Elliptic Curve Cryptography

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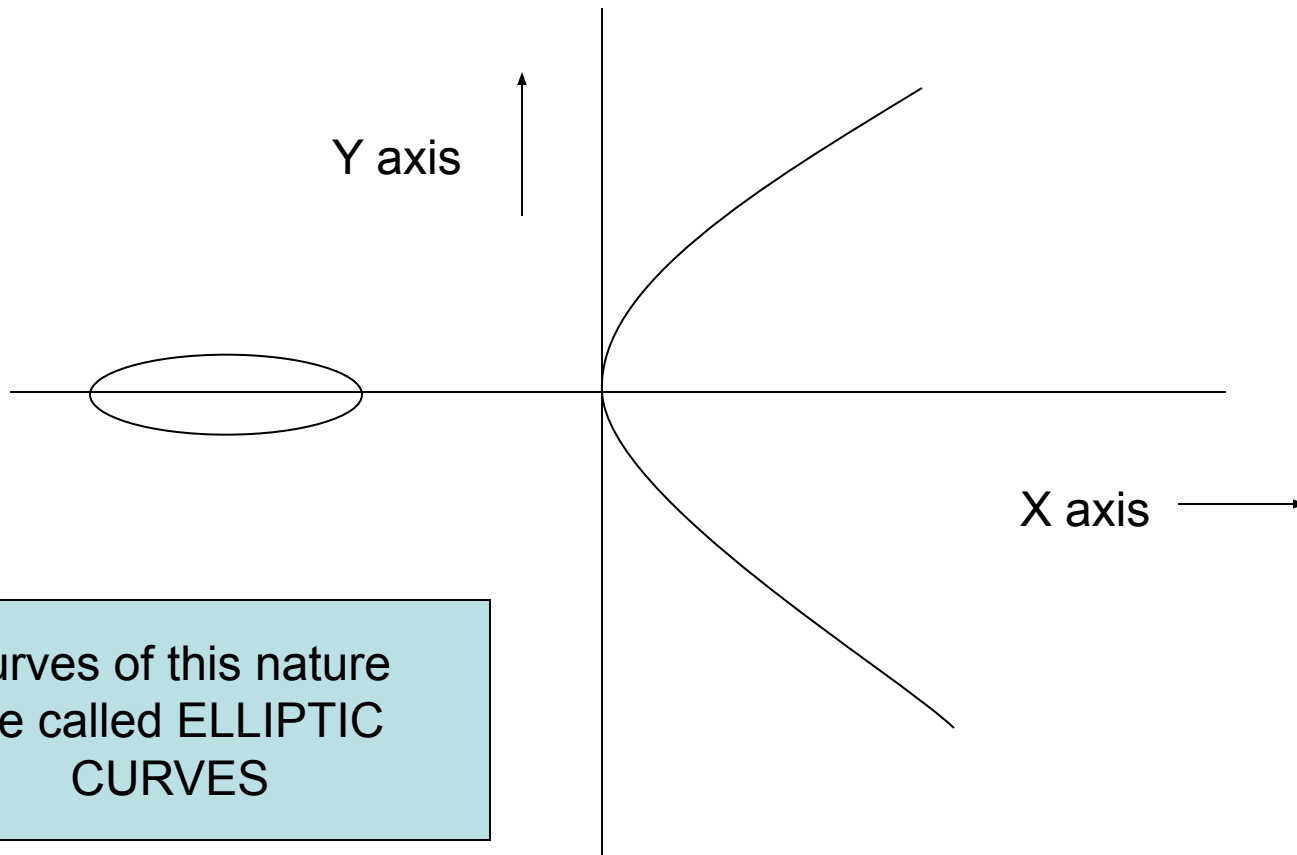
Credit:IIT Madras

Outline of the Talk…

- **Introduction to Elliptic Curves**

- Elliptic Curve Cryptosystems (ECC)

- Implementation of ECC in Binary Fields

Introduction to Elliptic Curves

Graphical Representation

Y axis

X axis

Curves of this nature are called ELLIPTIC CURVES

Elliptic curves in Cryptography

- Elliptic Curve (EC) systems as applied to cryptography were first proposed in 1985 independently by Neal Koblitz and Victor Miller.

- The discrete logarithm problem on elliptic curve groups is believed to be more difficult than the corresponding problem in (the multiplicative group of nonzero elements of) the underlying finite field.

Discrete Logarithms in Finite Fields
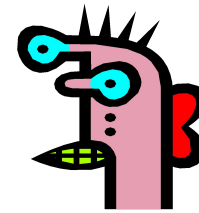
$F=\{1,2,3,\ldots,p-1\}$

Pick secret, random X from F
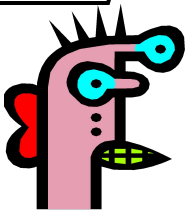
Pick secret, random Y from F

g^x mod p

g^y mod p

Ali ce

Bo b

Compute $k=(g^y)^x=g^{xy}$ mod p

Compute $k=(g^x)^y=g^{xy}$ mod p

Eve has to compute g^{xy} from g^x and g^y without knowing x and y…
She faces the Discrete Logarithm Problem in finite fields

Elliptic Curve on a finite set of Integers

- Consider $y^2 = x^3 + 2x + 3$ (**mod 5**)

  x = 0 $\Rightarrow$ y^2 = 3 $\Rightarrow$ no solution (mod 5)

  x = 1 $\Rightarrow$ y^2 = 6 = 1 $\Rightarrow$ y = 1,4 (mod 5)

  x = 2 $\Rightarrow$ y^2 = 15 = 0 $\Rightarrow$ y = 0 (mod 5)

  x = 3 $\Rightarrow$ y^2 = 36 = 1 $\Rightarrow$ y = 1,4 (mod 5)

  x = 4 $\Rightarrow$ y^2 = 75 = 0 $\Rightarrow$ y = 0 (mod 5)

- Then points on the elliptic curve are

  (1,1) (1,4) (2,0) (3,1) (3,4)
  (4,0) and the point at infinity: ∞

Using the finite fields we can form an Elliptic Curve Group where we also have a DLP problem which is harder to solve…

Definition of Elliptic curves

- An elliptic curve over a field **K** is a nonsingular cubic curve in two variables, *f(x,y) =0* with a rational point (which may be a point at infinity).

- The field **K** is usually taken to be the complex numbers, reals, rationals, algebraic extensions of rationals, p-adic numbers, or a **finite field**.

- Elliptic curves groups for cryptography are examined with the underlying fields of F_p (*where p>3 is a prime*) and F_2^m (*a binary representation with 2^m elements*).

General form of a EC

- An *elliptic curve* is a plane curve defined by an equation of the form

  Examples

Weierstrass Equation

- A two variable equation F(x,y)=0, forms a curve in the plane. We are seeking geometric arithmetic methods to find solutions

- Generalized Weierstrass Equation of elliptic curves:

$$y^2 + a_1 xy + a_3 y = x^2 + a_2 x^2 + a_4 x + a_6$$

Here, A, B, x and y all belong to a field of say rational numbers, complex numbers, finite fields (F_p) or Galois Fields ($GF(2^n)$).

- **If Characteristic field is not 2:**

$$(y + \frac{a_1 x}{2} + \frac{a_3}{2})^2 = x^3 + (a_2 + \frac{a_1^2}{4})x^2 + a_4 x + (\frac{a_3^2}{4} + a_6)$$

$$\Rightarrow y_1^2 = x^3 + a_2' x^2 + a_4' x + a_6'$$

- **If Characteristics of field is neither 2 nor 3:**

$$x_1 = x + a_2' / 3$$

$$\Rightarrow y_1^2 = x_1^3 + A x_1 + B$$

Points on the Elliptic Curve (EC)

- Elliptic Curve over field L

$$E(L) = \{\infty\} \cup \{(x, y) \in L \times L \mid y^2 + \ldots = x^3 + \ldots\}$$

- It is useful to add the point at infinity
- The point is sitting at the top of the y-axis and any line is said to pass through the point when it is vertical
- It is both the top and at the bottom of the y-axis

The Abelian Group

Given two points P,Q in *E(Fp)*, there is a third point, denoted by *P+Q* on *E(Fp)*, and the following relations hold for all P,Q,R in *E(Fp)*

- $P + Q = Q + P$ *(commutativity)*

- $(P + Q) + R = P + (Q + R)$ *(associativity)*

- $P + O = O + P = P$ *(existence of an identity element)*

- there exists $(-P)$ such that $-P + P = P + (-P) = O$ *(existence of inverses)*

