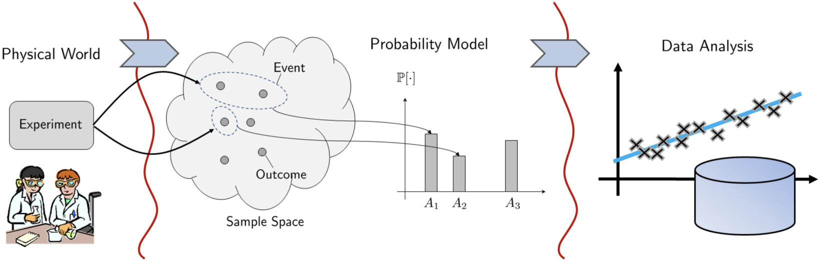
- Probability: it is the tool that allows us to describe our world with uncertainty.
- We typically provide an approximate description of phenomena in the real world using the tools of probablity, and then we use probability to allow us to quantify the uncertainty in our description
- It's the base at the foundation of our statistics/data science models and algorithms



Example (Coin tossing)

- A={obtain head when launching a coin} (event)
- How can we describe such event? First, we can look at the set containing all the results obtained when "launching a coin":

$$\Omega = \{H, T\} \qquad \text{(head, tail)} \Rightarrow \text{universal set}$$

- Then, the result that describes the event

$$A = \{\text{obtain head when launching a coin}\} = \{H\}$$

can be described by the simple element H.

- Of course, $A \subset \Omega$, A is a subset of Ω
- We will be interested in giving some sense/provide meaning to the following statement

$$P(A) = P(\{H\}) = \frac{1}{2}$$

Example

Die rolling

- B={Launch a regular die and obtain an event number}
- $\Omega = \{$⚀, ⚁, ⚂, ⚃, ⚄, ⚅$\} = \{1, 2, 3, 4, 5, 6\}$ ⇨ universal set (collection of all simple events when launching a die)
- Clearly, $B \subset \Omega$ is a complex event (formed by multiple simple elements) and $B = \{2, 4, 6\}$
- We will be interested in giving meaning to the following statement

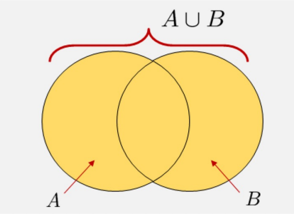
$$P(B) = P(\{\text{even number}\}) = \frac{3}{6} = \frac{1}{2}$$

Set theory

- In order to compute probabilities of complex events, it's often useful to be able to compute and represent an event through operations with sets
- We will recall 4 set operations:
  1. Union of sets
  2. Intersection of sets
  3. Complement of a set
  4. Difference between sets
- The textbook provides a more thorough discussion (invitation to read)

Union of two sets

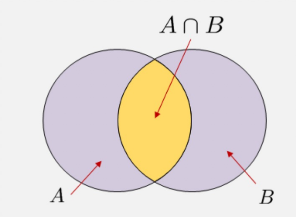
- The union
$$A \cup B = \{\xi \mid \xi \in A \text{ or } \xi \in B\}$$
of two sets contains all elements in A or B.



- Of course, $A \subseteq A \cup B$, $B \subseteq A \cup B$.
- Ex 1: $B = \{\text{even number when launching a die}\} = \{2\} \cup \{4\} \cup \{6\}$
- Ex 2: $A = \{1, 2\}$, $B = \{1, 5\} \Rightarrow A \cup B = \{1, 2, 5\}$
- Ex 3: $A = (3, 4]$, $B = [3.5, \infty)$, $\Rightarrow A \cup B = (3, \infty)$

Intersection of two sets

- The intersection
$$A \cap B = \{\xi \mid \xi \in A \text{ and } \xi \in B\}$$
of two sets contains all elements in A and B.



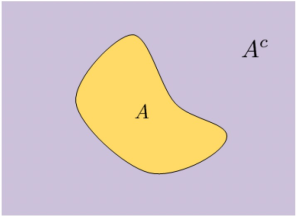
- Of course, $A \cap B \subseteq A$, $A \cap B \subseteq B$.
- Ex 1: $A = \{1, 2, 3, 4\}$, $B = \{1, 5, 6\} \Rightarrow A \cap B = \{1\}$
- Ex 2: $A = (3, 4]$, $B = [3.5, \infty)$, $\Rightarrow A \cap B = [3.5, 4]$
- Ex 3: $A = (3, 4)$, $B = \emptyset$ (empty set) $\Rightarrow A \cap B = \emptyset$

