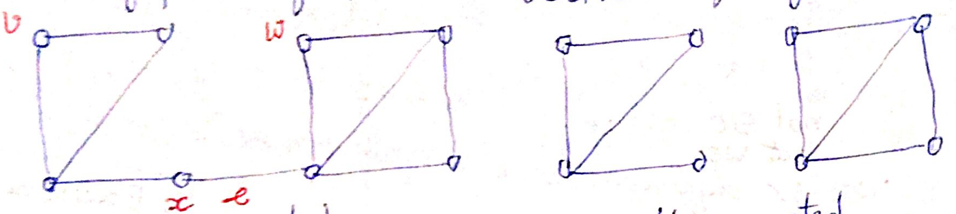
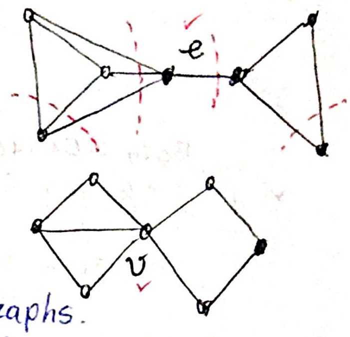
Connectedness in Undirected Graphs:

DEF. An undirected graph is called connected if there is a path between every pair of distinct vertices of the graph. Otherwise disconnected.



connected

disconnected



A disconnected graph consists of two or more connected graphs. Each of these connected subgraphs is called a component.

Theorem: A graph is disconnected if and only its vertex set V can be partitioned into two nonempty subsets V_1 and V_2 such that there exists no edge in G whose one end vertex is in V_1 and the other in V_2

Theorem: If a graph (connected or disconnected) has exactly two vertices of odd degree, there must be a path joining these two vertices.

Theorem: If G is a bipartite graph, then each cycle of G has even length.

Theorem: Let G be a simple graph on n vertices. If G has k components, then the number m of edges of satisfies

$$(n-k) \leq m \leq (n-k)(n-k+1)/2$$

Corollary: Any simple graph with n vertices and more than $(n-1)(n-2)/2$ edges is connected.

Q. Determine $k(G)$ and $\lambda(G)$ for each of: (i) C_6 (ii) W_6 (iii) $K_{4,7}$ (iv) Q_4.

A disconnecting set in a connected graph is a set of edges whose removal disconnects G.

A disconnecting set, no proper subset of which is a disconnecting set is called a cutset. A cutset with only one edge is called a bridge.

Edge connectivity, $\lambda(G)$ = size of the smallest cutset in G. G is k-edge connected if $\lambda(G) \geq k$.

A separating set in a connected graph G is a set of vertices whose deletion disconnects G. A separating set with only one vertex is called a CUT-VERTEX.

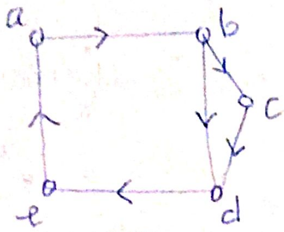
(Vertex) connectivity $k(G)$ = size of small separating set in G. G is k-connected if $k(G) \geq k$

Connectedness in Directed Graphs:

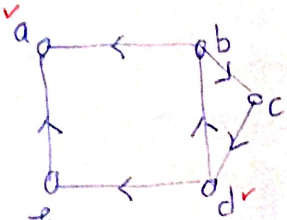
DEF. A directed graph is strongly connected if there is a path from a to b and from b to a whenever a and b are vertices in the graph.

DEF A directed graph is weakly connected if there is a path between every two vertices in the underlying undirected graph.

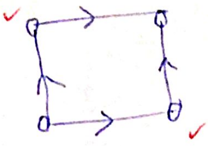
DEF. A digraph is unilaterally connected if for every pair of vertices one is reachable from the other



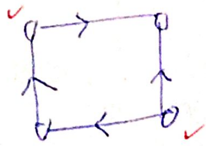
Both SC & WC



not SC, not WC
but WC



WC
not UC



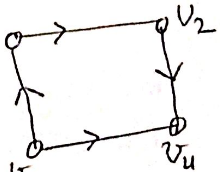
UC
not SC

✓ **THEOREM:** An n-vertex digraph is strongly connected if and only if the matrix M defined by

$$M = A + A^2 + A^3 + \cdots + A^n$$

has no zero entry, A is the adjacency matrix.

Q. Examine whether ----

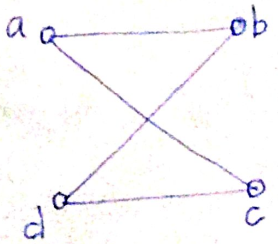


is strongly connected.

Counting Paths Between Vertices:

✓ **Theorem:** Let G be a graph with adjacency matrix A with respect to the ordering 1, 2, 3, ---, n (with directed or undirected edges, with multiple edges and loops allowed). The number of different paths of length r from i to j, where r is a positive integer, equals the (i, j)th entry of A^r.

EX. Determine the number of paths of length four from a to d in the simple graph:



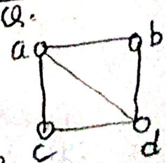
$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix};$$

The number of paths of length four from a to d is the (1, 4)th entry of A^4:

$$A^2 = \begin{bmatrix} & & & \\ & & & \end{bmatrix}; \quad A^3 = \begin{bmatrix} & & & \\ & & & \end{bmatrix}; \quad ✓ \; A^4 = \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix}$$

? $X = A + A^2 + A^3 = [\quad]$

Q.



