1 Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$

See my updated TA information in Carmen → Modules → Recitation notes and office hours.

Exercise 1.1. Let $\vec{u} = \langle 1,2,3\rangle$ and $\vec{v} = \langle 4,5,6\rangle$.

1. Find the norm of $\vec{v} - 3\vec{u}$.

   Solution. We have $\vec{v} - 3\vec{u} = \langle 4,5,6\rangle - 3\cdot\langle 1,2,3\rangle = \langle 4-3\cdot 1, 5-3\cdot 2, 6-3\cdot 3\rangle = \langle 1,-1,-3\rangle$.

   Thus, $|\vec{u} - 3\vec{v}| = \sqrt{1^2+(-1)^2+(-3)^2} = \sqrt{11}$.

2. Find the vector with opposite direction to $\vec{u}$ and with four times the magnitude of $\vec{u}$.

   Solution. Vectors in the opposite direction of $\vec{u}$ have the form $\lambda\vec{u}$, where λ is a negative constant. Our vector has four times the magnitude of $\vec{u}$, so it is

   $$-4\vec{u} = -4\langle 1,2,3\rangle = \langle -4,-8,-12\rangle .$$

3. What are the two unit vectors parallel to $\vec{v}$?

   Solution. The two unit vectors parallel to $\vec{v}$ are $\pm\frac{\vec{v}}{|\vec{v}|}$. In our case, $|\vec{v}| = \sqrt{4^2+5^2+6^2} = \sqrt{16+25+36} = \sqrt{77}$, so $\frac{\vec{v}}{|\vec{v}|} = \left\langle \frac{4}{\sqrt{77}}, \frac{5}{\sqrt{77}}, \frac{6}{\sqrt{77}}\right\rangle$.

   The two solutions are $\left\langle \frac{4}{\sqrt{77}}, \frac{5}{\sqrt{77}}, \frac{6}{\sqrt{77}}\right\rangle$ and $\left\langle \frac{-4}{\sqrt{77}}, \frac{-5}{\sqrt{77}}, \frac{-6}{\sqrt{77}}\right\rangle$.

4. What are the two unit vectors parallel to $\vec{v}$ and with norm 7?

   Solution. They are $\pm 7\frac{\vec{v}}{|\vec{v}|}$. So, $\left\langle \frac{28}{\sqrt{77}}, \frac{35}{\sqrt{77}}, \frac{42}{\sqrt{77}}\right\rangle$ and $\left\langle \frac{-28}{\sqrt{77}}, \frac{-35}{\sqrt{77}}, \frac{-42}{\sqrt{77}}\right\rangle$.

Exercise 1.2. Three people located at the points $A(1,2), B(2,1)$ and $C(0,0)$ in the plane are pulling on ropes tied to a ring in position $P(1,1)$.

If C pulls with a force of $100lb$, find the magnitude of the forces with which A and B pull so that the ring stays in place (the system is in equilibrium).

Solution. Let the vectors $\vec{F_A}, \vec{F_B}, \vec{F_C}$ be the forces with which A, B, C pull. If the system is in equilibrium, that means that $\vec{F_A} + \vec{F_B} + \vec{F_C} = 0$. Moreover, $|\vec{F_C}| = 100$. And the forces $\vec{F_A}, \vec{F_B}$ and $\vec{F_C}$ have the same directions as the vectors $\overrightarrow{PA} = \langle 0,1\rangle, \overrightarrow{PB} = \langle 1,0\rangle$ and $\overrightarrow{PC} = \langle -1,-1\rangle$ respectively.

That is, for some constants $a,b,c \geq 0$ we have

$$\begin{aligned}
\vec{F_A} &= a\cdot\langle 0,1\rangle = \langle 0,a\rangle \\
\vec{F_B} &= b\cdot\langle 0,1\rangle = \langle b,0\rangle \\
\vec{F_C} &= c\langle -1,-1\rangle = \langle -c,-c\rangle .
\end{aligned}$$

We can use that $|\vec{F_C}| = 100$ to find c: $100 = \sqrt{(-c)^2+(-c)^2} = \sqrt{2c^2} = c\sqrt{2}$; $c = \frac{100}{\sqrt{2}} = 50\sqrt{2}$.

So $\vec{F_C} = \langle -50\sqrt{2}, -50\sqrt{2} \rangle$. The total force $\vec{F_A} + \vec{F_B} + \vec{F_C}$ now becomes

$$\langle 0, a \rangle + \langle b, 0 \rangle + \left\langle -50\sqrt{2}, -50\sqrt{2} \right\rangle = \left\langle b - 50\sqrt{2}, a - 50\sqrt{2} \right\rangle.$$

As $\vec{F_A} + \vec{F_B} + \vec{F_C} = 0$, we necessarily have $b - 50\sqrt{2} = a - 50\sqrt{2} = 0$, so $a = b = 50\sqrt{2}$. Thus, the forces $\vec{F_A} = \langle 0, 50\sqrt{2} \rangle$ and $\vec{F_B} = \langle 50\sqrt{2}, 0 \rangle$ both have magnitude $50\sqrt{2}$ lb.

Exercise 1.3. 1. What are the equations of the plane parallel to the xy-plane and passing through the midpoint between $A(3, 2, 1)$ and $B(3, 4, 5)$?

Solution. The midpoint between two points $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ will have coordinates $\left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}, \frac{a_3+b_3}{2}\right)$. In our case the midpoint is $\left(\frac{3+3}{2}, \frac{2+4}{2}, \frac{1+5}{2}\right) = (3, 3, 3)$.

Planes parallel to the xy-plane have equations of the form $z =$ constant. If the plane passes through the point $(3, 3, 3)$, the equation must be $z = 3$.

2. Give a geometric description of the set of points in $\mathbb{R}^3$ given by the equation $x^2 + y^2 + z^2 - 2x - 4y - 6z = 2$.

Solution. We can complete squares to turn our equation into the usual equation for a sphere:

$$\begin{gathered}(x^2 - 2x) + (y^2 - 4y) + (z^2 - 6z) = 2;\\ (x-1)^2 - 1 + (y-2)^2 - 4 + (z-3)^2 - 9 = 2;\\ (x-1)^2 + (y-2)^2 + (z-3)^3 = 2 + 1^2 + 2^2 + 3^2 = 16.\end{gathered}$$

So our geometric object is a sphere with center $\langle 1, 2, 3 \rangle$ and radius $\sqrt{16} = 4$.

Exercise 1.4. A submarine climbs at an angle of 30° above the horizontal with a heading to the northeast. If its speed is 20 knots, find the components of the velocity in the east, north, and vertical directions.

Solution. If we call the velocity $\vec{v}$, we can split it as a sum of the horizontal and vertical components $\vec{v} = \vec{v_v} + \vec{v_h}$. The vertical component has norm $20\sin(30°) = 20 \cdot \frac{1}{2} = 10$, so $\vec{v_v} = \langle 0, 0, 10 \rangle$.

As the submarine is moving in the northeast direction, the horizontal component of the velocity will have form $\vec{v_h} = \langle a, a, 0 \rangle$, for some constant $a \geq 0$. So the velocity vector is $\vec{v} = \vec{v_v} + \vec{v_h} = \langle a, a, 10 \rangle$. As $|\vec{v}| = 20$, we have $\sqrt{a^2 + a^2 + 10^2} = 20$; $2a^2 = 300$; $a = \sqrt{150} = 5\sqrt{6}$.

Thus, the submarine is moving at $5\sqrt{6}$ knots in the east and north directions, and at 10 knots in the vertical direction.

2 Dot product

1. Dot product: $\vec{u} \bullet \vec{v} = |\vec{u}| \cdot |\vec{v}| \cdot \cos(\alpha) = \sum_i u_i v_i$.
2. Some properties:

- $\vec{u} \bullet (\vec{v} + \vec{w}) = \vec{u} \bullet \vec{v} + \vec{u} \bullet \vec{w}$
- $\vec{u} \bullet \vec{u} = |\vec{u}|^2$
- $\vec{u} \bullet \vec{v} = \vec{v} \bullet \vec{u}$, $(a\vec{u}) \bullet (b\vec{v}) = ab(\vec{u} \bullet \vec{v})$
- $\vec{u}, \vec{v}$ are orthogonal iff $\vec{u} \bullet \vec{v} = 0$

Exercise 2.1. A triangle has vertices $P(1,2), Q(3,4), R(5,0)$. Find the angle at Q.

Solution. The angle α at Q is the angle between the vectors $\overrightarrow{QP} = \langle -2, -2 \rangle$ and $\overrightarrow{QR} = \langle 2, -4 \rangle$. Thus,

$$\cos(\alpha) = \frac{\overrightarrow{QP} \bullet \overrightarrow{QR}}{|\overrightarrow{QP}| \cdot |\overrightarrow{QR}|} = \frac{-4+8}{(\sqrt{2^2+2^2}) \cdot \sqrt{4^2+2^2}} = \frac{4}{4\sqrt{10}} = \frac{1}{\sqrt{10}}.$$

Thus, $\alpha = \arccos\left(\frac{1}{\sqrt{10}}\right)$ (approximately $71.57°$).

3. Work produced by constant force $\vec{F}$ with displacement $\vec{d}$: $W = \vec{F} \bullet \vec{d}$.

   Exercise 2.2. Gravity pulls down a ball with a force of 10N. The ball rolls down a straight ramp, covering a total vertical height of 10 meters. What is the total work done by gravity?

   Solution. We will represent this problem in 2D.

   The force vector has a magnitude of 10 and points down, so it is $\vec{F} = \langle 0, -10 \rangle$.

   The vertical displacement is -10m. We do not know how steep the ramp is, so we will leave the horizontal displacement as an unknown a. Thus, the displacement vector is $\vec{d} = \langle a, -10 \rangle$.

   Thus, the total work done is $\vec{F} \bullet \vec{d} = \langle 0, -10 \rangle \bullet \langle a, -10 \rangle = 100J$.

4. Orthogonal projection of $\vec{u}$ onto $\vec{v}$: $\text{proj}_{\vec{v}}\vec{u} = \text{scal}_{\vec{v}}\vec{u} \cdot \dfrac{\vec{v}}{|\vec{v}|} = \dfrac{\vec{u} \bullet \vec{v}}{\vec{v} \bullet \vec{v}}\vec{v}$.

5. Scalar component of $\vec{u}$ in the direction of $\vec{v}$: $\text{scal}_{\vec{v}}\vec{u} = |\vec{u}| \cos(\alpha) = \dfrac{\vec{u} \bullet \vec{v}}{|\vec{v}|}$

6. Decomposition of $\vec{u}$ into parts parallel and orthogonal to a vector $\vec{v}$: $\vec{u} = \text{proj}_{\vec{v}}\vec{u} + (\vec{u} - \text{proj}_{\vec{v}}\vec{u})$

   Exercise 2.3. Decompose $\vec{u} = \langle 3, 4, 5 \rangle$ into parts parallel and orthogonal to $\vec{v} = \langle 0, 1, 2 \rangle$. What is the scalar component of $\vec{u}$ in the direction of $\vec{v}$?

   Solution. The part of $\vec{u}$ parallel to $\vec{v}$ is $\text{proj}_{\vec{v}}\vec{u} = \frac{\vec{u} \bullet \vec{v}}{\vec{v} \bullet \vec{v}}\vec{v} = \frac{14}{5}\langle 0, 1, 2 \rangle = \left\langle 0, \frac{14}{5}, \frac{28}{5} \right\rangle$.

   The part of $\vec{u}$ perpendicular to $\vec{v}$ is $\vec{u} - \text{proj}_{\vec{v}}\vec{u} = \langle 3, 4, 5 \rangle - \left\langle 0, \frac{14}{5}, \frac{28}{5} \right\rangle = \left\langle 3, \frac{6}{5}, \frac{-3}{5} \right\rangle$.

   The scalar component of $\vec{u}$ in the direction of $\vec{v}$ is $\frac{\vec{u} \bullet \vec{v}}{|v|} = \frac{14}{\sqrt{5}}$.

3 Cross product

1. The cross product $\vec{u} \times \vec{v}$ is a vector with magnitude $|\vec{u}| \cdot |\vec{v}| \cdot \sin(\alpha)$ and is orthogonal to $\vec{u}$ and $\vec{v}$, with direction given by the right hand rule.
2. $|\vec{u} \times \vec{v}|$ is the area of the parallelogram formed by $\vec{u}$ and $\vec{v}$.
3. Some properties:
   - $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$.
   - $(a\vec{u}) \times (b\vec{v}) = ab \cdot \vec{u} \times \vec{v}$
   - $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
   - $(\vec{v} + \vec{w}) \times \vec{u} = \vec{v} \times \vec{u} + \vec{w} \times \vec{u}$

   Exercise 3.1. Three points $P, Q, R \in \mathbb{R}^3$ satisfy $\overrightarrow{PQ} \times \overrightarrow{PR} = 0$. What is $\overrightarrow{QP} \times \overrightarrow{QR}$?

   If instead we have $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle 1, 0, 0 \rangle$, what is $\overrightarrow{QP} \times \overrightarrow{QR}$?

   Solution. If $\overrightarrow{PQ} \times \overrightarrow{PR} = 0$ that means that the points P, Q, R are collinear. Thus, we also have $\overrightarrow{QP} \times \overrightarrow{QR} = 0$.

   For the second part, where $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle 1, 0, 0 \rangle$, we may use the properties of cross product:

   $$\overrightarrow{QP} \times \overrightarrow{QR} = (-\overrightarrow{PQ}) \times (\overrightarrow{PR} - \overrightarrow{PQ}) = -\overrightarrow{PQ} \times \overrightarrow{PQ} + \overrightarrow{PQ} \times \overrightarrow{PQ} = -\langle 1, 0, 0 \rangle + 0 = \langle -1, 0, 0 \rangle .$$

4. Coordinate expression of cross product: $\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$.

   Exercise 3.2. Find a vector orthogonal to $\vec{u} = \langle 0, 1, 2 \rangle$ and $\vec{v} = \langle -2, -1, 0 \rangle$.

   Solution. A fast solution is to just take the cross product of $\vec{u}$ and $\vec{v}$:

   $$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 2 \\ -2 & -1 & 0 \end{vmatrix} = 2\vec{i} - 4\vec{j} + 2\vec{k} = \langle 2, -4, 2 \rangle .$$

5. Torque produced by a force $\vec{F}$: $\vec{\tau} = \vec{r} \times \vec{F}$, where $\vec{r}$ is the vector from the axis of rotation to the point where the force is applied.

   Exercise 3.3. [Exercise 52, online textbook] A person holds their left arm (which is 2 ft long) outstretched horizontally, holding a 20 lb weight in their hand. Find the magnitude and direction of the torque that the weight exerts on the shoulder.

   Solution. We can use the definition of cross product, but now I will just use coordinates.

   Suppose the arm starts at the origin and ends at the point $(0, 2, 0)$, so $\vec{r} = \langle 0, 2, 0 \rangle$.

The force points down with a magnitude of 20 lbs, so $\vec{F} = \langle 0, 0, -20 \rangle$. Thus,

$$\vec{\tau} = \vec{r} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 2 & 0 \\ 0 & 0 & -20 \end{vmatrix} = -40\vec{i}.$$

So the torque points towards the back of the person, and has a magnitude of 40 lbs·ft.

