Assignment-6

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Download all python codes from

```
https://github.com/satyasm45/Summer−Internship/
    tree/main/Assignment−6/Codes
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and latex-tikz codes from

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https://github.com/satyasm45/Summer−Internship/
    tree/main/Assignment−6
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1 Question No. 2.41

Find the equation of all lines having slope 2 and being tangent to the curve $y + \frac{2}{x-3} = 0$

2 Explanation

The equation of curve:

$$y + \frac{2}{x-3} = 0 \tag{2.0.1}$$

$$xy - 3y + 2 = 0 \tag{2.0.2}$$

Comparing with the standard equation :

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.3}$$

$$\mathbf{V} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -\frac{3}{2} \end{pmatrix}, f = 2 \tag{2.0.4}$$

$$\mathbf{V}^{-1} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \tag{2.0.5}$$

$\because$

$$|\mathbf{V}| = \frac{-1}{4} \tag{2.0.6}$$

$$\implies |\mathbf{V}| < 0 \tag{2.0.7}$$

$\therefore$ (2.0.2) represents a hyperbola .
Now,the characteristic equation of $\mathbf{V}$ is

$$|\mathbf{V} - \lambda\mathbf{I}| = \begin{vmatrix} -\lambda & \frac{1}{2} \\ \frac{1}{2} & -\lambda \end{vmatrix} = 0 \tag{2.0.8}$$

$$\implies \lambda^2 - \frac{1}{4} = 0 \tag{2.0.9}$$

$\therefore$ Eigen values are

$$\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{-1}{2} \tag{2.0.10}$$

Eigen vector $\mathbf{p}$ is

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{2.0.11}$$

$$\implies (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \tag{2.0.12}$$

Eigen vector $\mathbf{p}_1$ corressponding to λ_1 can be obtained as

$$(\mathbf{V} - \lambda_1\mathbf{I}) = \begin{pmatrix} \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{pmatrix} \xleftrightarrow[R_1 \leftarrow -2R_1]{R_2 = R_1 + R_2} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \tag{2.0.13}$$

$$\implies \mathbf{p_1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.0.14}$$

Similarly,

$$\mathbf{p_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{2.0.15}$$

$\therefore$

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \tag{2.0.16}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{-1}{2} \end{pmatrix} \tag{2.0.17}$$

Now,

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{2.0.18}$$

$$= -\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \frac{-3}{2} \end{pmatrix} \tag{2.0.19}$$

$$= \begin{pmatrix} 3 \\ 0 \end{pmatrix} \tag{2.0.20}$$

$$\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f < 0 \tag{2.0.21}$$

So axes need to be swapped.

$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_2}} = 2 \tag{2.0.22}$$

$$\sqrt{\frac{f - \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_1}} = 2 \tag{2.0.23}$$

$\therefore$ Equation of standard hyperbola can be expressed as

$$\frac{y^2}{4} - \frac{x^2}{4} = 1 \tag{2.0.24}$$

The direction and normal vectors of tangent with slope 2:

$$\mathbf{m} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{2.0.25}$$

From conics table in manual:

$$\kappa = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}} = \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2} \tag{2.0.26}$$

and the desired points of contact are

$$\mathbf{q} = \mathbf{V}^{-1}(\kappa\mathbf{n} - \mathbf{u}) \tag{2.0.27}$$

$$\mathbf{q}_1 = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \tag{2.0.28}$$

$$\mathbf{q}_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \tag{2.0.29}$$

The equation of tangents are given by:

$$\mathbf{n}^T(\mathbf{x} - \mathbf{q}) = 0 \tag{2.0.30}$$

$$\begin{pmatrix} -2 & 1 \end{pmatrix}(\mathbf{x} - \mathbf{q}_1) = 0 \tag{2.0.31}$$

$$\begin{pmatrix} -2 & 1 \end{pmatrix}(\mathbf{x} - \mathbf{q}_2) = 0 \tag{2.0.32}$$

where $\mathbf{q}_1$ and $\mathbf{q}_2$ are given by (2.0.28) and (2.0.29)

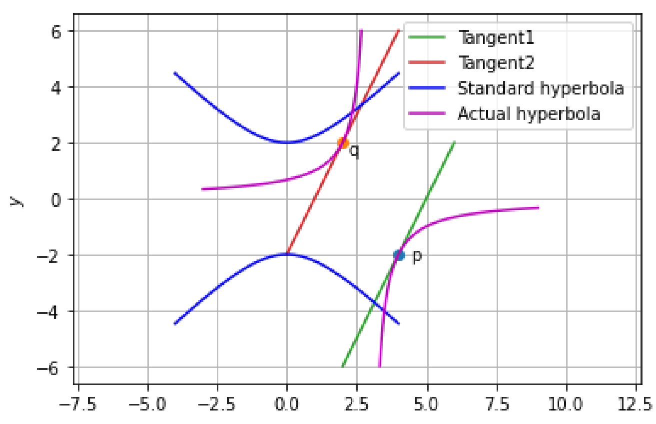

Fig. 2.1: The tangents to the curve with slope 2