Complement of a set

- The complement of a set A

$$A^c = \{\xi \mid \xi \in \Omega \text{ and } \xi \notin A\}$$

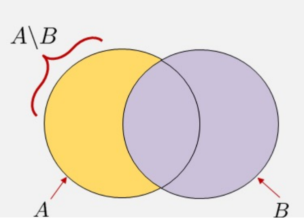
contains all elements that are in Ω but not in A



- Ex 1: Let $A = \{1, 2, 3\}$ and $\Omega = \{1, 2, 3, 4, 5, 6\} \Rightarrow A^c = \{4, 5, 6\}$.
- Ex 2: Let $B = \{\text{even number when launching a die}\} \Rightarrow$
  $B^c = \{\text{odd number when launching a die}\}$
- Ex 3: Let $A = [0, 5)$ and $\Omega = \mathbb{R} \Rightarrow A^c = (-\infty, 0) \cup [5, \infty)$.

Difference between two sets

- The difference
$$A \setminus B = \{\xi \mid \xi \in A \text{ and } \xi \notin B\}$$
contains all elements that are in A but not in B.



- Ex 1: Let $A = \{1, 3, 5, 6\}$ and $B = \{2, 3, 4\}$ ⇨ $A \setminus B = \{1, 5, 6\}$ and $B \setminus A = \{2, 4\}$.
- Ex 2: Let $A = [0, 1]$, B $= [2, 3]$ (non overlapping sets)⇨ $A \setminus B = [0, 1]$, and $B \setminus A = [2, 3]$.
- It can be shown that $A \setminus B = A \cap B^c$.

Sample space

- Sets corresponds to description of "events"⇨ we want to be able to express the probability of events

Sample space Ω

A sample space Ω is the set of all possible outcomes from an experiment. We denote ξ as an element in Ω.

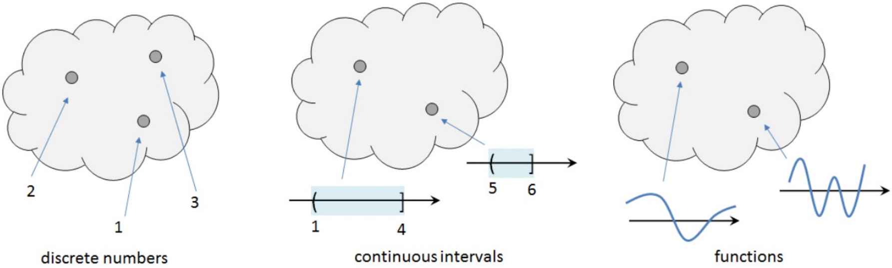
Example (Discrete outcomes)

- Coin flip: $\Omega = \{H, T\}$
- Throw a die: $\Omega = \{$⚀, ⚁, ⚂, ⚃, ⚄, ⚅$\}$
- Numnber of points by Lebron James in a match: $\Omega : \{0, 1, 2, \ldots\}$
- Recovery from a disease $\Omega = \{\text{No, Yes}\} = \{0, 1\}$.

Example (Continuous Outcomes)

- Waiting time for a bus in Vashi: $\Omega = \{t \mid 0 \leq t \leq 30 \text{ minutes } \}$
- Winnings of golfers on the PCGA (Professionals' golfers association) tour $\Omega : \{x \mid x \geq 0\}$
- Individual height: $\Omega = \{x \mid 0 \leq x \leq 2.5m\}$

Elements in the sample space can be anything.



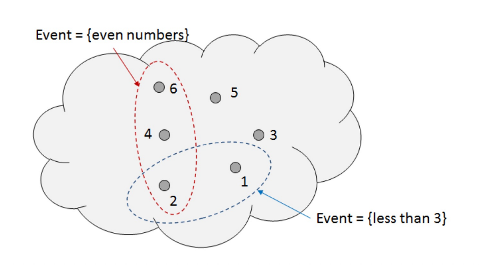
Event space $\mathcal{F}$

- The sample space contains all the possible outcomes. However, in many practical situations, we are not interested in each of the individual outcomes; we are interested in the combinations of the outcomes.

Die rolling

An event E is a subset in the sample space Ω. The set of all possible events is denoted as $\mathcal{F}$.

