PAPERS | AUGUST 01 2005

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M. G. Stewart

Am. J. Phys. 73, 730–734 (2005)
https://doi.org/10.1119/1.1949625

Precession of the perihelion of Mercury's orbit

M. G. Stewart[a)]
Department of Physics, Wayne State University, Detroit, Michigan 48202

(Received 23 July 2004; accepted 20 May 2005)

The precession of the perihelion of Mercury's orbit is calculated using the Laplace–Runge–Lenz vector. An approximate calculation that assumes the orbits of the perturbing planets are circular and coplanar with Mercury's orbit is within 4.4% of the correct value. A complete calculation that uses the correct elliptical orbit and orientation for each of the perturbing planets is then presented. The precession due to a perturbing planet is proportional to the mass of the planet and is approximately inversely proportional to the cube of its semimajor axis.
[DOI: 10.1119/1.1949625]

I. INTRODUCTION

Most classical mechanics textbooks mention that the precession of the perihelion of Mercury's orbit was one of the first tests of Einstein's general theory of relativity. The perihelion precesses about 575 s of arc per century (in a sun fixed coordinate system) of which 532 seconds of arc can be explained by the perturbations of Mercury's orbit by the other planets. From the mid-nineteenth century through the early twentieth century several workers calculated these perturbations.[1] The cause of the remaining 43 seconds of arc per century was not understood until the general theory of relativity was developed. Several texts calculate the relativistic contribution,[2,3] but none (to my knowledge) indicate how to calculate the precession due to the other planets. This problem was recognized by Price and Rush,[4] who calculated the precession using circular, coplanar orbits for the perturbing planets and approximated the perturbing force.

With the prevalence of computers, we revisit this problem and reexamine the Price and Rush calculation, utilizing the Laplace–Runge–Lenz vector (also see Ref. 5). We then do the calculation for perturbing orbits that are elliptical and inclined with respect to Mercury's orbit. The relativistic contribution also is calculated.

II. THE LAPLACE–RUNGE–LENZ VECTOR

For a Kepler orbit the Laplace–Runge–Lenz vector is defined as[6]

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - \mu \mathrm{k}\frac{\mathbf{r}}{r}, \tag{2.1}$$

where $\mathbf{p}$ and $\mathbf{L}$ are the linear and angular momentum vectors, respectively, and $\mu = mM/(M+m)$ is the reduced mass. For planetary motion, m is the mass of the planet, M is the mass of the sun, $k = GMm$, G is the gravitational constant, and $\mathbf{r}$ is the radius vector from the origin to the planet. For Mercury, $m/\mu = 1 + (1.7 \times 10^{-7})$ (see Table I), and, thus, m can be used instead of μ in the following calculations without affecting the final results.

The vector $\mathbf{A}$ lies in the plane of the elliptical orbit, is parallel to the major axis, and points in the direction of the perihelion. Its magnitude is $\mu k e$, where e is the eccentricity of the orbit. If there are no perturbing forces, $\mathbf{A}$ is conserved, and the major axis of the orbit is fixed in space. However, if there are perturbing forces, then $\mathbf{A}$ is not constant, and, in general, the perihelion will precess. If the perturbing force is $\mathbf{F}$, the time derivative of $\mathbf{A}$ is given by[7]

$$\frac{\dot{\mathbf{A}}}{\mu} = 2(\dot{\mathbf{r}} \cdot \mathbf{F})\mathbf{r} - (\mathbf{r} \cdot \dot{\mathbf{r}})\mathbf{F} - (\mathbf{r} \cdot \mathbf{F})\dot{\mathbf{r}}, \tag{2.2}$$

and the angular velocity of the Laplace–Runge–Lenz vector is

$$\boldsymbol{\omega} = \frac{\mathbf{A} \times \dot{\mathbf{A}}}{A^2} = \hat{a} \times \frac{\dot{\mathbf{A}}}{A}, \tag{2.3}$$

where $\hat{a}$ is the unit vector in the direction of $\mathbf{A}$.