Elliptic Curve Picture
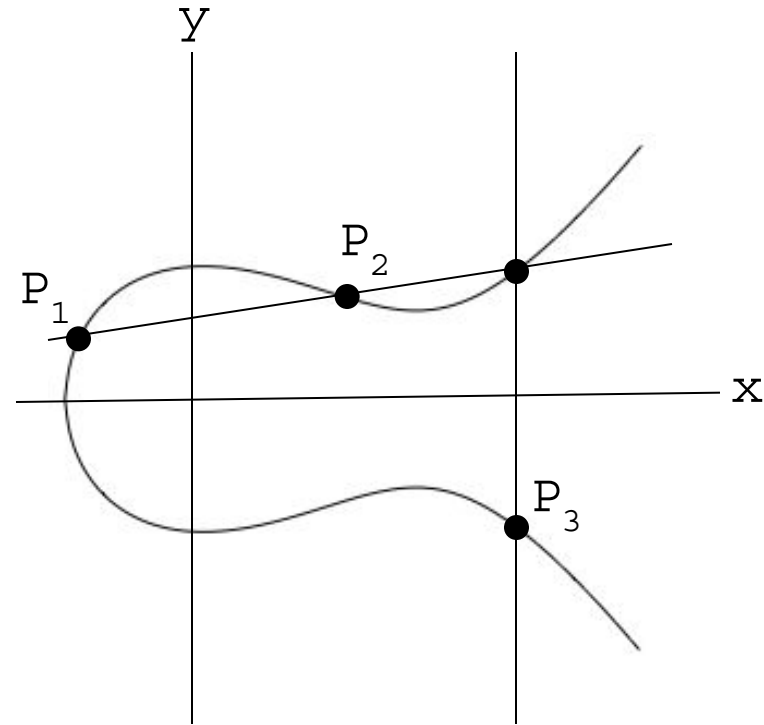
- Consider elliptic curve

$$E: y^2 = x^3 - x + 1$$

- If P_1 and P_2 are on E, we can define

$$P_3 = P_1 + P_2$$

as shown in picture

- Addition is all we need

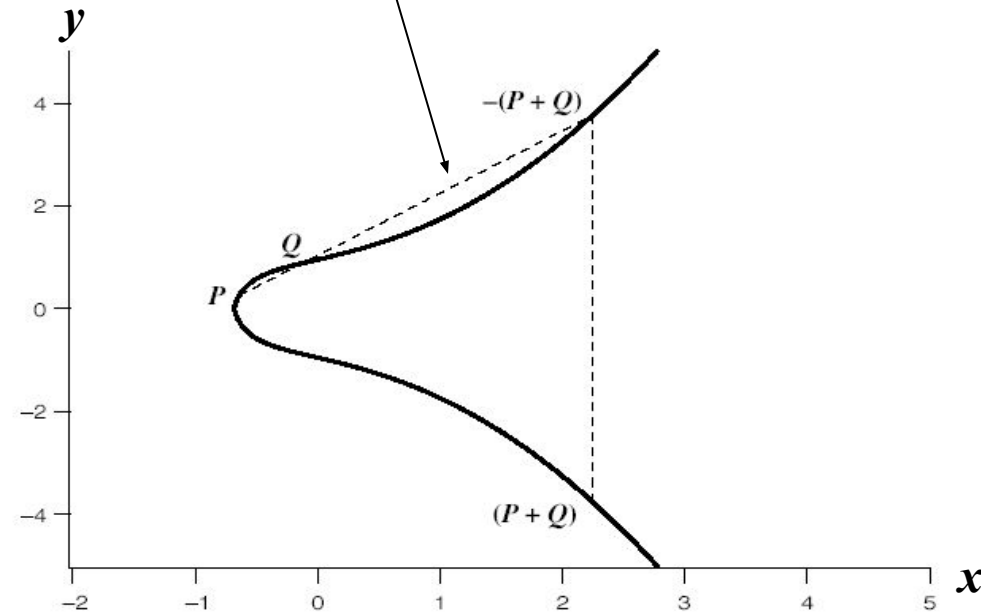# Addition in Affine Co-ordinates

$y=m(x-x_1)+y_1$

$$P = (x_1, y_1), Q = (x_2, y_2)$$
$$R = (P + Q) = (x_3, y_3)$$

Let, P≠Q,

$y^2=x^3+Ax+B$

Finding Third Point

- If P(Xp, Yp) and Q(Xq, Yq) be the point on the curve then R(Xr, Yr)=P+Q is given by

- $Xr= m^2-Xp-Xq$

- $Yr= m(Xp-Xr)-Yp$

- If P==Q then

- The value of m changes as

Doubling of a point

- Let, P=Q

$$2y\frac{dy}{dx} = 3x^2 + A$$

$$\Rightarrow m = \frac{dy}{dx} = \frac{3x_1^2 + A}{2y_1}$$

$$If, y_1 \neq 0 \text{ (since then } P_1 + P_2 = \infty):$$

$$\therefore 0 = x^3 - m^2 x^2 + \ldots$$

$$\Rightarrow x_3 = m^2 - 2x_1, y_3 = m(x_1 - x_3) - y_1$$

- What happens when $P_2 = \infty$?

Examples

- For E $_{11}$(1,6) consider the point G = (2,7). Compute the value of 2G and 3G.

- Solution:

- Given that E $_{11}$(1,6) = E $_P$(a,b)

- 2G=G+G

- m=($3x^2$+a)/2y=(3*4+1)/(2*7) mod 11
    =13/14 mod 11 =13*4 mod 11= 8

- Xr= m^2-Xp-Xq mod P= (64-2-2) %11=5

- Yr= m(Xp-Xr)-Yp mod P=(8(2-5)-7)%11=2

Multiplicative inverse

- 14=11*1+3
- 11=3*3+2
- 3=2*1+1
- 1=3-2*1=3-(11*1-3*3)*1=3*1-11*1+3*3
- =3*4-11*1
- =(14*1-11*1)*4-11*1
- =14*4-11*5

- 3G=2G+G
- 2G=(5,2), G=(2, 7)
- m=(y2-y1/x2-x1) mod P=(7-2)/(2-5) %11
-     = 5/(-3) %11=(-5*4) %11=-20%11=2
- Xr= m^2-Xp-Xq mod P=(4-5-2)%11= 8
- Yr= m(Xp-Xr)-Yp mod P=(2(5-8)-2)%11=-8 mod 11=3
- 3G=(8,3)
- Similarly we can find 4G, 5G, …91G, 92G….211G

- **let's take the EC point G = {15, 13} on the elliptic curve over finite field y2 ≡ x3 + 7 (mod 17) and multiply it by k = 6**.
- Given data: G=(15, 13), P=17, a=0, b=7
- KG=6G
- 2G=G+G=(2,10)
- m= (3x^2+a)/2y%P=(3*15^2+0)/(2*13) %17
- m= 675/26 %17 675%17=12, 1/26 %17=2
- m= 12*2%17=7
- Xr=m^2-2X %P=(49-30) %17=2
- Yr=m(x1-xr)-y %P=(7(15-2)-13)%17

=78%17=10

Multiplicative inverse of 26 wrt 17

- 26=17*1+9
- 17=9*1+8
- 9=8*1+1
- 1=9*1-8*1=(9*1-(17*1-9*1)*1)=9*2-17*1
- =(26*1-17*1)*2-17*1
- =26*2-17*2-17*1

- **4G**=2G+2G, 2G=(2,10)
- M=(3*4+0/20)%17=12*6%17=4
- Xr=(16-4)%17=12
- Yr=4(2-12)-10 %17=-50%17=1
- **6G**=4G+2G, 4G=(12,1)
- M=1-10/12-2 %P=-9/10 %17=9*5%17=11
- Xr=(121-2-12)%17=5
- Yr=11(2-5)-10 %17=8
- 6G=(5, 8)

Sum of two points

Define for two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the Elliptic curve

Then $P+Q$ is given by $R(x_3, y_3)$:
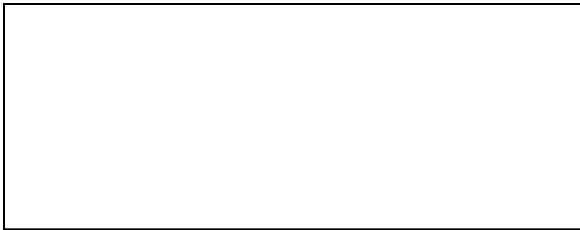
Point at infinity   *O*

P+P = 2P

As a result of the above case *P=O+P*

O **is called the additive identity of the elliptic curve group.**

Hence all elliptic curves have an additive identity *O*.

Projective Co-ordinates

- Two-dimensional projective space P_K^2 over *K* is given by the equivalence classes of triples (x,y,z) with x,y z in K and at least one of x, y, z nonzero.

