Exact Results in $\mathcal{N} = 4$ Super Yang-Mills

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Abstract

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1 Introduction

The title of this thesis is *Exact Results in $\mathcal{N} = 4$ Super Yang-Mills*. A reasonable question to ask is – why would anyone care about that ? After all $\mathcal{N} = 4$ is just a toy theory, it is not something that nature implements and thus we can not observe it in particle accelerators, as opposed to say the Standard Model of particle physics. And indeed those are all valid points, however there are very good reasons for studying $\mathcal{N} = 4$ SYM.

From a pragmatic point of view, it is the simplest non-trivial quantum field theory in four spacetime dimensions and since attempts at solving realistic QFTs such as the theory of strong interactions (QCD) have so far been futile, it seems like a good starting point – some go as far as calling it the harmonic oscillator of QFTs.

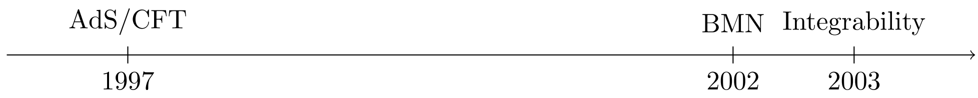
Another (and probably the main) reason why $\mathcal{N} = 4$ has been receiving so much attention in the last decades is the long list of mysterious and intriguing properties it seems to posses, making it almost an intelectual pursuit of understanding it. The theory has been surprising the theoretical physics community from the very beginning: it is a rare instance of a conformal theory in dimensions higher than two, it has a dual description in terms of a string theory and more recently it was discovered to be integrable. All of these properties give reasonable hope for actually solving the theory exactly, something that has never been achieved before for any four dimensional interacting QFT.

In the remainder of the section we give a proper introduction to the subject from a historic point of view focusing on its integrability aspect, for it is integrability that allows one to actually find exact results in the theory. We then give an overview of the thesis itself, emphasizing which parts of the text that are reviews of known material and which parts constitute original work.

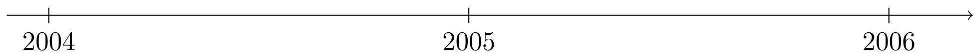
1.1 Brief history of the subject

Quantum field theory has been at the spot light of theoretical physics since the beginning of the century when it was found that electromagnetism is described by the theory of quantum electrodynamics (QED). Since then people have been trying to fit other forces of nature into the QFT framework. Ultimately it worked: the theory of strong interactions, quantum chromodynamics or QCD for short, together with the electroweak theory, spontaneously broken down to QED, collectively make up the *Standard Model* of particle physics, which has been extensively tested in particle accelerators since then.

However nature did not give away her secrets without a fight. For some time it was though that strong interactions were described by a theory of vibrating strings, as it seemed to incorporate the so-called Regge trajectories observed in experiments [1]. Even after discovering QCD as a Yang-Mills gauge theory, stringy aspects of it were still evident and largely mysterious. Most notably lattice gauge theory calculations at strong coupling suggested that surfaces of color-electric fluxes between quarks could be given the interpretation of stretched strings [2], thus an idea of a gauge-string duality was starting to emerge. It was strongly re-enforced by t'Hooft, who showed that the perturbative expansion of $U(N)$ gauge theories in the large N limit could be rearranged into a genus expansion of surfaces triangulated by the Feynman diagrams, which strongly resembles string theory genus expansions [3].



However it was the work of Maldacena in 1997 that sparked a true revolution [4]. He formulated the first concrete conjecture for a duality between a gauge theory, the maximally supersymmetric $\mathcal{N} = 4$ super Yang-Mills, and type IIB string theory on $AdS_5 \times S^5$, now universally referred to as the AdS/CFT duality.



Very rapid development.

Exact solutions. Bright future ahead.

1.2 Thesis overview

Maybe a nice picture for the structure of the thesis.

2 $\mathcal{N} = 4$ super Yang-Mills

Here we describe the theory that is the main interest of the thesis.

2.1 The theory and its action

Write the action, maybe also show dimensional reduction from 10d. Talk about observables: traces, Wilson lines.

Planar limit.

2.2 Symmetry

Talk about conformal symmetry, write down the algebra ? Oscillator representation ? Discuss subgroups of psu(2,2—4), closed sectors.

Since it's a CFT we want to find 2pt and 3pt function. The spectral problem.

2.3 Weak coupling

Take a simple operator, e.g. Konishi and calculate the anomalous dimension using perturbation theory ?

2.4 String description: AdS/CFT

Here we talk about the alternative description of the theory as strings moving in AdS.

2.4.1 Motivation

Planar diagrams are string interactions.

2.4.2 String theory and the duality

Give details of the string theory, what are the parameters on both sides, how they match up. What are the limits. Anomalous dimensions match string state energies.

2.5 Testing the duality: BMN, GKP, FT

Describe these limits, give first eveidence for the duality. Is this where one finds the first strong coupling coefficient to Konishi ?

3 Integrability

In this section we dive into the magical world of integrability.

3.1 Overview

Give picture summarizing all techniques and their ranges of applicability.

3.2 One loop at weak coupling

Roughly rederive the Minahan/Zarembo result.

3.2.1 $\mathfrak{su}(2)$ sector

Give the su(2) Hamiltonian, example states and energies.

3.2.2 $\mathfrak{sl}(2)$ sector

Same with sl(2).

3.3 Higher loops

Short example of a two loop Hamiltonian, perturbative corrections for the states found above with contact terms.

3.4 Asymptotic solution

Generalize su(2) BAE to all loops, maybe give a simple example.

3.4.1 A glimpse ahead: the slope function

Derive slope from ABA.

3.5 Strong coupling and the algebraic curve construction

Describe flat connections, monodromies, sheets etc.

3.6 Classical solutions

All finite gap solutions can be described this way.

3.6.1 BMN string

Give explicit solution.

3.6.2 Folded string

Something similar.

3.7 Quantization and semi-classics

Describe the quantization procedure. Derive next coefficient for Konishi.

3.8 Short strings

Combine with slope, derive next coeffient for Konishi.

3.9 Full solution to the spectral problem

Here we finally give the complete solution.

3.9.1 The full theory

Mention nested BAE, full psu(2,2—4) spin chain without going into much detail.

3.9.2 Finite length

Deprecated approches: TBA, Y-system.

3.9.3 The $\mathbf{P}\mu$ system

Define $\mathbf{P}\mu$ as if it was an axiom.

4 Exact results

Exact results are rare and important.

4.1 Folded string

Mention Frolov numerics. Volin's 8(9) ? loops with $\mathbf{P}\mu$.

4.2 Cusped Wilson line

Bremstahlung result from $\mathbf{P}\mu$.

4.2.1 Classical limit

Find the curve, matrix models.

4.3 Revisiting the slope function

Derive slope from $\mathbf{P}\mu$.

4.4 The curvature function

Derive curvature from $\mathbf{P}\mu$.

4.4.1 Weak coupling expansion

Mention weak coupling and how it matches ABA.

4.4.2 Strong coupling expansion

Mention strong coupling, be amazed how it matches string theory.

4.5 Update on short strings

Combine semiclassics with curvature and finally derive three-loop Konishi coefficient.

5 Conclusions

Conclude with a tearful and heroic description about the journey of Konishi through the land of integrability - from weak to strong coupling.

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