

# The limit of a sequence of probability spaces, from scratch.

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For each  $n \in \mathbb{N}$  let

- $X_n$  be a nonempty set.
- $\mathcal{B}_n \subseteq \mathcal{P}(X_n)$  an algebra, that is,  $\emptyset, X_n \in \mathcal{B}_n$  and if  $A, B \in \mathcal{B}_n$ , then  $A \cup B, A \cap B, X_n \setminus A \in \mathcal{B}_n$ .
- $\mu_n : \mathcal{B}_n \rightarrow [0, 1]$  a finitely additive probability measure.
- $T_n : X_n \rightarrow X_n$  measure-preserving, that is, for all  $A \in \mathcal{B}_n$ ,  $T_n^{-1}(A) \in \mathcal{B}_n$  and  $\mu_n(T_n^{-1}(A)) = \mu_n(A)$ .

In the following we explain a reasonable way to construct limits of the sequences  $X_n, \mathcal{B}_n, \mu_n$  and  $T_n$ , which we suggestively denote as  $X_\infty, \mathcal{B}_\infty, \mu_\infty$  and  $T_\infty$ . As a plus, the probability space  $(X_\infty, \mathcal{B}_\infty, \mu_\infty)$  will be countably additive. This allow us to, for any finitely additive probability space, construct a (countably additive) probability space with similar properties, see Proposition 11.

The results presented here are well-known; this document aims to be a self-contained, elementary exposition of them, without requiring previous knowledge of non-standard analysis. The construction of the ‘limit measure’ we use below is known as a Loeb measure, and was found by Loeb in [L]. For other mentions of Loeb measures in the literature, see for example [DW], [AB, Definition 7.5], or [AEHL, Section 3.1].

The main ingredient in our constructions is a non-principal ultrafilter  $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$  in  $\mathbb{N}$ , that is, a family of sets such that:

1.  $\emptyset \notin \mathcal{F}$ .
2. If  $A, B \in \mathcal{F}$ , then  $A \cap B \in \mathcal{F}$ . If  $A, B \notin \mathcal{F}$ , then  $A \cup B \notin \mathcal{F}$ .
3. If  $A \in \mathcal{F}$  and  $A \subseteq B \subseteq \mathbb{N}$ , then  $B \in \mathcal{F}$ .
4. For every  $A \subseteq \mathbb{N}$ , exactly one of  $A$  or  $\mathbb{N} \setminus A$  belongs to  $\mathcal{F}$ .
5.  $\mathcal{F}$  is non-principal, that is, it contains no finite set.

One can prove the existence of non-principal ultrafilters using the axiom of choice, see e.g. [W, Theorem 12.12]. The non-principal ultrafilter  $\mathcal{F}$  will be fixed throughout the whole document, so the constructions below depend on the choice of ultrafilter  $\mathcal{F}$ . One can check the following using the definition of ultrafilter:

**Proposition 1.** *Let  $\mu_{\mathcal{F}} : \mathcal{P}(\mathbb{N}) \rightarrow \{0, 1\}$ ;  $\mu_{\mathcal{F}}(A) = 1$  if  $A \in \mathcal{F}$  and  $\mu_{\mathcal{F}}(A) = 0$  if  $A \notin \mathcal{F}$ . Then  $\mu_{\mathcal{F}}$  is a finitely additive probability measure in  $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ .*  $\square$

So ultrafilters give rise to  $\{0, 1\}$ -valued, finitely additive probability measures. When we say ‘almost all  $n$  satisfies some property  $P$ ’, we mean that the set of numbers  $n \in \mathbb{N}$  that satisfy  $P$  is in  $\mathcal{F}$ .

**Definition 2** (Ultraproduct of sets). The *ultraproduct*  $X_\infty := \lim_{n \rightarrow \mathcal{F}} X_n$  is the quotient set  $\frac{\prod_{n \in \mathbb{N}} X_n}{\sim}$ , where  $(x_n)_n \sim (y_n)_n$  iff  $x_n = y_n$  for almost all  $n$ . We denote by  $[y_n]_n \in \lim_{n \rightarrow \mathcal{F}} X_n$  the class of an element  $(y_n)_n$ .

**Definition 3** (Internal subsets). Given a sequence of sets  $A_n \subseteq X_n$ , we denote  $\lim_{n \rightarrow \mathcal{F}} A_n = \{[x_n]_n \in X_\infty; x_n \in A_n \text{ for almost all } n\}$ . Subsets of  $X_\infty$  which can be obtained this way are called *internal*.

