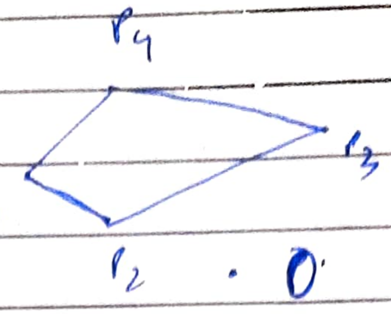
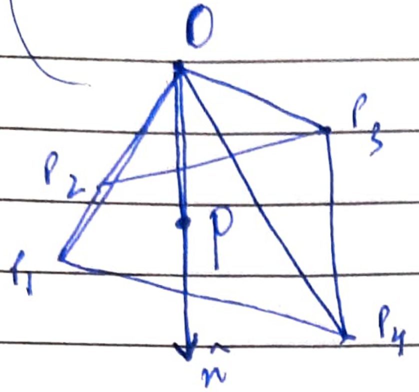
(here an example of 4 sided polygon)

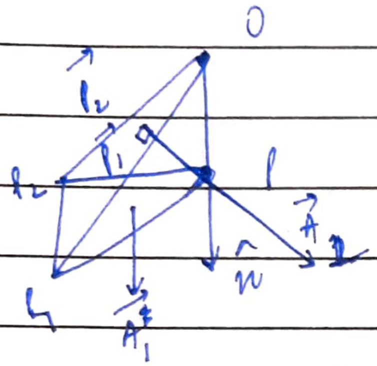


origin



taking apart of it [like divide this polyon of n-sides into n part according to point vector

normal vector along the plane.



$$\vec{A}_2' = \text{Area}(\Delta\, O, P_1, P_2) = \frac{1}{2}\left[\vec{P}_1 \times \vec{P}_2\right]$$

$$\vec{A}_1 = \hat{n} \cdot \vec{A}_2'$$

$$= \frac{1}{2}\left[\hat{n} \cdot (\vec{P}_1 \times \vec{P}_2)\right]$$

This will happen for each n part,

$$\text{Total Area} = \left| \sum_{i=1}^{n-1} \frac{1}{2}\left[\hat{n} \cdot (\vec{P}_i \times \vec{P}_{i+1})\right] + \frac{1}{2}\left[\hat{n} \cdot (\vec{P}_n \times \vec{P}_1)\right] \right|$$

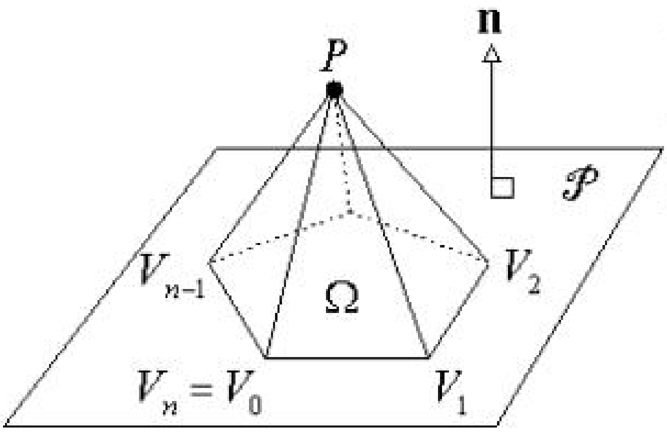
$$\boxed{\text{Total Area} = \frac{1}{2}\,\hat{n} \cdot \left[\sum \vec{P}_i \times \vec{P}_{i+1}\right]}$$

3D Planar Polygons

An important generalization is for planar polygons embedded in 3D space [Goldman, 1994]. We have already shown that the area of a 3D triangle $\Delta V_0 V_1 V_2$ is given by half the magnitude of the cross product of two edge vectors; namely, $|(V_1 - V_0) \times (V_2 - V_0)|/2$.

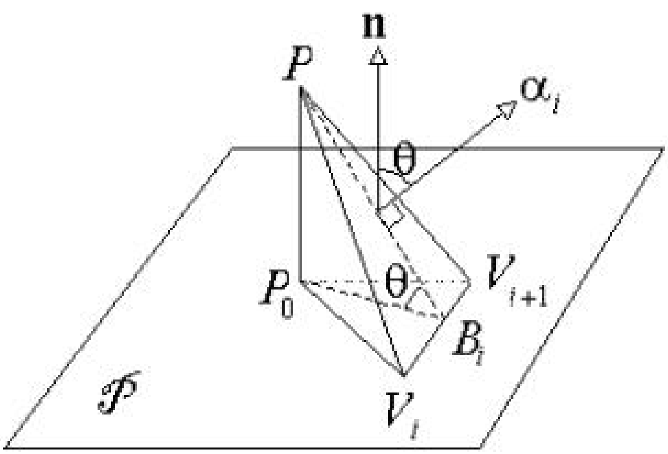
The Standard Formula

There is a classic standard formula for the area of a 3D polygon [Goldman, 1994] that extends the cross-product formula for a triangle. It can be derived from Stokes Theorem. However, we show here how to derive it from a 3D triangular decomposition that is geometrically more intuitive.