Example



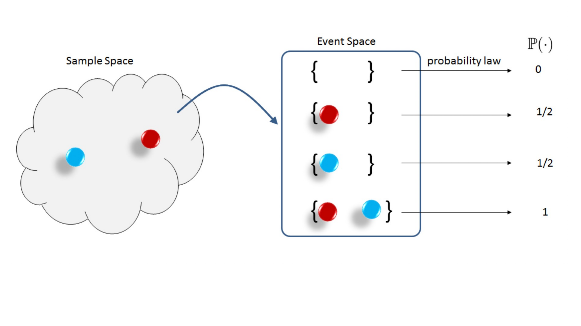
Probability law $\mathbb{P}$

Definition: Part 1

A probability law is a function $\mathbb{P}$ that associates to any event $E \in \mathcal{F}$ a real number in $[0, 1]$.

Example (Coin flip)

The event space $\mathcal{F} = \{\emptyset, \{H\}, \{T\}, \Omega\}$.



Probability law $\mathbb{P}$

Definition: Part 2

The function must satisfy three axioms (Kolmogorov, 1933):

① Non-negativity: $\mathbb{P}[A] \geq 0$, for any $A \subseteq \Omega$.

② Normalization: $\mathbb{P}[\Omega] = 1$

③ (Countable) Additivity: For any disjoint sets $\{A_1, A_2, \ldots\}$, it must be true that

$$\mathbb{P}\left[\bigcup_{i=1}^{\infty} A_i\right] = \sum_{i=1}^{\infty} \mathbb{P}\left[A_i\right]$$

Why these three axioms?

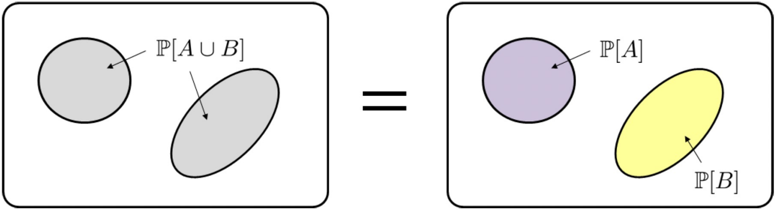
Axiom ① (Non-negativity) ensures that probability is never negative.

Axiom ② (Normalization) ensures that probability is never greater than 1.

Understanding additivity

If A and B are disjoint, then

$$\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B],$$



Why these three axioms?

Axiom ③ (Additivity) allows us to add probabilities when two events do not overlap.

Example (Throwing a fair die)

- $\Omega = \{\text{⚀}, \text{⚁}, \text{⚂}, \text{⚃}, \text{⚄}, \text{⚅}\} = \{1, 2, 3, 4, 5, 6\}$
- Let's consider the probability of getting $\{\text{⚁}, \text{⚃}, \}$:

$$P(\{\text{⚁}, \text{⚃}, \}) = P(\{\text{⚁}\} \cup \{\text{⚃}\}) = P((\{\text{⚁}\}) + P(\{\text{⚃}\}) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6}$$

Example (Baseball)

- In Baseball, a single is the most common type of base hit, accomplished through the act of a batter safely reaching first base by hitting a fair ball.

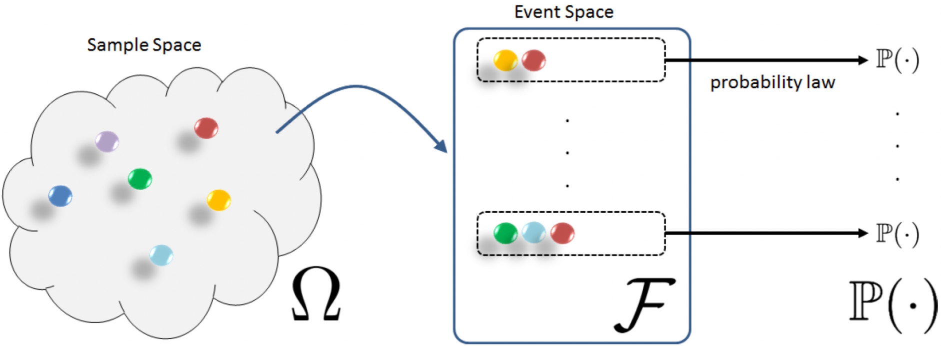
  A double is the act of a batter striking the pitched ball and safely reaching second base.
- Suppose that for a baseball player, $P(S) = 0.2$ and $P(D) = 0.05$. Then

$$P(S \text{ or } D) = P(S \cup D) = P(S) + P(D) = 0.25$$

Probability Space

A probability space consists of a triplet:

$$(\Omega, \mathcal{F}, \mathbb{P})$$



Properties of Probability

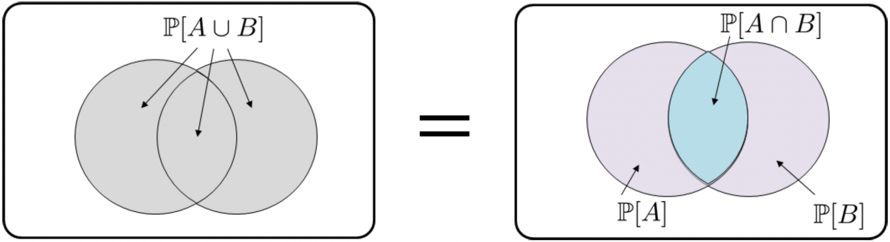
1. $\mathbb{P}[A^c] = 1 - \mathbb{P}[A]$