4 Lines and planes in $\mathbb{R}^3$

1. Line with direction $\vec{v} = \langle a, b, c \rangle$ containing the vector $\vec{r_0} = \langle x_0, y_0, z_0 \rangle$:

$$\vec{r} = \vec{r_0} + \vec{v}t = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle, t \in \mathbb{R}.$$

2. Distance from a point Q to a line $\vec{r} = \vec{r_0} + t\vec{v}$ passing through a point P:

$$d = \frac{\left|\vec{v} \times \overrightarrow{PQ}\right|}{|\vec{v}|}.$$

Exercise 4.1. Given the points $P(0, 1, 2), Q(3, 4, 5)$ and $R(1, 3, 5)$, find the line l passing through P and parallel to $\overrightarrow{QR}$. What is the distance from Q to l?

Solution. The line passes through $P(0, 1, 2)$ and has direction $\vec{v} = \overrightarrow{QR} = \langle -2, -1, 0 \rangle$, so it is

$$\vec{r} = \langle 0 - 2t, 1 - t, 2 \rangle = \langle -2t, 1 - t, 2 \rangle, t \in \mathbb{R}.$$

The distance from Q to the line l is, using the formula above,

$$d = \frac{\left|\vec{v} \times \overrightarrow{PQ}\right|}{|\vec{v}|} = \frac{|\langle -2, -1, 0 \rangle \times \langle 3, 3, 3 \rangle|}{|\langle -2, -1, 0 \rangle|} = \frac{|\langle -3, 6, -3 \rangle|}{|\langle -2, -1, 0 \rangle|} = \frac{\sqrt{54}}{\sqrt{5}}.$$

3. Equation of a plane passing through $P(x_0, y_0, z_0)$ orthogonal to $\vec{n} = \langle a, b, c \rangle$:

$$ax + by + cz = ax_0 + by_0 + cz_0.$$

4. Two planes are orthogonal iff their normal vectors are orthogonal.

5. Two planes are parallel iff their normal vectors are parallel.

Exercise 4.2. Find the equation of the plane which:

- Is orthogonal to the plane $x + y + z = 2$.
- Is parallel to the line $\vec{r} = \langle 4 + 3t, 5 + 2t, 6 + t \rangle$.
- Contains the point $P(1, 2, 3)$.

Solution. In order to find the equation of the plane, we first find a vector $\vec{n}$ normal to it.

Our plane being orthogonal to $x+y+z=2$ means that $\vec{n}$ is orthogonal to $\langle 1,1,1\rangle$. And the plane being parallel to the line $\vec{r}=\langle 4+3t, 5+2t, 6+t\rangle$ means that $\vec{n}$ is orthogonal to the direction vector $\langle 3,2,1\rangle$. Thus, we can take $\vec{n}$ to be the cross-product $\langle 1,1,1\rangle \times \langle 3,2,1\rangle = \langle -1,2,-1\rangle$.

The plane with normal vector $\vec{n}=\langle -1,2,-1\rangle$ and passing through $P(1,2,3)$ will have equation

$$-x+2y-z=-1+2\cdot 2-3=0.$$

5 Cylinders & quadric surfaces

1. Cylinder: Surface in $\mathbb{R}^3$ whose equation depends on two variables.

2. Trace of a surface in a plane is the intersection between the surface and the plane.

   Exercise 5.1. Find the trace of the cylinder $x^2+4y^2=4$ in the xy plane. Sketch said cylinder.

   Solution. The given cylinder is parallel to the z axis, as the variable z does not appear in the equation of the cylinder.

   The trace is just the intersection of $x^2+4y^2=4$ with the xy plane ($z=0$). So it is just the ellipse with equation $x^2+4y^2=4$ within the plane $z=0$. This ellipse has a major axis of length 4 (in the x-axis) and minor axis of length 2 (in the y-axis).

   (Sketch of the cylinder given in the recitation, as the rest of the drawings)

3. Quadric surface in $\mathbb{R}^3$: Its equation is of the form

$$Ax^2+By^2+Cz^2+Dxy+Exz+Fyz+Gx+Hy+Iz+J=0.$$

4. Classification: Not counting degenerate quadrics (e.g. cylinders, planes, pairs of planes, lines, points), we have studied 6 types of quadrics in the lectures.

   Exercise 5.2. Find the traces of the following surfaces in the xy, xz, yz planes, and identify the surfaces.

   - $x^2-\frac{y^2}{4}+z^2=0.$

     Solution. Traces:

     - $x=0$: $z^2-\frac{y^2}{4}=0$, pair of lines $y=\pm 2z$.
     - $y=0$: $x^2+z^2=0$; just the point 0.
     - $z=0$: $x^2-\frac{y^2}{4}=0$; pair of lines $y=\pm 2x$.

     This surface is a cone centered around the y-axis.

- $x = y^2 + \frac{z^2}{3}$.

  Solution.
  - $x = 0$: $y^2 + \frac{z^2}{3} = 0$, just the point 0.
  - $y = 0$: $x = \frac{z^2}{3}$ a parabola.
  - $z = 0$: $x^2 = y^2$ another parabola.

  This surface is an elliptic paraboloid centered around the x-axis (note that the section with planes of the form $x = c$, $c > 0$, are ellipses).

6 Vector-valued functions

1. Vector valued function: $\vec{F}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$.

   Exercise 6.1. Find all points in which the curve $\vec{F}(t) = \langle 3t, t^2 - 2, 4t - 1\rangle$ intersects the plane $x = 2y + 3z$.

   Solution. We can just substitute the coordinates $3t, t^2-2$ and $4t-1$ into the equation $x = 2y + 3z$, obtaining:

   $$3t = 2(t^2 - 2) + 3(4t - 1); 0 = 2t^2 + 9t - 7; t = \frac{-9 \pm \sqrt{9^2 + 4 \cdot 7 \cdot 2}}{4} = \frac{-9 \pm \sqrt{137}}{4}.$$

   So the curve intersects the plane two times, at $t = \frac{-9+\sqrt{137}}{4}$ and $t = \frac{-9-\sqrt{137}}{4}$. The points of intersection are thus $\left(3\left(\frac{-9+\sqrt{137}}{4}\right), \left(\frac{-9+\sqrt{137}}{4}\right)^2 - 2, 4\left(\frac{-9+\sqrt{137}}{4}\right) - 1\right)$ and $\left(3\left(\frac{-9-\sqrt{137}}{4}\right), \left(\frac{-9-\sqrt{137}}{4}\right)^2 - 2, 4\left(\frac{-9-\sqrt{137}}{4}\right) - 1\right)$.

2. Domain of a vector-valued function: $\mathrm{dom}(\vec{F}) = \mathrm{dom}(f) \cap \mathrm{dom}(g) \cap \mathrm{dom}(h)$.

3. Limits: $\lim_{t\to t_0} \vec{F}(t) = \langle \lim_{t\to t_0} f(t), \lim_{t\to t_0} g(t), \lim_{t\to t_0} h(t)\rangle$.

4. $\vec{F}$ is continuous at t_0 iff $\lim_{t\to t_0} F(t) = F(t_0)$ iff f, g, h are continuous at t_0.

   Exercise 6.2. Let $\vec{F}(t) = \left\langle \ln(t + 100), t + 10, \frac{1}{t+1}\right\rangle$. Find the domain of $\vec{F}$ and $\lim_{t\to 10} \vec{F}(t)$.

   Solution. For $\ln(t + 100)$ and $\frac{1}{t+1}$ to be defined, we need $t > -100$ and $t \neq -1$. So the domain is $(-100, -1) \cup (-1, \infty)$.

   $$\lim_{t\to 10} \vec{F}(t) = \left\langle \lim_{t\to 10} \ln(t + 100), \lim_{t\to 10} t + 10, \lim_{t\to 10} \frac{1}{t + 1}\right\rangle = \left\langle \ln(110), 20, \frac{1}{11}\right\rangle.$$

   Exercise 6.3. Represent the curve $\vec{F}(t) = \langle \sin(t), t, -\sin(t)\rangle$ and indicate its orientation.

   Solution. The curve is contained in the plane $x = -z$, and also satisfies $x = \sin(y)$, $z = -\sin(y)$. The curve looks like the graph of the function $y = \sin(x)$, but tilted so that it is inside the plane $z = -x$. The curve points towards the positive y direction.

7 Calculus of vector-valued functions

1. Derivative of $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$:
$$\vec{r}\,'(t) = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} = \langle f'(t), g'(t), h'(t) \rangle .$$

2. Differentiation rules:
   (a) $\frac{d}{dt}\vec{C} = 0$ if $\vec{C}$ is constant.
   (b) $(\vec{u} + \vec{v})'(t) = \vec{u}\,'(t) + \vec{v}\,'(t)$.
   (c) $(f(t) \cdot \vec{u}(t))' = f'(t)\vec{u}(t) + f(t)\vec{u}\,'(t)$.
   (d) $\frac{d}{dt}\vec{u}(f(t))' = \vec{u}\,'(f(t)) \cdot f'(t)$.
   (e) $(\vec{u} \bullet \vec{v})'(t) = \vec{u}\,'(t) \bullet \vec{v}(t) + \vec{u}(t) \bullet \vec{v}\,'(t)$.
   (f) $(\vec{u} \times \vec{v})'(t) = \vec{u}\,'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}\,'(t)$.

3. Tangent vector to $\vec{r}(t)$ at time t: $\vec{r}\,'(t)$ (when $\vec{r}(t) \neq 0$, i.e. the curve is *smooth* at t).
   Unit tangent vector: $\frac{\vec{r}(t)}{|\vec{r}(t)|}$.
   Exercise 7.1. Let $\vec{r}(t) = \langle 1, t, t^2 \rangle$. Find the derivative of $\vec{r}(t) \times \vec{r}\,'(t)$ at $t = 1$.
   Solution. Using the product rule, we have
$$(\vec{r}(t) \times \vec{r}\,'(t))' = \vec{r}\,'(t) \times \vec{r}\,'(t) + \vec{r}(t) \times \vec{r}\,''(t) = \vec{r}(t) \times \vec{r}\,''(t).$$
   We have $\vec{r}\,'(t) = \langle 0, 1, 2t \rangle$ and $\vec{r}\,''(t) = \langle 0, 0, 2 \rangle$, so
$$\vec{r}(t) \times \vec{r}\,''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & t & t^2 \\ 0 & 0 & 2 \end{vmatrix} = 2t\,\vec{i} - 2\vec{j}.$$
   Alternatively, we can first compute $\vec{r}(t) \times \vec{r}\,'(t)$ and then take the derivative of that.

4. Indefinite integral of $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$:
$$\int \langle f(t), g(t), h(t) \rangle \, dt = \left\langle \int f(t)dt, \int g(t)dt, \int h(t)dt \right\rangle + \vec{C} \text{ (where } \vec{C} \in \mathbb{R}^3 \text{ is constant)}$$

5. Definite integrals: $\displaystyle\int_a^b \vec{r}(t)dt = \left\langle \int_a^b f(t)dt, \int_a^b g(t)dt, \int_a^b h(t)dt \right\rangle$.
   Exercise 7.2. Find $\displaystyle\int_0^1 2te^{t^2} \cdot \langle 0, 1, 2 \rangle \, dt$.
   Solution. We first compute the indefinite integral, which we will call $\vec{F}(t)$:
$$\vec{F}(t) = \int 2te^{t^2} \cdot \langle 0, 1, 2 \rangle \, dt = \int \left\langle 0, 2te^{t^2}, 4te^{t^2} \right\rangle dt = \left\langle \int 0dt, \int 2te^{t^2} dt, \int 4te^{t^2} dt \right\rangle$$
$$= \left\langle 0, e^{t^2}, 2e^{t^2} \right\rangle + \vec{C}.$$

Thus, the definite integral will be

$$\int_0^1 2te^{t^2} \cdot \langle 0, 1, 2 \rangle \, dt = \vec{F}(1) - \vec{F}(0) = \langle 0, e, 2e \rangle - \langle 0, 1, 2 \rangle = \langle 0, e-1, 2(e-1) \rangle .$$

8 Motion in space

1. An object has position $\vec{r}(t)$.

   Velocity: $\vec{v}(t) = \vec{r}\,'(t)$.

   Speed: $|\vec{v}(t)| = |\vec{r}\,'(t)|$.

   Acceleration: $\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$.

Exercise 8.1. A child standing on the ground throws a ball, with initial velocity $\vec{v}_0 = \langle 10, 10 \rangle \, \frac{\mathrm{m}}{\mathrm{s}}$. The only force acting on the ball is gravity, which exerts a constant downward acceleration of $10 \frac{\mathrm{m}}{\mathrm{s}^2}$. Find the position function $\vec{r}(t)$ of the ball and the maximum height it reaches.

Solution. We will assume that the initial position of the object is $\langle 0, 0 \rangle$.

The acceleration vector is $\vec{a}(t) = \langle 0, -10 \rangle$. Velocity is an anti-derivative of acceleration, so it will have the form

$$\vec{v}(t) = \int \vec{a}(t) dt = \int \langle 0, -10 \rangle \, dt = \langle 0, -10t \rangle + \vec{C},$$

for some constant $\vec{C}$. We can use that $\vec{v}(0) = \langle 10, 10 \rangle$ to find the constant $\vec{C}$: $\langle 10, 10 \rangle = \langle 0, -10 \cdot 0 \rangle + \vec{C}$, so $\vec{C} = \langle 10, 10 \rangle$. So $\vec{v}(t) = \langle 10, 10 - 10t \rangle$. We integrate again to find $\vec{r}(t)$:

$$\vec{r}(t) = \int \vec{v}(t) dt = \int \langle 10, 10 - 10t \rangle \, dt = \left\langle 10t, 10t - 5t^2 \right\rangle + \vec{C}.$$

As the initial position is $\langle 0, 0 \rangle$, we have $\vec{C} = \langle 0, 0 \rangle$, so $\vec{r}(t) = \langle 10t, 10t - 5t^2 \rangle$.

Let us find the maximum height now. The height at time t is $h(t) = 10t - 5t^2$. We have $h'(t) = 10 - 10t$, so $h'(t) = 0 \implies t = 1$, and indeed $h(t)$ is increasing for $t < 1$ and decreasing for $t > 1$, so the maximum height will be is $h(1) = 10 - 5 = 5$.

(Equivalently, we can use that when the object reaches its maximum height, the vertical component of the velocity must be 0).

9 Arc length

1. Arc length of the curve $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ from time $t = a$ to b:

$$\int_a^b |\vec{v}(t)|dt = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}dt.$$

2. Arc length function starting at $t = a$: $s(t) = \int_a^t |\vec{v}(s)|ds$.

   If $|\vec{v}(t)| = 1$ for all t, then the parameter t corresponds to arc length.

Exercise 9.1. Consider the helix $\vec{r}(t) = \langle \cos(t), \sin(t), t\rangle$. Find the length of the curve between the points $(1, 0, 0)$ and $(1, 0, 2\pi)$. Find a description of the curve that uses arc length as a parameter.