III. CIRCULAR, COPLANAR ORBITS

If the orbit of the perturbing planet is assumed to be circular and coplanar with Mercury's orbit, then from symmetry considerations, the perturbing force will have only a radial component. If we substitute $\mathbf{F} = F_r \hat{r}$ and $\dot{\mathbf{r}} = \dot{r}\hat{r} + r\dot{\gamma}\hat{\gamma}$ into Eq. (2.2), where the true anomaly γ is the angle between the line from the origin to the perihelion and $\mathbf{r}$, we find

$$\dot{\mathbf{A}} = -mr^2 F_r \dot{\gamma}\hat{\gamma}. \tag{3.1}$$

The substitution of Eq. (3.1) into Eq. (2.3) gives

$$\boldsymbol{\omega} = -\frac{r^2 F_r \dot{\gamma}}{ke}(\hat{a} \times \hat{\gamma}) = -\frac{r^2 F_r \dot{\gamma}}{ke}\cos\gamma \hat{z}'. \tag{3.2}$$

Instead of using a point mass for a perturbing planet, we will replace it by a uniform ring of radius a_P (equal to the semimajor axis of the planet's elliptical orbit) and mass M_P, and will calculate the force on Mercury from this mass ring. The gravitational potential of a mass ring at a point that is a distance r from the origin and in the plane of the ring is[7]

$$\Phi(r) = \frac{2GM_P}{\pi a_P}K(\alpha), \tag{3.3}$$

where $K(\alpha)$ is the complete elliptic integral of the first kind and $\alpha = r/a_P$. The gravitational force on a mass m that lies in the plane of the ring is

$$F_r = m\frac{\partial \Phi}{\partial r} = \frac{2GmM_P}{\pi a_P^2}\frac{1}{\alpha}\left[\frac{E(\alpha)}{1-\alpha^2} - K(\alpha)\right], \tag{3.4}$$

where $E(\alpha)$ is the complete elliptic integral of the second kind.

Because $\mathbf{A}$ points in the direction of the perihelion, the contribution to the precession of the perihelion of Mercury's

Table I. The ratio of the sun's mass to the planet's mass, the semimajor axis a_p, and the contribution to the precession of the perihelion of Mercury are given for each planet.

Planet	M/M_P	a_P (AU)	$\delta\gamma$ (arcsec/century) from Eq. (3.5)	$\delta\gamma$ (arcsec/century) from Eq. (4.14)	Doolittle[a] (arcsec/cent.)
Mercury	6 023 600	0.387 098 93	⋯	⋯	⋯
Venus	408 523.5	0.723 331 99	292.84	277.42	277.37
Earth+Moon	328 900.55	1.000 000 11	95.89	90.88	90.92
Mars	3 098 710	1.523 662 31	2.38	2.48	2.48
Jupiter	1 047.350	5.203 363 01	156.94	153.95	154.09
Saturn	3 498.0	9.537 070 32	7.57	7.32	7.32
Uranus	22 960	19.191 263 93	0.14	0.14	0.14
Neptune	19 314	30.068 963 48	0.04	0.04	0.04
Total			555.80	532.23	532.36

[a]Reference 1, p. 179, but corrected for current values of M/M_P.

orbit, $\delta\gamma$ due to the perturbing planet P, can be found by calculating the angle of rotation of $\mathbf{A}$ for one revolution of Mercury,

$$\begin{aligned}\delta\gamma &= \int_0^\tau \omega dt \\ &= -\frac{2M_P}{\pi e a_P M}\int_0^{2\pi} r\left[\frac{E(\alpha)}{1-\alpha^2}-K(\alpha)\right]\cos\gamma\, d\gamma. \qquad (3.5)\end{aligned}$$

The magnitude of the radius vector from the origin to Mercury is

$$r=\frac{a(1-e^2)}{1+e\cos\gamma},$$

where a is the semimajor axis, and e is the eccentricity of the orbit of Mercury.

The values[8] of M/M_P and a_P (in astronomical units)[9] for the planets are listed in Table I, along with each planet's contribution to the precession in seconds of arc per century as calculated from Eq. (3.5)[10] using numerical integration. The values of e are listed in Table II. For comparison, the values calculated by Doolittle,[11] corrected for current mass values, also are listed.

The approximation of circular, coplanar orbits for the seven perturbing planets gives a result for the precession of Mercury's perihelion due to these planets that is 4.4% too large. In Sec. IV we extend the above methods to compute the perihelion precession for elliptical and inclined orbits.

