Exact Results in Supersymmetric Gauge Theories

Saulius Valatka

Department of Mathematics, King's College London,

Strand, London WC2R 2LS, U.K.

Thesis supervisor Dr. Nikolay Gromov

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Abstract

In this thesis we discuss supersymmetric gauge theories, focusing on exact results achieved using methods of integrability. For the larger part of the work we focus solely on the best known example of an integrable supersymmetric gauge theory, namely the $\mathcal{N} = 4$ super Yang-Mills theory. After motivating the problem of solving it exactly and briefly reviewing the historical developments in the subject we begin rigorous analysis by firstly defining the theory. We discuss it from the gauge theory point of view and give the alternative string theoretic formulation via the AdS/CFT correspondence.

We devote a large portion of the thesis for perturbative integrability, which addresses the spectral problem in perturbation theory both in weak and strong coupling. As we develop integrability technology we keep on applying new concepts to the simplest non-trivial observable in the theory, the Konishi operator. We find its anomalous dimension using perturbative integrability methods at both ends of the range of the coupling constant. At this point we find first hints of exact results by introducing the slope function and exploiting it to extend perturbative results for the Konishi operator.

Finally we address the most recent developments in solving the spectral problem exactly. To that end we introduce the $\mathbf{P}\mu$-system after shortly discussing its motivation using the now arguably obsolete TBA and Y-system approaches. We then demonstrate the applicability of this construction by rederiving the slope function and deriving the so called curvature function. We show how these exact analytical results can be used to extract further information about the Konishi anomalous dimension.

We devote a separate section to the ABJM theory, which is another example of an integrable supersymmetric gauge theory closely related to $\mathcal{N} = 4$ supersymmetric Yang-Mills. As most of the techniques in ABJM are analogous to the ones describes while covering $\mathcal{N} = 4$ SYM, we only provide a very brief review of results with references to other work containing the details.

Contents

1 Introduction

I didn't choose the thug life, the thug life chose me.
– Tupac Shakur

The title of this thesis is *Exact Results in Supersymmetric Gauge Theories.* A reasonable question to ask is – why would anyone care about that ? After all there is no evidence that supersymmetry is a true symmetry of nature and supersymmetric theories are mostly toy theories, we can not observe them in particle accelerators, as opposed to the Standard Model of particle physics. And indeed these are all valid points, however there are very good reasons for studying them.

Consider $\mathcal{N} = 4$ super Yang-Mills, from a pragmatic point of view it is the simplest non-trivial quantum field theory in four spacetime dimensions and since attempts at solving realistic QFTs such as the theory of strong interactions (QCD) have so far been futile, it seems like a good starting point – some go as far as calling it the harmonic oscillator of QFTs.

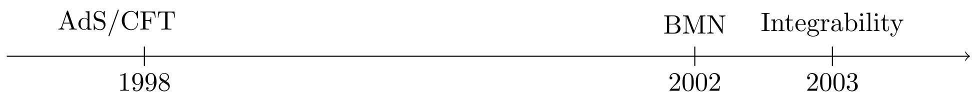
Another (and probably the main) reason why $\mathcal{N} = 4$ has been receiving so much attention in the last decades is the long list of mysterious and intriguing properties it seems to posses, making it almost an intelectual pursuit of understanding it. The theory has been surprising the theoretical physics community from the very beginning: it is a rare instance of a conformal theory in dimensions higher than two, it has a dual description in terms of a string theory and more recently it was discovered to be integrable. All of these properties give reasonable hope for actually solving the theory exactly, something that has never been achieved before for any four dimensional interacting QFT.

In the remainder of the section we give a proper introduction to the subject from a historic point of view focusing on $\mathcal{N} = 4$ SYM and its integrability aspect, for it is integrability that allows one to actually find exact results in the theory. We then give an overview of the thesis itself, emphasizing which parts of the text are reviews of known material and which parts constitute original work.

1.1 Brief history of the subject

Quantum field theory has been at the spot light of theoretical physics since the beginning of the century when it was found that electromagnetism is described by the theory of quantum electrodynamics (QED). Since then people have been trying to fit other forces of nature into the QFT framework. Ultimately it worked: the theory of strong interactions, quantum chromodynamics or QCD for short, together with the electroweak theory, spontaneously broken down to QED, collectively make up the *Standard Model* of particle physics, which has been extensively tested in particle accelerators since then.

However nature did not give away her secrets without a fight. For some time it was though that strong interactions were described by a theory of vibrating strings, as it seemed to incorporate the so-called Regge trajectories observed in experiments [1]. Even after discovering QCD, which is a Yang-Mills gauge theory, stringy aspects of it were still evident and largely mysterious. Most notably lattice gauge theory calculations at strong coupling suggested that surfaces of color-electric fluxes between quarks could be given the interpretation of stretched strings [2], thus an idea of a gauge-string duality was starting to emerge. It was strongly re-enforced by t'Hooft, who showed that the perturbative expansion of $U(N)$ gauge theories in the large N limit could be rearranged into a genus expansion of surfaces triangulated by the double-line Feynman graphs, which strongly resembles string theory genus expansions [3].



However it was the work of Maldacena in the end of 1997 that sparked a true revolution in the field [4]. He formulated the first concrete conjecture, now universally referred to as *AdS/CFT*, for a duality between a gauge theory, the maximally supersymmetric $\mathcal{N}=4$ super Yang-Mills, and type IIB string theory on $AdS_5 \times S^5$. Polyakov had already shown that non-critical string theory in four-dimensions describing gauge fields should be complemented with an extra Liouville-like direction thus enriching the space to a curved five dimensional manifold [5]. Furthermore the gauge theory had to be defined on the boundary of this manifold. Maldacena's conjecture was consistent with this view, as the gauge theory was defined on the boundary of AdS_5, whereas the S^5 was associated with the internal symmetries of the gauge fields. The idea of a higher dimensional theory being fully described by a theory living on the boundary was also considered before in the context of black hole physics [6, 7] and goes by the name of holography, thus AdS/CFT is also refered to as a holographic duality.

The duality can be motivated by considering a stack of N parallel D3 branes in type IIB string theory. Open strings moving on the branes can be described by $\mathcal{N}=4$ SYM with the gauge group $SU(N)$. Roughly the idea is that there are six extra dimensions transverse to the stack of branes, thus a string stretching between two of them can be viewed as a set of six scalar fields $(\Phi^i)^a{}_b$ defined in four dimensional spacetime carrying two extra indices denoting the branes it is attached to. These are precisely the indices of the adjoint representation of $SU(N)$. A similar argument can be put forward for other fields thus recovering the field content of $\mathcal{N}=4$ SYM. Far away from the branes we have closed strings propagating in empty space. In the low energy limit these systems decouple and far away from the branes we are left with ten dimensional supergravity.

Another way of looking at this system is considering the branes as a defect in spacetime, which from the point of view of supergravity is a source of curvature. The supergravity solution carrying D3 brane charge can be written down explicity [8]. Far away from the branes it is obviously once again the usual flat space ten dimensional supergravity. However the near horizon geometry of the brane system becomes $AdS_5 \times S^5$. Since both points of view end up with supergravity far away from the branes, one is tempted to identify the theories close to the branes, $\mathcal{N} = 4$ SYM and type IIB string theory on $AdS_5 \times S^5$. This is exactly what Maldacena did in his seminal paper [4].

By studying the supergravity solution one can identify the parameters of the theories, namely $\mathcal{N} = 4$ SYM is parameterized by the coupling constant g_{YM} and the number of colors N, whereas string theory has the string coupling constant g_s and the string length squared α'. These are identified in the following way

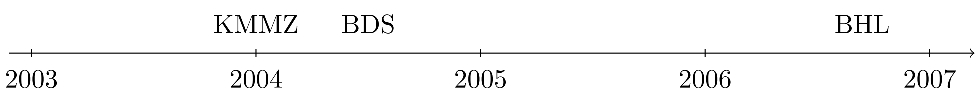
$$4\pi g_s = g_{YM}^2 \equiv \frac{\lambda}{N}, \qquad \frac{R^4}{\alpha'^2} = \lambda, \tag{1.1}$$

where λ is the t'Hooft coupling and R is the radius of both AdS_5 and S_5, which is fixed as only the ratio R^2/α' is measurable. A few things are to be noted here. First of all, the identification directly implements t'Hooft's idea of large N expansion of gauge theory, since $g_s \sim 1/N$. In fact in the large N limit only planar Feynman graphs survive and everything simplifies dramatically, a fact that we will take advantage of a lot in this thesis. In this limit the effective coupling constant of the gauge theory is λ.

The supergravity approximation is valid when $\alpha' \ll R^2$, which corresponds to strongly coupled gauge theory, thus the conjecture is of the weak-strong type. This fact is a blessing in disguise, since initially it seems very restrictive as one can not easily compare results of the theories. However it provides a possibility to access strongly coupled regimes of both theories, which was beyond reach before. Prescriptions for matching up observables on both sides of the correspondence were given in [9, 10]. However because of the weak-strong nature of the duality initial tests were performed only for BPS states, which are protected from quantum corrections. The first direct match was observed in [10] where it was shown that the spectrum of half-BPS single trace operators matches the Kaluza-Klein modes of type IIB supergravity. Another indirect confirmation of the conjecture was the formulation of type IIB string theory as a super-coset sigma model on the target space $PSU(2,2|4)/SO(2,4) \otimes SO(6)$, which has the same global symmetries as $\mathcal{N} = 4$ SYM [15].

The situation changed dramatically in 2002 when Berenstein, Maldacena and Nastase devised a way to go beyond BPS checks [16]. The idea was to take an operator in gauge theory with large R-charge J and add some some impurities, effectively making it "near-BPS". The canonical example of such an operator is $\mathrm{Tr}(Z^J X^S)$, where Z and X are two complex scalar fields of $\mathcal{N} = 4$ SYM, with X being the impurities ($S \ll J$). Since anomalous dimensions are suppressed like λ/J^2, perturbative gauge theory calculations are valid even at large λ, as long as $\lambda' \equiv \lambda/J^2 \ll 1$ and N is large. It is thus possible to compare gauge theory calculations with string theory results. From the string theory point of view this limit corresponds to excitations of point-like strings with angular momentum J moving at the speed of light around the great circle of S^5. The background seen by this string is the so-called pp-wave geometry and string theory in this background is tractible.

The discovery of the BMN limit was arguably the first time it was explicitly demonstrated how the world sheet theory of a string can be reconstructed by a physical picture of scalar fields dubbed as "impurities" propagating in a closed single trace operator of "background" scalar fields of the gauge theory. Shortly after this discovery Minahan and Zarembo revolutionized the subject once again by discovering Integrability at the end of 2002 [17]. They showed that in the large N limit single-trace operators of scalar fields can be identified with spin chains and their anomalous dimensions at one-loop in weak coupling are given by the energies of the corresponding spin chain states. These spin-chain systems are known to be integrable, which in practice allows one to solve the problem exactly using techniques such as the Bethe ansatz [18]. This discovery sparked a very rapid development of integrability methods in AdS/CFT during the coming years.



Solving a quantum field theory in principle means finding all n-point correlation functions of all physical observables. Since $\mathcal{N} = 4$ SYM is conformal it is enough to find all 2-point and 3-point correlators, as all higher point correlation functions can be decomposed in terms of these basic constituents. Due to conformal symmetry the two-point functions only depend on the scaling dimensions of operators, whereas for three point functions one also needs the so called structure constants C_{ijk} in addtion to the scaling dimensions. Integrability methods from the very beginning mainly focused on solving the spectral problem in the large N limit, that is finding the spectrum of operators with definite anomalous dimensions and their exact numeric values. The initial discovery of [17] was that the spectral problem was analogous to diagonalizing a spin chain Hamiltonian, which was identified with the dilatation operator of the superconformal symmetry of the theory. The eigenstates correspond to operators in the gauge theory and the eigenvalues are their anomalous dimensions.

Soon after the initial discovery of integrability a spin chain formulation at one-loop was found for the full $PSU(2,2|4)$ theory, not only the scalar sector [19]. The result was also extended to two and three loops [20]. Integrability was also discovered at strong coupling as it was shown that the Metsaev-Tseytlin sigma model is classicaly integrable [21]. With integrability methods now being available at both weak and strong coupling it was possible to compare results in the BMN limit. As expected, comparisons in the first two orders of the BMN coupling constant λ' showed promising agreement [23, 24, 25], however an order of limits problem emerged at three loops [20].

All of these results seemed to suggest that integrability may be an all loop phenomenon, only the surface of it being scratched so far. This notion was strongly reinforced when classical string integrability was reformulated in the elegant language of algebraic curves by Kazakov, Marshakov, Minahan and Zarembo (KMMZ), which made the connection with weak coupling more manifest [22]. The algebraic curve was interpreted as the continuum limit of Bethe equations, which made it possible to speculate about all loop equations. The first such attempt was made by Beisert, Dippel and Staudacher (BDS), who conjectured a set of Bethe equations and a dispresion relation which together sucessfully showcased some all-loop features [26]. This result was later extended to all sectors of the theory [27]. The BDS result was quickly shown to be incomplete as it was lacking a so-called dressing phase [28], a scalar function not constrained by symmetry of the problem. It was found to leading order at strong coupling in [28] and later to one-loop in [29]. A crossing equation satisfied by the dressing phase was soon found [30] and eventually solved by Beisert, Hernandez and Lopez (BHL) [31]. Collectively these results are often refered to as the asymptotic Bethe ansatz (ABA), reminding that they are valid only for asymptotically long spin chains. When the states are short so-called wrapping effects become relevant. At weak coupling they manifest as long-range spin chain interactions wrapping around the chain, whereas at strong coupling they are due to virtual particles self-interacting across the circumreference of the worldsheet [32, 33].

Once the asymptotic solution was found attention shifted to finite size corrections, which once resolved would in principle complete the solution to the spectral problem for single trace operators. Scattering corrections in finite volume for arbitrary QFTs were first addressed by Lüscher [34], who derived a set of universal formulas. This approach, while very general and not directly related to integrability, was employed to calculate four [35] and five loop anomalous dimension coefficients [36] of the simplest non-BPS operator with length two, the Konishi operator. The results agreed with available diagramatic four-loop calculations [37] and gave a new prediction for five loops.

An alternative approach more in line with integrability is the Thermodynamic Bete Ansatz (TBA). Its origins can be traced back to Yang and Yang [38], however it was the work of Alexey Zamolodchikov [39, 40] that brought it to the mainstream. The idea is to consider the partition function of a two dimensional integrable CFT and it's "mirror" image found after exchanging length and time with a modular transformation. At large imaginary times the partition function will be dominated by the ground state energy, whereas in the mirror theory large time means asymptotic length, which is under control using the asymptotic Bethe ansatz techniques. Thus one can evaluate the partition function using the saddle point method and after rotating back to the original theory compute the exact ground state energy. Excited states can then be reached using analytic continuation. This approach was already proposed as an option for the AdS/CFT system in [33] and was first discussed in depth in [41]. The TBA approach crystalized in 2009 with multiple groups publishing results almost simultaneously [42, 43, 44, 45]. The Konishi anomalous dimension was initially checked at four [44] and five [47, 48] loops by linearizing the TBA equations, showing precise agreement with results obtained using the Lüscher method. Ultimately the Konishi anomalous dimension was calculated numerically for a wide range of values of the t'Hooft coupling constant [49].

And so the spectral problem seemed to be solved, at least in the case of Konishi an exact and complete result was finally found, even if only numerically. However it was increasingly becoming clear that the solution was not in its final and most elegant form. Indeed the TBA equations are an infinite set of coupled integral equations, obviously one has to employ various numerical tricks to actually solve them and this mostly works in a case-by-case basis. Cases such as the $\mathfrak{sl}(2)$ sector of the theory, containing the Konishi operator [44] and cusped Wilson lines [50, 51] have been worked out explicitly, however it still remains a hard problem in general. From the very beginning alternative formulations of the solution were being proposed. An infinite set of non-linear functional equations, the so called Y-system was proposed already in [44], later completed with analytical constraints comming from the TBA equations [46]. Connections of the Y-system with the Hirota bilinear relation were later explored in [52] and the Y-system was reduced to a finite set of non-linear integral equations (FiNLIE). The long sought beauty of the solution to the spectral problem was arguably uncovered with the formulation of the $\mathbf{P}\mu$-system [53], also refered to as the quantum spectral curve. The whole TBA construction was ultimately reduced to a Riemann-Hilbert problem for eight Q functions, which can be though of as the quantum analogues of quasimomenta found in the algebraic curve construction. The $\mathbf{P}\mu$-system quickly showed its potential as previously known results were rederived almost without any effort and new results were being rapidly discovered [54, 55].

Thus one can safely say that the spectral problem in $\mathcal{N} = 4$ SYM is by now very well understood with numerical results readilly available and deeper understanding of the structure being within reach. Integrabillity methods have also been useful in other areas such as three point functions [56, 57] and scattering amplitudes [58, 59], however the situation there is still not as complete. Having witnessed the successful resolution of the spectral problem it appears that $\mathcal{N} = 4$ SYM is within reach of being solved completely. If this programme were to be successfuly carried out it would be the first example of a four dimensional interacting quantum field theory being solved exactly. Undoubtadly this would provide a huge boost to our understanding of QFTs in general and hopefully bring us closed to solving QCD.

It turns out that $\mathcal{N} = 4$ SYM is not the only example of an integrable supersymmetric gauge theory having a dual string description. Probably the most famous example involves the so called ABJM theory, proposed by Aharony, Bergman, Jafferis and Maldacena [11], following [12, 13, 14]. It is a three-dimensional superconformal Chern-Simons gauge theory with $\mathcal{N} = 6$

supersymmetry. This theory was conjectured to be the effective theory for a stack of M2 branes at a Z_k orbifold point. In the large N limit its gravitational dual turns out to be M-theory on $AdS_4 \times S^7/Z_k$. For large k and N with $\lambda = N/k$ fixed, the dual theory becomes type IIA superstring theory in $AdS_4 \times CP^3$. This duality is also integrable and all of the developments outlined above have been reworked for it almost in parallel.

1.2 Thesis overview

Maybe a nice picture for the structure of the thesis.

2 $\mathcal{N}=4$ super Yang-Mills

For the most part of this thesis we will be dealing with $\mathcal{N}=4$ super Yang-Mills theory. In this section we start off by defining it via its action and discussing its symmetries and observables. We also give an alternative formulation of the theory as a string theory, which is the core idea of the AdS/CFT correspondence. This formulation will later prove to be incredibly useful when discussing integrability and exact solutions.

2.1 Action

$\mathcal{N}=4$ super Yang-Mills theory is a quantum field theory much like the Standard Model of particle physics with a certain field content and interaction pattern. It was first discovered by considering $\mathcal{N}=1$ super Yang-Mills theory in $9+1$ spacetime dimensions [60], its action is given by

$$S=\int d^{10}x\,\mathrm{Tr}\left(-\frac{1}{4}F_{MN}F^{MN}+\frac{1}{2}\bar{\Psi}\Gamma^{M}\mathcal{D}_{M}\Psi\right),\qquad M=1\ldots 10,\tag{2.1}$$

where Ψ is a Majorana-Weyl spinor in $9+1$ dimensions with 16 real components and Γ^M are the appropriate gamma matrices. The covariant derivative $\mathcal{D}_M$ is defined as

$$\mathcal{D}_M=\partial_M-ig_{YM}\,[A_M,\,],\tag{2.2}$$

where g_{YM} is the Yang-Mills coupling constant. The gauge group is in principle arbitrary, but we choose $SU(N)$ in anticipation of the AdS/CFT correspondence. By dimensionally reducing this theory on a flat torus T^6 one recovers the maximally supersymmetric $\mathcal{N}=4$ Yang-Mills gauge theory in $3+1$ spacetime dimensions. The reduced action reads

$$\begin{aligned}S=\int d^4x\,\mathrm{Tr}\bigg(&-\frac{1}{2}\mathcal{D}_\mu\Phi_I\mathcal{D}^\mu\Phi^I+\frac{g_{YM}^2}{4}[\Phi_I,\Phi_J][\Phi^I,\Phi^J]-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\\&-\bar{\psi}^a\sigma^\mu\mathcal{D}_\mu\psi_a+\frac{ig_{YM}}{2}\sigma_I^{ab}\psi_a[\Phi^I,\psi_b]+\frac{ig_{YM}}{2}\sigma^I_{ab}\bar{\psi}^a[\Phi_I,\bar{\psi}^b]\bigg).\end{aligned}\tag{2.3}$$

After dimensional reduction the gauge field A_M decomposes to the four dimensional gauge field A_μ and to six real scalar fields Φ_I whereas the Majorana-Weyl spinor Ψ_A breaks up into four copies of the left and right Weyl spinors in four dimensions

$$\Psi_A\ (A=1,...,16)\ \rightarrow\ \bar{\psi}^a_{\dot{\alpha}},\ \psi_{a\alpha}\ (\alpha,\dot{\alpha}=1,2,\ \ a=1,...,4).\tag{2.4}$$

It also gives rise to the $SO(6)\simeq SU(4)$ symmetry called *R-symmetry*, which originally was part of the ten dimensional Poincare group, but now acts as an internal symmetry of the supercharges. It permutes the scalars, which live in the fundamental **6** of $SO(6)$ and the spinors, which live in the fundamental of $SU(4)$, namely the lower index a in $\psi_{a\alpha}$ transforms in **4**, while $\bar{\psi}^a_{\dot{\alpha}}$ transforms in $\bar{\mathbf{4}}$. From this it follows that we can combine the six real scalars Φ^I

into three complex scalars Φ^{ab}, often denoted as X, Y and Z, which then transform under the second rank antisymmetric **6** of $SU(4)$. The gauge field is a singlet under R-symmetry.

It is now a straightforward but rather tedious task to calculate the beta function for this theory. For any $SU(N)$ gauge theory at one loop level it is given by [61]

$$\beta(g) = -\frac{g_{YM}^3}{16\pi^2}\left(\frac{11}{3}N - \frac{1}{6}\sum_s C_s - \frac{1}{3}\sum_f \tilde{C}_f\right) \tag{2.5}$$

where the first sum is over the real scalars and the second one over the fermions. C_s and $\tilde{C}_f$ are the quadratic Casimirs, which in our case are equal to N since all fields are in the adjoint representation of the group. It is then easy to see that at least at one loop level the theory is conformally invariant. In fact the β function was shown to be identically zero to all orders in perturbation theory [62, 63], hence $\mathcal{N}=4$ super Yang-Mills is fully conformally invariant even after quantization. After discussing the full symmetry algebra of the theory and its representations we will give an elegant argument why this is true.

2.2 Observables

The theory has 16 on-shell degrees of freedom which make up the gauge multiplet of $\mathcal{N}=4$ supersymmetry, namely (Φ_I, ψ_a, A_μ). Gauge invariant operators are then formed by taking traces over the gauge group. An important class of operators are the *local operators*, which are traces of fields all evaluated at the same spacetime point. They have the general form

$$\begin{aligned}\mathcal{O}_{i_1\mu i_2\alpha\ldots i_n\ldots j_1\nu\beta\ldots j_n}(x) &= \mathrm{Tr}\left[\Phi_{i_1}(x)\mathcal{D}_\mu\Phi_{i_2}(x)\psi_\alpha(x)\ldots\Phi_{i_n}(x)\right]\times\ldots\\ &\cdots\times\mathrm{Tr}\left[\Phi_{j_1}(x)\mathcal{D}_\nu\psi_\beta(x)\ldots\Phi_{j_n}(x)\right].\end{aligned} \tag{2.6}$$

In this thesis we will be exclusively focusing on the planar limit, which is the limit when the number of colors N is sent to infinity. Diagrams involving multi-trace operators are non-planar, hence suppressed in the large N limit and therefore we will only be considering single trace operators. An example of a non-local operator is the Wilson loop, given by

$$W_L = \mathrm{Tr}\left(\mathcal{P}\exp\oint_C dt\left(iA\cdot\dot{x} + \vec{\Phi}\cdot\vec{n}\,|\dot{x}|\right)\right), \tag{2.7}$$

which depends on the path $x^\mu(t)$ in spacetime, hence it is known as a *line operator*. It also depends on the coupling to the scalar fields, which is encoded in the six-dimensional unit vector $\vec{n}(t)$. The scalar field term can also be understood by recalling that the scalar fields are a result of dimensional reduction from $9+1$ dimensions, thus the coupling vector $\vec{n}(t)$ together with the curve $x^\mu(t)$ make up a path $x^M(t)$ in $9+1$ dimensional spacetime. In later sections of the text we will be considering cusped Wilson lines with other operators inserted at the cusp.

We will be mostly working in these two classes of operators, however in principle one could go on and define surface operators, etc.

2.3 Symmetry

Conformal symmetry, supersymmetry and R-symmetry are a part of a bigger group $PSU(2,2|4)$, which is also known as the $\mathcal{N}=4$ *superconformal group*. It is the full symmetry group of $\mathcal{N}=4$ super Yang-Mills and is unbroken by quantum corrections. It is an example of a *supergroup*, i.e. a graded group containing bosonic and fermionic generators. The theory of supergroups is highly developed (see [64]) and much of the techniques from studying bosonic groups carry over to supergroups with some additional complications, i.e. Dynkin diagrams, root spaces, weights etc.