- Two triples (x_1,y_1,z_1) and (x_2,y_2,z_2) are said to be equivalent if there exists a non-zero element λ in K, st:
  - $(x_1,y_1,z_1) = (λx_2, λy_2, λz_2)$
  - The equivalence class depends only the ratios and hence is denoted by (x:y:z)

Projective Co-ordinates

- If z≠0, (x:y:z)=(x/z:y/z:1)
- What is z=0? We obtain the point at infinity.
- The two dimensional affine plane over K:

$$A_K^2 = \{(x, y) \in K \times K\}$$

Hence using,

$$(x, y) \rightarrow (X : Y : 1)$$

$$\Rightarrow A_K^2 = P_K^2$$

There are advantages with projective co-ordinates
from the implementation point of view

Singularity

- For an elliptic curve $y^2 = f(x)$, define $F(x,y) = y^2 - F(x)$. A singularity of the EC is a pt (x_0, y_0) such that:

$$\frac{\partial F}{\partial x}(x_0, y_0) = \frac{\partial F}{\partial y}(x_0, y_0) = 0$$

$$or, 2y_0 = -f'(x_0) = 0$$

$$or, f(x_0) = f'(x_0)$$

$$\therefore \ f \text{ has a double root}$$

It is usual to assume the EC has no singular points

If Characteristics of field is not 3:

$$y^2 = f(x) = x^3 + Ax + B$$

1. **Hence condition for no singularity is 4A³+27B²≠0**
2. **Generally, EC curves have no singularity**

$$\frac{\partial F}{\partial x}(x_0, y_0) = \frac{\partial F}{\partial y}(x_0, y_0) = 0$$

$$or, 2y_0 = -f'(x_0) = 0$$

$$or, f(x_0) = f'(x_0)$$

$\therefore$ f has a double root

$$y^2 = x^3 + Ax + B$$

For double roots,

$$x^3 + Ax + B = 3x^2 + A = 0$$

$$\Rightarrow x^2 = -A/3.$$

Also, $x^4 + Ax^2 + Bx = 0$,

$$\Rightarrow \frac{A^2}{9} - \frac{A^2}{3} + Bx = 0$$

$$\Rightarrow x = \frac{2A^2}{9B}$$

$$\Rightarrow 3(\frac{2A^2}{9B})^2 + A = 0$$

$$\Rightarrow 4A^3 + 27B^2 = 0$$

Elliptic Curves in Characteristic 2

- Generalized Equation:

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

- If a_1 is not 0, this reduces to the form:

$$y^2 + xy = x^3 + Ax^2 + B$$

- If a_1 is 0, the reduced form is:

$$y^2 + Ay = x^3 + Bx + C$$

- Note that the form cannot be:

$$y^2 = x^3 + Ax + B$$

Outline of the Talk…

- **Introduction to Elliptic Curves**

- **Elliptic Curve Cryptosystems**

- **Implementation of ECC in Binary Fields**

Elliptic Curve Cryptosystems (ECC)

Public-Key Cryptosystems

Public-Key Cryptosystem: Secrecy and Authentication
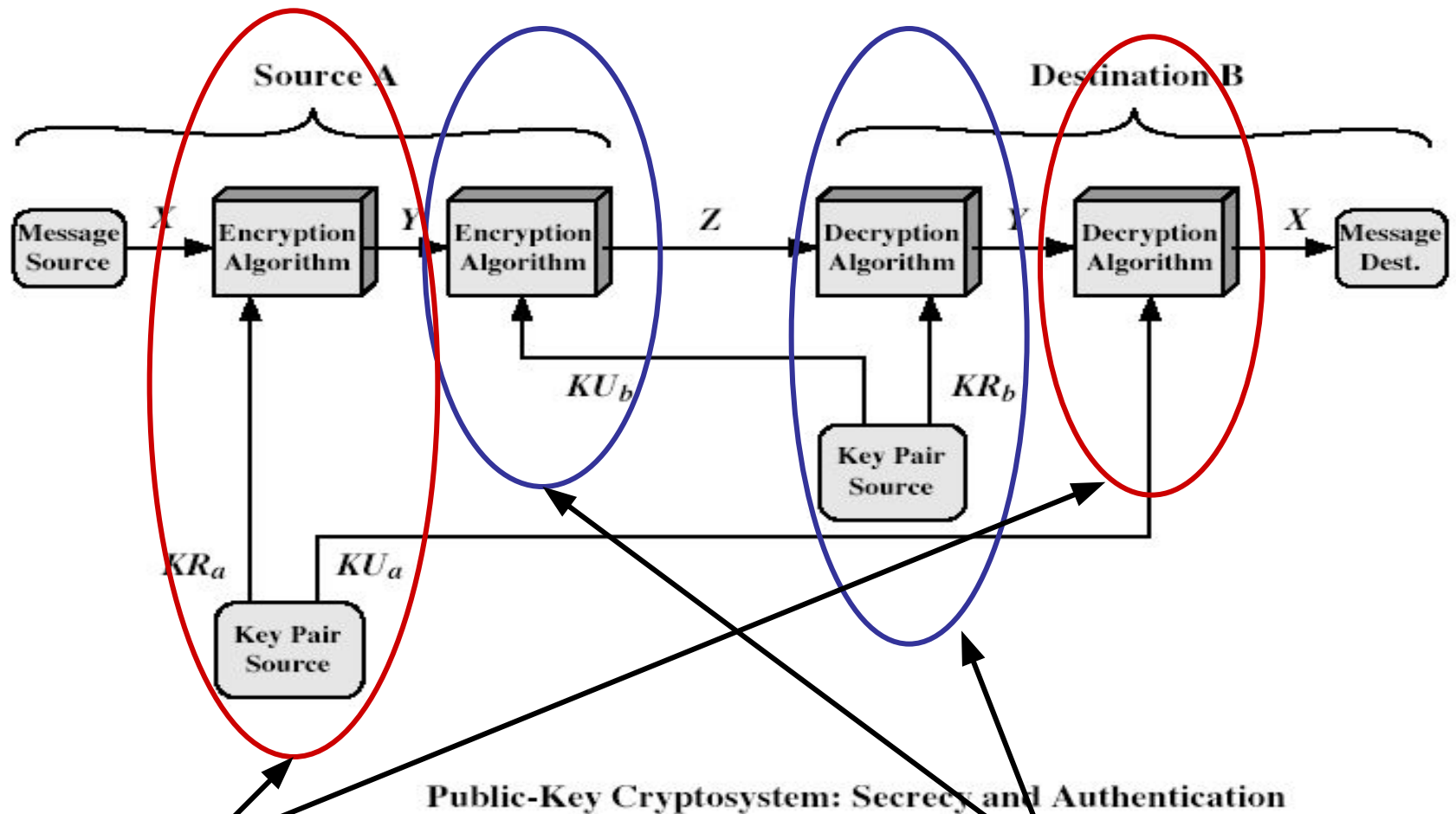
Authentication: Only **A** can generate the encrypted message