Solution. The points $(1, 0, 0)$ and $(1, 0, 2\pi)$ correspond to times $t = 0$, and $t = 2\pi$ respectively (as the third coordinate of $\vec{r}(t)$ is the time t). So the length of the curve is

$$\begin{aligned}\int_0^{2\pi} |\vec{v}(t)|dt &= \int_0^{2\pi} |\langle -\sin(t), \cos(t), 1\rangle|dt = \int_0^{2\pi} |\langle -\sin(t), \cos(t), 1\rangle|dt \\ &= \int_0^{2\pi} \sqrt{\sin(t)^2 + \cos(t)^2 + 1}dt = \int_0^{2\pi} \sqrt{2}dt = 2\pi\sqrt{2}.\end{aligned}$$

Let us now find a description of the curve using arc length. The arc length function (starting at $t = 0$) is

$$s(t) = \int_0^t |\vec{v}(t)|dt = \int_0^t \sqrt{2}dt = t\sqrt{2}.$$

To express the curve in terms of the arclength s, we use that $t = \frac{s}{\sqrt{2}}$, so thinking of t as a function $t(s)$ of arclength, we have

$$\vec{r}(s) = \langle \cos(t(s)), \sin(t(s)), t(s)\rangle = \left\langle \cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}}\right\rangle.$$

10 Curvature and normal vectors

1. Curvature of a curve $\vec{r}(t)$ with arclength s and **unit** tangent vector $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$:

$$\kappa(t) = \left|\frac{d\vec{T}}{ds}\right| = \frac{\left|d\vec{T}/dt\right|}{|ds/dt|} = \frac{1}{|\vec{v}|}\cdot\left|\frac{d\vec{T}}{dt}\right|.$$

Exercise 10.1. Compute the unit tangent vector $\vec{T}(t)$ and the curvature $\kappa(t)$ of the parabola $\vec{r}(t) = \langle t, t^2\rangle$ at $t = 1$.

Solution. We have $\vec{r}\,'(t) = \langle 1, 2t\rangle$, so $\vec{T}(t) = \frac{\langle 1,2t\rangle}{\sqrt{1^2+(2t)^2}} = \left\langle \frac{1}{\sqrt{1+4t^2}}, \frac{2t}{\sqrt{1+4t^2}}\right\rangle$.

The curvature will be

$$\begin{aligned}
\kappa(t) = \frac{1}{|\vec{v}|}\cdot\left|\frac{d\vec{T}}{dt}\right| &= \frac{1}{\sqrt{1+4t^2}}\left|\left\langle \frac{-1}{2}(1+4t^2)^{-3/2}\cdot 8t, \frac{2\sqrt{1+4t^2} - 2t\frac{8t}{2\sqrt{1+4t^2}}}{1+4t^2}\right\rangle\right| \\
&= \left|\left\langle \frac{-4t}{(1+4t^2)^2}, \frac{2(1+4t^2)-8t^2}{(1+4t^2)^2}\right\rangle\right| \\
&= \frac{1}{(1+4t^2)^2}|\langle -4t, 2\rangle| = \frac{\sqrt{16t^2+4}}{(1+4t^2)^2} = \frac{2\sqrt{1+4t^2}}{(1+4t^2)^2} = \frac{2}{(1+4t^2)^{3/2}}.
\end{aligned}$$

2. Alternate formula for curvature in $\mathbb{R}^3$: if $\vec{v}(t) = \vec{r}\,'(t)$ and $\vec{a}(t) = \vec{v}\,'(t)$,

$$\kappa(t) = \frac{|\vec{v}\times\vec{a}|}{|\vec{v}|^3}.$$

Exercise 10.2. Using the alternate formula, compute the curvature of the parabola $\vec{r}(t) = \langle t, t^2, 0\rangle$ at $t = 1$.

Solution. Notice that the parabola is essentially the same as in the previous exercise. But this computation will be easier!

We have $\vec{v}(t) = \langle 1, 2t, 0\rangle$ and $\vec{a}(t) = \langle 0, 2, 0\rangle$. So $\vec{v}(t)\times\vec{a}(t) = \langle 0, 0, 2\rangle$. Thus,

$$\kappa(t) = \frac{|\vec{v}\times\vec{a}|}{|v|^3}\frac{2}{\sqrt{1+4t^2}^3} = \frac{2}{(1+4t^2)^{3/2}}.$$

3. Principal unit normal vector (when $\kappa \neq 0$): $\vec{N}(s) = \dfrac{d\vec{T}/ds}{\left|d\vec{T}/ds\right|} = \dfrac{1}{\kappa}\dfrac{d\vec{T}}{ds}$.

If t is not arclength, $\vec{N}(t) = \dfrac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$.

4. Tangential/normal components of acceleration:

$$\vec{a}(t) = a_N\vec{N} + a_T\vec{T} = a_N(t)\vec{N}(t) + a_T(t)\vec{T}(t),$$

where

- $a_N = \vec{a} \cdot \vec{N} \left(= \kappa|\vec{v}|^2 = \frac{|\vec{v}\times\vec{a}|}{\vec{v}}\right)$.
- $a_T = \vec{a} \cdot \vec{T} \left(= \frac{d^2s}{dt^2}\right)$.

Exercise 10.3. Find the curvature and the tangential and normal accelerations of an object which rotates faster and faster, with motion $\vec{r}(t) = \langle\cos(t^2), \sin(t^2)\rangle$ for $t > 0$.

Solution. We first find the tangential and normal components of acceleration; we will find the curvature at the end.

The unit tangent and normal vectors have simple expressions for this curve:

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{\langle -2t\sin(t^2), 2t\cos(t^2)\rangle}{\sqrt{(2t\sin(t^2))^2 + (2t\cos(t^2))^2}} = \frac{\langle -2t\sin(t^2), 2t\cos(t^2)\rangle}{2t} = \left\langle -\sin(t^2), \cos(t^2)\right\rangle.$$

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\left|\vec{T}'(t)\right|} = \frac{\langle -2t\cos(t^2), -2t\sin(t^2)\rangle}{2t} = \left\langle -\cos(t^2), -\sin(t^2)\right\rangle.$$

We now compute the acceleration vector:

$$\vec{v}(t) = \left\langle -2t\sin(t^2), 2t\cos(t^2)\right\rangle = 2t\left\langle -\sin(t^2), \cos(t^2)\right\rangle.$$

$$\begin{aligned}\vec{a}(t) = \vec{v}\,'(t) &= 2\left\langle -\sin(t^2), \cos(t^2)\right\rangle + 2t\left\langle -2t\cos(t^2), -2t\sin(t^2)\right\rangle \\ &= 2\left\langle -\sin(t^2), \cos(t^2)\right\rangle + 4t^2\left\langle -\cos(t^2), -\sin(t^2)\right\rangle.\end{aligned}$$

In this case, just from our computation we see that $\vec{a} = 2\vec{T} + 4t^2\vec{N}$, so

- Tangential component of acceleration: $2\vec{T} = 2\langle -\sin(t^2), \cos(t^2)\rangle$.
- Normal component of acceleration: $4t^2\vec{N} = 4t^2\langle -\cos(t^2), -\sin(t^2)\rangle$.

The curvature is $\dfrac{\left|\vec{T}'(t)\right|}{|\vec{v}(t)|} = \dfrac{|\langle -2t\cos(t^2), -2t\sin(t^2)\rangle|}{|\langle -2t\sin(t^2), 2t\cos(t^2)\rangle|} = \dfrac{2t}{2t} = 1$. This makes sense because our point moves inside the unit circle, which has curvature 1.

11 Graphs and level curves

1. Domain of a function $z = f(x, y)$: set of points (x, y) in the plane such that $f(x, y)$ is defined.

   Exercise 11.1. Find the domain of the following functions $f(x, y) = \ln(x + y) - x^2 + \frac{1}{x+2}$.

   (a) $f(x, y) = x^2 - 4y + 7$.
   Solution. The domain is the entire plane, $\mathbb{R}^2$, as f is a polynomial.

   (b) $f(x, y) = \frac{1}{x+y}$.
   Solution. $f(x, y)$ is defined iff $x + y \neq 0$. So the domain is the entire plane, except for the line $x + y = 0$.

   (c) $f(x, y) = \frac{1}{xy} + \sqrt{1 - x^2 - y^2}$.
   Solution. $f(x, y)$ will be defined when $xy \neq 0$ (that is, the coordinate axis are not in the domain) and $1 - x^2 - y^2 \geq 0$, so $x^2 + y^2 \leq 1$.
   That is, the domain is obtained from removing the coordinate axes from the ball centered at the origin with radius 1 (so the domain is divided into 4 regions).

2. Contour curves of $f(x, y)$: intersection of the surface $z = f(x, y)$ with a horizontal plane.

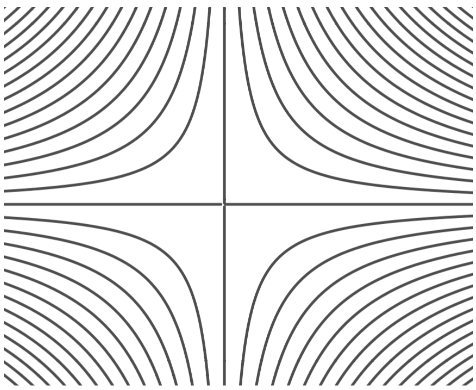
3. Level curves of $f(x, y)$: curves of the form $f(x, y) = C$, for some constant C / projection of contour curves to the xy plane.

   Exercise 11.2. Draw the level curves of $f(x, y) = x \cdot y$.

   Solution. The level curves are of the form $xy = C$, for some constant C.

   - If $C > 0$, then $xy = C$ is a hyperbola contained in the second and third quadrants of the coordinate plane.
   - If $C < 0$, then $xy = C$ is a hyperbola contained in the first/fourth quadrants of the coordinate plane.
   - If $C = 0$, the curve $xy = 0$ is the union of the two coordinate axes.

   Here is a picture with many of the level curves:



12 Limits and continuity

Let $f(x,y)$ be a two-variable function. We can define the limit $\lim_{(x,y)\to P} f(x,y)$ when P is a point in the plane, and there are points of $\mathrm{dom}(f)$ as close to P as we want:

1. Mathematical definition: We say $\lim_{(x,y)\to P} f(x,y) = L$ if for all $\varepsilon > 0$ there is $\delta > 0$ such that, for all points Q in $\mathrm{dom}(f)$ at distance $< \delta$ of P and not equal to P, we have $|f(Q) - L| < \varepsilon$.

2. Intuition: $\lim_{(x,y)\to P} f(x,y) = L$ when the value of $f(x,y)$ gets arbitrarily close to L as (x,y) gets closer and closer to P along any continuous curve.

3. Limit laws (see lecture notes) are useful when computing limits.

4. Two paths test: If $f(x,y)$ approaches two different values along two paths as (x,y) approaches P, then $\lim_{(x,y)\to P} f(x,y)$ does not exist. Note: this test only allows us to prove that a limit does not exist. We cannot use it to prove that a limit exists.

   Exercise 12.1. Do the following limits exist? If so, give their value.

   (a) $\lim_{(x,y)\to(3,5)} \ln(3x+y^2)$.

   Solution. As the function $\ln(3x+y^2)$ is continuous near the point $(3,5)$, we can just substitute:

   $$\lim_{(x,y)\to(3,5)} \ln(3x+y^2) = \ln(3\cdot 3 + \cdot 5^2) = \ln(9+25) = \ln(34).$$

   (b) $\lim_{(x,y)\to(5,1)} \frac{\sqrt{x-y}-2}{x-y-4}$.

   Solution. Note that $\lim_{(x,y)\to(5,1)} \sqrt{x-y}-2 = \lim_{(x,y)\to(5,1)} x-y-4 = 0$, so we have a 'limit of form $\frac{0}{0}$' (we cannot use the quotient rule). We compute:

   $$\begin{aligned}\lim_{(x,y)\to(5,1)} \frac{\sqrt{x-y}-2}{x-y-4} &= \lim_{(x,y)\to(5,1)} \frac{\sqrt{x-y}-2}{(\sqrt{x-y}+2)(\sqrt{x-y}-2)} \\ &= \lim_{(x,y)\to(5,1)} \frac{1}{\sqrt{x-y}+2} \\ &= \frac{1}{\sqrt{5-1}+2} = \frac{1}{4}.\end{aligned}$$

   We were able to substitute $(x,y) = (5,1)$ into the last limit because the function $\frac{1}{\sqrt{x-y}+2}$ is continuous in the point $(5,1)$.

   (c) $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$.

   Solution. We can use the two paths test to check that the function is not continuous. We compute the limit when $(x,y)\to 0$ of $\frac{xy}{x^2+y^2}$ along two lines:

   - $y = 0$; $\lim_{x\to 0} \frac{x0}{x^2+0^2} = \lim_{x\to 0} \frac{0}{x^2} = 0$.
   - $y = x$; $\lim_{x\to 0} \frac{xx}{x^2+x^2} = \frac{1}{2}$.

As the function approaches different values when (x, y) goes to 0 along the lines $y = 0$ and $y = x$, the limit at $(0, 0)$ does not exist.

(d) (Bonus/More difficult than required for this course) $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}$.

Solution. The limit exists and is 0. To prove it, note that for all x, y, we have $-(x^2 + y^2) \leq 2xy \leq x^2 + y^2$ (this follows from the fact that $(x + y)^2$ and $(x - y)^2$ are ≥ 0). So for all x, y (not both equal to 0) we always have $\left|\frac{xy}{x^2+y^2}\right| \leq \frac{1}{2}$.

Thus, for all $\varepsilon > 0$, whenever the point (x, y) is at distance $< \varepsilon$ of 0 (in particular $|x|, |y| < \varepsilon$), we have

$$\left|\frac{x^2y^2}{x^2 + y^2}\right| = |x| \cdot |y| \cdot \left|\frac{xy}{x^2 + y^2}\right| < \varepsilon^2 \cdot \frac{1}{2}.$$

We can make the value $\varepsilon^2 \cdot \frac{1}{2}$ as small as we want by letting $\varepsilon \to 0$. So we can make the value of $\frac{x^2y^2}{x^2+y^2}$ arbitrarily close to 0 by making ε be small (that is, by making x, y be in a small ball centered at 0). This proves that $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} = 0$.

5. We say a point P in $\mathbb{R}^2$ (or $\mathbb{R}, \mathbb{R}^3$) is in the boundary of a region R if every open disk centered at P intersects R but is not contained in R.

6. A set O in $\mathbb{R}^2$ (or $\mathbb{R}, \mathbb{R}^3$) is open if it does not contain any of its boundary points.

   A set C in $\mathbb{R}^2$ (or $\mathbb{R}, \mathbb{R}^3$) is closed if it contains all its boundary points.

   Exercise 12.2. Find the boundary of the following sets. Determine whether they are open, closed, both, or neither.

   (a) The interval $(0, 1)$ in $\mathbb{R}$.

   Solution. The boundary of the interval are the points 0 and 1. They are not in the interval, so the interval is open (and it is not closed).

   (b) The square $[0, 1]^2$ in $\mathbb{R}^2$.

   Solution. The boundary of the set $[0, 1]^2$ is a square of side 1, contained in the set $[0, 1]^2$. So the set $[0, 1]^2$ is closed (and it is not open).

   (c) (Bonus) The entire space $\mathbb{R}^3$.

   Solution. There are no points in the boundary of this set (the boundary is empty). The set is both open and closed.

   (d) (Bonus) The set of rational numbers $\mathbb{Q}$ in $\mathbb{R}$.

   Solution. The boundary of this set is the entire set of real numbers, $\mathbb{R}$. It is neither open nor closed.

7. A function $f(x, y)$ is continuous at P if the two values $f(P)$ and $\lim_{(x,y)\to P} f(x, y)$ exist and $f(P) = \lim_{(x,y)\to P} f(x, y)$.

8. Polynomials/rational functions of several variables are continuous in their domains.
9. If f, g are continuous at P, then $f \pm g$, fg are continuous at P, and so is f/g if $g(P) \neq 0$.
10. If f is continuous at P and g is continuous at $f(P)$, then $g \circ f$ is continuous at P.