$PSU(2,2|4)$ has the bosonic subgroup of $SU(2,2)\times SU(4)$, where $SU(2,2)\simeq SO(2,4)$ is the conformal group in four dimensions and $SU(4)\simeq SO(6)$ is the R-symmetry. The conformal group has the Poincaré group as a subgroup, which has a total of 10 generators including four translations P_μ and six Lorentz transformations $M_{\mu\nu}$, in addition there is the generator for dilatations D and four special conformal generators K_μ. Their commutation relations read

$$\begin{gathered}
[D,M_{\mu\nu}]=0 \quad [D,P_\mu]=-iP_\mu \quad [D,K_\mu]=+iK_\mu,\\
[M_{\mu\nu},P_\lambda]=-i(\eta_{\mu\nu}P_\nu-\eta_{\lambda\nu}P_\mu) \quad [M_{\mu\nu},K_\lambda]=-i(\eta_{\mu\lambda}K_\nu-\eta_{\lambda\nu}K_\mu),\\
[P_\mu,K_\nu]=2i(M_{\mu\nu}-\eta_{\mu\nu}D).
\end{gathered}\tag{2.8}$$

$\mathcal{N}=4$ supersymmetry has 16 supercharges $Q_{a\alpha}$ and $\tilde{Q}^a_{\dot\alpha}$ where $\alpha,\dot\alpha=1,2$ are the Weyl spinor indices and $a=1,...,4$ are the R-symmetry indices. These generators have the usual commutation and anti-commutation relations with the Poincaré generators given by

$$\begin{gathered}
\{Q_{\alpha a},\tilde{Q}^b_{\dot\alpha}\}=\gamma^\mu_{\alpha\dot\alpha}\delta^b_a P_\mu \quad \{Q_{\alpha a},Q_{\alpha b}\}=\{\tilde{Q}^a_{\dot\alpha},\tilde{Q}^b_{\dot\alpha}\}=0,\\
[M^{\mu\nu},Q_{\alpha a}]=i\gamma^{\mu\nu}_{\alpha\beta}\epsilon^{\beta\gamma}Q_{\gamma a} \quad [M^{\mu\nu},\tilde{Q}^a_{\dot\alpha}]=i\gamma^{\mu\nu}_{\dot\alpha\dot\beta}\epsilon^{\dot\beta\dot\gamma}\tilde{Q}^a_{\dot\gamma},\\
[P_\mu,Q_{\alpha a}]=[P_\mu,\tilde{Q}^b_{\dot\alpha}]=0,
\end{gathered}\tag{2.9}$$

where $\gamma^{\mu\nu}_{\alpha\beta}=\gamma^{[\mu}_{\alpha\dot\alpha}\gamma^{\nu]}_{\beta\dot\beta}\epsilon^{\dot\alpha\dot\beta}$. Commutators between supercharges and the conformal generators are also non trivial and introduce new supercharges,

$$\begin{gathered}
[D,Q_{\alpha a}]=-\tfrac{i}{2}Q_{\alpha a} \quad [D,\tilde{Q}^a_{\dot\alpha}]=-\tfrac{i}{2}\tilde{Q}^a_{\dot\alpha},\\
[K^\mu,Q_{\alpha a}]=\gamma^\mu_{\alpha\dot\alpha}\epsilon^{\dot\alpha\dot\beta}\tilde{S}_{\dot\beta a} \quad [K^\mu,\tilde{Q}^a_{\dot\alpha}]=\gamma^\mu_{\alpha\dot\alpha}\epsilon^{\alpha\beta}S^a_\beta,
\end{gathered}\tag{2.10}$$

where $\tilde{S}_{\dot\alpha a}$ and S^a_α are the *special conformal supercharges*. They have opposite R-symmetry representations compared to the usual supercharges. The special supercharges bring the total of supercharges to 32. The commutation and anti-commutation relations for the special conformal supercharges are very much like the ones for normal supercharges,

$$\begin{gathered}
\{S^a_\alpha,\tilde{S}_{\dot\alpha b}\}=\gamma^\mu_{\alpha\dot\alpha}\delta^a_b K_\mu \quad \{S^a_\alpha,S^b_\alpha\}=\{\tilde{S}_{\dot\alpha a},\tilde{S}_{\dot\alpha b}\}=0,\\
[M^{\mu\nu},S^a_\alpha]=i\gamma^{\mu\nu}_{\alpha\beta}\epsilon^{\beta\gamma}S^a_\gamma \quad [M^{\mu\nu},\tilde{S}_{\dot\alpha a}]=i\gamma^{\mu\nu}_{\dot\alpha\dot\beta}\epsilon^{\dot\beta\dot\gamma}\tilde{S}_{\dot\gamma a},\\
[K_\mu,S^a_\alpha]=[K_\mu,\tilde{S}_{\dot\alpha a}]=0.
\end{gathered}\tag{2.11}$$

Finally the anti-commutation relations between the special conformal and usual supercharges close the algebra,

$$\begin{aligned}
\{Q_{\alpha a}, S^b_\beta\} &= -i\epsilon_{\alpha\beta}\sigma^{IJ}{}_a{}^b R_{IJ} + \gamma^{\mu\nu}_{\alpha\beta}\delta_a{}^b M_{\mu\nu} - \frac{1}{2}\epsilon_{\alpha\beta}\delta_a{}^b D \\
\{\tilde{Q}^a_{\dot\alpha}, \tilde{S}_{\dot\beta b}\} &= +i\epsilon_{\dot\alpha\dot\beta}\sigma^{IJa}{}_b R_{IJ} + \gamma^{\mu\nu}_{\dot\alpha\dot\beta}\delta^a{}_b M_{\mu\nu} - \frac{1}{2}\epsilon_{\dot\alpha\dot\beta}\delta^a{}_b D \\
\{Q_{\alpha a}, \tilde{S}_{\dot\beta b}\} &= \{\tilde{Q}^a_{\dot\alpha}, S^b_\beta\} = 0
\end{aligned} \tag{2.12}$$

where R_{IJ} are the generators of R-symmetry with $I, J = 1, ..., 6$. All supercharges transform under the two spinor representations of the R-symmetry group and all other generators commute with it. All of the generators can be organized as follows

$$\left(\begin{array}{c|c} K^\mu, P^\mu, M^{\mu\nu}, D & Q_{a\alpha}, \bar{S}_{a\dot\alpha} \\ \hline S^a_\alpha, \bar{Q}^a_{\dot\alpha} & R_{IJ} \end{array}\right) \tag{2.13}$$

where the generators in the diagonal blocks are bosonic and the ones in the anti-diagonal blocks are fermionic. They have a definite dimensions, which are not modified by the radiative corrections

$$[D] = [L] = [\bar{L}] = [R] = 0\ , \quad [P] = 1\ ,\ [K] = -1\ , \quad [Q] = 1/2\ ,\ [S] = -1/2\ . \tag{2.14}$$

In contrast, the classical dimensions of fields

$$[\Phi^I] = [A_\mu] = 1\ , \quad [\psi_a] = \frac{3}{2}, \tag{2.15}$$

do receive radiative corrections and acquire *anomalous dimensions*, which together with the bare dimension make up the conformal dimension

$$\Delta = \Delta_0 + \gamma(g_{YM}). \tag{2.16}$$

The name is justified by the fact that in conformal field theories all two point functions are determined by the scaling dimensions of the fields. More than that, together with the knowledge of all three point functions they are enough to determine any n-point function. This is why finding conformal dimensions of all operators, i.e. the spectrum of the theory is a very important step in solving it.

2.3.1 Superconformal multiplets

Fields of the theory can be organized in unitary representations of the superconformal symmetry group, which are labeled by quantum numbers of the bosonic subgroup

$$\begin{array}{ccc} SO(1,3) \times & SO(1,1) \times & SU(4) \\ (s_+, s_-) & \Delta & [r_1, r_2, r_3] \end{array} \tag{2.17}$$

where (s_+, s_-) are the usual positive half-integer spin labels of the Lorentz group, Δ is the positive conformal dimension that can depend on the coupling and $[r_1, r_2, r_3]$ are Dynkin labels of the R-symmetry. All unitary representations of the superconformal group have been classified into four families [65, 66], here we give a short description of the classification.

Looking at the commutation relations of the conformal subgroup (2.8), we see that the operators P_μ and K_μ act as raising and lowering operators for the dilatation operator D – this gives a hint as to how we could construct representations of the group. The dilatation operator D is the generator of scalings, i.e. upon a rescaling $x \to \lambda x$ a local operator in a field theory scales as

$$\mathcal{O}(x) \to \lambda^{-\Delta}\mathcal{O}(\lambda x) \tag{2.18}$$

where Δ is the conformal dimension of the operator $\mathcal{O}(x)$. Restricting to the point $x = 0$, which is a fixed point of scalings, we see that the conformal dimension is the eigenvalue of the dilatation operator,

$$[D, \mathcal{O}(0)] = -i\Delta\mathcal{O}(0). \tag{2.19}$$

It is now clear that acting on a field with K_μ should lower the dimension by one and acting with P_μ – raise it by one. We can show this explicitly using the Jacobi identity as

$$[D, [K_\mu, \mathcal{O}(0)]] = [[D, K_\mu], \mathcal{O}(0)] + [K_\mu, [D, \mathcal{O}(0)]] = -i(\Delta - 1)\ [K_\mu, \mathcal{O}(0)]. \tag{2.20}$$

Since operators in a unitary quantum field theory should have positive dimensions (aside from the identity operator), we should not be able to keep lowering the dimension indefinitely, i.e. there should always be an operator that satisfies

$$[K_\mu, \tilde{\mathcal{O}}(0)] = 0. \tag{2.21}$$

We call such operators *conformal primary operators*. Acting on these with P_μ keeps producing operators with a dimension one higher – we call these the *descendants* of $\tilde{\mathcal{O}}(0)$. We can also act with the supercharges and looking at the commutators in (2.10) we see that they raise the dimension by 1/2, while the special conformal supercharges lower it by 1/2. Operators annihilated by special conformal supercharges are called *superconformal primaries*, which is a stronger condition that being a conformal primary.

(Super-)conformal primaries and their descendants make up multiplets that constitute the three families of discrete representations in the classification. They are further distinguished by the number of supercharges the primary commutes with. One example is a class of operators that satisfy the condition

$$\Delta = r_1 + r_2 + r_3, \tag{2.22}$$

a canonical representative would be a single-trace symmetrized scalar field operator such as

$$\mathcal{O}^{ij\ldots k}(x) = \mathrm{Tr}\left(\Phi(x)^{(i}\Phi(x)^{j}\ldots\Phi(x)^{k)}\right). \tag{2.23}$$

These operators commute with half of the supercharges, thus they are referred to as half-BPS. A key fact is that operators in the same representation must have the same anomalous dimension, because the generators of the group can only change it by half integer steps and there's only a finite number of generators. What is more, operators in the discrete BPS representations are protected from quantum corrections, because at any coupling the total dimension is always algebraically related to the Dynkin labels of the R-symmetry, e.g. as in (2.22). Since charges of compact groups are quantized it must meant that the dimension can't continuously depend on the coupling and hence the anomalous dimension must vanish. This is however not true for the fourth continuous non-BPS family of representations, hence operators from these multiplets do acquire anomalous dimensions.

Let us conclude the section with an elegant argument for why the beta function of $\mathcal{N}=4$ super Yang-Mills is zero. One can use the algebra and shown that the operators $\operatorname{Tr} F_+F_+$ and $\operatorname{Tr} F_-F_-$, where $F_\pm$ are the (anti-)self-dual field strengths, belong to the same multiplet as a superconformal primary [67], meaning that the $\operatorname{Tr} F_{\mu\nu}F^{\mu\nu}$ term in the Lagrangian is protected from quantum corrections, hence so is the coupling constant g_{YM}. This argument is valid to all orders in perturbation theory, which means that $\mathcal{N}=4$ super Yang-Mills is conformally invariant to all orders in perturbation theory.

2.4 String description at strong coupling

As already briefly explained in the introduction, the AdS/CFT conjecture states that $\mathcal{N}=4$ super Yang-Mills is exactly dual to type IIB string theory on $AdS_5\times S^5$, [4, 9, 10]. To be more precise, the gauge group of the Yang-Mills theory is taken to be $SU(N)$ and the coupling constant g_{YM}. The string theory is defined on $AdS_5\times S^5$ where both AdS_5 and S^5 have radius R. The self-dual five-form field F_5^+ has integer flux through the sphere

$$\int_{S^5} F_5^+ = N, \tag{2.24}$$

and N is identified with the number of colors in the gauge theory. The string theory is further parametrized by the string coupling g_s and the string string length squared α'. The following relations are conjectured to hold

$$4\pi g_s = g_{YM}^2 \equiv \frac{\lambda}{N}, \qquad \frac{R^4}{\alpha'^2} = \lambda, \tag{2.25}$$

where λ is the t'Hooft coupling. We will be working in the planar limit $N\to\infty$ with λ fixed. It is easy to see that in this limit $g_s\to 0$ and we are left with freely propagating strings. Furthermore, the regime of strongly coupled gauge theory when $\lambda\to\infty$ corresponds to the regime of string theory where the supegravity approximation is valid, namely $\alpha'\ll R^2$. The takeaway here is that one can formulate strongly coupled planar $\mathcal{N}=4$ super Yang-Mills as a classical theory of free strings on $AdS_5\times S^5$.

2.4.1 Sigma model formulation

A very useful formulation of string theory on $AdS_5 \times S^5$ is the coset space sigma model [15] with the target superspace of

$$\frac{PSU(2,2|4)}{SO(1,4) \times SO(5)}. \tag{2.26}$$

The bosonic part of the supercoset where the string moves is given by

$$\frac{SO(2,4) \times SO(6)}{SO(1,4) \times SO(5)} = AdS_5 \times S^5, \tag{2.27}$$

which is constructed as the coset between the isometry and isotropy groups of $AdS_5 \times S^5$. The action is then written in terms of the algebra of $PSU(2,2|4)$.

The superalgebra $\mathfrak{psu}(2,2|4)$ has no realization in terms of matrices, instead it is the quotient of $\mathfrak{su}(2,2|4)$ by matrices proportional to the identity. On the other hand $\mathfrak{su}(2,2|4)$ is a matrix superalgebra spanned by 8×8 supertraceless matrices

$$M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right), \tag{2.28}$$

where the supertrace is defined as

$$\mathrm{STr}\, M = \mathrm{Tr} A - \mathrm{Tr} D. \tag{2.29}$$

A and D are elements of $\mathfrak{su}(2,2)$ and $\mathfrak{su}(4)$ respectively, whereas the fermionic components are related by

$$C = \left(\begin{array}{c|c} +\mathbb{1}_{2\times 2} & 0 \\ \hline 0 & -\mathbb{1}_{2\times 2} \end{array}\right) B^\dagger. \tag{2.30}$$

An important feature of this algebra is the following automorphism

$$\Omega \circ M = \left(\begin{array}{c|c} EA^T E & -EC^T E \\ \hline EB^T E & ED^T E \end{array}\right), \qquad E = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{2.31}$$

which endows the algebra with a $\mathbb{Z}_4$ grading, since one can easily check that $\Omega^4 = 1$. This in turn means that any element of the algebra can be decomposed under this grading as

$$M = \sum_{i=0}^{3} M^{(i)}, \tag{2.32}$$

where

$$\begin{aligned} M^{(0,2)} &= \tfrac{1}{2} \left(\begin{array}{c|c} A \pm EA^T E & 0 \\ \hline 0 & D \pm ED^T E \end{array}\right) \\ M^{(1,3)} &= \tfrac{1}{2} \left(\begin{array}{c|c} 0 & B \pm iEC^T E \\ \hline C \mp iEB^T E & 0 \end{array}\right) \end{aligned} \tag{2.33}$$

and the morphism then acts on the elements of the decomposition as

$$\Omega \circ M^{(n)} = i^n M^{(n)}. \tag{2.34}$$

The Metsaev-Tseytlin action for the Green-Schwarz superstring is then given by

$$S = \frac{\sqrt{\lambda}}{4\pi} \int \mathrm{STr}\left(J^{(2)} \wedge *J^{(2)} - J^{(1)} \wedge J^{(3)} + \Lambda \wedge J^{(2)} \right), \tag{2.35}$$

which is written down in terms of the graded elements of the algebra current

$$J = -g^{-1} \mathrm{d} g \tag{2.36}$$

where $g(\sigma, \tau) \in PSU(2,2|4)$ is a map from the string worldsheet to the supergroup $PSU(2,2|4)$. The last term contains a Lagrange multiplier Λ, which ensures that $J^{(2)}$ is supertraceless, whereas all other components are manifestly traceless as seen from (2.33). Since the target space is the coset of $PSU(2,2|4)$ by $SO(1,4) \times SO(5)$, the map g has an extra gauge symmetry

$$g \to gH, \quad H \in SO(1,4) \times SO(5) \tag{2.37}$$

under which the components of the supercurrent transform as

$$J^{(0)} \to H^{-1} J^{(0)} H - H^{-1} \mathrm{d} H \tag{2.38}$$

$$J^{(i)} \to H^{-1} J^{(i)} H, \quad i = 1, 2, 3 \tag{2.39}$$

The equations of motion read

$$d * k = 0, \tag{2.40}$$

where $k = gKg^{-1}$ and

$$K = J^{(2)} + \frac{1}{2} * J^{(1)} - \frac{1}{2} * J^{(3)} - \frac{1}{2} * \Lambda. \tag{2.41}$$

They are equivalent to the conservation of the Noether current associated to the global left $PSU(2,2|4)$ multiplication symmetry.

Finally let us briefly remark on how the action reduces to the usual sigma model action if one restricts to bosonic fields. A purely bosonic representative of $PSU(2,2|4)$ has the form

$$g = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array} \right), \tag{2.42}$$

where $A \in SO(6) \simeq SU(4)$ and $D \in SO(2,4) \simeq SU(2,2)$. Then we see that AEA^T is a good parametrization of

$$\frac{SO(6)}{SO(5)} \simeq \frac{SU(4)}{SP(4)} = S^5, \tag{2.43}$$

since it is invariant under $A \to AH$ with $H \in SP(4)$. Similarly DED^T parametrizes AdS_5. If we now define the coordinates u^i and v^i in the following way

$$u^i \Gamma_i^S = AEA^T, \qquad v^i \Gamma_i^A = DED^T, \tag{2.44}$$

with Γ^S and Γ^A being the gamma matrices of $SO(6)$ and $SO(2,4)$ respectively, then by construction they will satisfy the following constraints

$$\begin{aligned} 1 &= u \cdot u \equiv +u_1^2 + u_2^2 + u_3^2 + u_4^2 + u_5^2 + u_6^2 \\ 1 &= v \cdot v \equiv -v_1^2 - v_2^2 - v_3^2 - v_4^2 + v_5^2 + v_6^2, \end{aligned} \tag{2.45}$$

and the action (2.35) will read

$$S_b = \frac{\sqrt{\lambda}}{4\pi} \int_0^{2\pi} d\sigma \int d\tau \, \sqrt{h} \left(h^{\mu\nu} \partial_\mu u \cdot \partial_\nu u + \lambda_u \left(u \cdot u - 1 \right) - \left(u \to v \right) \right), \tag{2.46}$$

which is of course just the usual non-linear sigma model for a string moving in $AdS_5 \times S^5$.

3 Perturbative Integrability

In this section we start attacking the problem of finding the spectrum and as expected we begin with perturbation theory. Starting at weak coupling we quickly stumble upon an amazing feature of the theory, so-called integrability, which allows one to apply numerous techniques that greatly simplify the problem. We demonstrate integrability from the string theoretic perspective at strong coupling as well, which suggests a unified picture of the integrable structure embedded in the theory persisting to all loops. After discussing results achievable via perturbative integrability we finish off with our first exact result, the slope function, which in turn allows one to extract novel information about the spectrum.

3.1 One loop at weak coupling

We begin with two point correlation functions of local operators. In any conformal field theory they are constrained by symmetry, namely for operators that are eigenvalues of dilatations they have the following form at all loop levels

$$\left\langle \mathcal{O}(x)\,\tilde{\mathcal{O}}(y) \right\rangle \approx \frac{1}{|x-y|^{2\Delta}}, \tag{3.1}$$

where Δ is the scaling dimension of the operator and we ignore unphysical normalization factors. Classically $\Delta = \Delta_0$ is simply the mass dimension, but at the quantum level it receives radiative corrections and acquires an anomalous dimension γ, such that $\Delta(g_{YM}) = \Delta_0 + \gamma(g_{YM})$, where the anomalous dimension depends on the coupling. Usually the corrections are small and the correlator can be expanded perturbatively. Of course one has to be careful here, as expanding in γ would result in expressions like $\log|x-y|$, which do not make sense. To that end we introduce a scale μ and expand the following quantity instead

$$\mu^{-2\gamma} \left\langle \mathcal{O}(x)\,\tilde{\mathcal{O}}(y) \right\rangle \approx \frac{1}{|x-y|^{2\Delta_0}} \left(1 - \gamma \log \mu^2 |x-y|^2\right), \tag{3.2}$$

however we will formally assume that the factor $\mu^{-\gamma}$ is absorbed into the field definition and thus we will ignore it from now on. We can now take some explicit local operator $\mathcal{O}(x)$, calculate the correlator using perturbation theory and read off the anomalous dimension γ. Let us start with a very simple chiral primary operator

$$\Psi = \operatorname{Tr} Z^L = Z^a{}_b Z^b{}_c \dots Z^l{}_a, \tag{3.3}$$

where the complex scalar field Z and its conjugate $\tilde{Z}$ have the standard tree level correlators

$$\left\langle Z^a{}_b(x) \tilde{Z}^{b'}{}_{a'}(y) \right\rangle_{\text{tree}} \approx \frac{\delta^a{}_{a'} \delta_b{}^{b'}}{|x-y|^2}. \tag{3.4}$$

In order to find the anomalous dimension of the chiral primary operator Ψ we must calculate $< \Psi(x)\tilde{\Psi}(x) >$. We do this by using Wick's theorem and plugging in the two-point correlator

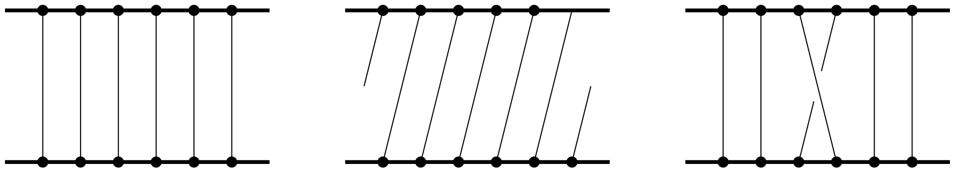


Figure 1: Possible types of Wick contractions (vertical lines) between single trace operators. The constituent scalar fields are represented by dots in the horizontal lines, which represent the successive index contractions due to the trace. First two figures are examples of planar contractions while the last one is an example of a non-planar contraction.

(3.4), which produces a lot of terms with delta function contractions between the adjoint indices. Some examples are

$$\ldots \delta^{a'}{}_{a}\, \delta^{a}{}_{a'}\, \delta^{b'}{}_{b}\, \delta^{b}{}_{b'}\, \delta^{c'}{}_{c}\, \delta^{c}{}_{c'}\, \ldots \tag{3.5a}$$

$$\ldots \delta^{a'}{}_{c}\, \delta^{c}{}_{a'}\, \delta^{b'}{}_{a}\, \delta^{a}{}_{b'}\, \delta^{c'}{}_{b}\, \delta^{b}{}_{c'}\, \ldots \tag{3.5b}$$

$$\ldots \delta^{a'}{}_{a}\, \delta^{a}{}_{b'}\, \delta^{c'}{}_{b}\, \delta^{b}{}_{a'}\, \delta^{b'}{}_{c}\, \delta^{c}{}_{c'}\, \ldots \tag{3.5c}$$

These contractions have a graphical interpretation. Consider the scalar field $Z^{a}{}_{b}$ as a dot and each contraction of the adjoint indices as a line connecting these dots, then the chiral primary operator Ψ is simply a circle due to the trace. Wick's theorem says that in order to find the correlator $< \Psi(x)\tilde{\Psi}(x) >$ we must sum all possible ways we can connect the dots in the circle of Ψ to the dots in the circle of $\tilde{\Psi}$. All the delta function contractions that we get after expanding the correlator represent precisely all the possible ways we can contract the dots in the circles. The three excerpts of contractions shown in (3.5) can be represented graphically as shown in fig. 1. One can immediately notice that the first two are planar, while the third one is intersecting itself. Evaluating the three contractions we immediately see that planar ones produce a factor of N^3 while the non-planar one produces a factor of N, i.e. non-planar diagrams are suppressed and we can discard them once we take the planar limit $N \to \infty$. All that's left then are cyclic permutations of lines by shifting all of them as seen in fig. 1 while going from (a) to (b). There are $L - 1$ shifts that can be done in this way, since after making a full circle we return to the initial configuration. Thus finally for the chiral primary correlator at tree level we find

$$\left\langle \Psi(x)\, \tilde{\Psi}(y) \right\rangle_{\text{tree}} \approx \frac{LN^L}{|x-y|^{2L}}, \tag{3.6}$$

where N^L comes from the contractions and L from all the possible planar ways we can contract. This can easily be generalized for correlators of operators with arbitrary scalar fields $\Phi_{I_1 I_2 \ldots I_L}(x) = \mathrm{Tr}\,[\Phi_{I_1}(x)\Phi_{I_2}(x)\ldots\Phi_{I_L}(x)]$ to

$$\left\langle \Phi_{I_1 I_2 \ldots I_L}(x)\, \tilde{\Phi}^{J_1 J_2 \ldots J_L}(y) \right\rangle_{\text{tree}} \approx \frac{1}{|x-y|^{2L}} \left(\delta^{J_1}_{I_1} \delta^{J_2}_{I_2} \ldots \delta^{J_L}_{I_L} + \text{cycles} \right), \tag{3.7}$$

where "cycles" refers to terms with the J indices pushed. I and J are flavor indices, the color indices are suppressed.

So far so good, but in order to calculate anomalous dimensions we have to go beyond tree level. This may seen like a highly non-trivial thing to do, since we expect not only scalar interactions, but also gluon exchanges and fermion loops appearing. Luckily the symmetry of the theory allows one to calculate all gluon and fermion effects in one go. First let's concentrate on the bosonic sector of the theory ignoring gluons. The action (2.3) contains a single scalar-only interaction term

$$\begin{aligned} S_\Phi &= \frac{g_{YM}^2}{4}\sum_{I,J}\int d^4x \,\mathrm{Tr}\,[\Phi_I,\Phi_J][\Phi_I,\Phi_J] \\ &= -\frac{g_{YM}^2}{4}\sum_{I,J}\int d^4x \,\left(\mathrm{Tr}\,[\Phi_I\Phi_I\Phi_J\Phi_J] - \mathrm{Tr}\,[\Phi_I\Phi_J\Phi_I\Phi_J]\right). \end{aligned} \tag{3.8}$$

In order to calculate the correlator (3.7) at one-loop level, one should insert this term and Wick contract. Just like in tree level, we only have to keep planar diagrams. For the interaction terms this means that only neighbouring fields can interact. This drastically reduces the number of terms we get after Wick contracting. Because of that it is enough to consider a length two operator $\Phi_{I_k I_{k+1}}$ and with a bit of work one can show that at one-loop level we get

$$\begin{aligned} &\left\langle \Phi_{I_k I_{k+1}}(x)\,\tilde{\Phi}^{J_k J_{k+1}}\right\rangle_{\mathrm{one-loop}} = \tfrac{\lambda}{16\pi^2}\tfrac{\log(\mu^2|x-y|^2)}{|x-y|^{2L}}\times \\ &\times\left(2\delta_{I_k}{}^{J_{k+1}}\delta_{I_{k+1}}{}^{J_k} - \delta_{I_k I_{k+1}}\delta^{J_k J_{k+1}} - \delta_{I_k}{}^{J_k}\delta_{I_{k+1}}{}^{J_{k+1}}\right), \end{aligned} \tag{3.9}$$

where $\lambda = g_{YM}^2 N$ is the t'Hooft coupling. Comparing this to (3.7) we see that effectively the interactions permute and contract the delta function indices. We can introduce exchange and trace operators to make this explicit. The permutation operator, also called the exchange operator, $\mathcal{P}_{l,l+1}$ is defined by it's action on a set of delta functions as

$$\mathcal{P}_{l,l+1}\,\delta_{I_1}{}^{J_1}\ldots\delta_{I_l}{}^{J_l}\delta_{I_{l+1}}{}^{J_{l+1}}\ldots\delta_{I_L}{}^{J_L} = \delta_{I_1}{}^{J_1}\ldots\delta_{I_l}{}^{J_{l+1}}\delta_{I_{l+1}}{}^{J_l}\ldots\delta_{I_L}{}^{J_L} \tag{3.10}$$

and the trace operator $\mathcal{K}_{l,l+1}$ is defined as

$$\mathcal{K}_{l,l+1}\,\delta_{I_1}{}^{J_1}\ldots\delta_{I_l}{}^{J_l}\delta_{I_{l+1}}{}^{J_{l+1}}\ldots\delta_{I_L}{}^{J_L} = \delta_{I_1}{}^{J_1}\ldots\delta_{I_l I_{l+1}}\delta^{J_l J_{l+1}}\ldots\delta_{I_L}{}^{J_L}. \tag{3.11}$$

Using these operators we can rewrite the correlator in (3.9) in a more compact notation

$$\begin{aligned} &\left\langle \Phi_{I_k I_{k+1}}(x)\,\tilde{\Phi}^{J_k J_{k+1}}\right\rangle_{\mathrm{one-loop}} = \\ &\quad = \frac{\lambda}{16\pi^2}\frac{\log(\mu^2|x-y|^2)}{|x-y|^{2L}}\quad\left(2\,\mathcal{P}_{k,k+1} - \mathcal{K}_{k,k+1} - 1\right)\delta_{I_k}^{J_k}\delta_{I_{k+1}}^{J_{k+1}}. \end{aligned} \tag{3.12}$$

This result includes four scalar interactions only, however as mentioned before at one-loop level we can also have gluon interactions and fermion loops in scalar propagators. The nice thing about these is that such interactions don't alter the flavor index structure, i.e. there are no permutations or traces. Basically this happens because the gluon transforms trivially under R-symmetry and hence can't change the flavor index (which transforms under R-symmetry). Similarly, fermions can only appear in loops altering scalar self-energies, hence they also leave the flavor structure intact. Thus all of these interactions contribute a constant term C, which we can determine later. We can generalize our one-loop result with all interactions included for operators of arbitrary length,

$$\begin{aligned}\left\langle \Phi_{I_1 I_2 \dots I_L}(x)\, \tilde{\Phi}^{J_1 J_2 \dots J_L}(y) \right\rangle_{\text{one-loop}} &= \frac{\lambda}{16\pi^2} \frac{\log(\mu^2|x-y|^2)}{|x-y|^{2L}} \times \\ \times \sum_{l=1}^{L} \left(2\, \mathcal{P}_{l,l+1} - \mathcal{K}_{l,l+1} - 1 + C\right) &\left(\delta_{I_1}^{J_1} \delta_{I_2}^{J_2} \dots \delta_{I_L}^{J_L} + \text{cycles}\right).\end{aligned}$$

Combining this with the tree level result (3.7) and comparing to the general expression of a two-point function at one-loop level (3.2) we can deduce the anomalous dimension γ, which now becomes an operator Γ because of the flavor mixing. It is given by

$$\Gamma = \frac{\lambda}{16\pi^2} \sum_{l=1}^{L} \left(-2\, \mathcal{P}_{l,l+1} + \mathcal{K}_{l,l+1} + 1 - C\right). \tag{3.13}$$

At first sight it may seem strange that what was supposed to be a number, i.e. a correction to the mass dimension of an operator has turned out to be an operator acting on the flavor space, i.e. a matrix. But this is very natural and in fact expected, since interactions can change the flavor of fields and we can't be sure that an operator at the quantum level has the same flavor indices as it does at the classical level. This line of thinking may lead to a natural question, why do we have mixing between the scalars only and not between all the fields in the theory including fermions, which miraculously do not appear. It turns out that this is a one-loop feature only and mixing becomes a problem at higher loop levels [17].