2. For any $A \subseteq \Omega, \mathbb{P}[A] \mid\leq 1$

3. $\mathbb{P}[\emptyset] = 0$

④ For any A and B

$$\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B]$$



This statement is different from Axiom 3 because A and B are not necessarily disjoint.

Example (Fair coin tossing)

Let $\Omega = \{\text{⚀}, \text{⚁}, \text{⚂}, \text{⚃}, \text{⚄}, \text{⚅}\} = \{1, 2, 3, 4, 5, 6\}$ be the sample space of a fair die.

Let $A = \{\text{⚀}, \text{⚁}, \text{⚂}\}$ and $B = \{\text{⚂}, \text{⚃}, \text{⚄}\}$.

Then

$$P(A \cup B) = P(\{\text{⚀}, \text{⚁}, \text{⚂}, \text{⚃}, \text{⚄}\}) = \frac{5}{6}$$

but also

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= P(\{\text{⚀}, \text{⚁}, \text{⚂}\}) + P(\{\text{⚂}, \text{⚃}, \text{⚄}\}) - P(\{\text{⚂}\}) \\ &= \frac{3}{6} + \frac{3}{6} - \frac{1}{6} = \frac{5}{6} \end{aligned}$$

Properties of Probability

5. For any A and B

$$\mathbb{P}[A \cup B] \leq \mathbb{P}[A] + \mathbb{P}[B]$$

Properties of Probability

⑤ For any A and B

$$\mathbb{P}[A \cup B] \leq \mathbb{P}[A] + \mathbb{P}[B]$$

⑥ If $A \subseteq B$, then

$$\mathbb{P}[A] \leq \mathbb{P}[B]$$

Example

$A = \{t \leq 5\}$, and $B = \{t \leq 10\}$, then $\mathbb{P}[A] \leq \mathbb{P}[B]$

Probability & Statistics for DS & AI

Conditional Probability

Michele Guindani

Jio Institute

Summer 2022

Conditional Probability

- In many practical data science problems, we are interested in the relationship between two or more events.
- For example, an event A may make B more likely or less likely to happen, and B may make C more or less likely to happen.
- A legitimate question in probability is then: If A has happened, what is the probability that B also happens?
- Of course, if A and B are correlated events, then knowing one event can tell us something about the other event.
- If the two events have no relationship (they are "independent"), knowing one event will not tell us anything about the other.

Conditional Probability

Consider two events A and B. Assume $\mathbb{P}[B] \neq 0$. The conditional probability of A given B is

$$\mathbb{P}[A \mid B] \stackrel{\text{def}}{=} \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$$

- Intuitively, the conditional probability of A given B is the probability that A happens when we know that B has already happened. Since B has already happened, the event that A has also happened is represented by $A \cap B$. However, since we are only interested in the relative probability of A with respect to B
- The difference between $\mathbb{P}[A \mid B]$ and $\mathbb{P}[A \cap B]$ is the denominator they carry:

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]} \text{ and } \mathbb{P}[A \cap B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[\Omega]}$$

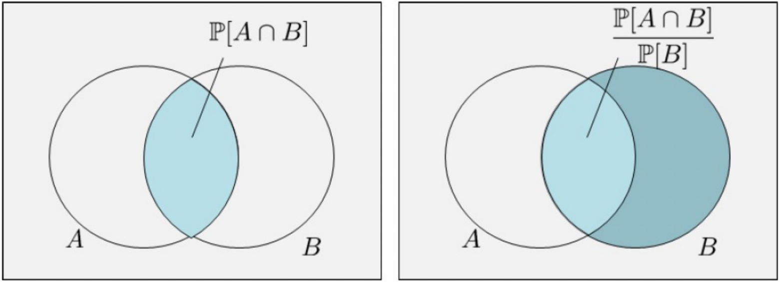


Figure: Illustration of conditional probability and its comparison with $\mathbb{P}[A \cap B]$.