IV. ELLIPTICAL AND INCLINED ORBITS

If the perturbing planet's orbit is elliptical and inclined with respect to Mercury's orbit, we have the geometry depicted in Fig. 1; the perturbing planet's orbit lies in the xy plane, and Mercury's orbit lies in the $x'y'$ plane. The angle Ω is measured from the perihelion of the perturbing planet's orbit, π_P, to the ascending node of Mercury's orbit.[12] The mutual inclination of the two orbits is i, and ω is the angle measured from the ascending node to Mercury's perihelion, π_M.[13] These three angles are the three Euler angles which specify the rotation of the axes from (x,y,z) to (x',y',z').[14]

To find the gravitational potential at $\mathbf{r}$, we replace a perturbing planet by an elliptical ring of mass M_P that coincides with its orbit. Doolittle has shown that this replacement gives the same results as those derived from the moving planet.[15] However, because a planet does not move with constant speed, the mass element dM_P is proportional to the time the planet spends in the line element ds. The gravitational potential of this elliptical ring at $\mathbf{r}$ is[7]

Table II. The orbital parameters for the planets. Columns 2–7 are in degrees. The quantities Ω_P, i_P, and ω_P are the longitude of the ascending node, the inclination, and the argument of the perihelion, respectively, for each planet measured with respect to the ecliptic (Ref. 9). The quantities Ω, i, and ω are the orbital parameters for Mercury measured with respect to the plane of the perturbing planet's orbit. (See Fig. 1.) The angle Ω is measured from the perihelion of the perturbing planet's orbit to the ascending node of Mercury's orbit. The angle i is the mutual inclination of the two orbits, and the angle ω is measured from the ascending node to Mercury's perihelion. Column 8 lists the eccentricities of the planetary orbits (Ref. 9).

Planet	Ω_P	i_P	ω_P	Ω	i	ω	e
Mercury	48.331 67	7.004 87	29.124 87	⋯	⋯	⋯	0.205 630 69
Venus	76.680 69	3.394 71	54.846 08	255.023 19	4.327 27	51.004 69	0.006 773 23
Earth+ Moon	−11.260 64	0.000 05	114.207 83	305.384 06	7.004 96	29.124 87	0.016 710 22
Mars	49.578 54	1.850 61	286.462 30	71.844 12	5.154 97	29.573 62	0.093 412 33
Jupiter	100.556 15	1.305 30	274.197 70	24.181 27	6.290 18	38.584 12	0.048 392 66
Saturn	113.715 04	2.484 46	338.716 90	−64.730 81	6.381 11	49.893 18	0.054 150 60
Uranus	74.229 88	0.769 86	96.734 36	234.331 83	6.321 35	32.180 62	0.047 167 71
Neptune	131.721 69	1.769 17	273.249 66	4.558 53	7.023 65	43.650 17	0.008 585 87

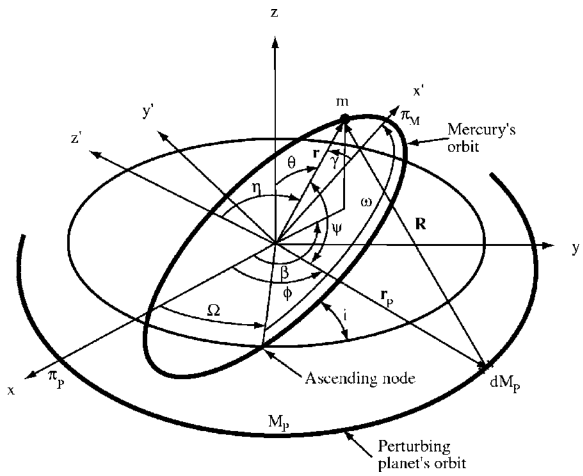

Fig. 1. The orbits of Mercury and the perturbing planet.

$$\Phi(\mathbf{r}) = \frac{GM_P}{2\pi a_P^2\sqrt{1-e_P^2}}\int_0^{2\pi}\frac{r_P^2 d\phi}{\sqrt{r^2+r_P^2-2rr_P\sin\theta\cos(\beta-\phi)}}, \tag{4.1}$$

where $r_P=a_P(1-e_P^2)/(1+e_P\cos\phi)$.