The following properties follow from the definitions:

1.  $\lim_{n \rightarrow \mathcal{F}} A_n \cup B_n = (\lim_{n \rightarrow \mathcal{F}} A_n) \cup (\lim_{n \rightarrow \mathcal{F}} B_n)$ .
2.  $\lim_{n \rightarrow \mathcal{F}} A_n \cap B_n = (\lim_{n \rightarrow \mathcal{F}} A_n) \cap (\lim_{n \rightarrow \mathcal{F}} B_n)$ .

3.  $\lim_{n \rightarrow \mathcal{F}} A_n \setminus B_n = (\lim_{n \rightarrow \mathcal{F}} A_n) \setminus (\lim_{n \rightarrow \mathcal{F}} B_n)$ .
4.  $\lim_{n \rightarrow \mathcal{F}} A_n = \lim_{n \rightarrow \mathcal{F}} B_n$  iff  $A_n = B_n$  for  $\mathcal{F}$ -almost all  $n$ .
5.  $\lim_{n \rightarrow \mathcal{F}} A_n = \emptyset$  iff  $A_n = \emptyset$  for almost all  $n$ .

In particular, internal subsets are an algebra of subsets of  $X_\infty$ . In order to define our measure  $\mu_\infty$  we need to define what  $\lim_{n \rightarrow \omega} x_n$  means, when  $(x_n)_{n \in \mathbb{N}}$  is a bounded sequence of real/complex numbers.

**Proposition 4.** *For any sequence  $(x_n)_{n \in \mathbb{N}}$  in a compact Hausdorff space  $X$ , there exists a unique  $x \in X$ , which we denote*

$$\lim_{n \rightarrow \mathcal{F}} x_n,$$

*such that for all neighborhoods  $U$  of  $x$ , the set  $\{n \in \mathbb{N}; x_n \in U\}$  is in  $\mathcal{F}$ .*

*Proof.* Existence of the limit: Suppose for contradiction that for all  $x \in X$  there is a nhood  $U_x$  such that  $\mu_{\mathcal{F}}(\{n \in \mathbb{N}; x_n \in U_x\}) = 0$ . Take a finite cover  $X = U_1 \cup \dots \cup U_n$  of  $X$  by such neighborhoods. Then, we have  $\mathbb{N} = \bigcup_{i=1}^n \{n \in \mathbb{N}; x_n \in U_i\}$ , a contradiction as  $\mu_{\mathcal{F}}(\mathbb{N}) > 0$ .

Uniqueness of the limit: Suppose there are two ‘limit points’  $x \neq y$  in  $X$  satisfying the property above. Let  $U_x, U_y$  be disjoint neighborhoods of  $x, y$ . Then the sets  $\{n \in \mathbb{N}; x_n \in U_x\}$  and  $\{n \in \mathbb{N}; x_n \in U_y\}$  have  $\mu_{\mathcal{F}}$ -measure 1, a contradiction as they are disjoint.  $\square$

If a sequence  $(x_n)$  of complex numbers is bounded, then we can interpret it as a sequence in a compact subset of  $\mathbb{C}$  and define  $\lim_{n \rightarrow \omega} x_n$  according to Proposition 4; the limit will only depend on the sequence, not on the compact set we choose.

We also need the following technical result.

**Proposition 5.** *For each  $k \in \mathbb{N}$  let  $A^k = \lim_{n \rightarrow \mathcal{F}} A_n^k$  be an internal subset of  $X_\infty$ . If the sets  $A^k$  are nonempty and pairwise disjoint, then  $A := \bigcup_{k \in \mathbb{N}} A^k$  is not internal.*

*Proof.* Suppose that  $A = \lim_n A_n$  for some sets  $A_n \subseteq X_n$ . For each  $k, n \in \mathbb{N}$  let

$$B_n^k := A_n \cap (A_n^k \setminus (A_n^1 \cup \dots \cup A_n^{k-1})).$$

Then for each fixed  $k$ , the fact that  $\emptyset \neq A \cap (A^k \setminus (A^1 \cup \dots \cup A^{k-1}))$  implies that  $B_n^k \neq \emptyset$  for almost all  $n$ . We define a point  $[x_n]_n \in A$  by letting  $x_n \in X_n$  be some point of  $B_n^{k_n}$ , where  $k_n$  is given by:

1. If  $B_n^j \neq \emptyset$  for finitely many values of  $j$ , let  $k_n$  be the maximum such  $j$ .
2. If  $B_n^j \neq \emptyset$  for infinitely many  $j$ , let  $k_n$  satisfy  $k_n > n$  and  $B_n^{k_n} \neq \emptyset$ .

