Assignment-5

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Download all python codes from

```
https://github.com/satyasm45/Summer−Internship/
    tree/main/Assignment−5/Codes
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and latex-tikz codes from

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https://github.com/satyasm45/Summer−Internship/
    tree/main/Assignment−5
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1 Question No. 2.37

Find the equation of the ellipse, with major axis along the x-axis and passing through the points $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$

2 Explanation

Let

$$\mathbf{p} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \mathbf{q} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} \tag{2.0.1}$$

Let there be a standard ellipse that satisfies the constraints given by :

$$\mathbf{x}^T\mathbf{D}\mathbf{x} = 1, \quad \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \lambda_1, \lambda_2 > 0 \tag{2.0.2}$$

$\because$ $\mathbf{p}, \mathbf{q}$ satisfy (2.0.2),

$$\mathbf{p}^T\mathbf{D}\mathbf{p} = 1, \tag{2.0.3}$$

$$\mathbf{q}^T\mathbf{D}\mathbf{q} = 1 \tag{2.0.4}$$

which then can be expressed as:

$$\begin{aligned} \mathbf{p}^T\mathbf{P}\mathbf{d} &= 1, \\ \mathbf{q}^T\mathbf{Q}\mathbf{d} &= 1 \end{aligned} \tag{2.0.5}$$

where:

$$\mathbf{d} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \mathbf{P} = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}. \tag{2.0.6}$$

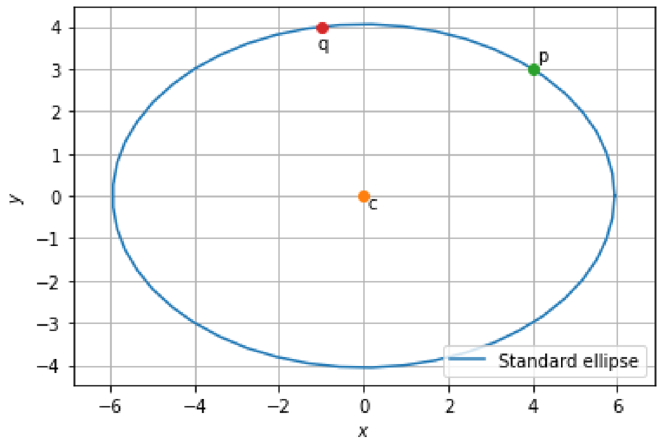

Fig. 2.1: Standard Ellipse

(2.0.5) can then be expressed as:

$$\begin{pmatrix} \mathbf{p}^T\mathbf{P} \\ \mathbf{q}^T\mathbf{Q} \end{pmatrix} \mathbf{d} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.0.7}$$

$$\begin{pmatrix} 16 & 9 \\ 1 & 16 \end{pmatrix} \mathbf{d} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.0.8}$$

Forming the augmented matrix and performing row reduction,

$$\begin{pmatrix} 16 & 9 & 1 \\ 1 & 16 & 1 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_2 - 16R1, R_2 \leftarrow -R_2]{R_2 \leftrightarrow R1} \begin{pmatrix} 1 & 16 & 1 \\ 0 & 247 & 15 \end{pmatrix} \tag{2.0.9}$$

$$\xleftrightarrow{R_1 \leftarrow 247R_1 - 16R_2} \begin{pmatrix} 247 & 0 & 7 \\ 0 & 247 & 15 \end{pmatrix} \tag{2.0.10}$$

$$\implies \mathbf{d} = \frac{1}{247}\begin{pmatrix} 7 \\ 15 \end{pmatrix}, \text{ or, } \mathbf{D} = \frac{1}{247}\begin{pmatrix} 7 & 0 \\ 0 & 15 \end{pmatrix} \tag{2.0.11}$$

So, equation of ellipse is given by:

$$\mathbf{x}^T \begin{pmatrix} 7 & 0 \\ 0 & 15 \end{pmatrix} \mathbf{x} = 247 \tag{2.0.12}$$

The ellipse parameters center and axes are obtained from conics table in manual as

$$\mathbf{c} = \mathbf{0}; \frac{1}{\sqrt{\lambda_1}} = \sqrt{\frac{247}{7}}, \frac{1}{\sqrt{\lambda_2}} = \sqrt{\frac{247}{15}}. \quad (2.0.13)$$

Clearly the (2.0.12) satisfies all the constraints. So,it is one of the possible answers. If standard ellipse is not considered then center $\mathbf{c} = \begin{pmatrix} \beta \\ 0 \end{pmatrix}$ can be taken anywhere on X axis in principle.The equation is given by:

$$(\mathbf{x} - \mathbf{c})^T \mathbf{D}(\mathbf{x} - \mathbf{c}) = 1 \quad (2.0.14)$$

$\because$ $\mathbf{p}, \mathbf{q}$ satisfy (2.0.14),

$$(\mathbf{p} - \mathbf{c})^T \mathbf{D}(\mathbf{p} - \mathbf{c}) = 1, \quad (2.0.15)$$
$$(\mathbf{q} - \mathbf{c})^T \mathbf{D}(\mathbf{q} - \mathbf{c}) = 1, \quad (2.0.16)$$

which can then be written as:

$$2 \times (\mathbf{p}^T\mathbf{D} - \mathbf{q}^T\mathbf{D})\mathbf{c} = \mathbf{p}^T\mathbf{D}\mathbf{p} - \mathbf{q}^T\mathbf{D}\mathbf{q} \quad (2.0.17)$$

Now $\mathbf{c}$ can be expressed as:

$$2\mathbf{c} = (\mathbf{p} + \mathbf{q}) + K\mathbf{D}^{-1}\mathbf{m} \quad (2.0.18)$$

Here K is any arbitrary constant and $\mathbf{m}$ satisfies $(\mathbf{p}^T - \mathbf{q}^T)\mathbf{m} = 0$ Now,

$$2\begin{pmatrix} 0 & 1 \end{pmatrix}\mathbf{c} = \begin{pmatrix} 0 & 1 \end{pmatrix}(\mathbf{p} + \mathbf{q} + K\mathbf{D}^{-1}\mathbf{m}) \quad (2.0.19)$$
$$0 = \begin{pmatrix} 0 & 1 \end{pmatrix}(\mathbf{p} + \mathbf{q} + K\mathbf{D}^{-1}\mathbf{m}) \quad (2.0.20)$$

Similarly,

$$2\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{c} = \begin{pmatrix} 1 & 0 \end{pmatrix}(\mathbf{p} + \mathbf{q} + K\mathbf{D}^{-1}\mathbf{m}) \quad (2.0.21)$$
$$2\beta = \begin{pmatrix} 1 & 0 \end{pmatrix}(\mathbf{p} + \mathbf{q} + K\mathbf{D}^{-1}\mathbf{m}) \quad (2.0.22)$$

(2.0.20) and (2.0.22) can be written as:

$$\begin{pmatrix} 2\beta \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}(\mathbf{p} + \mathbf{q} + K\mathbf{D}^{-1}\mathbf{m}) \quad (2.0.23)$$
$$\begin{pmatrix} 2\beta \\ 0 \end{pmatrix} = (\mathbf{p} + \mathbf{q} + K\mathbf{D}^{-1}\mathbf{m}) \quad (2.0.24)$$

Using values $\mathbf{p}$,$\mathbf{q}$ and $\mathbf{D}$:

$$\begin{pmatrix} 2\beta \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix} + T\begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{pmatrix}\begin{pmatrix} 1 \\ 5 \end{pmatrix} \quad (2.0.25)$$

Here T=$\frac{K}{\lambda_1\lambda_2}$.Now,X axis is major axis so $\frac{\lambda_2}{\lambda_1} > 1$

$$\frac{2\beta - 3}{\frac{-7}{5}} > 1 \quad (2.0.26)$$
$$\beta < 0.8 \quad (2.0.27)$$

Once we fix β,let

$$\mathbf{p} = \begin{pmatrix} 4 - \beta \\ 3 \end{pmatrix}, \mathbf{q} = \begin{pmatrix} -1 - \beta \\ 4 \end{pmatrix} \quad (2.0.28)$$

Then $\mathbf{p}$,$\mathbf{q}$ will again satisfy (2.0.2). Proceeding in a similar manner (2.0.8) becomes:

$$\begin{pmatrix} (4-\beta)^2 & 9 \\ (-1-\beta)^2 & 16 \end{pmatrix}\mathbf{d} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (2.0.29)$$

The augmented matrix:

$$\begin{pmatrix} (4-\beta)^2 & 9 & 1 \\ (-1-\beta)^2 & 16 & 1 \end{pmatrix} \quad (2.0.30)$$

RRE of augmented matrix:

$$\begin{pmatrix} 1 & 0 & \frac{7}{7\beta^2-146\beta+247} \\ 0 & 1 & \frac{5(3-2\beta)}{7\beta^2-146\beta+247} \end{pmatrix} \quad (2.0.31)$$

So,

$$\lambda_1 = \frac{7}{7\beta^2 - 146\beta + 247} \quad (2.0.32)$$
$$\lambda_2 = \frac{5(3 - 2\beta)}{7\beta^2 - 146\beta + 247} \quad (2.0.33)$$
$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (2.0.34)$$

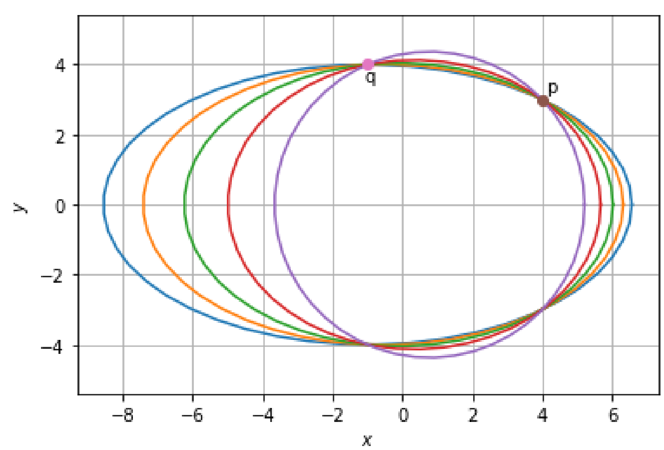

Fig. 2.2: Ellipses passing through the two points with X axis as major axis