? $A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

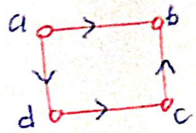
✓ **THEOREM:** If A is the adjacency matrix of an undirected graph G with n vertices, and

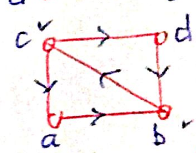
$$X = A + A^2 + A^3 + \cdots + A^{n-1}$$

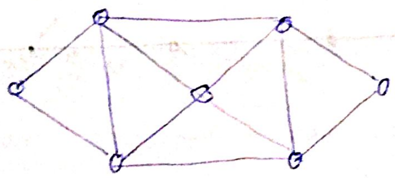
Then, G is disconnected iff and only if there exists at least one entry in matrix that is zero.

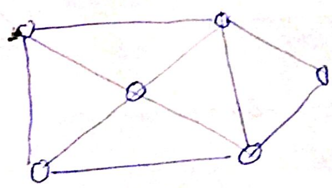
EULER GRAPHS

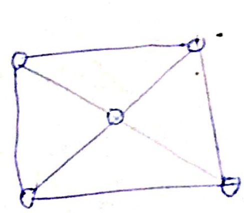
- A closed walk in a graph that contains every edge of the graph exactly once is called an Euler line/Euler circuit, and a graph that consists of an Euler is called an Euler graph.
- A open walk in a graph that includes (or traces or covers) all edges of the graph without retracing any edge is called a unicursal line or an open Euler or an Euler path.
  A (connected) graph that has a unicursal line will be termed/called a unicursal graph or semi-Euler graph.
- A graph that has neither Euler line nor unicursal line is called non-Euler graph.

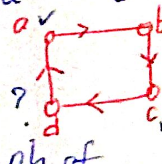












Q. Discuss Königsberg bridges problem.

Q. Which of the following graphs are Eulerian? semi-Eulerian?
(i) K_5 (ii) $K_{2,3}$ (iii) the graph of the cube (iv) the graph of the octahedron (v) the Petersen graph.

Q. Examine each of the following for an Euler graph.
(i) K_n (ii) $K_{m,n}$ (iii) W_n (iv) Q_k (v) Platonic graphs.

__Theorem__: A connected graph G an Euler graph if and only if all vertices of G are of even degree.

__Theorem__. A connected graph G is an Euler graph if and only if it can be decomposed into circuits / its set of edges can be split up into disjoint cycles.

__Corollary__: A connected graph is semi-Eulerian if and only if it has exactly two vertices of odd degree.

__Theorem__: In a connected graph G with exactly 2k odd vertices, there exist k edge-disjoint subgraphs such that they together contain all edges of G and that each is a unicursal graph.

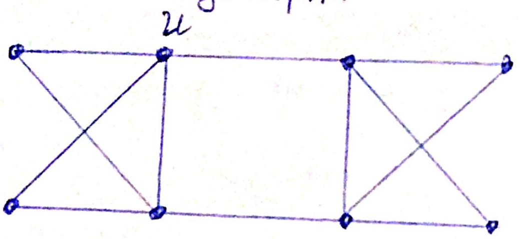
Fleury's Algorithm :

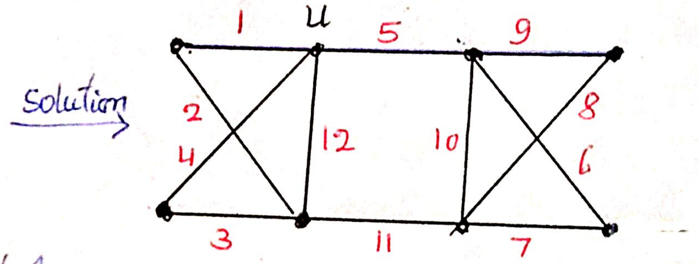
Theorem: Let G be an Eulerian graph. Then the following construction is always possible, and produces an Eulerian line of G.
Start at any vertex u and traverse the edges in an arbitrary manner, subject only to the following rules.:

(i) erase the edges as they are traversed, and if any isolated vertices result, erase them too;

(ii) at each stage, use a bridge only if there is no alternative.

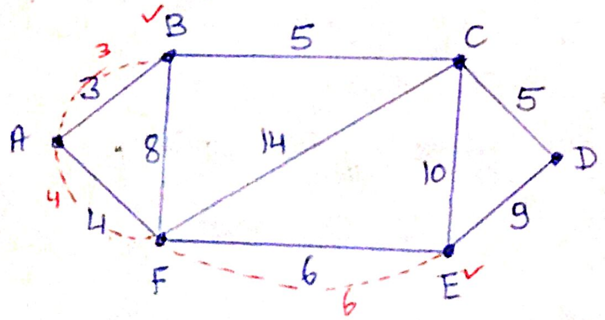
EX: Use Fleury's algorithm to produce an Eulerian trail / line / circuit for the graph.





Solution: There are many possible solutions; for example, traverse the edges in the order indicated by the adjoining diagram.

EX: Solve the Chinese postman for the weighted graph:



EX: Find a closed walk in a graph G that passes through every vertex exactly once.
Find an open walk in a graph G that passes through every vertex exactly once.