If we use spherical polar coordinates (r,η,γ) in the primed coordinate system, the gravitational force on m due to the elliptical mass ring is $F=F_r\hat{r}+F_\eta\hat{\eta}+F_\gamma\hat{\gamma}$, and because the precession of the perihelion occurs in the plane of the orbit, we have $\dot{\mathbf{r}}=\dot{r}\hat{r}+\dot{\gamma}\hat{\gamma}$. If we substitute for $\mathbf{F}$ and $\dot{\mathbf{r}}$ in Eq. (2.3), we obtain

$$\boldsymbol{\omega} = \hat{a}\times\frac{\dot{\mathbf{A}}}{A} = \frac{1}{ke}[2r^2F_r(\hat{a}\times\hat{r})\dot{\gamma} - r\dot{r}F_\eta(\hat{a}\times\hat{\eta}) - (r\dot{r}F_\gamma + r^2F_r\dot{\gamma})(\hat{a}\times\hat{\gamma})], \tag{4.2}$$

where $\hat{a}\times\hat{r}=\sin\gamma\hat{z}'$, $\hat{a}\times\hat{\eta}=\hat{y}'$, and $\hat{a}\times\hat{\gamma}=\cos\gamma\hat{z}'$.

The term $\hat{a}\times\hat{\eta}$ gives rise to rotation about the y' axis and does not contribute to the precession of the perihelion.[16] If we substitute for $\hat{a}\times\hat{r}$, $\hat{a}\times\hat{\gamma}$, and

$$\dot{r} = \frac{r^2 e\sin\gamma}{a(1-e^2)}\dot{\gamma},$$

we have

$$\boldsymbol{\omega} = \frac{1}{ke}\left[-r^2\cos\gamma F_r + r^2\sin\gamma\left(\frac{2+e\cos\gamma}{1+e\cos\gamma}\right)F_\gamma\right]\dot{\gamma}\hat{z}', \tag{4.3}$$

where

$$F_r = m\frac{\partial\Phi}{\partial r},\quad F_\gamma = \frac{m}{r}\frac{\partial\Phi}{\partial\gamma} = \frac{m}{r}\left(\frac{\partial\Phi}{\partial\theta}\frac{\partial\theta}{\partial\gamma} + \frac{\partial\Phi}{\partial\beta}\frac{\partial\beta}{\partial\gamma}\right).$$

To express θ in terms of the variable γ we write

$$\cos\theta = \hat{r}\cdot\hat{z} = \cos\gamma(\hat{x}'\cdot\hat{z}) + \sin\gamma(\hat{y}'\cdot\hat{z}). \tag{4.4}$$

The relation between the primed and the unprimed coordinates is $\mathbf{x}'=\mathbf{T}\mathbf{x}$, where the matrix $\mathbf{T}$ is[17]

$$\mathbf{T} = \begin{pmatrix} \cos\omega\cos\Omega - \cos i\sin\Omega\sin\omega & \cos\omega\sin\Omega + \cos i\cos\Omega\sin\omega & \sin\omega\sin i \\ -\sin\omega\cos\Omega - \cos i\sin\Omega\cos\omega & -\sin\omega\sin\Omega + \cos i\cos\Omega\cos\omega & \cos\omega\sin i \\ \sin i\sin\Omega & -\sin i\cos\Omega & \cos i \end{pmatrix}. \tag{4.5}$$

Then

$$\cos\theta = \cos\gamma\sin\omega\sin i + \sin\gamma\cos\omega\sin i = \sin i\sin(\omega+\gamma). \tag{4.6}$$

If we differentiate $\cos\theta$ with respect to γ and solve for $\partial\theta/\partial\gamma$, we find

$$\frac{\partial\theta}{\partial\gamma} = -\frac{\sin i\cos(\omega+\gamma)}{\sin\theta}. \tag{4.7}$$

To find the relation between β and γ we have

$$\cos(\omega+\gamma) = \hat{r}\cdot\hat{n}, \tag{4.8}$$

where $\hat{n}$ is the unit vector pointing to the ascending node. Then

$$\cos(\omega+\gamma) = (\sin\theta\cos\beta\hat{x} + \sin\theta\sin\beta\hat{y} + \cos\theta\hat{z})\cdot(\cos\Omega\hat{x} + \sin\Omega\hat{y}) = \sin\theta\cos(\beta-\Omega). \tag{4.9}$$

