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Recitation 1

Exercise 1.1. A smooth function $f(x,y)$ satisfies $f(0,0) = 1$ and $\nabla f(0,0) = (1,0)$.

(a) Let $\delta > 0$ be a small number. Which of the following values is the biggest?

  (i) $f(\delta, 0)$

  (ii) $f(0, \delta)$

  (iii) $f(-\delta, 0)$

(b) Let c be the level curve of f containing the point $(0,0)$. Is c tangent to the x-axis, the y-axis, or neither of them?

(c) What linear function $L(x,y)$ approximates $f(x,y)$ best near the point $(0,0)$?

Exercise 1.2. Let $f(x,y) = x + 4y$.

(a) What is the equation of the level curve c of f passing through the point $(0,1)$? Draw c.

(b) Draw the values of the gradient of f at several points in the level curve c above.

(c) What is the directional derivative of f at the point $(0,1)$ in the direction of $\vec{v} = (0,2)$? And the rate of change?

Exercise 1.3. A function $f(x, y, z)$, whose formula we don't know, has $\{(x, y, z); x^2 + y^2 + z^2 = 1\}$ as one of its level sets. What are the possible values of $\nabla f(0, 0, 1)$?

Exercise 1.4. The directional derivatives of a function $f(x, y)$ at the point $(1, 1)$, in the directions $\vec{u} = (1, 1)$ and $\vec{v} = (1, -1)$, have been determined to be $D_{\vec{u}} f(1, 1) = 1$ and $D_{\vec{v}} f(1, 1) = 2$. What is $\partial_x f(1, 1)$?

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Recitation 2

For points $p = (a, b)$ in the interior of the domain of a smooth function $f(x, y)$:

Local maximum. f has a local maximum at p if there is a disk U around p such that $f(p) \geq f(q)$ for all points $q \in U$.

Local minimum. f has a local minimum at p if there is a disk U around p such that $f(p) \leq f(q)$ for all points $q \in U$.

All local maximums and minimums p of f are critical points, that is, $\nabla f(p) = 0$.

Exercise 2.1. Find all the critical points of the following functions:

(a) $f(x, y) = 3x^2 + 3y - y^3$ with domain $\mathbb{R}^2$.

Solution. We take $0 = \nabla f(x, y) = (\partial_x f, \partial_y f)$, so $\partial_x f(x, y) = 0$ and $\partial_y f(x, y) = 0$.

$$0 = \partial_x f(x, y) = 6x; x = 0.$$
$$0 = \partial_y f(x, y) = 3 - 3y^2; 3y^2 = 3; y^2 = 1; y = \pm 1.$$

So f has two critical points: $(0, 1)$ and $(0, -1)$.

(b) $f(x, y, z) = x^2 + y^2 + z^2 + xy$, with domain $\mathbb{R}^3$.

Solution. We find the critical points:

$$0 = \partial_x f(x, y, z) = 2x + y$$
$$0 = \partial_y f(x, y, z) = 2y + x$$
$$0 = \partial_z f(x, y, z) = 2z.$$

So $z = 0$, and from the equations $2x + y = 0$ and $2y + x = 0$ we deduce (e.g. by taking $x = -2y$ and substituting in the equation $2x + y = 0$) that $x = y = 0$. The only critical point is thus $(0, 0, 0)$.

(c) $f(x, y) = \sqrt{1 - x^2 - y^2}$, with domain the unit disk $\{(x, y) \in \mathbb{R}^2; x^2 + y^2 \leq 1\}$.

Solution. We take $\nabla f = 0$ to find the critical points:

$$0 = \partial_x f(x, y, z) = \frac{-x}{\sqrt{1 - x^2 - y^2}}$$
$$0 = \partial_y f(x, y, z) = \frac{-y}{\sqrt{1 - x^2 - y^2}}$$

So $x = 0, y = 0$; the only critical point is $(0, 0)$.

Second order partial derivatives We denote $\partial_{xx} f = \partial_x(\partial_x f)$, $\partial_{xy} f = \partial_x(\partial_y f)$ and so on.
We always have $\partial_{xy} f = \partial_{yx} f$ if f is smooth.

Discriminant The discriminant of $f(x, y)$ at p is given by

$$D(p) = \begin{vmatrix} \partial_{xx} f(p) & \partial_{xy} f(p) \\ \partial_{yx} f(p) & \partial_{yy} f(p) \end{vmatrix}, \text{or just } D(p) = \partial_{xx} f(p) \cdot \partial_{yy} f(p) - \partial_{xy} f(p)^2.$$

Second derivative test. At a critical point p of a two-variable function $f(x,y)$, several things can happen:

- $D(p) > 0$, $\partial_{xx}f(p) < 0 \to$ Local maximum.
- $D(p) > 0$, $\partial_{xx}f(p) > 0 \to$ Local minimum.
- $D(p) < 0 \to$ Saddle point (saddle point just means 'critical point but not local max or min').
- $D(p) = 0 \to$ Inconclusive; we need more information to know whether it is max, min or saddle.

Exercise 2.2. Find the critical points of the following functions. Use the Second Derivative Test to determine whether each critical point corresponds to a local maximum, minimum, or saddle point.

(a) $f(x,y) = x^4 + 2y^2 - 4xy$.

Solution. Critical points: $\begin{cases} 0 &= \partial_x f(x,y) &= 4x^3 - 4y; \quad y = x^3. \\ 0 &= \partial_y f(x,y) &= 4y - 4x; \quad y = x. \end{cases}$

So $x = x^3; x(x^2-1) = 0; x = 0, 1$ or -1.

As $y = x$, we have three critical points: $(0,0), (1,1), (-1,-1)$.

Second order partial derivatives: $\partial_{xx}f(x,y) = \partial_x(4x^3 - 4y) = 12x^2$. $\partial_{xy}f(x,y) = \partial_x(4y - 4x) = 4$. $\partial_{yy}f(x,y) = \partial_y(4y-4x) = 4$.

So the discriminant is $D(x,y) = (\partial_{xx}f)\cdot(\partial_{yy}f) - (\partial_{xy}f)^2 = 12x^2 \cdot 4 - 4^2 = 48x^2 - 16$. So, applying the second derivative test,

- Point $(0,0)$: $D(0,0) = -16 < 0$: saddle point.
- Point $(1,1)$: $D(1,1) = 32 > 0$, and $\partial_{xx}f(1,1) = 12 \cdot 1^2 > 0$, local minimum.
- Point $(-1,-1)$: $D(-1,-1) = 32 > 0$, and $\partial_{xx}f(1,1) = 12 \cdot 1^2 > 0$, local minimum.

(b) $f(x,y) = (4x-1)^2 + (2y+4)^2 + 1$.

Solution. Critical points: $\begin{cases} 0 &= \partial_x f(x,y) &= 8(4x-1); \quad x = \frac{1}{4}. \\ 0 &= \partial_y f(x,y) &= 4(2y+4); \quad y = -2. \end{cases}$

So the only critical point is $\left(\frac{1}{4}, -2\right)$. To apply the second derivative test, we first need to compute the second order partial derivatives:

$$\partial_{xx}f(x,y) = \partial_x(8(4x-1)) = 32; \partial_{xy}f(x,y) = 0 = \partial_x(4(2y+4)) = 0; \partial_{yy}f(x,y) = \partial_y(4(2y+4)) = 8.$$

So the discriminant is $D(x,y) = (\partial_{xx}f)\cdot(\partial_{yy}f) - (\partial_{xy}f)^2 = 32 \cdot 8 = 256 > 0$. Meaning that at the critical point, we have $D\left(\frac{1}{4}, -2\right) = 256 > 0$ and $\partial_{xx}f(x,y) = 32 > 0$, so it is a local minimum.

(c) $f(x,y) = x^4 + y^4$.

Solution. Critical points: $\begin{cases} 0 &= \partial_x f(x,y) &= 4x^3; \quad x = 0. \\ 0 &= \partial_y f(x,y) &= 4y^3; \quad y = 0. \end{cases}$

Just one critical point, $(0,0)$.

The second derivatives are $\partial_{xx}f(x,y) = 12x^2, \partial_{xy}f(x,y) = 0, \partial_{yy}f(x,y) = 12y^2$. So the discriminant is $D(x,y) = (\partial_{xx}f)\cdot(\partial_{yy}f) - (\partial_{xy}f)^2 = (12x^2)\cdot(12y^2) = 144x^2y^4$. At our point, $D(0,0) = 0$, so the test is inconclusive.

Using methods other than the second derivative test (namely, directly checking that $f(x,y) \geq f(0,0)$ for all $x, y \in \mathbb{R}$), one can check that the point $(0,0)$ is a minimum.

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Recitation 3

Let $f : D \to \mathbb{R}$ be a smooth function with domain $D \subseteq \mathbb{R}^2$ (or $\mathbb{R}^3$), and let p be a point of D.

Absolute maximum. f has an absolute maximum at $p \in D$ if $f(p) \geq f(q)$ for all $q \in D$. The absolute maximum is the 'highest point in the graph of f'.

Now consider a curve $C = \{(x, y); g(x, y) = k\}$ inside D.

Lagrange multipliers. If $p = (x, y) \in C$ is an absolute maximum of f in C and $\nabla g(p) \neq 0$, then

$$\nabla f(x, y) = \lambda \nabla g(x, y) \text{ for some number } \lambda.$$

The condition $\nabla g \neq 0$ ensures that the curve C is smooth near the point p (and orthogonal to ∇g).

Finding the absolute maximum of f in the curve C. The equations $g(x, y) = 0$ and $\nabla f(x, y) = \lambda \nabla g(x, y)$ give us a system of equations in x, y, λ, whose solutions are candidates for absolute maximum of f in C. If the maximum exists, it is the candidate in which f has the biggest value.

If the curve C has endpoints (e.g. if C is a segment), corners (e.g. C is a rectangle) or points where $\nabla g = 0$ (which hopefully won't happen in this course), we should add those points as candidates, just in case.