Now that we have acknowledged that the anomalous dimension is a matrix and found an expression for it, the next logical step would be diagonalizing it and finding the flavor eigenstates. One example of such an eigenstate is the chiral primary operator Ψ. Since it contains scalar fields of only one type, the permutation and trace operators act trivially on it. Thus we see that

$$\Gamma\, \Psi = \frac{\lambda}{16\pi^2} \sum_{l=1}^{L} \left(-2 + 1 - C\right) \Psi, \tag{3.14}$$

but we already saw that a chiral primary has an anomalous dimension of zero, which then fixes the constant C to -1. And finally we get

$$\Gamma = \frac{\lambda}{16\pi^2} \sum_{l=1}^{L} \left(2 - 2\, \mathcal{P}_{l,l+1} + \mathcal{K}_{l,l+1}\right). \tag{3.15}$$

A keen eye might already notice that this expression resembles a Hamiltonian of a spin chain. In fact, this is hardly surprising, since from the very beginning we were talking about fields as points in some closed line, which indeed resembles a spin chain. Furthermore the correlators that we were calculating are nothing more that propagators from one state of the chain to another, hence no wonder that the operator describing this evolution looks like a Hamiltonian for a spin chain. This identification is very useful, because the spin chains that appear in AdS/CFT are integrable and can be solved exactly, which gives us hope that we can apply the same techniques here and solve the spectral problem in $\mathcal{N}=4$ exactly. The first steps towards this goal were outlined in the seminal paper [17], which launched the integrability program in AdS/CFT. However saying that the spectral problem can be solved exactly in this particular case is too strong, since we are only at one-loop level. Nevertheless one can apply the same techniques going beyond one-loop level, as we shall soon see in the coming sections.

3.1.1 The $\mathfrak{su}(2)$ sector

In the previous section we considered single trace operators potentially containing all six scalar fields, we also mentioned that at higher loops the remaining fields of the theory start mixing in, i.e. the scalar sector is only closed at one-loop level. However it is easy to see that there exist sectors that are closed at all loops. From the algebra of the theory we know that dilatations commute with Lorentz and R-symmetries at any value of the coupling, hence it must follow that all the coefficients in

$$D = \sum_n \lambda^n D^{(2n)}, \tag{3.16}$$

where in particular $D^{(2)} \equiv \Gamma$, commute with Lorentz and R-symmetry generators. Thus we conclude that only operators with the same bare dimensions, Lorentz charges and R-charges can mix when acting with the anomalous dimension matrix.

Arguably the simplest possible closed sector is the so-called $\mathfrak{su}(2)$ sector, containing only two scalar fields X and Z. An operator with M and $L-M$ scalars X and Z has the charges $(0,0;L;M,L-M,0)$, the only other operators with these charges are permutations of this operator, hence the sector is closed. The anomalous dimension operator in this sector is given by

$$\Gamma = \frac{\lambda}{8\pi^2}\sum_{l=1}^{L}\left(1-\mathcal{P}_{l,l+1}\right). \tag{3.17}$$

Up to a constant factor this is the same as the Hamiltonian for the Heisenberg spin chain (also called the XXX spin chain), which is a quantum description of a one dimensional magnet. The Hamiltonian is given by

$$\mathbf{H} = \sum_{l=1}^{L}\left(1-\mathcal{P}_{l,l+1}\right), \tag{3.18}$$

which can also be rewritten in terms of Pauli matrices as

$$\mathbf{H} = 2\sum_{l=1}^{L}\left(\frac{1}{4}-\vec{S}_l\cdot\vec{S}_{l+1}\right), \qquad \vec{S}_l = \frac{1}{2}\vec{\sigma}_l. \tag{3.19}$$

Hence solving the spectral problem in $\mathcal{N}=4$ SYM translates into solving the Schrödinger equation

$$\mathbf{H}\left|\psi\right\rangle = E\left|\psi\right\rangle, \tag{3.20}$$

where we now seek to find the energy eigenvalues for the Hamiltonian of the spin chain. If the chain is short, this is a trivial diagonalization problem that can be easily solved by a present day computer. However this problem was first solved analytically by Hans Bethe in a time when computers were still in their infancy. The original solution now goes by the name of *coordinate Bethe ansatz* and it is by far one of the most important and beautiful solutions in physics in the past century, which is still very widely used even to this day. The idea is to make an educated guess for the wave function $|\psi\rangle$, plug it in to the Schrödinger equation and determine when does it actually hold. This produces a set of algebraic Bethe ansatz equations for a set of variables unimaginatively called the Bethe roots. All observables can then be expressed in terms of these numbers as simple algebraic functions, thus transforming a diagonalization problem to an algebraic problem. This has an enormous advantage, since in the asymptotic limit, when the spin chain is very large, instead of diagonalizing an infinite matrix, the set of algebraic equations actually simplify and produce integral equations, which can be solved.

In the spin chain language the scalar fields can be treated as up and down spin states, i.e.

$$|\uparrow\rangle = Z = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle = X = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{3.21}$$

thus local single trace operators can be treated as states of a spin chain, e.g.

$$\mathrm{Tr}\,(XXZXXZX) = |\downarrow\downarrow\uparrow\downarrow\downarrow\uparrow\downarrow\rangle. \tag{3.22}$$

Due to the cyclicity of the trace all rotations of the chain are equivalent. We should also specify the periodicity boundary condition

$$\vec{S}_{L+1} = \vec{S}_1. \tag{3.23}$$

The operators $\vec{S}_l$ act as Pauli matrices on the l'th spin site and trivially on all the others. Since a spin "chain" with a single site would have a state space $\mathbb{C}^2$, a spin chain of length L has a state space $\mathbb{C}^{\otimes_L 2}$, which has 2^L basis vectors and the Hamiltonian is then a $2^L \times 2^L$ matrix, which we need to diagonalize. Working directly with Pauli matrices one can find some simple results directly, e.g. it is trivial to show that the chiral primary operator

$$|\Psi\rangle = \mathrm{Tr}\, Z^L = |\uparrow\uparrow \dots \uparrow\rangle \tag{3.24}$$

is an eigenstate of the Hamiltonian with zero energy, i.e. it is the ferromagnetic ground state of the spin chain, which we will denote as $|0\rangle$ from now on. This is expected, since we know that chiral primaries have zero anomalous dimensions. Another eigenstate of the Hamiltonian is the *single magnon* state, defined as

$$|p\rangle = \sum_{n=1}^{L} e^{ipn} |n\rangle, \tag{3.25}$$

where $|n\rangle$ is the ground state with the n'th spin flipped,

$$|n\rangle = S_n^- \,|0\rangle = |\uparrow\uparrow\uparrow \dots \downarrow \dots \uparrow\uparrow\uparrow\rangle, \tag{3.26}$$

here p is formally just a parameter, but it can be interpreted as the momentum of the excitation travelling in the spin chain. Due to the cyclicity of the chain the momentum is quantized,

$$p = \frac{2\pi}{L} n, \quad n \in \mathbb{Z}, \tag{3.27}$$

where n is the mode number. The energy of the excitation is given by the dispersion relation

$$E(p) = 4\sin^2\frac{p}{2}. \tag{3.28}$$

Now consider a two magnon state

$$|\psi\rangle = \sum_{n<m} \psi(n,m)\,|n,m\rangle, \quad |n,m\rangle = S_n^-\,S_m^-\,|0\rangle. \tag{3.29}$$

The situation is not so trivial this time, since the two magnons might scatter among themselves. We now plug this into (3.20) and find the conditions for $\psi(n,m)$, which are

$$\begin{aligned} E\,\psi(n,m) = 4\,\psi(n,m) - \psi(n+1,m) - \psi(n-1,m) \\ - \psi(n,m+1) - \psi(n,m-1) \end{aligned} \tag{3.30}$$

when $m > n+1$ and

$$E\,\psi(n,n+1) = 2\,\psi(n,n+1) - \psi(n-1,n+1) - \psi(n,n+2) \tag{3.31}$$

when $m = n+1$, i.e. when the two magnons scatter. The solution is now a superposition of single magnon states

$$\psi(n,m) = e^{ikn+ipm} + S(k,p)e^{ipn+ikm}, \tag{3.32}$$

where

$$S(p,k) = \frac{\frac{1}{2}\cot\frac{k}{2} - \frac{1}{2}\cot\frac{p}{2} - i}{\frac{1}{2}\cot\frac{k}{2} - \frac{1}{2}\cot\frac{p}{2} + i} \tag{3.33}$$

is the scattering matrix. As required, such a state is an eigenstate and the energy is given by

$$E = E(p) + E(k), \tag{3.34}$$

i.e. it is simply the sum of the single magnon energies. Finally the spin chain periodicity condition imposes the following equations

$$e^{ikL}\,S(p,k) = e^{ipL}\,S(k,p) = 1. \tag{3.35}$$

It is now straightforward to generalize this procedure, which is exactly what Bethe did. The wave function for M spins down can be written as

$$|\psi\rangle = \sum_{1\le l_1<l_2<\cdots<l_M\le L} \psi(l_1,l_2,\ldots,l_M)\,S_{l_1}^-\,S_{l_2}^-\,\ldots\,S_{l_M}^-\,|0\rangle. \tag{3.36}$$

The sum is chosen in a way so as not to over count states. The Bethe ansatz is the educated guess of the wave function

$$\psi(l_1,l_2,\ldots,l_M) = \sum_{\sigma\,\in\,perm(1,2,\ldots\,M)} A(p)\,e^{ip_{\sigma_1}l_1+ip_{\sigma_2}l_2+\cdots+ip_{\sigma_M}l_M}, \tag{3.37}$$

where the sum runs over all permutations of the down spin labels $1,2,\ldots,M$. p_i are the momenta of the down spins, which can be treated as excitations moving in the vacuum state of the spin chain. The ansatz then looks like a superposition of plane waves. As in the two magnon case, one should now plug in the ansatz and find the conditions that make it work. The result is a set of algebraic equations, called the *Bethe equations*

$$e^{ip_k L} = -\prod_{\substack{j=1 \\ j\neq k}}^{M} \frac{e^{ip_j} - e^{ip_k} + 1}{e^{ip_k} - e^{ip_j} + 1} \quad \text{for } k = 1, 2, \ldots, M \tag{3.38}$$

and the amplitude is given by

$$A(r) = \operatorname{sign}(\sigma) \prod_{j<k} \left(e^{ip_j} - e^{ip_k} + 1\right). \tag{3.39}$$

These equations can be interpreted physically once rewritten as

$$e^{ip_k L} \prod_{\substack{j=1 \\ j\neq k}}^{M} S(p_j, p_k) = 1, \quad \text{where } S(p_j, p_k) = -\frac{e^{ip_k} - e^{ip_j} + 1}{e^{ip_j} - e^{ip_k} + 1}. \tag{3.40}$$

This is simply saying that if we take a magnon, carry it around the spin chain, the total phase change which is a result of free propagation (represented by $e^{ip_k L}$) and scattering with other magnons (due to $S(p_j, p_k)$) must be trivial. Changing variables to

$$e^{ip_k} = \frac{u_k + i/2}{u_k - i/2}, \quad u_k = \frac{1}{2} \cot \frac{p_k}{2}, \tag{3.41}$$

brings the Bethe equations (3.38) to a more familiar form

$$\left(\frac{u_k + i/2}{u_k - i/2}\right)^L = \prod_{\substack{j=1 \\ j\neq k}}^{M} \frac{u_k - u_j + i}{u_k - u_j - i}, \tag{3.42}$$

where now one solves for the Bethe roots u_k, also known as magnon rapidities. It is now straightforward to see that this general solution reproduces the two magnon scenario we discussed earlier. The energy of the M magnon state is given by

$$E = \sum_{k=1}^{M} \frac{1}{u_k^2 + 1/4}, \tag{3.43}$$

which also agrees with the single and two magnon examples.

They key thing worth noting in (3.40) is that the spin chain can be fully described in terms of the scattering matrix for just two particles, i.e. the full M particle scattering matrix factorizes. This is the defining property of integrability, since factorized scattering means that individual momenta are conserved in each two particle scattering producing a tower of conserved quantities – just the thing one would want in an integrable system.

3.1.2 $\mathfrak{sl}(2)$ and other sectors

sl2

3.2 Higher loops

The next step in solving the spectral problem is increasing the loop level. For the $\mathfrak{su}(2)$ sector this has first been done for two-loops using mainly diagrammatic methods and by fixing the structure of the operator by symmetry. The resulting dilatation operator is given by [?]

$$\Gamma_{2-loop} = \frac{\lambda}{8\pi^2} \sum_{l=1}^{L} \left(-4 + 6\,\mathcal{P}_{l,l+1} - \left(\mathcal{P}_{l,l+1}\,\mathcal{P}_{l+1,l+2} + \mathcal{P}_{l+1,l+2}\,\mathcal{P}_{l,l+1}\right)\right). \tag{3.44}$$

In the spin chain picture this corresponds to a Hamiltonian for a long range spin chain with two nearest neighbour interactions. This spin chain has been shown to be integrable [?]. This result has been extended to three, four and five loops [?]. The explicit expressions for the dilatation operator at higher loops get more and more lengthy and complicated, but a pattern emerges that at loop level l the dilatation operator can be identified with a Hamiltonian of a long range spin chain where at most l nearest neighbours in the chain interact. What is even more remarkable is that these spin chains also turn out to be integrable [?], which hints that integrability may be an all loop phenomenon. This was in part verified by solving the spectral problem in the asymptotic limit, i.e. when the spin chain length L becomes infinite, but the number of excitations M is kept finite. The solution is given by conjecturing a set of *asymptotic Bethe ansatz equations*, which since their original inception have been extensively verified [?, ?]. The equations have the same form as in the one-loop case (3.40), but the scattering function for two magnons gets modified to [?]

$$S(p_i, p_j) = \frac{u(p_i) - u(p_j) + i}{u(p_i) - u(p_j) - i} \times S_D(p_i, p_j), \tag{3.45}$$

where $S_D(p_i, p_j)$ is the so called dressing factor (an explicit expression for it can be found in [?]) and the rapidities are now defined as

$$u(p) = \frac{1}{2} \cot\frac{p}{2} \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2\frac{p}{2}}. \tag{3.46}$$

The outcome is that all one-loop results get slightly modified, e.g. the magnon dispersion relation (3.28) becomes

$$E(p) = \frac{8\pi^2}{\lambda} \left(\sqrt{1 + \frac{\lambda}{\pi^2} \sin^2\frac{p}{2}} - 1 \right), \tag{3.47}$$

which in the low coupling limit $\lambda \to 0$ agrees with the one-loop result as it should. For many magnon states the energy is still given by the sum of individual magnon energies. It is truly remarkable that such a simple solution exists even though the Hamiltonian of the all-loop spin chain is not known. But even though such an easy generalization to an all-loop solution looks promising, it is only the first step towards the full solution of the spectral problem in $\mathcal{N} = 4$ SYM.

Short example of a two loop Hamiltonian, perturbative corrections for the states found above with contact terms.

3.3 Asymptotic solution

$$\left(\frac{x_k^+}{x_k^-}\right)^J = \prod_{j \neq k}^{S} \frac{x_k^- - x_j^+}{x_k^+ - x_j^-} \frac{1 - 1/(x_k^+ x_j^-)}{1 - 1/(x_k^- x_j^+)} \sigma^2(u_k, u_j), \quad k = 1, \ldots S \tag{3.48}$$

where

$$\Delta = J + S + \gamma(g), \quad \gamma(g) = \frac{i\sqrt{\lambda}}{2\pi} \sum_{j=1}^{S} \left(\frac{1}{x_j^+} - \frac{1}{x_j^-} \right) \tag{3.49}$$

with

$$u(x) = g\left(x + \frac{1}{x}\right) \tag{3.50}$$

3.3.1 A glimpse ahead: the slope function

The asymptotic Bethe ansatz (3.48) is the first non-trivial exact result we encountered so far, even if only valid in the asymptotic limit. In this short paragraph we will demonstrate how it can be used to find the exact *slope function* $\gamma^{(1)}(g)$, which is defined as the linear term in the small S expansion of the anomalous dimension, namely

$$\gamma(g) = \gamma^{(1)}(g)\, S + \gamma^{(2)}(g)\, S^2 + \mathcal{O}\left(S^3\right). \tag{3.51}$$

The subleading coefficient $\gamma^{(2)}(g)$ is called the *curvature function* and it will be the main study object of section ??, where we also address the question of what it actually means to send an integer quantity S to zero.

The slope function was initially conjectured in [68] and later independently proved in [69] and [70], our derivation will follow the former reference. The starting point is the logarithm of the asymptotic Bethe ansatz (3.48), given by

$$\frac{J}{i}\log\left(\frac{x_k^+}{x_k^-}\right) - \sum_{j\neq k}^{S} \frac{1}{i}\log\left(\frac{x_k^- - x_j^+}{x_k^+ - x_j^-}\,\frac{1 - 1/(x_k^+ x_j^-)}{1 - 1/(x_k^- x_j^+)}\,\sigma^2(u_k, u_j)\right) = 2\pi n_k, \tag{3.52}$$

where n_k is the mode number of the k'th Bethe root. In the small S limit the number of Bethe roots also tends to zero and in this regime they stop interacting [68], thus we will consider the case when $n_k = n$ and the general result will simply be a linear combination of terms with different values of n_k. The key idea of the derivation is assuming that the result only depends on the combination $\Lambda \equiv n\sqrt{\lambda}$ and taking the small n limit. Obviously this limit is also the strong coupling limit, as $\lambda \sim 1/n^2 \to \infty$. This considerably simplifies the derivation, for starters we only need the strong coupling expansion of the dressing phase, which is given by [??]

$$\log\sigma(u_k, u_j) \simeq -\log\left(\frac{1 - 1/(x_k^+ x_j^-)}{1 - 1/(x_k^- x_j^+)}\right) + i\,(u_j - u_k)\log\left(\frac{x_j^- x_k^- - 1}{x_j^- x_k^+ - 1}\,\frac{x_j^+ x_k^+ - 1}{x_j^+ x_k^- - 1}\right). \tag{3.53}$$

Also, since $u_k \sim 1/n$ we can simplify the shifts in the rapidities u_k, namely

$$x_k^\pm = x\left(u_k \pm \frac{i}{2}\right) = x\left(\frac{1}{g}\left(x_k + \frac{1}{x_k}\right) \pm \frac{i}{2}\right) = x_k \pm \frac{i}{2g}\frac{x_k^2}{x_k^2 - 1} + \mathcal{O}\left(\frac{1}{g^2}\right). \tag{3.54}$$

Plugging in the leading order dressing phase expansion and getting rid of the shifts in the rapidities reduces the asymptotic Bethe ansatz equations (3.52) to

$$\sum_{j\neq k} \frac{2}{x_k - x_j} + \frac{1}{x_k}\left(J + \gamma + \frac{2}{1 - x_k^2}\right) = \frac{\Lambda(x_k^2 - 1)}{2x_k^2} \tag{3.55}$$

$$\gamma = G(1) - G(-1) \tag{3.56}$$

$$G^2(x) + G'(x) + \left(\frac{J+\gamma+2}{x} - \frac{2x}{x^2-1} + \frac{\Lambda}{2}\frac{1-x^2}{x^2}\right) G(x) = F(x) \tag{3.57}$$

$$F(x) = \frac{\Lambda}{2}\frac{G(0)+G'(0)x}{x^2} + (J+\gamma+2)\,\frac{G(0)}{x} - \frac{G(1)}{x-1} - \frac{G(-1)}{x+1} \tag{3.58}$$

$$\Lambda G'(0) = 2G(1) + 2G(-1) - 2G(0)(J+\gamma+2) - \Lambda S \tag{3.59}$$

$$F(x) = \frac{\Lambda}{2}\left(\frac{G(0)}{x^2} - \frac{S}{x}\right) + \frac{G(-1)}{x(x+1)} - \frac{G(1)}{x(x-1)} \tag{3.60}$$

$$G(x) = \frac{x^2-1}{x^{J+2}} e^{\Lambda\frac{x^2+1}{2x}} \int_{x_0}^{x} dy\; F(y)\frac{y^{J+2}}{y^2-1} e^{-\Lambda\frac{y^2+1}{2y}} \tag{3.61}$$

$$\Lambda(G(0) - S) - G(+1)(2J+1) + G(-1) = 0 \tag{3.62a}$$

$$\Lambda(G(0) + S) + G(-1)(2J+1) - G(+1) = 0 \tag{3.62b}$$

$$G(x) = -\frac{\Lambda}{2}\frac{S}{J} - \frac{\gamma}{2x} - \frac{\Lambda}{4J}\frac{x^2-1}{x^{J+2}} e^{\Lambda\frac{x^2+1}{2x}} \int_0^x dy\left(\gamma J y^{J-1} + \Lambda S y^J\right) e^{-\Lambda\frac{y^2+1}{2y}} \tag{3.63}$$

$$I_\nu(\Lambda) = \frac{(-1)^{-\nu}}{2\pi i}\oint dy\; y^{\nu-1} e^{\Lambda\frac{y^2+1}{2y}} \tag{3.64}$$

$$\gamma J(-1)^J I_J(\Lambda) + \Lambda S(-1)^{J+1} I_{J+1}(\Lambda) = 0 \tag{3.65}$$

$$\gamma^{(1)}(\Lambda) = \frac{\Lambda}{J}\frac{I_{J+1}(\Lambda)}{I_J(\Lambda)} \tag{3.66}$$

3.4 Strong coupling and the algebraic curve

In the previous section we defined strings on $AdS_5 \times S^5$ in terms of the algebra current J, given in (2.36). This current has the property of being flat,

$$dJ - J \wedge J = 0, \tag{3.67}$$

and what is more, one can even define a one parameter family of connections from it by [?]

$$\begin{aligned} L(x) = J^{(0)} &+ \frac{x^2+1}{x^2-1}\,J^{(2)} - \frac{2x}{x^2-1}\left(*J^{(0)} - \Lambda\right) \\ &+ \sqrt{\frac{x+1}{x-1}}\,J^{(1)} + \sqrt{\frac{x-1}{x+1}}\,J^{(3)}, \end{aligned} \tag{3.68}$$

which are flat for all x,

$$dL(x) - L(x) \wedge L(x) = 0. \tag{3.69}$$

Here $L(x)$ is the *Lax connection* and x is the spectral parameter. The existence of such a set of connections signals that the theory is at least classically integrable. This can be shown by constructing the monodromy matrix

$$\Omega(x) = \mathcal{P} \exp \oint_\gamma L(x), \tag{3.70}$$

where γ is any path wrapping the worldsheet cylinder. Since the connection is flat, by definition it is path independent and we can evaluate the integral along any $\tau = const$ loop. Furthermore, shifting the τ value corresponds to doing a similarity transformation on the monodromy matrix [?], meaning that the eigenvalues must be time independent. Thus we have an infinite tower of conserved charges, hinting that the theory may be integrable. Denote the eigenvalues of the monodromy matrix as

$$\{e^{i\hat{p}_1(x)},\ e^{i\hat{p}_2(x)},\ e^{i\hat{p}_3(x)},\ e^{i\hat{p}_4(x)} \mid e^{i\tilde{p}_1(x)},\ e^{i\tilde{p}_2(x)},\ e^{i\tilde{p}_3(x)},\ e^{i\tilde{p}_4(x)}\}. \tag{3.71}$$

Here the quantities $p(x)$ are called *quasi-momenta*. The bar as always denotes the $\mathbb{Z}_2$ grading and we use the convention that hatted quantities correspond to AdS_5 variables and quantities with tildes correspond to S^5. From elementary algebraic geometry we know that the zeroes of any polynomial define an algebraic curve and since the eigenvalues $e^{ip(x)}$ are the zeroes of the characteristic polynomial of the monodromy matrix, they must also define an algebraic curve. A key idea in the development of integrability was the realization that this algebraic curve also known as the *spectral curve* can be used to define classical string solutions [?]. It is highly nontrivial to reconstruct a classical string solution given a set of quasi-momenta, yet they provide a very convenient way of describing solutions and they are very useful for solving the spectral problem. In that sense the spectral curve is the string analogue of the Bethe equations, which can also be used to find explicit solutions, but the their true power lies in their ability to efficiently solve the spectral problem.

The characteristic equation for the monodromy matrix is of order eight, meaning that the algebraic curves it defines can be thought of as cuts connecting eight sheets of a Riemann surface. A cut connecting sheets i and j is denoted as $\mathcal{C}^{ij}$ and the quasi-momenta on these sheets have discontinuities

$$p_i(x + i\epsilon) - p_j(x - i\epsilon) = 2\pi n_{ij}, \tag{3.72}$$

where n_{ij} is an integer. Four of the eight sheets correspond to the AdS_5 part of the string target space and the other four to the S^5 part, hence the indices i and j take on values

$$i \in \{\,\tilde{1},\, \tilde{2},\, \hat{1},\, \hat{2}\,\}, \quad j \in \{\,\tilde{3},\, \tilde{4},\, \hat{3},\, \hat{4}\,\} \tag{3.73}$$

and we define p to have either a hat or a tilde based on the index, i.e.

$$p_{\hat{i}}(x) \equiv \hat{p}_i(x) \quad \text{and} \quad p_{\tilde{i}}(x) \equiv \tilde{p}_i(x). \tag{3.74}$$

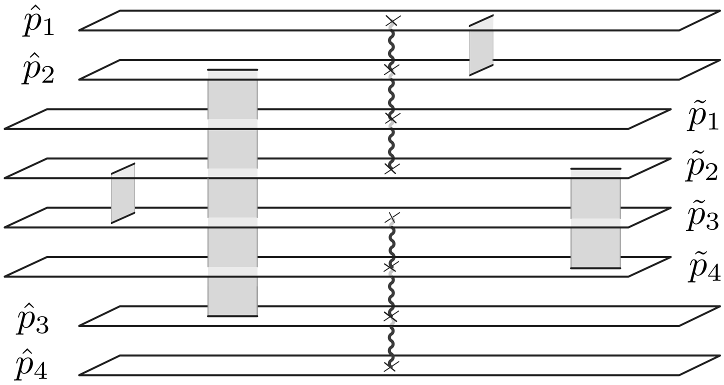


Figure 2: Examples of cuts connecting the eight sheets of the Riemann surface corresponding to the spectral curve for strings in $AdS_5 \times S^5$. The wavy line corresponds to the pole at $x = 1$.

