Assignment-5

Satya Sangram Mishra

Download all python codes from

```
https://github.com/satyasm45/Summer−Internship/
    tree/main/Assignment−5/Codes
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and latex-tikz codes from

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https://github.com/satyasm45/Summer−Internship/
    tree/main/Assignment−5
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1 Question No. 2.37

Find the equation of the ellipse, with major axis along the x-axis and passing through the points $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$

2 Explanation

Let

$$\mathbf{p} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \mathbf{q} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} \tag{2.0.1}$$

Let there be a standard ellipse that satisfies the constraints given by :

$$\mathbf{x}^T\mathbf{D}\mathbf{x} = 1, \quad \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \lambda_1, \lambda_2 > 0 \tag{2.0.2}$$

$\because$ $\mathbf{p}, \mathbf{q}$ satisfy (2.0.2),

$$\mathbf{p}^T\mathbf{D}\mathbf{p} = 1, \tag{2.0.3}$$
$$\mathbf{q}^T\mathbf{D}\mathbf{q} = 1 \tag{2.0.4}$$

which then can be expressed as:

$$\begin{aligned} \mathbf{p}^T\mathbf{P}\mathbf{d} &= 1, \\ \mathbf{q}^T\mathbf{Q}\mathbf{d} &= 1 \end{aligned} \tag{2.0.5}$$

where:

$$\mathbf{d} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \mathbf{P} = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}. \tag{2.0.6}$$

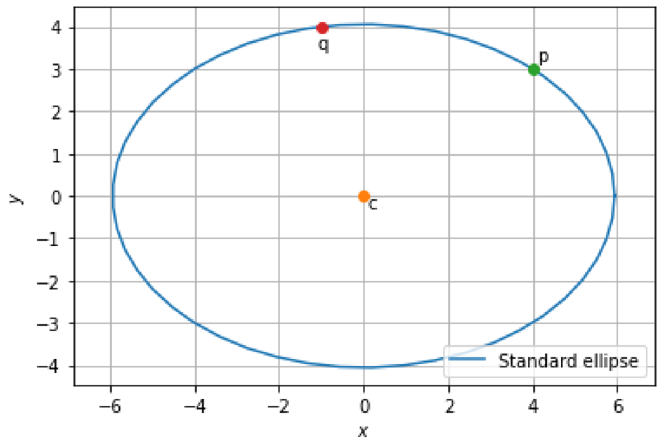

Fig. 2.1: Standard Ellipse

(2.0.5) can then be expressed as:

$$\begin{pmatrix} \mathbf{p}^T\mathbf{P} \\ \mathbf{q}^T\mathbf{Q} \end{pmatrix} \mathbf{d} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.0.7}$$

$$\begin{pmatrix} 16 & 9 \\ 1 & 16 \end{pmatrix} \mathbf{d} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.0.8}$$

Forming the augmented matrix and performing row reduction,

$$\begin{pmatrix} 16 & 9 & 1 \\ 1 & 16 & 1 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_2 - 16R1, R_2 \leftarrow -R_2]{R_2 \leftrightarrow R1} \begin{pmatrix} 1 & 16 & 1 \\ 0 & 247 & 15 \end{pmatrix} \tag{2.0.9}$$

$$\xleftrightarrow{R_1 \leftarrow 247R_1 - 16R_2} \begin{pmatrix} 247 & 0 & 7 \\ 0 & 247 & 15 \end{pmatrix} \tag{2.0.10}$$

$$\implies \mathbf{d} = \frac{1}{247}\begin{pmatrix} 7 \\ 15 \end{pmatrix}, \text{ or, } \mathbf{D} = \frac{1}{247}\begin{pmatrix} 7 & 0 \\ 0 & 15 \end{pmatrix} \tag{2.0.11}$$

So, equation of ellipse is given by:

$$\mathbf{x}^T \begin{pmatrix} 7 & 0 \\ 0 & 15 \end{pmatrix} \mathbf{x} = 247 \tag{2.0.12}$$

The ellipse parameters center and axes are obtained from conics table in manual as

$$\mathbf{c} = \mathbf{0}; \frac{1}{\sqrt{\lambda_1}} = \sqrt{\frac{247}{7}}, \frac{1}{\sqrt{\lambda_2}} = \sqrt{\frac{247}{15}}. \tag{2.0.13}$$

Clearly the (2.0.12) satisfies all the constraints. So,it is one of the possible answers. If standard ellipse is not considered then center $\mathbf{c} = \begin{pmatrix} \beta \\ 0 \end{pmatrix}$ can be taken anywhere on X axis in principle.The equation is given by:

$$(\mathbf{x} - \mathbf{c})^T \mathbf{M} (\mathbf{x} - \mathbf{c}) = 1 \tag{2.0.14}$$

where $\mathbf{M}$ is a diagonal matrix.