Exercise 3.1. Find the maximum of $f(x, y) = x + y$ in the unit circumference $C = \{(x, y); x^2 + y^2 = 1\}$.
Solution. In this case the equation $g(x, y) = k$ is just $x^2 + y^2 = 1$, so we can take $g(x, y) = x^2 + y^2$. So $\nabla g(x, y) = (2x, 2y)$ and $\nabla f(x, y) = (1, 1)$. Our system of equations is:

$$x^2 + y^2 = 1$$

$$\nabla f(x, y) = \lambda \nabla g(x, y); (1, 1) = \lambda(2x, 2y); \begin{cases} 1 & = 2\lambda x \\ 1 & = 2\lambda y \end{cases}$$

As $2\lambda y = 2\lambda x = 1$, we have $2\lambda(x - y) = 0$, so either $\lambda = 0$ (impossible, as $2\lambda x = 1$) or $x = y$. So $x = y$. Substituting $x = y$ in the first equation $x^2 + y^2 = 1$, we obtain $2x^2 = 1; x = \pm\frac{1}{\sqrt{2}}$, so using $y = x$ we have two possible maximums: $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$.

As $f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \sqrt{2}$ and $f\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) = -\sqrt{2}$, the maximum of f in C is $f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \sqrt{2}$.

Finding the absolute maximum of f in the domain D. Absolute maximums can be:

1) In the interior of D. These are critical points.

2) In the boundary of D. We can find these using Lagrange multipliers, or other methods like parameterizing the boundary. If the boundary has vertices (e.g. if it's a rectangle), we can include them as candidates for absolute maximum just in case.

Parts $(a), (b)$ above give us some candidates for the absolute maximum. If the absolute maximum of f exists, then it is the candidate point p for which $f(p)$ is the biggest.

Exercise 3.2. Find the absolute maximum of $f(x,y) = x^2 + y^3 - xy$, if the variable x can take values between 0 and 2 and y can take values between 0 and 1.

Solution. The domain is the rectangle $D = [0,2] \times [0,1]$, with vertices $(0,0), (0,1), (2,0), (2,1)$.

We now look for candidates for the absolute maximum.

1) In the interior of D: let's find critical points of f.

$$0 = \partial_x f(x,y) = 2x - y; y = 2x.$$
$$0 = \partial_y f(x,y) = 3y^2 - x = 3(2x)^2 - x = 12x^2 - x = x(12x-1); x = 0 \text{ or } x = \frac{1}{12}.$$

If $x = 0$ then $y = 2x = 0$, if $x = \frac{1}{12}$ then $y = 2x = \frac{1}{6}$. So there are two critical points: $(0,0)$ and $\left(\frac{1}{12}, \frac{1}{6}\right)$. $(0,0)$ is not in the interior of D (it is in the boundary), so we ignore it. $\left(\frac{1}{12}, \frac{1}{6}\right)$ is in the interior of D, so it is the first candidate for absolute maximum.

2) In the boundary ∂D. There are four segments in ∂D, so we have to find the maximum of f in each of them. Recall that a segment from a point a to b can be parameterized as $(1-t) \cdot a + t \cdot b$, for t in $[0,1]$.

   (i) Segment s_1 between $(0,0)$ and $(0,1)$. This segment consists of the points $(1-t)\cdot(0,0)+t\cdot(0,1) = (0,0) + (0,t) = (0,t)$, $t \in [0,1]$. In this segment, the function f is

   $$f_1(t) = f(0,t) = t^3, t \in [0,1].$$

   The maximum of $f_1(t) = t^3$ for t in $[0,1]$ is $t = 1$ (point $(0,1)$), as the function t^3 is increasing. So the maximum of f in the segment s_1 is at the point $(0,1)$.

   (ii) Segment s_2 between $(0,1)$ and $(2,1)$: $(1-t)\cdot(0,1) + t(2,1) = (0, 1-t) + (2t, t) = (2t, 1)$, $t \in [0,1]$. So $f_2(t) = f(2t, 1) = 4t^2 + 1 - 2t$, $t \in [0,1]$. The function f_2 has maximum at $t = 1$ (this follows from arguments seen e.g. in Calc 1), so the maximum of f in s_2 is at $(2,1)$.

   (iii) Segment s_3 between $(2,1)$ and $(2,0)$: $(1-t)\cdot(2,1)+t(2,0) = (2(1-t), 1-t)+(2t,0) = (2, 1-t)$, $t \in [0,1]$. So $f_3(t) = f(2, 1-t) = 4+(1-t)^3 - 2(1-t)$, $t \in [0,1]$. The function f_3 has maximum at $t = 1$, so the maximum of f in s_3 is at $(2,0)$.

   (iv) Segment s_4 between $(0,0)$ and $(2,0)$: $(1-t)\cdot(0,0) + t(2,0) = (2t, 0)$, $t \in [0,1]$. So $f_4(t) = f(2t, 0) = 4t^2$, $t \in [0,1]$. The function $f_3(t)$ has maximum at $t = 1$, so the maximum of f in s_4 is at $(2,0)$.

   The candidates for absolute maximum are $\left(\frac{1}{12}, \frac{1}{6}\right), (0,1), (2,1)$ and $(2,0)$. We have $f\left(\frac{1}{12}, \frac{1}{6}\right) = \frac{-1}{432}$, $f(0,1) = 1$, $f(2,1) = 3$, $f(2,0) = 4$. So the maximum of f in the rectangle D is $f(2,0) = 4$.

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Recitation 4

Double integral $\iint_R f(x,y)dA$ in the region $R = \{(x,y); a \le x \le b; g(x) \le y \le h(x)\}$ (assuming $g(x) \le h(x)$ for $x \in [a,b]$):

$$\int_a^b \int_{g(x)}^{h(x)} f(x,y)dydx$$

Double integral $\iint_R f(x,y)dA$ in the region $R = \{(x,y); a \le y \le b; g(y) \le x \le h(y)\}$:

$$\int_a^b \int_{g(y)}^{h(y)} f(x,y)dxdy$$

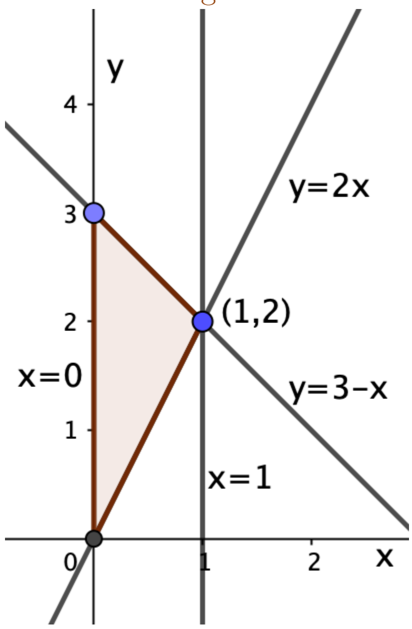
Exercise 4.1. Sketch the region of integration corresponding to the following integrals, and fully label your sketch

(a) $\displaystyle\int_0^1 \int_{2x}^{3-x} f(x,y)dydx.$

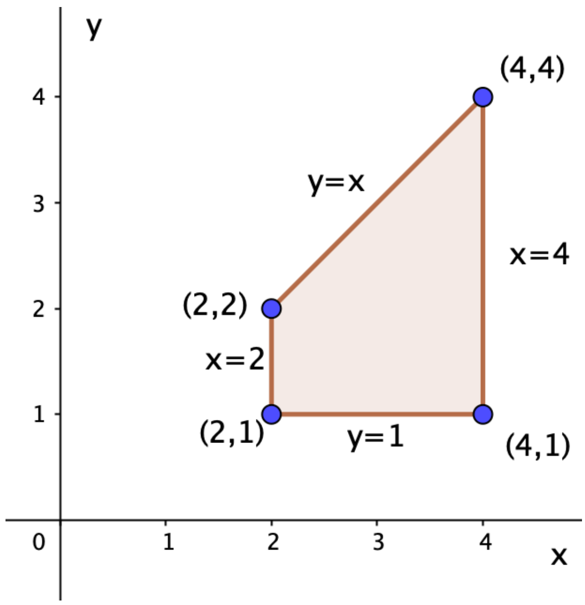
(b) $\displaystyle\iint_R f dA$, where

$$R = \{(x,y); 2 \le x \le 4; 1 \le y \le x\}.$$

Solution. Note that in the sketches below we have marked the x and y axes, we have shaded the regions and marked the boundary clearly, and we have marked the coordinates of the vertices of the regions (this can also be done using lines to the x and y axes).



(a)



(b)

Exercise 4.2. Reverse the order of integration in the following integrals (that is, change them from $dxdy$ to $dydx$ or viceversa).

(a) $\displaystyle\int_0^2 \int_{y-1}^1 f(x,y)dxdy.$

Solution. The region of integration is the triangle with vertices $(-1,0), (1,0), (1,2)$. So the integral $dydx$ is

$$\int_{-1}^1 \int_0^{x+1} f(x,y)dydx$$

(b) $\displaystyle\int_{-1}^{1}\int_{x^2}^{1} f(x,y)dydx.$

Solution.

In this case we have the region between the parabola $y = x^2$ and $y = 1$. The integral $dxdy$ is

$$\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)dxdy$$

Exercise 4.3. Find the value of the following integrals.

(a) $\displaystyle\int_0^4 \int_{-1}^{1} x^2 y \, dxdy.$

Solution.

$$\begin{aligned}\int_0^4 \left(\int_{-1}^{1} x^2 y \, dx\right) dy &= \int_0^4 \left[\frac{x^3}{3} y \, dx\right]_{-1}^{1} dy \\ &= \int_0^4 \frac{y}{3} - \left(-\frac{y}{3}\right) dy \\ &= \int_0^4 \frac{2y}{3} dy = \left[\frac{y^2}{3}\right]_0^4 = \frac{16}{3}.\end{aligned}$$

(b) $\displaystyle\int_0^1 \int_0^y x^2 \cos\left(y^4\right) dxdy$

Solution.

$$\begin{aligned}\int_0^1 \left(\int_0^y x^2 \cos\left(y^4\right) dx\right) dy &= \int_0^1 \left[\frac{x^3}{3} \cos\left(y^4\right) dx\right]_0^y dy \\ &= \int_0^1 \frac{y^3}{3} \cos\left(y^4\right) dy \text{ (Substituting } u = y^4) \\ &= \left[\frac{1}{12} \sin(y^4)\right]_0^1 = \frac{1}{12} \sin(1).\end{aligned}$$

(c) $\displaystyle\iint_R 1dA$, where $R = \{(x,y); x^2 + y^2 \leq 1\}$

Solution. The integral $\iint_R 1dA$ is just the area of R, in this case R is the unit disk, so its area is π.