Exercise 12.3. Find the set of points where the function $\ln(1 - x^2 - y^2)$ is continuous.

Solution. Functions like this one, whose expressions involve only polynomials, trigonometric functions, exponentials and logarithms (and which are not piecewise defined), will be continuous in their entire domain. This is because polynomials, trigonometric functions, e^x, $\ln(x)$, $\sqrt{x}$ and $\frac{1}{x}$ are continuous in their domains, and the sum/product/composition of continuous functions is continuous.

The function $\ln(1 - x^2 - y^2)$ is continuous for all x, y such that $1 - x^2 - y^2 > 0$, the open ball of center $(0, 0)$ and radius 1.

Exercise 12.4. What is the region of points where the given function f is continuous?

$$f(x, y, z) = \sqrt{1 - (|x| - 7)^2 - y^2 - z^2}$$

A. A closed ball

B. Two closed balls

C. An open ball

D. Two open balls

Solution. It is two closed balls. Recall that $|x|$ is equal to x when $x \geq 0$ and $-x$ when $x < 0$. So for $x > 0$, the function f is

$$f(x, y, z) = \sqrt{1 - (x - 7)^2 - y^2 - z^2},$$

which is defined and continuous in the closed ball centered at $(7, 0, 0)$ with radius 1.

And for $x < 0$, the function f is given by

$$f(x, y, z) = \sqrt{1 - (-x - 7)^2 - y^2 - z^2},$$

which is defined and continuous in the closed ball centered at $(-7, 0, 0)$ with radius 1.

13 Partial derivatives

1. Partial derivative of $f(x, y, z)$ with respect to x at a point a, b:

$$\frac{\partial f}{\partial x}(a, b) = \left.\frac{\partial f}{\partial x}\right|_{(a,b)} = f_x(a, b).$$

2. To find partial derivatives with respect to x, we fix the rest of the variables $(y, z, \dots)$ as constants and differenciate with respect to x.

   Exercise 13.1. Find the partial derivatives of $f(x, y) = 5 + 7e^{x^2-y}$ at $(2, 2)$.

   Solution. To find f_x, we take the derivative with respect to x considering y as a constant:

$$f_x(x, y) = \frac{\partial}{\partial x}(5 + 7e^{x^2-y}) = 7e^{x^2-y} \cdot 2x.$$

   To find f_x, we take the derivative with respect to y considering x as a constant:

$$f_y(x, y) = \frac{\partial}{\partial y}(5 + 7e^{x^2-y}) = -7e^{x^2-y}.$$

3. Second partial derivatives: $f_{xx}, f_{xy}, f_{yx}, f_{yy}$, where

$$f_{xy}(a, b) = \frac{\partial^2 f}{\partial y \partial x}(a, b) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(a, b).$$

4. $f_{xy}(a, b) = f_{yx}(a, b)$ (when f_{xy}, f_{yx} are continuous close to (a, b)).

   Exercise 13.2. Letting $f(x, y, z) = xz^2e^{2y}$, find $f_{xz}(1, 2, 3)$ and $f_{yz}(4, 5, 6)$.

   Solution. We have

$$f_{xz} = \frac{\partial^2}{\partial z \partial x}(xz^2e^{2y}) = \frac{\partial}{\partial z}(z^2e^{2y}) = 2ze^{2y}.$$

   So $f_{xz}(1, 2, 3) = 2 \cdot 3e^{2\cdot 2} = 6e^4$.

   We have

$$f_{yz} = \frac{\partial^2}{\partial z \partial y}(xz^2e^{2y}) = \frac{\partial}{\partial z}(2xz^2e^{2y}) = 4xze^{2y}.$$

   So $f_{yz}(4, 5, 6) = 4 \cdot 4 \cdot 6e^{2\cdot 5} = 96e^{10}$.

14 Chain rule

We will assume all functions are differentiable in the following sections.

1. If we have a function $z = f(x, y)$, where x, y depend on a variable t, then we have

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

2. For more complicated derivatives with more variables/dependencies between variables, we can use tree diagrams (see the lecture notes).

Exercise 14.1. The temperature at a point (x, y, z) of a region is given by $T(x, y, z) = 3x^2 + 2yz$. If a particle moves along the curve with position $(t, 2t, e^t)$, use the chain rule to compute $\frac{\partial T}{\partial t}$ at time $t = 1$.

Solution. We have

$$\begin{aligned}\frac{\partial T}{\partial t}(t) &= \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt} \\ &= (6x)\cdot 1 + (2z)\cdot 2 + (2y)\cdot e^t.\end{aligned}$$

For $t = 1$, we have $x = 1, y = 2$ and $z = e$, so

$$\frac{\partial T}{\partial t}(1) = 6 + 4e + 4e = 6 + 8e.$$

3. If two variables x, y satisfy $F(x, y) = 0$, then $\frac{dy}{dx} = \frac{-F_x}{F_y}$.

Exercise 14.2. A cylinder with changing radius r and height h has fixed volume 20π. Find $\frac{dh}{dr}$ when $r = 2$.

Solution. We know that $\pi r^2 h = 20\pi$, or equivalently, $\pi r^2 h - 20\pi = 0$. So our function $F(r, h)$ is $\pi r^2 h - 20\pi$. So we have

$$\frac{dh}{dr} = \frac{-F_r}{F_h} = \frac{-2\pi r h}{\pi r^2} = \frac{-2h}{r}.$$

When $r = 2$, we have $h = \frac{20\pi}{\pi r^2} = 5$, so $\frac{dh}{dr} = \frac{-10}{2} = -5$.

15 Gradient and directional derivatives

1. Gradient of $f(x,y)$: $\nabla f(x,y) = \langle f_x(a,b), f_y(a,b)\rangle$.
2. Directional derivative of $f(x,y)$ at (a,b) in the direction of a unit vector $\vec{u}$:
$$D_{\vec{u}}f(a,b) = \nabla f(a,b) \cdot \langle u_1, u_2\rangle .$$
3. - $f(x,y)$ increases fastest in the direction of $\nabla f(x,y)$. That is, among all unit vectors $\vec{v}$, $D_{\vec{v}}f(P)$ takes the biggest value when $\vec{v} = \frac{\nabla f(x,y)}{|\nabla f(x,y)|}$.
   - $f(x,y)$ decreases fastest in the direction of $-\nabla f(x,y)$.
   - $f(x,y)$ 'does not change much' (directional derivative derivative is 0) in directions orthogonal to $\nabla f(x,y)$.
4. Gradient is 'orthogonal to the level curves' of f (it is orthogonal to the tangent lines to the level curves).

Exercise 15.1. Consider the function $f(x,y,z) = xy^2e^z$. At the point $(1,2,3)$, is the rate of change of f bigger if we increase the x, y or z coordinate?

A. x-coordinate (direction $\langle 1,0,0\rangle$).

B. y-coordinate (direction $\langle 0,1,0\rangle$).

C. z-coordinate (direction $\langle 0,0,1\rangle$).

Solution. We first compute the gradient of f:
$$\begin{aligned}\nabla f(x,y,z) &= \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle \\ &= \left\langle y^2e^z, 2xye^z, xy^2e^z\right\rangle .\end{aligned}$$
So for $(x,y,z) = (1,2,3)$, we have $\nabla f(1,2,3) = \langle 4e^3, 4e^3, 4e^3\rangle$.

So the rate of change of f when we increase the coordinates is as follows:
$$\begin{aligned}x \text{ coordinate: } D_{\langle 1,0,0\rangle}f(1,2,3) &= \left\langle 4e^3, 4e^3, 4e^3\right\rangle \cdot \langle 1,0,0\rangle = 4e^3. \\ y \text{ coordinate: } D_{\langle 1,0,0\rangle}f(1,2,3) &= \left\langle 4e^3, 4e^3, 4e^3\right\rangle \cdot \langle 0,1,0\rangle = 4e^3. \\ z \text{ coordinate: } D_{\langle 1,0,0\rangle}f(1,2,3) &= \left\langle 4e^3, 4e^3, 4e^3\right\rangle \cdot \langle 0,0,1\rangle = 4e^3.\end{aligned}$$
It seems in this case (by coincidence! this does not happen in general), the rate of change of f when increasing the three coordinates is exactly the same.

Exercise 15.2. The gradient of a function $f(x,y,z)$ at a point P has norm 7. Compute the directional derivative of f in the direction of $\nabla f(P)$.

Solution. The unit vector in the direction of $\nabla f(P)$ is $\frac{\nabla f(P)}{|\nabla f(P)|}$. So the answer will be
$$D_{\frac{\nabla f(P)}{|\nabla f(P)|}}f(P) = \nabla f(P) \cdot \frac{\nabla f(P)}{|\nabla f(P)|} = \frac{|\nabla f(P)|^2}{|\nabla f(P)|} = |\nabla f(P)| = 7.$$

16 Tangent planes and linear approximation

1. Tangent plane to a surface $F(x, y, z) = 0$ at a point (a, b, c) is orthogonal to the gradient vector of F, $\nabla F(a, b, c)$. We can express the equation in several ways:

$$F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) = 0.$$

This is equivalent to the equation $\nabla F(a, b, c) \cdot \langle x, y, z\rangle = \nabla F(a, b, c) \cdot \langle a, b, c\rangle$.
Exercise 16.1. What is the equation of the plane tangent to the sphere $x^2+y^2+z^2 = 10$ at the point $(3, 1, 0)$?

$$\_\_\_\_ x + \_\_\_\_ y + \_\_\_\_ z = \_\_\_\_$$

Solution. We have $F(x, y, z) = x^2 + y^2 + z^2 - 10$ in this case, so $\nabla F(x, y, z) = \langle 2x, 2y, 2z\rangle$, so $\nabla F(3, 1, 0) = \langle 6, 2, 0\rangle$. So the tangent plane will have equation

$$\langle 6, 2, 0\rangle \cdot \langle x, y, z\rangle = \langle 6, 2, 0\rangle \cdot \langle 3, 1, 0\rangle ;$$
$$6x + 2y = 20.$$

2. Linear approximation for $f(x)$ at $x = a$:

$$L(x) = f(a) + f'(a)(x - a).$$

Linear approximation to $f(x, y)$ at $(x, y) = (a, b)$:

$$L(x, y) = f(a, b) + \nabla f(a, b) \cdot \langle x - a, y - b\rangle .$$

Linear approximation to $f(x, y, z)$ at $(x, y, z) = (a, b, c)$:

$$L(x, y, z) = f(a, b, c) + \nabla f(a, b) \cdot \langle x - a, y - b, z - c\rangle .$$

3. The graph of $L(x, y)$ is the tangent plane to the graph of $f(x, y)$ at the point $(a, b, f(a, b))$.
Exercise 16.2. Find the linear approximation to $f(x, y) = \ln(7x + e^y)$ at $(1, 1)$.

$$L(x, y) = \_\_\_\_ x + \_\_\_\_ y + \_\_\_\_$$

Solution. We have $\nabla f(x, y) = \left\langle \frac{7}{7x+e^y}, \frac{e^y}{7x+e^y} \right\rangle$. So at $(a, b) = (1, 1)$, $f(1, 1) = \ln(7+e)$ and $\nabla f(1, 1) = \left\langle \frac{7}{7+e}, \frac{e}{7+e} \right\rangle$. So the linear approximation to f at $(1, 1)$ is:

$$L(x, y) = \ln(7 + e) + \frac{7}{7 + e}(x - 1) + \frac{e}{7 + e}(y - 1) = \frac{7}{7 + e}x + \frac{e}{7 + e}y + \ln(7 + e) - 1.$$

4. We define the differential dz as $\nabla f(x, y) \cdot \langle dx, dy\rangle = f_x(x, y)dx + f_y(x, y)dy$.

5. If f is differentiable at (x, y) and dx, dy are small:

$$\begin{aligned} f(x+dx, y+dy) \approx L(x+dx, y+dy) &= f(x,y) + f_x(x,y)dx + f_y(x,y)dy \\ &= f(x,y) + dz. \end{aligned}$$

Exercise 16.3. Use linear approximation to approximate $f(x,y) = \sqrt{x+y^2}$ at $(x,y) = (8.9, 4.1)$.

Solution. We take the linear approximation of f at the point $(9,4)$. We have

$$\nabla f(x,y) = \left\langle \frac{1}{2\sqrt{x+y^2}}, \frac{y}{\sqrt{x+y^2}} \right\rangle,$$

so the linear approximation is

$$f(9+dx, 4+dy) \approx f(9,4) + \frac{1}{2\sqrt{9+4^2}}dx + \frac{4}{\sqrt{9+4^2}}dy = 5 + \frac{dx}{10} + \frac{4}{5}dy.$$

So for $dx = -0.1$ and $dy = 0.1$, we obtain $f(8.9, 4.1) \approx 5 - \frac{0.1}{10} + \frac{4}{5}0.1 = 5.07$.

Note: the actual value of $\sqrt{8.9 + 4.1^2}$ is $5.07050\ldots$.

17 Maximum and minimum problems

1. For a two variable function $f(x,y)$, global/local extrema of f are defined as follows:
   - Global maximum: point (a,b) in $\mathrm{dom}(f)$ such that $f(a,b) \geq f(x,y)$ for all (x,y) in $\mathrm{dom}(f)$.
   - Local maximum: point (a,b) in **the interior of** $\mathrm{dom}(f)$ such that $f(a,b) \geq f(x,y)$ for all (x,y) in some open disk around (a,b).
   - Global/local minimum: Same but changing $\geq$ for $\leq$.

   Exercise 17.1. What does the function $f(x,y) = 2$ have in the point $(3,5)$? Select all that apply.

   A. Local maximum

   B. Local minimum

   C. Global maximum

   D. Global minimum

   Solution. A,B,C,D are all correct, e.g. $(3,5)$ is a global maximum because $f(3,5) \geq f(x,y)$ for all (x,y).

2. Critical points of f: Points (x,y) in the interior of $\mathrm{dom}(f)$ such that at least one of the following is true:
   - $f_x(x,y)$ is not defined.
   - $f_y(x,y)$ is not defined.
   - $f_x(x,y) = f_y(x,y) = 0$.
3. Every global extremum is either a local extremum or a boundary point.
4. Every local extremum is a critical point.
5. A continuous function defined in a *compact* (closed and bounded) subset of $\mathbb{R}, \mathbb{R}^2$ or $\mathbb{R}^3$, always has global max/min.

   Exercise 17.2. Find the absolute extremums of the function $f(x,y) = \sqrt{1-x^2-y^2} + 2x$ in the closed unit disk $\{x^2+y^2 \leq 1\}$.

   Solution. Absolute extremums are either local extrema or boundary points.

   - Boundary (circle $x^2+y^2=1$). Here $1-x^2-y^2=0$, so $f(x,y)=2x$. Thus, the function attains its maximum values in the points $(\pm 1, 0)$, where $f(x,y) = \pm 2$.
   - Local extrema (in interior disk $x^2+y^2<1$). Local extrema are critical points, so we take $f_x = f_y = 0$.

$$0 = f_x(x,y) = \frac{-x}{\sqrt{1-x^2-y^2}} + 2$$
$$0 = f_y(x,y) = \frac{-y}{\sqrt{1-x^2-y^2}}.$$

So $y = 0$, and $\frac{-x}{\sqrt{1-x^2}} + 2 = 0$, so $\frac{x}{\sqrt{1-x^2}} = 2$, so $x = 2\sqrt{1-x^2}$, so $x^2 = 4(1-x^2)$, so $5x^2 = 4$, so $x = \pm\sqrt{4/5}$. So we obtain the points $(\pm\sqrt{4/5}, 0)$. We have

$$f\left(\sqrt{4/5}, 0\right) = \sqrt{1-4/5} + 2\sqrt{4/5} = \sqrt{1/5} + 2\sqrt{4/5} = \sqrt{5}.$$
$$f\left(-\sqrt{4/5}, 0\right) = \sqrt{1/5} - 2\sqrt{4/5} = \frac{-3}{\sqrt{5}}.$$

So at our for candidates for global extrema, the function f takes the values $-2, 2, \sqrt{5}, \frac{-3}{\sqrt{5}}$. The biggest of these values is $\sqrt{5}$, and the smallest of these values is -2.