Secrecy: Only **B** can Decrypt the message

Public-Key Cryptography
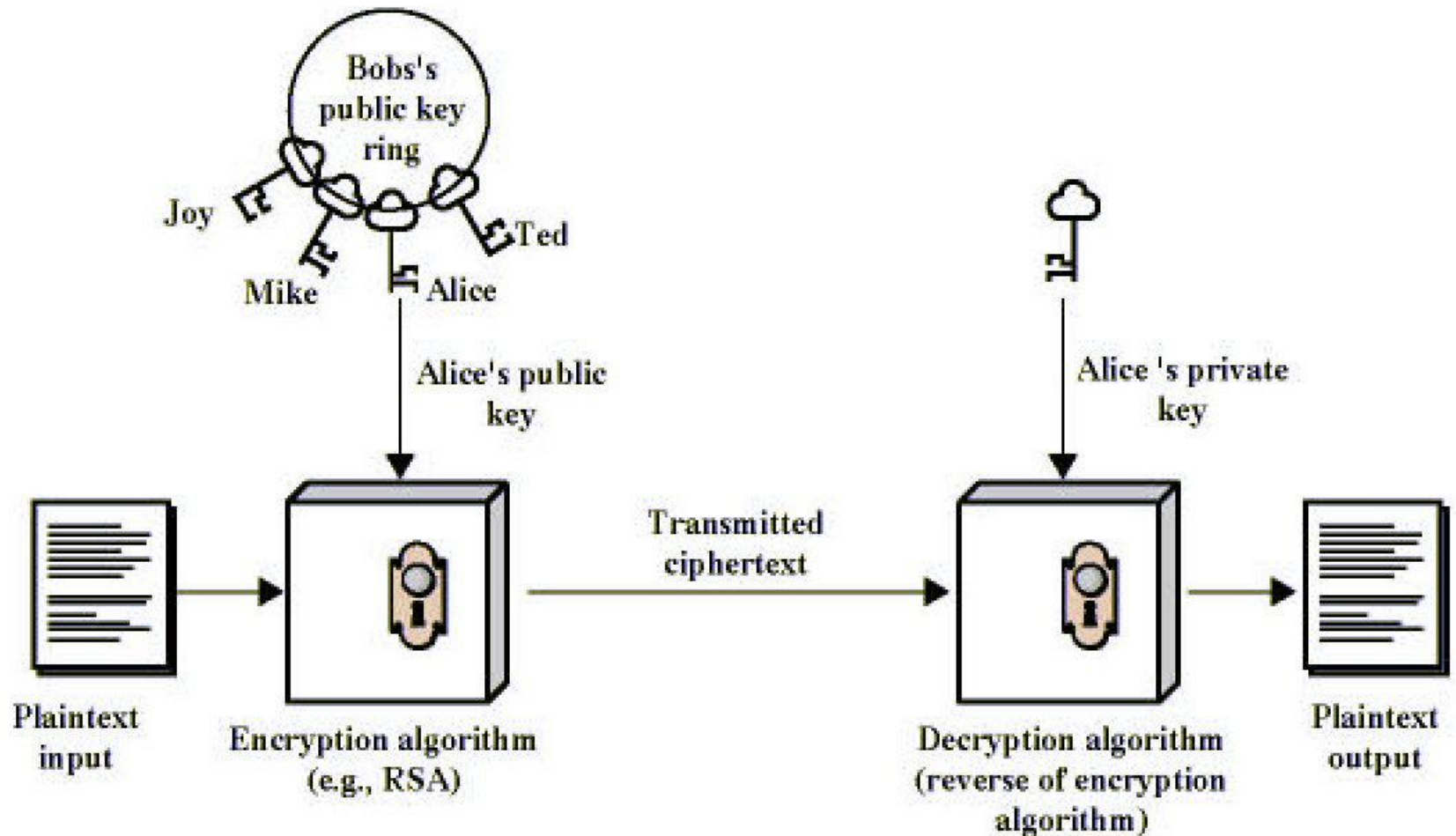
(a) Encryption

Public-Key Cryptography
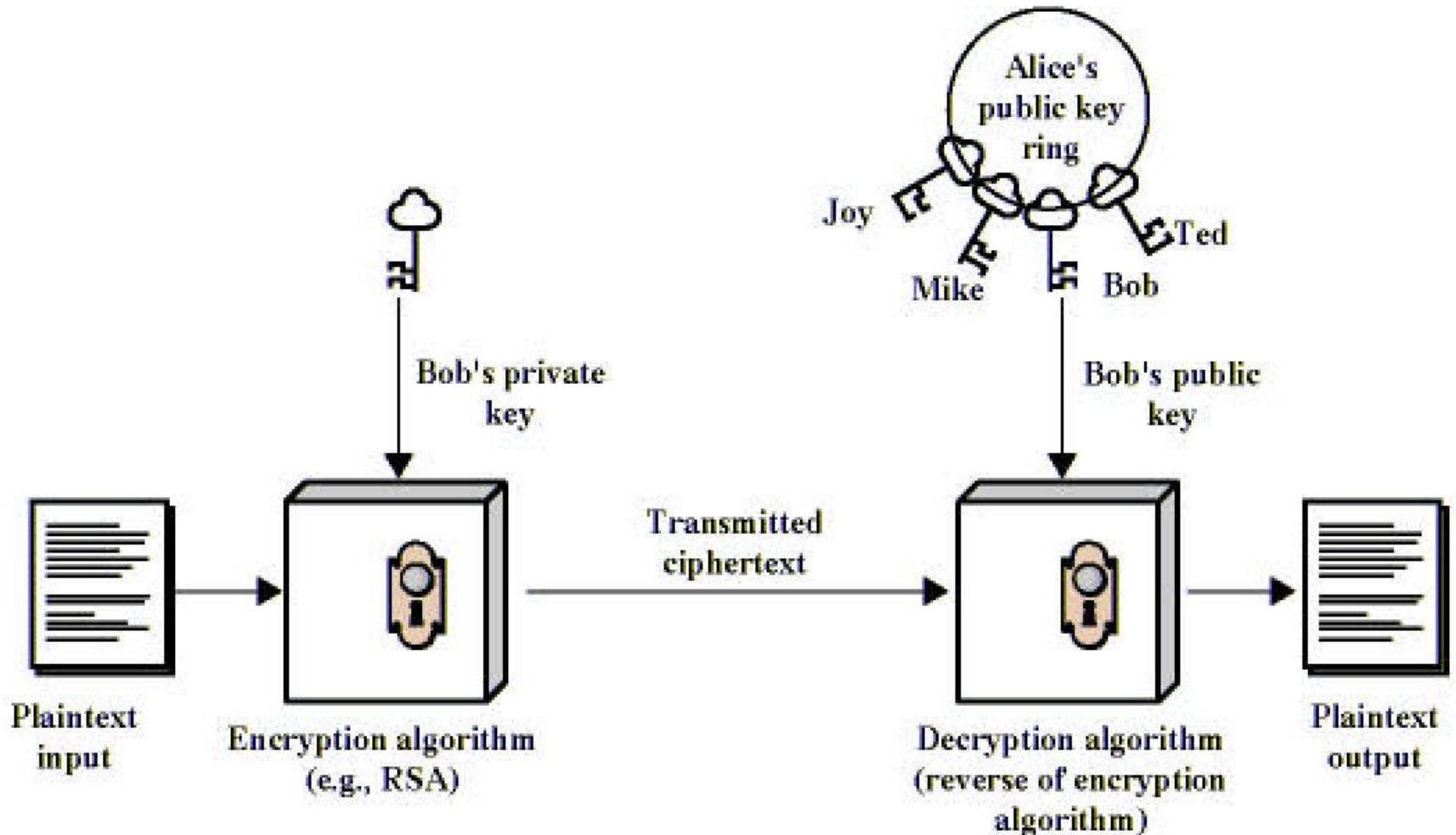
(b) Authentication

What Is Elliptic Curve Cryptography (ECC)?

- Elliptic curve cryptography [ECC] is a **public-key** cryptosystem just like RSA, Rabin, and El Gamal.

- Every user has a **public** and a **private** key.
  - Public key is used for encryption/signature verification.
  - Private key is used for decryption/signature generation.

- Elliptic curves are used as an extension to other current cryptosystems.
  - Elliptic Curve Diffie-Hellman Key Exchange
  - Elliptic Curve Digital Signature Algorithm

Using Elliptic Curves In Cryptography

- The central part of any cryptosystem involving elliptic curves is the **elliptic group**.

- All public-key cryptosystems have some underlying mathematical operation.

  - RSA has exponentiation (raising the message or ciphertext to the public or private values)

  - ECC has point multiplication (repeated addition of two points).

Generic Procedures of ECC

- Both parties agree to some publicly-known data items
  - The **elliptic curve equation**
    - values of *a* and *b*
    - prime, *p*
  - The **elliptic group** computed from the elliptic curve equation
  - A **base point**, B, taken from the elliptic group
    - Similar to the generator used in current cryptosystems
- Each user generates their public/private key pair
  - Private Key = an integer, x, selected from the interval [1, p-1]
  - Public Key = product, Q, of private key and base point
    - (Q = x*B)

Example – Elliptic Curve Cryptosystem Analog to El Gamal

- Suppose Alice wants to send to Bob an encrypted message.
    - Both agree on a base point, B.
    - Alice and Bob create public/private keys.
        - Alice
            - Private Key = a
            - Public Key = P_A = a * B
        - Bob
            - Private Key = b
            - Public Key = P_B = b * B
    - Alice takes plaintext message, M, and encodes it onto a point, P_M, from the elliptic group

Example – Elliptic Curve Cryptosystem Analog to El Gamal

– Alice chooses another random integer, k from the interval [1, p-1]

– The ciphertext is a pair of points
  - $P_C = [(kB), (P_M + kP_B)]$

– To decrypt, Bob computes the product of the first point from P_C and his private key, b
  - $b * (kB)$

– Bob then takes this product and subtracts it from the second point from P_C
  - $(P_M + kP_B) - [b(kB)] = P_M + k(bB) - b(kB) = P_M$

– Bob then decodes P_M to get the message, M.