Thus

$$\cos(\beta-\Omega) = \frac{\cos(\omega+\gamma)}{\sin\theta}. \tag{4.10}$$

We then have

$$\sin(\beta-\Omega) = \frac{\cos i\sin(\omega+\gamma)}{\sin\theta}, \tag{4.11}$$

and

$$\tan(\beta-\Omega) = \cos i\tan(\omega+\gamma). \tag{4.12}$$

If we differentiate $\tan(\beta-\Omega)$ with respect to γ and solve for $\partial\beta/\partial\gamma$, we obtain

$$\frac{\partial\beta}{\partial\gamma} = \frac{\cos i}{\sin^2\theta}. \tag{4.13}$$

The precession, in radians per revolution, is given by integrating Eq. (4.3) over one revolution. We substitute for $\partial\Phi/\partial r$, $\partial\Phi/\partial\theta$, $\partial\Phi/\partial\beta$, which are obtained by differentiating Eq. (4.1), $\partial\theta/\partial\gamma$ from Eq. (4.7) and $\partial\beta/\partial\gamma$ from Eq. (4.13), and write the precession as

$$\delta\gamma = \int_0^\tau \omega dt = -\frac{1}{2\pi}\frac{M_P}{M}\frac{1}{a_P^2\sqrt{1-e_P^2}}\frac{1}{e}\int_0^{2\pi} r^2 \int_0^{2\pi} \frac{r_P^3}{[r^2+r_P^2-2rr_P\sin\theta\cos(\beta-\phi)]^{3/2}} \cdot \left\{\cos\gamma\left[\sin\theta\cos(\beta-\phi)-\frac{r}{r_P}\right] + \frac{\sin\gamma}{\sin\theta}\left(\frac{2+e\cos\gamma}{1+e\cos\gamma}\right) \times \left[\begin{matrix}\sin^2 i\sin(\omega+\gamma)\cos(\omega+\gamma)\\ \cos(\beta-\phi)+\cos i\sin(\beta-\phi)\end{matrix}\right]\right\} d\phi d\gamma, \tag{4.14}$$

where

$$r = \frac{a(1-e^2)}{1+e\cos\gamma}, \quad r_P = \frac{a_P(1-e_P^2)}{1+e_P\cos\phi},$$
$$\cos\theta = \sin i\sin(\omega+\gamma),$$
$$\cos\beta = \frac{\cos\Omega\cos(\omega+\gamma)-\sin\Omega\cos i\sin(\omega+\gamma)}{\sin\theta}.$$

The constants Ω, i, and ω (actually $\Omega+\omega$) are given in astronomical tables for planetary orbits with respect to the ecliptic, but in Eq. (4.14) these quantities must be specified for Mercury's orbit with respect to the perturbing planet's orbit. In the Appendix the appropriate formulas are derived for determining these quantities as defined in Fig. 1. In Table II the quantities Ω, i, and ω, for each planet are given with respect to the ecliptic[9] and are denoted by the subscript P. The same quantities are given for Mercury with respect to the perturbing planet's orbit as calculated from Eqs. (A3), (A1), and (A2), respectively, and are used to evaluate Eq. (4.14).

Each planet's contribution to the precession of the perihelion of Mercury's orbit, in seconds of arc per century, calculated from Eq. (4.14) using the orbital parameters listed in Table II is listed in column 5 of Table I. The small differences in the precession calculated from Eq. (4.14) compared to Doolittle's calculations are due mainly to small differences in the orbital parameters used in the two calculations.

The magnitude of the tangential force, F_γ, on Mercury is always less than 3% of the radial force, F_r, on Mercury due to all the planets (see Fig. 2).

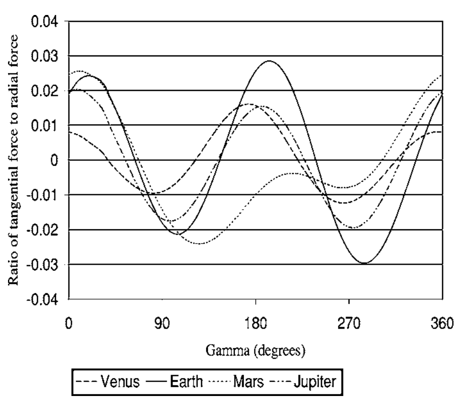

Fig. 2. Ratio of the tangential force to the radial force on Mercury due to Venus, Earth, Mars, and Jupiter as a function of gamma.