Example (Throwing a die)

Consider throwing a die. Let

$$A = \{ \text{ getting a } 3\} \quad \text{and} \quad B = \{ \text{ getting an odd number } \}$$

The two probabilities are easy to compute:

$$P(A) = \frac{1}{6} \quad P(B) = \frac{3}{6}.$$

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Then,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$$

And,

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{6}}{\frac{1}{6}} = 1$$

Independence

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- Conditional probability deals with situations where two events A and B are related. What if the two events are unrelated? In probability, we have a technical term for this situation: statistical independence.

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Example (Throwing a dice)

Throw a dice twice. Let

$$A = \{ \text{ 1st dice is 3}\} \text{ and } B = \{ \text{ 2nd dice is 4}\}$$

Independence

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Example (Throwing a dice)

Throw a dice twice. Let

$$A = \{\text{ 1st dice is } 3\} \text{ and } B = \{\text{ 2nd dice is } 4\}$$

- What is independence?
- One event does not affect the other event!
- Are A and B independent then?

Independence

Two events A and B are statistically independent if

$$\mathbb{P}[A \cap B] = \mathbb{P}[A]\,\mathbb{P}[B]$$

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Independence

Two events A and B are statistically independent if

$$\mathbb{P}[A \cap B] = \mathbb{P}[A]\,\mathbb{P}[B]$$

- Why define independence in this way? Recall that $\mathbb{P}[A \mid B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$.

  If A and B are independent, then $\mathbb{P}[A \cap B] = \mathbb{P}[A]\mathbb{P}[B]$ and so

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]} = \frac{\mathbb{P}[A]\,\mathbb{P}[B]}{\mathbb{P}[B]} = \mathbb{P}[A]$$

- Intuitively, if the occurrence of B provides no additional information about the occurrence of A, then A and B are independent.

Example (Throwing a dice)

Throw a dice twice. Let

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Throw a dice twice. Let

$$A = \{\text{ 1st dice is 3}\} \text{ and } B = \{\text{ 2nd dice is 4}\}$$

Are A and B independent?
Let's compute

$$P(A) = \frac{1}{6} \qquad P(B) = \frac{1}{6}$$

We can also compute

$$P(A \cap B) = P(\text{ 1st dice is 3 and 2nd dice is 4 out of 36 possible combinations}) =$$

Since

$$P(A \cap B) = P(A)\,P(B)$$

then the two events are statistically independent.

Example (Throwing a dice twice)

Let

$$A = \{1 \text{ st dice is } 1\} \quad \text{and } B = \{ \text{ sum is } 7\}$$

Are A and B independent?

Example (Throwing a dice twice)

Let

$$A = \{1 \text{ st dice is } 1\} \quad \text{and } B = \{ \text{ sum is } 7\}$$

Are A and B independent?

Let's compute

$$P(A) = \frac{1}{6} \quad P(B) = \frac{6}{36} = \frac{1}{6}$$

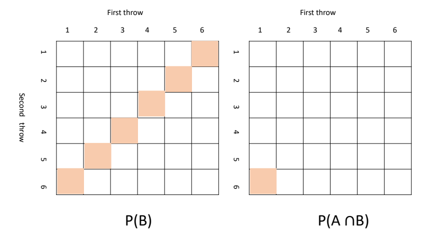
We can also check that

$$P(A \cap B) = \frac{1}{36}$$

So,

$$P(A \cap B) = P(A)\, P(B)$$

and the two events are independent.



Example (Throwing a dice twice)

Let

$$A = \{1 \text{ st dice is } 2\} \quad \text{and } B = \{ \text{ sum is } 8\}$$

Are A and B independent?

Example (Throwing a dice twice)

Let

$$A = \{1 \text{ st dice is } 2\} \quad \text{and } B = \{ \text{ sum is } 8\}$$

Are A and B independent?

Let's compute

$$P(A) = \frac{1}{6} \quad P(B) = \frac{5}{36}$$

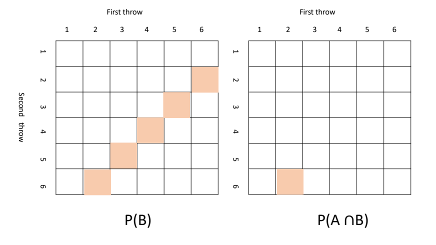
We can also check that

$$P(A \cap B) = \frac{1}{36}$$

So,

$$P(A \cap B) \neq P(A)\, P(B)$$

and the two events are NOT independent.



