Assignment-5

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Download all python codes from

```
https://github.com/satyasm45/Summer−Internship/
  tree/main/Assignment−5/Codes
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and latex-tikz codes from

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https://github.com/satyasm45/Summer−Internship/
  tree/main/Assignment−5
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1 Question No. 2.37

Find the equation of the ellipse, with major axis along the x-axis and passing through the points $\begin{pmatrix}4\\3\end{pmatrix}$ and $\begin{pmatrix}-1\\4\end{pmatrix}$

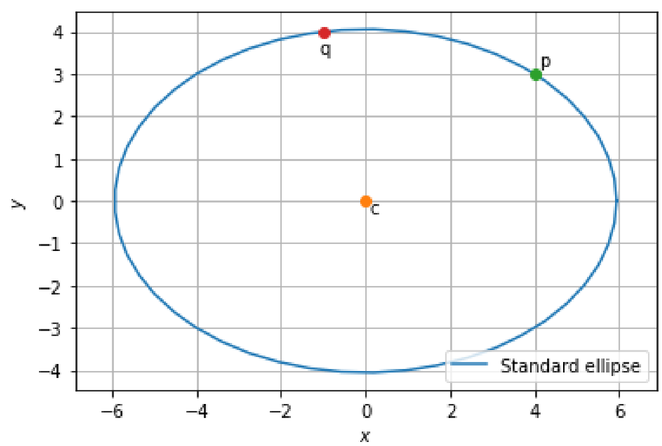

Fig. 2.1: Standard Ellipse

2 Explanation

Let

$$\mathbf{p} = \begin{pmatrix}4\\3\end{pmatrix}, \mathbf{q} = \begin{pmatrix}-1\\4\end{pmatrix} \tag{2.0.1}$$

This is a standard ellipse given by :

$$\mathbf{x}^T\mathbf{D}\mathbf{x} = 1, \quad \mathbf{D} = \begin{pmatrix}\lambda_1 & 0\\0 & \lambda_2\end{pmatrix}, \lambda_1, \lambda_2 > 0 \tag{2.0.2}$$

$\because$ $\mathbf{p}, \mathbf{q}$ satisfy (2.0.2),

$$\mathbf{p}^T\mathbf{D}\mathbf{p} = 1, \tag{2.0.3}$$

$$\mathbf{q}^T\mathbf{D}\mathbf{q} = 1 \tag{2.0.4}$$

which then can be expressed as:

$$\begin{aligned}\mathbf{p}^T\mathbf{P}\mathbf{d} &= 1,\\ \mathbf{q}^T\mathbf{Q}\mathbf{d} &= 1\end{aligned} \tag{2.0.5}$$

where:

$$\mathbf{d} = \begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix}, \mathbf{P} = \begin{pmatrix}4 & 0\\0 & 3\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}-1 & 0\\0 & 4\end{pmatrix}. \tag{2.0.6}$$

(2.0.5) can then be expressed as:

$$\begin{pmatrix}\mathbf{p}^T\mathbf{P}\\\mathbf{q}^T\mathbf{Q}\end{pmatrix}\mathbf{d} = \begin{pmatrix}1\\1\end{pmatrix} \tag{2.0.7}$$

$$\begin{pmatrix}16 & 9\\1 & 16\end{pmatrix}\mathbf{d} = \begin{pmatrix}1\\1\end{pmatrix} \tag{2.0.8}$$

Forming the augmented matrix and performing row reduction,

$$\begin{pmatrix}16 & 9 & 1\\1 & 16 & 1\end{pmatrix} \xleftrightarrow[R_2\leftarrow R_2-16R1, R_2\leftarrow -R_2]{R_2\leftrightarrow R1} \begin{pmatrix}1 & 16 & 1\\0 & 247 & 15\end{pmatrix} \tag{2.0.9}$$

$$\xleftrightarrow{R_1\leftarrow 247R_1-16R_2} \begin{pmatrix}247 & 0 & 7\\0 & 247 & 15\end{pmatrix} \tag{2.0.10}$$

$$\implies \mathbf{d} = \frac{1}{247}\begin{pmatrix}7\\15\end{pmatrix}, \text{ or, } \mathbf{D} = \frac{1}{247}\begin{pmatrix}7 & 0\\0 & 15\end{pmatrix} \tag{2.0.11}$$

So, equation of ellipse is given by:

$$\mathbf{x}^T\begin{pmatrix}7 & 0\\0 & 15\end{pmatrix}\mathbf{x} = 247 \tag{2.0.12}$$

The ellipse parameters center and axes are obtained from conics table in manual as

$$\mathbf{c} = \mathbf{0}; \frac{1}{\sqrt{\lambda_1}} = \sqrt{\frac{247}{7}}, \frac{1}{\sqrt{\lambda_2}} = \sqrt{\frac{247}{15}}. \tag{2.0.13}$$

If standard ellipse is not considered and center is taken anywhere on x axis i.e $\mathbf{c} = \begin{pmatrix}\beta\\0\end{pmatrix}$ infinite ellipses are possible. Once we fix β,let

$$\mathbf{p} = \begin{pmatrix} 4-\beta \\ 3 \end{pmatrix}, \mathbf{q} = \begin{pmatrix} -1-\beta \\ 4 \end{pmatrix} \tag{2.0.14}$$

Then $\mathbf{p}$,$\mathbf{q}$ will again satisfy (2.0.2). Proceeding in a similar manner (2.0.8) becomes:

$$\begin{pmatrix} (4-\beta)^2 & 9 \\ (-1-\beta)^2 & 16 \end{pmatrix} \mathbf{d} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.0.15}$$

The augmented matrix:

$$\begin{pmatrix} (4-\beta)^2 & 9 & 1 \\ (-1-\beta)^2 & 16 & 1 \end{pmatrix} \tag{2.0.16}$$

RRE of augmented matrix:

$$\begin{pmatrix} 1 & 0 & \frac{7}{7\beta^2-146\beta+247} \\ 0 & 1 & \frac{5(3-2\beta)}{7\beta^2-146\beta+247} \end{pmatrix} \tag{2.0.17}$$

So,

$$\lambda_1 = \frac{7}{7\beta^2 - 146\beta + 247} \tag{2.0.18}$$

$$\lambda_2 = \frac{5(3-2\beta)}{7\beta^2 - 146\beta + 247} \tag{2.0.19}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{2.0.20}$$

Now to find the appropriate range of β:

$$\lambda_1 > 0 \implies \beta\epsilon(-\infty, 1.86) \cup (19, \infty) \tag{2.0.21}$$

$$\lambda_2 > 0 \implies \beta\epsilon(-\infty, 1.5) \cup (1.86, 19) \tag{2.0.22}$$

Also X axis is major axis:

$$\lambda_2 > \lambda_1 \implies \beta\epsilon(-\infty, 0.8) \tag{2.0.23}$$

Taking $\cap$ of (2.0.21),(2.0.22),(2.0.23)

$$\beta\epsilon(-\infty, 0.8) \tag{2.0.24}$$

Therefore equation of ellipse is

$$(\mathbf{x}-\mathbf{c})^T \mathbf{D}(\mathbf{x}-\mathbf{c}) = 1 \tag{2.0.25}$$

where values of λ_1 and λ_2 given by (2.0.18) and (2.0.19) and range of β given by (2.0.24) and ellipse parameters:

$$\mathbf{c} = \begin{pmatrix} \beta \\ 0 \end{pmatrix}; \frac{1}{\sqrt{\lambda_1}}, \frac{1}{\sqrt{\lambda_2}}. \tag{2.0.26}$$

Fig. 2.2: Ellipses passing through the two points with X axis as major axis