We determine the polarization of a solution by the type of sheets the corresponding cut connects, e.g. if it connects two hatted sheets, the string is polarized in the AdS_5 part of the background and if it connects mixed sheets it is a fermionic excitation. Solutions in closed sectors, e.g. strings moving in the $\mathbb{R} \times S^3$ submanifold of the target space will be limited to cuts between a subset of the eight sheets. Some examples of cuts are shown in fig. 2. For each cut we associate the so called *filling fraction* defined by

$$S_{ij} = \pm \frac{\lambda}{8\pi^2 i} \oint_{\mathcal{C}^{ij}} \left(1 - \frac{1}{x^2}\right) p_i(x) dx, \tag{3.75}$$

where a plus sign is used for indices with a hat and a minus for indices with a tilde. These are the action angle variables for the theory, which is another concept from classical integrability [?]. Roughly they measure the length of the cut, it is also known that they correspond to the excitation numbers of strings or the number of Bethe roots in the spin chain picture [?], hence they are integers.

Since the Lax connection has poles at $x = \pm 1$, so do the quasi-momenta. Due to the Virasoro constraint, which comes about from the diffeomorphism invariance of the worldsheet, the residues of the quasi-momenta are constrained to

$$\{\hat{p}_1,\ \hat{p}_2,\ \hat{p}_3,\ \hat{p}_4 \,|\, \tilde{p}_1,\ \tilde{p}_2,\ \tilde{p}_3,\ \tilde{p}_4\} = \frac{\{\,\alpha_\pm,\, \alpha_\pm,\, \beta_\pm,\, \beta_\pm,\, |\, \alpha_\pm,\, \alpha_\pm,\, \beta_\pm,\, \beta_\pm\}}{x \pm 1}. \tag{3.76}$$

An additional constraint on the quasi-momenta comes from the fact that the algebra $\mathfrak{psu}(2,2|4)$ has an automorphism, which is the cause for an additional $\mathbb{Z}_4$ grading. The constraints are given by [?]

$$\begin{aligned}
\tilde{p}_{1,2}(x) &= -\tilde{p}_{2,1}(1/x) - 2\pi m \\
\tilde{p}_{3,4}(x) &= -\tilde{p}_{4,3}(1/x) - 2\pi m \\
\hat{p}_{1,2,3,4}(x) &= -\hat{p}_{2,1,4,3}(1/x).
\end{aligned} \tag{3.77}$$

These relations define an inversion symmetry. Finally one can look at the asymptotics of the quasi-momenta as the spectral parameter becomes infinite. In this limit the Lax connection becomes related to the Noether currents of the theory and hence one can relate the quasi-momenta to the charges of the global symmetry algebra by [?]

$$\begin{pmatrix} \hat{p}_1 \\ \hat{p}_2 \\ \hat{p}_3 \\ \hat{p}_4 \\ \hline \tilde{p}_1 \\ \tilde{p}_2 \\ \tilde{p}_3 \\ \tilde{p}_4 \end{pmatrix} = \frac{2\pi}{x} \begin{pmatrix} +\mathcal{E} - \mathcal{S}_1 + \mathcal{S}_2 \\ +\mathcal{E} + \mathcal{S}_1 - \mathcal{S}_2 \\ -\mathcal{E} - \mathcal{S}_1 - \mathcal{S}_2 \\ -\mathcal{E} + \mathcal{S}_1 + \mathcal{S}_2 \\ \hline +\mathcal{J}_1 + \mathcal{J}_2 - \mathcal{J}_3 \\ +\mathcal{J}_1 - \mathcal{J}_2 + \mathcal{J}_3 \\ -\mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 \\ -\mathcal{J}_1 - \mathcal{J}_2 - \mathcal{J}_3 \end{pmatrix}, \tag{3.78}$$

where the charges are rescaled by $\mathcal{Q} = Q/\sqrt{\lambda}$. Thus we see that we can characterize the quasi-momenta by describing their behaviour at poles and under symmetries, by their asymptotics and their filling fractions.

Let us now revisit the simplest string solution we know, the BMN string and describe it using the spectral curve. Not surprisingly it is the simplest algebraic curve possible, containing no poles or cuts except the trivial ones at $x = \pm 1$. The quasi-momenta are given by [?]

$$\tilde{p}_{1,2} = -\tilde{p}_{3,4} = \hat{p}_{1,2} = -\hat{p}_{3,4} = \frac{2\pi \mathcal{J} x}{x^2 - 1}. \tag{3.79}$$

From the asymptotic behaviour as $x \to \infty$ we can determine the charges of this solution by comparing to (4.21) and we find

$$\mathcal{J}_1 = \mathcal{J}, \qquad \mathcal{E} = \kappa = \mathcal{J}, \tag{3.80}$$

all other charges being zero. Many other solutions can be characterized this way, e.g. the giant magnon corresponds to a two cut solution [?].

As already mentioned, the key advantage of describing solutions using algebraic curves is the ability to solve the spectral problem without actually solving the equations of motion. We already saw this at the classical level, where finding the energy of a solution amounts to looking at the asymptotic behaviour of the quasi-momenta. Quasi-classical analysis of the spectral curve enables one to go beyond the classical theory and find quantum corrections to the energy levels of classical solutions. The idea of quasi-classical analysis and the spectral curve for that matter traces back to the early days of quantum mechanics. Consider a particle in a smooth one dimensional potential described by the wave function $\psi(x)$. Define the quasi-momentum by

$$p(x) \equiv \frac{\hbar}{i} \frac{\psi'(x)}{\psi(x)}, \tag{3.81}$$

the Schrödinger equation then looks like

$$p^2(x) - i\hbar\, p'(x) = 2m(E - V), \tag{3.82}$$

which would be the classic energy momentum relation if it were not for the $\hbar$ term. The quasi-momentum has a pole for each zero of the wave function, so for a highly excited state this will be some big number $N \to \infty$ and we would recover the classical solution. What is more, the poles get closer and closer to each other and in the classical limit they condense to form a cut connecting two sheets in a Riemann surface. Thus in the classical limit we recover the spectral curve of this system. We also know that the number of poles is given by

$$\frac{1}{2\pi\hbar}\oint_{\mathcal{C}} p(x)\,dx = N, \tag{3.83}$$

which is also the Bohr-Sommerfield quantization condition. This integral effectively measures the size of the cut when the poles condense to a cut, thus this is the filling fraction. This simplified discussion illustrates how one could go from a classical system to a quantum one. The idea of quasi-classical analysis is to start with a classical solution and perturb it by adding microscopic cuts to the Riemann surface, which effectively describe some quantum excitations. This is exactly what has been done for various string solutions in $AdS_5 \times S^5$ [**?**]. One can then proceed with comparing the spectra of string solutions beyond the classical level with spectra of spin chain states at higher loop levels and the results so far have been encouraging [**?**].

Describe flat connections, monodromies, sheets etc.

3.4.1 Folded string

Give solution

3.5 String quantization and semi-classics

Describe the quantization procedure. Derive next coefficient for Konishi.

3.6 Short strings

The folded string is the strong coupling counterpart of the Wilson operators $\mathrm{Tr}(D^S Z^J)$. This class of operators in particular contains the Konishi operator that has been receiving a lot of attention recently. The classical energy of the folded string is a function of the Lorentz spin S, twist J and the mode number n. This function can be written in a parametric form in terms of the branch points a and b [**?**, **?**, 22, **?**]:

$$\begin{aligned} 2\pi\mathcal{S} &= \frac{ab+1}{ab}\left[bE\left(1-\tfrac{a^2}{b^2}\right) - aK\left(1-\frac{a^2}{b^2}\right)\right], \\ 2\pi\mathcal{J} &= \frac{2\sqrt{(a^2-1)(b^2-1)}}{b}K\left(1-\frac{a^2}{b^2}\right), \\ 2\pi\mathcal{D}_{\text{tree}} &= \frac{ab-1}{ab}\left[bE\left(1-\tfrac{a^2}{b^2}\right) + aK\left(1-\frac{a^2}{b^2}\right)\right]. \end{aligned} \tag{3.84}$$

where $\mathcal{S},\mathcal{J},\mathcal{D}=\frac{S}{n\sqrt{\lambda}},\frac{J}{n\sqrt{\lambda}},\frac{\Delta}{n\sqrt{\lambda}}$. In this paper we will concentrate on a special limit when S is sent to zero. In this limit one can write a more explicit expression for the square of the scaling dimension:

$$\mathcal{D}^2_{\rm tree} = \mathcal{J}^2+2\,\mathcal{S}\,\sqrt{\mathcal{J}^2+1}+\mathcal{S}^2\,\frac{2\mathcal{J}^2+3}{2\mathcal{J}^2+2}-\mathcal{S}^3\,\frac{\mathcal{J}^2+3}{8\left(\mathcal{J}^2+1\right)^{5/2}}+\mathcal{O}\left(\mathcal{S}^4\right)\,. \qquad (3.85)$$

One can easily see that the coefficients in the expansion of $\mathcal{D}^2_{\rm tree}$ are considerably simpler than the same coefficients in the expansion of $\mathcal{D}_{\rm tree}$.

One can further notice [68] that the re-expansion of the function Δ^2 in the large $\mu\equiv\lambda n^2$ limit with S and J fixed has a particularly nice structure

$$\Delta^2_{\rm tree}= J^2+S\left(2\,\sqrt{\mu}+\frac{J^2}{\sqrt{\mu}}+\dots\right)+S^2\left(\frac{3}{2}-\frac{J^2}{2\mu}+\dots\right)-S^3\left(\frac{3}{8\sqrt{\mu}}-\frac{13J^2}{16\sqrt[3]{\mu}}+\dots\right)+\mathcal{O}(S^4) \qquad (3.86)$$

where each next term in S gets more and more suppressed for large λ. This structure indicates that the expansion in large λ and small S should be easily computable, which is very important in the study of short operators. The structure in (3.86) is a purely classical result. In the next section we discuss whether it is preserved when quantum corrections are taken into account.

Using the algebraic curve technique [**?**, **?**, **?**, **?**, **?**] the result (3.86) at one loop can be shown to be just a little bit more involved than the classical energy. The derivation is described in [**?**] so we only quote the result here (see appendix F for more details).

Again, in the limit when $\mathcal{S}$ is sent to zero the result simplifies significantly. Up to two orders in $\mathcal{S}$ we found the following expansion

$$\Delta_{1-{\rm loop}} \simeq \frac{-\mathcal{S}}{2\left(\mathcal{J}^3+\mathcal{J}\right)}+\mathcal{S}^2\left[\frac{3\mathcal{J}^4+11\mathcal{J}^2+17}{16\mathcal{J}^3\left(\mathcal{J}^2+1\right)^{5/2}}-\sum_{m>0,m\neq n}\frac{n^3m^2\left(2m^2+n^2\mathcal{J}^2-n^2\right)}{\mathcal{J}^3\left(m^2-n^2\right)^2\left(m^2+n^2\mathcal{J}^2\right)^{3/2}}\right] \qquad (3.87)$$

The next term in this expansion can be found in (G.1), (G.2). The sum is nothing but a sum over the fluctuation energies, whereas the remaining terms originate from the "zero"-modes $m=n$, which have to be treated separately. The sum can be very easily expanded for small $\mathcal{J}$. It is easy to see that the expansion coefficients will be certain combinations of zeta-functions. It is also easy to see that the dependence on the mode number n is rather nontrivial.

The expansion of the one loop energy first in small $\mathcal{S}$ up to a second order and then in small $\mathcal{J}$ reads

$$\Delta_{1-{\rm loop}}\simeq\begin{cases} -\frac{\mathcal{S}}{2\mathcal{J}}+\mathcal{S}^2\left(+\frac{1}{2\mathcal{J}^3}-\frac{3\zeta_3}{2\mathcal{J}}-\frac{1}{16\mathcal{J}}\right) & ,\ n=1\\ -\frac{\mathcal{S}}{2\mathcal{J}}+\mathcal{S}^2\left(+\frac{1}{2\mathcal{J}^3}-\frac{12\zeta_3}{\mathcal{J}}-\frac{17}{16\mathcal{J}}\right) & ,\ n=2\\ -\frac{\mathcal{S}}{2\mathcal{J}}+\mathcal{S}^2\left(-\frac{5}{8\mathcal{J}^3}-\frac{81\zeta_3}{2\mathcal{J}}-\frac{7}{4\mathcal{J}}\right) & ,\ n=3\end{cases} \qquad (3.88)$$

Expansions up to four orders in $\mathcal{S}$ and then in $\mathcal{J}$ are given in appendix G. We note that the contributions $\mathcal{S}^2/\mathcal{J}^3$ are universal for $n=1$ and $n=2$, however starting from $n=3$ we get some nasty coefficient. As we will discuss in the next section this could imply that the naive generalization of the conjecture in [68] is not fully correct for $n>2$. Also for $n=2$ we found a similar anomaly at the order S^3. Let us take a close look at the conjecture in [68]. It says that making expansions of the scaling dimension squared first in $\mathcal{S} \to 0$ and then in $\mu \to \infty$ should reveal the following structure

$$\Delta^2 = J^2 + S\left(A_1\sqrt{\mu} + A_2 + \ldots\right) + S^2\left(B_1 + \frac{B_2}{\sqrt{\mu}} + \ldots\right) + \mathcal{O}(S^3)\ , \tag{3.89}$$

where the coefficients $A_i,\ B_i,\ C_i$ are some functions of J. This is, as can be easily seen, a nontrivial constraint on Δ itself as

$$\begin{aligned}\Delta &= J + \frac{S}{2J}\left(A_1\sqrt{\mu} + A_2 + \frac{A_3}{\sqrt{\mu}} + \ldots\right) \\ &+ S^2\left(-\frac{A_1^2}{8J^3}\,\mu - \frac{A_1A_2}{4J^3}\,\sqrt{\mu} + \left[\frac{B_1}{2J} - \frac{A_2^2 + 2A_1A_3}{8J^3}\right] + \left[\frac{B_2}{2J} - \frac{A_2A_3 + A_1A_4}{4J^3}\right]\frac{1}{\sqrt{\mu}} + \ldots\right) + \mathcal{O}(S^3)\ .\end{aligned} \tag{3.90}$$

One of the results of [68] is the exact formula for all the coefficients A_i. They can be found easily by expanding a simple combination of Bessel functions, called the “slope”, around infinity and it produces [68]:

$$A_1 = 2\ ,\ \ A_2 = -1\ ,\ \ A_3 = J^2 - \frac{1}{4}\ ,\ \ A_4 = J^2 - \frac{1}{4}\ldots\ . \tag{3.91}$$

Comparing with our one-loop result we get[1]

$$B_1 = \frac{3}{2}\ ,\ \ B_2 = \begin{cases} -3\,\zeta_3 + \frac{3}{8} & ,\ n = 1 \\ -24\,\zeta_3 - \frac{13}{8} & ,\ n = 2 \\ -81\,\zeta_3 - \frac{24}{8} & ,\ n = 3 \end{cases}\ . \tag{3.92}$$

We should, however, notice that for $n > 1$ we were not able to fully satisfy (3.90). One example is the coefficient in front of S^2/J^3, which for $n = 3$ is $-5/8$, whereas (3.90) predicts $1/2$. We observe that only for S^2, S^3 and higher order terms do we find such disagreements and it is interesting to note that the coefficients for S order terms seem to be correct for any n[2]. These observations imply that the generalization of the original slope function, which is done by a naive replacement $\lambda \to n^2\lambda$, is not correct for the cases when $n > 1$ and thus either the coefficients in (3.91) or the conjecture itself should be modified to accommodate this. We discuss this in details in the next section **??**.

The analysis in the previous sections was done only up to second order in the small S expansion. The appendix G contains our result for the one-loop quantization of the n-times folded string up to the order S^4. For $n = 1$ our result is in perfect agreement with the conjectured structure (3.89), yet for cases with $n > 1$ there are inconsistencies. For $n = 2$ the first inconsistency appears in the $\frac{S^3\mu}{J^4}$ term and for $n = 3$ there are already inconsistencies at order S^2. We found that for $n > 1$ one has to modify the structure in (3.89) by including

[1] $B_2 = -b$ in the notations of [68]. The $-3\zeta_3$ term also arises in the formalism of [**?**] when formally extended to two loops. A very similar ζ_3 term can be also extracted from [**?**]. This gives extra support to our results. We would like to thank L.Mazzucato and A.Tseytlin for pointing this out.

[2] We indeed verified numerically that the naive replacement $\lambda \to n^2\lambda$ works at weak coupling at least to two loops.

negative coefficients in order for it to be consistent with our one-loop results. E.g. for $n = 2$ the structure has to be modified starting with the S^3 term, which now becomes

$$\left(C_{-2}\,\mu + \frac{C_1}{\sqrt{\mu}} + \frac{C_2}{\mu} + \dots \right) S^3 \tag{3.93}$$

with $C_{-2} = \frac{12}{J^4}$. To the next order in S we find

$$\left(D_{-4}\,\mu^{3/2} + D_{-2}\,\sqrt{\mu} + \frac{D_0}{\sqrt{\mu}} + \frac{D_1}{\mu} + \dots \right) S^4 \tag{3.94}$$

where $D_{-4} = -\frac{78}{J^6},\ D_{-2} = -\frac{36}{J^4},\ D_0 = \frac{21}{2J^2}$.

For $n = 3$ the first modification already occurs at order S^2 and it can be resolved if the term $-\frac{9S^2\sqrt{\mu}}{4J^2}$ is added to (3.89). Thus effectively the conjectured structure (3.89) has to be modified as in the $n = 2$ case by including negative coefficients, which now depend on n in a nontrivial way. It is also worth noticing that since inconsistencies start appearing at orders of $\frac{S^2}{J^2}$ and $\frac{S^3}{J^4}$ for $n = 3$ and $n = 2$ respectively, one might guess that there should be an inconsistency at order $\frac{S^4}{J^6}$ for $n = 1$, however we found no such thing.

This study of inconsistencies reveals that the proposed modifications to the structure of (3.89) have growing powers of μ, thus one should resum them together with similar singular terms which may arise in higher loop levels before being able to make justified predictions for short operators ($S \sim J \sim 1$) at strong coupling when $n > 1$.

The equation (3.89) allows one to make a very nontrivial prediction for the strong coupling expansion of operators with fixed length J and the number of derivatives S. For that end we simply fix S and J in (3.89) and expand for large λ or, equivalently, μ. This procedure gives:

$$\Delta_{S,J,n} \simeq \sqrt{2S}\mu^{1/4} + \frac{2J^2 + 3S^2 - 2S}{4\,(2S)^{1/2}\,\mu^{1/4}} + \frac{-21S^4 + (32B_2 + 12)S^3 + (20J^2 - 12)S^2 + 8J^2S - 4J^4}{32\,(2S)^{3/2}\,\mu^{3/4}} \tag{3.95}$$

where B_2 is given in (3.92). Note that according to our observations there are some inconsistencies in the conjecture that this derivation relies on when $n > 1$ and thus this result should be treated with great care.[3]

Let us write the result more explicitly for a particular important case of two magnons

$$\Delta_{2,J,1} = 2\,\lambda^{1/4} + \frac{\frac{J^2}{4} + 1}{\lambda^{1/4}} + \frac{-\frac{J^4}{64} + \frac{3J^2}{8} - 3\,\zeta(3) - \frac{3}{4}}{\lambda^{3/4}}\,. \tag{3.96}$$

In the next section we compare our prediction with the available TBA data.

In order to extract strong coupling asymptotics from available TBA data, we performed numerical fits of Padé type. First we changed variables from λ to

$$y(\lambda) = \sqrt{\lambda}\frac{\partial}{\partial\sqrt{\lambda}} \log I_2(\sqrt{\lambda}) - 2, \tag{3.97}$$

[3]We assume that the results of [68] for the slope function can be lifted by generalizing with the simple replacement $\lambda \to n^2\lambda$ when $n > 1$. We indeed verified this numerically with high precision at weak coupling up to two loops and this is also in agreement with our one loop strong coupling results. I.e. the slope function and hence the coefficients A_i in (3.91) are still correct after the replacement, but as argued before, the structure of the expansion (3.89) may need to be modified.

(S,J,n)	$(n^2\lambda)^{-3/4}$ prediction		$(n^2\lambda)^{-3/4}$ fit	error	g^2 analytical	g^2 fit	fit order
$(2,2,1)$	$1/2-\zeta_3$	$=-3.1062$	-3.0739	1.0%	12	12.0108	6
$(2,3,1)$	$87/64-3\,\zeta_3$	$=-2.2468$	-2.2296	0.8%	8	8.0039	5
$(2,4,2)$	$-3/4-24\,\zeta_3$	$=-29.5994$	-30.0547	1.5%	14.4721	14.4428	5

Table 1: Comparisons of strong coupling expansion coefficients for $\lambda^{-3/4}$ obtained from fits to TBA data versus our predictions for various operators. The weak coupling expansion coefficients for g^2 show how well the fit approximates the data. The fit order is the order of polynomials used for the rational fit function.

which seems arbitrary, but nevertheless is convenient because scaling dimension dependence on y looks nearly linear and automatically captures some important analytical features. We then represent the scaling dimension as the square root of a rational function of two polynomials in y with some of the unknown coefficients chosen so as to fix the leading order weak and strong coupling behaviours. So for example, for the Konishi operator we chose

$$\Delta_{2,2,1}=\sqrt{18+4y+\frac{-2+\sum_{i=1}^{P}a_iy^i}{1+\sum_{i=1}^{P+1}b_iy^i}},$$

because one can easily verify that the weak coupling expansion of this function is given by

$$\Delta_{2,2,1}=4+\mathcal{O}(g^2),$$

and the strong coupling expansion is given by

$$\Delta_{2,2,1}=2\lambda^{1/4}+\frac{2}{\lambda^{1/4}}+\mathcal{O}(\lambda^{-3/4}).$$

This way the leading order behaviour is fixed and next to leading order coefficients are combinations of the unknowns a_i and b_i, which we then find by the method of least squares. The number of fit coefficients P is chosen so that their values after fitting would be of order one, which would imply that the fit is reasonable. Though the procedure seems ad hoc, it produces incredibly good fits, which agree very well with both weak and strong coupling expansions. Fits to available TBA numerical data are shown in Fig. **??**, where dots represent numerical values and the solid lines are our fits[4]. Expanding our fits in powers of λ at strong coupling we were able to compare the $\lambda^{-3/4}$ coefficients in the expansions to our predictions. These are summarized in Table 2 for various operators. We see that our predictions agree with numerical data very well. The table also lists the weak coupling expansion coefficients of g^2 (tree level is fixed by hand), which agree with remarkable precision to Bethe ansatz predictions, once again indicating that the fits work well in both ends of the coupling range.

We also tried comparing our predictions to numerical data for the operator $S=2, J=4, n=2$ (see Fig. **??** and Table 2). As argued before, since this operator has $n>1$, we cannot

[4]For some of the fits we took the first 50 points from the corresponding data set, since we suspected the precission to be lower for higher values of λ. Also, these points were enough to get stable fits.

fully trust our result in this case, nevertheless the result agrees well with the numerical fits we get and the error is only slightly bigger than for the $n = 1$ states. It is hard to draw conclusions about this, as there is not a lot of numerical data available for such operators.

3.7 Towards the full solution

Mention nested BAE, full psu(2,2—4) spin chain without going into much detail.

4 Exact results

In this section we leave the perturbative regime (weak and strong coupling) behind and move on towards exact results in $\mathcal{N} = 4$ super Yang-Mills, where by "exact" we mean available at any value of the coupling constant. In principle such results are the ultimate goal of the whole AdS/CFT programme and being able to extract them is a remarkable achievement.

One can argue that the spectral problem has been worked out exactly, at least conceptually, as we will review shortly. However in practice one immediately runs into technical difficulties with finding solutions, thus at the moment only certain calculations have been carried out explicitly. In this section we will present a few examples of exact results, most notably the slope and curvature functions, which are the two leading coefficients in the small spin expansion of the anomalous dimension of the folded string. As mentioned before, they are indeed exact in the coupling constant, yet they are somewhat abstract, one might even say not physical and thus of little use. We will argue the opposite by demonstrating how one can use them in order to find new information about phyiscally relevant quantities such as the Konishi anomalous dimension.

We will start the section by reviewing the exact solution to the spectral problem of AdS/CFT, first discussing historic approaches and quickly moving on to the novel $\mathbf{P}\mu$ approach. We then devote the rest of the section to various exact results, mostly achieved using the $\mathbf{P}\mu$-system.

4.1 Solution to the spectral problem

Historically the first solution to the spectral problem that fully incorporated finite length corrections was the thermodynamic Bethe ansatz (TBA). Being a set of infinite integral equations it was obviously very hard to use in practice and was thus soon reformulated as an infinite set of functional relations, the so called Y-system. Both of these formulations can now be seen as intermediary steps towards the more elegant $\mathbf{P}\mu$-system, which we cover in more depth in the next subsection.

4.1.1 TBA

The idea here is to consider the mirror theory of the Euclidean version of the original theory, where by Euclidean we mean a theory that is defined by the analytical continuation in the $y = it$ complex time variable. The mirror theory is then defined by exchanging time and size coordinates with $x = i\tau$ being the complex version of the mirror time coordinate τ and y being the space coordinate. The mirror theory is obviously different from the original, e.g. for the asymptotic $SU(2)$ spin chain the original dispersion relation (3.47) gets mirror inverted to [**?**]

$$\tilde{E}(\tilde{p}) = 2 \arcsin \frac{\sqrt{\tilde{p}\,(16\,\pi^2 + \tilde{p}\,\lambda)}}{8\pi}, \tag{4.1}$$

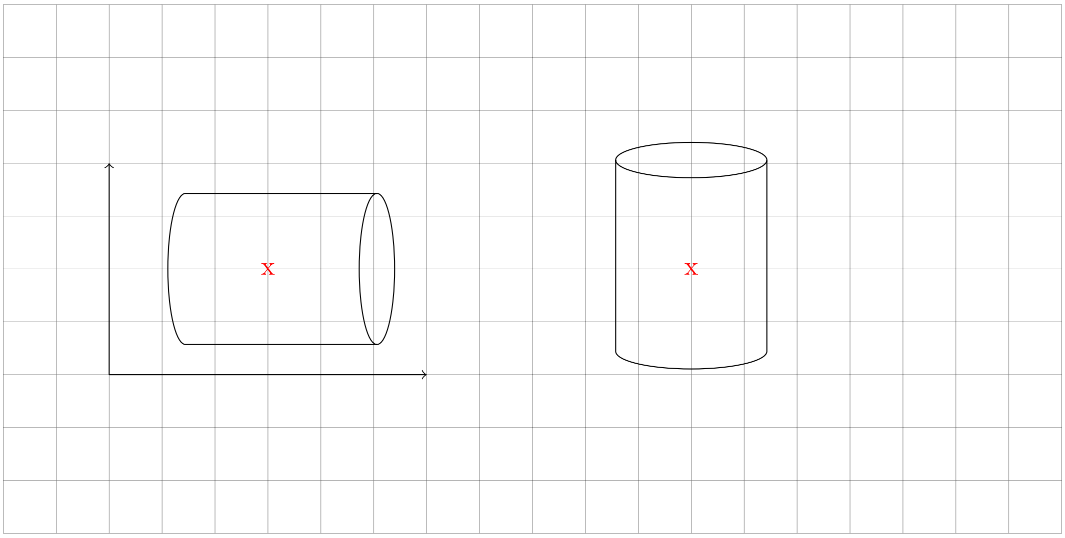


Figure 3: A high level illustration of the idea behind the TBA method: by double Wick rotating the theory one can exchange time with length and thus finite volume scattering with the ground state scattering at finite length.

the scattering matrix also has a different pole structure, meaning that the theory has different bound states from the original ones. But the remarkable thing is that mirroring a theory preserves integrability, meaning that one can solve it with an asymptotic Bethe ansatz [?]. One can use this fact, since the partition functions for these theories satisfy the obvious identity

$$Z(L,R) = \tilde{Z}(R,L), \tag{4.2}$$

where L is the length scale of the original theory and R is the time scale. At asymptotic time scales the partition function is dominated by contributions from the ground state, this applies to any length scale of the system, thus in the asymptotic time limit

$$Z(L,R) = \mathrm{Tr} e^{-R\,H(L)} \xrightarrow[R\to\infty]{} e^{-R\,E_0(L)}. \tag{4.3}$$

This limit corresponds to the infinite length limit for the mirror model, which we can solve using the asymptotic Bethe ansatz, thus

$$\tilde{Z}(R,L) = \mathrm{Tr} e^{-L\,\tilde{H}(R)} \xrightarrow[L\to\infty]{} \sum_n e^{-L\,\tilde{E}_n(R)}, \tag{4.4}$$

where $\tilde{H}$ is the Hamiltonian of the mirror theory. Now we simply identify the partition functions and solve for the ground state energy in the original theory at any length L. The result is then a simple integral given by [?]

$$E_0(L) = -\frac{1}{2\pi}\sum_r \int du\,(\partial_u \tilde{p}) \log\left(1 + e^{-\epsilon_r(u)}\right), \tag{4.5}$$

where $\epsilon_r(u)$ is the so called pseudo energy, defined in terms of the density of solutions with some charge r in the mirror theory and we sum over all charges that describe the solutions. Energies of excited states at finite length can then be found by analytic continuation [?].