Example (Throwing a dice twice)

Let

$$A = \{1 \text{ st dice is } 3, 4, \text{ or } 5\} \quad \text{and } B = \{ \text{ sum is } 9\}$$

Are A and B independent?

Example (Throwing a dice twice)

Let

$$A = \{1 \text{ st dice is } 3, 4, \text{ or } 5\} \quad \text{and } B = \{ \text{ sum is } 9\}$$

Are A and B independent?

Let's compute

$$P(A) = \frac{3}{6} = \frac{1}{2} \quad P(B) = \frac{4}{36} = \frac{1}{9}$$

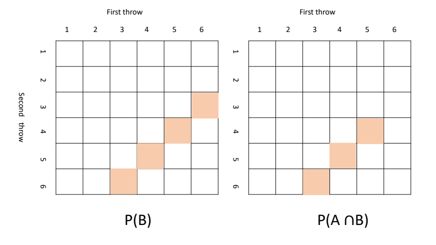
We can also check that

$$P(A \cap B) = \frac{3}{36} = \frac{1}{12}$$

So,

$$P(A \cap B) \neq P(A)\, P(B)$$

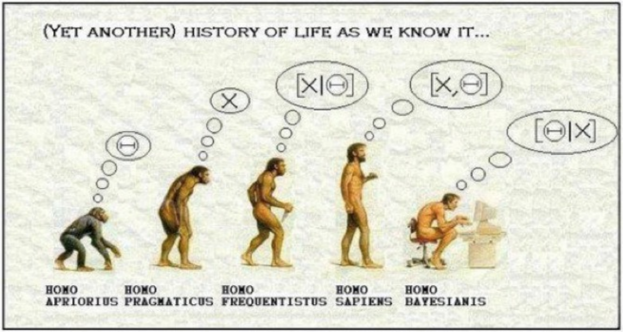
and the two events are NOT independent.



Bayesian Analysis!

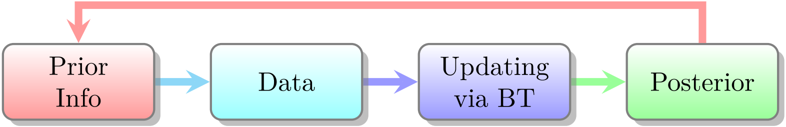
Bayesian Analysis!

- The Bayes Theorem is at the basis of a specific field of Statistics: Bayesian Analysis!

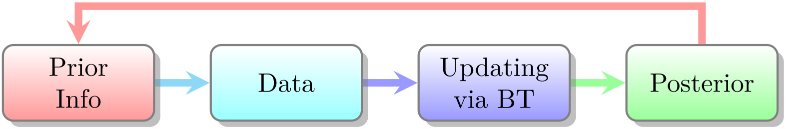


- The next evolution of Statistics! 🤪 😆

The tenets of Bayesian analysis

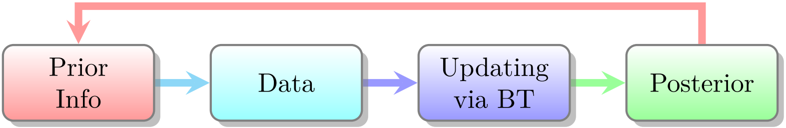


The tenets of Bayesian analysis



- Bayesian statistics starts by using (prior) probabilities to describe your current state of knowledge, say $P(\text{(available) info})$
- It then incorporates information through the collection of data, say $P(\text{data} \mid \text{(available) info})$

The tenets of Bayesian analysis



- Bayesian statistics starts by using (prior) probabilities to describe your current state of knowledge, say $P(\text{(available) info})$
- It then incorporates information through the collection of data, say $P(\text{data} \mid \text{(available) info})$
- By combining the prior probabilities with the data, you can obtain new (posterior) probabilities to describe an updated state of knowledge:

$$P(\text{(updated) info}|\text{data}) = \frac{P(\text{data} \mid \text{(available) info}) \times P(\text{(available) info})}{P(\text{data})}.$$

Bayes Theorem

- Sometimes, we may be interested in finding $P(A|B)$ but we only know $P(B|A)$ and the marginal probability $P(B)$

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Example (Medical Testing)

- Let D indicate someone who has a disease and H a healthy subject
- Let $+$ indicate that someone tested *positive* for the disease in a screening test, and $-$ that they tested *negative*
- We are interested to know $P(D|+)$. But we may know:

Bayes Theorem

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Prevalence of disease in the population: $Pr(D) = 0.01$

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- We are interested to know $P(D|+)$. But we may know:

  Prevalence of disease in the population: $Pr(D) = 0.01$

  Sensitivity of the test: how good is the test at identifying people who have the disease, $Pr(+|D) = 0.98$

Bayes Theorem

- Sometimes, we may be interested in finding $P(A|B)$ but we only know $P(B|A)$ and the marginal probability $P(B)$

Example (Medical Testing)

- Let D indicate someone who has a disease and H a healthy subject
- Let $+$ indicate that someone tested *positive* for the disease in a screening test, and $-$ that they tested *negative*
- We are interested to know $P(D|+)$. But we may know:

  Prevalence of disease in the population: $Pr(D) = 0.01$

  Sensitivity of the test: how good is the test at identifying people who have the disease, $Pr(+|D) = 0.98$

  Specificity of the test: how good is the test at correctly identifying healthy individuals: $Pr(-|H) = 0.95$

Bayes Theorem

For any two events A and B such that $\mathbb{P}[A] > 0$ and $\mathbb{P}[B] > 0$,

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[B \mid A]\,\mathbb{P}[A]}{\mathbb{P}[B]}$$

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So, substitute in (*)

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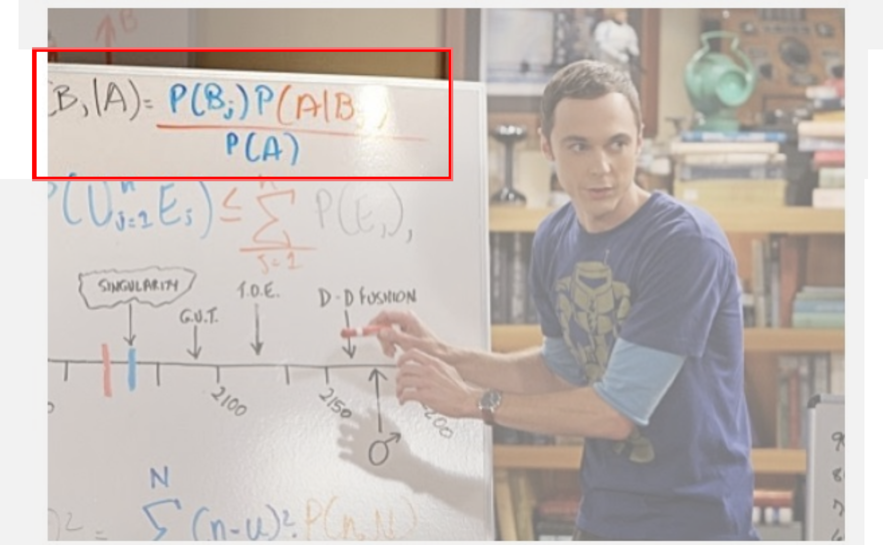
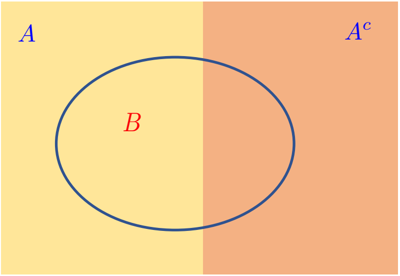
Of course, this is a very important Theorem...



- If we don't know the denominator, i.e. the marginal probability $P(B)$, we can compute it using the law of total probability, in this way:



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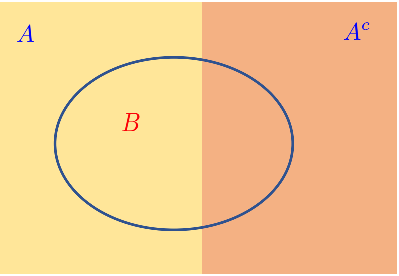

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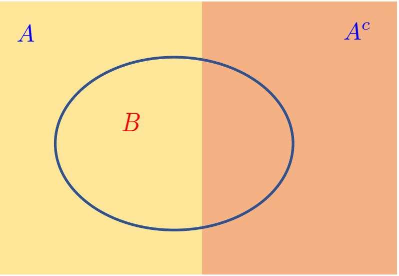

$$P(B) = P(B \cap A) + P(B \cap A^c) = P(B|A) \times P(A) + P(B|A^c) \times P(A^c)$$

Let's go back to the medical testing example

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\begin{aligned}
Pr(D|+) &= \frac{Pr(+|D)Pr(D)}{Pr(+)} \\
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&= \frac{0.0098}{0.0098 + 0.0495} \\
&= 0.165
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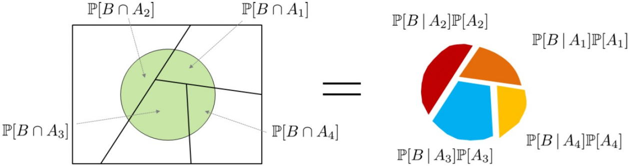
&= \frac{0.0098}{0.0098 + 0.0495} \\
&= 0.165
\end{aligned}
$$

- Even after testing positive, there's still a 83.5% chance that the person is healthy!
- Notice the effect of the prevalence: if $Pr(D) = 0.5$, then $Pr(D|+) = 0.95$!!!

