CS446 / ECE 449: Machine Learning, Fall 2020, Homework 2

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Worked individually

Problem (1):

Solution:

Problem (1.1):
We will show that the Gram Matrix is positive semidefinite using the definintion. We will show that, for any $v \in \mathbf{R}^d$, we have that $v^T K v \geq 0$.

$$v^T K v = \sum_i \sum_j v_i K_{i,j} v_i = \sum_i \sum_j v_i k(x^{(i)}, x^{(j)}) v_j$$

Then, since $k(x^{(i)}, x^{(j)})$ defines inner product, the linearity of inner products implies that,

$$\sum_i \sum_j v_i k(x^{(i)}, x^{(j)}) v_j = \sum_j k(\sum_i v_i x^{(i)}, x^{(j)}) v_j = k(\sum_i v_i x^{(i)}, \sum_j v_j x^{(j)})$$

But, $\sum_i v_i x^{(i)} = \sum_j v_j x^{(j)}$. Thus, we have that,

$$v^T K v = k(\sum_i v_i x^{(i)}, \sum_j v_j x^{(j)}) = k(z, z)$$

We have that $k(z, z) \geq 0$ for all z. Therefore we obtain that,

$$v^T K v = k(\sum_i v_i x^{(i)}, \sum_j v_j x^{(j)}) = k(z, z) \geq 0$$

K is positive semidefinite. $\square$

Problem (1.2.a):
Let K_1, K_2 be the Gram Matrices for k_1, k_2 respectively. Then by (Problem 1.1), both K_1, K_2 are positive semidefinite. Now, let $k(x, y) = ck_1(x, y)$. Thus, we have that for any $v \in \mathbf{R}^d$,

$$v^T K v = v^T (cK_1) v = cv^T (K_1) v$$

Since $c > 0$ and K_1 is positive definite, we get that,

$$v^T K v = cv^T (K_1) v \geq 0$$

Problem (1.2.b):
Now, let $k(x,y) = k_1(x,y) + k_2(x,y)$. Thus, we have that for any $v \in \mathbf{R}^d$,

$$v^T K v = v^T (K_1 + K_2) v = v^T K_1 v + v^T K_2 v \geq 0$$

Since both K_1, K_2 are positive definite.

Problem (1.2.c):

Let a, b, c, d be $\phi_1(x), \phi_1(y), \phi_2(x), \phi_2(y)$ respectively, and let $k(x,y) = k_1(x,y)k_2(x,y)$. Then,

$$k(x,y) = k_1(x,y)k_2(x,y) = \phi_1(x)^T \phi_1(y) \phi_2(x)^T \phi_2(y) = \sum_{i,j} a_i b_i c_j d_j = \sum_j d_j (c_j \sum_i a_i b_i)$$

Thus, one can easily observe that,

$$k(x,y) = k_1(x,y)k_2(x,y) =<< \phi_1(x), \phi_1(y) > \phi_2(x), \phi_2(y) >$$
$$\implies =<< \phi_1(x), \phi_1(y) >^{1/2} \phi_2(x), < \phi_1(x), \phi_1(y) >^{1/2} \phi_2(y) >$$

Thus, we have shown that k can be written as inner product. $\square$

Problem (1.2.d):
We know that the Taylor series of e^x is,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

Thus,

$$e^{k_1(x,y)} = 1 + k_1(x,y) + \frac{k_1(x,y)^2}{2!} + \frac{k_1(x,y)^3}{3!} + ...$$

That's clearly is a combination of kernel addition, multiplication and scalar multiplication. Thus, by Problem (a), (b) and (c), we have that k is a valid kernel.

Problem (2.1): $k_1(x,x') = 0$
Problem (2.2): $k_1(x,x') = 1$

Problem (2.3):

Let d be length of unit grid. One can have that,

$$k_s(x,x') = \sum_{z_1} \sum_{z_2} 1[z_1 \in [x_1 - s, x_1 + s] \cap z_2 \in [x_2 - s, x_2 + s]] \;\; 1[z_2 \in [x'_1 - s, x'_1 + s] \cap z_2 \in [x'_2 - s, x'_2 + s]] d$$

Thus, as d decreases we have that,

$$k_s(x,x') = \int\int_z 1[z_1 \in [x_1 - s, x_1 + s] \cap z_2 \in [x_2 - s, x_2 + s]] \;\; 1[z_1 \in [x'_1 - s, x'_1 + s] \cap z_2 \in [x'_2 - s, x'_2 + s]]\, dz$$

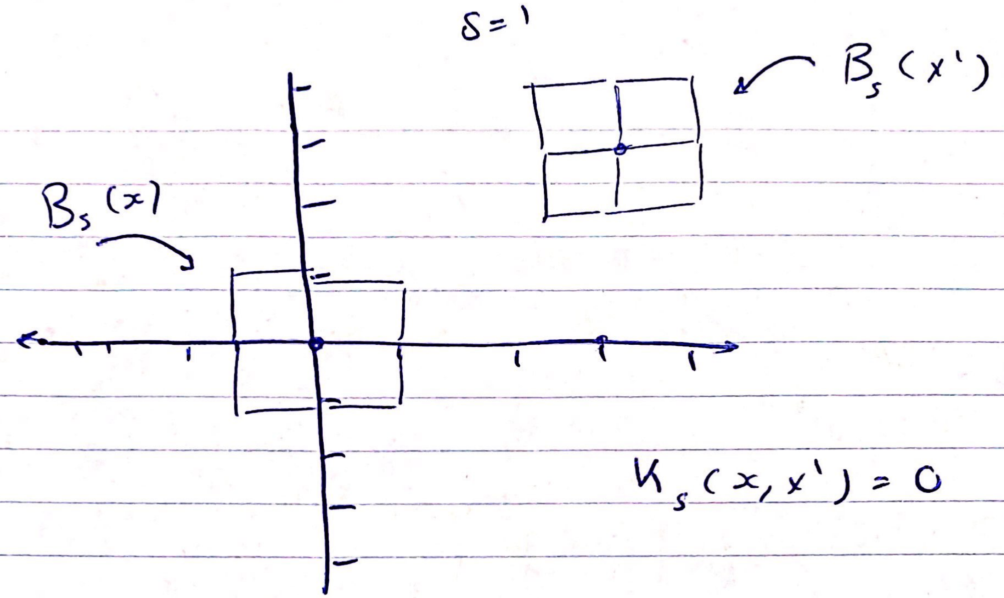
Problem (2.4.a): The feature map describe the length of the intersection in the one dimensional space.

2.1

(a)

(b)

(c)



2.2

(a)

(b)

(c)