The fact that the radial force is dominant, that the eccentricities of the orbits of the perturbing planets are small, and that the inclinations of the perturbing planets' orbits with respect to Mercury's orbit are small, explains why the calculation of Sec. III for circular, coplanar orbits gives a result that is only 4.4% in error.

Equations (3.5) and (4.14) show that the precession is proportional to the mass of the perturbing planet, and, as can be seen in Fig. 3, the precession per unit mass is approximately inversely proportional to the cube of the semimajor axis of the perturbing planet's orbit. This inverse cubic relationship also can be seen by inspection of Eqs. (3.5) and (4.14). If the elliptic integrals in Eq. (3.5) are expanded in series form in terms of the parameter (a/a_P), the leading term of $\delta\gamma$ is proportional to $(a/a_P)^3$. The higher order terms cannot be neglected, of course, if (a/a_P) is not small compared to one. A least-squares fit to the data in Fig. 3 gives a slope equal to -3.1.

V. RELATIVISTIC CONTRIBUTION

The general relativistic force correction to the Newtonian central gravitational force is[18]

$$\mathbf{F}_{\text{gr}} = -\frac{3GML^2}{mc^2r^4}\hat{r}, \tag{5.1}$$

where c is the speed of light. Because there is only a radial component, we can substitute for F_r in Eq. (3.2), which gives

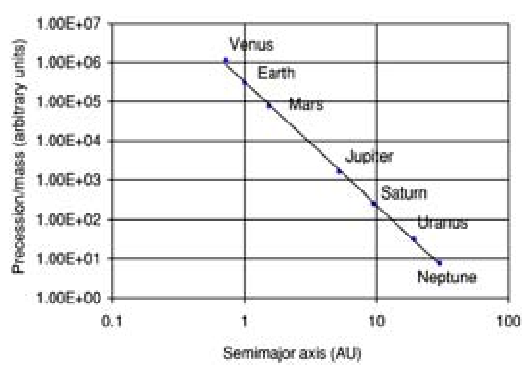

Fig. 3. Precession of Mercury's orbit divided by the mass of the perturbing planet vs the semimajor axis of the perturbing planet. The precession/mass is approximately proportional to the inverse cube of the perturbing planet's semimajor axis.

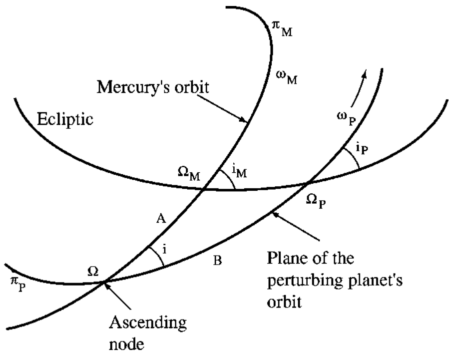

Fig. 4. Relationship between the ecliptic, Mercury's orbit, and the perturbing planet's orbit.

$$\boldsymbol{\omega}_{\rm gr} = \frac{GML^2}{mkec^2r^2}\dot{\gamma}\cos\gamma\hat{z}'. \tag{5.2}$$

If we substitute $mka(1-e^2)$ for L^2 in Eq. (5.2) and integrate $\omega_{\rm gr}$ for one revolution of Mercury's orbit, the precession due to the relativistic force is

$$\begin{aligned}\delta\gamma_{\rm gr} &= \int_0^\tau \omega_{\rm gr}dt \\ &= \frac{3GM}{ec^2a(1-e^2)}\int_0^{2\pi}(1+e\cos\gamma)^2\cos\gamma\, d\gamma \\ &= \frac{6\pi GM}{c^2a(1-e^2)} \\ &= 5.0191\times10^{-7}\ \text{rad/revolution} \\ &= 42.98\ \text{seconds of arc/century}. \end{aligned}\tag{5.3}$$

As noted in Sec. I, this 43 seconds of arc/century was the missing contribution that completed the explanation of the precession of the perihelion of Mercury's orbit.

ACKNOWLEDGMENTS

I would like to thank H. V. Bohm, H. H. Denman, and G. B. Beard for their helpful comments and suggestions.