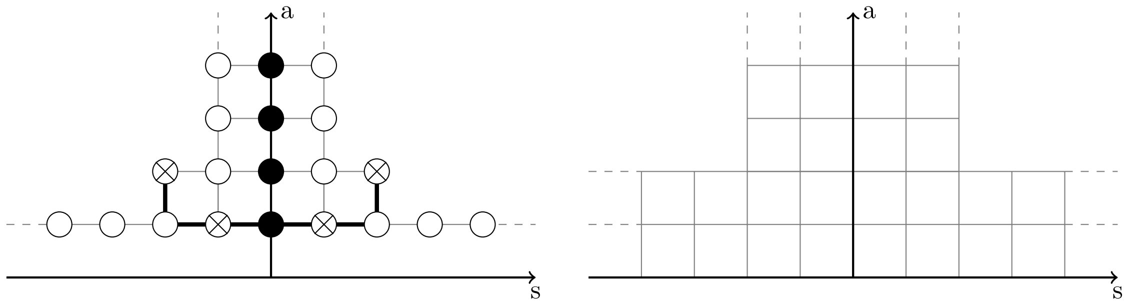


Figure 4: The domains of the $Y_{a,s}$ and $T_{a,s}$ functions (left and right respectively). The $Y_{a,s}$ functions are defined on the nodes, where the type of node signifies the type of excitation. The $T_{a,s}$ functions are defined on the latttice points of the grid.

The argument for the finite size solution presented here is very sketchy and for the full $PSU(2,2|4)$ theory the story is obviously way more complicated, but everything that we discussed has indeed been done for the full theory. The solution is written in terms of a Y-system [?], which is a set of algebraic equations frequently found in integrable systems [?]. The Y-system equations have been verified on numerous occasions and have thus far passed every test [?].

$$\log Y_{a,s}(u) = \delta_s^0 \, iL \, p_a(u) + \int dv \, K_{a,s}^{a',s'}(u,v) \log\left(1 + Y_{a',s'}(v)\right) \tag{4.6}$$

$$E = \sum_j \epsilon_1(u_{4,j}) + \sum_{a=1}^{\infty} \int_{-\infty}^{\infty} \frac{du}{2\pi i} \frac{\partial \epsilon_a^*}{\partial u} \log\left(1 + Y_{a,0}^*(u)\right) \tag{4.7}$$

4.1.2 The Y-system

The Y-system

$$\frac{Y_{a,s}^+ Y_{a,s}^-}{Y_{a+1,s} Y_{a-1,s}} = \frac{(1 + Y_{a,s+1})\,(1 + Y_{a,s-1})}{(1 + Y_{a+1,s})\,(1 + Y_{a-1,s})} \tag{4.8}$$

$$T_{a,s}^+ T_{a,s}^- = T_{a+1,s} T_{a-1,s} + T_{a,s+1} T_{a,s-1} \tag{4.9}$$

$$Y_{a,s} = \frac{T_{a,s+1} T_{a,s-1}}{T_{a+1,s} T_{a-1,s}} \tag{4.10}$$

$$T_{a,s} \to g_1^{[a+s]} g_2^{[a-s]} g_3^{[s-1]} g_4^{[-a-s]} \, T_{a,s} \tag{4.11}$$

4.2 The $\mathbf{P}\mu$-system

In this section we review the formulation of the $\mathbf{P}\mu$-system, and also discuss its symmetries which will be useful later. Below, we will restrict the discussion to states in the $sl(2)$ sector as presented in [?]. Remarkably, the general case is not much more complicated and will appear soon in [?].

The $\mathbf{P}\mu$-system is a nonlinear system of functional equations for a four-vector $\mathbf{P}_a(u)$ and a 4×4 antisymmetric matrix $\mu_{ab}(u)$ depending on the spectral parameter u. For full details about the origin of the construction we refer the reader to [?]. As functions of u, both $\mathbf{P}_a$ and μ_{ab} have prescribed analyticity properties which play a key role. First, $\mathbf{P}_a$ must have only a single branch cut in u going between $-2g$ and $2g$, being analytic in the rest of the complex plane. We call this cut the *short* cut, while the cut on the real line connecting the same two points through infinity is called the *long* cut. The functions μ_{ab} have an infinite set of short branch cuts going between $-2g + in$ and $2g + in$ for all $n \in \mathbb{Z}$ (see Fig. 2). Most importantly, the analytic continuation of $\mathbf{P}_a$ and μ_{ab} through these cuts is again expressed in terms of these functions, according to the following equations:

$$\tilde{\mathbf{P}}_a = -\mu_{ab}\chi^{bc}\mathbf{P}_c, \quad \text{with} \quad \chi^{ab} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{4.12}$$

and

$$\tilde{\mu}_{ab} - \mu_{ab} = \mathbf{P}_a\tilde{\mathbf{P}}_b - \mathbf{P}_b\tilde{\mathbf{P}}_a \,. \tag{4.13}$$

Here we denote by $\tilde{\mathbf{P}}_a$ and $\tilde{\mu}_{ab}$ the analytic continuation of $\mathbf{P}_a$ and μ_{ab} through the cut on the real axis. In addition, we have a pseudo-periodicity condition

$$\tilde{\mu}_{ab}(u) = \mu_{ab}(u + i) \tag{4.14}$$

which, actually, means that $\mu_{ab}(u)$ would be an i-periodic function if defined with long cuts instead of the short cuts.

The functions μ_{ab} are also constrained by the relations

$$\mu_{12}\mu_{34} - \mu_{13}\mu_{24} + \mu_{14}^2 \quad = \quad 1 \,, \tag{4.15}$$

$$\mu_{14} = \mu_{23} \,, \tag{4.16}$$

the first of which states that the Pfaffian of the matrix μ_{ab} is equal to 1. Let us also write the equations (4.12) explicitly:

$$\tilde{\mathbf{P}}_1 = -\mathbf{P}_3\mu_{12} + \mathbf{P}_2\mu_{13} - \mathbf{P}_1\mu_{14} \tag{4.17}$$

$$\tilde{\mathbf{P}}_2 = -\mathbf{P}_4\mu_{12} + \mathbf{P}_2\mu_{14} - \mathbf{P}_1\mu_{24} \tag{4.18}$$

$$\tilde{\mathbf{P}}_3 = -\mathbf{P}_4\mu_{13} + \mathbf{P}_3\mu_{14} - \mathbf{P}_1\mu_{34} \tag{4.19}$$

$$\tilde{\mathbf{P}}_4 = -\mathbf{P}_4\mu_{14} + \mathbf{P}_3\mu_{24} - \mathbf{P}_2\mu_{34} \,. \tag{4.20}$$

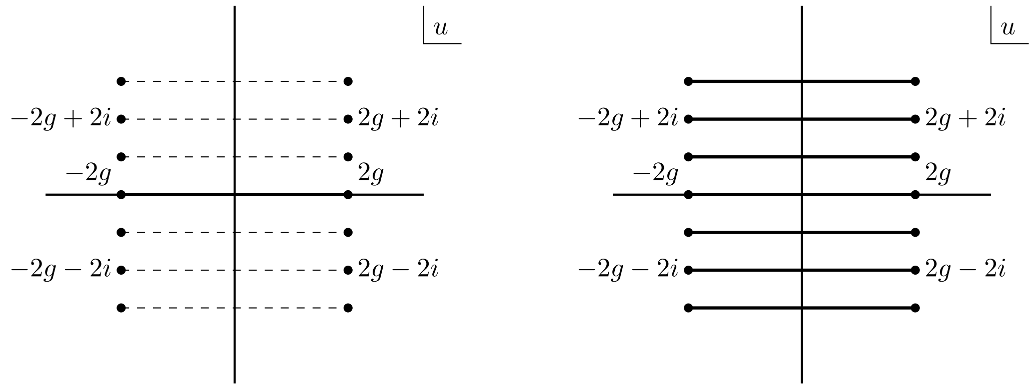


Figure 5: **Cuts in the u plane.** We show the location of branch cuts in u for the functions $\mathbf{P}_a(u)$ (left) and $\mu_{ab}(u)$ (right). The infinitely many cuts of $\tilde{\mathbf{P}}_a$ are shown on the left picture with dashed lines.

The above equations ensure that the branch points of $\mathbf{P}_a$ and μ_{ab} are of the square root type, i.e. $\tilde{\tilde{\mathbf{P}}}_a = \mathbf{P}_a$ and $\tilde{\tilde{\mu}}_{ab} = \mu_{ab}$.

Finally, we require that $\mathbf{P}_a$ and μ_{ab} do not have any singularities except these branch points[5].

The quantum numbers and the energy of the state are encoded in the asymptotics of the functions $\mathbf{P}_a$ and μ_{ab} at large real u. The generic case is described in [**?**], while here we are interested in the states in the $sl(2)$ sector, for which the relations read [**?**]

$$\mathbf{P}_a \sim (A_1 u^{-J/2}, A_2 u^{-J/2-1}, A_3 u^{J/2}, A_4 u^{J/2-1}) \tag{4.21}$$

$$(\mu_{12},\ \mu_{13},\ \mu_{14},\ \mu_{24},\ \mu_{34}) \sim \left(u^{\Delta-J},\ u^{\Delta+1},\ u^{\Delta},\ u^{\Delta-1},\ u^{\Delta+J}\right) \tag{4.22}$$

where J is the twist of the gauge theory operator, and Δ is its conformal dimension. With these asymptotics, the equations (4.12)-(4.16) form a closed system which fixes $\mathbf{P}_a$ and μ_{ab}.

Lastly, the spin S of the operator is related [**?**] to the leading coefficients A_a of the $\mathbf{P}_a$ functions (see (4.21)):

$$A_1 A_4 = \frac{\left((J+S-2)^2-\Delta^2\right)\left((J-S)^2-\Delta^2\right)}{16iJ(J-1)} \tag{4.23}$$

$$A_2 A_3 = \frac{\left((J-S+2)^2-\Delta^2\right)\left((J+S)^2-\Delta^2\right)}{16iJ(J+1)}\ . \tag{4.24}$$

The $\mathbf{P}\mu$-system enjoys a symmetry preserving all of its essential features. It has the form of a linear transformation of $\mathbf{P}_a$ and μ_{ab} which leaves the system (4.12)-(**??**) and the asymptotics (4.21), (4.22) invariant. Indeed, consider a general linear transformation $\mathbf{P}'_a = R_a{}^b \mathbf{P}_b$ with a non-degenerate constant matrix R. In order to preserve the system (4.12), μ should at the same time be transformed as

$$\mu' = -R\mu\chi R^{-1}\chi. \tag{4.25}$$

[5]For odd values of J the functions $\mathbf{P}_a$ may have an additional branch point at infinity. However, it should cancel in any product of two $\mathbf{P}_a$'s, and therefore it will not appear in any physically relevant quantity (see [**?**], [**?**]). We will discuss some explicit examples in the text.

Such a transformation also preserves the form of (4.13) if

$$R^T \chi R \chi = -1 \ , \tag{4.26}$$

which also automatically ensures antisymmetry of μ_{ab} and (4.15), (4.16). In general, this transformation will spoil the asymptotics of $\mathbf{P}_a$. These asymptotics are ordered as $|\mathbf{P}_2| < |\mathbf{P}_1| < |\mathbf{P}_4| < |\mathbf{P}_3|$, which implies that the matrix R must have the following structure[6]

$$R = \begin{pmatrix} * & * & 0 & 0 \\ 0 & * & 0 & 0 \\ * & * & * & * \\ * & * & 0 & * \end{pmatrix} . \tag{4.27}$$

The general form of R which satisfies (4.26) and does not spoil the asymptotics generates a 6-parametric transformation, which we will call a γ-transformation. The simplest γ-transformation is the following rescaling:

$$\mathbf{P}_1 \to \alpha \mathbf{P}_1 \ \ , \ \ \mathbf{P}_2 \to \beta \mathbf{P}_2 \ \ , \ \ \mathbf{P}_3 \to 1/\beta \mathbf{P}_3 \ \ , \ \ \mathbf{P}_4 \to 1/\alpha \mathbf{P}_4 \ \ , \tag{4.28}$$

$$\mu_{12} \to \alpha\beta\mu_{12} \ \ , \ \ \mu_{13} \to \frac{\alpha}{\beta}\mu_{13} \ \ , \ \ \mu_{14} \to \mu_{14} \ \ , \ \ \mu_{24} \to \frac{\beta}{\alpha}\mu_{24} \ \ , \ \ \mu_{34} \to \frac{1}{\alpha\beta}\mu_{34} \ \ , \tag{4.29}$$

with α, β being constants.

In all the solutions we consider in this paper all functions $\mathbf{P}_a$ turn out to be functions of definite parity, so it makes sense to consider γ-transformations which preserve parity. $\mathbf{P}_1$ and $\mathbf{P}_2$ always have opposite parity (as one can see from from (4.21)) and thus should not mix under such transformations; the same is true about $\mathbf{P}_3$ and $\mathbf{P}_4$. Thus, depending on parity of J the parity-preserving γ-transformations are either

$$\mathbf{P}_3 \to \mathbf{P}_3 + \gamma_3 \mathbf{P}_2, \ \mathbf{P}_4 \to \mathbf{P}_4 + \gamma_2 \mathbf{P}_1, \tag{4.30}$$

$$\mu_{13} \to \mu_{13} + \gamma_3\mu_{12}, \ \mu_{24} \to \mu_{24} - \gamma_2\mu_{12}, \ \mu_{34} \to \mu_{34} + \gamma_3\mu_{24} - \gamma_2\mu_{13} - \gamma_2\gamma_3\mu_{12}$$

for odd J or

$$\mathbf{P}_3 \to \mathbf{P}_3 + \gamma_1 \mathbf{P}_1, \ \mathbf{P}_4 \to \mathbf{P}_4 - \gamma_1 \mathbf{P}_2, \tag{4.31}$$

$$\mu_{14} \to \mu_{14} - \gamma_1\mu_{12}, \ \mu_{34} \to \mu_{34} + 2\gamma_1\mu_{14} - \gamma_1^2\mu_{12} \ ,$$

for even J.

4.3 Folded string

Mention Frolov numerics. Volin's 8(9) ? loops with $\mathbf{P}\mu$.

[6]This matrix would of course be lower triangular if we ordered $\mathbf{P}_a$ by their asymptotics.

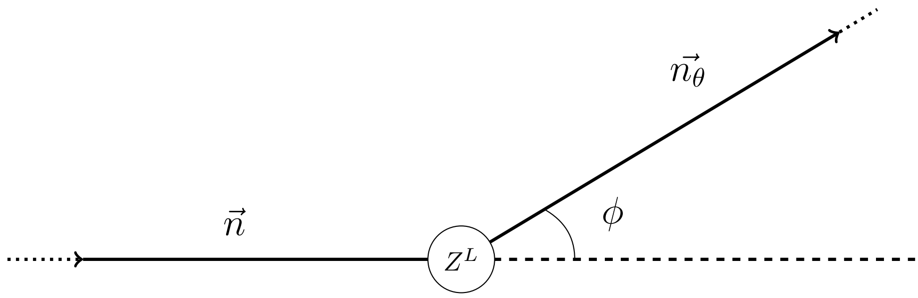


Figure 6: The cusped Wilson line with an operator insertion.

4.4 Cusped Wilson line

The observable which we will be considering is the same as in [51],[50] and [?]: it consists of two rays of a supersymmetric Wilson line forming a cusp with the angle ϕ and an operator Z^L inserted at the cusp, where Z is a scalar of $\mathcal{N}=4$ SYM (see figure ??). To completely define a supersymmetric Wilson line we should also specify the coupling to scalars, which is parameterized by a six-dimensional unit vector $\vec{n}(t)$ at each point of the line (t being a parameter on the line). In our case $\vec{n}(t)$ is constant and equal $\vec{n}$ on one ray and $\vec{n}_\theta$ on another ray, so that $\vec{n}\cdot\vec{n}_\theta=\cos\theta$. Due to the R-symmetry the observable depends on $\vec{n},\vec{n}_\theta$ only through θ.

Explicitly the observable is defined as

$$W_L = \mathrm{P}\exp\int_{-\infty}^{0} dt\left(iA\cdot\dot{x}_q+\vec{\Phi}\cdot\vec{n}\,|\dot{x}_q|\right)\times Z^L\times\mathrm{P}\exp\int_{0}^{\infty} dt\left(iA\cdot\dot{x}_{\bar{q}}+\vec{\Phi}\cdot\vec{n}_\theta\,|\dot{x}_{\bar{q}}|\right). \quad (4.32)$$

Due to the cusp the expectation value of such an observable diverges as

$$\langle W_L\rangle\sim\left(\frac{\Lambda_{IR}}{\Lambda_{UV}}\right)^{\Gamma_L(\lambda)}, \quad (4.33)$$

where Λ_{IR} and Λ_{UV} are the infra-red and ultraviolet cut-offs respectively [?],[?]. The quantity Γ_L, which we will call the cusp anomalous dimension, will be the main object of our studies. When $\theta^2-\phi^2=0$ the observable W_L becomes BPS and the cusp anomalous dimension vanishes [?]. In [51] the anomalous dimension in the near-BPS limit $\theta=0$, $\phi\to 0$ was calculated using the method of Y-system or Thermodynamical Bethe Ansatz [42, 43, 45]. In [54] the calculation was generalized to the case of arbitrary, but close to each other angles $\theta^2-\phi^2\to 0$. In this general case the cusp anomalous dimension was found to be

$$\Gamma_L(\lambda)=\frac{\phi-\theta}{4}\partial_\theta\log\frac{\det\mathcal{M}_{2L+1}}{\det\mathcal{M}_{2L-1}}+\mathcal{O}((\phi-\theta)^2), \quad (4.34)$$

where $\mathcal{M}_N$ is an $(N+1)\times(N+1)$ sized matrix defined as

$$(\mathcal{M}_N)_{ij}=I^\theta_{i-j+1}, \quad (4.35)$$

$$I^\theta_n=i^{n+1}I_n\left(\frac{\sqrt{\lambda}}{\sin\beta}\right)\sin n\beta,\quad\text{with}\quad\sin\beta=\frac{1}{\sqrt{1-\theta^2/\pi^2}},$$

and $I_n(x)$ are modified Bessel functions of the first kind. In [53] the result for the case $L = 0$, $\theta = 0$ was re-derived in a significantly shorter way by using the novel $P - \mu$ system.

The AdS/CFT duality allows one to relate the observable in $\mathcal{N} = 4$ SYM described above to an open string in AdS which ends on a cusped line on the boundary of AdS. In particular, in the classical scaling limit when L and λ are both taken to infinity with $L/\sqrt{\lambda}$ fixed, we can match Γ_L with the energy of the classical string. However, since the result contains determinants of $(2L+1) \times (2L+1)$ sized matrices it is not obvious how to take the large L limit. In the subsequent sections we develop the apparatus for this, describe the classical string solution and finally compare the results for the energy.

Bremstahlung result from $\mathbf{P}\mu$.

4.4.1 Classical limit

Taking the classical limit $L \to \infty$ keeping $L/\sqrt{\lambda}$ fixed becomes considerably easier once we realize that the cusp anomalous dimension (4.34) can be expressed in terms of an expectation value of some operator in a matrix model. In this section we will show how to use this approach to find the large N expansion of the determinant of $\mathcal{M}_N$ defined in (4.35).

One can check that the quantities I_n^θ defined in the previous section can be rewritten in the following integral representation

$$I_n^\theta = \frac{1}{2\pi i} \oint \frac{dx}{x^{n+1}} \sinh(2\pi g\,(x + 1/x))\, e^{2g\theta(x-1/x)}, \tag{4.36}$$

where the integration contour is the unit circle and $g = \frac{\sqrt{\lambda}}{4\pi}$. This makes it possible to write the determinant of $\mathcal{M}_N$ as

$$\det \mathcal{M}_N = \oint \prod_{i=1}^{N+1} \frac{dx_i}{2\pi i} e^{2g\,\theta\left(x_i - \frac{1}{x_i}\right)} \sinh\left(2\pi g \left(x_i + \frac{1}{x_i}\right)\right) \times \det X, \tag{4.37}$$

where

$$\det X = \begin{vmatrix} x_1^{-2} & x_1^{-1} & \dots & x_1^{N-1} & x_1^{N-2} \\ x_2^{-3} & x_2^{-2} & \dots & x_2^{N} & x_2^{N-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_N^{-N-1} & x_N^{-N} & \dots & x_N^{-2} & x_N^{-1} \\ x_{N+1}^{-N-2} & x_{N+1}^{-N-1} & \dots & x_{N+1}^{-3} & x_{N+1}^{-2} \end{vmatrix} = \frac{\prod_{i<j}^{N+1}(x_i - x_j)}{\prod_{i=1}^{N+1} x_i^{i+1}}, \tag{4.38}$$

and we recognize the numerator as the Vandermonde determinant $\Delta(x_i)$. We can further simplify the final result by anti-symmetrizing the denominator, which we can do because everything else in the integrand is anti-symmetric and the integration measure is symmetric w.r.t x_i, thus under the integral we can replace $\det X$ by

$$\det X' = \frac{\Delta^2(x_i)}{(N+1)!} \prod_{i=1}^{N+1} \frac{1}{x_i^{N+2}}. \tag{4.39}$$

Thus finally we get the following expression

$$\det \mathcal{M}_N = \frac{1}{(2\pi i)^{N+1}} \oint \prod_{i=1}^{N+1} \frac{dx_i}{x_i^{N+2}} \frac{\Delta^2(x_i)}{(N+1)!} \sinh(2\pi g\,(x_i + 1/x_i))\, e^{2g\theta(x_i - 1/x_i)}, \tag{4.40}$$

which indeed has the structure of a partition function of some matrix model[7]. It now becomes a matter of simple algebra to convince oneself that the cusp anomalous dimension (4.34) can be written in terms of expectation values in this matrix model, namely

$$\Gamma_L(g) = g\,\frac{\phi-\theta}{2}\left[\left\langle\sum_{i=1}^{2L+1}\left(x_i-\frac{1}{x_i}\right)\right\rangle_{2L+1} - \left\langle\sum_{i=1}^{2L-1}\left(x_i-\frac{1}{x_i}\right)\right\rangle_{2L-1}\right], \tag{4.41}$$

where $\langle\ldots\rangle_N$ denotes the normalized expectation value in the matrix model of size N with the partition function defined in (4.40). Note that this formula is exact and we have not yet taken any limits.

In this section we will explore the classical $L \sim \sqrt{\lambda} \to \infty$ limit of the matrix model (4.40). As usual in matrix models, when the size of matrices becomes large, the partition function is dominated by the solution of the saddle point equations. In the leading order it is just equal to the value of the integrand at the saddle point. Here we work in this approximation, leaving the corrections (beyond the first one calculated in section **??**) for future work.

The partition function (4.40) can be recast in the form [8]

$$\det \mathcal{M}_{2L} = \frac{1}{(2\pi i)^{2L+1}}\frac{1}{(2L+1)!}\oint\prod_{i=1}^{2L+1}dx_i\, e^{-S(x_1,x_2,\ldots,x_{2L+1})}, \tag{4.42}$$

where the action is given by

$$\begin{aligned} S \;&=\; \sum_{i=1}^{2L+1}\left[2g\theta\left(x_i-\frac{1}{x_i}\right)-(2L+2)\log x_i\right]+2\sum_{i<j}\log(x_i-x_j)+ \qquad (4.43)\\ &+\;\sum_{i=1}^{2L+1}\log\sinh\left(2\pi g\left(x_i+\frac{1}{x_i}\right)\right). \end{aligned}$$

The saddle point equations $\partial S/\partial x_j = 0$ now read[9]

$$g\theta\left(1+\frac{1}{x_j^2}\right)-\frac{L}{x_j}+\sum_{i\neq j}^{2L+1}\frac{1}{x_j-x_i}+\pi g\left(1-\frac{1}{x_j^2}\right)\coth\left(2\pi g\left(x_j+\frac{1}{x_j}\right)\right)=0. \tag{4.44}$$

We can further simplify them by noting that a large coupling constant g appears inside the cotangent and since the roots x_i are expected to be of order 1, with exponential precision it is possible to replace

$$\coth\left(2\pi g\left(x_j+\frac{1}{x_j}\right)\right)\approx \mathrm{sgn}(\mathrm{Re}(x_j)). \tag{4.45}$$

Finally we bring the equations to a more canonical and convenient form and get the following result,

$$-\,\theta\,\frac{x_j^2+1}{x_j^2-1}+\frac{L}{g}\frac{x_j}{x_j^2-1}-\frac{1}{g}\frac{x_j^2}{x_j^2-1}\sum_{i\neq j}^{2L+1}\frac{1}{x_j-x_i}=\pi\,\mathrm{sgn}(\mathrm{Re}(x_j)). \tag{4.46}$$

[7] Namely, it is equal to the partition function of a two-matrix model. We thank I.Kostov for discussions related to this question.

[8] we take $N = 2L$.

[9] Technically the x_j^{-1} term has a coefficient of $L+1$, but since we are taking $L\to\infty$ we chose to neglect it for simplicity.

An alternative way of finding these values x_i is to consider the following quantity $P_L(x)$, which played an important role in [51],

$$P_L(x) = \frac{1}{\det \mathcal{M}_{2L}} \begin{vmatrix} I_1^\theta & I_0^\theta & \cdots & I_{2-2L}^\theta & I_{1-2L}^\theta \\ I_2^\theta & I_1^\theta & \cdots & I_{3-2L}^\theta & I_{2-2L}^\theta \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ I_{2L}^\theta & I_{2L-1}^\theta & \cdots & I_1^\theta & I_0^\theta \\ x^{-L} & x^{1-L} & \cdots & x^{L-1} & x^L \end{vmatrix}. \tag{4.47}$$

The numerator is the same as $\det \mathcal{M}_{2L}$ except in the last line x_{2L+1} is replaced by x which is not integrated over. In the classical limit all integrals are saturated by their saddle point values, i.e. one can remove the integrals by simply replacing $x_i \to x_i^{cl}$. If we replace x with any saddle point value x_i^{cl} the determinant will contain two identical rows and will automatically become zero, thus the zeros of $P_L(x)$ are the saddle point values. On the complex plane they are distributed on two arcs as shown in figure **??**. As expected, for the case $\theta = 0$ we recover two symmetric arcs on the unit circle [51].

