \text{ V}}}$$

Prob. 15.4

$$2A = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 1 & 4 \end{vmatrix} = 9$$

$$\alpha_1 = \frac{1}{9}[(0-0) + (4-0)x + (0-1)y] = \frac{1}{9}(4x-y)$$

$$\alpha_2 = \frac{1}{9}[(0-0) + (0+1)x + (2-0)y] = \frac{1}{9}(x+2y)$$

$$\alpha_3 = \frac{1}{9}[(8+1) + (-1-4)x + (1-2)y] = \frac{1}{9}(9-5x-y)$$

$$V_e = \alpha_1 V_{e1} + \alpha_2 V_{e2} + \alpha_3 V_{e3}$$

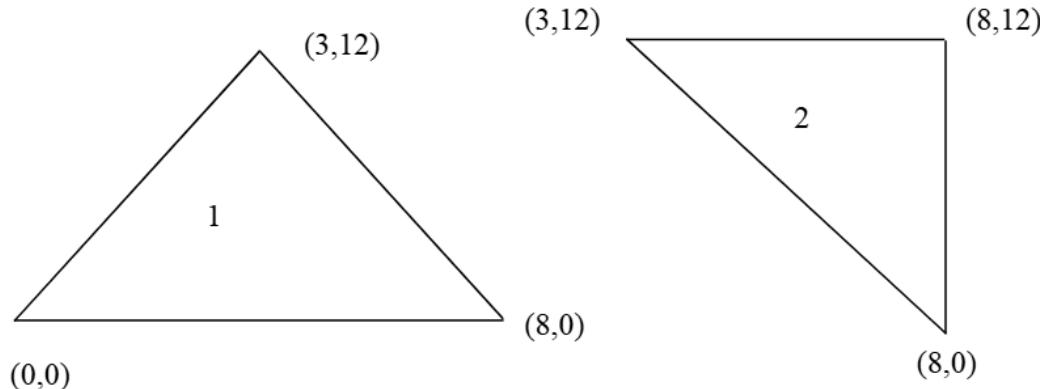
$$V(1,2) = 8(4-2)/9 + 12(1+4)/9 + 10(9-5-1)/9 = 96/9 = \underline{\underline{10.667 \text{ V}}}$$

At the center $\alpha_1 = \alpha_2 = \alpha_3 = 1/3$ so that

$$V(\text{center}) = (8+12+10)/3 = 10$$

$$\text{Or at the center, } (x, y) = (0+1+2, 0+4-1)/3 = (1,1)$$

$$V(1,1) = 8(3)/9 + 12(3)/9 + 10(3)/9 = \underline{\underline{10 \text{ V}}}$$

Prob. 15.5

For element 1, local numbering 1-2-3 corresponds to global numbering 4-2-1.

$$P_1 = 12, P_2 = 0, P_3 = -12, Q_1 = -3, Q_2 = 8, Q_3 = -5,$$

$$A = (0 + 12 \times 8)/2 = 48$$

$$C_{ij} = \frac{1}{4x48}[P_j P_i + Q_j Q_i]$$

$$C^{(1)} = \begin{bmatrix} 0.7969 & -0.125 & -0.6719 \\ -0.125 & 0.3333 & -0.2083 \\ -0.6719 & -0.2083 & 0.8802 \end{bmatrix}$$

For element 2, local numbering 1-2-3 corresponds to global numbering 2-4-3.

