

Assignment-4

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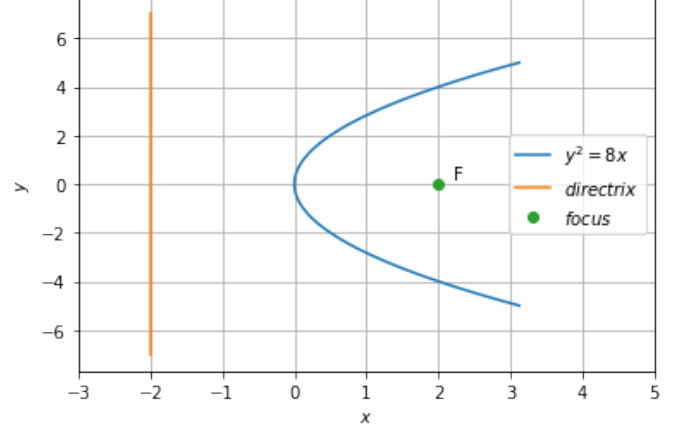


Fig. 2.1: Parabola $y^2 = 8x$

1 QUESTION No. 2.30

Find the equation of the parabola with focus $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and directrix $\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = -2$.

2 EXPLANATION

Definition 1. A conic is set of all points \mathbf{P} on the plane such that the ratio of distance of \mathbf{P} from: a fixed point (the focus \mathbf{F}), and, a fixed straight line (the directrix $\mathbf{n}^T \mathbf{x} = c$) is constant. This ratio is known as eccentricity(e).

Definition 2. Parabola is a conic with $e=1$.

Lemma 2.1. The distance of a point \mathbf{P} from a line $\mathbf{n}^T \mathbf{x} = c$ is given by:

$$\frac{|c - \mathbf{P}^T \mathbf{n}|}{\|\mathbf{n}\|} \quad (2.0.1)$$

Theorem 2.1. The equation of any conic with focus \mathbf{F} , directrix $\mathbf{n}^T \mathbf{x} = c$, eccentricity ' e ' and $t = \frac{\|\mathbf{n}\|^2}{e^2}$ is given by:

$$\mathbf{x}^T (t\mathbf{I} - \mathbf{nn}^T) \mathbf{x} + 2(c\mathbf{n} - t\mathbf{F})^T \mathbf{x} + t\|\mathbf{F}\|^2 - c^2 = 0 \quad (2.0.2)$$

Proof. Using Definition 1 and Lemma 2.1 for any point \mathbf{x} on conic we have:

$$\|\mathbf{x} - \mathbf{F}\|^2 = e^2 \times \frac{(c - \mathbf{x}^T \mathbf{n})^2}{\|\mathbf{n}\|^2} \quad (2.0.3)$$

$$t(\mathbf{x} - \mathbf{F})^T (\mathbf{x} - \mathbf{F}) = (c - \mathbf{x}^T \mathbf{n})^2 \quad (2.0.4)$$

$$t(\mathbf{x}^T \mathbf{x} - 2\mathbf{F}^T \mathbf{x} + \|\mathbf{F}\|^2) = c^2 + (\mathbf{x}^T \mathbf{n})^2 - 2c\mathbf{x}^T \mathbf{n} \quad (2.0.5)$$

$$t\mathbf{x}^T \mathbf{x} - (\mathbf{x}^T \mathbf{n})^2 - 2t\mathbf{F}^T \mathbf{x} + 2c\mathbf{n}^T \mathbf{x} = c^2 - t\|\mathbf{F}\|^2 \quad (2.0.6)$$

$$t\mathbf{x}^T \mathbf{I} \mathbf{x} - \mathbf{x}^T \mathbf{nn}^T \mathbf{x} + 2(c\mathbf{n} - t\mathbf{F})^T \mathbf{x} = c^2 - t\|\mathbf{F}\|^2 \quad (2.0.7)$$

$$\mathbf{x}^T (t\mathbf{I} - \mathbf{nn}^T) \mathbf{x} + 2(c\mathbf{n} - t\mathbf{F})^T \mathbf{x} + t\|\mathbf{F}\|^2 - c^2 = 0 \quad (2.0.8)$$

□

Corollary 2.2. The equation for parabola assuming $\lambda = \|\mathbf{n}\|^2$ will be:

$$\mathbf{x}^T (\lambda \mathbf{I} - \mathbf{nn}^T) \mathbf{x} + 2(c\mathbf{n} - \lambda \mathbf{F})^T \mathbf{x} + \lambda \|\mathbf{F}\|^2 - c^2 = 0 \quad (2.0.9)$$

Given information:

$$\mathbf{F} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, c = -2, \lambda = 1 \quad (2.0.10)$$