Example – Compare to El Gamal

- – The ciphertext is a pair of points
  - $P_C = [\,(kB),\,(P_M + kP_B)\,]$
- – The ciphertext in El Gamal is also a pair.
  - $C = (g^k \bmod p,\ mP_B^{\,k} \bmod p)$

- – Bob then takes this product and subtracts it from the second point from P_C
  - $(P_M + kP_B) - [b(kB)] = P_M + k(bB) - b(kB) = P_M$
- – In El Gamal, Bob takes the quotient of the second value and the first value raised to Bob's private value
  - $m = mP_B^{\,k} / (g^k)^b = mg^{k*b} / g^{k*b} = m$

Diffie-Hellman (DH) Key Exchange

User A

Generate
random $X_A < q$;
Calculate
$Y_A = \alpha^{X_A} \bmod q$

Y_A

Y_B

Calculate
$K = (Y_B)^{X_A} \bmod q$

User B

Generate
random $X_B < q$;
Calculate
$Y_B = \alpha^{X_B} \bmod q$;
Calculate
$K = (Y_A)^{X_B} \bmod q$
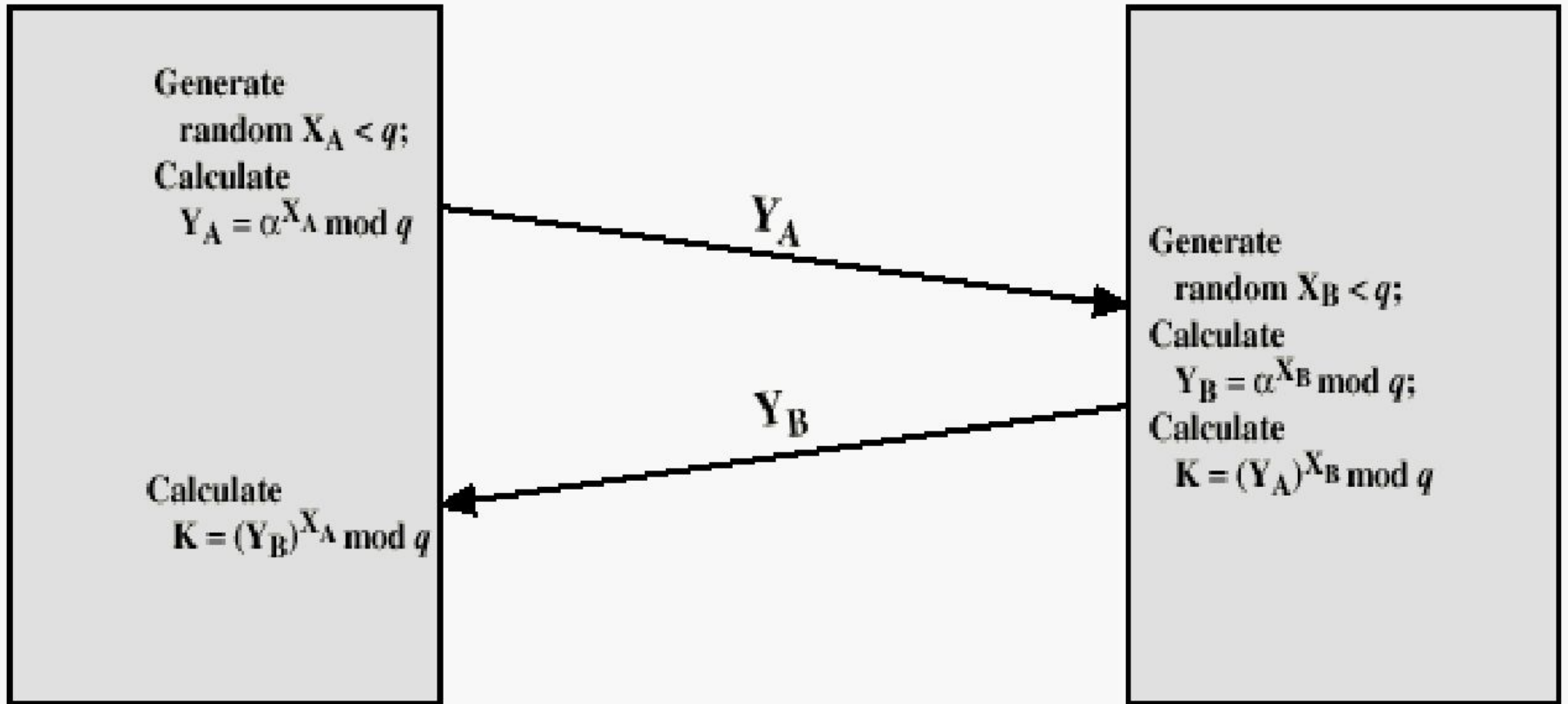
ECC Diffie-Hellman

- **Public:** Elliptic curve and point $B=(x,y)$ on curve
- **Secret:** Alice's a and Bob's b

$$a(x,y) \longrightarrow$$

$$b(x,y) \longleftarrow$$

Alice, A Bob, B
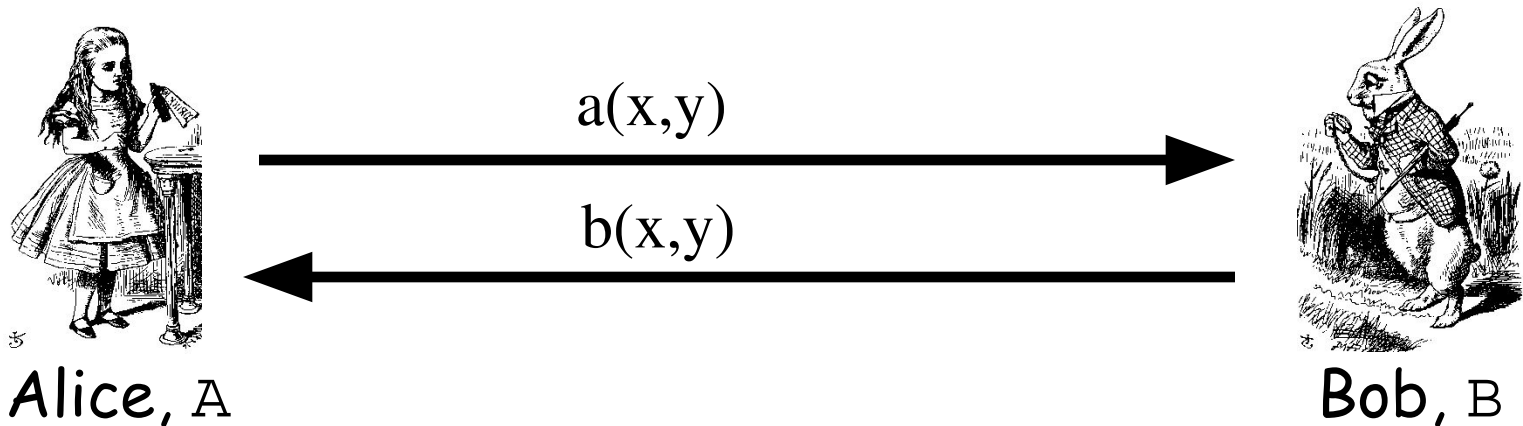
- Alice computes $a(b(x,y))$
- Bob computes $b(a(x,y))$
- These are the same since $ab = ba$

Example – Elliptic Curve Diffie-Hellman Exchange

- Alice and Bob want to agree on a shared key.
  - Alice and Bob compute their public and private keys.
    - Alice
      - » Private Key = a
      - » Public Key = P_A = a * B
    - Bob
      - » Private Key = b
      - » Public Key = P_B = b * B
  - Alice and Bob send each other their public keys.
  - Both take the product of their private key and the other user's public key.
    - Alice ☐ K_{AB} = a(bB)
    - Bob ☐ K_{AB} = b(aB)
    - **Shared Secret Key = K_{AB} = abB**

Why use ECC?

- How do we analyze Cryptosystems?
  - How difficult is the underlying problem that it is based upon
    - RSA – Integer Factorization
    - DH – Discrete Logarithms
    - ECC - Elliptic Curve Discrete Logarithm problem
  - How do we measure difficulty?
    - We examine the algorithms used to solve these problems

Security of ECC

- To **protect** a 128 bit AES key it would take a:
  - RSA Key Size: 3072 bits
  - ECC Key Size: 256 bits
- How do we strengthen RSA?
  - Increase the key length
- **Impractical?**

NIST guidelines for public key sizes for AES

ECC KEY SIZE (Bits)	RSA KEY SIZE (Bits)	KEY SIZE RATIO	AES KEY SIZE (Bits)
163	1024	1:6	
256	3072	1:12	128
384	7680	1:20	192
512	15 360	1:30	256

Supplied by NIST to ANSI X9F1

Applications of ECC

- Many devices are small and have limited storage and computational power
- Where can we apply ECC?
  - **Wireless communication devices**
  - Smart cards
  - Web servers that need to handle many encryption sessions
  - Any application where security is needed but lacks the power, storage and computational power that is necessary for our current cryptosystems

Benefits of ECC

- Same benefits of the other cryptosystems: confidentiality, integrity, authentication and non-repudiation but…

- Shorter key lengths
  – Encryption, Decryption and Signature Verification speed up
  – Storage and bandwidth savings

Summary of ECC

- "**Hard problem**" analogous to discrete log
  - $Q=kP$, where Q,P belong to a prime curve
    - given k,P ☐ "easy" to compute Q
    - given Q,P ☐ "hard" to find k
  - known as the elliptic curve logarithm problem
    - k must be large enough

- ECC security relies on elliptic curve logarithm problem
  - compared to factoring, can use much smaller key sizes than with RSA etc
    - ☐ for similar security ECC offers significant computational advantages

