Assignment-5

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1 QUESTION No. 2.37

Find the equation of the ellipse, with major axis along the x-axis and passing through the points $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$

Fig. 2.1: Standard Ellipse

2 EXPLANATION

Let

$$\mathbf{p} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \mathbf{q} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} \qquad (2.0.1)$$

Let there be a standard ellipse that satisfies the constraints given by :

$$\mathbf{x}^T \mathbf{D} \mathbf{x} = 1, \quad \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \lambda_1, \lambda_2 > 0 \qquad (2.0.2)$$

$\because \mathbf{p}, \mathbf{q}$ satisfy (2.0.2),

$$\mathbf{p}^T \mathbf{D} \mathbf{p} = 1, \qquad (2.0.3)$$

$$\mathbf{q}^T \mathbf{D} \mathbf{q} = 1 \qquad (2.0.4)$$

which then can be expressed as:

$$\mathbf{p}^T \mathbf{P} \mathbf{d} = 1, \\ \mathbf{q}^T \mathbf{Q} \mathbf{d} = 1 \qquad (2.0.5)$$

where:

$$\mathbf{d} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \mathbf{P} = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}. \qquad (2.0.6)$$
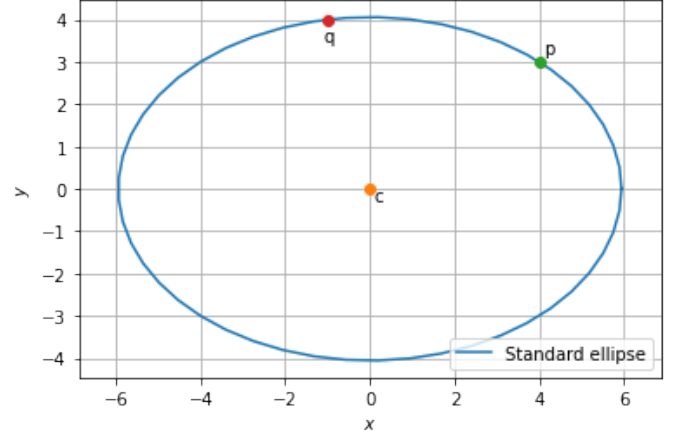
(2.0.5) can then be expressed as:

$$\begin{pmatrix} \mathbf{p}^T \mathbf{P} \\ \mathbf{q}^T \mathbf{Q} \end{pmatrix} \mathbf{d} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad (2.0.7)$$

$$\begin{pmatrix} 16 & 9 \\ 1 & 16 \end{pmatrix} \mathbf{d} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad (2.0.8)$$

Forming the augmented matrix and performing row reduction,

$$\begin{pmatrix} 16 & 9 & 1 \\ 1 & 16 & 1 \end{pmatrix} \xrightarrow[R_2 \leftarrow R_2 - 16R1, R_2 \leftarrow -R_2]{R_2 \leftrightarrow R1} \begin{pmatrix} 1 & 16 & 1 \\ 0 & 247 & 15 \end{pmatrix} \qquad (2.0.9)$$

$$\xrightarrow{R_1 \leftarrow 247R_1 - 16R_2} \begin{pmatrix} 247 & 0 & 7 \\ 0 & 247 & 15 \end{pmatrix} \qquad (2.0.10)$$

$$\implies \mathbf{d} = \frac{1}{247} \begin{pmatrix} 7 \\ 15 \end{pmatrix}, \text{ or, } \mathbf{D} = \frac{1}{247} \begin{pmatrix} 7 & 0 \\ 0 & 15 \end{pmatrix} \qquad (2.0.11)$$

So, equation of ellipse is given by:

$$\mathbf{x}^T \begin{pmatrix} 7 & 0 \\ 0 & 15 \end{pmatrix} \mathbf{x} = 247 \qquad (2.0.12)$$

The ellipse parameters center and axes are ob-

tained from conics table in manual as

$$\mathbf{c} = \mathbf{0}; \frac{1}{\sqrt{\lambda_1}} = \sqrt{\frac{247}{7}}, \frac{1}{\sqrt{\lambda_2}} = \sqrt{\frac{247}{15}}. \quad (2.0.13)$$

Clearly the (2.0.12) satisfies all the constraints. So,it is one of the possible answers. If standard ellipse is not considered then center $\mathbf{c} = \begin{pmatrix} \beta \\ 0 \end{pmatrix}$ can be taken anywhere on X axis in principle.The equation is given by:

$$(\mathbf{x} - \mathbf{c})^T \mathbf{M}(\mathbf{x} - \mathbf{c}) = 1 \quad (2.0.14)$$

where $\mathbf{M}$ is a diagonal matrix.