APPENDIX: CALCULATION OF Ω, i, AND ω

In astronomical publications the longitude of the ascending node, Ω, the inclination, i, and the argument of the perihelion, ω, are given with respect to the ecliptic, and Ω is measured from the vernal equinox. In Eq. (4.14) these quantities refer to the plane of the perturbing planet's orbit, and Ω is the angle measured from the perihelion of the perturbing planet to the ascending node. In Fig. 4 the orbital parameters with respect to the ecliptic are shown with subscripts. The subscripts M and P refer to Mercury and the perturbing planet, respectively. The point π_M is the perihelion of Mercury's orbit, and the point π_P lies on the line from the origin to the perturbing planet's perihelion.

From spherical trigonometry the cosine of the mutual inclination is

$$\cos i = \cos i_M \cos i_P + \sin i_M \sin i_P \cos(\Omega_P - \Omega_M). \tag{A1}$$

The angle from the ascending node to Mercury's perihelion, π_M, is $\omega = A + \omega_M$, where

$$\sin A = \frac{\sin i_P}{\sin i}\sin(\Omega_P - \Omega_M). \tag{A2}$$

The angle from the perturbing planet's perihelion, π_P, to the ascending node is $\Omega = 2\pi - B - \omega_P$, where

$$\sin B = \frac{\sin i_M}{\sin i}\sin(\Omega_P - \Omega_M). \tag{A3}$$

[a]Electronic mail: stewart@physics.wayne.edu

[1]Eric Doolittle, "The secular variations of the elements of the orbits of the four inner planets computed for the epoch 1850.0 G.M.T.," Trans. Am. Phil. Soc. **22**, 37–189 (1912), and references therein.

[2]Herbert Goldstein, Charles Poole, and John Safko, *Classical Mechanics* (Addison–Wesley, San Francisco, CA, 2002), 3rd ed., p. 538.

[3]Jerry B. Marion and Stephen T. Thornton, *Classical Dynamics of Particles and Systems* (Harcourt Brace, Orlando, FL, 1995), 4th ed., pp. 318–321.

[4]Michael P. Price and William F. Rush, "Nonrelativistic contribution to Mercury's perihelion precession," Am. J. Phys. **47**, 531–534 (1979).

[5]B. Davies, "Elementary theory of perihelion precession," Am. J. Phys. **51**, 909–911 (1983).

[6]Reference 2, pp. 102–106. For more information about the history of the Laplace–Runge–Lenz vector, see Herbert Goldstein, Am. J. Phys. **43**, 737–738 (1975); **44**, 1123–1124 (1976).

[7]See EPAPS Document No. E-AJPIAS-73-011508 for details of the calculations. A direct link to this document may be found in the online article's HTML reference section. The document may also be reached via the EPAPS homepage ⟨http://www.aip.org/pubservs/epaps.html⟩ or from ftp.aip.org in the directory/epaps. See the EPAPS homepage for more information.

[8]*Explanatory Supplement to the Astronomical Almanac* edited by P. Kenneth Seidelmann (University Science Books, Mill Valley, CA, 1992), p. 697, Table 15.2.20.

[9]See Ref. 8, p. 316, Table 5.8.1.

[10]The sidereal period of Mercury is 0.240 844 45 years (Ref. 8, p. 704, Table 15.6), which gives a conversion factor from radians per revolution to seconds of arc per century of $8.564\,233\times10^7$.

[11]Reference 1, p. 179. The quantity in Ref. 1 that corresponds to $\delta\gamma$ is $[d\chi/dt]_{00}$.

[12]When measured from the vernal equinox of earth's orbit along the ecliptic to the ascending node, Ω is called the longitude of the ascending node.

[13]When measured from the ecliptic to the perihelion, ω is called the argument of the perihelion. The longitude of the perihelion, which is defined as $\varpi = \Omega + \omega$ is usually given in astronomical tables instead of ω.

[14]Reference 2, pp. 150–154.

[15]Reference 1, p. 41.

[16]The normal force, F_η, gives rise to changes in i and Ω.

[17]Reference 2, p. 153. Their (ϕ,θ,ψ) correspond to the angles (Ω,i,ω) used here.

[18]Reference 3, p. 318.