Substituting values of $\mathbf{F}, \mathbf{n}, c, \lambda$ from (2.0.10):

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -4 & 0 \end{pmatrix} \mathbf{x} + 0 = 0 \quad (2.0.11)$$

Replacing \mathbf{x} by $\begin{pmatrix} x \\ y \end{pmatrix}$ in (2.0.11) gives:

$$y^2 = 8x \quad (2.0.12)$$

The general equation of parabola we got in (2.0.9) is of form:

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.0.13)$$

$$\mathbf{V} = \lambda \mathbf{I} - \mathbf{nn}^T \quad (2.0.14)$$

$$\mathbf{u} = c\mathbf{n} - \lambda \mathbf{F} \quad (2.0.15)$$

$$f = \lambda \|\mathbf{F}\|^2 - c^2 \quad (2.0.16)$$

$$\mathbf{n} = \begin{pmatrix} x \\ y \end{pmatrix}, \lambda = \|\mathbf{n}\|^2 = x^2 + y^2; \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \text{Now,}$$

$$|\mathbf{V}| = \begin{vmatrix} \lambda - x^2 & -xy \\ -xy & \lambda - y^2 \end{vmatrix} \quad (2.0.17)$$

$$= \begin{vmatrix} y^2 & -xy \\ -xy & x^2 \end{vmatrix} \quad (2.0.18)$$

$$= 0 \quad (2.0.19)$$

Also characteristic equation of \mathbf{V} is given by:

$$|\beta \mathbf{I} - \mathbf{V}| = 0 \quad (2.0.20)$$

$$\begin{vmatrix} \beta - \lambda + x^2 & xy \\ xy & \beta - \lambda + y^2 \end{vmatrix} = 0 \quad (2.0.21)$$

$$\begin{vmatrix} \beta - y^2 & xy \\ xy & \beta - x^2 \end{vmatrix} = 0 \quad (2.0.22)$$

$$\beta^2 - \beta(x^2 + y^2) = 0 \quad (2.0.23)$$

$$\beta(\beta - \lambda) = 0 \quad (2.0.24)$$

$$\beta_1 = 0 \quad (2.0.25)$$

$$\beta_2 = \lambda = x^2 + y^2 = \|\mathbf{n}\|^2 \quad (2.0.26)$$

So, (2.0.19) and (2.0.25) indicate that (2.0.9) is indeed an equation of parabola.

The eigen vector \mathbf{p} corresponding to $\beta_1 = 0$ is given by:

$$\mathbf{V}\mathbf{p} = 0 \quad (2.0.27)$$

Row reducing \mathbf{V} gives:

$$\begin{pmatrix} y^2 & -xy \\ -xy & x^2 \end{pmatrix} \xrightarrow[R_2=R_2+x.R_1]{R_1=\frac{R_1}{y}} \begin{pmatrix} y & -x \\ 0 & 0 \end{pmatrix} \quad (2.0.28)$$

$$\Rightarrow \mathbf{p}_1 = \frac{1}{\sqrt{x^2 + y^2}} \begin{pmatrix} x \\ y \end{pmatrix} \quad (2.0.29)$$

$$\Rightarrow \mathbf{p}_1 = \frac{\mathbf{n}}{\|\mathbf{n}\|} \quad (2.0.30)$$

Similarly, the eigen vector \mathbf{p} corresponding to $\beta_2 = \lambda$ is given by:

$$(\mathbf{V} - \lambda \mathbf{I})\mathbf{p} = 0 \quad (2.0.31)$$

And a similar computation gives:

$$\mathbf{p}_2 = \frac{1}{\sqrt{x^2 + y^2}} \begin{pmatrix} -y \\ x \end{pmatrix} \quad (2.0.32)$$

$$\Rightarrow \mathbf{p}_2 = \frac{\mathbf{m}}{\|\mathbf{m}\|} \quad (2.0.33)$$

$$\mathbf{m}^T \mathbf{n} = 0, \|\mathbf{m}\| = \|\mathbf{n}\| \quad (2.0.34)$$

Let

$$\mathbf{P} = \frac{1}{\|\mathbf{n}\|} \begin{pmatrix} \mathbf{n} & \mathbf{m} \end{pmatrix} \quad (2.0.35)$$

$$\mathbf{D} = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \|\mathbf{n}\|^2 \end{pmatrix} \quad (2.0.36)$$

It is now easy to verify that:

$$\mathbf{P}^{-1} = \mathbf{P}^T \quad (2.0.37)$$

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} \quad (2.0.38)$$

All the above exercises indicate that the conic equation (2.0.9) is indeed of parabola.