$$\mathbf{M} = \begin{pmatrix} l_1 & 0 \\ 0 & l_2 \end{pmatrix} \quad (2.0.15)$$

Comparing (2.0.14) with the general equation of ellipse:

$$\mathbf{V} = \mathbf{M} \quad (2.0.16)$$
$$\mathbf{u} = -\mathbf{Mc} \quad (2.0.17)$$
$$f = \mathbf{c}^T \mathbf{Mc} - 1 \quad (2.0.18)$$

Ellipse parameters for major axis and minor axis lengths:

$$\sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{l_1}} = \frac{1}{\sqrt{l_1}} \quad (2.0.19)$$

$$\sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{l_2}} = \frac{1}{\sqrt{l_2}} \quad (2.0.20)$$

$\because \mathbf{p}, \mathbf{q}$ satisfy (2.0.14),

$$(\mathbf{p} - \mathbf{c})^T \mathbf{M}(\mathbf{p} - \mathbf{c}) = 1, \quad (2.0.21)$$
$$(\mathbf{q} - \mathbf{c})^T \mathbf{M}(\mathbf{q} - \mathbf{c}) = 1, \quad (2.0.22)$$

which can then be written as:

$$2 \times (\mathbf{p}^T \mathbf{M} - \mathbf{q}^T \mathbf{M})\mathbf{c} = \mathbf{p}^T \mathbf{Mp} - \mathbf{q}^T \mathbf{Mq} \quad (2.0.23)$$

Now we have:

$$(\mathbf{p}^T - \mathbf{q}^T)\mathbf{M}(\mathbf{p} + \mathbf{q}) \quad (2.0.24)$$
$$= \mathbf{p}^T \mathbf{Mp} - \mathbf{q}^T \mathbf{Mq} + \mathbf{p}^T \mathbf{Mq} - \mathbf{q}^T \mathbf{Mp} \quad (2.0.25)$$
$$= \mathbf{p}^T \mathbf{Mp} - \mathbf{q}^T \mathbf{Mq} \quad (2.0.26)$$

Let $\mathbf{m}$ satisfy $(\mathbf{p}^T - \mathbf{q}^T)\mathbf{m} = 0$.So,

$$(\mathbf{p}^T - \mathbf{q}^T)\mathbf{MM}^{-1}\mathbf{m} = 0 \quad (2.0.27)$$

Now $\mathbf{c}$ can be expressed as:

$$2\mathbf{c} = (\mathbf{p} + \mathbf{q}) + K\mathbf{M}^{-1}\mathbf{m} \quad (2.0.28)$$

where K is an arbitrary constant Using values $\mathbf{c},\mathbf{p},\mathbf{q}$ and $\mathbf{M}$:

$$\begin{pmatrix} 2\beta \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix} + \frac{K}{l_1 \times l_2} \begin{pmatrix} l_2 & 0 \\ 0 & l_1 \end{pmatrix} \begin{pmatrix} 1 \\ 5 \end{pmatrix} \quad (2.0.29)$$

Now,X axis is major axis so $\frac{l_2}{l_1} > 1$

$$\frac{2\beta - 3}{\frac{-7}{5}} > 1 \quad (2.0.30)$$
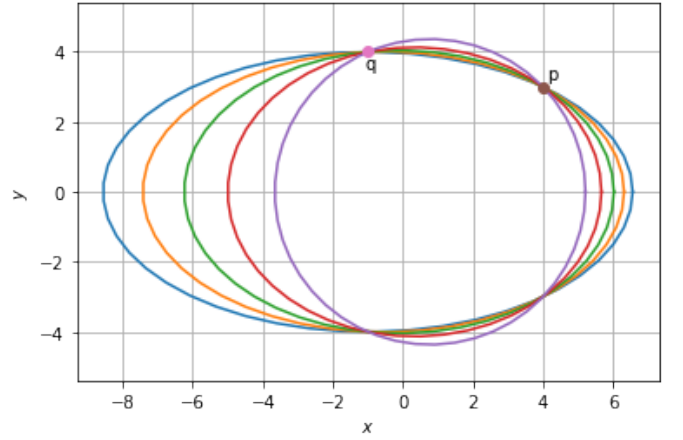
$$\beta < 0.8 \quad (2.0.31)$$

Fig. 2.2: Ellipses passing through the two points with X axis as major axis