(d) $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2} dydx$ (hint: the volume of the unit ball is $\frac{4}{3}\pi$).

Solution. The integral above corresponds to the volume of the region A between the graph of $\sqrt{1-x^2-y^2}$ and the xy plane, and above the region $R = \left\{(x,y); 0 \leq x \leq 1, 0 \leq y \leq \sqrt{1-x^2}\right\}$. But this region A is just one octant of the unit ball $\{(x,y,z); x^2 + y^2 + z^2 \leq 1\}$, so the volume of A is $\frac{1}{8} \cdot \frac{4}{3}\pi = \frac{\pi}{6}$.

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Recitation 5

Polar coordinates. $x = r\cos(\theta)$, $y = r\sin(\theta)$. If $(x,y) \neq 0$, the polar coordinates of (x,y) are:

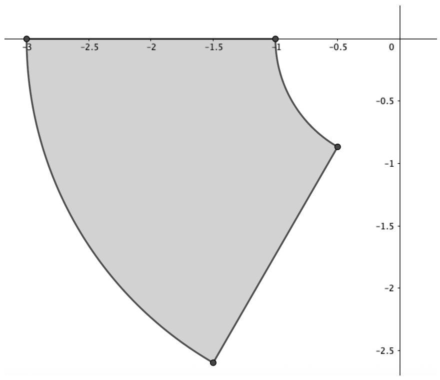
- $r = \sqrt{x^2+y^2}$
- θ is the angle between the positive x-axis and the vector (x,y). It can be expressed as $\arctan\left(\frac{y}{x}\right)$ plus some multiple of π.

Integrals in polar coordinates We have $dA = dxdy = r \cdot drd\theta$.

Exercise 5.1. Sketch the region of points R with polar coordinates r, θ such that $1 \leq r \leq 3$ and $\theta \in \left[\pi, \frac{4\pi}{3}\right]$, and compute the following integral:

$$\iint_R r^2 \sin(\theta) dA.$$

Solution.



To compute the integral, using $dA = rdrd\theta$ we have that

$$\int_R r^2 \sin(\theta) dA = \int_R r^3 \sin(\theta) drd\theta = \int_\pi^{4\pi/3}\int_1^3 r^3 \sin(\theta) drd\theta = \int_\pi^{4\pi/3} \sin(\theta)\int_1^3 r^3 drd\theta$$
$$= \int_\pi^{4\pi/3} \sin(\theta)\left(\frac{3^4}{4} - \frac{1}{4}\right) d\theta = 20\int_\pi^{4\pi/3} \sin(\theta) d\theta = 20\left(-\cos(4\pi/3) + \cos(\pi)\right) = 20\left(\frac{1}{2} - 1\right) = -10.$$

Now some short exercises as midterm review.

Exercise 5.2. True or false. If all the absolute extremums of a smooth function are in the boundary of the domain, then there cannot be any critical points in the interior.

(a) True

(b) False

Solution. False. For example a function can have just one critical point in the domain which is a saddle point (so not a local max/min, so not an absolute max/min). An example of this is the function $f(x,y) = xy$ with domain the square $\{(x,y); -1 \leq x, y \leq 1\}$.

Exercise 5.3. A function $f(x,y)$ has a critical point at a point p in the interior of the domain, and we only know that $\partial_{xx} f(x,y) > 0$ and $\partial_{yy} f(x,y) < 0$. Which of the following could f have at p?

(a) Local maximum

(b) Local minimum

(c) Saddle point

Solution. Is it a saddle point by the second derivative test, because the discriminant is negative:

$$D(x,y) = \partial_{xx} f(x,y) \cdot \partial_{yy} f(x,y) - \partial_{xy} f(x,y)^2 < \partial_{xx} f(x,y) \cdot \partial_{yy} f(x,y) < 0.$$

Exercise 5.4. A smooth function $f(x,y)$ with domain $\{(x,y); 1 \leq x \leq 3; 1 \leq y \leq x\}$ satisfies $\partial_x f(x,y) > 0$ and $\partial_y f(x,y) < 0$ for all x, y. Which point is the absolute maximum of $f(x,y)$?

(a) $(1,1)$

(b) $(3,1)$

(c) $(1,3)$

(d) $(3,3)$

Solution. The absolute maximum is at $(3,1)$, the bottom-right vertex of the triangle. This is because $f(x,y)$ increases towards the right and towards the bottom, due to the conditions $\partial_x f(x,y) > 0$ and $\partial_y f(x,y) < 0$.

Exercise 5.5. In which of the following cases is it true that the integral $\int_R f(x,y) - g(x,y) dA$ represents the (positive) volume between the graphs of smooth functions f and g and above the region R? Select all correct choices

(a) f, g are both positive in R.

(b) $f(p) > 0$ and $g(p) < 0$ for all $p \in R$.

(c) $f(p) \geq g(p)$ for all $p \in R$.

(d) $|f(p)| \geq |g(p)|$ for all $p \in R$.

Solution. In general, the given integral represents the volume between the graphs of f and g whenever $f(p) \geq g(p)$ for all $p \in R$ (if not part of the volume would count as negative when computing the integral). So c is correct, and b is also correct because if $f(p)$ is positive and $g(p)$ is negative, then $f(p) \geq g(p)$. The other two answers, a and c, are not correct.

Exercise 5.6. Find the rate of change of $f(x,y) = x^2y + y^2x$ at $p = (1,2)$ in the direction of $\vec{v} = (3,4)$.
Solution. The gradient of f is $\nabla f(x,y) = (2xy + y^2, x^2 + 2xy)$, so $\nabla f(1,2) = (8,5)$. So the directional derivative is

$$D_p f(\vec{v}) = (\nabla f(p)) \bullet \frac{\vec{v}}{|\vec{v}|} = (8,5) \bullet \frac{(3,4)}{\sqrt{3^2+4^2}} = (8,5) \bullet \left(\frac{3}{5}, \frac{4}{5}\right) = 8 \cdot \frac{3}{5} + 5 \bullet \frac{4}{5} = \frac{44}{5}.$$

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Recitation 6

Triple integrals in cuboids The integral of $f(x,y,z)$ in the cuboid $C = \{(x,y,z); x_0 \le x \le x_1, y_0 \le y \le y_1, z_0 \le z \le z_1\}$ is denoted

$$\iiint_C f(x,y,z)dV = \int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} f(x,y,z)dzdydx.$$

Orders of integration There are 6 orders of integration for triple integrals: $dxdydz, dxdzdy, \dots$ Integral of f in a region does not depend on the order of integration (but limits of integration do depend).

Exercise 6.1. Compute $\int_1^3\int_0^2\int_{-1}^0 xyz\, dxdydz$

Solution.

$$\int_1^3\int_0^2\left(\int_{-1}^0 xyz\, dx\right)dydz = \int_1^3\int_0^2\left[\frac{x^2}{2}yz\right]_{x=-1}^0 dydz = \int_1^3\int_0^2 -\frac{yz}{2}dydz$$
$$= \int_1^3\left[-\frac{y^2z}{4}\right]_{y=0}^2 dz = \int_1^3 -zdz = \left[-\frac{z^2}{2}\right]_1^3 = \frac{-9}{2}+\frac{1}{2} = -4.$$

Triple integrals in general We can iteratively compute integrals of the following kind:

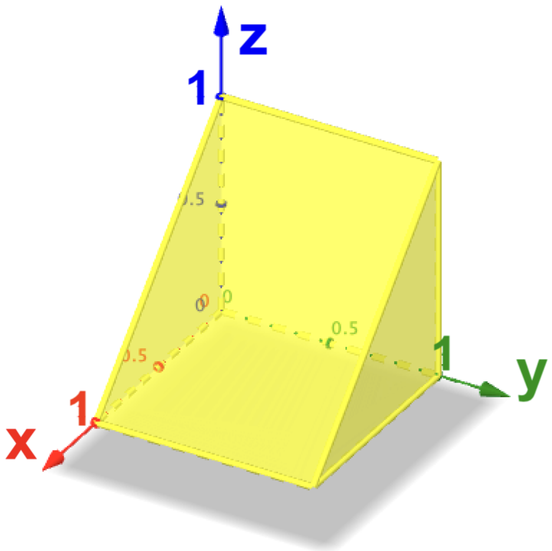
$$\int_{x_0}^{x_1}\int_{g_0(x)}^{g_1(x)}\int_{h_0(x,y)}^{h_1(x,y)} f(x,y,z)dzdydx.$$

This is the integral of $f(x,y,z)$ in the following region:

$$D = \{(x,y,z); a \le x \le b, g_0(x) \le y \le g_1(x), h_0(x,y) \le z \le h_1(x,y)\}.$$

In other words, D is the region between the graphs of $z = h_0(x,y)$ and $z = h_1(x,y)$ and above the 2-dimensional region $R = \{(x,y); a \le x \le b, g_0(x) \le y \le g_1(x)\}$.

Exercise 6.2. Find the limits of integration for the following region D, in the given orders of integration.



$$\int_0^1\int_0^1\int_0^{1-x} f(x,y,z)dzdydx$$

$$\int_0^1\int_0^{1-x}\int_0^1 f(x,y,z)dydzdx$$

$$\int_0^1\int_0^1\int_0^{1-z} f(x,y,z)dxdzdy$$

Solution. To find limits of integration in the first part, we first find the two outer variables (in the first case, $dydx$), so we draw the projection R of D to the xy plane (a $2D$ region, in this case a 1×1 square). The limits associated to the region R in order $dydx$ give us the outer limits of integration, $\int_0^1 \int_0^1$.

