Assignment-2

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EE20MTECH14015

***Abstract*—This assignment finds the equation of the two straight lines from second degree equation.**

Download all python codes from

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https://github.com/satyam463/matrix−theory/blob/
    master/assignment2.py
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1 Problem Statement

Prove that the following equations represents two straight lines also find their point of intersection and angle between them.

$$y^2 + xy - 2x^2 - 5x - y - 2 = 0 \tag{1.0.1}$$

2 Solution

The general equation of second degree equation is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.0.1}$$

and can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.2}$$

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix}; \mathbf{u} = \begin{pmatrix} d & e \end{pmatrix} \tag{2.0.3}$$

represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u}^T \\ \mathbf{u} & f \end{vmatrix} = 0 \tag{2.0.4}$$

On comparing equation(1.0.1) and equation(2.0.1) we get

$$a = -2, b = \frac{1}{2}, c = 1, d = \frac{-5}{2}, e = \frac{-1}{2}, f = -2 \tag{2.0.5}$$

substituting above value in equation(2.0.4) we get

$$\begin{vmatrix} -2 & \frac{1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 1 & \frac{-1}{2} \\ \frac{-5}{2} & \frac{-1}{2} & -2 \end{vmatrix} \xleftrightarrow[R_1 \to R_1 + R_3]{R_1 \to R_1 - R_2} \begin{vmatrix} 0 & 0 & 0 \\ \frac{1}{2} & 1 & \frac{-1}{2} \\ \frac{-5}{2} & \frac{-1}{2} & -2 \end{vmatrix} = 0 \tag{2.0.6}$$

Hence this represents pair of two straight lines. Now two intersecting lines are obtained when

$$\left|V\right| < 0 \tag{2.0.7}$$

$$\implies \begin{vmatrix} -2 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{vmatrix} = \frac{-9}{4} \text{ i.e. } < 0 \tag{2.0.8}$$

Let the pair of straight of lines be given by

$$\mathbf{n_1}^T\mathbf{x} = c_1 \tag{2.0.9}$$

$$\mathbf{n_2}^T\mathbf{x} = c_2 \tag{2.0.10}$$

The slopes of the lines are given by the roots of the polynomial

$$cm^2 + 2bm + a = 0 \tag{2.0.11}$$

$$\implies m_i = \frac{-b \pm \sqrt{-\left|V\right|}}{c} \text{ i = 1,2} \tag{2.0.12}$$

$$\implies m_i = \frac{-\frac{1}{2} \pm \sqrt{\frac{9}{4}}}{1} \tag{2.0.13}$$

$$\implies m_1 = 1 \text{ and } m_2 = -2 \tag{2.0.14}$$

$$\text{Hence } \mathbf{n_1} = k_i \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \tag{2.0.15}$$

$$\implies \mathbf{n_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ and } \mathbf{n_2} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{2.0.16}$$

Equating the product of lines to the equation(2.0.1)

$$(\mathbf{n_1}^T\mathbf{x} - c_1)(\mathbf{n_2}^T\mathbf{x} - c_2) = \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f \tag{2.0.17}$$

$$\mathbf{n_1} * \mathbf{n_2} = \{c, 2b, a\} \text{ * represents convolution} \tag{2.0.18}$$

$$c_2\mathbf{n_1} + c_1\mathbf{n_2} = -2\mathbf{u} \implies c_2\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} = -2\begin{pmatrix} \frac{-5}{2} & \frac{-1}{2} \end{pmatrix} \tag{2.0.19}$$

$$-c_2 + 2c_1 = 5 \text{ and } c_2 + c_1 = 1 \implies c_1 = 2, c_2 = -1 \tag{2.0.20}$$

$$c_1c_2 = f \implies c_1c_2 = -2 \tag{2.0.21}$$

Therefore the equation of lines is given by

$$\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 2 \tag{2.0.22}$$

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = -1 \tag{2.0.23}$$

consider augmented matrix

$$\begin{pmatrix} -1 & 1 & 2 \\ 2 & 1 & -1 \end{pmatrix} \xleftrightarrow[R_2 \to R_2 - 2R_1]{R_1 \to -R_1} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \xleftrightarrow[R_1 \to R_1 + R_2]{R_1 \to R_1/3} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \tag{2.0.24}$$

Therefore point of intersection is $\mathbf{A} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Now Angle between two lines can be expressed in terms of normal vector

$$\mathbf{n_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ and } \mathbf{n_2} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{2.0.25}$$

Angle between two lines θ can be given by

$$\cos\theta = \frac{\mathbf{n_1}^T \mathbf{n_2}}{||\mathbf{n_1}||||\mathbf{n_2}||} = \frac{\begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}}{\sqrt{(1)^2 + 1} \times \sqrt{(2)^2 + 1}} = \frac{-1}{\sqrt{10}} \tag{2.0.26}$$

$$\theta = \cos^{-1}(\frac{-1}{\sqrt{10}}) \implies \theta = tan^{-1}3 \tag{2.0.27}$$

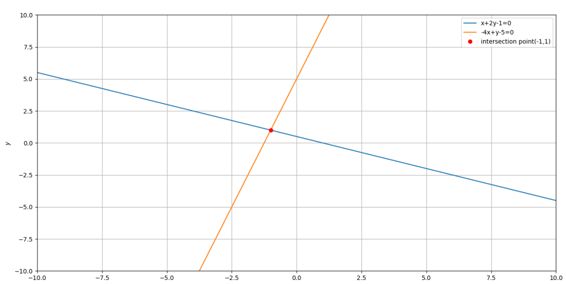

Fig. 0: plot showing intersection of lines