$$P_1 = -12, P_2 = 0, P_3 = 12, Q_1 = 0, Q_2 = -5, Q_3 = 5,$$

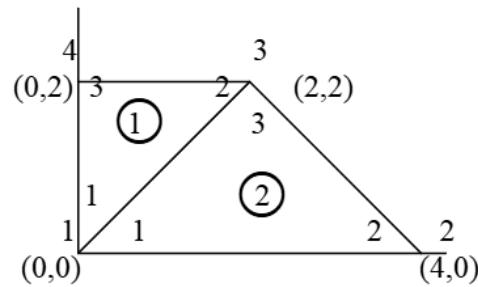
$$A = (0 + 60)/2 = 30$$

$$C_{ij} = \frac{1}{4x48}[P_j P_i + Q_j Q_i]$$

$$C^{(2)} = \begin{bmatrix} 1.2 & 0 & -1.2 \\ 0 & 0.208 & -0.208 \\ -1.2 & -0.208 & 1.408 \end{bmatrix}$$

$$C = \begin{bmatrix} C_{33}^{(1)} & C_{23}^{(1)} & 0 & C_{31}^{(1)} \\ C_{23}^{(1)} & C_{22}^{(1)} + C_{11}^{(2)} & C_{13}^{(2)} & C_{21}^{(1)} + C_{12}^{(2)} \\ 0 & C_{31}^{(2)} & C_{33}^{(2)} & C_{32}^{(2)} \\ C_{13}^{(1)} & C_{21}^{(1)} + C_{21}^{(2)} & C_{23}^{(2)} & C_{22}^{(2)} + C_{11}^{(1)} \end{bmatrix}$$

$$= \begin{bmatrix} 0.8802 & -0.2083 & 0 & -0.6719 \\ -0.2083 & 1.533 & -1.2 & -0.125 \\ 0 & -1.2 & 1.4083 & -0.2083 \\ -0.6719 & -0.125 & -0.2083 & 1.0052 \end{bmatrix}$$

Prob. 15.6

For element 1,

$$P_1 = 0, P_2 = 2, P_3 = -2, Q_1 = -2, Q_2 = 0, Q_3 = 2$$

$$A = \frac{1}{2}(4-0) = 2, \quad 4A = 8$$

$$C^{(1)} = \frac{1}{8} \begin{bmatrix} 4 & 0 & -4 \\ 0 & 4 & -4 \\ -4 & -4 & 8 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 & -0.5 \\ 0 & 0.5 & -0.5 \\ -0.5 & -0.5 & 1 \end{bmatrix}$$

For element 2,

$$P_1 = -2, P_2 = 2, P_3 = 0, Q_1 = -2, Q_2 = -2, Q_3 = 4$$

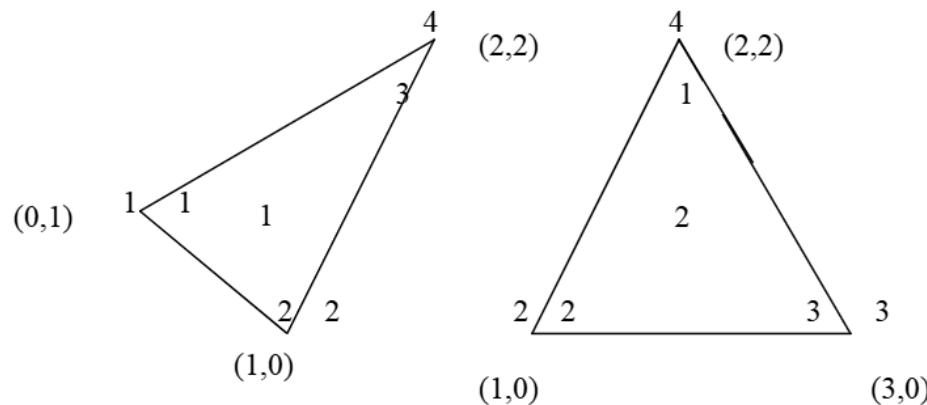
$$A = \frac{1}{2}(8-0) = 4, \quad 4A = 16$$

$$C^{(2)} = \frac{1}{16} \begin{bmatrix} 8 & 0 & -8 \\ 0 & 8 & -8 \\ -8 & -8 & 16 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 & -0.5 \\ 0 & 0.5 & -0.5 \\ -0.5 & -0.5 & 1 \end{bmatrix}$$

