Assignment-15

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***Abstract*—This assignment deals with linear transformation.**

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https://github.com/satyam463/Assignment−15/blob/main/Assignment%2015.tex

1 Problem Statement

Let V be a vector space over the field of complex numbers , and suppose there is an isomorphism T of V onto C^3. Let α_1 , α_2 , α_3 , α_4 be vectors in V such that

$$\begin{aligned} T(\alpha_1) &= \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}, T(\alpha_2) = \begin{pmatrix} -2 \\ 1+i \\ 0 \end{pmatrix}, \\ T(\alpha_3) &= \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, T(\alpha_4) = \begin{pmatrix} \sqrt{2} \\ i \\ 3 \end{pmatrix} \end{aligned} \tag{1.0.1}$$

Is α_1 in the subspace spanned by α_2 and α_3.

2 Isomorphism

2.1 Properties

T : V → W is an isomorphism if
(1) T is one one.
(2) T is onto.

3 Solution

$$\begin{pmatrix} 1 & -2 & -1 & \sqrt{2} \\ 0 & 1+i & 1 & i \\ i & 0 & 1 & 3 \end{pmatrix} \xrightarrow{ref} \begin{pmatrix} 1 & 0 & -i & -3i \\ 0 & 2 & 1-i & i+1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.0.1}$$

T is one one over C^3 if

$$T(\alpha) = 0 \implies \alpha = 0 \tag{3.0.2}$$

now,

$$\begin{pmatrix} 1 & -2 & \sqrt{2} \\ 0 & 1+i & i \\ i & 0 & 3 \end{pmatrix} \alpha = 0 \tag{3.0.3}$$

consider the row reduced matrix

$$\begin{pmatrix} 1 & -2 & \sqrt{2} \\ 0 & 1+i & i \\ i & 0 & 3 \end{pmatrix} \xleftrightarrow[R_3 \to iR_3]{R_3 \to R_3 - iR_1} \begin{pmatrix} 1 & -2 & \sqrt{2} \\ 0 & 1+i & i \\ 0 & -2 & \sqrt{2}+3i \end{pmatrix} \tag{3.0.4}$$

$$\xleftrightarrow[R_3 \leftarrow R_3 + R_2]{R_2 \leftarrow (1-i)R_2} \begin{pmatrix} 1 & -2 & \sqrt{2} \\ 0 & 2 & i+1 \\ 0 & 0 & \sqrt{2}+4i+1 \end{pmatrix} \tag{3.0.5}$$

$$\alpha = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.0.6}$$

Therefore it holds the condition of one one and the rank = no. of pivot columns = 3 (equal to no. of columns).Thus the vectors are linearly independent hence it is onto . Since T is an isomorphoism onto C^3.

$$T(\alpha_1) = c_1 T(\alpha_2) + c_2 T(\alpha_3) \tag{3.0.7}$$

c_1 and c_2 are scalar .

$$\begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix} = c_1 \begin{pmatrix} -2 \\ 1+i \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \tag{3.0.8}$$

$$\begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ 1+i & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \tag{3.0.9}$$

Now we find c_i by row reducing augmented matrix.

$$\begin{pmatrix} -2 & -1 & 1 \\ 1+i & 1 & 0 \\ 0 & 1 & i \end{pmatrix} \xleftrightarrow[R_2 \to R_3]{R_1 \to -R_1/2} \begin{pmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & i \\ 1+i & 1 & 0 \end{pmatrix} \tag{3.0.10}$$

$$\xleftrightarrow[R_3 \leftarrow R_3-(1+i)R_1]{R_1 \leftarrow R_1 - R_2/2} \begin{pmatrix} 1 & 0 & \frac{-1-i}{2} \\ 0 & 1 & i \\ 0 & \frac{1-i}{2} & \frac{1+i}{2} \end{pmatrix} \tag{3.0.11}$$

$$\xleftrightarrow{R_3 \leftarrow R_3-(1-i)/2R_2} \begin{pmatrix} 1 & 0 & \frac{-1-i}{2} \\ 0 & 1 & i \\ 0 & 0 & 0 \end{pmatrix} \tag{3.0.12}$$

Therefore the coordinate matrix of the vector is

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{-1-i}{2} \\ i \end{pmatrix} \tag{3.0.13}$$

substituting the c_i in 3.0.7

$$T(\alpha_1) = -\frac{1+i}{2} T(\alpha_2) + iT(\alpha_3) \tag{3.0.14}$$

Hence α_1 belongs to the subspace spanned by α_2 and α_3.