Challenge Problem 5

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Download all python codes from

```
https://github.com/satyasm45/Summer−Internship/
    tree/main/Challenge5/Codes
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and latex-tikz codes from

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https://github.com/satyasm45/Summer−Internship/
    tree/main/Challenge5
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1 Challenge Question 5

Express the axis of a parabola in terms of $\mathbf{V}$,$\mathbf{u}$,f in general.

2 Explanation

We have seen in earlier assignment that the equation for parabola having focus $\mathbf{F}$ and directrix $\mathbf{n}^T\mathbf{x} = c$ assuming $\lambda = ||\mathbf{n}||^2$ will be:

$$\mathbf{x}^T(\lambda\mathbf{I} - \mathbf{n}\mathbf{n}^T)\mathbf{x} + 2(c\mathbf{n} - \lambda\mathbf{F})^T\mathbf{x} + \lambda||\mathbf{F}||^2 - c^2 = 0 \tag{2.0.1}$$

Comparing with standard equation

$$\mathbf{V} = \lambda\mathbf{I} - \mathbf{n}\mathbf{n}^T \tag{2.0.2}$$

$$\mathbf{u} = c\mathbf{n} - \lambda\mathbf{F} \tag{2.0.3}$$

$$f = \lambda||\mathbf{F}||^2 - c^2 \tag{2.0.4}$$

Now we have

$$\lambda\mathbf{I} - \mathbf{n}\mathbf{n}^T = \mathbf{V} \tag{2.0.5}$$

$$\mathbf{n}^T||\mathbf{n}||^2\mathbf{I} - \mathbf{n}^T\mathbf{n}\mathbf{n}^T = \mathbf{n}^T\mathbf{V} \tag{2.0.6}$$

$$||\mathbf{n}||^2(\mathbf{n}^T - \mathbf{n}^T) = \mathbf{n}^T\mathbf{V} \tag{2.0.7}$$

$$\mathbf{n}^T\mathbf{V} = 0 \tag{2.0.8}$$

Let:

$$\mathbf{V} = \begin{pmatrix}\mathbf{v_1} & \mathbf{v_2}\end{pmatrix} \tag{2.0.9}$$

$$\implies \mathbf{n}^T\mathbf{v_1} = 0, \mathbf{n}^T\mathbf{v_2} = 0 \tag{2.0.10}$$

It is also known that atleast one of $\mathbf{v_1}, \mathbf{v_2} \neq \begin{pmatrix}0\\0\end{pmatrix}$. WLOG let $\mathbf{v_1} \neq \begin{pmatrix}0\\0\end{pmatrix}$ The axis of parabola is perpendicular to directrix. $\therefore$ normal of axis is given as $\mathbf{v_1}$. The axis passes through focus,so equation of axis is:

$$\mathbf{v_1}^T\mathbf{x} = \mathbf{v_1}^T\mathbf{F} \tag{2.0.11}$$

We have:

$$c\mathbf{n} - \lambda\mathbf{F} = \mathbf{u} \tag{2.0.12}$$

$$c\mathbf{n}^T - \lambda\mathbf{F}^T = \mathbf{u}^T \tag{2.0.13}$$

$$c\mathbf{n}^T\mathbf{V} - \lambda\mathbf{F}^T\mathbf{V} = \mathbf{u}^T\mathbf{V} \tag{2.0.14}$$

$$0 - \lambda\mathbf{F}^T\mathbf{V} = \mathbf{u}^T\mathbf{V} \tag{2.0.15}$$

$$-\lambda\begin{pmatrix}\mathbf{F}^T\mathbf{v_1} & \mathbf{F}^T\mathbf{v_2}\end{pmatrix} = \begin{pmatrix}\mathbf{u}^T\mathbf{v_1} & \mathbf{u}^T\mathbf{v_2}\end{pmatrix} \tag{2.0.16}$$

$$\mathbf{v_1}^T\mathbf{F} = -\frac{\mathbf{u}^T\mathbf{v_1}}{\lambda} \tag{2.0.17}$$

Also

$$\lambda\mathbf{I} - \mathbf{n}\mathbf{n}^T = \mathbf{V} \tag{2.0.18}$$

$$\lambda\mathbf{V} - \mathbf{n}\mathbf{n}^T\mathbf{V} = \mathbf{V}^2 \tag{2.0.19}$$

$$\lambda\mathbf{V} = \mathbf{V}^2 \tag{2.0.20}$$

From 2.0.11,2.0.17 and 2.0.20 the equation of axis is given by:

$$\mathbf{v_1}^T\mathbf{x} = -\frac{\mathbf{u}^T\mathbf{v_1}}{\lambda} \tag{2.0.21}$$

$$\text{where } ,\lambda\mathbf{V} = \mathbf{V}^2 \tag{2.0.22}$$

It is interesting to note that equation is independent of 'f'.