The global coefficient matrix is

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{bmatrix} = \begin{bmatrix} C_{11}^{(1)} + C_{11}^{(2)} & C_{12}^{(2)} & C_{12}^{(1)} + C_{13}^{(2)} & C_{13}^{(1)} \\ C_{21}^{(2)} & C_{22}^{(2)} & C_{23}^{(2)} & 0 \\ C_{21}^{(1)} + C_{31}^{(2)} & C_{32}^{(2)} & C_{22}^{(1)} + C_{33}^{(2)} & C_{23}^{(1)} \\ C_{31}^{(1)} & 0 & C_{32}^{(1)} & C_{33}^{(1)} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -0.5 & -0.5 \\ 0 & 0.5 & -0.5 & 0 \\ -0.5 & -0.5 & 1.5 & -0.5 \\ -0.5 & 0 & -0.5 & 1 \end{bmatrix}$$

Prob. 15.7

For element 1, local numbering 1-2-3 corresponds to global numbering 1-2-4.

$$P_1 = -2, P_2 = 1, P_3 = 1, Q_1 = 1, Q_2 = -2, Q_3 = 1,$$

$$A = (P_2 Q_3 - P_3 Q_2)/2 = 3/2, \text{ i.e. } 4A = 6$$

$$C_{ij} = \frac{1}{4A} [P_j P_i + Q_j Q_i]$$

$$C^{(1)} = \frac{1}{6} \begin{bmatrix} 5 & -4 & -1 \\ -4 & 5 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

For element 2, local numbering 1-2-3 corresponds to global numbering 4-2-3.

$$P_1 = 0, P_2 = -2, P_3 = 2, Q_1 = 2, Q_2 = -1, Q_3 = -1,$$

$$A = 2, 4A = 8$$

$$C^{(2)} = \frac{1}{8} \begin{bmatrix} 4 & -2 & -2 \\ -2 & 5 & -3 \\ -2 & -3 & 5 \end{bmatrix}$$

The global coefficient matrix is

$$C = \begin{bmatrix} C_{11}^{(1)} & C_{12}^{(1)} & 0 & C_{13}^{(1)} \\ C_{12}^{(1)} & C_{22}^{(1)} + C_{22}^{(2)} & C_{23}^{(2)} & C_{23}^{(1)} + C_{21}^{(2)} \\ 0 & C_{23}^{(2)} & C_{33}^{(2)} & C_{31}^{(2)} \\ C_{13}^{(1)} & C_{23}^{(1)} + C_{21}^{(2)} & C_{31}^{(2)} & C_{33}^{(1)} + C_{11}^{(2)} \end{bmatrix}$$

$$= \begin{bmatrix} 0.8333 & -0.667 & 0 & -0.1667 \\ -0.6667 & 1.4583 & -0.375 & -0.4167 \\ 0 & -0.375 & 0.625 & -0.25 \\ -0.1667 & -0.4167 & -0.25 & 0.833 \end{bmatrix}$$

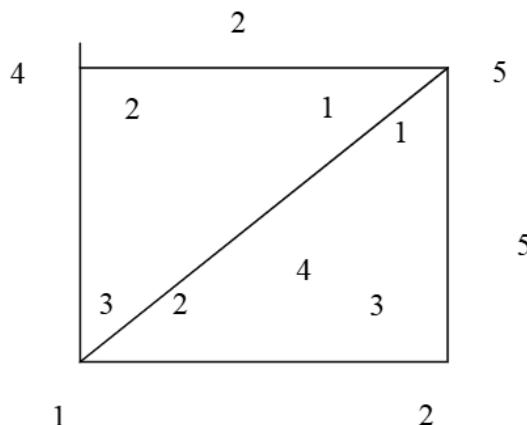
Prob. 15.8 We can do it by hand as in Example 15.2. However, it is easier to prepare an input file and use the program in Fig. 15.8. The MATLAB input data is

```
NE = 2;
ND = 4;
NP = 2;
NL = [1 2 4
       4 2 3];
X = [ 0.0 1.0 3.0 2.0];
Y = [ 1.0 0.0 0.0 2.0];
NDP = [ 1 3 ];
VAL = [ 10.0 30.0]
```