Then, the lower bound for z is obtained by finding the lower z-bound of D (i.e. the face of the region D that we can see when we look from the negative z-axis). This is just the xy-plane, i.e. $z = 0$, so the lower limit of integration for z is 0.

Then, the upper bound for z is obtained by finding the upper z-bound of D (i.e. the face of the region D that we can see when we look from the positive z-axis). This is the plane $z + x = 1$, so solving for z we obtain $z = 1 - x$, so out upper limit of integration for the variable z is $1 - x$.

Volumes. The volume of a region D in space is $\iiint_D 1dV$.

Exercise 6.3. Sketch the region D of integration of $\displaystyle\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dzdydx$, rewrite it in the order of integration $dxdzdy$, and compute the integral. Check it coincides with the volume of D.

Solution. The region of integration is the tetrahedron with vertices $(0,0,0), (1,0,0), (0,1,0)$ and $(0,0,1)$ (sketch drawn in recitation). The new integral $dxdzdy$ is:

$$\int_0^1 \int_0^{1-y} \int_0^{1-y-z} dxdzdy = \int_0^1 \int_0^{1-y} (1-z-y)dzdy = \int_0^1 \left[z - \frac{z^2}{2} - yz\right]_{z=0}^{1-y} dy$$
$$= \int_0^1 (1-y) - \frac{(1-y)^2}{2} - y(1-y)dy = \int_0^1 \frac{y^2}{2} - y + \frac{1}{2}dy = \left[\frac{y^3}{6} - \frac{y^2}{2} + \frac{y}{2}\right]_{y=0}^1 = \frac{1}{6}.$$

Exercise 6.4. Sketch and label the region D of points enclosed by the half-cone $z = \sqrt{x^2+y^2}$, and the plane $z = 1$. Write the associated limits of integration in the order $dzdxdy$.

Solution. Sketch drawn in recitation.

The region is between the graphs of $\sqrt{x^2+y^2}$ and $z = 1$ (so, in the region, $\sqrt{x^2+y^2} \le z \le 1$), and its projection to the xy plane is the unit disk, which in the order of integration $dxdy$ has limits $\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dxdy$.

So the new limits of integration are $\displaystyle\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{\sqrt{1-x^2-y^2}}^1 f(x,y,z)dzdxdy$.

Exercise 6.5. Compute the following integrals.

(a) $$\int_0^2 \int_0^4 \int_{y^2}^4 \sqrt{x}\, dzdxdy = \int_0^2 \int_0^4 \left[z\sqrt{x}\right]_{z=y^2}^4 dxdy = \int_0^2 \int_0^4 \left[z\sqrt{x}\right]_{z=y^2}^4 dxdy = \int_0^2 \int_0^4 (4-y^2)xdxdy$$
$$= \int_0^2 \left[\frac{x^2}{2}(4-y^2)\right]_{x=0}^4 = \int_0^2 8(4-y^2)dy = 8\left[(4y - y^3/3)\right]_0^2 = 8(8 - 8/3) = \frac{16}{3}.$$

(b) (Bonus) $\displaystyle\iiint_D 2 + 5x\, dV$, where $D = \{(x,y,z); x^2+y^2+z^2 \le 1\}$ is the unit ball in $\mathbb{R}^3$.

Hint: the volume of the 3D unit ball is $\frac{4}{3}\pi$.

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Recitation 7

Average of a function. The average of a function $f(x, y, z)$ in a region $D \subseteq \mathbb{R}^3$ is

$$\text{Average}(f) = \frac{1}{\text{Vol}(D)} \iiint_D f dV.$$

Cylindrical coordinates. Cylindrical coordinates, $r \in (0, \infty)$, $\theta \in (0, 2\pi)$, $z \in \mathbb{R}$ are like polar coordinates but adding z:

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$
$$z = z$$

Similarly to polar coordinates, we have $r = \sqrt{x^2 + y^2}$ and $dV = rdrd\theta dz$.

Cylindrical coordinates tend to be useful for regions bounded by surfaces of revolution, that is, surfaces whose equations $g(r, z) = 0$ are independent of θ.

Exercise 7.1. Sketch the given regions, and write the corresponding integrals in cylindrical coordinates, in orders of integration $drdzd\theta$ and $dzdrd\theta$. Then, compute the volumes of the regions.

(a) $D = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 - 2 \leq z \leq 0\}$.

Solution. Sketch done in recitation. The sketch can be done in cartesian coordinates as in previous recitations (as long as we know how to draw the surface $z = x^2 + y^2 - 2$). Otherwise, we can use the cylindrical coordinate expression of the surface: using that $r^2 = x^2 + y^2$, our region is given by

$$r^2 - 2 \leq z \leq 0.$$

So we could also sketch our region by first sketching it in 2D in the rz 'half-plane' (where we only include the positive r-axis) and draw the 3D region by taking the corresponding solid of revolution (rotating the 2D region around the z-axis).

The integral we can use to compute the volume are:

$$\int_0^{2\pi}\int_{-2}^{0}\int_0^{\sqrt{z+2}} rdrdzd\theta \qquad \int_0^{2\pi}\int_0^{\sqrt{2}}\int_{r^2-2}^{0} rdzdrd\theta$$

The second expression looks less scary as it has no square roots, so we use it to compute the volume:

$$\int_0^{2\pi}\int_0^{\sqrt{2}}\int_{r^2-2}^{0} rdzdrd\theta = \int_0^{2\pi}\int_0^{\sqrt{2}} r(2-r^2)drd\theta = \int_0^{2\pi}\left[r^2 - \frac{r^4}{4}\right]_0^{\sqrt{2}} d\theta = \int_0^{2\pi}(2-4/4)d\theta = 2\pi.$$

(b) $1 + r \leq z \leq 10$.

Solution. The sketch can be done similarly to part a. The integrals to compute the volume are:

$$\int_0^{2\pi}\int_1^{10}\int_0^{z-1} rdrdzd\theta \qquad \int_0^{2\pi}\int_0^{9}\int_{1+r}^{10} rdzdrd\theta$$

We may use the first integral to compute the volume:

$$\int_0^{2\pi}\int_1^{10}\int_0^{z-1} rdrdzd\theta = \int_0^{2\pi}\int_1^{10}\frac{(z-1)^2}{2}dzd\theta = \int_0^{2\pi}\left[\frac{(z-1)^3}{6}\right]_1^{10} dzd\theta = \int_0^{2\pi}\frac{9^3}{6}d\theta = 2\pi\cdot\frac{9^3}{6} = 243\pi.$$

Spherical coordinates. They are $r \in (0, \infty), \varphi \in [0, \pi], \theta \in [0, 2\pi)$, their relation to cartesian coordinates is

$$\begin{aligned} x &= r \sin(\varphi) \cos(\theta) \\ y &= r \cos(\varphi) \sin(\theta) \\ z &= r \cos(\theta). \end{aligned}$$

For a vector (x, y, z), r is $|v| = \sqrt{x^2 + y^2 + z^2}$, θ is the angle between the positive x-axis and the vector (x, y) (as in polar coords) and φ is the angle (x, y, z) forms with the z axis.

In spherical coordinates, $dV = r^2 \sin(\varphi) dr d\varphi dz$.

Exercise 7.2. A ball, which we model as the unit ball $B = \{(x, y, z); x^2 + y^2 + z^2 \leq 1\}$ has density $f(x, y, z) = 2 - \frac{3}{2}\sqrt{x^2 + y^2 + z^2}$, with units $\frac{g}{cm^3}$ (our units of length are also cm). Does this ball float in water, which has density $\sim 1\frac{g}{cm^3}$?

Hint: the unit ball has volume $\frac{4\pi}{3}$. You could also compute the volume of the ball easily by integrating in spherical coordinates.

Solution. The (average) density of the ball is, (recall that mass is integral of the density)

$$\rho(B) = \frac{\text{Mass}(B)}{\text{Vol(B)}} = \frac{1}{4\pi/3} \iiint_B f dV.$$

We will use spherical coordinates r, φ, θ; note that the ball is given by $r \leq 1$ in spherical coordinates, and $f(x, y, z) = 2 - \frac{3}{2}r$. So,

$$\begin{aligned} \iiint_B f dV &= \int_0^{2\pi} \int_0^{\pi} \int_0^1 (2 - r) \cdot r^2 \sin(\varphi) dr d\varphi d\theta \\ &= \int_0^{2\pi} \int_0^{\pi} \sin(\varphi) \int_0^1 (2r^2 - r^3) dr d\varphi d\theta \\ &= \int_0^{2\pi} \int_0^{\pi} \sin(\varphi) \left[\frac{2r^3}{3} - \frac{r^4}{4}\right]_{r=0}^1 d\varphi d\theta \\ &= \int_0^{2\pi} \int_0^{\pi} \frac{5}{12} \sin(\varphi) d\varphi d\theta = \int_0^{2\pi} 2 \cdot \frac{5}{12} d\theta = 2\pi \cdot 2 \cdot \frac{5}{12} = \frac{5\pi}{4}. \end{aligned}$$

So $\rho(B) = \frac{\frac{5\pi}{4}}{\frac{4\pi}{3}} = \frac{15}{16} = 0.9375\frac{g}{cm^3}$. So our ball is slightly less dense than water.

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Recitation 8

Change of variables in $\mathbb{R}^2$ Let $R, \tilde{R}$ be open regions in $\mathbb{R}^2$. A change of variables is a smooth map with smooth inverse T^{-1}, which transforms the region R into $\tilde{R}$:

$$\begin{array}{rcl} T : R & \to & \tilde{R}; \\ T(u,v) & = & (x(u,v), y(u,v)) = (x,y). \end{array}$$

Jacobian of T, $\mathfrak{J} = \mathfrak{J}_T$ The Jacobian matrix $\mathfrak{J}$ of a smooth map $T(u,v) = (x(u,v), y(u,v))$ at a point (u,v) in R is

$$\mathfrak{J}_T(u,v) = \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix}$$

One can think of the Jacobian as a 'derivative' of a function $T : R \to \tilde{R}$ at the point (u,v).

