

Assignment-6

Satya Sangram Mishra

Download all python codes from

<https://github.com/satyasm45/Summer-Internship/tree/main/Assignment-6/Codes>

and latex-tikz codes from

<https://github.com/satyasm45/Summer-Internship/tree/main/Assignment-6>

1 QUESTION No. 2.41

Find the equation of all lines having slope 2 and being tangent to the curve $y + \frac{2}{x-3} = 0$

2 EXPLANATION

The equation of curve:

$$y + \frac{2}{x-3} = 0 \quad (2.0.1)$$

$$xy - 3y + 2 = 0 \quad (2.0.2)$$

Comparing with the standard equation :

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.0.3)$$

$$\mathbf{V} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -\frac{3}{2} \end{pmatrix}, f = 2 \quad (2.0.4)$$

$$\mathbf{V}^{-1} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \quad (2.0.5)$$

\therefore

$$|\mathbf{V}| = \frac{-1}{4} \quad (2.0.6)$$

$$\Rightarrow |\mathbf{V}| < 0 \quad (2.0.7)$$

\therefore (2.0.2) represents a hyperbola .

Now, the characteristic equation of \mathbf{V} is

$$|\mathbf{V} - \lambda \mathbf{I}| = \begin{vmatrix} -\lambda & \frac{1}{2} \\ \frac{1}{2} & -\lambda \end{vmatrix} = 0 \quad (2.0.8)$$

$$\Rightarrow \lambda^2 - \frac{1}{4} = 0 \quad (2.0.9)$$

\therefore Eigen values are

$$\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{-1}{2} \quad (2.0.10)$$

Eigen vector \mathbf{p} is

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \quad (2.0.11)$$

$$\Rightarrow (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \quad (2.0.12)$$

Eigen vector \mathbf{p}_1 corresponding to λ_1 can be obtained as

$$(\mathbf{V} - \lambda_1 \mathbf{I}) = \begin{pmatrix} \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{pmatrix} \xrightarrow[R_1 \leftarrow -2R_1]{R_2 = R_1 + R_2} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad (2.0.13)$$

$$\Rightarrow \mathbf{p}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (2.0.14)$$

Similarly,

$$\mathbf{p}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad (2.0.15)$$

\therefore

$$\mathbf{P} = (\mathbf{p}_1 \ \mathbf{p}_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad (2.0.16)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{-1}{2} \end{pmatrix} \quad (2.0.17)$$

Now,

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \quad (2.0.18)$$

$$= -\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -\frac{3}{2} \end{pmatrix} \quad (2.0.19)$$

$$= \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad (2.0.20)$$

$$\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f < 0 \quad (2.0.21)$$

So axes need to be swapped.

$$\sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_2}} = 2 \quad (2.0.22)$$

$$\sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_1}} = 2 \quad (2.0.23)$$

∴ Equation of standard hyperbola can be expressed as

$$\frac{y^2}{4} - \frac{x^2}{4} = 1 \quad (2.0.24)$$

The direction and normal vectors of tangent with slope 2:

$$\mathbf{m} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad (2.0.25)$$

From conics table in manual:

$$\kappa = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\mathbf{n}^T \mathbf{V}^{-1} \mathbf{n}}} = \pm \sqrt{\frac{1}{4}} = \pm \frac{1}{2} \quad (2.0.26)$$

and the desired points of contact are

$$\mathbf{q} = \mathbf{V}^{-1}(\kappa \mathbf{n} - \mathbf{u}) \quad (2.0.27)$$

$$\mathbf{q}_1 = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \quad (2.0.28)$$

$$\mathbf{q}_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad (2.0.29)$$

The equation of tangents are given by:

$$\mathbf{n}^T \mathbf{x} = \mathbf{n}^T \mathbf{q} = c \quad (2.0.30)$$

$$\begin{pmatrix} -2 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -2 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ -2 \end{pmatrix} = -10 \quad (2.0.31)$$

$$\begin{pmatrix} -2 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = -2 \quad (2.0.32)$$

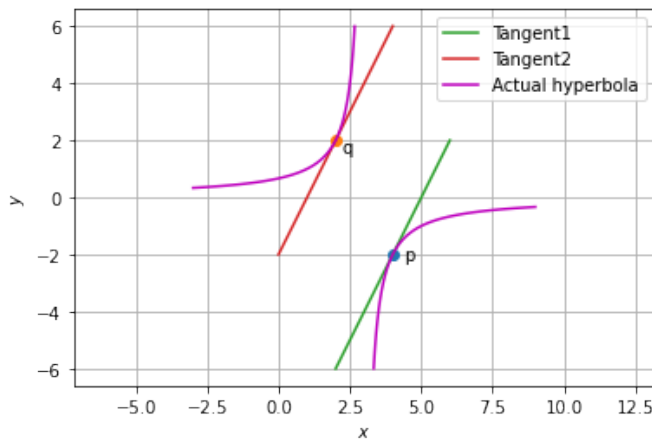


Fig. 2.1: The tangents to the curve with slope 2