The result is $V = \begin{bmatrix} 10 \\ 18 \\ 30 \\ 20 \end{bmatrix}$

From this,

$$\underline{V_2 = 18 \text{ V}}, \quad \underline{V_4 = 20 \text{ V}}$$

Prob. 15.9

The local numbering 1-2-3 in element 3 corresponds with the global numbering 5-4-1, while the local number 1-2-3 in element 4 corresponds with the global numbering 5-1-2.

$$C_{5,5} = C_{11}^{(2)} + C_{11}^{(3)} + C_{11}^{(4)} + C_{11}^{(5)}, \quad A = 2,$$

$$C_{11}^{(2)} = (2 \times 2 + 2 \times 2)/8 = 1 = C_{11}^{(5)}$$

$$C_{11}^{(3)} = (2 \times 2 + 0)/8 = \frac{1}{2} = C_{11}^{(4)}$$

$$C_{5,5} = 1 + 1 + \frac{1}{2} + \frac{1}{2} = \underline{\underline{3}}$$

$$C_{5,1} = C_{31}^{(3)} + C_{21}^{(4)}$$

$$\text{But } C_{31}^{(3)} = \frac{1}{8}(P_3 P_1 + Q_3 Q_1) = 0 \text{ since } P_3 = 0 = Q_3$$

$$C_{21}^{(4)} = \frac{1}{8}(P_2 P_1 + Q_2 Q_1) = 0 \text{ since } P_3 = 0 = Q_3$$

$$\underline{\underline{C_{5,1}} = 0}$$

Prob. 15.10 As in P. E. 15.3, we use the program in Fig. 15.8. The input data based on Fig. 15.19 is as follows.

$$NE = 50; \quad ND = 36; \quad NP = 20;$$

$$NL = \begin{bmatrix} 1 & 8 & 7 \\ 1 & 2 & 8 \\ 2 & 9 & 8 \\ 2 & 3 & 9 \\ 3 & 10 & 9 \\ 3 & 4 & 10 \end{bmatrix}$$

4	11	10
4	5	11
5	12	11
5	6	12
7	14	13
7	8	14
8	15	14
8	9	15
9	16	15
9	10	16
10	17	16
10	11	17
11	18	17
11	12	18
13	20	19
13	14	20
14	21	20
14	15	21
15	22	21
15	16	22
16	23	22
16	17	23
17	24	23
17	18	24
19	26	25
19	20	26
20	27	26
20	21	27
21	28	27
21	22	28
22	29	28
22	23	29
23	30	29
23	24	30
25	32	31
25	26	32
26	33	32
26	27	33
27	34	33
27	28	34
28	35	34
28	29	35
29	36	35
29	30	36];
$X = [0.0 \ 0.2 \ 0.4 \ 0.6 \ 0.8 \ 1.0 \ 0.0 \ 0.2 \ 0.4 \ 0.6 \ 0.8 \ 1.0 \ 0.0 \ 0.2 \ 0.4 \ 0.6 \ 0.8 \ 1.0 \ 0.0 \ 0.2 \ 0.4 \ 0.6 \ 0.8 \ 1.0]$		
$[0.0 \ 0.2 \ 0.4 \ 0.6 \ 0.8 \ 1.0 \ 0.0 \ 0.2 \ 0.4 \ 0.6 \ 0.8 \ 1.0 \ 0.2 \ 0.4 \ 0.6 \ 0.8 \ 1.0];$		

```

Y = [0.0 0.0 0.0 0.0 0.0 0.0 0.2 0.2 0.2 0.2 0.2 0.2 0.2 0.2 0.4 0.4 0.4 0.4
0.4 0.6 0.6 0.6 0.6 0.6 0.8 0.8 0.8 0.8 0.8 0.8 1.0 1.0 1.0 1.0 1.0 1.0];
NDP = [ 1 2 3 4 5 6 12 18 24 30 36 35 34 33 32 31 25 19 13 7];
VAL = [ 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 50.0 100.0 100.0 100.0
100.0 50.0 0.0 0.0 0.0];

```

With this data, the potentials at the free nodes are compared with the exact values as shown below.