There may be some values of  $n$  such that  $B_n^k = \emptyset$  for all  $k$ . But the set of such values  $n$  is  $\mathcal{F}$ -small, so we can choose  $x_n$  however we want in that case, without changing  $[x_n]_n$ .

We then have  $[x_n]_n \in A$ , because  $x_n \in B_n^{k_n} \subseteq A_n$  for almost all  $n$ . But this leads to contradiction, as for each fixed value of  $k$ ,  $[x_n]_n \notin A^k$ . Indeed, for almost all  $n$  we have  $k_n > k$  (this is obvious in Item 2, and in Item 1 it follows from the fact that  $B_n^{k+1} \neq \emptyset$  for almost all  $n$ ), so  $x_n \in B_n^{k_n} \subseteq A_n \setminus A_n^k$ , so  $x_n \notin A_n^k$ .  $\square$

**Definition 6.** Let  $\mathcal{A}_\infty$  be the algebra of all internal sets of the form  $\lim_{n \rightarrow \mathcal{F}} A_n$ , where  $A_n \in \mathcal{B}_n$  for all  $n$ . Let  $\mathcal{B}_\infty \subseteq \mathcal{P}(X_\infty)$  be the  $\sigma$ -algebra generated by  $\mathcal{A}_\infty$ .

**Proposition 7.** *The map  $\mu : \mathcal{A}_\infty \rightarrow [0, 1]$ ;  $\mu(\lim_{n \rightarrow \mathcal{F}} A_n) = \lim_{n \rightarrow \mathcal{F}} \mu_n(A_n)$  is a pre-measure, thus it extends to a probability measure  $\mu_\infty : \mathcal{B}_\infty \rightarrow [0, 1]$ .*

*Proof.* We omit checking that  $\mu$  is well defined. Now suppose we have disjoint sets  $A^1, A^2, \dots$  in  $\mathcal{A}_\infty$ , with  $A_i = \lim_{n \rightarrow \mathcal{F}} A_n^i$ . Note that for  $i \neq j$  we have  $A_n^i \cap A_n^j = \emptyset$  for almost all  $n$ . And suppose that  $A := \bigcup_{k \in \mathbb{N}} A^k \in \mathcal{A}_\infty$  (say,  $A = \lim_{n \rightarrow \mathcal{F}} A_n$ ), then by Proposition 5,  $A$  must be a finite union of the sets  $A^k$ , say  $A = \bigcup_{k=1}^K A^k$ , and in particular  $A^n = \emptyset$  for  $n > K$ . So  $\mu$  is a pre-measure:

$$\begin{aligned} \mu(A) &= \mu\left(\bigcup_{k=1}^K A^k\right) = \mu\left(\lim_{n \rightarrow \mathcal{F}} (A_n^1 \cup \dots \cup A_n^K)\right) = \lim_{n \rightarrow \mathcal{F}} \mu_n(A_n^1 \cup \dots \cup A_n^K) \\ &= \sum_{k=1}^K \lim_{n \rightarrow \mathcal{F}} \mu_n(A_n^k) = \sum_{k=1}^K \mu(A^k) = \sum_{k=1}^\infty \mu(A^k). \end{aligned}$$

Thus, by Caratheodory’s extension theorem,  $\mu$  extends to a probability measure  $\mu_\infty$  in  $\mathcal{B}_\infty$ .  $\square$

The following again follows from the definitions.

**Proposition 8** (Limits of measure preserving actions). *If  $T_n : X_n \rightarrow X_n$  is a measure preserving map  $\forall n$ , then the map  $T_\infty : X_\infty \rightarrow X_\infty; [x_n]_n \mapsto [Tx_n]_n$  satisfies  $T_\infty^{-1}(\lim_{n \rightarrow \mathcal{F}} A_n) = \lim_{n \rightarrow \mathcal{F}} T_n^{-1}(A_n)$ , so it is measurable and measure preserving.*  $\square$

**Remark 9.** The limit of a sequence of ergodic measure-preserving systems need not be ergodic. E.g. let  $\mathbb{T}$  be the torus, i.e. the quotient  $\frac{\mathbb{R}}{\mathbb{Z}}$  with Lebesgue measure, and  $T_n : \mathbb{T} \rightarrow \mathbb{T}$  be given by  $T_n(x) = x + \frac{1}{n} \bmod 1$ . Consider the map  $T_\infty : \mathbb{T}_\infty \rightarrow \mathbb{T}_\infty$ , and  $A = \lim_{n \rightarrow \mathcal{F}} [0, 1/2]$ . Then  $\mu_\infty(A) = 0$ , and  $\mu_\infty(A \Delta T_\infty^{-1}(A)) = 0$ , so  $T_\infty$  is not ergodic.  $T_\infty$  is not trivial either, in fact there exist internal sets  $B$  such that  $\mu_\infty(B \Delta T_\infty^{-1}(B)) = 1$ .

