Assignment-6

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***Abstract*—This assignment deals with single value decomposition .**

Download all python codes from

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https://github.com/satyam463/Assignment−6/blob/
    master/code.py
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1 Problem Statement

Find the foot of the perpendicular using svd drawn from $\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ to the plane

$$\begin{pmatrix} 2 & -1 & 2 \end{pmatrix} \mathbf{x} + 3 = 0 \tag{1.0.1}$$

2 solution

First we find orthogonal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the given normal vector $\mathbf{n}$. Let, $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$\mathbf{m^T n} = 0 \tag{2.0.1}$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = 0 \tag{2.0.2}$$

$$\implies 2a - b + 2c = 0 \tag{2.0.3}$$

Putting a=1 and b=0 we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \tag{2.0.4}$$

Putting a=0 and b=1 we get,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \tag{2.0.5}$$

Now we solve the equation,

$$\mathbf{Mx} = \mathbf{b} \tag{2.0.6}$$

Putting values in 2.0.6,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \tag{2.0.7}$$

Now, to solve 2.0.7, we perform Singular Value Decomposition on $\mathbf{M}$ as follows,

$$\mathbf{M} = \mathbf{USV}^T \tag{2.0.8}$$

Where the columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{MM}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \tag{2.0.9}$$

$$\mathbf{MM}^T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \tag{2.0.10}$$

From 2.0.6 putting 2.0.8 we get,

$$\mathbf{USV}^T\mathbf{x} = \mathbf{b} \tag{2.0.11}$$

$$\implies \mathbf{x} = \mathbf{VS_+U^Tb} \tag{2.0.12}$$

Where $\mathbf{S_+}$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$.Now, calculating eigen value of $\mathbf{MM}^T$,

$$\left|\mathbf{MM}^T - \lambda\mathbf{I}\right| = 0 \tag{2.0.13}$$

$$\implies \begin{pmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{pmatrix} = 0 \tag{2.0.14}$$

$$\implies \lambda^3 - 4\lambda^2 + 3\lambda = 0 \tag{2.0.15}$$

Hence eigen values of $\mathbf{MM}^T$ are,

$$\lambda_1 = 3 \tag{2.0.16}$$

$$\lambda_2 = 1 \tag{2.0.17}$$

$$\lambda_3 = 0 \tag{2.0.18}$$

Hence the eigen vectors of $\mathbf{M}\mathbf{M}^T$ are,

$$\mathbf{u}_1 = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \quad (2.0.19)$$

Normalizing the eigen vectors we get,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{-1}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \sqrt{\frac{2}{3}} \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} -\sqrt{\frac{1}{3}} \\ -\sqrt{\frac{1}{3}} \\ \sqrt{\frac{1}{3}} \end{pmatrix} \quad (2.0.20)$$

Hence we obtain $\mathbf{U}$ of 2.0.8 as follows,

$$\mathbf{U} = \begin{pmatrix} \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\sqrt{\frac{1}{3}} \\ \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & -\sqrt{\frac{1}{3}} \\ \sqrt{\frac{2}{3}} & 0 & \sqrt{\frac{1}{3}} \end{pmatrix} \quad (2.0.21)$$

After computing the singular values from eigen values $\lambda_1, \lambda_2, \lambda_3$ we get $\mathbf{S}$ of 2.0.8 as follows,

$$\mathbf{S} = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (2.0.22)$$

Now, calculating eigen value of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \quad (2.0.23)$$

$$\implies \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} = 0 \quad (2.0.24)$$

$$\implies \lambda^2 - 4\lambda + 3 = 0 \quad (2.0.25)$$

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_4 = 3 \quad (2.0.26)$$

$$\lambda_5 = 1 \quad (2.0.27)$$

Hence the eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (2.0.28)$$

Normalizing the eigen vectors we get,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} \quad (2.0.29)$$

Hence we obtain $\mathbf{V}$ of 2.0.8 as follows,

$$\mathbf{V} = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \quad (2.0.30)$$

Finally from 2.0.8 we get the Singualr Value Decomposition of $\mathbf{M}$ as follows,

$$\mathbf{M} = \begin{pmatrix} \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\sqrt{\frac{1}{3}} \\ \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & -\sqrt{\frac{1}{3}} \\ \sqrt{\frac{2}{3}} & 0 & \sqrt{\frac{1}{3}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}^T \quad (2.0.31)$$

Now, Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (2.0.32)$$

From 2.0.12 we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{3}{\sqrt{6}} \\ \frac{5}{\sqrt{2}} \\ 0 \end{pmatrix} \quad (2.0.33)$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{3}{\sqrt{18}} \\ \frac{5}{\sqrt{2}} \end{pmatrix} \quad (2.0.34)$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad (2.0.35)$$

Verifying the solution of 2.0.35 using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \quad (2.0.36)$$

Evaluating the R.H.S in 2.0.36 we get,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} 4 \\ -1 \end{pmatrix} \quad (2.0.37)$$

$$\implies \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 4 \\ -1 \end{pmatrix} \quad (2.0.38)$$

Solving the augmented matrix of 2.0.38 we get,

$$\begin{pmatrix} 2 & 1 & 4 \\ 1 & 2 & -1 \end{pmatrix} \xleftrightarrow[R_2 \to R_2 - 2R_1]{R_2 \to R_1} \begin{pmatrix} 1 & 2 & -1 \\ 0 & -3 & 6 \end{pmatrix} \quad (2.0.39)$$

$$\xleftrightarrow[R_1 \leftarrow R_1 - 2R_2]{R_2 \leftarrow R_2/-3} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \end{pmatrix} \quad (2.0.40)$$

Hence, Solution of 2.0.36 is given by,

$$\mathbf{x} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad (2.0.41)$$

Comparing results of $\mathbf{x}$ from 2.0.35 and 2.0.41 we conclude that the solution is verified.