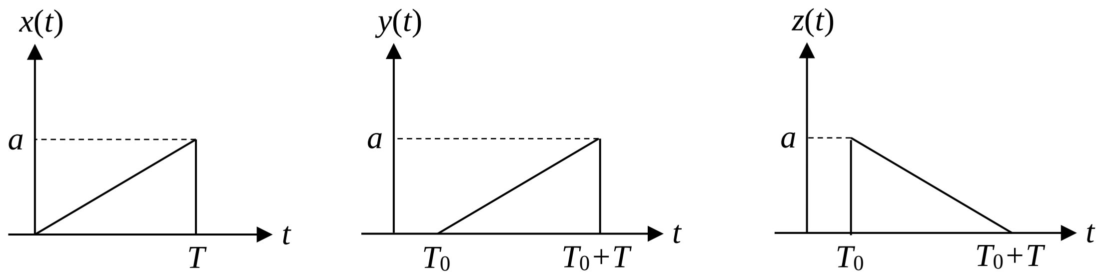
COMM.SYS.300 COMMUNICATION THEORY, Exercise 3, Fall 2023

1. Calculate the autocorrelation function of the following finite-energy signals.



   Hint: For signals $y(t)$ and $z(t)$, you can avoid additional calculations by recalling how the aurocorrelation function and the spectral density function are related by the Fourier transform.

2. Explain why the following time-domain functions can or cannot be the autocorrelation function of a finite-energy signal:

   (a) a rectangular pulse $\Pi(t/\tau)$

   (b) a triangular pulse $\Lambda(t/\tau)$

3. Two random variables Z_1 and Z_2 are defined in terms of two other random variables X and Y as

   $Z_1 = 3X - Y$ $\qquad\qquad$ $Z_2 = X(1 - Y)$

   Verify if the expectation (i.e. statistical average) of Z_1 and Z_2 can be written respectively as

   $\mathrm{E}[Z_1] = 3\mathrm{E}[X] - \mathrm{E}[Y]$ $\qquad\qquad$ $\mathrm{E}[Z_2] = \mathrm{E}[X](1 - \mathrm{E}[Y])$

4. Consider the following random sequence, known as *"random walk"*:

$$X[n] = \sum_{i=1}^{n} W[i]$$

   The individual terms $W[i]$ are binary i.i.d. (independent and identically distributed) random variables, with probabilities $\mathrm{P}(W[i] = +s) = p$ and $\mathrm{P}(W[i] = -s) = 1 - p$.

   a) Draw a possible sample function of this random sequence, for example up to $n = 10$.

   b) Give that the statistical mean and autocovariance (or discrete-time autocorrelation) of the random sequence $X[n]$ can be written as:

$$\mu_X[n] = E\big(X[n]\big) = ns(2p-1)$$

$$K_X(m,n) = E\Big(\big(X[m] - \mu_X[m]\big)\big(X[n] - \mu_X[n]\big)\Big) = \min(m,n)(2s)^2\,p(1-p)$$

   Explain why the random sequence $X[n]$ is Wide-Sense-Stationary or not.

5. Review the case of a random-phase sinusoid in the lecture notes (pages 142-143), then consider the following two random signals:

   i. $w(t) = a_0 \cos(2\pi F t)$ "random-frequency sinusoid"

      a_0 constant amplitude

      F r.v. uniformly distributed between f_1 and f_2 i.e.

      $$p_F(f) = \begin{cases} 1/(f_2 - f_1) & \text{for } 0 < f_1 < f < f_2 \\ 0 & \text{otherwise} \end{cases}$$

   ii. $z(t) = A \cos(2\pi f_0 t)$ "random-amplitude sinusoid"

      f_0 constant frequency

      A r.v. uniformly distributed between a_1 and a_2 i.e.

      $$p_A(a) = \begin{cases} 1/(a_2 - a_1) & \text{for } 0 < a_1 < a < a_2 \\ 0 & \text{otherwise} \end{cases}$$

For both these random signals:

a) Derive the statistical averages $E[w(t)]$ and $E[z(t)]$

b) Derive the autocorrelations $R_w(t_1,t_2)$ and $R_z(t_1,t_2)$

c) Explain why the random signal is Wide-Sense-Stationary or not

Hint: Try to derive expressions for correlations as a function of a time difference $t_1 - t_2$. During the derivations, you may find the following trigonometric identity useful:

$$\cos(x)\cos(y) = 1/2 \cos(x-y) + 1/2 \cos(x+y)$$