A general 3D planar polygon $\boldsymbol{\Omega}$ has vertices $V_i = (x_i, y_i\ z_i)$ for $i = 0, n$ with $V_n = V_0$, where all the vertices lie on the same 3D plane $\mathscr{P}$ which has a ***unit normal*** vector $\mathbf{n}$. Now, as in the 2D case, let P be any 3D point (not generally on the plane $\mathscr{P}$); and for each edge $\mathbf{e}_i = V_i V_{i+1}$ of $\boldsymbol{\Omega}$, form the 3D triangle $\boldsymbol{\Delta}_i = \boldsymbol{\Delta} P V_i V_{i+1}$. We would like to relate the sum of the areas of all these triangles to the area of the polygon $\boldsymbol{\Omega}$ in the plane $\mathscr{P}$. But what we have is a pyramidal cone with P as an apex over the polygon $\boldsymbol{\Omega}$ as a base. We are going to project the triangular sides of this cone onto the plane $\mathscr{P}$ of the base polygon, and compute signed areas of the projected triangles. Then the sum of the projected areas will equal the total area of the planar polygon.

To achieve this, start by associating to each triangle $\boldsymbol{\Delta}_i = \boldsymbol{\Delta} P V_i V_{i+1}$ an area vector $\boldsymbol{\alpha}_i = [(V_i - P) \times (V_{i+1} - P)]/2$, which is perpendicular to $\boldsymbol{\Delta}_i$, and whose magnitude we know is equal to that triangle's area. Next, drop a perpendicular from $P$to a point P_0 on $\mathscr{P}$, and consider the projected triangle $\mathbf{T}_i = \boldsymbol{\Delta} P_0 V_i V_{i+1}$. Then drop a



perpendicular P_0B_i from P_0 to B_i on the edge $\mathbf{e}_i = V_iV_{i+1}$. Since PP_0 is also perpendicular to $\mathbf{e}_i$, the three points PP_0B_i define a plane that is perpendicular to $\mathbf{e}_i$, and thus PB_i is a perpendicular from P to $\mathbf{e}_i$. Thus $|PB_i|$ is the height of $\mathbf{\Delta}_i$, and $|P_0B_i|$ is the height of $\mathbf{T}_i$. Further, the angle between these two altitudes = θ = the angle between $\mathbf{n}$ and $\boldsymbol{\alpha}_i$ since a 90° rotation (in the PP_0B_i plane) results in congruence. This gives:

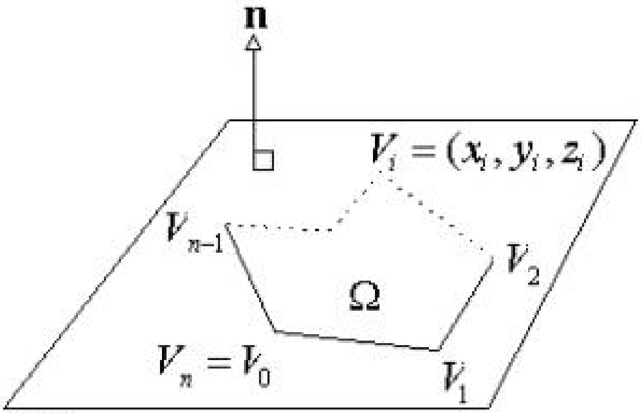
$$\mathrm{A}(\mathbf{T}_i) = \tfrac{1}{2}|V_iV_{i+1}||P_0B_i| = \tfrac{1}{2}|V_iV_{i+1}||PB_i|\cos\theta = \mathrm{A}(\mathbf{\Delta}_i)\cos\theta = |\mathbf{n}||\boldsymbol{\alpha}_i|\cos\theta = \mathbf{n}\cdot\boldsymbol{\alpha}_i$$

This signed area computation is positive if the vertices of $\mathbf{T}_i$ are oriented counterclockwise when we look at the plane $\mathscr{P}$ from the side pointed to by $\mathbf{n}$. As in the 2D case, we can now add together the signed areas of all the triangles $\mathbf{T}_i$ to get the area of the polygon $\mathbf{\Omega}$. Writing this down, we have:

$$\mathrm{A}(\mathbf{\Omega}) = \sum_{i=0}^{n-1}\mathrm{A}(\mathbf{T}_i) = \sum_{i=0}^{n-1}\mathbf{n}\cdot\boldsymbol{\alpha}_i = \frac{\mathbf{n}}{2}\cdot\sum_{i=0}^{n-1}(PV_i \times PV_{i+1})$$

Finally, by selecting $P = (0,0,0)$, we have $PV_i = V_i$ and this produces the concise formula:

$$2\mathrm{A}(\mathbf{\Omega}) = \mathbf{n}\cdot\sum_{i=0}^{n-1}(V_i \times V_{i+1})$$



which uses $6n+3$ multiplications and $4n+2$ additions

Similar to the 2D case, this is a *signed* area which is positive when the vertices are oriented counterclockwise around the polygon when viewed from the side of $\mathscr{P}$ pointed to by $\mathbf{n}$.