So the global maximum of f is $f\left(\sqrt{4/5}, 0\right) = \sqrt{5}$ and the global minimum is $f(-1, 0) = -2$.

18 Critical points, second derivative test

1. There are three types of critical points of a two variable function $f(x, y)$:

   (a) Local maximum.

   (b) Local minimum.

   (c) Saddle point: (a, b) is a saddle point if inside every disk centered at (a, b) there are points (x_1, y_1) and (x_2, y_2) such that $f(x_1, y_1) > f(a, b) > f(x_2, y_2)$.

2. Second derivative test: Let (a, b) be a critical point such that $f_x(a, b) = f_y(a, b) = 0$ and $f_{xx}, f_{xy}, f_{yx}, f_{yy}$ are continuous in a disk around (a, b). The *discriminant* of f is

$$D(a,b) = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix} = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}(a,b)^2.$$

   If $D(x, y) > 0$, f has a local extremum at (a, b).

   - Local maximum if $f_{xx}(a, b) < 0$.
   - Local minimum if $f_{xx}(a, b) > 0$.

   If $D(x, y) < 0$, f has a saddle point at (a, b).

   If $D(x, y) = 0$, the test is inconclusive (could be local max, min or saddle point).

Exercise 18.1. Find and classify the critical points of the following functions. Determine their types.

1. $f(x, y) = y \sin(x)$.

   Solution. f is differentiable in the interior of the domain (as will be usual when f is not a piecewise function). So we can just set $f_x = f_y = 0$.

$$0 = f_x(x, y) = y\cos(x).$$
$$0 = f_y(x, y) = \sin(x).$$

   So $\sin(x) = 0$ (which implies $\cos(x) \neq 0$) and $y = 0$. So the critical points are $(0, \pi n)$, for any integer n. To find the type of critical points, we compute $D(x, y)$

$$D(x,y) = \begin{vmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,x) \end{vmatrix} = \begin{vmatrix} -y\sin(x) & \cos(x) \\ \cos(x) & 0 \end{vmatrix} = -\cos(x)^2.$$

   At all points $(0, \pi n)$ we have $-\cos(x)^2 = -1 < 0$, so f has saddle points.

2. $f(x, y) = y^3x^3$.

   Solution. Again f is differentiable so we take $f_x = f_y = 0$. $f_x = 3x^2y^3$ and $f_y = 3y^2x^3$. So we have $f_x = f_y = 0$ whenever either x or y is 0, that is, all the points in the x and y axes are critical points.

   In fact all the critical points are saddle points. This is because: $f(x, y) > 0$ in the first and third cuadrants, and $f(x, y) < 0$ in the second and fourth quadrants. So for any critical point (a, b) of f, we can find points arbitrarily close to (a, b) where f takes positive and negative values.

19 Integrals in rectangular regions

1. Integral of $f(x,y)$ in a rectangle $R=[a,b]\times[c,d]$:
$$\iint_R f(x,y)dA = \int_a^b \left(\int_c^d f(x,y)dy\right)dx = \int_c^d \left(\int_a^b f(x,y)dx\right)dy.$$

2. Integral gives you the (signed) volume between the xy plane and the graph of the function $f(x,y)$ in the rectangle R.

3. Some rules of integration:
   (a) $\iint_R f(x,y)+g(x,y)dA = \iint_R f(x,y)dA + \iint_R g(x,y)dA$.
   (b) If k is a constant, $\iint_R k\cdot f(x,y)dA = k\iint_R f(x,y)dA$.

   Exercise 19.1. Compute these integrals.

   (a) Integral of $f(x,y)=x^3+y^5$ in $[0,3]\times[0,5]$.
   Solution. We can compute the integrals of x^3 and y^5 in the rectangle and then take their sum.
$$\int_0^3\left(\int_0^5 x^3dy\right)dx = \int_0^3 5x^3dx = \left[\frac{5}{4}x^4\right]_{x=0}^3 = \frac{5}{4}3^4.$$
$$\int_0^5\left(\int_0^3 y^5dx\right)dy = \int_0^5 3y^5dy = \left[\frac{y^6}{2}\right]_{y=0}^5 = \frac{5^6}{2}.$$
   So the answer is $\frac{5^6}{2}+\frac{5}{4}3^4$.

   (b) $\iint_R x^5e^{x^3y}dA$ in $R=[0,1]\times[0,2]$.
   Solution. In this particular case, it is much easier to integrate using $dydx$, instead of $dxdy$.
$$\int_0^1\int_0^2 x^5e^{x^3y}dydx = \int_0^1\left(\left[x^2e^{x^3y}\right]_{y=0}^2\right)dx = \int_0^1\left(x^2e^{2x^3}-x^2\right)dx = \left[\frac{e^{2x^3}}{6}-\frac{x^3}{3}\right]_{x=0}^1$$
$$= \frac{e^2}{6}-\frac{1}{3}-\frac{1}{6} = \frac{e^2}{6}-\frac{1}{2}.$$

4. Average value of $f(x,y)$ in a region R:
$$\overline{f} = \frac{1}{\text{Area of } R}\iint_R f(x,y)dA.$$

5. Area of any region R (not only rectangles) can be computed as $\iint_R 1dA$.

   Exercise 19.2. Find the average value of $f(x,y)=xy$ in the rectangle $[0,2]\times[0,2]$.

   Solution. The rectangle has area 4, so the answer is
$$\frac{1}{4}\int_0^2\int_0^2 xydxdy = \frac{1}{4}\int_0^2 y\left(\int_0^2 xdx\right)dy = \frac{1}{4}\int_0^2 2ydy = 1.$$

20 Integrals in general regions

We now consider integrals of the following form

$$\int_a^b \int_{f(x)}^{g(x)} h(x,y)dydx \qquad\qquad \int_a^b \int_{f(y)}^{g(y)} h(x,y)dxdy.$$

Limits of the inner integral can depend on the outer variable, but not the other way around!

Exercise 20.1. For the following integrals of a function $f(x,y)$, draw the region in which we are integrating the function f.

1. $\displaystyle\int_0^1 \int_{y^2}^1 f(x,y)dxdy.$

   Solution. Region bounded by the lines $y=0, x=1$ and the parabola $x=y^2$.

2. $\displaystyle\int_3^5 \int_{-x}^{\sqrt{x}} f(x,y)dydx + \int_1^2 \int_0^{x+1} f(x,y)dydx.$

   Solution. Union of two regions R_1 and R_2, where

   - R_1 is bounded by $x=3, x=5, y=\sqrt{x}, y=-x$.
   - R_2 is bounded by $x=1, x=2, y=0, y=x+1$.

Exercise 20.2. For the following regions R, write iterated integrals with order of integration $dydx$ that will allow us to compute $\iint_R f(x,y)dA$.

1. R is a triangle with vertices $(0,0),(2,2),(3,1)$.

   Solution. There are several possible solutions, here is one.

$$\int_0^2 \int_{x/3}^{2x} f(x,y)dydx + \int_2^3 \int_{x/3}^{4-x} f(x,y)dydx$$

2. R is the region between the unit circle $x^2+y^2=1$ and the square $\{-2 \le x, y \le 2\}$.

   Solution. There are several possible solutions, here is one.

$$\int_{-2}^2 \int_{-2}^2 f(x,y)dydx - \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)dydx.$$

Exercise 20.3. Find the area of the region R enclosed between the graphs of $y = x^2$ and $y = 18 - x^2$. Find the average of the function $f(x, y) = xy^2$ in the region R.

Solution. Note that the graphs of $y = x^2$ and $y = 18 - x^2$ intersect at the points where $x^2 = 18 - x^2$, that is, $x = \pm 3$. So the area of the region is

$$\int_{-3}^{3}\int_{x^2}^{18-x^2} 1dydx = \int_{-3}^{3} 18 - 2x^2 dx = \left[18x - 2\frac{x^3}{3}\right]_{-3}^{3} = 18\cdot 3 - 18 - (18\cdot(-3) + 18) = 72.$$

So the average of the function will be

$$\begin{aligned}
\frac{1}{72}\int_{-3}^{3}\int_{x^2}^{18-x^2} xy^2dydx &= \frac{1}{72}\int_{-3}^{3}\left[\frac{xy^3}{3}\right]_{x^2}^{18-x^2} dx & (1)\\
&= \frac{1}{72}\int_{-3}^{3}\frac{x(18-x^2)^3}{3} - \frac{x^7}{3}dx & (2)\\
&= \frac{1}{72}\left[\frac{-(18-x^2)^4}{24} - \frac{x^8}{24}\right]_{-3}^{3} & (3)\\
&= 0. & (4)
\end{aligned}$$

The last term is zero because the expression inside the brackets is the same for $x = -3$ and $x = 3$. There is also a faster solution once one notices that $f(x, y)$ is odd as a function of x, and the domain is symmetric around the y axis.

Exercise 20.4. Two of the following four integrals correspond to integrating $f(x, y)$ in the same region of the xy plane. Circle both of them.

A. $\displaystyle\int_{1}^{2}\int_{0}^{\ln(x)} f(x, y)dydx$

B. $\displaystyle\int_{1}^{2}\int_{\ln(x)}^{\ln(2)} f(x, y)dydx$

C. $\displaystyle\int_{0}^{\ln(2)}\int_{e^y}^{2} f(x, y)dxdy$

D. $\displaystyle\int_{0}^{\ln(2)}\int_{0}^{e^y} f(x, y)dxdy$

Solution. Both A and C correspond to the region enclosed by $y = \ln(x)$ (or equivalently, $x = e^y$, $y = 0$ and $x = 2$).

Exercise 20.5. Express the integral $\int_1^2\int_{\sqrt{y}}^{y}\frac{1}{y}dxdy + \int_2^4\int_{\sqrt{y}}^{2}\frac{1}{y}dxdy$ as a $dydx$ integral, and compute it.

Solution. The region corresponding to the integral above is the region between the graphs of $y = x$ and $y = x^2$, between $x = 1$ and $x = 2$. So the integral is

$$\int_{1}^{2}\int_{x}^{x^2}\frac{1}{y}dydx = \int_{1}^{2}\left[\ln(y)\right]_{x}^{x^2} dx = \int_{1}^{2}\ln(x^2) - \ln(x)dx = \int_{1}^{2}\ln(x)dx$$

Now we integrate by parts, with $u = \ln(x)$ so $du = \frac{1}{x}dx$ and $dv = dx$, so $v = x$.

$$\int_1^2 \ln(x)dx = [x\ln(x)]_1^2 - \int_1^2 1dx = 2\ln(2) - 1\ln(1) - 1 = 2\ln(2) - 1.$$

Exercise 20.6. (Bonus) Find the volume of the region enclosed by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = x + y + 5$.

Solution. The volume will be the integral in the unit disk of the function $x+y+5$, that is,

$$\begin{aligned}
\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(x+y+5)dydx &= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(x+5)dydx \\
&= \int_{-1}^{1}[y(x+5)]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\,dx \\
&= \int_{-1}^{1}2(x+5)\sqrt{1-x^2}dx \\
&= 2\int_{-1}^{1}x\sqrt{1-x^2}dx + 10\int_{-1}^{1}\sqrt{1-x^2}dx \\
&= 2\cdot 0 + 10\cdot\frac{\pi}{2} = 5\pi.
\end{aligned}$$

In the first and next to last steps we used that the integral of an odd function in an interval of the form $[-a, a]$ is 0.

21 Integration in polar coordinates

We can use polar coordinates r, θ to integrate. Coordinate change:

1. $r = \sqrt{x^2 + y^2}; \tan(\theta) = y/x$ (θ is the direction of the vector (x, y)).

2. $x = r\cos(\alpha), y = r\sin(\alpha)$.

   Exercise 21.1. Find the polar coordinates of the points $(2, -3), (-2, 3), (0, -7)$.

   Solution. For $(2, -3)$, we have $r = \sqrt{2^2 + 3^2} = \sqrt{13}$, and as $(2, -3)$ is in the fourth quadrant, $\theta = \arctan\left(\frac{2}{-3}\right)$.

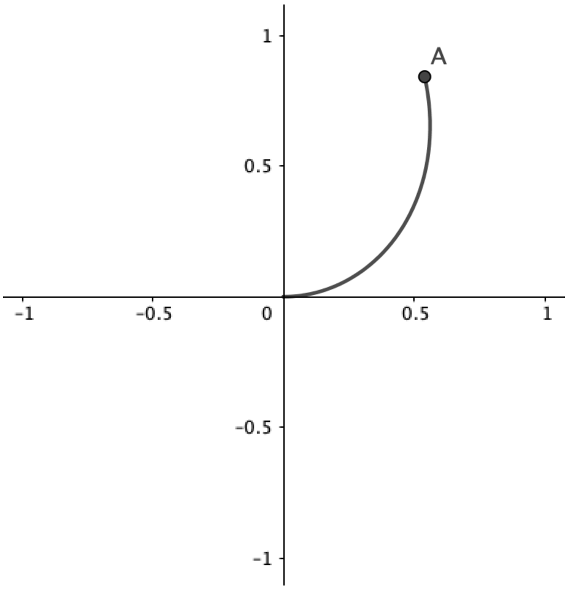
   For $(-2, 3)$, we have $r = \sqrt{2^2 + 3^2} = \sqrt{13}$, and as $(-2, 3)$ is in the second quadrant, $\theta = \arctan\left(\frac{2}{-3}\right) + \pi$.

   For $(0, -7)$, we have $r = \sqrt{0^2 + 7^2} = 7$, and as $(0, -7)$ points down, $\theta = -\pi/2$.

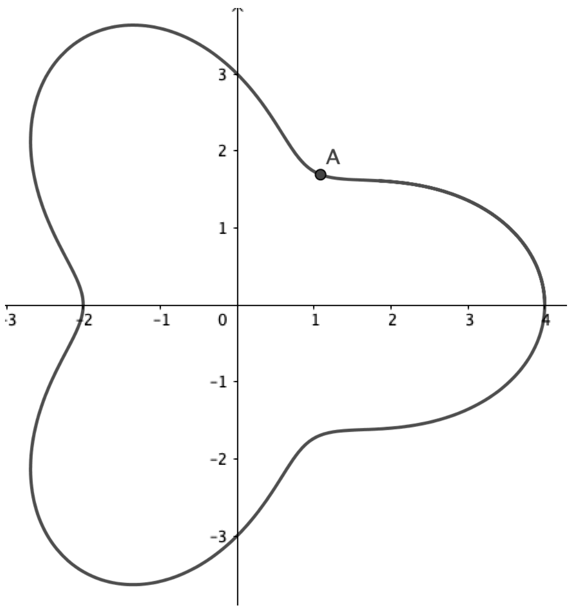
   Exercise 21.2. Sketch the following curves in polar coordinates.

   (a) $r = \theta; 0 \leq \theta \leq 1$.
   (b) $r = 3 + \cos(3\theta)$.
   (c) (Bonus) $\theta = \ln(r)$.