Outline of the Talk…

- Introduction to Elliptic Curves

- Elliptic Curve Cryptosystems

- **Implementation of ECC in Binary Fields**

Implementation of ECC in Binary Fields

Sub-Topics

1. Scalar Multiplication: LSB first vs MSB first
2. Montgomery Technique of Scalar Multiplication
3. Fast Scalar Multiplication without pre-computation.
4. Lopez and Dahab Projective Transformation to Reduce Inverters
5. Mixed Coordinates
6. Parallelization Techniques
7. Half and Add Technique for Scalar Multiplication

ECC operations: Hierarchy

ECC

Point multiplication: kP

Group operation: point add/double

Finite field arithmetic: multiplication, addition, subtraction, inversion, …

Parallelize the architectures

Level 0

Level 1

Level 2

Level 3
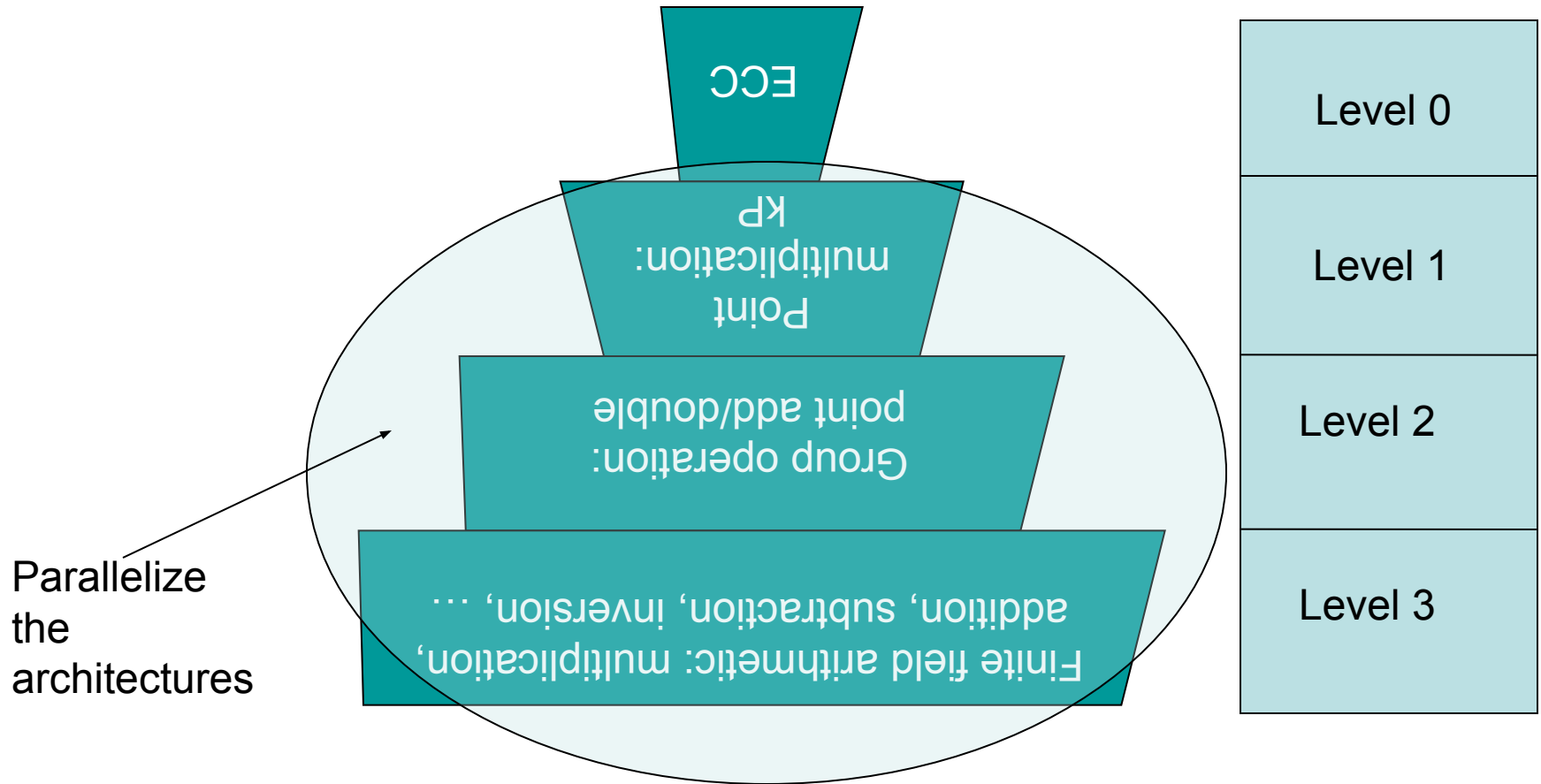
Scalar Multiplication: MSB first

- Require $k=(k_{m-1}, k_{m-2}, \ldots, k_0)_2$, $k_m=1$
- Compute $Q=kP$
  - $Q=P$
  - For $i=m-2$ to 0
    - $Q=2Q$
    - If $k_i=1$ then
      - $Q=Q+P$
    - End if
  - End for
  - Return Q

Sequential Algorithm

Requires m point doublings and $(m-1)/2$ point additions on the average

Example

- **Compute 7P:**
  - $7 = (111)_2$
  - $7P = 2(2(P)+P)+P \Rightarrow$ 2 iterations are required
  - Principle: First double and then add (accumulate)
- **Compute 6P:**
  - $6 = (110)_2$
  - $6P = 2(2(P)+P)$

Scalar Multiplication: LSB first

- Require $k=(k_{m-1}, k_{m-2}, \ldots, k_0)_2$, $k_m=1$
- Compute $Q=kP$
  - $Q=0$, $R=P$
  - For $i=0$ to $m-1$
    - If $k_i=1$ then
      - $Q=Q+R$
    - End if
    - $R=2R$
  - End for
  - Return Q

Can **Parallelize**…

What you are doubling and what you are accumulating are different…

On the average $m/2$ point Additions and $m/2$ point doublings

Example

- **Compute 7P**, $7 = (111)_2$, $Q=0$, $R=P$
  - $Q = Q + R = 0 + P = P$, $R = 2R = 2P$
  - $Q = P + 2P = 3P$, $R = 4P$
  - $Q = 7P$, $R = 8P$
- **Compute 6P**, $6 = (110)_2$, $Q=0$, $R=P$
  - $Q = 0$, $R = 2R = 2P$
  - $Q = 0 + 2P = 2P$, $R = 4P$
  - $Q = 2P + 4P = 6P$, $R = 8P$

Compute 31P…

$31 = (11111)_2$

MSB First	LSB First
1. Q=2P	1. Q=P, R=2P
2. Q=3P	2. Q=3P, R=4P
3. Q=6P	3. Q=7P, R=8P
4. Q=7P	4. Q=15P, R=16P
5. Q=14P	5. Q=31P, R=32P
6. Q=15P	
7. Q=30P	
8. Q=31P	

Weierstrass Point Addition

$$y^2 + xy = x^3 + ax^2 + b, \ (x, y) \in GF(2^m) \times GF(2^m)$$

- Let, P=(x_1, y_1) be a point on the curve.
- -P=$(x_1, x_1 + y_1)$
- Let, R=P+Q=(x_3, y_3)

$$x_3 = \begin{cases} \left(\dfrac{y_1 + y_2}{x_1 + x_2} \right)^2 + \dfrac{y_1 + y_2}{x_1 + x_2} + x_1 + x_2 + a; P \neq Q \\[2em] x_1^2 + \dfrac{b}{x_1^2}; P = Q \end{cases}$$

$$y_3 = \begin{cases} \left(\dfrac{y_1 + y_2}{x_1 + x_2} \right)(x_1 + x_3) + x_3 + y_1; P \neq Q \\[2em] x_1^2 + (x_1 + \dfrac{y_1}{x_1})x_3 + x_3; P = Q \end{cases}$$

1. Point addition and doubling each require 1 inversion & 2 multiplications
2. We neglect the costs of squaring and addition
3. *Montgomery noticed that the x-coordinate of 2P does not depend on the y-coordinate of P*

Montgomery's method to perform scalar multiplication

- Input: k>0, P
- Output: Q=kP

1. Set $k \leftarrow (k_{l-1}, \ldots, k_1, k_0)_2$
2. Set $P_1 = P$, $P_2 = 2P$
3. For i from *l-2* to 0

    If $k_i = 1$,

    Set $P_1 = P_1 + P_2$, $P_2 = 2P_2$

    else

    Set $P_2 = P_2 + P_1$, $P_1 = 2P_1$

4. Return $Q = P_1$

Invariant Property:
$P = P_2 - P_1$

Question: How to implement the Operation efficiently?