Determinants The determinant of a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is just $ad - bc$. For any invertible square matrix M, we have $\det\left(M^{-1}\right) = \frac{1}{\det(M)}$.

Inverse function theorem in 2 variables. If $(x,y) = T(u,v)$, then the Jacobian matrix of T^{-1} at (x,y) is the inverse of the Jacobian matrix of T at (u,v):

$$\mathfrak{J}_T(u,v) = \mathfrak{J}_{T^{-1}}(x,y)^{-1} = \mathfrak{J}_{T^{-1}}(T(u,v))^{-1} = \mathfrak{J}_{T^{-1}}(x(u,v), y(u,v))^{-1}$$

Taking determinants in both sides, we obtain that

$$\det(\mathfrak{J}_T(u,v)) = \frac{1}{\det\left(\mathfrak{J}_{T^{-1}}(x,y)\right)}.$$

Exercise 8.1. A transformation T of the plane satisfies $T^{-1}(x,y) = (u,v) = (x - y, x + 2y)$. Find the transformation T and the Jacobians of T, T^{-1}. Check that the inverse function theorem is true for this transformation and use the two methods seen in the lectures to compute the Jacobian determinant $\det(\mathfrak{J}_T(u,v))$.

Solution. We begin by finding the transformation T. This just means solving for x, y in terms of u, v, as we have $T(u,v) = (x(u,v), y(u,v))$.

So, as $u = x - y, v = x + 2y$, we have $y = x - u$, so substituting in the second equation, $v = x + 2(x - u); v = 3x - 2u; x = \frac{v+2u}{3}$. And then $y = x - u = \frac{v-u}{3}$. So

$$(x,y) = T(u,v) = \left(\frac{v+2u}{3}, \frac{v-u}{3}\right).$$

Now we find the Jacobians.

$$\mathfrak{J}_T(u,v) = \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{-1}{3} & \frac{1}{3} \end{pmatrix}; \qquad \mathfrak{J}_{T^{-1}}(x,y) = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}.$$

In this case the Jacobians are constant, so we just have to check that the matrices $\begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{-1}{3} & \frac{1}{3} \end{pmatrix}$ and $\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$ are inverse. Indeed,

$$\begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{-1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{-1}{3} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Here are two methods to determine the determinant of the Jacobian, $\det(\mathfrak{J}_T(u,v))$:

(a) Compute directly the Jacobian of the transformation T, as we have the formula of T:

$$\mathfrak{J}_T(u,v) = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{-1}{3} & \frac{1}{3} \end{pmatrix}.$$

(b) Use the inverse function theorem, which in this case says that

$$\det(\mathfrak{J}_T(u,v)) = \frac{1}{\det(\mathfrak{J}_{T^{-1}}(x(u,v), y(u,v)))} = \frac{1}{\det \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}} = \frac{1}{3}.$$

Change of variables theorem We can compute an integral $dxdy$ in terms of other variables u, v:

For a change of variables $T : R \to \tilde{R}; T(u,v) = (x,y)$ and a function $f : \tilde{R} \to \mathbb{R}$, we have

$$\iint_{\tilde{R}} f(x,y)dxdy = \iint_R f(x(u,v), y(u,v))|\det \mathfrak{J}(u,v)|dudv.$$

So the area elements are transformed as $dxdy = |\det \mathfrak{J}(u,v)|dudv$.

Exercise 8.2. Compute $\iint_{\tilde{R}} x^2 + 1dxdy$ using the change of variables $u = x - y$, $v = x + 2y$, where $\tilde{R}$ is the region $\{(x,y); 0 \le x - y \le 1; 1 \le x + 2y \le 2\}$. Start by sketching the region $\tilde{R}$ in the xy-plane and the corresponding region R in the uv-plane.

Solution. Sketches done in the recitation. In coordinates u, v, the region R is just

$$R = \{(u,v); 0 \le u \le 1, 1 \le v \le 2\}.$$

In the previous exercise we saw that $\det(\mathfrak{J}(u,v)) = \frac{1}{3}$. We also checked that $x = \frac{v+2u}{3}$, so the function $x^2 + 1$ is, in terms of u, v, $f(u,v) = \left(\frac{v+2u}{3}\right)^2 + 1 = \frac{v^2+4uv+4u^2}{9} + 1$. We now compute:

$$\begin{aligned}
\iint_{\tilde{R}} x^2 + 1\, dxdy &= \iint_R \left(\frac{v^2 + 4uv + 4u^2}{9} + 1\right) \cdot \left|\frac{1}{3}\right| dudv \\
&= \frac{1}{3} \int_1^2 \int_0^1 \frac{v^2}{9} + \frac{4uv}{9} + \frac{4u^2}{9} + 1dudv \\
&= \frac{1}{3} \int_1^2 \left[\frac{uv^2}{9} + \frac{2u^2v}{9} + \frac{4u^3}{27} + u\right]_{u=0}^1 dv \\
&= \frac{1}{3} \int_1^2 \frac{v^2}{9} + \frac{2v}{9} + \frac{4}{27} + 1\, dv \\
&= \frac{1}{3} \left[\frac{v^3}{27} + \frac{v^2}{9} + \frac{4v}{27} + v\right]_1^2 \\
&= \frac{1}{3}\left(\frac{8}{27} + \frac{4}{9} + \frac{8}{27} + 2\right) - \frac{1}{3}\left(\frac{1}{27} + \frac{1}{9} + \frac{4}{27} + 1\right) = \frac{47}{81}.
\end{aligned}$$

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Recitation 9

Parameterization of a curve C. $c(t) = (x(t), y(t))$, $t \in [a, b]$. Or in $\mathbb{R}^3$, $c(t) = (x(t), y(t), z(t))$.

Speed of a smooth curve c at time t: $|c'(t)| = \sqrt{x'(t)^2 + y'(t)^2}$, or $\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$ in $\mathbb{R}^3$.

Exercise 9.1. Draw the arc C of the parabola $y = x^2$ between the points $(-2, 4)$ and $(-1, 1)$. Give two parameterizations of C, one of the form $c_1(t) = (t, f(t))$ and one of the form $c_2(t) = (g(t), t)$. In what interval does t take values for each parameterization? Compute $|c_1'(t)|$ and $|c_2'(t)|$.

Solution. Drawing done in recitation.

For $c_1(t) = (t, f_1(t))$, substituting $y = x^2$ we get $f_1(t) = t^2$, so $c_1(t) = (t, t^2)$, for $t \in [1, 2]$.

For $c_2(t) = (f_2(t), t)$, as $y = x^2$ we have $f_2(t)^2 = t$, so as $x < 0$ we have $f_2(t) = -\sqrt{t}$. So $c_2(t) = (-\sqrt{t}, t)$, with $t \in [1, 4]$ (because t is the y coordinate).

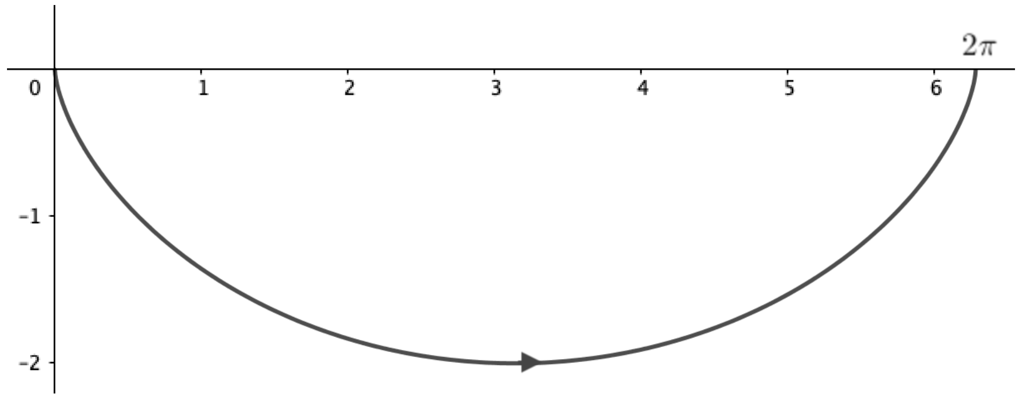
Now we compute the speeds. Note that $c_1'(t) = (1, 2t)$ and $c_2'(t) = \left(\frac{-1}{2\sqrt{t}}, 1\right)$

$$|c_1'(t)| = \sqrt{1^2 + (2t)^2} = \sqrt{1 + 4t^2}; \qquad |c_2'(t)| = \sqrt{\left(\frac{-1}{2\sqrt{t}}\right)^2 + 1^2} = \sqrt{\frac{1}{4t} + 1}.$$

Scalar line integral of f along the curve C. $\int_C f ds = \int_a^b f(t) \cdot |c'(t)| dt$. Scalar line integral does not change if we reparameterize the curve (even if we change its orientation).

Length of a curve C: $\text{len}(C) = \int_C 1 ds = \int_a^b |c'(t)| dt$.

Exercise 9.2. The following curve $C : c(t) = (t - \sin(t), \cos(t) - 1)$ for $t \in [0, 2\pi]$, is called a cycloid.



Find the length of C and the average of $f(x, y) = y$ in C as an integral dt. Hint: $1 - \cos(t) = 2\sin^2\left(\frac{t}{2}\right)$.