   Solution. The curve from c is a logarithmic spiral.



(a)



(b)

3. Integral in a region $R = \{(r\cos(\theta), r\sin(\theta)); \alpha \leq \theta \leq \beta, 0 \leq g(\theta) \leq r \leq h(\theta)\}$:

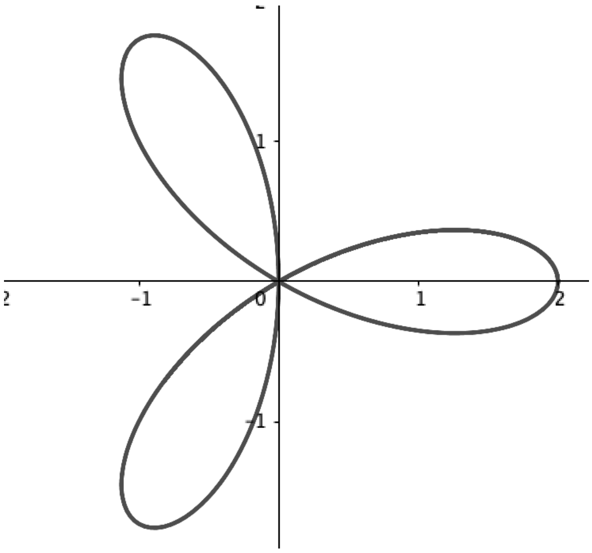
$$\iint_R f(x, y)dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} r \cdot f(r\cos(\alpha), r\sin(\alpha))drd\theta.$$

4. Area of the region R:

$$\iint_R 1A = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} r dr d\theta.$$

Exercise 21.3. [Ex. 49, textbook] Find the area of the region bounded by all leaves of the rose $r = 2\cos(3\theta)$.

Solution. The rose will look as follows:



Note that the curve is only defined for the values of θ for which $2\cos(3\theta)$ is positive. The rose is composed by three petals P_1, P_2, P_3.

Note that $r = 0$ when $\cos(3\theta) = 0$. In particular, $r = 0$ when $\theta = \pm\pi/6$, so the first petal P_1 is between $\theta = -\pi/6$ and $\theta = \pi/6$. We can compute the area now:

$$\begin{aligned}
\iint_{P_1} 1dA &= \int_{-\pi/6}^{\pi/6} \int_0^{2\cos(3\theta)} r \cdot dr d\theta \\
&= \int_{-\pi/6}^{\pi/6} 2\cos^2(3\theta) d\theta \\
&= \int_{-\pi/6}^{\pi/6} (1 + \cos(6\theta)) d\theta \\
&= [\theta + \sin(6\theta)/6]_{-\pi/6}^{\pi/6} \\
&= \frac{\pi}{3}.
\end{aligned}$$

So the area of the rose is $3\frac{\pi}{3} = \pi$.

22 Triple integrals

1. Integral of $f(x, y, z)$ over a box: $D = \{x_0 \le x \le x_1; y_0 \le y \le y_1; z_0 \le z \le z_1\}$

$$\iiint_D f(x, y, z)dV = \int_{x_0}^{x_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} f(x, y, z)dzdydx$$

2. More generally,

$$\iiint_D f(x, y, z)dV = \int_a^b \int_{g(x)}^{f(x)} \int_{G(x,y)}^{H(x,y)} f(x, y, z)dzdydx$$

   Is the integral of $f(x, y, z)$ over the region

$$D = \{(x, y, z) \in \mathbb{R}^3; a \le x \le b; g(x) \le y \le h(x); G(x, y) \le z \le H(x, y)\}.$$

3. As in 2D, the limits of integration of the inner integrals can only depend on the variables from the outer integrals.

4. There are six orders of integration, and if f is continuous, the integral in a given region is the same for all of them. (This is one formulation of Fubini's theorem)

   Exercise 22.1. [Exercise 44, textbook] Find the following integral.

$$\int_0^1 \int_y^{2-y} \int_0^{2-x-y} 15xy \; dzdxdy$$

Solution.

$$\begin{aligned}
\int_0^1\int_y^{2-y}\int_0^{2-x-y} 15xy\ dzdxdy &= \int_0^1\int_y^{2-y} [15xyz]_{z=0}^{2-x-y}\, dxdy \\
&= \int_0^1\int_y^{2-y} 15xy(2-x-y)dxdy \\
&= 15\int_0^1\int_y^{2-y} 2xy - x^2y - xy^2 dxdy \\
&= 15\int_0^1 \left[x^2y - \frac{x^3y}{3} - \frac{x^2y^2}{2}\right]_y^{2-y} dy \\
&= 15\int_0^1 (2-y)^2y - \frac{(2-y)^3y}{3} - \frac{(2-y)^2y^2}{2} - \left(y^3 - \frac{y^4}{3} - \frac{y^4}{2}\right) dy \\
&= 15\int_0^1 \frac{2y^4}{3} - 2y^2 + \frac{4y}{3} dy \\
&= 15 \cdot \frac{2}{15} = 2.
\end{aligned}$$

5. Volume of the region D is $\iiint_D 1dV$.
6. Average of a function in the region R:

$$\frac{1}{\text{Vol}(D)}\iiint_D f(x,y,z)dV$$

Exercise 22.2. Write the integral of a function $f(x,y,z)$ in the region $\{x,y,z; z^2+y^2+z^2 \leq 1, x \geq 0, y \geq 0\}$, in the orders of integration $dxdydz$, $dydzdx$ and $dzdxdy$.
Solution. The region is between a sphere and two planes. It is a 'quarter of a sphere'.

$$\int_{-1}^1\int_0^{\sqrt{1-z^2}}\int_0^{\sqrt{1-y^2-z^2}} f(x,y,z)dxdydz$$
$$\int_0^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-z^2}} f(x,y,z)dydzdx$$
$$\int_0^1\int_0^{\sqrt{1-y^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} f(x,y,z)dzdxdy.$$

23 Integrals in cylindrical/spherical coordinates

1. Cylindrical coordinates: r, θ, z. Change of coordinates to rectangular coordinates:

$$\begin{aligned} r &= \sqrt{x^2+y^2} & x &= r\cos(\theta) \\ \tan(\theta) &= y/x & y &= r\sin(\theta) \\ z &= z \end{aligned}$$

2. Integrals in a region D using cylindrical coordinates:

$$\iiint_D f(x,y,z)dV = \iiint r \cdot f(r\cos(\theta), r\sin(\theta), z)drd\theta dz.$$

3. As usual, the limits of integration of the inner integrals can only depend on the variables from the outer integrals.

   Exercise 23.1. Find the average of the function $f(x,y,z) = z + x^2 + y^2$ in the cylinder $\{(x,y,z) \in \mathbb{R}^3; x^2+y^2 \leq 1, -1 \leq z \leq 1\}$.

   Solution. The average is $\frac{1}{\mathrm{Vol}(D)} \iiint_D f(x,y,z)dV$. In our case, the cylinder has radius 1 and height 2, so its volume is 2π. In cylindrical coordinates, the function is $z + r^2$. So the integral is

$$\begin{aligned} \iiint_D (z+r^2)dV &= \int_{-1}^{1}\int_0^{2\pi}\int_0^1 r\cdot(z+r^2)drd\theta dz \\ &= \int_{-1}^{1}\int_0^{2\pi}\left[\frac{r^2 z}{2} + \frac{r^4}{4}\right]_0^1 d\theta dz \\ &= \int_{-1}^{1}\int_0^{2\pi} \frac{z}{2} + \frac{1}{4} d\theta dz \\ &= \int_{-1}^{1} \pi z + \frac{\pi}{2} dz \\ &= \left[\pi\frac{z^2}{2} + \frac{\pi}{2}z\right]_{-1}^{1} = \pi. \end{aligned}$$

   So the average is $\frac{1}{2\pi} \times \pi = \frac{1}{2}$.

4. Spherical coordinates: ρ (in $[0,\infty)$), φ (in $[0,\pi]$),θ (in $[0,2\pi]$). Change of coordinates:

$$\begin{aligned} x &= \rho\sin(\varphi)\cos(\theta) \\ y &= \rho\sin(\varphi)\sin(\theta) \\ z &= \rho\cos(\varphi). \end{aligned}$$

5. Inverse change of coordinates: $\rho = \sqrt{x^2+y^2+z^2}$, $\varphi = \arccos\left(\frac{z}{\rho}\right) = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$, $\tan(\theta) = y/x$.

6. Integrals in a region D using spherical coordinates:

$$\iiint_D f(x,y,z)dV = \iiint \rho^2\sin(\varphi)\cdot f(\rho\sin(\varphi)\cos(\theta), \rho\cos(\varphi)\cos(\theta), \rho\sin(\theta))d\rho d\varphi d\theta.$$

Exercise 23.2. Compute the integral in the unit ball of the function $f(x,y,z) = x^2+y^2+z^2$.

Solution. Note that $x^2+y^2+z^2$ is just ρ^2, so we have

$$\begin{aligned}
\iiint_D x^2+y^2+z^2 dV &= \int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^2\sin(\varphi)\cdot\rho^2 d\rho d\varphi d\theta \\
&= \int_0^{2\pi}\int_0^{\pi}\left[\frac{\rho^5}{5}\sin(\varphi)\right]_0^1 d\varphi d\theta \\
&= \int_0^{2\pi}\int_0^{\pi}\frac{\sin(\varphi)}{5} d\varphi d\theta \\
&= \int_0^{2\pi}\left[\frac{-\cos(\varphi)}{5}\right]_0^{\pi} d\theta \\
&= \int_0^{2\pi}\frac{2}{5}d\theta = \frac{4\pi}{5}.
\end{aligned}$$

24 Change of variables theorem

DEFINITION Jacobian Determinant of a Transformation of Two Variables
Given a transformation T: $x = g(u, v)$, $y = h(u, v)$, where g and h are differentiable on a region of the uv-plane, the **Jacobian determinant** (or **Jacobian**) of T is

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

Exercise 24.1. [Ex. 15, textbook] Compute the Jacobian of the transformation $T : x = u/v, y = v$. Find the image S in the xy-plane of the region $R = \{(u, v); 1 \leq u \leq 3, 2 \leq v \leq 4\}$ under T.

Solution. Let us compute the Jacobian first.

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & \frac{-u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}.$$

The region R is the set of points $(x, y) = (u/v, v)$ in the plane such that $1 \leq u \leq 3, 2 \leq v \leq 4$.

As $y = v$, we have $2 \leq y \leq 4$. Note that $u = xy$, so the condition $1 \leq u \leq 3$ translates to $1 \leq xy \leq 3$.

So our region is the region bounded by the lines $y = 2$ and $y = 4$, and the hyperbolas $xy = 1$ and $xy = 3$.

THEOREM 16.8 Change of Variables for Double Integrals
Let T: $x = g(u, v)$, $y = h(u, v)$ be a transformation that maps a closed bounded region S in the uv-plane to a region R in the xy-plane. Assume T is one-to-one on the interior of S and g and h have continuous first partial derivatives there. If f is continuous on R, then

$$\iint_R f(x, y)\, dA = \iint_S f(g(u, v), h(u, v))\, |J(u, v)|\, dA.$$

(*) Examples: **Double integrals—transformation given** *To evaluate the following integrals, carry out these steps.*

a. *Sketch the original region of integration R in the xy-plane and the new region S in the uv-plane using the given change of variables.*
b. *Find the limits of integration for the new integral with respect to u and v.*
c. *Compute the Jacobian.*
d. *Change variables and evaluate the new integral.*

Exercise 24.2. [Ex. 31, textbook] Evaluate the integral $\int_0^1 \int_y^{y+2} \sqrt{x-y}dxdy$ using an appropriate change of variables.

Solution. We can use the change of variables $u = x - y, v = y$ (so $x = u + v$). Then the region $\{(x, y); 0 \leq y \leq 1, y \leq x \leq y + 2\}$ becomes $\{(u, v); 0 \leq v \leq 1, 0 \leq u \leq 2\}$, we have $\sqrt{x-y} = \sqrt{u}$ and the Jacobian is

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1.$$

So by the change of variables theorem, we have

$$\int_0^1 \int_y^{y+2} \sqrt{x-y}dxdy = \int_0^1 \int_0^2 \sqrt{u} \cdot 1dudv = \int_0^1 \left[\frac{u^{3/2}}{3/2}\right]_0^2 dv = \int_0^1 \frac{4\sqrt{2}}{3}dv = \frac{4\sqrt{2}}{3}.$$

25 Vector fields

Vector Fields in Two Dimensions

DEFINITION Vector Fields in Two Dimensions

Let f and g be defined on a region R of $\mathbb{R}^2$. A **vector field** in $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point in R a vector $\langle f(x, y), g(x, y) \rangle$. The vector field is written as

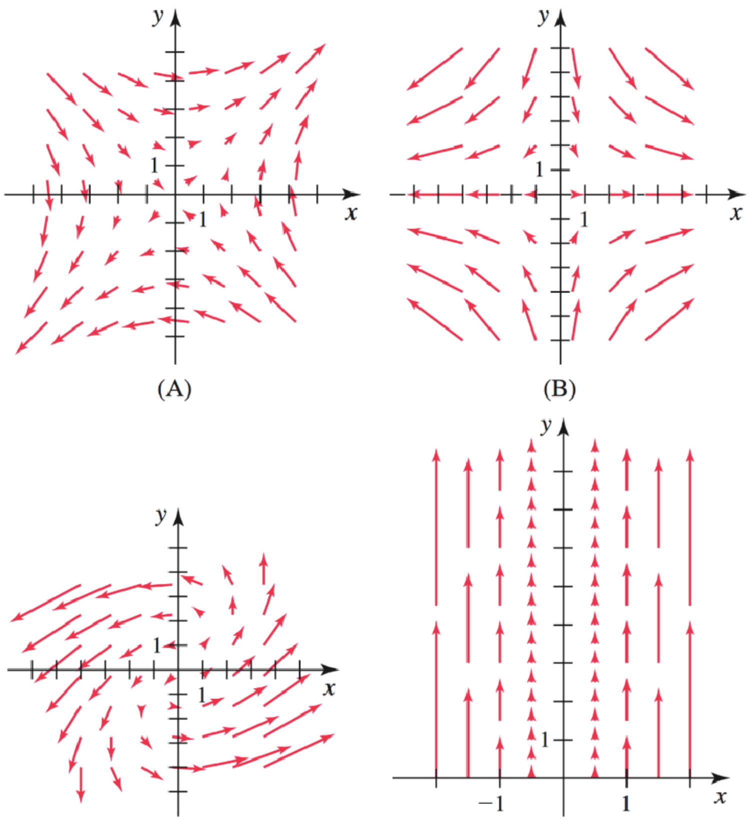
$$\mathbf{F}(x, y) = \langle f(x, y), g(x, y) \rangle \quad \text{or}$$
$$\mathbf{F}(x, y) = f(x, y)\,\mathbf{i} + g(x, y)\,\mathbf{j}.$$

A vector field $\mathbf{F} = \langle f, g \rangle$ is continuous or differentiable on a region R of $\mathbb{R}^2$ if f and g are continuous or differentiable on R, respectively.

Exercise 25.1. [Ex. 24, textbook] Match the vector fields a-d with the graphs A-D.

(a) $F = \langle 0, x^2 \rangle$ (b) $F = \langle x - y, x \rangle$

(c) $F = \langle 2x, -y \rangle$ (d) $F = \langle y, x \rangle$



(A) (B) (C) (D)

Solution. a-D, b-C, c-B, d-A.