Example

Compute 7P

- $7=(111)_2$
- Initialization:

  $P_1=P; P_2=2P$

- Steps:
  - $P_1=3P, P_2=4P$
  - $\mathbf{P_1=7P}, P_2=8P$

Compute 6P

- $7=(110)_2$
- Initialization:

  $P_1=P; P_2=2P$

- Steps:
  - $P_1=3P, P_2=4P$
  - $P_2=7P, \mathbf{P_1=6P}$

Fast Multiplication on EC without pre-computation

Result-1

- Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be elliptic points. Then the x-coordinate of $P_1 + P_2$, x_3 can be computed as:

$$x_3 = \frac{x_1 y_2 + x_2 y_1 + x_1 x_2^2 + x_2 x_1^2}{(x_1 + x_2)^2}$$

Hint: Remember that the field has a characteristic 2 and that P_1 and P_2 are points on the curve

Result-2

- Let P=(x,y), P_1 = (x_1,y_1) and P_2=(x_2,y_2) be elliptic points. Let *P=P₂-P₁* be an invariant.

  Then **the x-coordinate of P_1+P_2, x_3 can be computed in terms of the x-coordinates** as:

$$x_3 = \begin{cases} x + \left(\dfrac{x_1}{x_1 + x_2}\right)^2 + \dfrac{x_1}{x_1 + x_2} ; P_1 \neq P_2 \\ \\ x_1^2 + \dfrac{b}{x_1^2} ; P_1 = P_2 \end{cases}$$

Result-3

Let P=(x,y), P_1=(x_1,y_1) and P_2=(x_2,y_2) be elliptic points. Assume that *P_2-P_1=P* and x is not 0. Then **the y-coordinates of P_1 can be expressed in terms of P, and the x-coordinates of P_1 and P_2** as follows:

$$y_1 = (x_1 + x)\{(x_1 + x)(x_2 + x) + x^2 + y\} / x + y$$

Final Algorithm

Input: k>0, P=(x,y)
Output: Q=kP

1. If k=0 or x=0 then output(0,0)
2. Set $k = (k_{l-1}, k_{l-2}, \ldots, k_0)_2$
3. Set $x_1 = x$, $x_2 = x^2 + b/x^2$
4. For i from *l-2* to 0
    1. Set $t = x_1/(x_1 + x_2)$
    2. If $k_i = 1$,
        $x_1 = x + t^2 + t$, $x_2 = x_2^2 + b/x_2^2$
       else
        $x_1 = x_1^2 + b/x_1^2$, $x_2 = x + t^2 + t$
5. $r_1 = x_1 + x$, $r_2 = x_2 + x$
6. $y_1 = r_1(r_1 r_2 + x^2 + y)/x + y$
7. Return $Q = (x_1, y_1)$

- #INV: 2(l-2)+1;
- #MULT: 2(l-2)+4
- #ADD: 4(l-2)+6
- #SQR: 2(l-2)+2

How to reduce inversions?

1. In affine coordinates Inverses are very expensive

2. For $n \geq 128$ each inversion requires around 7 multipliers (in hardware designs)

3. Lopez Dahab Projective coordinates:

    *   (X,Y,Z), Z≠0, maps to (X/Z,Y/Z^2)

    *   Motivation is to replace inversions by the multiplication operations and then perform one inversion at the end (to obtain back the affine coordinates)

Doubling

- **Remember:**

$$x_3 = \begin{cases} x + \left(\dfrac{x_1}{x_1 + x_2}\right)^2 + \dfrac{x_1}{x_1 + x_2} \, ; P_1 \neq P_2 \\ \\ x_1^2 + \dfrac{b}{x_1^2} \, ; P_1 = P_2 \end{cases}$$

- 2 inverses
- 1 general field multiplication
- 4 additions
- 2 squarings

- **In Projective Coordinates**:

$$P_1 = P_2, \ X_3 = X_1^4 + b.Z_1^4$$

$$Z_3 = Z_1^2.X_1^2$$

$$P_1 \neq P_2, \ Z_3 = (X_1.Z_2 + X_2.Z_1)^2$$

$$X_3 = x.Z_3 + (X_1.Z_2).(X_2.Z_1)$$

- 0 inverses
- 4 general field multiplications
- 3 additions
- 5 squarings

Montgomery Algorithm

- Input: k>0, P=(x,y)
- Output: Q=kP
- Set k<-$(k_{l-1},\ldots,k_1,k_0)_2$
- Set $X_1=x$, $Z_1=1$; $X_2=x^4+b$, $Z_2=x^2$
- For i from *l-2* to 0

  –   If $k_i =1$,

            Madd(X_1,Z_1,X_2,Z_2), Mdouble(X_2,Z_2)

          else

            Madd(X_2,Z_2,X_1,Z_1), Mdouble(X_1,Z_1)

- Return Q=(Mxy(X_1,Y_1,X_2,Y_2))

Mxy: Projective to Affine

$$x_3 = X_1 / Z_1$$

$$y_3 = (x + X_1 / Z_1)[(X_1 + xZ_1)(X_2 + xZ_2) + (x^2 + y)(Z_1 Z_2)](xZ_1 Z_2)^{-1} + y$$

Requires 10 multiplications and one inverse operation

Final Comparison

Affine Coordinates	*Projective Coordinates*
Inv: 2logk + 1 **Mult: 2logk + 4** Add: 4logk + 6 Sqr: 2logk + 2	**Inv: 1** **Mult: 6logk + 10** Add: 3logk + 7 Sqr: 5logk + 3

Hence, final decision depends upon the I:M ratio of the finite field operators

Addition in Mixed Coordinates

- **Theorem:** Let $P_1=(X_1/Z_1, Y_1/Z_1^2)$ and $P_2=(X_2/Z_2, Y_2/Z_2^2)$ be two points on the curve. If $Z_1=1$, then $P_1+P_2=(X_3/Z_3, Y_3/Z_3^2)$ st.

$$U = Z_2^2 Y_1 + Y_2, S = Z_2 X_1 + X_2, T = Z_2 S, Z_3 = T^2,$$

$$V = Z_3 X_1, X_3 = U^2 + T(U + S^2 + Ta),$$

$$Y_3 = (V + X_3)(TU + Z_3) + Z_3^2 C$$

Number of multiplications are further reduced.
Squaring is increased a bit, but they are cheap in GF(2^n)
Improvement by 10 % if a≠0, otherwise 12 %...

Parallel Strategies for Scalar Point Multiplication

- **Point Doubling**
  - Cycle 1: $T = X_1^2$, $M = cZ_1^2$, $Z_2 = T.Z_1^2$
  - Cycle 1a: $X_2 = T^2 + M^2$

  1 multiplier

- Point Addition
  - Cycle 1: $t_1 = (X_1.Z_2)$; $t_2 = (Z_1.X_2)$
  - Cycle 1a: $M = (t_1 + t_2)$, $Z_1 = M^2$
  - Cycle 2: $N = t_1.t_2$, $M = xZ_1$
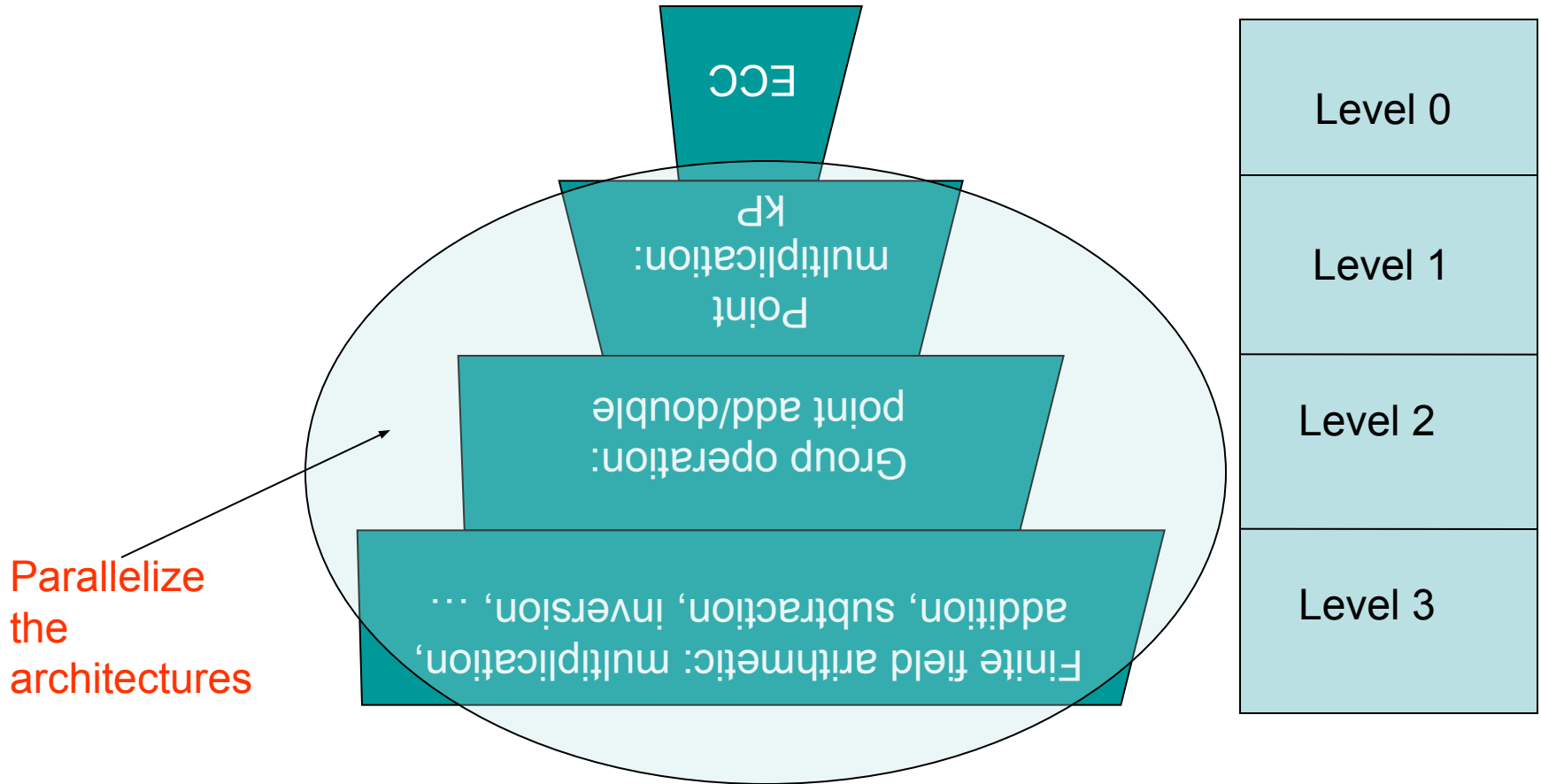 - Cycle 2a: $X_1 = M + N$