Solution. First we compute the speed. Note that $c'(t) = (1 - \cos(t), -\sin(t))$, so

$$|c'(t)| = \sqrt{(1 - \cos(t))^2 + \sin(t)^2} = \sqrt{1 - 2\cos(t) + \cos^2(t) + \sin^2(t)} = \sqrt{2 - 2\cos(t)} = \sqrt{4\sin^2(t/2)}.$$

As $\sin(t/2) \geq 0$ for $t \in [0, 2\pi]$, we have $|c'(t)| = 2\sin\left(\frac{t}{2}\right)$. So the length of the curve C is

$$\int_C 1 ds \int_0^{2\pi} |c'(t)| dt = \int_0^{2\pi} 2\sin(t/2) dt = [-4\cos(t/2)]_0^{2\pi} = -4\cos(\pi) + 4\cos(0) = 4 + 4 = 8.$$

The average of y along the curve C is $\frac{\int_C y ds}{\int_C 1 ds} = \frac{1}{8}\int_C y ds$. We compute:

$$\int_C y ds = \int_0^{2\pi} (\cos(t) - 1) \cdot 2\sin(t/2) dt = 2\int_0^{2\pi} (-2\sin^2(t/2)) \cdot \sin(t/2) dt = 2\int_0^{2\pi} (2\cos^2(t/2) - 2) \cdot \sin(t/2) dt$$

Using $u = -\cos(t/2)$, so $du = \frac{1}{2}\sin(t/2)dt$, we continue:

$$\int_{-1}^{1} 8(u^2-1)du = 8\left[\frac{1}{3}u^3 - u\right]_{-1}^{1} = 8\left((1/3-1)-(-1/3+1)\right) = 8\cdot\frac{-4}{3}.$$

So the average coordinate y in the curve C is $\frac{1}{8}\cdot 8\cdot\frac{-4}{3} = \frac{-4}{3}$.

Vector fields. $\vec{F}(x,y) = (f(x,y), g(x,y))$. Or in $\mathbb{R}^3$, $\vec{F}(x,y,z) = (f(x,y,z), g(x,y,z), h(x,y,z))$.

Line integral of $\vec{F}$ along C. $\int_C \vec{F}\cdot d\vec{s} = \int_a^b \vec{F}(c(t))\cdot c'(t)dt$

The line integral changes sign if we change the orientation of the curve.

Work. Work is the line integral of a force vector field $\vec{F}$ along a curve C, $\int_a^b \vec{F}(c(t))\cdot c'(t)dt$. Informally, work is positive when the force pushes the particle in the same direction as it moves (i.e. $\vec{F}\cdot c' > 0$), and negative when the force opposes the motion (i.e. $\vec{F}\cdot c' < 0$).

Exercise 9.3. The gravitational force that an object A in the origin exerts on an object B with position (x,y) is $\vec{F}(x,y) = \left(\frac{-x}{(x^2+y^2)^{3/2}}, \frac{-y}{(x^2+y^2)^{3/2}}\right)$ (SI units are assumed during the exercise).

(a) Will the work performed by $\vec{F}$ when B moves from $(1,0)$ to $(2,1)$ along the curve $c(t) = (t,(t-1)^2)$ be positive or negative? Why?

Solution. Negative, because the object moves away from the origin and $\vec{F}$ pulls towards the origin. (To formalize this, one would have to check that $\vec{F}\cdot c' < 0$ at every point, i.e. $\vec{F}$ forms an angle of $> 90°$ with c' at all points of the curve.

(b) What is the work done by $\vec{F}$ when the object moves along the curve $(\cos(t), \sin(t))$, $t\in[0,2\pi]$?

Solution. The work will be 0 because the force field and the velocity are perpendicular at all points of the curve. But let's compute the work in detail.

We have $c'(t) = (-\sin(t), \cos(t))$, and

$$\vec{F}(c(t)) = \vec{F}(\cos(t),\sin(t)) = \left(\frac{-\cos(t)}{(\cos^2(t)+\sin^2(t))^{3/2}}, \frac{-\sin(t)}{(\cos^2(t)+\sin^2(t))^{3/2}}\right) = (-\cos(t), -\sin(t))$$

so the result is

$$\int_C \vec{F}d\vec{s} = \int_0^{2\pi} \vec{F}(c(t))\cdot c'(t) = \int_0^{2\pi} (-\cos(t),-\sin(t))\cdot(-\sin(t),\cos(t))dt = \int_0^{2\pi} 0dt = 0.$$

Exercise 9.4. Draw the vector field $\vec{F}(x,y) = (-y,x)$ and the curves $C_1 : c_1(t) = (\cos(t),\sin(t))$ and $C_2 : c_2(t) = (\cos(t), -\sin(t))$, $t\in[0,2\pi]$, indicating their orientation. Compute the line integrals of $\vec{F}$ along C_1 and C_2.

Solution. Both curves are the same, except that they are oriented differently. So the line integral of the vector field along both curves will be opposite. Let's check that.

For $c_1(t)$ we have $\vec{F}(\cos(t),\sin(t)) = (-\sin(t),\cos(t))$ and $c_1'(t) = (-\sin(t),\cos(t))$. So the integral is

$$\int_0^{2\pi} (-\sin(t),\cos(t))\cdot(-\sin(t),\cos(t))dt = \int_0^{2\pi} \cos^2(t)+\sin^2(t)dt = \int_0^{2\pi} 1dt = 2\pi.$$

For $c_1(t)$ we have $\vec{F}(\cos(t),-\sin(t)) = (\sin(t),\cos(t))$ and $c_1'(t) = (-\sin(t),-\cos(t))$. So the integral is

$$\int_0^{2\pi} (\sin(t),\cos(t))\cdot(-\sin(t),-\cos(t))dt = \int_0^{2\pi} cos^2(t)-\sin^2(t)dt = \int_0^{2\pi} -1dt = -2\pi.$$

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Recitation 10

Potential. Let $\vec{F}$ be a vector field in an open set $U \subseteq \mathbb{R}^2$. We say a function $\phi(x,y)$ is a *potential* for $\vec{F}$ if

$$\vec{F}(x,y) = \nabla\phi(x,y) \text{ for all } (x,y) \in U.$$

Conservative vector field. Not all smooth vector fields have a potential! We say $\vec{F}$ is conservative if it has a potential, that is, there is a function ϕ such that $\vec{F} = \nabla\phi$.

Exercise 10.1. Find a potential function for the vector field $\vec{F}(x,y) = (y + \cos(x), x - \sin(y))$.

Solution. We want a function $\phi(x,y)$ such that $\vec{F}(x,y) = \nabla\phi(x,y) = (\partial_x\phi, \partial_y\phi)$, that is,

$$y + \cos(x) = \partial_x\phi(x,y)$$
$$x - \sin(y) = \partial_y\phi(x,y).$$

Integrating the first equation with respect to x, we obtain:

$$\phi(x,y) = xy + \sin(x) + C(y).$$

Now we find $C(y)$ by differentiating with respect to y and using the second equation:

$$x - \sin(y) = \partial_y\phi(x,y) = \partial_y(xy + \sin(x) + C(y)) = x + 0 + C'(y).$$

So $x - \sin(y) = x + C'(y); C'(y) = -\sin(y)$; integrating with respect to y we obtain that $C(y) = \cos(y) + C$, for some constant C. The constant C does not matter, let's just take $C = 0$, so $C(y) = \cos(y)$.

So finally,

$$\phi(x,y) = xy + \sin(x) + C(y) = xy + \sin(x) + \cos(y).$$

Start point, end point. For a curve C with parameterization $c(t) = (x(t), y(t))$, $a \le t \le b$, the start point of C is $c(a)$, and the end point of C is $c(b)$.

Fundamental theorem of calculus for line integrals. Let ϕ be a potential for a vector field $\vec{F}$ in an open set U. For any curve $C : c(t) = (x(t), y(t)); a \le t \le b$ with start point $\underline{a} = c(a)$ and $\underline{b} = c(b)$, we have

$$\int_C \vec{F} \cdot d\vec{s} = \phi\left(\underline{b}\right) - \phi\left(\underline{a}\right) = \phi(c(b)) - \phi(c(a)).$$

Exercise 10.2. A conservative vector field $\vec{F}$, defined in all $\mathbb{R}^2$, has potential function $\phi(x,y) = xy + \cos(x) + \cos(y)$. Find the line integral $\int_C \vec{F} ds$ along the curve $C : c(t) = (t, t^2); 0 \le t \le 3$. What about the curve $C_2 : c_2(t) = (t^2 - 1, \sin(\pi t))$ for $t \in [-1, 1]$?

Solution. By the FTC for line integrals we have $\int_C \vec{F} \cdot d\vec{s} = \phi(c(b)) - \phi(c(a))$. So for the first curve,

$$\int_C \vec{F} \cdot d\vec{s} = \phi(c(3)) - \phi(c(0)) = \phi(3,9) - \phi(0,0) = (3 \cdot 9 + \cos(3) + \cos(9)) - (0 + 1 + 1) = 25 + \cos(3) + \cos(9).$$

And for the second curve,

$$\int_{C_2} \vec{F} \cdot d\vec{s} = \phi(c_2(1)) - \phi(c_2(-1)) = \phi(0,0) - \phi(0,0) = 0.$$

Indeed, the curve is a loop, so as the field is conservative, the integral had to be 0.

Closed curve A curve $c(t)$, $a \leq t \leq b$, is a closed curve (or 'loop') if $c(a) = c(b)$.

Checking that $\vec{F}(x, y) = (f(x, y), g(x, y))$ is *not* conservative

- If $\partial_x g \neq \partial_y f$, then $\vec{F}$ is not conservative.
- If $\int_C \vec{F} d\vec{s} \neq 0$ for a *closed* curve C, then $\vec{F}$ is not conservative.

(Examples of these methods to check $\vec{F}$ is not conservative are in lecture 17, we don't have time in the recitation)

Simply connected set. A set $U \subseteq \mathbb{R}^n$ is simply connected if every loop inside U can be continuously shrunk to a single point, without ever leaving U.

In $\mathbb{R}^2$, informally, a set $U \subseteq \mathbb{R}^2$ is simply connected if 'it has no holes'. E.g. $\mathbb{R}^2$ is simply connected, the unit disk $\{(x, y); x^2 + y^2 \leq 1\}$ is simply connected. The plane without the origin, $\mathbb{R}^2 - \{(0, 0)\}$, and the annulus $\{(x, y); 1 \leq x^2 + y^2 \leq 2\}$, are not simply connected.

Checking that $\vec{F}(x, y) = (f(x, y), g(x, y))$ is conservative without the potential.

- If $\partial_x g = \partial_y f$ and the domain of $\vec{F}$ is simply connected, then $\vec{F}$ is conservative.

Exercise 10.3. Check that the vector field $\vec{F}(x, y) = (y + \cos(x), x - \sin(y))$ is conservative, without computing its potential.