Now, following [**?**],[**?**],[51], we introduce the quasimomentum $p(x)$ as

$$p(x) = -\theta \frac{x^2+1}{x^2-1} + \frac{L}{g}\frac{x}{x^2-1} - \frac{2L}{g}\frac{x^2}{x^2-1} G_L(x), \tag{4.48}$$

where the resolvent $G_L(x)$ is

$$G_L(x) = \frac{1}{2L} \sum_{k=1}^{2L+1} \frac{1}{x - x_k}. \tag{4.49}$$

The motivation for introducing $p(x)$ is that the saddle point equations (4.46) expressed through $p(x)$ take a very simple form

$$\frac{1}{2}\left(p(x_i + i\epsilon) + p(x_i - i\epsilon)\right) = \pi \operatorname{sgn}(\operatorname{Re}(x_i)). \tag{4.50}$$

In the classical limit the poles in the quasimomentum condense and form two cuts. The shifts $\pm i\epsilon$ in the equation above refer to taking the argument of the quasimomentum to one or the other side of the cut.

The quasimomentum (4.48) introduced in the previous section is a convenient object to consider when taking the classical limit $L \sim \sqrt{\lambda} \to \infty$, because in this limit it is related to the algebraic curve of the corresponding classical solution. In this section we will construct this curve explicitly.

In the classical limit the poles of $p(x)$, which we denote as x_i, are governed by the saddle-point equation and condense on two cuts in the complex plane, as shown in figure **??**. The saddle-point equation (4.46) has a symmetry $x \to -1/x$, so does the set of poles x_i. For the quasimomentum (4.48) this symmetry manifests as the identity $p(x) = -p(-1/x)$. Thus we conclude that the two cuts are related by an $x \to -1/x$ transformation. This and the invariance of the saddle-point equation under complex conjugation implies that the four branch points can be parameterized as $\{r\,e^{i\psi}, r\,e^{-i\psi}, -1/r\,e^{i\psi}, -1/r\,e^{-i\psi}\}$. Note that in the case $\theta = 0$

the symmetry is enhanced to $p(x) = -p(-x)$ and $p(1/x) = p(x)$, which is not true for arbitrary θ.

The crucial point to notice is that while $p(x)$ satisfies the equation (4.50) which has different constants on the right hand side for the two different cuts, the corresponding equation for $p'(x)$ has a zero on the right hand side for both cuts, thus we expect $p'(x)$ to have a simpler form than $p(x)$. Our strategy is to write down an ansatz for the derivative $p'(x)$ using the symmetries and analytical properties of $p(x)$ and then integrate it. The form of the expression we get is analogous to the curve constructed in [22], which also helps us to construct the ansatz.

First, $p(x)$ has four branch points and according to (4.50) its derivative changes sign on each cut, hence all the cuts are of square-root type. One can write $p'(x) \propto 1/y(x)$, where

$$y(x) = \sqrt{x - re^{i\psi}}\sqrt{x - re^{-i\psi}}\sqrt{x + \frac{1}{r}e^{i\psi}}\sqrt{x + \frac{1}{r}e^{-i\psi}}. \tag{4.51}$$

Second, since the algebraic curve is obtained from (4.48) in the classical limit, $p(x)$ should have simple poles at $x = \pm 1$. Finally, from (4.48) we can get the behaviour at infinity:

$$p'(x) \approx \frac{L}{g}\frac{1}{x^2} + \mathcal{O}\left(\frac{1}{x^3}\right). \tag{4.52}$$

By using the knowledge about these singularities and asymptotics we can fix $p(x)$ completely. Based on what we know up to now we write down our ansatz for the derivative

$$p'(x) = \frac{A_1 x^4 + A_2 x^3 + A_3 x^2 + A_4 x + A_5}{(x^2-1)^2\sqrt{x - re^{i\psi}}\sqrt{x - re^{-i\psi}}\sqrt{x + \frac{1}{r}e^{i\psi}}\sqrt{x + \frac{1}{r}e^{-i\psi}}}. \tag{4.53}$$

The polynomial in the numerator is of order four in order to maintain the correct asymptotics, and below we fix its coefficients using the properties of the quasimomentum.[10]

The $x \to -1/x$ symmetry for the derivative implies that $A_1 = A_5$ and $A_2 = -A_4$. Next, simple poles at $x = \pm 1$ in $p(x)$ require zero residues of $p'(x)$ at $x = \pm 1$, which fixes A_2 to be

$$A_2 = -\frac{(2A_1 + A_3)\, r\, (r^2-1)\cos\psi}{r^4 - 2\,r^2\,\cos 2\psi + 1}. \tag{4.54}$$

We fix the two remaining unknowns A_1 and A_3 after integrating the $p'(x)$. We don't write the intermediate results of the integration as the expressions are enormous without any apparent structure. Looking back at (4.50) we see that at the branchpoints

$$p(x_{bp}) = \pm\pi. \tag{4.55}$$

We use this condition to fix A_1 and we get

$$A_1 = \frac{A_3}{2}\frac{K_1 - E_1}{E_1 + K_1 - 2\,a^2\,K_1\cos^2(\psi)}, \tag{4.56}$$

where

$$E_1 = \mathbb{E}\left(a^2\sin^2(\psi)\right),\ K_1 = \mathbb{K}\left(a^2\sin^2(\psi)\right),\ a = \frac{2r}{r^2+1}. \tag{4.57}$$

[10]Comparing with the asymptotic one can immediately see that $A_1 = L/g$, however our objective is to express $p(x)$ solely in terms of r and ψ, which parameterize the algebraic curve.

Finally we can use the $x \to -1/x$ symmetry on the quasimomentum itself, as before we only used it on the derivative. Imposing the symmetry yields

$$A_3 = \frac{8}{a}\left(E_1 + K_1 - 2\,a^2\,\cos^2(\psi) K_1\right). \tag{4.58}$$

As expected, after plugging these coefficients into $p(x)$ (and using the identities from the Appendix **??**) the whole expression simplifies enormously and we are left with our main result

$$p(x) = \pi - 4\,i\,E_1\,\mathbb{F}_1(x) + 4\,i\,K_1\,\mathbb{F}_2(x) - a\left(\frac{x + re^{-i\psi}}{x + \frac{1}{r}e^{i\psi}}\right)\left(\frac{2/r\,e^{i\psi}}{x^2-1}\right) y(x)\,K_1, \tag{4.59}$$

where

$$\mathbb{F}_1(x) = \mathbb{F}\left(\sin^{-1}\sqrt{\left(\frac{x - re^{-i\psi}}{x + \frac{1}{r}e^{i\psi}}\right)\left(\frac{e^{i\psi}}{ia\,r\sin\psi}\right)}\,\middle|\,a^2\sin^2(\psi)\right), \tag{4.60}$$

$$\mathbb{F}_2(x) = \mathbb{E}\left(\sin^{-1}\sqrt{\left(\frac{x - re^{-i\psi}}{x + \frac{1}{r}e^{i\psi}}\right)\left(\frac{e^{i\psi}}{ia\,r\sin\psi}\right)}\,\middle|\,a^2\sin^2(\psi)\right). \tag{4.61}$$

We verified this result numerically by comparing it to the extrapolation of the discrete quasimomentum (4.48) at large L and got an agreement up to thirty digits. We also compared this expression at $\theta = 0$ with the quasimomentum obtained in [51] and the expressions agree perfectly.

The resulting quasimomentum is parameterized in terms of the branchpoints, i.e. the parameters are the radius r and angle ψ. They are determined in terms of L/g and θ, which are parameters of the matrix model. We already mentioned that L/g is simply the constant A_1, which we found to be

$$\frac{L}{g} = 4\,\frac{K_1 - E_1}{a}, \tag{4.62}$$

and looking back at (4.48) we see that $\theta = p(0) = -p(\infty)$, hence

$$\begin{aligned} \theta &= -\pi + \frac{2a}{r}\,e^{i\psi} K_1 \\ &- 4\,i\,K_1\,\mathbb{E}\left(\sin^{-1}\sqrt{\frac{e^{i\psi}}{ia\,r\sin\psi}}\,\middle|\,a^2\sin^2(\psi)\right) \\ &+ 4\,i\,E_1\,\mathbb{F}\left(\sin^{-1}\sqrt{\frac{e^{i\psi}}{ia\,r\sin\psi}}\,\middle|\,a^2\sin^2(\psi)\right). \end{aligned} \tag{4.63}$$

In the next section the two equations above will be matched with two analogous equations following from the classical string equations of motion.

As we have mentioned before, in the classical $L \sim \sqrt{\lambda} \to \infty$ limit $\Gamma_L(\lambda)$ can be matched with the energy of an open string. In this section we will describe the corresponding string solution and find the classical energy.

The class of string solutions we are interested in was introduced in [50] and generalized in [51]. It is a string in $AdS_3 \times S^3$ governed by the parameters θ, ϕ, AdS_3 charge E and S^3 charge

L; the four parameters are restricted by the Virasoro constraint. The ansatz for the embedding coordinates of AdS^3 and S^3 is

$$y_1 + iy_2 = e^{i\kappa\tau}\sqrt{1+r^2(\sigma)}, \;\; y_3 + iy_4 = r(\sigma)e^{i\phi(\sigma)}, \tag{4.64}$$

$$x_1 + ix_2 = e^{i\gamma\tau}\sqrt{1+\rho^2(\sigma)}, \;\; x_3 + ix_4 = r(\sigma)e^{if(\sigma)}. \tag{4.65}$$

The range of the worldsheet coordinate is $-s/2 < \sigma < s/2$, where s is to be found dynamically. The angles θ and ϕ parameterizing the cusp enter the string solution through the boundary conditions $\phi(\pm s/2) = \pm(\pi - \phi)/2$ and $f(\pm s/2) = \pm\theta/2$. The equations of motion and Virasoro constraints lead to the following system of equations (see Appendix E of [51] for more details, also [**?**]):

$$f(\gamma, l_\theta) = f(\kappa, l_\phi), \tag{4.66}$$

$$h(\gamma, l_\theta) = \theta, \;\; h(\kappa, l_\phi) = \phi, \tag{4.67}$$

$$g(\gamma, l_\theta) = L, \;\; g(\kappa, l_\phi) = E, \tag{4.68}$$

where

$$f(\gamma, l) = \frac{2\sqrt{2}}{\sqrt{\gamma^2+k^2+1}}\,\mathbb{K}\left(\frac{-k^2+\gamma^2+1}{k^2+\gamma^2+1}\right), \tag{4.69}$$

$$\begin{aligned} h(\gamma, l) = {} & \frac{2l}{k(1+k^2-\gamma^2)}\left[(1+\gamma^2+k^2)\,\Pi\left(\frac{k^2-2l^2-\gamma^2+1}{2k^2}\middle|\frac{k^2-\gamma^2-1}{2k^2}\right) - \right. \\ & \left. -2\gamma^2\,\mathbb{K}\left(\frac{k^2-\gamma^2-1}{2k^2}\right)\right], \end{aligned} \tag{4.70}$$

$$g(\gamma, l) = -2\sqrt{2}\,\frac{\sqrt{\gamma^2+k^2+1}}{\gamma}\left[\mathbb{E}\left(\frac{-k^2+\gamma^2+1}{k^2+\gamma^2+1}\right) - \mathbb{K}\left(\frac{-k^2+\gamma^2+1}{k^2+\gamma^2+1}\right)\right], \tag{4.71}$$

$$k^4 = \gamma^4 - 2\gamma^2 + 4\,\gamma^2 l^2 + 1.$$

One can see that the variables θ, l_θ, γ and L are responsible for the S^3 part of the solution, while ϕ, l_ϕ, κ and E are their analogues for AdS_3. The two parts of the solution are connected only by the Virasoro condition which leads to (4.66). We are interested in the limit when $\theta \approx \phi$. In this limit the two groups of variables responsible for S^3 and AdS_3 parts of the solution become close to each other, namely $l_\theta \approx l_\phi$ and $E \approx L$. The cusp anomalous dimension should be compared with the difference $E - L$, because L is the classical part of the dimension of the observable W_L. To find $E - L$ we linearize the system (4.69),(4.70),(4.71) around $\phi \approx \theta$, which yields

$$E - L = (\phi - \theta)\left|\frac{\partial(g, f)}{\partial(l, \kappa)}\right| / \left|\frac{\partial(h, f)}{\partial(l, \kappa)}\right|. \tag{4.72}$$

Plugging in here the explicit form of g, f and h one gets as a result an extremely complicated expression with a lot of elliptic functions. However, there exists a parametrization in which the result looks surprisingly simple: this parametrization comes from comparison of the string conserved charges with the corresponding quantities of the algebraic curve. One can notice that the equations for θ and L/g in the end of the last section have the same structure as the equations (4.67) and (4.68). Indeed, it is possible to match them precisely if one chooses the correct identification of parameters of the string solution l_θ, γ with the parameters of the algebraic curve r, ψ. We used the elliptic identities presented in the appendix **??** to bring the equations to identical form after the following identifications

$$\gamma = \frac{2r}{\sqrt{r^4 - 2r^2 \cos 2\psi + 1}}, \; l_\theta = \frac{(r^2-1)\cos\psi}{\sqrt{r^4 - 2r^2 \cos 2\psi + 1}}. \tag{4.73}$$

As another confirmation of correctness of this identification, after plugging it into (4.72) the complicated expression reduces to the following simple formula for the classical energy

$$E - L = g(\phi - \theta)(r - 1/r)\cos\psi. \tag{4.74}$$

Notice that this can be rewritten as a sum over the branch points of the algebraic curve

$$E - L = \frac{g}{2}(\phi - \theta)\sum_i a_i, \tag{4.75}$$

where $a_i = \{r\, e^{i\psi}, r\, e^{-i\psi}, -1/r\, e^{i\psi}, -1/r\, e^{-i\psi}\}$.

In this section we will find the classical limit of the cusp anomalous dimension from the algebraic curve. At large L the formula (4.34) can be rewritten as

$$\Gamma_L(g) = \frac{\phi - \theta}{4}\partial_\theta \partial_L \det \mathcal{M}_{2L}. \tag{4.76}$$

Use the integral representation (4.40) for $\det \mathcal{M}_L$ we can notice that

$$\partial_\theta \log \det \mathcal{M}_L = \left\langle 2g \sum_{i=1}^{2L} (x_i - 1/x_i) \right\rangle, \tag{4.77}$$

where by the angular brackets we denoted an expectation value in the matrix model with the partition function (4.40). In the quasiclassical approximation the expectation value is determined by the saddle-point, i.e. the previous expression is equal to $2g \sum_{i=1}^{2L}(x_i - 1/x_i)$, where the roots x_i are the solutions of the saddle-point equation (4.46). Since the set of the roots has a $x \to -1/x$ symmetry, the two terms in the sum give the same contribution. Thus

$$\partial_\theta \log \det \mathcal{M}_L = -4g \sum_{i=1}^{2L} \frac{1}{x_i} = 8\, g\, L\, G(0), \tag{4.78}$$

where we used the resolvent (4.49).

Using the relation (4.48) between the resolvent and the quasimomentum we find $G(0) = \frac{g}{L}\left(p''(0)/4 - \theta\right)$, so the final expression for the cusp anomalous dimension in terms of the quasimomentum is

$$\Gamma_L(g) = -\frac{g^2}{2}\partial_L p''_L(0). \tag{4.79}$$

The formula for $p(x)$ presented in the previous section is given in terms of the parameters of the branch points r and ψ. They are implicitly defined through L/g and θ by the equations

(4.62) and (4.63). In order to get Γ_L we express ∂_L though ∂_r and ∂_ψ and then apply (4.79) to (4.59). Finally we obtain a very simple result in terms of r and ψ

$$\Gamma_L(g) = g(\phi - \theta)\,(r - 1/r)\cos\psi \tag{4.80}$$

which exactly coincides with the calculation from the string solution!

Here we will check our formula (4.80) in the limit $\phi = 0$ and $\theta \to 0$ considered in section E.2 of [51]. As the angles go to zero, the branch points approach the unit circle: $r \to 1$, thus the formula (4.80) gives

$$\Gamma_L(g) = 2\,g\,\theta(r-1)\cos\psi. \tag{4.81}$$

In this limit $r - 1 \propto \theta$, and the coefficient of proportionality can be found by expanding[11] the equation (4.63) for θ around $r = 1$:

$$2(1-r)\frac{\mathbb{E}\left(\sin^2\psi\right)}{\cos\psi} = \theta/2. \tag{4.82}$$

Plugging it into the formula above we get

$$\Gamma_L(g) = g\,\theta^2\,\frac{\cos^2\psi}{2\mathbb{E}\left(\sin^2\psi\right)} \tag{4.83}$$

which perfectly agrees with (190) of [51].

Now that the classical limit of the cusp anomalous dimension is calculated, we can consider corrections to it. In the limit $L \sim \sqrt{\lambda} \to \infty$ which we are studying here the perturbative expansion around the classical value can be written as

$$\Gamma_L(g) = \sum_{n=0}^{\infty} g^{1-n} b_n(L/g) + \text{non-perturbative terms}. \tag{4.84}$$

The classical energy is $g\,b_0(L/g)$ and other corrections are suppressed by powers of g. A symmetry of the formula for $\Gamma_L(g)$ found in [?] allows one to express the even terms in the expansion (4.84) through the odd ones and the other way round. In particular, b_1 can be obtained from b_0 by differentiating with respect to L/g. Since the classical energy is

$$\Gamma_L^{cl}(g) = g\,(\phi - \theta)\,(r - 1/r)\cos\psi \tag{4.85}$$

by differentiating it with respect to L/g we find that the perturbative part of energy in the first two orders in the classical expansion is

$$\Gamma_L(g) = g\,(\phi - \theta)\,(r - 1/r)\cos\psi\left(1 + \frac{1}{g} f(r,\psi)\right), \tag{4.86}$$

where

$$f(r,\psi) = \frac{r + 1/r}{4}\,\frac{\left|r^2 e^{2i\psi} + 1\right|^2 K_1 - r^2\left|r + \frac{1}{r} + e^{i\psi} - e^{-i\psi}\right|^2 E_1}{\left|\left(r + \frac{1}{r}\right)\left(r^2 e^{2i\psi} - 1\right)E_1 - \left(r - \frac{1}{r}\right)\left(r^2 e^{2i\psi} + 1\right)K_1\right|^2}, \tag{4.87}$$

[11] The equation (4.63) is written in the approximation $\phi \approx \theta$ and now on the top of it we want to take a limit $\theta \to 0$. Since before we have neglected the terms $\mathcal{O}(\theta - \phi)^2$, the result, which is now of the order $\mathcal{O}(\theta)^2$ will not generally be reproduced. However, we found that here and in several other formulas correct small angle limit is reproduced if before taking θ, ϕ to zero we replace θ and ϕ by the middle angle $\phi_0 = \frac{\phi+\theta}{2}$, which is in our case equal to $\theta/2$.

and E_1, K_1 are defined in (4.57). We have checked this formula and the classical energy (4.74) against a numerical extrapolation of the exact expression (4.34) and found an agreement up to more than thirty digits.

Find the curve, matrix models.

4.5 Revisiting the slope function

In this section we will find the solution of the $\mathbf{P}\mu$-system (4.12)-(4.16) corresponding to the $sl(2)$ sector operators at leading order in small S. Based on this solution we will compute the slope function $\gamma^{(1)}(g)$ for any value of the coupling.

The solution of the $\mathbf{P}\mu$-system is a little simpler for even J, because for odd J extra branch points at infinity will appear in $\mathbf{P}_a$ due to the asymptotics (4.21). Let us first consider the even J case.

The description of the $\mathbf{P}\mu$-system in the previous section was done for physical operators. Our goal is to take some peculiar limit when the (integer) number of covariant derivatives S goes to zero. As we will see this requires some extension of the asymptotic requirement for μ functions. In this section we will be guided by principles of naturalness and simplicity to deduce these modifications which we will summarize in section **??**. There we also give a concrete prescription for analytical continuation in S, which we then use to derive the curvature function.

We will start by finding μ_{ab}. Recalling that $\Delta = J + \mathcal{O}(S)$, from (4.23), (4.24) we see that $A_1 A_4$ and $A_2 A_3$ are of order S for small S, so we can take the functions $\mathbf{P}_a$ to be of order $\sqrt{S}$. This is a key simplification, because now (4.13) indicates that the discontinuities of μ_{ab} on the cut are small when S goes to zero. Thus at leading order in S all μ_{ab} are just periodic entire functions without cuts. For power-like asymptotics of μ_{ab} like in (4.22) the only possibility is that they are all constants. However, we found that in this case there is only a trivial solution, i.e. $\mathbf{P}_a$ can only be zero. The reason for this is that for physical states S must be integer and thus cannot be arbitrarily small, nevertheless, it is a sensible question how to define an analytical continuation from integer values of S.[12]

Thus we have to relax the requirement of power-like behavior at infinity. The first possibility is to allow for $e^{2\pi u}$ asymptotics at $u \to +\infty$. We should, however, remember about the constraints (4.15) and (4.16) which restrict our choice and the fact that we can also use γ-symmetry. Let us show that by allowing μ_{24} to have exponential behavior and setting it to $\mu_{24} = C \sinh(2\pi u)$ we arrive to the correct result. We analyze the reason for this choice in detail in section **??**.

To simplify the constant part of μ_{ab} let us now make use of the γ-transformation, described in section **??**. This allows us to set $\mu_{12} = 1,\ \mu_{34} = 0$ and the constant C to 1 then the constraint (4.15) imposes $\mu_{13} = 0$ and $\mu_{14} = -1$.

[12] Restricting the large positive S behavior one can achieve uniqueness of the continuation.

Having fixed all μ's at leading order we get the following system of equations for $\mathbf{P}_a$:

$$\tilde{\mathbf{P}}_1 = -\mathbf{P}_3 + \mathbf{P}_1, \tag{4.88}$$

$$\tilde{\mathbf{P}}_2 = -\mathbf{P}_4 - \mathbf{P}_2 - \mathbf{P}_1 \sinh(2\pi u), \tag{4.89}$$

$$\tilde{\mathbf{P}}_3 = -\mathbf{P}_3, \tag{4.90}$$

$$\tilde{\mathbf{P}}_4 = +\mathbf{P}_4 + \mathbf{P}_3 \sinh(2\pi u). \tag{4.91}$$

Recalling that the functions $\mathbf{P}_a$ only have a single short cut, we see from these equations that $\tilde{\mathbf{P}}_a$ also have only this cut! This means that we can take all $\mathbf{P}_a$ to be infinite Laurent series in the Zhukovsky variable $x(u)$, which rationalizes the Riemann surface with two sheets and one cut. It is defined as

$$x + \frac{1}{x} = \frac{u}{g} \tag{4.92}$$

where we pick the solution with a short cut, i.e.

$$x(u) = \frac{1}{2}\left(\frac{u}{g} + \sqrt{\frac{u}{g} - 2}\,\sqrt{\frac{u}{g} + 2}\right) \quad . \tag{4.93}$$

Solving the equations (4.89) and (4.90) with the asymptotics (4.21) we uniquely fix $\mathbf{P}_1 = \epsilon x^{-J/2}$ and $\mathbf{P}_3 = \epsilon\left(x^{-J/2} - x^{+J/2}\right)$, where ϵ is a constant yet to be fixed; we expect it to be proportional to $\sqrt{S}$. Thus the equations (4.89) and (4.91) become

$$\tilde{\mathbf{P}}_2 + \mathbf{P}_2 = -\mathbf{P}_4 - \epsilon x^{-J/2} \sinh(2\pi u) \ , \tag{4.94}$$

$$\tilde{\mathbf{P}}_4 - \mathbf{P}_4 = \epsilon(x^{-J/2} - x^{+J/2}) \sinh(2\pi u) \ . \tag{4.95}$$

We will first solve the second equation. It is useful to introduce operations $[f(x)]_+$ and $[f(x)]_-$, which take parts of Laurent series with positive and negative powers of x respectively. Taking into account that

$$\sinh(2\pi u) = \sum_{n=-\infty}^{\infty} I_{2n+1} x^{2n+1}, \tag{4.96}$$

where $I_k \equiv I_k(4\pi g)$ is the modified Bessel function of the first kind, we can write $\sinh(2\pi u)$ as

$$\sinh(2\pi u) = \sinh_+ + \sinh_-, \tag{4.97}$$

where explicitly

$$\sinh_+ = [\sinh(2\pi u)]_+ = \sum_{n=1}^{\infty} I_{2n-1} x^{2n-1} \tag{4.98}$$

$$\sinh_- = [\sinh(2\pi u)]_- = \sum_{n=1}^{\infty} I_{2n-1} x^{-2n+1} \ . \tag{4.99}$$

We now take the following ansatz for $\mathbf{P}_4$

$$\mathbf{P}_4 = \epsilon(x^{J/2} - x^{-J/2}) \sinh_- + Q_{J/2-1}(u), \tag{4.100}$$

where $Q_{J/2-1}$ is a polynomial of degree $J/2 - 1$ in u. It is easy to see that this ansatz solves (4.95) and has correct asymptotics. The polynomial $Q_{J/2-1}$ can be fixed from the equation

(4.94) for $\mathbf{P}_2$. Indeed, from the asymptotics of $\mathbf{P}_2$ we see that the lhs of (4.94) does not have powers of x from $-J/2+1$ to $J/2-1$. This fixes

$$Q_{J/2-1}(x) = -\epsilon \sum_{k=1}^{J/2} I_{2k-1} \left(x^{\frac{J}{2}-2k+1} + x^{-\frac{J}{2}+2k-1} \right). \tag{4.101}$$

Once $Q_{J/2-1}$ is found, we set $\mathbf{P}_2$ to be the part of the right hand side of (4.94) with powers of x less than $-J/2$, which gives

$$\mathbf{P}_2 = -\epsilon x^{+J/2} \sum_{n=\frac{J}{2}+1}^{\infty} I_{2n-1} x^{1-2n}. \tag{4.102}$$

This completes the solution for even J, we summarize it below:

$$\mu_{12} = 1,\ \mu_{13} = 0,\ \mu_{14} = -1,\ \mu_{24} = \sinh(2\pi u),\ \mu_{34} = 0, \tag{4.103}$$

$$\mathbf{P}_1 = \epsilon x^{-J/2} \tag{4.104}$$

$$\mathbf{P}_2 = -\epsilon x^{+J/2} \sum_{n=J/2+1}^{\infty} I_{2n-1} x^{1-2n} \tag{4.105}$$

$$\mathbf{P}_3 = \epsilon \left(x^{-J/2} - x^{+J/2} \right) \tag{4.106}$$

$$\mathbf{P}_4 = \epsilon \left(x^{J/2} - x^{-J/2} \right) \sinh_- -\epsilon \sum_{n=1}^{J/2} I_{2n-1} \left(x^{\frac{J}{2}-2n+1} + x^{-\frac{J}{2}+2n-1} \right) . \tag{4.107}$$

In the next section we fix the remaining parameter ϵ of the solution in terms of S and find the energy, but now let us briefly discuss the solution for odd J. As we mentioned above the main difference is that the functions $\mathbf{P}_a$ now have a branch point at $u = \infty$, which is dictated by the asymptotics (4.21). In addition, the parity of μ_{ab} is different according to the asymptotics of these functions (4.22). The solution is still very similar to the even J case, and we discuss it in detail in Appendix B. Let us present the result here:

$$\mu_{12} = 1,\ \mu_{13} = 0,\ \mu_{14} = 0,\ \mu_{24} = \cosh(2\pi u),\ \mu_{34} = 1 \tag{4.108}$$

$$\mathbf{P}_1 = \epsilon x^{-J/2}, \tag{4.109}$$

$$\mathbf{P}_2 = -\epsilon x^{J/2} \sum_{k=-\infty}^{-\frac{J+1}{2}} I_{2k} x^{2k}, \tag{4.110}$$

$$\mathbf{P}_3 = -\epsilon x^{J/2}, \tag{4.111}$$

$$\mathbf{P}_4 = \epsilon x^{-J/2} \cosh_- -\epsilon x^{-J/2} \sum_{k=1}^{\frac{J-1}{2}} I_{2k} x^{2k} - \epsilon I_0 x^{-J/2}. \tag{4.112}$$

Note that now $\mathbf{P}_a$ include half-integer powers of x.

