

Assignment-5

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Download all python codes from

<https://github.com/satyasm45/Summer-Internship/tree/main/Assignment-5/Codes>

and latex-tikz codes from

<https://github.com/satyasm45/Summer-Internship/tree/main/Assignment-5>

1 QUESTION No. 2.37

Find the equation of the ellipse, with major axis along the x-axis and passing through the points $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$

2 EXPLANATION

Let

$$\mathbf{p} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \mathbf{q} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} \quad (2.0.1)$$

This is a standard ellipse given by :

$$\mathbf{x}^T \mathbf{D} \mathbf{x} = 1, \quad \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \lambda_1, \lambda_2 > 0 \quad (2.0.2)$$

$\therefore \mathbf{p}, \mathbf{q}$ satisfy (2.0.2),

$$\mathbf{p}^T \mathbf{D} \mathbf{p} = 1, \quad (2.0.3)$$

$$\mathbf{q}^T \mathbf{D} \mathbf{q} = 1 \quad (2.0.4)$$

which then can be expressed as:

$$\mathbf{p}^T \mathbf{P} \mathbf{d} = 1, \quad (2.0.5)$$

$$\mathbf{q}^T \mathbf{Q} \mathbf{d} = 1$$

where:

$$\mathbf{d} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \mathbf{P} = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}. \quad (2.0.6)$$

(2.0.5) can then be expressed as:

$$\begin{pmatrix} \mathbf{p}^T \mathbf{P} \\ \mathbf{q}^T \mathbf{Q} \end{pmatrix} \mathbf{d} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (2.0.7)$$

$$\begin{pmatrix} 16 & 9 \\ 1 & 16 \end{pmatrix} \mathbf{d} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (2.0.8)$$

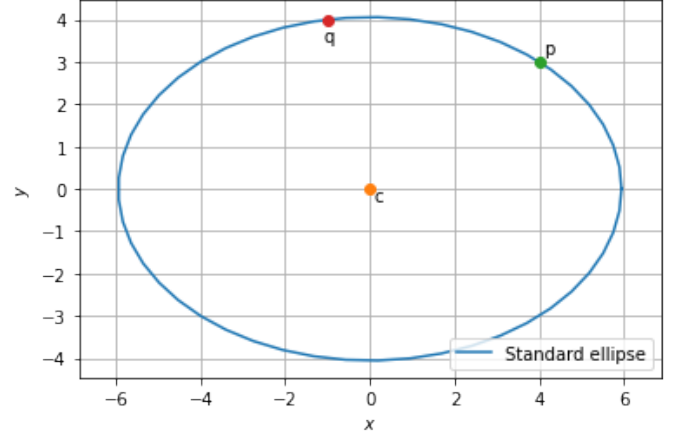


Fig. 2.1: Standard Ellipse

Forming the augmented matrix and performing row reduction,

$$\begin{pmatrix} 16 & 9 & 1 \\ 1 & 16 & 1 \end{pmatrix} \xrightarrow[R_2 \leftarrow R_2 - 16R_1, R_2 \leftarrow -R_2]{R_2 \leftrightarrow R_1} \begin{pmatrix} 1 & 16 & 1 \\ 0 & 247 & 15 \end{pmatrix} \quad (2.0.9)$$

$$\xrightarrow{R_1 \leftarrow 247R_1 - 16R_2} \begin{pmatrix} 247 & 0 & 7 \\ 0 & 247 & 15 \end{pmatrix} \quad (2.0.10)$$

$$\Rightarrow \mathbf{d} = \frac{1}{247} \begin{pmatrix} 7 \\ 15 \end{pmatrix}, \text{ or, } \mathbf{D} = \frac{1}{247} \begin{pmatrix} 7 & 0 \\ 0 & 15 \end{pmatrix} \quad (2.0.11)$$

So, equation of ellipse is given by:

$$\mathbf{x}^T \begin{pmatrix} 7 & 0 \\ 0 & 15 \end{pmatrix} \mathbf{x} = 247 \quad (2.0.12)$$

The ellipse parameters center and axes are obtained from conics table in manual as

$$\mathbf{c} = \mathbf{0}; \frac{1}{\sqrt{\lambda_1}} = \sqrt{\frac{247}{7}}, \frac{1}{\sqrt{\lambda_2}} = \sqrt{\frac{247}{15}}. \quad (2.0.13)$$

If standard ellipse is not considered and center is taken anywhere on x axis i.e $\mathbf{c} = \begin{pmatrix} \beta \\ 0 \end{pmatrix}$ infinite ellipses

are possible. Once we fix β , let

$$\mathbf{p} = \begin{pmatrix} 4 - \beta \\ 3 \end{pmatrix}, \mathbf{q} = \begin{pmatrix} -1 - \beta \\ 4 \end{pmatrix} \quad (2.0.14)$$

Then \mathbf{p}, \mathbf{q} will again satisfy (2.0.2). Proceeding in a similar manner (2.0.8) becomes:

$$\begin{pmatrix} (4 - \beta)^2 & 9 \\ (-1 - \beta)^2 & 16 \end{pmatrix} \mathbf{d} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (2.0.15)$$

The augmented matrix:

$$\begin{pmatrix} (4 - \beta)^2 & 9 & 1 \\ (-1 - \beta)^2 & 16 & 1 \end{pmatrix} \quad (2.0.16)$$

RRE of augmented matrix:

$$\begin{pmatrix} 1 & 0 & \frac{7}{7\beta^2 - 146\beta + 247} \\ 0 & 1 & \frac{5(3 - 2\beta)}{7\beta^2 - 146\beta + 247} \end{pmatrix} \quad (2.0.17)$$

So,

$$\lambda_1 = \frac{7}{7\beta^2 - 146\beta + 247} \quad (2.0.18)$$

$$\lambda_2 = \frac{5(3 - 2\beta)}{7\beta^2 - 146\beta + 247} \quad (2.0.19)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (2.0.20)$$

Now to find the appropriate range of β :

$$\lambda_1 > 0 \implies \beta \in (-\infty, 1.86) \cup (19, \infty) \quad (2.0.21)$$

$$\lambda_2 > 0 \implies \beta \in (-\infty, 1.5) \cup (1.86, 19) \quad (2.0.22)$$

Also X axis is major axis:

$$\lambda_2 > \lambda_1 \implies \beta \in (-\infty, 0.8) \quad (2.0.23)$$

Taking \cap of (2.0.21), (2.0.22), (2.0.23)

$$\beta \in (-\infty, 0.8) \quad (2.0.24)$$

Therefore equation of ellipse is

$$(\mathbf{x} - \mathbf{c})^T \mathbf{D} (\mathbf{x} - \mathbf{c}) = 1 \quad (2.0.25)$$

where \mathbf{D} can be obtained from (2.0.18), (2.0.19) and (2.0.20) and $\mathbf{c} = \begin{pmatrix} \beta \\ 0 \end{pmatrix}$ and range of β given by (2.0.24) and ellipse parameters:

$$\mathbf{c} = \begin{pmatrix} \beta \\ 0 \end{pmatrix}; \frac{1}{\sqrt{\lambda_1}}, \frac{1}{\sqrt{\lambda_2}}. \quad (2.0.26)$$



Fig. 2.2: Ellipses passing through the two points with X axis as major axis