Solution. We need to check two things:

- The domain of $\vec{F}$ is simply connected: in this case the domain of $\vec{F}$ is the entire plane, $\mathbb{R}^2$, so indeed it is simply connected.
- $\partial_x g = \partial_y f$; indeed,
$$\partial_x(x - \sin(y)) = 1 = \partial_y(y + \cos(x)).$$

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Recitation 11

The square root of -1: i. No real numbers x satisfy $x^2 = -1$. But such a solution would be very useful. So we define i to be an 'imaginary number' such that $i^2 = -1$.

Complex numbers, $\mathbb{C}$. A complex number is a number of the form $z = x + yi$, where x, y are real numbers. We can see complex numbers as points $(x, y) \sim x + yi$ in the plane. We can sum, multiply, divide (not by 0) complex numbers, using $i^2 = -1$.

- Sum of complex numbers: $(x_1 + y_1 i) + (x_2 + y_2 i) = (x_1 + x_2) + (y_1 + y_2)i$.
- Product of complex numbers: using the distributive property of the product

$$(x_1 + y_1 i)(x_2 + y_2 i) = x_1 x_2 + x_1 y_2 i + y_1 x_2 i + y_1 y_2 i^2 = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + y_1 x_2)i.$$

Exercise 11.1. Express the following complex numbers as $x + yi$, for real x, y. The constants a, b below are real numbers.

(a) $(1 - 2i) \cdot 3 = 3 - 6i.$

(b) $(1 + i)^2 = 1 + 2i + i^2 = 2i.$

(c) $(a + bi)(a - bi) = a^2 - (bi)^2 = a^2 - b^2 i^2 = a^2 + b^2.$

(d) $\dfrac{1}{a + bi} = \dfrac{1}{a + bi} \cdot \dfrac{a - bi}{a - bi} = \dfrac{a - bi}{a^2 + b^2} = \dfrac{a}{a^2 + b^2} + i\dfrac{-b}{a^2 + b^2}.$

Square roots of a negative number. Any complex number z has two opposite 'square roots' (i.e. complex numbers whose square is z).

- If $z = x \geq 0$ is a positive real number, then the square roots are $\pm\sqrt{x}$.
- If $z = -x \leq 0$ is a negative real number, then its square roots are $\pm i\sqrt{x}$.

Roots of a polynomial of degree 2. If a, b, c are complex numbers with $a \neq 0$, then the complex solutions of $az^2 + bz + c = 0$ are

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Here, by $\pm\sqrt{b^2 - 4ac}$ we mean the two square roots of $b^2 - 4ac$.

Exercise 11.2. Find all complex solutions (in form $a + bi$) to the following equations.

(a) $z^2 + 1 = 0$. $z^2 = -1; z = \pm i.$

(b) $z^2 + z + 10 = 0$. $z = \dfrac{-1 \pm \sqrt{1 - 40}}{2} = \dfrac{-1 \pm \sqrt{-39}}{2} = \dfrac{-1 \pm i\sqrt{39}}{2}.$

(c) $z^2 + iz - 1 = 0$. $z = \dfrac{-i \pm \sqrt{i^2 + 4}}{2} = \dfrac{-i \pm \sqrt{3}}{2}.$

The complex exponential function. When x is a real number, we define

$$e^{ix} = \cos(x) + i\sin(x).$$

More generally, for any complex number $z = x + iy$, we define

$$e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos(y) + i\sin(y)) = e^x\cos(y) + ie^x\sin(y).$$

Exercise 11.3. Check the following equalities, where x is a real number:

(a) $\cos(x) = \dfrac{e^{ix} + e^{-ix}}{2}$.

Solution. Recall that $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$. So we have

$$\begin{aligned} e^{ix} + e^{-ix} &= (\cos(x) + i\sin(x)) + (\cos(-x) + i\sin(-x)) \\ &= (\cos(x) + i\sin(x)) + (\cos(x) - i\sin(x)) = 2\cos(x). \end{aligned}$$

(b) $\sin(x) = \dfrac{e^{ix} - e^{-ix}}{2i}$.

Solution. $e^{ix} - e^{-ix} = (\cos(x) + i\sin(x)) - (\cos(x) - i\sin(x)) = 2i\sin(x)$.

(c) $e^{\pi i} + 1 = 0$.

Solution. $e^{\pi i} = \cos(\pi) + i\sin(\pi) = -1 + i \cdot 0 = -1$.

Ordinary differential equations (ODEs). An ODE is an equation that relates a function $x(t)$ with its derivatives, $x'(t), x''(t)$, etc. We say the ODE has order n if the highest order derivative that appears in the equation is the n^{th} derivative, $x^{(n)}(t)$.

Some examples:

(a) $x'(t) \cdot x(t) + 1 = t^2$. First order ODE.

(b) $x'(t) = x''(t)^7\sqrt{x(t) + 5t^2\ln(t)}$. Second order ODE.

Sometimes we can find the function $x(t)$ by using a differential equation and **initial conditions**; let's see an example of a first order ODE (in this course we will learn how to solve some second order ODEs).

Exercise 11.4. In a cold, winter day there are 0° outside. We know that, if we leave an object outside, then its temperature will decrease at a rate of $T(t)$ degrees per hour, where $T(t)$ is the temperature of the object at time t. If we leave an object outside and its initial temperature is $T(0) = 70°$, what will be the temperature of the object after 2 hours?

Solution. We have the equations $T'(t) = -T(t)$ and $T(0) = 70$. The first equation means that $\frac{T'(t)}{T(t)} = -1$, which integrating with respect to t gives

$$\ln(T(t)) = -t + C; T(t) = e^{-t+C}$$

for some constant C. Then we find the constant C by substituting $t = 0$; $70 = T(0) = e^{-0+C} = e^C$, so $C = \ln(70)$.

So $T(t) = e^{\ln(70)-t} = 70e^{-2}$. Thus, after two hours, the temperature will be $T(2) = 70e^{-2} \approx 9.47°$.

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Recitation 12

Second order homogeneous linear ODE with constant coefficients. That means an equation of the form
$$ax''(t) + bx'(t) + cx(t) = 0,$$
for some constants a, b, c with $a \neq 0$. A solution to the ODE is a function $x(t)$. ODEs normally have many solutions.

Finding the general solution of these ODEs

Step 1) Solve the second degree equation $a\omega^2 + b\omega + c = 0$, obtaining two solutions ω_1, ω_2 given by
$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Step 2) If the solutions $\omega_1 \neq \omega_2$ are real, then all solutions of the ODE are of the form
$$x(t) = Ae^{\omega_1 t} + Be^{\omega_2 t},$$
where $A, B \in \mathbb{R}$. Each choice of constants A, B gives a different solution.

The cases $\omega_1 = \omega_2$ and when $\omega_1 \neq \omega_2$ but they are not real, will be seen in future lectures.

Exercise 12.1. Find the general solution to the following ODEs.

(a) $x''(t) = x'(t)$.

Solution. We can rewrite the ODE as $x''(t) - x'(t) = 0$. So the associated second degree equation is $\omega^2 - \omega = 0; \omega(\omega - 1) = 0; \omega = 0$ or 1. So the general solution to our equation will be
$$x(t) = Ae^{0 \cdot t} + Be^{1 \cdot t} = A + Be^t, \text{ for constants } A, B.$$
For example, some solutions to the equation are $x(t) = 3e^t + 5$, $x(t) = e^t - 7$ or $x(t) = 1$.

(b) $x''(t) = 3x(t) + x'(t)$.

Solution. We can rewrite the ODE as $x''(t) - x'(t) - 3x(t) = 0$. So the associated second degree equation is $\omega^2 - \omega - 3 = 0; \omega = \frac{1 \pm \sqrt{1+12}}{2} = \frac{1 \pm \sqrt{13}}{2}$. So the general solution is
$$x(t) = Ae^{\frac{1+\sqrt{13}}{2}t} + Be^{\frac{1-\sqrt{13}}{2}t}, \text{ for constants } A, B.$$

Using initial conditions to obtain a specific solution. In order to determine a unique solution to the ODE (that is, the values of A, B), we need two extra conditions, usually values of $x(t)$ and $x'(t)$. For example,

- The values of $x(0)$ and $x'(0)$.
- The values of $x(0)$ and $x(1)$.

Exercise 12.2. Find the function $x(t)$ which satisfies $x''(t)+2x'(t) = 3x(t)$ and with $x(0) = 4$, $x'(0) = 5$.
Solution. We first find the general solution $x(t)$: $x''(t) + 2x'(t) - 3x(t) = 0$, so the associated equation is $\omega^2 + 2\omega - 3 = 0$, so $\omega = \frac{-2\pm\sqrt{4+12}}{2} = -1 \pm 2$. So the general solution is of the form

$$x(t) = Ae^t + Be^{-3t}.$$

Now we have to find A, B using the equations $x(0) = 4$ and $x'(0) = 5$. Note that $x'(t) = Ae^t - 3Be^{-3t}$

$$\begin{aligned} 4 &= x(0) = Ae^0 + Be^{-3\cdot 0} = A + B, \\ 5 &= x'(0) = A - 3B. \end{aligned}$$

So $B = 4 - A$, substituting into the bottom equation we obtain $5 = A - 3(4 - A) = A + 3A - 12$; $4A = 17$; $A = \frac{17}{4}$; $B = 4 - A = -\frac{1}{4}$. So our solution must be

$$x(t) = \frac{17}{4}e^t - \frac{1}{4}e^{-3t}.$$

Exercise 12.3. An object moving vertically inside a force field, with position $x(t)$, has acceleration given by $x''(t) = -6x'(t) - 5x(t)$. We observed that the position of the object at time 0 is 5, and at time 1 it is 1. What position do we expect the object to have at time 2?
Solution. We find first the general solution for $x(t)$; as $x''(t) + 6x'(t) + 5x(t) = 0$, we need to solve $\omega^2 + 6\omega + 5 = 0$, so $\omega = \frac{-6\pm\sqrt{36-20}}{2} = -3 \pm 2$. So the general solution is of the form

$$x(t) = Ae^{-t} + Be^{-5t}.$$

Let's find A, B now. We are given the conditions $x(0) = 5$ and $x(1) = 1$; so using the formula for $x(t)$,

$$\begin{aligned} 5 &= x(0) = A + B, \\ 1 &= x(1) = Ae^{-1} + Be^{-5}. \end{aligned}$$