Fixing the global charges of the solution. Finally, to fix our solution completely we have to find the value of ϵ and find the energy in terms of the spin using (**??**) and (4.24). For this we first extract the coefficients A_a of the leading terms for all $\mathbf{P}_a$ (see the asymptotics (4.21)).

From (4.104)-(4.107) or (4.109)-(4.112) we get

$$A_1 = g^{J/2}\epsilon, \tag{4.113}$$

$$A_2 = -g^{J/2+1}\epsilon I_{J+1}, \tag{4.114}$$

$$A_3 = -g^{-J/2}\epsilon, \tag{4.115}$$

$$A_4 = -g^{-J/2+1}\epsilon I_{J-1}. \tag{4.116}$$

Expanding (4.23), (4.24) at small S with $\Delta = J + S + \gamma$, where $\gamma = \mathcal{O}(S)$, we find at linear order

$$\gamma = i(A_1A_4 - A_2A_3) \tag{4.117}$$

$$S = i(A_1A_4 + A_2A_3)\ . \tag{4.118}$$

Plugging in the coefficients (4.113)-(4.116) we find that

$$\epsilon = \sqrt{\frac{2\pi i S}{J I_J(\sqrt{\lambda})}} \tag{4.119}$$

and we obtain the anomalous dimension at leading order,

$$\gamma = \frac{\sqrt{\lambda} I_{J+1}(\sqrt{\lambda})}{J I_J(\sqrt{\lambda})} S + \mathcal{O}(S^2), \tag{4.120}$$

which is precisely the slope function of Basso [68].

While the above discussion concerned the ground state, i.e. the $sl(2)$ sector operator with the lowest anomalous dimension at given twist J, it can be generalized for higher mode numbers. In the asymptotic Bethe ansatz for such operators we have two symmetric cuts formed by Bethe roots, with corresponding mode numbers being $\pm n$ (for the ground state $n = 1$). To describe these operators within the $\mathbf{P}\mu$-system we found that we should take $\mu_{24} = C\sinh(2\pi n u)$ instead of $\mu_{24} = C\sinh(2\pi u)$ (and for odd J we similarly use $\mu_{24} = C\cosh(2\pi n u)$ instead of $\mu_{24} = C\cosh(2\pi u)$). Then the solution is very similar to the one above, and we find

$$\gamma = \frac{n\sqrt{\lambda} I_{J+1}(n\sqrt{\lambda})}{J I_J(n\sqrt{\lambda})} S\ , \tag{4.121}$$

which reproduced the result of [68] for non-trivial mode number n. In Appendix E.1 we also show how using the $\mathbf{P}\mu$-system one can reproduce the slope function for a configuration of Bethe roots with arbitrary mode numbers and filling fractions.

In summary, we have shown how the $\mathbf{P}\mu$-system correctly computes the energy at linear order in S. In section **??** we will compute the next, S^2 term in the anomalous dimension.

4.6 The curvature function

In this section we use the $\mathbf{P}\mu$-system to compute the S^2 correction to the anomalous dimension, which we call the curvature function $\gamma^{(2)}(g)$. First we will discuss the case $J = 2$ in detail and then describe the modifications of the solution for the cases $J = 3$ and $J = 4$, more details on which can be found in appendix C.

4.6.1 Iterative procedure for the small S expansion of the $\mathbf{P}\mu$-system

For convenience let us repeat the leading order solution of the $\mathbf{P}\mu$-system for $J = 2$ (see (4.103)-(4.107))

$$\mathbf{P}_1^{(0)} = \epsilon\frac{1}{x} \quad , \quad \mathbf{P}_2^{(0)} = +\epsilon I_1 - \epsilon x[\sinh(2\pi u)]_- \ , \tag{4.122}$$

$$\mathbf{P}_3^{(0)} = \epsilon\left(\frac{1}{x} - x\right) \quad , \quad \mathbf{P}_4^{(0)} = -2\epsilon I_1 - \epsilon\left(\frac{1}{x} - x\right)[\sinh(2\pi u)]_-. \tag{4.123}$$

Here ϵ is a small parameter, proportional to $\sqrt{S}$ (see (4.119)), and by $\mathbf{P}_a^{(0)}$ we denote the $\mathbf{P}_a$ functions at leading order in ϵ.

The key observation is that the $\mathbf{P}\mu$-system can be solved iteratively order by order in ϵ. Let us write $\mathbf{P}_a$ and μ_{ab} as an expansion in this small parameter:

$$\mathbf{P}_a = \epsilon\mathbf{P}_a^{(0)} + \epsilon^3\mathbf{P}_a^{(1)} + \epsilon^5\mathbf{P}_a^{(2)} + \dots \tag{4.124}$$

$$\mu_{ab} = \mu_{ab}^{(0)} + \epsilon^2\mu_{ab}^{(1)} + \epsilon^4\mu_{ab}^{(2)} + \dots \ . \tag{4.125}$$

This structure of the expansion is dictated by the equations (4.12), (4.13) of the $\mathbf{P}\mu$-system (as we will soon see explicitly). Since the leading order $\mathbf{P}_a$ are of order ϵ, equation (4.13) implies that the discontinuity of μ_{ab} on the cut is of order ϵ^2. Thus to find μ_{ab} in the next to leading order (NLO) we only need the functions $\mathbf{P}_a$ at leading order. After this, we can find the NLO correction to $\mathbf{P}_a$ from equations (4.13). This will be done below, and having thus the full solution of the $\mathbf{P}\mu$-system at NLO we will find the energy at order S^2.

4.6.2 Correcting μ_{ab}...

In this subsection we find the NLO corrections $\mu_{ab}^{(1)}$ to μ_{ab}. As follows from (4.13) and (4.14), they should satisfy the equation

$$\mu_{ab}^{(1)}(u+i) - \mu_{ab}^{(1)}(u) = \mathbf{P}_a^{(0)}\tilde{\mathbf{P}}_b^{(0)} - \mathbf{P}_b^{(0)}\tilde{\mathbf{P}}_a^{(0)}, \tag{4.126}$$

in which the right hand is known explicitly. For that reason let us define an apparatus for solving equations of this type, i.e.

$$f(u+i) - f(u) = h(u). \tag{4.127}$$

More precisely, we consider functions $f(u)$ and $h(u)$ with one cut in u between $-2g$ and $2g$, and no poles. Such functions can be represented as infinite Laurent series in the Zhukovsky variable $x(u)$, and we additionally restrict ourselves to the case where for $h(u)$ this expansion does not have a constant term[13].

One can see that the general solution of (**??**) has a form of a particular solution plus an arbitrary i-periodic function, which we also call a zero mode. First we will describe the

[13] The r.h.s. of (4.126) has the form $F(u) - \tilde{F}(u)$ and therefore indeed does not have a constant term in its expansion, as the constant in F would cancel in the difference $F(u) - \tilde{F}(u)$.

construction of the particular solution and later deal with zero modes. The linear operator which gives the particular solution of (**??**) described below will be denoted as Σ.

Notice that given the explicit form (**??**) of $\mathbf{P}_a^{(0)}$, the right hand side of (4.126) can be represented in a form

$$\alpha(x)\sinh(2\pi u)+\beta(x), \tag{4.128}$$

where $\alpha(x),\beta(x)$ are power series in x growing at infinity not faster than polynomially. Thus for such α and β we define

$$\Sigma\cdot[\alpha(x)\sinh(2\pi u)+\beta(x)]\equiv\sinh(2\pi u)\Sigma\cdot\alpha(x)+\Sigma\cdot\beta(x). \tag{4.129}$$

We also define $\Sigma\cdot x^{-n}=\Gamma'\cdot x^{-n}$ for $n>0$, where the integral operator Γ' defined as

$$\left(\Gamma'\cdot h\right)(u)\equiv\oint_{-2g}^{2g}\frac{dv}{4\pi i}\partial_u\log\frac{\Gamma[i(u-v)+1]}{\Gamma[-i(u-v)]}h(v). \tag{4.130}$$

This requirement is consistent because of the following relation [14]

$$\left(\Gamma'\cdot h\right)(u+i)-\left(\Gamma'\cdot h\right)(u)=-\frac{1}{2\pi i}\oint_{-2g}^{2g}\frac{h(v)}{u-v}dv=h_-(u)-\widetilde{h_+}(u). \tag{4.131}$$

What is left is to define Σ on positive powers of x. We do it by requiring

$$\Sigma\cdot[x^a+1/x^a]\equiv p_a'(u) \tag{4.132}$$

where $p_a'(u)$ is a polynomial in u of degree $a+1$, which is a solution of

$$p_a'(u+i)-p_a'(u)=\frac{1}{2}\left(x^a+1/x^a\right) \tag{4.133}$$

and satisfies the following additional properties: $p_a'(0)=0$ for odd a and $p_a'(i/2)=0$ for even a. One can check that this definition is consistent and defines of $p_a'(u)$ uniquely. Explicit form of the first few $p_a'(u)$, which we call periodized Chebyshev polynomials, can be found in appendix A.

From this definition of Σ one can see that the result of its action on expressions of the form (**??**) can again be represented in this form - what is important for us is that no exponential functions other than $\sinh(2\pi u)$ appear in the result.

A good illustration of how the definitions above work would be the following two simple examples. Suppose one wants to calculate $\Sigma\cdot\left(x-\frac{1}{x}\right)$, then it is convenient to split the argument of Σ in the following way:

$$\Sigma\cdot\left(x-\frac{1}{x}\right)=\Sigma\cdot\left(x+\frac{1}{x}\right)-2\Sigma\cdot\frac{1}{x}. \tag{4.134}$$

In the first term we recognize $p_1'(u)=\frac{iu(u-i)}{2g}$, whereas in the second the argument of Σ is decaying at infinity, thus Σ is equivalent to Γ' in this context. Notice also that $\Gamma'\cdot\frac{1}{x}=-\Gamma'\cdot x$. All together, we get

$$\Sigma\cdot\left(x-\frac{1}{x}\right)=\Sigma\cdot\left(x+\frac{1}{x}\right)-2\Sigma\cdot\frac{1}{x}=2p_1'(u)+2\Gamma'\cdot x \tag{4.135}$$

[14]We remind that f_+ and f_- stand for the part of the Laurent expansion with, respectively, positive and negative powers of x, while $\tilde{f}$ is the analytic continuation around the branch point at $u=2g$ (which amounts to replacing $x\to\frac{1}{x}$)

In a similar way, in order to calculate $\Sigma\cdot\frac{\sinh_- - \sinh_+}{2}$, one can write $\frac{\sinh_- - \sinh_+}{2} = \sinh_- - \frac{1}{2}\sinh(2\pi u)$. Notice that since $\sinh_-$ decays at infinity,

$$\Sigma \cdot \sinh_- = \Gamma' \cdot \sinh_- \,. \tag{4.136}$$

Also, since i-periodic functions can be factored out of Σ,

$$\Sigma \cdot \sinh(2\pi u) = \sinh(2\pi u)\Sigma \cdot 1 = \sinh(2\pi u)p_0'(u)/2. \tag{4.137}$$

Finally,

$$\Sigma \cdot \frac{\sinh_- - \sinh_+}{2} = \Gamma' \cdot (\sinh_-) - \frac{1}{2}\sinh(2\pi u)p_0'(u). \tag{4.138}$$

As an example we present the particular solution for two components of μ_{ab} (below we will argue that π_{12} and π_{13} can be chosen to be zero, see (4.147))

$$\mu_{13}^{(1)} - \pi_{13} = \Sigma \cdot \left(\mathbf{P}_1\tilde{\mathbf{P}}_3 - \mathbf{P}_3\tilde{\mathbf{P}}_1\right) = \epsilon^2\Sigma \cdot \left(x^2 - \frac{1}{x^2}\right) = \epsilon^2 \left(\Gamma' \cdot x^2 + p_2'(u)\right), \tag{4.139}$$

$$\begin{aligned}\mu_{12}^{(1)} - \pi_{12} &= \Sigma \cdot \left(\mathbf{P}_1\tilde{\mathbf{P}}_2 - \mathbf{P}_2\tilde{\mathbf{P}}_1\right) = \\ &= -\epsilon^2 \left[2I_1\Gamma' \cdot x - \sinh(2\pi u)\,\Gamma' \cdot x^2 - \Gamma' \cdot \left(\sinh_- \left(x^2 + \frac{1}{x^2}\right)\right)\right].\end{aligned} \tag{4.140}$$

Now let us apply Σ defined above to (4.126), writing that its general solution is

$$\mu_{ab}^{(1)} = \Sigma \cdot (\mathbf{P}_a^{(0)}\tilde{\mathbf{P}}_b^{(0)} - \mathbf{P}_b^{(0)}\tilde{\mathbf{P}}_a^{(0)}) + \pi_{ab}, \tag{4.141}$$

where the zero mode π_{ab} is an arbitrary i-periodic entire function, which can be written similarly to the leading order as $c_{1,ab}\cosh 2\pi u + c_{2,ab}\sinh 2\pi u + c_{3,ab}$. Again, many of the coefficients $c_{i,ab}$ can be set to zero. First, the prescription from section **??** implies that non-vanishing at infinity part of coefficients of $\sinh(2\pi u)$ and $\cosh(2\pi u)$ in μ_{12} is zero. As one can see from the explicit form (**??**) of the particular solution which we choose for μ_{12}, it does not contain $\cosh(2\pi u)$ and the coefficient of $\sinh(2\pi u)$ is decaying at infinity. So in order to satisfy the prescription, we have to set $c_{2,12}$ and $c_{2,12}$ to zero. Second, since the coefficients $c_{n,ab}$ are of order S, we can remove some of them by making an infinitesimal γ-transformation, i.e. with $R = 1 + \mathcal{O}(S)$ (see section **??** and Eq. (**??**)). Further, the Pfaffian constraint (**??**) imposes 5 equations on the remaining coefficients, which leaves the following 2-parametric family of zero modes

$$\pi_{12} = 0,\ \pi_{13} = 0,\ \pi_{14} = \frac{1}{2}c_{1,34}\cosh 2\pi u, \tag{4.142}$$

$$\pi_{24} = c_{1,24}\cosh 2\pi u,\ \pi_{34} = c_{1,34}\cosh 2\pi u. \tag{4.143}$$

Let us now look closer at the exponential part of μ_{14} and μ_{24}. Combining the leading order (4.103) and the perturbation (4.141) and taking into account the fact that operator Σ does not produce terms proportional to $\cosh 2\pi u$, we obtain

$$\mu_{14} = \frac{1}{2}c_{1,34}\cosh 2\pi u + \mathcal{O}(\epsilon)\sinh 2\pi u + \mathcal{O}(\epsilon^2) + \dots, \tag{4.144}$$

$$\mu_{24} = \frac{1}{2}c_{1,24}\cosh 2\pi u + (1 + \mathcal{O}(\epsilon))\sinh 2\pi u + \mathcal{O}(\epsilon^2) + \dots, \tag{4.145}$$

where dots stand for powers-like terms or exponential terms suppressed by powers of u.

As we remember from section **??**, only the 2nd solution of the 5th order Baxter equation (**??**) can contribute to the exponential part of μ_{14} and μ_{24}, which means that μ_{14} and μ_{24} are proportional to the same linear combination of $\sinh 2\pi u$ and $\cosh 2\pi u$. From the second equation one can see that this linear combination can be normalized to be $\frac{1}{2}c_{1,24}\cosh 2\pi u + (1+\mathcal{O}(\epsilon))\sinh 2\pi u$. Then $\mu_{14} = C\left(\frac{1}{2}c_{1,24}\cosh 2\pi u + (1+\mathcal{O}(\epsilon))\sinh 2\pi u\right)$, where C is some constant, which is of order $\mathcal{O}(\epsilon)$, because the coefficient of $\sinh 2\pi u$ in the first equation is $\mathcal{O}(\epsilon)$. Taking into account that $c_{1,24}$ is $\mathcal{O}(\epsilon)$ itself, we find that $c_{1,34} = \mathcal{O}(\epsilon^2)$, i.e. it does not contribute at the order which we are considering. So the final form of the zero mode in (4.141) is

$$\pi_{12} = 0,\ \pi_{13} = 0,\ \pi_{14} = 0, \tag{4.146}$$

$$\pi_{24} = c_{1,24}\cosh 2\pi u,\ \pi_{34} = 0. \tag{4.147}$$

In this way, using the particular solution given by Σ and the form of zero modes (4.147) we have computed all the functions $\mu_{ab}^{(1)}$. The details and the results of the calculation can be found in appendix C.1.

4.6.3 Correcting $\mathbf{P}_a$...

In the previous section we found the NLO part of μ_{ab}. Now, according to the iterative procedure described in section 4.6.1, we can use it to write a closed system of equations for $\mathbf{P}_a^{(1)}$. Indeed, expanding the system (4.20) to NLO we get

$$\tilde{\mathbf{P}}_1^{(1)} - \mathbf{P}_1^{(1)} = -\mathbf{P}_3^{(1)} + r_1, \tag{4.148}$$

$$\tilde{\mathbf{P}}_2^{(1)} + \mathbf{P}_2^{(1)} = -\mathbf{P}_4^{(1)} - \mathbf{P}_1^{(1)}\sinh(2\pi u) + r_2, \tag{4.149}$$

$$\tilde{\mathbf{P}}_3^{(1)} + \mathbf{P}_3^{(1)} = r_3, \tag{4.150}$$

$$\tilde{\mathbf{P}}_4^{(1)} - \mathbf{P}_4^{(1)} = \mathbf{P}_3^{(1)}\sinh(2\pi u) + r_4, \tag{4.151}$$

where the free terms are given by

$$r_a = -\mu_{ab}^{(1)}\chi^{bc}\mathbf{P}_c^{(0)}. \tag{4.152}$$

Notice that r_a does not change if we add a matrix proportional to $\mathbf{P}_a^{(0)}\tilde{\mathbf{P}}_b^{(0)} - \mathbf{P}_b^{(0)}\tilde{\mathbf{P}}_a^{(0)}$ to $\mu_{ab}^{(1)}$, due to the relations

$$\mathbf{P}_a\chi^{ab}\mathbf{P}_b = 0,\ \mathbf{P}_a\chi^{ab}\tilde{\mathbf{P}}_b = 0, \tag{4.153}$$

which follow from the $\mathbf{P}\mu$-system equations. In particular we can use this property to replace $\mu_{ab}^{(1)}$ in (4.152) by $\mu_{ab}^{(1)} + \frac{1}{2}\left(\mathbf{P}_a^{(0)}\tilde{\mathbf{P}}_b^{(0)} - \mathbf{P}_b^{(0)}\tilde{\mathbf{P}}_a^{(0)}\right)$. This will be convenient for us, since in expressions for $\mu_{ab}^{(1)}$ in terms of p_a and Γ (see (4.139), (4.140) and appendix C.1) this change amounts to simply replacing Γ' by a convolution with a more symmetric kernel:

$$\Gamma' \to \Gamma, \tag{4.154}$$

$$(\Gamma \cdot h)(u) \equiv \oint_{-2g}^{2g} \frac{dv}{4\pi i} \partial_u \log \frac{\Gamma[i(u-v)+1]}{\Gamma[-i(u-v)+1]} h(v), \tag{4.155}$$

while at the same time replacing

$$p'_a(u) \to p_a(u), \tag{4.156}$$

$$p_a(u) = p'_a(u) + \frac{1}{2}\left(x^a(u) + x^{-a}(u)\right). \tag{4.157}$$

Having made this comment, we will now develop tools for solving the equations (4.148) - (4.151). Notice first that if we solve them in the order (4.150), (4.148), (4.151), (4.149), substituting into each subsequent equation the solution of all the previous, then at each step the problem we have to solve has a form

$$\tilde{f} + f = h \ \text{ or } \ \tilde{f} - f = h \ \ , \tag{4.158}$$

where h is known, f is unknown and both the right hand side and the left hand side are power series in x. It is obvious that equations (4.158) have solutions only for h such that $h = \tilde{h}$ and $h = -\tilde{h}$ respectively. On the class of such h a particular solution for f can be written as

$$f = [h]_- + [h]_0/2 \equiv H \cdot h \ \ \Rightarrow \ \ \tilde{f} + f = h \tag{4.159}$$

and

$$f = [h]_- \equiv K \cdot h \ \ \Rightarrow \ \ \tilde{f} - f = h, \tag{4.160}$$

where $[h]_0$ is the constant part of Laurent expansion of h (it does not appear in the second equation, because h such that $h = -\tilde{h}$ does not have a constant part). The operators K and H introduced here can be also defined by their integral kernels

$$H(u,v) = -\frac{1}{4\pi i} \frac{\sqrt{u-2g}\sqrt{u+2g}}{\sqrt{v-2g}\sqrt{v+2g}} \frac{1}{u-v}, \tag{4.161}$$

$$K(u,v) = +\frac{1}{4\pi i} \frac{1}{u-v}, \tag{4.162}$$

which are equivalent to (4.159),(4.160) of the classes of h such that $h = \tilde{h}$ and $h = -\tilde{h}$ respectively[15]. The particular solution $f = K \cdot h$ of the equation $\tilde{f} + f = h$ is unique in the class of functions f decaying at infinity, and the solution $f = H \cdot h$ of $\tilde{f} - f = h$ is unique for non-growing f. In all other cases the general solution will include zero modes, which, in our case are fixed by asymptotics of $\mathbf{P}_a$.

[15] We denote e.g. $K \cdot h = \oint_{-2g}^{2g} K(u,v) h(v) dv$ where the integral is around the branch cut between $-2g$ and $2g$.

Now it is easy to write the explicit solution of the equations (4.148)-(4.151):

$$\mathbf{P}_3^{(1)} = H \cdot r_3, \tag{4.163}$$

$$\mathbf{P}_1^{(1)} = \frac{1}{2}\mathbf{P}_3^{(1)} + K \cdot \left(r_1 - \frac{1}{2}r_3\right), \tag{4.164}$$

$$\mathbf{P}_4^{(1)} = K \cdot \left(-\frac{1}{2}\left(\tilde{\mathbf{P}}_3^{(1)} - \mathbf{P}_3^{(1)}\right)\sinh(2\pi u) + \frac{2r_4 + r_3\sinh(2\pi u)}{2}\right) - 2\delta, \tag{4.165}$$

$$\mathbf{P}_2^{(1)} = H \cdot \left(-\frac{1}{2}\left(\mathbf{P}_4^{(1)} + \sinh(2\pi u)\mathbf{P}_1^{(1)} + \tilde{\mathbf{P}}_4^{(1)} + \sinh(2\pi u)\tilde{\mathbf{P}}_1^{(1)}\right) + \right. \\ \left. + \frac{r_4 + \sinh(2\pi u) r_1 + 2r_2}{2}\right) + \delta, \tag{4.166}$$

where δ is a constant fixed uniquely by requiring $\mathcal{O}(1/u^2)$ asymptotics for $\mathbf{P}_2$. This asymptotic also sets the last coefficient $c_{1,24}$ left in π_{12} to zero. Thus in the class of functions with asymptotics (4.21) the solution for μ_{ab} and $\mathbf{P}_a$ is unique up to a γ-transformation.

4.6.4 Result for $J = 2$

In order to obtain the result for the anomalous dimension, we again use the formulas (4.23), (4.24) which connect the leading coefficients of $\mathbf{P}_a$ with Δ, J and S. After plugging in A_i which we find from our solution, we obtain the result for the S^2 correction to the anomalous dimension:

$$\begin{aligned}\gamma_{J=2}^{(2)} = \; & \frac{\pi}{g^2(I_1 - I_3)^3} \oint \frac{du_x}{2\pi i} \oint \frac{du_y}{2\pi i} \left[\frac{8I_1^2(I_1 + I_3)\left(x^3 - \left(x^2+1\right)y\right)}{\left(x^3 - x\right)y^2} \right. \\ & + \frac{8\text{sh}_-^x\text{sh}_-^y\left(x^2y^2 - 1\right)\left(I_1(x^4y^2+1) - I_3x^2(y^2+1)\right)}{x^2\left(x^2-1\right)y^2} \\ & - \frac{4(\text{sh}_-^y)^2x^2\left(y^4-1\right)\left(I_1(2x^2-1) - I_3\right)}{\left(x^2-1\right)y^2} \\ & + \frac{8I_1^2\text{sh}_-^y x\left(2\left(x^3-x\right)\left(y^3+y\right) - 2x^2\left(y^4+y^2+1\right) + y^4 + 4y^2 + 1\right)}{\left(x^2-1\right)y^2} \\ & - \frac{8(I_1 - I_3)I_1\text{sh}_-^y x(x-y)(xy-1)}{\left(x^2-1\right)y} \\ & \left. - \frac{4(I_1 - I_3)(\text{sh}_-^x)^2\left(x^2+1\right)y^2}{\left(x^2-1\right)} \right] \frac{1}{4\pi i}\partial_u \log\frac{\Gamma(iu_x - iu_y + 1)}{\Gamma(1 - iu_x + iu_y)} .\end{aligned} \tag{4.167}$$

Here the integration contour goes around the branch cut at $(-2g, 2g)$. We also denote $\text{sh}_-^x = \sinh_-(x)$, $\text{sh}_-^y = \sinh_-(y)$ (recall that $\sinh_-$ was defined in (4.99)). This is our final result for the curvature function at any coupling.

It is interesting to note that our result contains the combination $\log\frac{\Gamma(iu_x - iu_y+1)}{\Gamma(1-iu_x+iu_y)}$ which plays an essential role in the construction of the BES dressing phase. We will use this identification in section **??** to compute the integral in (4.167) numerically with high precision.

In the next subsections we will describe generalizations of the $J = 2$ result to operators with $J = 3$ and $J = 4$.