$$\mathbf{M} = \begin{pmatrix} l_1 & 0 \\ 0 & l_2 \end{pmatrix} \tag{2.0.15}$$

Comparing (2.0.14) with the general equation of ellipse:

$$\mathbf{V} = \mathbf{M} \tag{2.0.16}$$
$$\mathbf{u} = -\mathbf{M}\mathbf{c} \tag{2.0.17}$$
$$f = \mathbf{c}^T \mathbf{M} \mathbf{c} - 1 \tag{2.0.18}$$

Ellipse parameters for major axis and minor axis lengths:

$$\sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{l_1}} = \frac{1}{\sqrt{l_1}} \tag{2.0.19}$$
$$\sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{l_2}} = \frac{1}{\sqrt{l_2}} \tag{2.0.20}$$

$\because$ $\mathbf{p}, \mathbf{q}$ satisfy (2.0.14),

$$(\mathbf{p} - \mathbf{c})^T \mathbf{M} (\mathbf{p} - \mathbf{c}) = 1, \tag{2.0.21}$$
$$(\mathbf{q} - \mathbf{c})^T \mathbf{M} (\mathbf{q} - \mathbf{c}) = 1, \tag{2.0.22}$$

which can then be written as:

$$2 \times (\mathbf{p}^T \mathbf{M} - \mathbf{q}^T \mathbf{M}) \mathbf{c} = \mathbf{p}^T \mathbf{M} \mathbf{p} - \mathbf{q}^T \mathbf{M} \mathbf{q} \tag{2.0.23}$$

Now we have:

$$(\mathbf{p}^T - \mathbf{q}^T) \mathbf{M} (\mathbf{p} + \mathbf{q}) = \mathbf{p}^T \mathbf{M} \mathbf{p} - \mathbf{q}^T \mathbf{M} \mathbf{q} + \mathbf{p}^T \mathbf{M} \mathbf{q} - \mathbf{q}^T \mathbf{M} \mathbf{p} \tag{2.0.24}$$
$$= \mathbf{p}^T \mathbf{M} \mathbf{p} - \mathbf{q}^T \mathbf{M} \mathbf{q} \tag{2.0.25}$$

Let $\mathbf{m}$ satisfy $(\mathbf{p}^T - \mathbf{q}^T)\mathbf{m} = 0$.So,

$$(\mathbf{p}^T - \mathbf{q}^T) \mathbf{M} \mathbf{M}^{-1} \mathbf{m} = 0 \tag{2.0.26}$$

Now $\mathbf{c}$ can be expressed as:

$$2\mathbf{c} = (\mathbf{p} + \mathbf{q}) + K \mathbf{M}^{-1} \mathbf{m} \tag{2.0.27}$$

where K is an arbitrary constant Using values $\mathbf{c}$,$\mathbf{p}$,$\mathbf{q}$ and $\mathbf{M}$:

$$\begin{pmatrix} 2\beta \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix} + \frac{K}{l_1 \times l_2} \begin{pmatrix} l_2 & 0 \\ 0 & l_1 \end{pmatrix} \begin{pmatrix} 1 \\ 5 \end{pmatrix} \tag{2.0.28}$$

Now,X axis is major axis so $\frac{l_2}{l_1} > 1$

$$\frac{2\beta - 3}{\frac{-7}{5}} > 1 \tag{2.0.29}$$
$$\beta < 0.8 \tag{2.0.30}$$

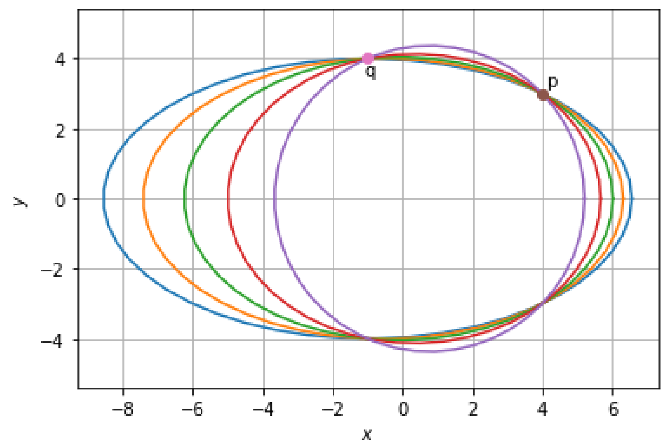

Fig. 2.2: Ellipses passing through the two points with X axis as major axis