Assignment-4

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Download all python codes from

```
https://github.com/satyasm45/Summer−Internship/
    tree/main/Assignment−4/Codes
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and latex-tikz codes from

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https://github.com/satyasm45/Summer−Internship/
    tree/main/Assignment−4
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1 Question No. 2.30

Find the equation of the parabola with focus $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and directrix $\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = -2$.

2 Explanation

Definition 1. *A parabola is a curve where any point is at an equal distance from: a fixed point (the focus* $\mathbf{F}$ *), and. a fixed straight line (the directrix* $\mathbf{n}^T\mathbf{x} = c$*).*

Lemma 2.1. *The distance of a point* $\mathbf{P}$ *from a line* $\mathbf{n}^T\mathbf{x} = c$ *is given by:*

$$\frac{|c - \mathbf{P}^T\mathbf{n}|}{||\mathbf{n}||} \tag{2.0.1}$$

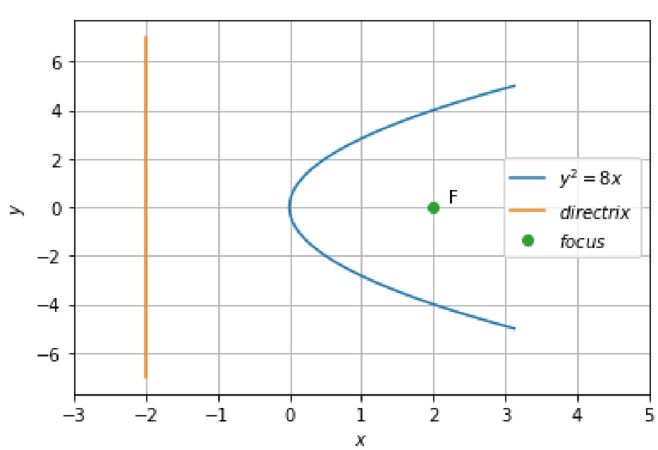

Fig. 2.1: Parabola $y^2 = 8x$

Theorem 2.1. *The equation of a parabola with focus* $\mathbf{F}$ *,directrix* $\mathbf{n}^T\mathbf{x} = c$ *and* $\lambda = ||\mathbf{n}||^2$ *is given by:*

$$\mathbf{x}^T(\lambda\mathbf{I} - \mathbf{n}\mathbf{n}^T)\mathbf{x} + 2(c\mathbf{n} - \lambda\mathbf{F})^T\mathbf{x} + \lambda||\mathbf{F}||^2 - c^2 = 0 \tag{2.0.2}$$

Proof. Using Definition 1 and Lemma 2.1 for any point $\mathbf{x}$ on parabola we have:

$$||\mathbf{x} - \mathbf{F}||^2 = \frac{(c - \mathbf{x}^T\mathbf{n})^2}{||\mathbf{n}||^2} \tag{2.0.3}$$

$$\lambda(\mathbf{x} - \mathbf{F})^T(\mathbf{x} - \mathbf{F}) = (c - \mathbf{x}^T\mathbf{n})^2 \tag{2.0.4}$$

$$\lambda(\mathbf{x}^T\mathbf{x} - 2\mathbf{F}^T\mathbf{x} + ||\mathbf{F}||^2) = c^2 + (\mathbf{x}^T\mathbf{n})^2 - 2c\mathbf{x}^T\mathbf{n} \tag{2.0.5}$$

$$\lambda\mathbf{x}^T\mathbf{x} - (\mathbf{x}^T\mathbf{n})^2 - 2\lambda\mathbf{F}^T\mathbf{x} + 2c\mathbf{n}^T\mathbf{x} = c^2 - \lambda||\mathbf{F}||^2 \tag{2.0.6}$$

$$\lambda\mathbf{x}^T\mathbf{I}\mathbf{x} - \mathbf{x}^T\mathbf{n}\mathbf{n}^T\mathbf{x} + 2(c\mathbf{n} - \lambda\mathbf{F})^T\mathbf{x} = c^2 - \lambda||\mathbf{F}||^2 \tag{2.0.7}$$

$$\mathbf{x}^T(\lambda\mathbf{I} - \mathbf{n}\mathbf{n}^T)\mathbf{x} + 2(c\mathbf{n} - \lambda\mathbf{F})^T\mathbf{x} + \lambda||\mathbf{F}||^2 - c^2 = 0 \tag{2.0.8}$$

□

Given information:

$$\mathbf{F} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, c = -2, \lambda = 1 \tag{2.0.9}$$

Substituting values of $\mathbf{F}, \mathbf{n}$,c,λ from(2.0.9):

$$\mathbf{x}^T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} + 2\begin{pmatrix} -4 & 0 \end{pmatrix}\mathbf{x} + 0 = 0 \tag{2.0.10}$$

Replacing $\mathbf{x}$ by $\begin{pmatrix} x \\ y \end{pmatrix}$ in (2.0.10) gives:

$$y^2 = 8x \tag{2.0.11}$$

The general equation of parabola we got in (2.0.8) is of form:

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.12}$$

$$\mathbf{V} = \lambda\mathbf{I} - \mathbf{n}\mathbf{n}^T \tag{2.0.13}$$