26 Line integrals

Let C be a smooth (differentiable) curve given by $\vec{r}(t) = \langle x(t), y(t)\rangle$ for $a \le t \le b$.

1. Line integral of a scalar function $f(x, y)$ defined in a region containing the curve C:

   $$\int_C f ds = \int_C f(x(s), y(s))ds = \int_a^b f(x(t), y(t)) \cdot |\vec{r}\,'(t)|dt = \int_a^b f(x(t), y(t))\sqrt{x'(t)^2 + y'(t)^2}dt,$$

   where $s = s(t)$ is arclength (so $ds = |\vec{r}\,'(t)|dt$).

   Exercise 26.1. Compute the line integral of $f(x, y, z) = x^2 + y + z$ over the curve C with equation $\vec{r}(t) = (t, t, t)$, $0 \le t \le 1$. If we traverse C in the opposite direction, is the line integral of f still the same or does it change signs?

   Solution. Note that we have $\vec{r}\,'(t) = (1, 1, 1)$ for all t, so $|\vec{r}\,'(t)| = \sqrt{3}$ for all t, so

   $$\int_C f ds = \int_0^1 f(t, t, t) \cdot |\vec{r}\,'(t)|dt = \int_0^1 (t^2 + 2t) \cdot \sqrt{3}dt = \sqrt{3}\left[\frac{t^3}{3} + t^2\right]_0^1 = \frac{4\sqrt{3}}{3}.$$

   The line integral is the same is the same in the reverse direction.

2. Line integral of a vector field $\vec{F}(x, y)$ defined in a region containing the curve C:

   $$\int_C \vec{F} d\vec{r} = \int_C \vec{F} \bullet \vec{T} ds = \int_C \vec{F}(\vec{r}(t)) \bullet \vec{r}\,'(t)dt,$$

   where T is the unit tangent vector to C.

3. The work of a force $\vec{F}$ when moving along a curve C is $\int_C \vec{F} \bullet \vec{T} ds$.

4. Line integral of vector field does not depend on parametrization of the curve, but it depends on the orientation, unlike scalar line integrals.

   Exercise 26.2. An object A in the origin of $\mathbb{R}^3$ exerts a gravitational force of $\vec{F}(x, y, z) = \frac{-\langle x,y,z\rangle}{(x^2+y^2+z^2)^{3/2}}$ on an object B in position (x, y, z). Find the work needed to move B from $(1, 0, 0)$ to $(10, 0, 0)$ along the curve C with equation $\vec{r}(t) = (t, 0, 0)$.

   Solution. The work is just the line integral of f along the curve:

   $$\int_C \vec{F} d\vec{r} = \int_1^{10} \vec{F}(t, 0, 0) \bullet \vec{r}\,'(t)dt = \int_1^{10} \frac{-\langle t, 0, 0\rangle}{(t^2)^{3/2}} \bullet \langle 1, 0, 0\rangle\, dt = \int_1^{10} \frac{-1}{t^2}dt = \left[\frac{1}{t}\right]_1^{10} = -\frac{9}{10}.$$

5. Circulation of $\vec{F}$ on a **closed** curve C is the line integral $\int_C \vec{F}(\vec{r}(t)) \bullet \vec{r}\,'(t)dt$.

6. Flux of $\vec{F} = \langle f, g\rangle$ across a **simple** curve C:

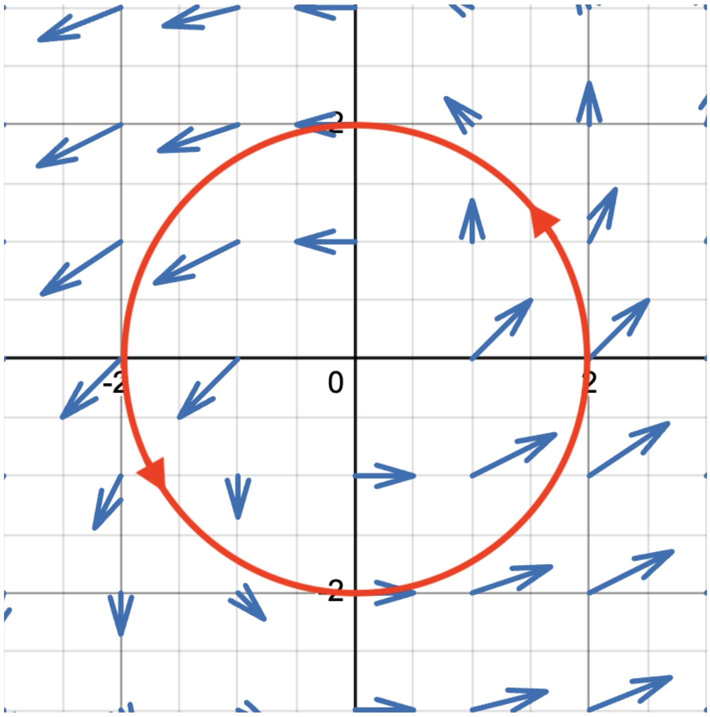
   $$\int_C \vec{F} \bullet \vec{n} ds = \int_a^b f(t)y'(t) - g(t)x'(t)dt,$$

   here $\vec{n}$ is a unit normal vector.

7. C oriented counterclockwise $\to$ outward flux (normal vector $\vec{n}$ points outward).

C oriented clockwise $\to$ inward flux (normal vector $\vec{n}$ points inward).

Exercise 26.3. Given the vector field $\vec{F}(x,y) = \langle x-y, x\rangle$ and the curve C from the picture (boundary of a ball of radius 2), what can we say about the circulation and flux of $\vec{F}$ on/through C?



A. Both are positive.

B. Both are negative.

C. The flux is positive, the circulation is negative.

D. The flux is negative, the circulation is positive.

Solution. Both are positive.

Intuition: In all points of the curve, vector field points in roughly same direction as the unit tangent vector (forming an angle of $< 90°$), so circulation is positive.

At all points of the curve, vector field points outside of the unit disk (or tangent to it, but not inside), so outward flux is positive.

Rigorous proof: We parametrize the curve as $\vec{r}(t) = \langle \cos(t), \sin(t)\rangle$, $0 \le t \le 2\pi$. $\vec{r}\,'(t) = \langle -\sin(t), \cos(t)\rangle$. Then the circulation is

$$\begin{aligned}\int_0^{2\pi} \vec{F}(\cos(t),\sin(t)) \cdot \vec{r}\,'(t)dt &= \int_0^{2\pi} \langle \cos(t)-\sin(t), \cos(t)\rangle \cdot \langle -\sin(t), \cos(t)\rangle\, dt \\ &= \int_0^{2\pi} 1 - \sin(t)\cos(t)dt,\end{aligned}$$

which is positive because $\sin(t)\cos(t) \leq 1$.

The flux

$$\int_0^{2\pi} (x(t) - y(t))y'(t) - x(t)x'(t)dt = \int_0^{2\pi} (\cos(t) - \sin(t))\cos(t) - \cos(t)(-\sin(t))dt$$
$$= \int_0^{2\pi} \cos^2(t)dt > 0.$$

27 Potential, conservative vector fields

1. A vector field $\vec{F}$ in $\mathbb{R}^2$ is conservative if it is the gradient field of some smooth function (called *potential*) φ, that is,
$$\vec{F}(x, y) = \nabla\varphi(x, y).$$

2. Line integral of a conservative vector field $\vec{F}$ with potential φ along a curve from A to B:
$$\int_C \vec{F} \bullet \vec{T} ds = \varphi(B) - \varphi(A).$$

Exercise 27.1. [Part 2 of Exercise 26.2] Check that the vector field $\vec{F}(x, y, z) = \frac{-\langle x,y,z\rangle}{(x^2+y^2+z^2)^{3/2}}$ has potential $\varphi(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$. What is the work needed to move the object B from the point $(1, 1, 1)$ to $(2, 1/2, 2)$ along the curve $\vec{r}(t) = \left\langle t^2 + 1, \frac{1}{t+1}, \sqrt{1 + 3t}\right\rangle$, $0 \leq t \leq 1$?

Solution. Indeed, $\vec{F}$ is equal to the gradient of φ:

$$\nabla\varphi(x, y, z) = \langle \varphi_x, \varphi_y, \varphi_z\rangle = \left\langle \frac{-x}{(x^2 + y^2 + z^2)}, \frac{-y}{(x^2 + y^2 + z^2)}, \frac{-z}{(x^2 + y^2 + z^2)} \right\rangle = \vec{F}(x, y, z).$$

The work needed to move B from the point $(1, 1, 1)$ to $(2, 1/2, 2)$ (along any path) is $\varphi(2, 1/2, 2) - \varphi(1, 1, 1) = \frac{1}{\sqrt{2^2+(1/2)^2+2^2}} - \frac{1}{\sqrt{1+1+1}} = \frac{1}{\sqrt{8+1/4}} - \frac{1}{\sqrt{3}}$.

3. If $\vec{F} = \langle f, g\rangle$ is defined in any (open) region of $\mathbb{R}^2$:
$$\vec{F} \text{ is conservative } \implies f_y = g_x.$$
If $\vec{F} = \langle f, g\rangle$ is defined in a simply connected[1] region of $\mathbb{R}^2$,
$$\vec{F} \text{ is conservative } \iff f_y = g_x.$$

4. For regions of $\mathbb{R}^3$, the condition '$f_y = g_x$' changes to '$f_y = g_x, f_z = h_x$ and $g_z = h_y$' Equivalently, the curl of the vector field is 0, as we see below.

5. $\vec{F}$ is conservative $\iff$ the line integral of $\vec{F}$ along any simple closed curve is 0.

[1] A region is simply connected if we any closed curve in the region can be continuously deformed to a single point (without leaving the region at any moment during the deformation).

Exercise 27.2. Let R be the region formed by all points of the plane except the origin.

(a) (Bonus) Is R simply connected?

Solution. No. Intuitively, this is because the region has a hole in the origin, and to deform a loop surrounding the origin to one single point we need the curve to cross the origin at some point.

(b) Is the field $\vec{F} = \frac{\langle 2x, 2y \rangle}{x^2+y^2}$ conservative?

Solution. We have $\vec{F}(x,y) = \langle f(x,y), g(x,y) \rangle = \left\langle \frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} \right\rangle$. Note that $f_y(x,y) = g_x(x,y) = \frac{2y^2-2x^2}{(x^2+y^2)^2}$, but as the region is not simply connected, this does **not** imply that the field is conservative.

Setting $\vec{F} = \nabla\varphi$, we obtain $\varphi_x = \frac{2x}{x^2+y^2}$ and $\varphi_y = \frac{2y}{x^2+y^2}$. We can try to integrate φ_x with respect to x to obtain φ, using the substitution $u = x^2$, so $du = 2dx$:

$$\int \frac{2x}{x^2+y^2} dx = \int \frac{1}{u+y^2} du = \ln(u+y^2) + C = \ln(x^2+y^2) + C.$$

And indeed, the function $\varphi(x,y) = \ln(x^2+y^2)$ satisfies $\nabla\varphi = \vec{F}$.

(c) Is the field $\vec{F} = \frac{\langle -y, x \rangle}{x^2+y^2}$ conservative?

Solution. We have $\vec{F}(x,y) = \langle f(x,y), g(x,y) \rangle = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$. Note that $f_y(x,y) = g_x(x,y) = \frac{y^2-x^2}{(x^2+y^2)^2}$, but as the region is not simply connected, this does **not** imply that the field is conservative.

And in fact the field is not conservative; considering the simple closed curve C given by $\vec{r}(t) = \langle x(t), y(t) \rangle = \langle \cos(t), \sin(t) \rangle$, $0 \leq t \leq 2\pi$ (so $\vec{r}\,'(t) = \langle -y(t), x(t) \rangle$), we have

$$\int_C \vec{F} d\vec{r} = \int_0^{2\pi} \frac{\langle -y(t), x(t) \rangle}{x(t)^2 + y(t)^2} \bullet \langle -y(t), x(t) \rangle = \int_0^{2\pi} \frac{x(t)^2 + y(t)^2}{x(t)^2 + y(t)^2} dt = 2\pi > 0.$$

28 Green's Theorem

1. Green's Theorem: If a simple, closed, piecewise smooth, counterclockwise curve C encloses a region R and $\vec{F} = \langle f, g \rangle$ is smooth, then

$$\int_C \vec{F} d\vec{r} = \iint_R \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} dA. \text{ (Circulation)}$$

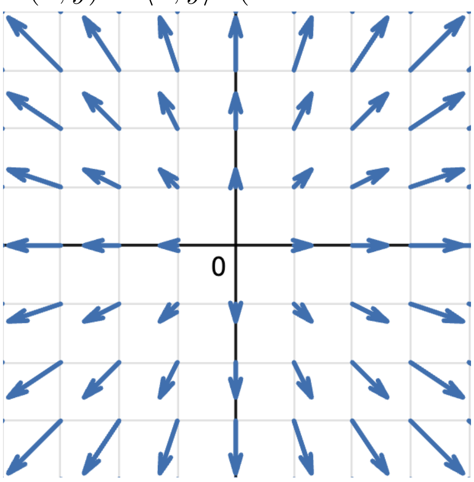
$$\int_C \vec{F} \cdot \vec{n} ds = \iint_R \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} dA. \text{ (Outward flux)}$$

2. We call $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y}$ the 'two-dimensional curl'. If it is 0, the field is 'irrotational' (and conservative, if domain is simply connected).

3. We call $\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y}$ the 'two-dimensional divergence'. If it is 0, the field is 'source-free'.

Exercise 28.1. Compute the two-dimensional curl/divergence of the following vector fields. Find the circulation/outward flux of the fields on/through the unit circle $C : \vec{r}(t) = (\cos(t), \sin(t)), 0 \leq t \leq 2\pi$.

1. $\vec{F}(x, y) = \langle x, y \rangle$. (The field has a source at $(0, 0)$)



Solution. Let R be the unit ball (disk) centered at 0.

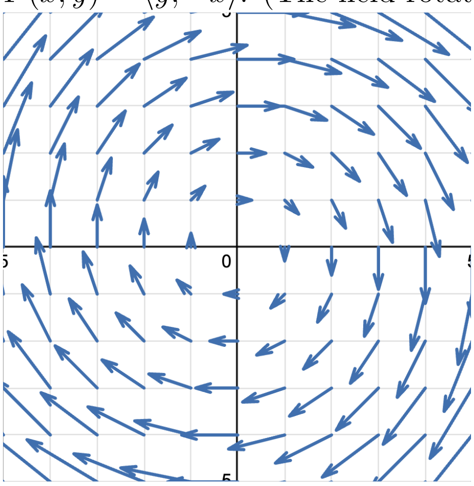
Two-dimensional curl: $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} = 0 - 0 = 0$.

So $\vec{F}$ is irrotational and conservative, the circulation on any closed curve is 0.

Divergence: $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 1 + 1 = 2$.
Outward flux is, by Green's theorem,

$$\iint_R 2dA = 2 \cdot \text{Area}(R) = 4\pi.$$

2. $\vec{F}(x, y) = \langle y, -x \rangle$. (The field rotates around $(0, 0)$)



Solution. Two-dimensional curl: $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} = -1 - 1 = -2$.
So circulation is, by Green's theorem,

$$\iint_R -2dA = -2 \cdot \text{Area}(R) = -4\pi.$$

Divergence: $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0 + 0 = 0$.
The field is 'source-free'.
Outward flux is, by Green's theorem, 0.