  2 multipliers

We assume that squarings and multiplications with constants can be performed without multipliers…

Parallelizing Montgomery Algorithm

1. Input: $k > 0$, $P = (x, y)$
2. Output: $Q = kP$
3. Set $k \leftarrow (k_{l-1}, \ldots, k_1, k_0)_2$
4. Set $X_1 = x$, $Z_1 = 1$; $X_2 = x^4 + b$, $Z_2 = x^2$
5. For i from *l-2* to 0

   If $k_i = 1$,

   5a) Madd(X_1, Z_1, X_2, Z_2), Mdouble(X_2, Z_2)

   else

   5b) Madd(X_2, Z_2, X_1, Z_1), Mdouble(X_1, Z_1)
6. Return $Q = (Mxy(X_1, Y_1, X_2, Y_2))$

Looking back at our Design Hierarchy

ECC

Point multiplication: kP

Group operation: point add/double

Finite field arithmetic: multiplication, addition, subtraction, inversion, …

Parallelize the architectures

Level 0

Level 1

Level 2

Level 3

Parallelizing Strategies

- **Parallelize level 1:** If we allocate one multiplier to each of Madd and Mdouble, then we can parallelize steps 5a and 5b. Thus 4 clock cycles are required for each iteration. Total time is nearly 4l.

- **Parallelize level 2:** If we can parallelize the underlying Madd and Mdouble, then we cannot parallelize level 1, if we have constraint of 2 multipliers. So, we have a sequential step 5a and 5b. Total time is 3l.

Parallelizing Strategies

- **Parallelize both the levels:** Total time is $2l$ clock cycles. Require 3 multipliers.

- Thus Montgomery algorithm is highly parallelizable

- Helpful in high performance designs (*low power, high thoughput etc*)

Point Halving

- In 1999 Scroeppel and Knudsen proposed further speed up

- Idea is to replace point doubling by halving

- Point Halving is three times as fast than doubling

- The scalar k, has to be expressed in the negative powers of 2

Computing the Half

- **Problem:** Let E be the Elliptic Curve, defined by the equation: $y^2 + xy = x^3 + ax^2 + b, b \neq 0$

- Let Q=(u,v)=2P

- Compute P=(x,y)

- Remember :

$$u = x^2 + \frac{b}{x^2}$$

$$v = x^2 + (x + \frac{y}{x})u + u$$

Halving (contd.)

$$Let, \lambda = x + \frac{y}{x}$$

$$\therefore v = x^2 + (\lambda + 1)u \Rightarrow x = \sqrt{v + (\lambda + 1)u}$$

$$Note : \lambda^2 + \lambda = u + a$$

| Square Root |
| Solving Quadratics |

- Thus, we have to solve the above equations

- λ-representation: (x, λ_x)

Trace of a point

- Define: $Tr(C) = C + C^2 + \ldots + C^{2^{m-1}}$

- Properties of Trace:
    - $Tr(c)=Tr(c^2)=Tr(c)^2$, $Tr(c)$ can be 0 or 1
    - $Tr(c+d)=Tr(c)+Tr(d)$
    - NIST Curves : $Tr(a)=1$
    - If x,y belongs to the Elliptic Curve, $Tr(x)=Tr(a)$

Computing λ

- The roots of $\lambda^2 + \lambda = u + a$ are $\lambda_1 = \lambda$ or $\lambda+1$

- **Theorem:**

  Let, $P = (x, y), Q = (u, v) \in G, st.\ Q = 2P$

  and denote $\lambda = x + y / x$. Let $\hat{\lambda}$ be a solution

  to $\lambda^2 + \lambda = u + a$ and $t = v + u\hat{\lambda}$. Suppose that

  $Tr(a) = 1$. Then $\hat{\lambda} = \lambda$ if and only if $Tr(t) = 0$.

Halving Algorithm

- Input: (u,v) , Output: (x,y)
1. Solve $\lambda^2 + \lambda = u + a$ for λ. Let the root be $\hat{\lambda}$
2. Compute $t = v + u\hat{\lambda}$
3. If Tr(t)=0, then $\lambda_P = \hat{\lambda}$, x=(t+u)$^{1/2}$
   else $\lambda_P = \hat{\lambda}$ +1,x=(t)$^{1/2}$
4. Return (x,λ_P)

Implementation of Trace

- **Trace :** $Tr(C) = Tr(\sum_{i=0}^{m-1} c_i x^i) = \sum_{i=0}^{m-1} c_i Tr(x^i)$

- Can be evaluated in O(1) time

- Example: GF(2^{163}), with reduction polynomial p(x)=x^{163}+x^7+x^6+x^3+1, Tr(x^i)=1, iff i=0 or 159.

- Thus, the implementation is only one xor gate to add the 0th and the 159th bits of the register storing C.

Solving a Quadratic over GF(2^m)

- Solve $x^2 + x = c + Tr(c)$, c is an element of GF(2^m)
- Define Half Trace:

$$H(C) = \sum_{i=0}^{(m-1)/2} C^{2^{2i}}$$

1. $H(C + D) = H(C) + H(D)$

2. $H(C)$ is a root for $x^2 + x = C + Tr(C)$, as

$$H(C) = H(C^2) + C + Tr(C)$$

H(C) gives a root for the quadratic equation. A simple method to find H(C) requires storage for m elements and m/2 field additions on an average

Obtaining Square Root

- Field squaring in binary field is linear
- Hence squaring can be rephrased as:
  - $C=MA=A^2$
- We require to compute D st. $D^2=A$
- Let, $D=M^{-1}A$ => A=MD
- $D^2=MD$ (as M is the squaring matrix)
    $=M(M^{-1}A)=A$
- Hence, $D=(A)^{1/2}$

An Example

Compute: $763R_7$, where order of $R_7 = 1013$

$\Rightarrow m = 10$

$2^{10-1}(763) = 651(\mathrm{mod}\ 1013) = (1010001011)_2$

$\therefore 763 = (\dfrac{1}{2^9} + \dfrac{1}{2^8} + \dfrac{1}{2^6} + \dfrac{1}{2^2} + 1)\,\mathrm{mod}(1013)$

$\therefore 763R_7$ may be computed using the following steps:

Step 1: $\dfrac{1}{2}R_7 + R_7$

Step 2: $\dfrac{1}{2}(\dfrac{1}{2}R_7 + R_7) + R_7$

Step 3: Similarly continue...

Half and Add Algorithm

1.   Input: 0<k<n, P=(x,y)
2.   Output: Q=kP
3.   Compute: $t=\lfloor \log_2 n \rfloor +1$, $k_1=(2^{t-1}k)\bmod n$
4.   Q=O
5.   for i=0 to m-1 do
   1.   Q=[1/2]Q
   2.   If, k_1^i=1, then Q=Q+P
6.   return Q

No method is currently known to perform point halving in projective Coordinates. Keep Q in affine coordinates and P in Projective Coordinates. Then step 5.2 is a mixed operation, giving further efficiency.

Key References

- **Papers:**
  - *J. Lopez and R. Dahab, "Fast Multiplication on Elliptic Curves over GF(2^m) without pre-computation", CHES 1999*
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  - *Guide to Elliptic Curve Cryptography, Alfred J. Menezes*
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Thank You