Node no.	FEM Solution	Exact Solution
8	4.546	4.366
9	7.197	7.017
10	7.197	7.017
11	4.546	4.366
14	10.98	10.60
15	17.05	16.84
16	17.05	16.84
17	10.98	10.60
20	22.35	21.78
21	32.95	33.16
22	32.95	33.16
23	22.35	21.78
26	45.45	45.63
27	59.49	60.60
28	59.49	60.60
29	45.45	45.63

Prob. 15.11 We use exactly the same input data as in the previous problem except that the last few lines are replaced by the following lines.

```

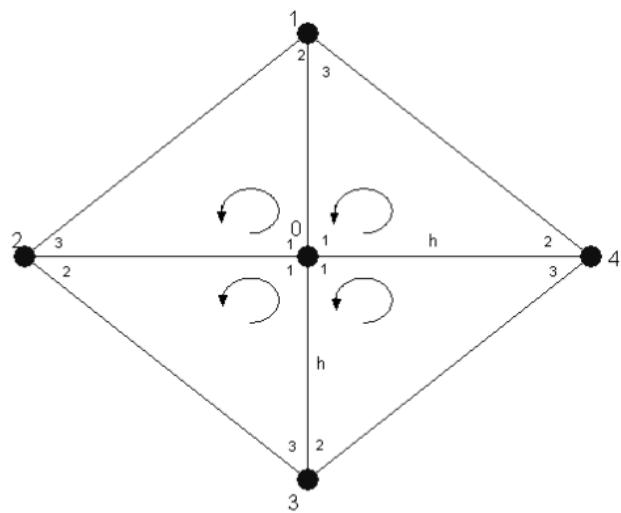
VAL = [0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 29.4 58.8 95.1 95.1
58.8 29.4 0.0 0.0 0.0 0.0];

```

The potential at the free nodes obtained with the input data are compared with the exact solution as shown below.

Node no.	FEM Solution	Exact Solution
8	3.635	3.412
9	5.882	5.521
10	5.882	5.521
11	3.635	3.412
14	8.659	8.217
15	14.01	13.30
16	14.01	13.30
17	8.659	8.217
20	16.99	16.37
21	27.49	26.49
22	27.49	26.49
23	16.99	16.37
26	31.81	31.21
27	51.47	50.5
28	51.47	50.5
29	31.81	31.21

Prob. 15.12



For element 1, the local numbering 1-2-3 corresponds with nodes with V_1 , V_2 , and V_3 .

$$V_o = -\frac{1}{C_{oo}} \sum_{i=1}^4 V_i C_{io}$$

$$C_{oo} = \sum_{j=1}^4 C_{oj}^{(e)} = \frac{1}{4h^2/2} (hh + hh) \times 2 + \frac{1}{4h^2/2} (hh + 0) \times 4 = 4$$

$$C_{o1} = \frac{2 \times 1}{2h^2} [P_3 P_1 + Q_3 Q_1] = \frac{2}{2h^2} [-hh - 0] = -1$$

$$C_{o2} = \frac{2 \times 1}{2h^2} [P_1 P_2 + Q_1 Q_2] = \frac{2}{2h^2} [-h \times 0 + h \times (-h)] = -1$$

Similarly, $C_{03} = -1 = C_{04}$. Thus

$$V_o = (V_1 + V_2 + V_3 + V_4)/4$$

which is the same result obtained using FDM.