Law of total probability

- A more general version of the law of total probability is the following (on partition of a set B)



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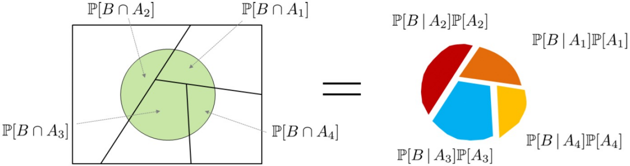

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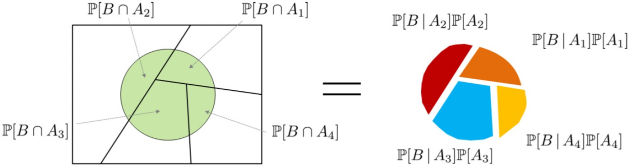

Theorem

Let $\{A_1, \ldots, A_n\}$ be a partition of Ω, i.e., $A_1, \ldots, A_n$ are disjoint and $\Omega = A_1 \cup \cdots \cup A_n$. Then, for any $B \subseteq \Omega$

$$\mathbb{P}[B] = \sum_{i=1}^{n} \mathbb{P}[B \mid A_i] \mathbb{P}[A_i]$$

Example (Tennis tournament)

Suppose your probability of winning the game is

- 0.3 against $\frac{1}{2}$ of the players (Event A).
- 0.4 against $\frac{1}{4}$ of the players (Event B).
- 0.5 against $\frac{1}{4}$ of the players (Event C).

① If you play a match in this tournament, what is the probability of your winning the match?

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   Let's see what we know already. We know that:

$$\mathbb{P}[A] = 0.5, \quad \mathbb{P}[B] = 0.25, \quad \mathbb{P}[C] = 0.25$$

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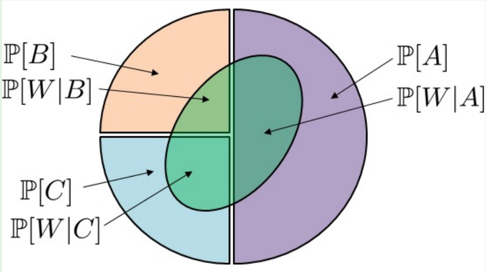
$$\mathbb{P}[A] = 0.5, \quad \mathbb{P}[B] = 0.25, \quad \mathbb{P}[C] = 0.25$$

   and we know that

$$\mathbb{P}[W \mid A] = 0.3, \quad \mathbb{P}[W \mid B] = 0.4, \quad \mathbb{P}[W \mid C] = 0.5$$

Example (Tennis tournament)

We can apply the law of total probability:



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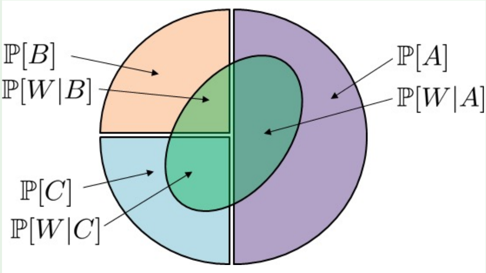

$$\begin{aligned}\mathbb{P}[W] &= \mathbb{P}[W \mid A]\mathbb{P}[A] + \mathbb{P}[W \mid B]\mathbb{P}[B] + \mathbb{P}[W \mid C]\mathbb{P}[C] \\ &= (0.3)(0.5) + (0.4)(0.25) + (0.5)(0.25) = 0.375\end{aligned}$$

Example (Tennis tournament - ctd)

2. Supposing that you have won a match, what is the probability that you played against an A player?

Example (Tennis tournament - ctd)

② Supposing that you have won a match, what is the probability that you played against an A player?
We can use the Bayes theorem to reply to this question. Given that you have won the match, the probability of A given W is

$$\mathbb{P}[A \mid W] = \frac{\mathbb{P}[W \mid A]\mathbb{P}[A]}{\mathbb{P}[W]} = \frac{(0.3)(0.5)}{0.375} = 0.4$$