$$\mathbf{u} = c\mathbf{n} - \lambda\mathbf{F} \tag{2.0.14}$$
$$f = \lambda\|\mathbf{F}\|^2 - c^2 \tag{2.0.15}$$

$\mathbf{n} = \begin{pmatrix} x \\ y \end{pmatrix}$,$\lambda = \|\mathbf{n}\|^2 = x^2 + y^2$; $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.Now,

$$\left|\mathbf{V}\right| = \begin{vmatrix} \lambda - x^2 & -xy \\ -xy & \lambda - y^2 \end{vmatrix} \tag{2.0.16}$$
$$= \begin{vmatrix} y^2 & -xy \\ -xy & x^2 \end{vmatrix} \tag{2.0.17}$$
$$= 0 \tag{2.0.18}$$

Also characteristic equation of $\mathbf{V}$ is given by:

$$\left|\beta\mathbf{I} - \mathbf{V}\right| = 0 \tag{2.0.19}$$
$$\begin{vmatrix} \beta - \lambda + x^2 & xy \\ xy & \beta - \lambda + y^2 \end{vmatrix} = 0 \tag{2.0.20}$$
$$\begin{vmatrix} \beta - y^2 & xy \\ xy & \beta - x^2 \end{vmatrix} = 0 \tag{2.0.21}$$
$$\beta^2 - \beta(x^2 + y^2) = 0 \tag{2.0.22}$$
$$\beta(\beta - \lambda) = 0 \tag{2.0.23}$$
$$\beta_1 = 0 \tag{2.0.24}$$
$$\beta_2 = \lambda = x^2 + y^2 = \|\mathbf{n}\|^2 \tag{2.0.25}$$

So, (2.0.18) and (2.0.25) indicate that (2.0.8) is indeed an equation of parabola.

The eigen vector $\mathbf{p}$ corresponding to $\beta_1 = 0$ is given by:

$$\mathbf{V}\mathbf{p} = 0 \tag{2.0.26}$$

Row reducing $\mathbf{V}$ gives:

$$\begin{pmatrix} y^2 & -xy \\ -xy & x^2 \end{pmatrix} \xleftrightarrow[R_2 = R_2 + x.R_1]{R_1 = \frac{R_1}{y}} \begin{pmatrix} y & -x \\ 0 & 0 \end{pmatrix} \tag{2.0.27}$$
$$\implies \mathbf{p}_1 = \frac{1}{\sqrt{x^2 + y^2}} \begin{pmatrix} x \\ y \end{pmatrix} \tag{2.0.28}$$
$$\implies \mathbf{p}_1 = \frac{\mathbf{n}}{\|\mathbf{n}\|} \tag{2.0.29}$$

Similarly,the eigen vector $\mathbf{p}$ corresponding to $\beta_2 = \lambda$ is given by:

$$(\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \tag{2.0.30}$$

And a similar computation gives:

$$\mathbf{p}_2 = \frac{1}{\sqrt{x^2 + y^2}} \begin{pmatrix} -y \\ x \end{pmatrix} \tag{2.0.31}$$
$$\implies \mathbf{p}_2 = \frac{\mathbf{m}}{\|\mathbf{n}\|} \tag{2.0.32}$$
$$\mathbf{m}^T\mathbf{n} = 0, \|\mathbf{m}\| = \|\mathbf{n}\| \tag{2.0.33}$$

Let

$$\mathbf{P} = \frac{1}{\|\mathbf{n}\|} \begin{pmatrix} \mathbf{n} & \mathbf{m} \end{pmatrix} \tag{2.0.34}$$
$$\mathbf{D} = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \|\mathbf{n}\|^2 \end{pmatrix} \tag{2.0.35}$$

It is now easy to verify that:

$$\mathbf{P}^{-1} = \mathbf{P}^T \tag{2.0.36}$$
$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} \tag{2.0.37}$$

All the above exercises indicate that the conic equation (2.0.8) is indeed of parabola. (2.0.8) can be generalised for any conic as for any conic we have:

$$\|\mathbf{x} - \mathbf{F}\|^2 = e^2 \times \frac{(c - \mathbf{x}^T\mathbf{n})^2}{\|\mathbf{n}\|^2} \tag{2.0.38}$$

where e=eccentricity of the conic and e=1 for parabola.In our earlier notation we used $\lambda = \|\mathbf{n}\|^2$.Replacing λ by t=$\frac{\lambda}{e^2}$ in (2.0.8) will give the generalised equation for any conic as:

$$\mathbf{x}^T(t\mathbf{I} - \mathbf{n}\mathbf{n}^T)\mathbf{x} + 2(c\mathbf{n} - t\mathbf{F})^T\mathbf{x} + t\|\mathbf{F}\|^2 - c^2 = 0 \tag{2.0.39}$$

where t=$\frac{\|\mathbf{n}\|^2}{e^2}$.