CS446 / ECE 449: Machine Learning, Fall 2020, Homework 3

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Worked individually

Problem (3):

Problem 3.1:

We have, according to the definition of $\mathcal{F}$, that,

$$\max_{f\in\mathcal{F}} \frac{1}{n}\sum_{i=1}^{n}\epsilon_i f(x_i) = \max_{w} \frac{1}{n}\sum_{i=1}^{n}\epsilon_i w^T x_i = \max_{w} \frac{w^T}{n}\sum_{i=1}^{n}\epsilon_i x_i = \max_{w} w^T x_\epsilon$$

Now, by Cauchy-Schwarz inequality and the fact the $||w|| \leq W$, we have that,

$$\max_{w} w^T x_\epsilon \leq \max_{w} ||w||||x_\epsilon|| = W||x_\epsilon||$$

Thus,

$$\max_{f\in\mathcal{F}} \frac{1}{n}\sum_{i=1}^{n}\epsilon_i f(x_i) \leq W||x_\epsilon||$$

Problem 3.2:

One can easily notice that, since $||x_i|| \leq R$, we have that,

$$(*) \qquad ||\sum_{i=1}^{n}\epsilon_i x_i|| \leq \sum_{i=1}^{n}||\epsilon_i||||x_i|| = nR$$

Thus,

$$\mathbf{E}_\epsilon||x_\epsilon||^2 = \mathbf{E}_\epsilon||\frac{1}{n}\sum_{i=1}^{n}\epsilon_i x_i||^2 = \frac{1}{n^2}\mathbf{E}_\epsilon||\sum_{i=1}^{n}\epsilon_i x_i||^2$$

Now, it clear that, by $(*)$, one can obtain that,

$$\mathbf{E}_\epsilon||x_\epsilon||^2 = \frac{1}{n^2}\mathbf{E}_\epsilon||\sum_{i=1}^{n}\epsilon_i x_i||^2 \leq \frac{nR^2}{n^2} = \frac{R^2}{n}$$

Problem 3.3:
We have that,

$$\begin{aligned}
Rad(\mathcal{F}) &= \mathbf{E}_\epsilon(\max_{f\in\mathcal{F}} \frac{1}{n}\sum_{i=1}^{n} \epsilon_i f(x_i)) \\
&\leq \mathbf{E}_\epsilon(W||x_e||) \quad \text{(by Problem 1)} \\
&= W\mathbf{E}_\epsilon(\sqrt{||x_e||^2}) \\
&= W\sqrt{\mathbf{E}_\epsilon(||x_e||^2)} \quad \text{(by Jensen's inequality)} \\
&\leq \frac{WR}{\sqrt{n}} \quad \text{(by problem 2)}
\end{aligned}$$

Thus, we are done. $\square$