29 Divergence and curl

1. Divergence of a vector field $\vec{F} = \langle f, g, h \rangle$: $\operatorname{div}(\vec{F}) = \nabla \bullet \vec{F} = \dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} + \dfrac{\partial h}{\partial z}$.

2. Curl of $\vec{F}$: $\operatorname{curl}(\vec{F}) = \nabla \times \vec{F} = \left\langle \dfrac{\partial h}{\partial y} - \dfrac{\partial g}{\partial z}, \dfrac{\partial f}{\partial z} - \dfrac{\partial h}{\partial x}, \dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} \right\rangle \left(= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} \right)$

3. Functions $\xrightarrow{\text{grad}}$ Vector fields $\xrightarrow{\text{curl}}$ Vector fields $\xrightarrow{\text{div}}$ Functions

   Exercise 29.1. For the function $f(x, y, z) = e^x y^2 z$, find the curl of the gradient of f.

   Solution. For any function f, it follows from the definitions of curl and gradient that $\operatorname{curl}(\nabla f) = 0$. But let's do the computation anyways.

$$\begin{aligned} \nabla f &= \left\langle e^x y^2 z, 2e^x yz, e^x y^2 \right\rangle \\ \operatorname{curl}(\nabla f) &= \left\langle 2ye^x - 2e^x y, e^x y^2 - e^x y^2, 2e^x yz - 2e^x yz \right\rangle = \langle 0, 0, 0 \rangle . \end{aligned}$$

   Exercise 29.2. Check that the divergence of the curl of $\vec{F} = \langle xy, y^2 e^z, xyz \rangle$ is 0.

   Solution.

$$\begin{aligned} \operatorname{curl}(\vec{F}) &= \left\langle xz - y^2 e^z, 0 - yz, 0 - x \right\rangle = \left\langle xz - y^2 e^z, -yz, -x \right\rangle . \\ \operatorname{div}(\operatorname{curl}(\vec{F})) &= z - z + 0 = 0. \end{aligned}$$

4. If a vector field $\vec{F}$ is conservative, then $\operatorname{curl}(\vec{F})$ is 0. Reciprocally, if $\operatorname{curl}(\vec{F}) = 0$ and $\vec{F}$ is defined in a simply connected (open) region, then $\vec{F}$ is conservative.

   Exercise 29.3. Is the vector field $\vec{F}(x, y, z) = \langle 2xy - y, x^2 + 3z^2 + x, 6yz \rangle$ conservative?

   Solution. The field is defined in all $\mathbb{R}^3$, which is simply connected, so we just need to check whether the curl is 0.

$$\operatorname{curl}(\vec{F}) = \langle 6z - 6z, 0 - 0, (2x + 1) - (2x - 1) \rangle = \langle 0, 0, 2 \rangle \neq 0,$$

   so the field is not conservative.

30 Surfaces

1. A surface S in $\mathbb{R}^3$ can be parametrized using two parameters u, v:

$$S : \vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle \, , \text{ where } (u,v) \text{ vary over a region } R.$$

2. Tangent vectors to the surface in directions given by u, v:

$$\vec{t_u} = \frac{\partial}{\partial u}\vec{r}(u,v) = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle, \text{ and } \vec{t_v} = \frac{\partial}{\partial v}\vec{r}(u,v) = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle.$$

3. Surface integral of scalar valued functions:

$$\iint_S f(x,y,z)dS = \iint_R f(x(u,v), y(u,v), z(u,v))|\vec{t_u} \times \vec{t_v}|dA.$$

4. Area of S: $\iint_S 1dS$. Average of $f(x,y,z)$ on S: $\frac{1}{\text{Area}(S)} \iint_S f(x,y,z)dS$.

Exercise 30.1. We parametrize a part S of the cone $z^2 = x^2 + y^2$ as $\vec{r}(u,v) = (u, v, \sqrt{u^2+v^2})$, $-1 \leq u, v \leq 1$. Find the area of S, and the average in S of $f(x,y,z) = z^2 - x$.

Solution. The tangent vectors are $\vec{t_u} = \left\langle 1, 0, \frac{u}{\sqrt{u^2+v^2}} \right\rangle$ and $\vec{t_v} = \left\langle 0, 1, \frac{v}{\sqrt{u^2+v^2}} \right\rangle$. So

$$\vec{t_u} \times \vec{t_v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{u}{\sqrt{u^2+v^2}} \\ 0 & 1 & \frac{v}{\sqrt{u^2+v^2}} \end{vmatrix} = \left\langle \frac{-u}{\sqrt{u^2+v^2}}, \frac{-v}{\sqrt{u^2+v^2}}, -1 \right\rangle.$$

So

$$|\vec{t_u} \times \vec{t_v}| = \sqrt{\left(\frac{-u}{\sqrt{u^2+v^2}}\right)^2 + \left(\frac{-v}{\sqrt{u^2+v^2}}\right)^2 + 1} = \sqrt{\frac{u^2+v^2}{u^2+v^2} + 1} = \sqrt{1+1} = \sqrt{2}.$$

Using this we can find the area:

$$\text{Area}(S) = \iint_S 1dS = \iint_R 1 \cdot \sqrt{2}dA = \int_{-1}^{1}\int_{-1}^{1} \sqrt{2}dudv = \int_{-1}^{1} 2\sqrt{2}dv = 4\sqrt{2}.$$

And we now find the average:

$$\begin{aligned}
\overline{f} &= \frac{1}{4\sqrt{2}} \iint_S (z^2 - x) dS \\
&= \frac{1}{4\sqrt{2}} \iint_R \left(\left(\sqrt{u^2 + v^2} \right)^2 - u \right) \cdot \sqrt{2} dA \\
&= \frac{1}{4\sqrt{2}} \int_{-1}^{1} \int_{-1}^{1} (u^2 + v^2 - u) \sqrt{2} du dv \\
&= \frac{1}{4} \int_{-1}^{1} \int_{-1}^{1} (u^2 + v^2 - u) du dv \\
&= \frac{1}{4} \int_{-1}^{1} \left[\frac{u^3}{3} + v^2 u - \frac{u^2}{2} \right]_{u=-1}^{1} dv \\
&= \frac{1}{4} \int_{-1}^{1} \left(\frac{1}{3} + v^2 - \frac{1}{2} \right) - \left(\frac{-1}{3} - v^2 - \frac{1}{2} \right) dv \\
&= \frac{1}{4} \int_{-1}^{1} \frac{2}{3} + 2v^2 dv \\
&= \frac{1}{4} \left(\frac{4}{3} + \frac{4}{3} \right) = \frac{2}{3}.
\end{aligned}$$

31 Surface integrals of vector fields

Let $S : \vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, where (u,v) varies over R, be an orientable surface.

1. Oriented surface: Surface with a choice of orientation of its unit normal vectors $\vec{n}$.

2. Each parametrization $\vec{r}(u,v)$ of a surface S defines an orientation of S, according to the unit normal vector $\vec{n}$ in the direction of $\vec{t}_u \times \vec{t}_v$. That is, $\vec{n} = \frac{\vec{t}_u \times \vec{t}_v}{|\vec{t}_u \times \vec{t}_v|}$.

   Exercise 31.1. (Bonus) Consider the rectangle $S : \vec{r}(u,v) = \langle u, v, 1\rangle \,; -1 \le u, v \le 1$. Do the parametrizations $\vec{r}_2(u,v) = \vec{r}(-u,v)$ and $\vec{r}_3(u,v) = \vec{r}(-u,-v)$ have the same, or different orientations from S?

   Solution. In $\vec{r}(u,v)$ the normal vector $\vec{n} = \vec{t}_u \times \vec{t}_v$ is just $\langle 0,0,1\rangle$, it points up from the rectangle.

   $\vec{r}_2(-u,v)$ has different orientation (the normal vector points down), while $\vec{r}_3(u,v)$ has the same orientation as $\vec{r}(u,v)$ (the vector again points up).

3. Surface integral of a vector field in S (measures flow of $\vec{F}$ through S in direction of $\vec{n}$):

$$\iint_S \vec{F} \bullet \vec{n} dS = \iint_R \vec{F}(\vec{r}(u,v)) \bullet (\vec{t}_u \times \vec{t}_v) dA = \iint_R \det \left[\vec{F}(\vec{r}(u,v)) \,\middle|\, \vec{t}_u(u,v) \,\middle|\, \vec{t}_v(u,v) \right] dA.$$

   Exercise 31.2. Express the surface integral in the oriented sphere

$$S : \vec{r}(u,v) = \langle \sin(u)\cos(v), \sin(u)\sin(v), \cos(u)\rangle \,;\; 0 \le u \le \pi, 0 \le v \le 2\pi$$

   of the vector field $\vec{F} = \langle x, 0, 0\rangle$, as a double integral $dudv$.

   Solution. Note that in this case, the tangent vectors are

$$\begin{aligned} \vec{t}_u &= \langle \cos(u)\cos(v), \cos(u)\sin(v), -\sin(u)\rangle \\ \vec{t}_v &= \langle -\sin(u)\sin(v), \sin(u)\cos(v), 0\rangle \,. \end{aligned}$$

$$\begin{aligned} \iint_S \vec{F} \bullet \vec{n} dS &= \iint_R \det \left[\vec{F}(\vec{r}(u,v)) \,\middle|\, \vec{t}_u(u,v) \,\middle|\, \vec{t}_v(u,v) \right] dA \\ &= \iint_R \begin{bmatrix} \sin(u)\cos(v) & \cos(u)\cos(v) & -\sin(u)\sin(v) \\ 0 & \cos(u)\sin(v) & \sin(u)\cos(v) \\ 0 & -\sin(u) & 0 \end{bmatrix} dA \\ &= \iint_R \sin(u)^3 \cos(v)^2 dA \\ &= \int_0^{2\pi} \int_0^{\pi} \sin(u)^3 \cos(v)^2 dudv. \end{aligned}$$

32 Divergence theorem

The outward flux of $\vec{F}$ is the integral of $\operatorname{div}(\vec{F})$.

More concretely, if D is the region enclosed by a surface S, and $\vec{n}$ is the outward unit normal vector to S, and $\vec{F}$ is a smooth vector field defined in all the region D, then

$$\int_S \vec{F} \bullet \vec{n} dS = \iiint_D \nabla \bullet \vec{F} dV.$$

Exercise 32.1. True or false?

1. If a smooth vector field $\vec{F}$ defined in all $\mathbb{R}^3$ has $\operatorname{div}(\vec{F}) = 0$ and S is the unit sphere, then $\iint_S \vec{F} \bullet \vec{n} dS = 0$.

   Solution. True, as letting D be the unit ball centered at the origin, we have

   $$\int_S \vec{F} \bullet \vec{n} dS = \iiint_D \nabla \bullet \vec{F} dV = \iiint_D 0 dV = 0.$$

2. If a smooth vector field $\vec{F}$ defined in all $\mathbb{R}^3$ has $\operatorname{div}(\vec{F}) = 0$ and $S : \vec{r}(u, v) = \langle u, v, 1\rangle$; $-1 \leq u, v \leq 1$, then $\iint_S \vec{F} \bullet \vec{n} dS = 0$.

   Solution. Not necessarily true, as S does not enclose a region.

Exercise 32.2. Find the outward flow of $\vec{F}(x, y, z) = \langle x, y + x^2, z - x\rangle$ across the boundary S of the cylinder $D : \{x^2 + y^2 \leq 1; -1 \leq z \leq 1\}$.
Solution. The divergence of the vector field is $\operatorname{div}(\vec{F}) = f_x + g_y + h_z = 1 + 1 + 1 = 3$. So

$$\int_S \vec{F} \bullet \vec{n} dS = \iiint_D \nabla \bullet \vec{F} dV = \iiint_D 3dV = 3\mathrm{Vol(D)} = 3 \cdot (2\pi) = 6\pi.$$

33 Stokes' theorem

We consider an oriented surface S, oriented by unit normal vectors $\vec{n}(u, v)$, and its boundary, a curve C, with orientation given by its unit tangent vector $\vec{T}(t)$.

1. Changing orientation of the surface $S \to$ changing normal vector $\vec{n}$ to its opposite, $-\vec{n}$.

   Changing orientation of the curve $C \to$ changing tangent vector $\vec{T}$ to its opposite.

2. We say $C : \vec{s}(t) = (x(t), y(t), z(t))$ is oriented consistently with S if at all points of C, the vector $\vec{T} \times \vec{n}$ points outside of the surface.

   Exercise 33.1. True or false? If we change the orientation of a surface, then the corresponding orientation of the boundary of the surface remains the same.

   Solution. Suppose that the curve C with tangent vector $\vec{T}(t)$, and the surface S, with normal vector $\vec{n}(u, v)$, are oriented consistently, so that $\vec{T} \times \vec{n}$ points outside of the surface.

   Changing orientation of S means that our new normal vector is $\vec{-n}$, so $\vec{T} \times (-\vec{n}) = -\vec{T} \times \vec{n}$ points inside the surface. So we have to change the orientation of the curve (to have tangent vector $-\vec{T}$) for the orientations to be consistent again.

3. Stokes' theorem: If C is oriented consistently with S and $\vec{F}$ is a smooth vector field,

$$\int_C \vec{F} d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n} dS.$$

   Exercise 33.2. [Ex 12, textbook] Find $\int_C \vec{F} d\vec{r}$, where $\vec{F}(x, y, z) = \langle y, xz, -y \rangle$ and C is the ellipse $x^2 + y^2/4 = 1$ in the plane $z = 1$, either directly or using Stokes' theorem.

   Solution. The answer depends on the orientation (so, on the parametrization) of the curve.

   A way to parametrize the ellipse is as $\vec{c}(t) = \langle \cos(t), 2\sin(t), 1 \rangle, 0 \leq t \leq 2\pi$. This ellipse is the boundary of the surface $S : \vec{r}(u, v) = \langle u, v, 1 \rangle ; u^2 + v^2/4 \leq 1$ (and their parametrizations are consistent).

   If we try to compute the line integral directly, we obtain:

$$\begin{aligned}
\int_C \vec{F} d\vec{r} &= \int_0^{2\pi} \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t) dt \\
&= \int_0^{2\pi} \langle 2\sin(t), \cos(t), -2\sin(t) \rangle \cdot \langle -\sin(t), 2\cos(t), 0 \rangle \\
&= \int_0^{2\pi} -2\sin^2(t) + 2\cos^2(t) dt \\
&= \int_0^{2\pi} 2\cos(2t) dt = [\sin(2t)]_0^{2\pi} = 0.
\end{aligned}$$

Using Stokes' theorem in the surface S, note that the curl is

$$\nabla \times \vec{F} = \langle h_y - g_z, f_z - h_x, g_x - f_y \rangle = \langle -1 - x, 0, z - 1 \rangle .$$

The tangent vectors are $\vec{t_u} = \langle 1, 0, 0 \rangle , \vec{t_v} = \langle 0, 1, 0 \rangle$, so $\vec{t_u} \times \vec{t_v} = \langle 0, 0, 1 \rangle$, so the normal vector to S is constant, $\vec{n} = \langle 0, 0, 1 \rangle$. Finally, we use Stokes' theorem:

$$\begin{aligned}
\int_C \vec{F} d\vec{r} &= \iint_S (\nabla \times \vec{F}) \cdot \vec{n} dS \\
&= \iint_S \langle -1 - u, 0, 1 - 1 \rangle \cdot \langle 0, 0, 1 \rangle \, dS \\
&= \iint_S 0 dS = 0.
\end{aligned}$$