We can also naturally define ultralimits of  $L^\infty$  functions. Let  $\mathbb{D} = \{z \in \mathbb{C}; |z| \leq 1\}$  be the unit disk.

**Proposition 10.** *Let  $f_n : X_n \rightarrow \mathbb{D}$  be measurable for all  $n$ . Then the limit function  $f : X_\infty \rightarrow \mathbb{D}$  given by  $f([x_n]_n) = \lim_{n \rightarrow \mathcal{F}} f_n(x_n)$  is measurable, and satisfies*

$$\int_{X_\infty} f d\mu_\infty = \lim_{n \rightarrow \mathcal{F}} \int_{X_n} f_n d\mu_n. \quad (1)$$

Note that the preimage of a measurable set need not be internal! But it is still measurable.

*Proof.* For the first part it is enough to check that the preimage of any closed set is measurable. And indeed, for any closed  $C \subseteq \mathbb{D}$  and letting  $C_k$  be the  $\frac{1}{k}$ -neighborhood of  $C$ , we have

$$f^{-1}(C) = \bigcap_{k \in \mathbb{N}} \lim_{n \rightarrow \mathcal{F}} f_n^{-1}(C_k).$$

To prove Equation (1) it is enough to prove the case where all the functions  $f_n$  take values in some finite set  $D_0 \subseteq \mathbb{D}$  (and then we approximate arbitrary functions in the  $L^\infty$  norm).

But if  $f_n : X_n \rightarrow D_0$  for all  $n$ , then  $f$  also takes values in  $D_0$ , and for each  $d \in D_0$  we have  $f^{-1}(d) = \lim_{n \rightarrow \infty} f_n^{-1}(d)$ . Thus,

$$\begin{aligned} \int_{X_\infty} f d\mu_\infty &= \sum_{d \in D_0} d \cdot \mu_\infty(f^{-1}(d)) = \sum_{d \in D_0} d \cdot \lim_{n \rightarrow \mathcal{F}} \mu_n(f_n^{-1}(d)) \\ &= \lim_{n \rightarrow \mathcal{F}} \sum_{d \in D_0} d \cdot \mu_n(f_n^{-1}(d)) = \lim_{n \rightarrow \mathcal{F}} \int_{X_n} f_n d\mu_n. \end{aligned} \quad \square$$

**From finitely additive to countably additive measures.** By letting  $(X_n, \mathcal{B}_n, \mu_n)$  be a fixed space  $(X, \mathcal{B}, \mu)$  for all  $n$ , the following follows from the results in the previous section:

**Proposition 11.** *Let  $(X, \mathcal{B}, \mu)$  be a finitely additive probability space. Then we have a (countably additive) probability space  $(\overline{X}, \overline{\mathcal{B}}, \overline{\mu}) := \lim_{n \rightarrow \mathcal{F}} (X, \mathcal{B}, \mu)$ , and an injective map  $\mathcal{B} \rightarrow \overline{\mathcal{B}}; A \mapsto \overline{A} := \lim_{n \rightarrow \mathcal{F}} A$ , which satisfies for all  $A, B \in \mathcal{B}$  that*

1.  $\overline{\mu}(\overline{A}) = \mu(A)$ .
2.  $\overline{A \cup B} = \overline{A} \cup \overline{B}$ ,  $\overline{A \cap B} = \overline{A} \cap \overline{B}$ .
3.  $\overline{\emptyset} = \emptyset$ .
4.  $\overline{X \setminus A} = \overline{X} \setminus \overline{A}$ .

*If  $(X, \mathcal{B}, \mu)$  and  $(Y, \mathcal{C}, \nu)$  are finitely additive probability spaces and  $T : X \rightarrow Y$  is measure preserving, then it has an associated map  $\overline{T} : \overline{X} \rightarrow \overline{Y}$ , given by  $\overline{T}([x_n]_n) = [Tx_n]_n$ , such that:*

1.  $\overline{T}^{-1}(\overline{A}) = \overline{T^{-1}(A)}$  for all  $A \in \mathcal{C}$ .
2. If  $T : X \rightarrow Y$  and  $S : Y \rightarrow Z$  are measure-preserving, then  $\overline{S \circ T} = \overline{S} \circ \overline{T}$ .

*In particular, the assignments  $(X, \mathcal{B}, \mu) \rightarrow (\overline{X}, \overline{\mathcal{B}}, \overline{\mu})$ ,  $T \mapsto \overline{T}$  can be seen as a functor.*  $\square$

## References

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