We substitute $A = 5 - B$ into the second equation;

$$\begin{aligned} 2 &= (5 - B)e^{-1} + Be^{-5} \\ 1 &= 5e^{-1} - Be^{-1} + Be^{-5} \\ B(e^{-1} - e^{-5}) &= 5e^{-1} - 1; \\ B &= \frac{5e^{-1} - 1}{e^{-1} - e^{-5}} = \frac{5 - e}{1 - e^{-4}}. \quad (B \approx 2.324) \end{aligned}$$

And $A = 5 - \frac{5-e}{1-e^{-4}} = \frac{5-5e^{-4}}{1-e^{-4}} - \frac{5-e}{1-e^{-4}} = \frac{e-5e^{-4}}{1-e^{-4}}$ $(A \approx 2.675)$. So

$$x(t) = \frac{e - 5e^{-4}}{1 - e^{-4}}e^{-t} + \frac{5 - e}{1 - e^{-4}}e^{-5t}.$$

So the position of the object at time 2 is

$$x(2) = \frac{e - 5e^{-4}}{1 - e^{-4}}e^{-2} + \frac{5 - e}{1 - e^{-4}}e^{-10} \approx 0.3622.$$

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Recitation 13

Recall the complex exponential function:

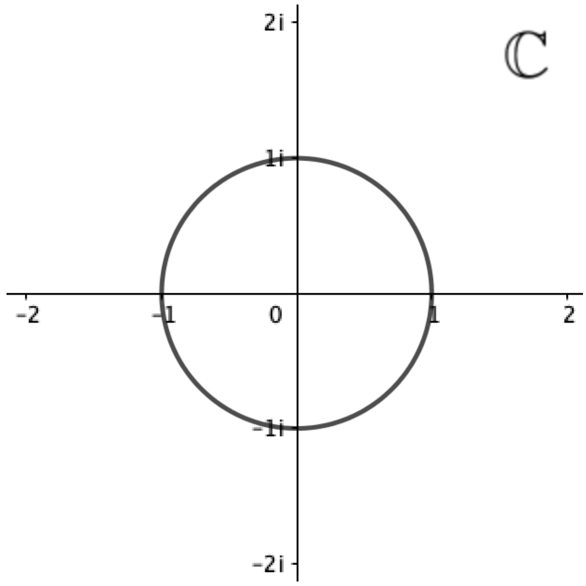
$$e^{ix} = \cos(x) + i\sin(x).$$
$$e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos(y) + i\sin(y)).$$

Exercise 13.1. Represent the following numbers in the complex plane by expressing them as $a + bi$, with $a, b \in \mathbb{R}$.

(a) $\dfrac{e^{\frac{7\pi i}{6}}}{i} = -ie^{\frac{7\pi i}{6}} = -i\left(\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)\right)$
$= -i\left(-\frac{\sqrt{3}}{2} + i\cdot\frac{-1}{2}\right) = \frac{i\sqrt{3}}{2} - \frac{1}{2}.$

(b) $\dfrac{e^{\pi i}}{1+i} = \frac{-1}{1+i} = \frac{-1}{1+i}\cdot\frac{1-i}{1-i} = \frac{i-1}{1^2-i^2} = \frac{i}{2} - \frac{1}{2}.$

(c) $e^{\frac{\pi i}{2}-1} = e^{-1}\cdot e^{\frac{\pi}{2}i} = e^{-1}\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right) = e^{-1}i.$



Formulas for sine and cosine in terms of the complex exponential:

$$e^{ix} + e^{-ix} = 2\cos(x) \qquad e^{ix} - e^{-ix} = 2i\sin(x)$$

We continue solving equations of the form

$$ax''(t)^2 + bx'(t) + cx(t) = 0, \text{ with } a, b, c \in \mathbb{R},\ a \neq 0.$$

General complex solution. Suppose the characteristic equation $a\omega^2 + b\omega + c = 0$ has two distinct roots ω_1, ω_2. Then the complex solutions of our ODE are

$$x(t) = Ae^{\omega_1 t} + Be^{\omega_2 t}, \text{ where } A, B \text{ are complex numbers.}$$

Exercise 13.2. (a) Find the general complex solution of the ODE $x''(x) + x'(t) + x(t) = 0$.

Solution. The characteristic equation is $\omega^2 + \omega + 1 = 0$, so

$$\omega = \frac{-1 \pm \sqrt{1^2 - 4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2}.$$

So the general complex solution to the ODE is

$$x(t) = Ae^{\frac{-1+i\sqrt{3}}{2}t} + Be^{\frac{-1-i\sqrt{3}}{2}t}.$$

(b) Find the solution $x(t)$ satisfying $x(0) = 1$ and $x'(0) = 1$.

Solution. We have the system of equations,

$$1 = x(0) = A + B; B = 1 - A.$$
$$\begin{aligned} 1 = x'(0) &= \frac{-1+i\sqrt{3}}{2}A + \frac{-1-i\sqrt{3}}{2}B \\ &= \frac{-1+i\sqrt{3}}{2}A + \frac{-1-i\sqrt{3}}{2}(1-A) \\ &= \frac{2i\sqrt{3}}{2}A + \frac{-1-i\sqrt{3}}{2} = i\sqrt{3}A + \frac{-1-i\sqrt{3}}{2}. \end{aligned}$$

So $i\sqrt{3}A = 1 + \frac{1+i\sqrt{3}}{2} = \frac{3+i\sqrt{3}}{2}$;

$$A = \frac{3+i\sqrt{3}}{2i\sqrt{3}} = \frac{-i(3+i\sqrt{3})}{2\sqrt{3}} = \frac{-3i+\sqrt{3}}{2\sqrt{3}} = \frac{1}{2} - \frac{\sqrt{3}}{2}i; \qquad B = 1 - A = \frac{1}{2} + \frac{\sqrt{3}}{2}i.$$

So the solution of the ODE is

$$x(t) = \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) e^{\frac{-1+i\sqrt{3}}{2}t} + \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) e^{\frac{-1-i\sqrt{3}}{2}t}.$$

(c) Express the solution without using the complex exponential (in terms of $\sin, \cos$).

Solution. We could do it by brute force using $e^{x+it} = e^x(\cos(t) + i\sin(t))$; here is an alternative way to compute it

$$\begin{aligned}
x(t) &= \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) e^{-\frac{t}{2}} e^{\frac{i\sqrt{3}}{2}t} + \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) e^{-\frac{t}{2}} e^{\frac{-i\sqrt{3}}{2}t} \\
&= e^{-\frac{t}{2}} \left(\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) e^{\frac{i\sqrt{3}}{2}t} + \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) e^{\frac{-i\sqrt{3}}{2}t}\right) \\
&= e^{-\frac{t}{2}} \left(\frac{1}{2}\left(e^{\frac{i\sqrt{3}}{2}t} + e^{\frac{-i\sqrt{3}}{2}t}\right) + \frac{\sqrt{3}}{2}i\left(e^{\frac{-i\sqrt{3}}{2}t} - e^{\frac{i\sqrt{3}}{2}t}\right)\right) \\
&= e^{-\frac{t}{2}} \left(\frac{1}{2}\left(2\cos\left(\frac{\sqrt{3}}{2}t\right)\right) + \frac{\sqrt{3}}{2}i\left(2i\sin\left(\frac{-\sqrt{3}}{2}t\right)\right)\right) \\
&= e^{-\frac{t}{2}} \left(\cos\left(\frac{\sqrt{3}}{2}t\right) - \sqrt{3}\sin\left(\frac{-\sqrt{3}}{2}t\right)\right) \\
&= e^{-\frac{t}{2}} \left(\cos\left(\frac{\sqrt{3}}{2}t\right) + \sqrt{3}\sin\left(\frac{\sqrt{3}}{2}t\right)\right).
\end{aligned}$$

What if $a\omega^2 + b\omega + c$ has repeated roots? If there is only one root ω_0, then the root is real, and the general real solution of the ODE is

$$x(t) = Ae^{\omega_0 t} + Bte^{\omega_0 t}, \text{ for real constants } A, B.$$

Exercise 13.3. (Extra) Find the solution to the ODE $x''(t) + 2x'(t) + x(t) = 0$ with initial conditions $x(0) = 1$ and $x'(0) = 2$.

Solution. The characteristic equation is $\omega^2 + 2\omega + 1 = 0$, which has as its only solution $\omega = -1$. Thus, the general solution is

$$x(t) = Ae^{-t} + Bte^{-t}.$$

The derivative is $x'(t) = -Ae^{-t} + Be^{-t} - Bte^{-t}$. So we get a system of equations in A, B:

$$\begin{aligned}
&1 = x(0) = A. \\
&2 = x'(0) = -A + B; B = 2 + A = 3.
\end{aligned}$$

So the solution is $x(t) = e^{-t} + 3te^{-t}$.

Exercise 13.4. (Extra 2) Find the general solution to the non-homogeneous ODE $x''(t)+2x'(t)+x(t) = 1$.

Solution. The general solution to the inhomogeneous equation is the solution to the homogeneous ODE $x''(t) + 2x'(t) + x(t) = 0$, plus a particular solution of $x''(t) + 2x'(t) + x(t) = 1$.

- General solution to $x''(t) + 2x'(t) + x(t) = 0$:

$$x(t) = Ae^{-t} + Bte^{-t}, \text{ where } A, B \text{ are real constants.}$$

- Particular solution to $x''(t) + 2x'(t) + x(t) = 1$: $x(t) = 1$ works.

So the general solution of $x''(t) + 2x'(t) + x(t) = 1$ will be

$$x(t) = 1 + Ae^{-t} + Bte^{-t}, \text{ where } A, B \text{ are real constants.}$$