4.6.5 Results for higher J

Solving the $\mathbf{P}\mu$-system for $J = 3$ is similar to the $J = 2$ case described above, except for several technical complications, which we will describe here, leaving the details for the appendix C.2. As in the previous section, the starting point is the LO solution of the $\mathbf{P}\mu$ system, which for $J = 3$ reads

$$\mathbf{P}_1 = \epsilon x^{-3/2},\ \mathbf{P}_3 = -\epsilon x^{3/2}, \tag{4.168}$$

$$\mathbf{P}_2 = -\epsilon x^{3/2}\cosh_- + \epsilon x^{-1/2} I_2, \tag{4.169}$$

$$\mathbf{P}_4 = -\epsilon x^{1/2} I_2 - \epsilon x^{-3/2} I_0 - \epsilon x^{-3/2}\cosh_-, \tag{4.170}$$

$$\mu_{12} = 1,\ \mu_{13} = 0,\ \mu_{14} = 0,\ \mu_{24} = \cosh(2\pi u),\ \mu_{34} = 1\ . \tag{4.171}$$

The first step is to construct $\mu_{ab}^{(1)}$ from its discontinuity given by the equation (4.126). The full solution consists of a particular solution and a general solution of the corresponding homogeneous equation, i.e. zero mode π_{ab}. In our case the zero mode can be an i-periodic function, i.e. a linear combination of $\sinh(2\pi u)$, $\cosh(2\pi u)$ and constants. As in the case of $J = 2$, we use a combination of the Pfaffian constraint, prescription from section **??** and a γ-transformation to reduce all the parameters of the zero mode to just one, sitting in μ_{24}:

$$\pi_{12} = 0,\ \pi_{13} = 0,\ \pi_{14} = 0,\ \pi_{24} = c_{24,2}\sinh(2\pi u),\ \pi_{34} = 0. \tag{4.172}$$

As in the previous section, the next step is to find $\mathbf{P}_a^{(1)}$ from the $P\mu$ system expanded to the first order, namely from

$$\tilde{\mathbf{P}}_1^{(1)} + \mathbf{P}_3^{(1)} = r_1, \tag{4.173}$$

$$\tilde{\mathbf{P}}_2^{(1)} + \mathbf{P}_4^{(1)} + \mathbf{P}_1^{(1)}\cosh(2\pi u) = r_2, \tag{4.174}$$

$$\tilde{\mathbf{P}}_3^{(1)} + \mathbf{P}_1^{(1)} = r_3, \tag{4.175}$$

$$\tilde{\mathbf{P}}_4^{(1)} + \mathbf{P}_2^{(1)} - \mathbf{P}_3^{(1)}\cosh(2\pi u) = r_4, \tag{4.176}$$

where r_a are defined by (4.152) and for $J = 3$ are given explicitly in appendix C.2. In attempt to solve this system, however, we encounter another technical complication. As one can see from (**??**)-(**??**), the LO solution contains half-integer powers of J, meaning that the $\mathbf{P}_a$ now have an extra branch point at infinity. However, the operations H and K defined by (4.162) work only for functions which have Laurent expansion in integer powers of x. In order to solve equations of the type (4.126) on the class of functions which allow Laurent-like expansion in x with only half-integer powers x, we introduce operations H^*, K^*:

$$H^* \cdot f \equiv \frac{x+1}{\sqrt{x}} H \cdot \frac{\sqrt{x}}{x+1} f, \tag{4.177}$$

$$K^* \cdot f \equiv \frac{x+1}{\sqrt{x}} K \cdot \frac{\sqrt{x}}{x+1} f. \tag{4.178}$$

In terms of these operations the solution of the system (4.173)-(4.176) is

$$\begin{aligned}
\mathbf{P}_1^{(1)} &= \frac{1}{2}\left(H^*(r_1+r_3)+K^*(r_1-r_3)\right)+\mathbf{P}_1^{\rm zm}, &(4.179)\\
\mathbf{P}_3^{(1)} &= \frac{1}{2}\left(H^*(r_1+r_3)-K^*(r_1-r_3)\right)+\mathbf{P}_2^{\rm zm}, &(4.180)\\
\mathbf{P}_2^{(1)} &= \frac{1}{2}\left(H^*(r_2+r_4)+K^*(r_2-r_4)-\right.\\
&- \left.H^*\left(\cosh(2\pi u)K^*(r_1-r_3)\right)-K^*\left(\cosh(2\pi u)H^*(r_1+r_3)\right)\right)+\mathbf{P}_3^{\rm zm}, &(4.181)\\
\mathbf{P}_4^{(1)} &= \frac{1}{2}\left(H^*(r_2+r_4)-K^*(r_2-r_4)-\right.\\
&- \left.H^*\left(\cosh(2\pi u)K^*(r_1-r_3)\right)+K^*\left(\cosh(2\pi u)H^*(r_1+r_3)\right)\right)+\mathbf{P}_4^{\rm zm}, &(4.182)
\end{aligned}$$

where $\mathbf{P}_a^{\rm zm}$ is a solution of the system (4.173)-(4.176) with right hand side set to zero, whose explicit form $\mathbf{P}_a^{\rm zm}$ is given in Appendix C.2 (see (??)-(??)) and which is parametrized by four constants L_1, L_2, L_3, L_4, e.g.

$$\mathbf{P}_1^{\rm zm} = L_1 x^{-1/2} + L_3 x^{1/2}. \tag{4.183}$$

These constants are fixed by requiring correct asymptotics of $\mathbf{P}_a$, which also fixes the parameter $c_{24,2}$ in the zero mode (4.172) of μ_{ab} [16]. Indeed, a priori $\mathbf{P}_2$ and $\mathbf{P}_1$ have wrong asymptotics. Imposing a constraint that $\mathbf{P}_2$ decays as $u^{-5/2}$ and $\mathbf{P}_1$ decays as $u^{-3/2}$ produces five equations, which fix all the parameters uniquely.

Skipping the details of the intermediate calculations, we present the final result for the anomalous dimension:

$$\begin{aligned}
\gamma^{(2)}_{J=3} &= \oint \frac{du_x}{2\pi i}\oint \frac{du_y}{2\pi i}\, i\frac{1}{g^2(I_2-I_4)^3}\left[\frac{2\left(x^6-1\right)y(\mathrm{ch}_-^y)^2(I_2-I_4)}{x^3\left(y^2-1\right)}-\right.\\
&-\frac{4\mathrm{ch}_-^x\mathrm{ch}_-^y\left(x^3y^3-1\right)\left(I_2x^5y^3+I_2-I_4x^2\left(xy^3+1\right)\right)}{x^3\left(x^2-1\right)y^3}+\\
&+\frac{(y^2-1)(\mathrm{ch}_-^y)^2 I_2\left(\left(x^8+1\right)\left(2y^4+3y^2+2\right)-\left(x^6+x^2\right)\left(y^2+1\right)^2\right)}{x^3\left(x^2-1\right)y^3}-\\
&-\frac{(y^2-1)(\mathrm{ch}_-^y)^2 I_4\left(\left(x^8+1\right)y^2+\left(x^6+x^2\right)\left(y^4+1\right)\right)}{x^3\left(x^2-1\right)y^3}-\\
&-\frac{4I_2\mathrm{ch}_-^y(x-y)(xy-1)\left(I_2\left(\left(x^6+1\right)\left(y^3+y\right)+\left(x^5+x\right)\left(y^4+y^2+1\right)-x^3\left(y^4+1\right)\right)+I_4x^3y^2\right)}{x^3\left(x^2-1\right)y^3}-\\
&-\left.\frac{I_2^2(y^2-1)(x-y)(xy-1)\left(I_2\left(\left(x^6+x^4+x^2+1\right)y+2x^3\left(y^2+1\right)\right)+I_4\left(x^5+x\right)\left(y^2+1\right)\right)}{x^3\left(x^2-1\right)y^3}\right]\\
&\frac{1}{4\pi i}\partial_u\log\frac{\Gamma(iu_x-iu_y+1)}{\Gamma(1-iu_x+iu_y)}.
\end{aligned} \tag{4.184}$$

We defined $\mathrm{ch}_-^x = \cosh_-(x)$ and $\mathrm{ch}_-^y = \cosh_-(y)$, where $\cosh_-(x)$ is the part of the Laurent expansion of $\cosh\left(g(x+1/x)\right)$ vanishing at infinity, i.e.

$$\cosh_-(x) = \sum_{k=1}^{\infty} I_{2k}x^{-2k}. \tag{4.185}$$

The result for $J=4$ is given in appendix C.4.

[16] Actually in this way $c_{24,2}$ is fixed to be zero.

4.6.6 Weak coupling expansion

Our results for the curvature function $\gamma^{(2)}(g)$ at $J = 2, 3, 4$ (Eqs. (4.167), (4.184), (C.32)) are straightforward to expand at weak coupling. We give expansions to 10 loops in appendix D. Let us start with the $J = 2$ case, for which we found

$$\begin{aligned}\gamma^{(2)}_{J=2} &= -8g^2\zeta_3 + g^4\left(140\zeta_5 - \frac{32\pi^2\zeta_3}{3}\right) + g^6\left(200\pi^2\zeta_5 - 2016\zeta_7\right) \\ &+ g^8\left(-\frac{16\pi^6\zeta_3}{45} - \frac{88\pi^4\zeta_5}{9} - \frac{9296\pi^2\zeta_7}{3} + 27720\zeta_9\right) \\ &+ g^{10}\left(\frac{208\pi^8\zeta_3}{405} + \frac{160\pi^6\zeta_5}{27} + 144\pi^4\zeta_7 + 45440\pi^2\zeta_9 - 377520\zeta_{11}\right) + \ldots\end{aligned} \tag{4.186}$$

Remarkably, at each loop order all contributions have the same transcendentality, and only simple zeta values (i.e. ζ_n) appear. This is also true for the $J = 3$ and $J = 4$ cases.

We can check this expansion against known results, as the anomalous dimensions of twist two operators have been computed up to five loops for arbitrary spin [?, ?, ?, ?, ?, ?, ?, ?] (see also [?] and the review [?]). To three loops they can be found solely from the ABA equations, while at four and five loops wrapping corrections need to be taken into account which was done in [?, ?] by utilizing generalized Luscher formulas. All these results are given by linear combinations of harmonic sums

$$S_a(N) = \sum_{n=1}^{N} \frac{(\text{sign}(a))^n}{n^{|a|}}, \quad S_{a_1,a_2,a_3,\ldots}(N) = \sum_{n=1}^{N} \frac{(\text{sign}(a_1))^n}{n^{|a_1|}} S_{a_2,a_3,\ldots}(n) \tag{4.187}$$

with argument equal to the spin S. To make a comparison with our results we expanded these predictions in the $S \to 0$ limit. For this lengthy computation, as well as to simplify the final expressions, we used the `Mathematica` packages HPL [?], the package [?] provided with the paper [?], and the HarmonicSums package [?].

In this way we have confirmed the coefficients in (4.186) to four loops. Let us note that expansion of harmonic sums leads to multiple zeta values (MZVs), which however cancel in the final result leaving only ζ_n.

Importantly, the part of the four-loop coefficient which comes from the wrapping correction is essential for matching with our result. This is a strong confirmation that our calculation based on the $\mathbf{P}\mu$-system is valid beyond the ABA level. Additional evidence that our result incorporates all finite-size effects is found at strong coupling (see section **??**).

For operators with $J = 3$, our prediction at weak coupling is

$$\begin{aligned}\gamma^{(2)}_{J=3} &= -2g^2\zeta_3 + g^4\left(12\zeta_5 - \frac{4\pi^2\zeta_3}{3}\right) + g^6\left(\frac{2\pi^4\zeta_3}{45} + 8\pi^2\zeta_5 - 28\zeta_7\right) \\ &+ g^8\left(-\frac{4\pi^6\zeta_3}{45} - \frac{4\pi^4\zeta_5}{15} - 528\zeta_9\right) + \ldots\end{aligned} \tag{4.188}$$

The known results for any spin in this case are available at up to six loops, including the wrapping correction which first appears at five loops [?, ?, ?]. Expanding them at $S \to 0$ we have checked our calculation to four loops.[17]

For future reference, in appendix D we present an expansion of known results for $J = 2, 3$ up to order S^3 at first several loop orders. In particular, we found that multiple zeta values appear in this expansion, which did not happen at lower orders in S.

Let us now discuss the $J = 4$ case. The expansion of our result reads:

$$\begin{aligned}\gamma^{(2)}_{J=4} &= g^2\left(-\frac{14\zeta_3}{5}+\frac{48\zeta_5}{\pi^2}-\frac{252\zeta_7}{\pi^4}\right) \\ &+ g^4\left(-\frac{22\pi^2\zeta_3}{25}+\frac{474\zeta_5}{5}-\frac{8568\zeta_7}{5\pi^2}+\frac{8316\zeta_9}{\pi^4}\right) \\ &+ g^6\left(\frac{32\pi^4\zeta_3}{875}+\frac{3656\pi^2\zeta_5}{175}-\frac{56568\zeta_7}{25}+\frac{196128\zeta_9}{5\pi^2}-\frac{185328\zeta_{11}}{\pi^4}\right) \\ &+ g^8\left(-\frac{4\pi^6\zeta_3}{175}-\frac{68\pi^4\zeta_5}{75}-\frac{55312\pi^2\zeta_7}{125}+\frac{1113396\zeta_9}{25}-\frac{3763188\zeta_{11}}{5\pi^2}\right. \\ &\quad\left.+\frac{3513510\zeta_{13}}{\pi^4}\right)+\ldots\end{aligned} \tag{4.189}$$

Unlike for the $J = 2$ and $J = 3$ cases, we could not find a closed expression for the energy at any spin S in literature even at one loop, however there is another way to check our result. One can expand the asymptotic Bethe ansatz equations at large J for fixed values of $S = 2, 4, 6, \ldots$ and then extract the coefficients in the expansion which are polynomial in S. This was done in [?] (see appendix C there) where at one loop the expansion was found up to order $1/J^6$:

$$\gamma(S,J) = g^2\left(\frac{S}{2\,J^2}-\left(\frac{S^2}{4}+\frac{S}{2}\right)\frac{1}{J^3}+\left[\frac{3S^3}{16}+\left(\frac{1}{8}-\frac{\pi^2}{12}\right)S^2+\frac{S}{2}\right]\frac{1}{J^4}+\ldots\right)+\mathcal{O}(g^4) \tag{4.190}$$

Now taking the part proportional to S^2 and substituting $J = 4$ one may expect to get a numerical approximation to the 1-loop coefficient in our result (4.189), i.e. $-\frac{14\zeta_3}{5}+\frac{48\zeta_5}{\pi^2}-\frac{252\zeta_7}{\pi^4}$. To increase the precision we extended the expansion in (4.190) to order $1/J^8$. Remarkably, in this way we confirmed the 1-loop part of the $\mathbf{P}\mu$ prediction (4.189) with about 1% accuracy! In Fig. **??** one can also see that the ABA result converges to our prediction when the order of expansion in $1/J$ is being increased.

Also, in contrast to $J = 2$ and $J = 3$ cases we see that negative powers of π appear in (4.189) (although still all the contributions at a given loop order have the same transcendentality). It would be interesting to understand why this happens from the gauge theory perspective, especially since expansion of the leading S term (**??**) has the same structure for all J,

$$\gamma^{(1)}_J = \frac{8\pi^2 g^2}{J(J+1)} - \frac{32\pi^4 g^4}{J(J+1)^2(J+2)} + \frac{256\pi^6 g^6}{J(J+1)^3(J+2)(J+3)} + \ldots \tag{4.191}$$

The change of structure at $J = 4$ might be related to the fact that for $J \geq 4$ the ground state anomalous dimension even at one loop is expected to be an irrational number for integer $S > 0$ (see [?], [?]), and thus cannot be written as a linear combination of harmonic sums with integer coefficients.

[17] As a further check it would be interesting to expand to order S^2 the known results for twist 2 operators at five loops, and for twist 3 operators at five and six loops – all of which are given by huge expressions.

In the next section we will discuss tests and applications of our results at strong coupling.

4.6.7 Strong coupling expansion

In this section we will present the strong coupling expansion of our results for the curvature function, and link these results to anomalous dimensions of short operators at strong coupling. We will also obtain new predictions for the BFKL pomeron intercept.

To obtain the strong coupling expansion of our exact results for the curvature function, we evaluated it numerically with high precision for a range of values of g and then made a fit to find the expansion coefficients. It would also be interesting to carry out the expansion analytically, and we leave this for the future.

For numerical study it is convenient to write our exact expressions (4.167), (4.184), (C.32) for $\gamma^{(2)}(g)$, which have the form

$$\gamma^{(2)}(g) = \oint du_x \oint du_y f(x,y) \partial_{u_x} \log \frac{\Gamma(iu_x - iu_y + 1)}{\Gamma(1 - iu_x + iu_y)} \tag{4.192}$$

where the integration goes around the branch cut between $-2g$ and $2g$, in a slightly different way (we remind that we use notation $x + \frac{1}{x} = \frac{u_x}{g}$ and $y + \frac{1}{y} = \frac{u_y}{g}$). Namely, by changing the variables of integration to x, y and integrating by parts one can write the result as

$$\gamma^{(2)}(g) = \oint dx \oint dy F(x,y) \log \frac{\Gamma(iu_x - iu_y + 1)}{\Gamma(iu_y - iu_x + 1)} \tag{4.193}$$

where $F(x,y)$ is some polynomial in the following variables: x, $1/x$, y, $1/y$, sh_-^x and sh_-^y (for $J = 3$ it includes ch_-^x, ch_-^y instead of the sh_- functions). The integral in (4.193) is over the unit circle. The advantage of this representation is that plugging in sh_-^x, sh_-^y as series expansions (truncated to some large order), we see that it only remains to compute integrals of the kind

$$C_{r,s} = \frac{1}{i} \oint \frac{dx}{2\pi} \oint \frac{dy}{2\pi} x^r y^s \log \frac{\Gamma(iu_x - iu_y + 1)}{\Gamma(iu_y - iu_x + 1)} \tag{4.194}$$

These are nothing but the coefficients of the BES dressing phase [**?**, **?**, 31, **?**]. They can be conveniently computed using the strong coupling expansion [**?**]

$$C_{r,s} = \sum_{n=0}^{\infty} \left[-\frac{2^{-n-1}(-\pi)^{-n} g^{1-n} \zeta_n \left(1 - (-1)^{r+s+4}\right) \Gamma\left(\frac{1}{2}(n - r + s - 1)\right) \Gamma\left(\frac{1}{2}(n + r + s + 1)\right)}{\Gamma(n-1) \Gamma\left(\frac{1}{2}(-n - r + s + 3)\right) \Gamma\left(\frac{1}{2}(-n + r + s + 5)\right)} \right] \tag{4.195}$$

However this expansion is only asymptotic and does not converge. For fixed g the terms will start growing with n when n is greater than some value N, and we only summed the terms up to $n = N$ which gives the value of $C_{r,s}$ with very good precision for large enough g.

Using this approach we computed the curvature function for a range of values of g (typically we took $7 \leq g \leq 30$) and then fitted the result as an expansion in $1/g$. This gave us only numerical values of the expansion coefficients, but in fact we found that with very high precision the coefficients are as follows. For $J = 2$

$$\begin{aligned} \gamma^{(2)}_{J=2} &= -\pi^2 g^2 + \frac{\pi g}{4} + \frac{1}{8} - \frac{1}{\pi g}\left(\frac{3\zeta_3}{16} + \frac{3}{512}\right) - \frac{1}{\pi^2 g^2}\left(\frac{9\zeta_3}{128} + \frac{21}{512}\right) \\ &+ \frac{1}{\pi^3 g^3}\left(\frac{3\zeta_3}{2048} + \frac{15\zeta_5}{512} - \frac{3957}{131072}\right) + \dots , \end{aligned} \tag{4.196}$$

then for $J=3$

$$\begin{aligned}\gamma^{(2)}_{J=3} &= -\frac{8\pi^2 g^2}{27}+\frac{2\pi g}{27}+\frac{1}{12}-\frac{1}{\pi g}\left(\frac{1}{216}+\frac{\zeta_3}{8}\right)-\frac{1}{\pi^2 g^2}\left(\frac{3\zeta_3}{64}+\frac{743}{13824}\right) \\ &+ \frac{1}{\pi^3 g^3}\left(\frac{41\zeta_3}{1024}+\frac{35\zeta_5}{512}-\frac{5519}{147456}\right)+\dots\,,\end{aligned} \tag{4.197}$$

and finally for $J=4$

$$\begin{aligned}\gamma^{(2)}_{J=4} &= -\frac{\pi^2 g^2}{8}+\frac{\pi g}{32}+\frac{1}{16}-\frac{1}{\pi g}\left(\frac{3\zeta_3}{32}+\frac{15}{4096}\right)-\frac{0.01114622551913}{g^2} \\ &+ \frac{0.004697583899}{g^3}+\dots\,.\end{aligned} \tag{4.198}$$

To fix coefficients for the first four terms in the expansion we were guided by known analytic predictions which will be discussed below, and found that our numerical result matches these predictions with high precision. Then for $J=2$ and $J=3$ we extracted the numerical values obtained from the fit for the coefficients of $1/g^2$ and $1/g^3$, and plugging them into the online calculator EZFace [?] we obtained a prediction for their exact values as combinations of ζ_3 and ζ_5. Fitting again our numerical results with these exact values fixed, we found that the precision of the fit at the previous orders in $1/g$ increased. This is a highly nontrivial test for the proposed exact values of $1/g^2$ and $1/g^3$ terms. For $J=2$ we confirmed the coefficients of these terms with absolute precision 10^{-17} and 10^{-15} at $1/g^2$ and $1/g^3$ respectively (at previous orders of the expansion the precision is even higher). For $J=3$ the precision was correspondingly 10^{-15} and 10^{-13}.

For $J=4$ we were not able to get a stable fit for the $1/g^2$ and $1/g^3$ coefficients from EZFace, so above we gave their numerical values (with uncertainty in the last digit). However below we will see that based on $J=2$ and $J=3$ results one can make a prediction for these coefficients, which we again confirmed by checking that precision of the fit at the previous orders in $1/g$ increases. The precision of the final fit at orders $1/g^2$ and $1/g^3$ is 10^{-16} and 10^{-14} respectively.

4.7 Update on short strings

Here we will find an analytic expression for the strong coupling expansion of the curvature function which generalizes the formulas (4.196) and (4.197) to any J. To this end it will be beneficial to consider the structure of classical expansions of the scaling dimension. A good entry point is considering the inverse relation $S(\Delta)$, frequently encountered in the context of BFKL. It satisfies a few basic properties, namely the curve $S(\Delta)$ goes through the points $(\pm J, 0)$ at any coupling, because at $S=0$ the operator is BPS. At the same time for non-BPS states one should have $\Delta(\lambda)\propto\lambda^{1/4}\to\infty$ [9] which indicates that if Δ is fixed, S should go to zero, thus combining this with the knowledge of fixed points $(\pm J, 0)$ we conclude that at infinite coupling $S(\Delta)$ is simply the line $S=0$. As the coupling becomes finite $S(\Delta)$ starts bending from the $S=0$ line and starts looking like a parabola going through the points $\pm J$, see fig. **??**.

Based on this qualitative picture and the scaling $\Delta(\lambda) \propto \lambda^{1/4}$ at $\lambda \to \infty$ and fixed J and S, one can write down the following ansatz,

$$\begin{aligned} S(\Delta) &= \left(\Delta^2 - J^2\right)\left(\alpha_1 \frac{1}{\lambda^{1/2}} + \alpha_2 \frac{1}{\lambda} + (\alpha_3 + \beta_3 \Delta^2)\frac{1}{\lambda^{3/2}} + (\alpha_4 + \beta_4 \Delta^2)\frac{1}{\lambda^2}\right. \\ &+ \left.(\alpha_5 + \beta_5 \Delta^2 + \gamma_5 \Delta^4)\frac{1}{\lambda^{5/2}} + (\alpha_6 + \beta_6 \Delta^2 + \gamma_6 \Delta^4)\frac{1}{\lambda^3} + \dots\right). \end{aligned} \tag{4.199}$$

We omit odd powers of the scaling dimension from the ansatz, as only the square of Δ enters the $\mathbf{P}\mu$-system. We can now invert the relation and express Δ in terms of S at strong coupling, which gives

$$\Delta^2 = J^2 + S\left(A_1\sqrt{\lambda} + A_2 + \dots\right) + S^2\left(B_1 + \frac{B_2}{\sqrt{\lambda}} + \dots\right) + S^3\left(\frac{C_1}{\lambda^{1/2}} + \frac{C_2}{\lambda} + \dots\right) + \mathcal{O}(S^4)\,, \tag{4.200}$$

where the coefficients A_i, B_i, C_i are some functions of J. There exists a one-to-one mapping between the coefficients α_i, β_i, etc. and A_i, B_i etc, which is rather complicated but easy to find. We note that this structure of Δ^2 coincides with Basso's conjecture in [68] for mode number $n = 1$ [18]. The pattern in (4.200) continues to higher orders in S with further coefficients D_i, E_i, etc. and powers of λ suppressed incrementally. This structure is a nontrivial constraint on Δ itself as one easily finds from (4.200) that

$$\begin{aligned} \Delta &= J + \frac{S}{2J}\left(A_1\sqrt{\lambda} + A_2 + \frac{A_3}{\sqrt{\lambda}} + \dots\right) \\ &+ S^2\left(-\frac{A_1^2}{8J^3}\lambda - \frac{A_1 A_2}{4J^3}\sqrt{\lambda} + \left[\frac{B_1}{2J} - \frac{A_2^2 + 2A_1A_3}{8J^3}\right] + \left[\frac{B_2}{2J} - \frac{A_2A_3 + A_1A_4}{4J^3}\right]\frac{1}{\sqrt{\lambda}} + \dots\right). \end{aligned} \tag{4.201}$$

By definition the coefficients of S and S^2 are the slope and curvature functions respectively, so now we have their expansions at strong coupling in terms of A_i, B_i, C_i, etc. Since the S coefficient only contains the constants A_i, we can find all of their values by simply expanding the slope function (4.120) at strong coupling. We get

$$A_1 = 2\ ,\quad A_2 = -1\ ,\quad A_3 = J^2 - \frac{1}{4}\ ,\quad A_4 = J^2 - \frac{1}{4}\dots\ . \tag{4.202}$$

Note that in this series the power of J increases by two at every other member, which is a direct consequence of omitting odd powers of Δ from (4.199). We also expect the same pattern to hold for the coefficients B_i, C_i, etc.

The curvature function written in terms of A_i, B_i, etc. is given by

$$\begin{aligned} \gamma_J^{(2)}(g) &= -\frac{2\pi^2 g^2 A_1^2}{J^3} - \frac{\pi g A_1 A_2}{J^3} - \frac{A_2^2 + 2A_1A_3 - 4B_1J^2}{8J^3} - \frac{A_2A_3 + A_1A_4 - 2B_2J^2}{16\pi g J^3} \\ &- \frac{A_3^2 + 2A_2A_4 + 2A_1A_5 - 4B_3J^2}{128\pi^2 g^2 J^3} - \frac{A_3A_4 + A_2A_5 + A_1A_6 - 2B_4J^2}{256\pi^3 g^3 J^3} + \mathcal{O}\left(\frac{1}{g^4}\right). \end{aligned} \tag{4.203}$$

The remaining unknowns here (up to order $1/g^4$) are B_1, B_2, which we expect to be constant due to the power pattern noticed above and B_3, B_4, which we expect to have the form $aJ^2 + b$ with

[18] The generalization of (4.200) for $n > 1$ is not fully clear, as noted in [?], and this case will be discussed in appendix E.2.

a and b constant. These unknowns are immediately fixed by comparing the general curvature expansion (4.203) to the two explicit cases that we know for $J = 2$ and $J = 3$. We find

$$B_1 = 3/2 \ , \ B_2 = -3\,\zeta_3 + \frac{3}{8}, \tag{4.204}$$

and

$$B_3 = -\frac{J^2}{2} - \frac{9\,\zeta_3}{2} + \frac{5}{16} \ , \ B_4 = \frac{3}{16}J^2(16\,\zeta_3 + 20\,\zeta_5 - 9) - \frac{15\,\zeta_5}{2} - \frac{93\,\zeta_3}{8} - \frac{3}{16}. \tag{4.205}$$

Having fixed all the unknowns we can write the strong coupling expansion of the curvature function for arbitrary values of J as

$$\begin{aligned} \gamma_J^{(2)}(g) &= -\frac{8\pi^2 g^2}{J^3} + \frac{2\pi g}{J^3} + \frac{1}{4J} + \frac{1 - J^2(24\,\zeta_3 + 1)}{64\pi g J^3} - \frac{8J^4 + J^2(72\,\zeta_3 + 11) - 4}{512 g^2\,(\pi^2 J^3)} \\ &+ \frac{3\left(8J^4(16\,\zeta_3 + 20\,\zeta_5 - 7) - 16J^2(31\,\zeta_3 + 20\,\zeta_5 + 7) + 25\right)}{16384\pi^3 g^3 J^3} + \mathcal{O}\left(\frac{1}{g^4}\right). \end{aligned} \tag{4.206}$$

Expanding $\gamma_{J=4}^{(2)}$ defined in (C.32) at strong coupling numerically we were able to confirm the above result with high precision.

In this section we will use the knowledge of slope functions $\gamma_J^{(n)}$ at strong coupling to find the strong coupling expansions of scaling dimensions of operators with finite S and J, in particular we will find the three loop coefficient of the Konishi operator by utilizing the techniques of [68, **?**]. What follows is a quick recap of the main ideas in these papers.

We are interested in the coefficients of the strong coupling expansion of Δ, namely

$$\Delta = \Delta^{(0)}\lambda^{\frac{1}{4}} + \Delta^{(1)}\lambda^{-\frac{1}{4}} + \Delta^{(2)}\lambda^{-\frac{3}{4}} + \Delta^{(3)}\lambda^{-\frac{5}{4}} + \dots \tag{4.207}$$

First, we use Basso's conjecture (4.200) and by fixing S and J we re-expand the square root of Δ^2 at strong coupling to find

$$\Delta = \sqrt{A_1 S}\,\sqrt[4]{\lambda} + \frac{\sqrt{A_1}\left(J^2 + A_2 S + B_1 S^2\right)}{2A_1\sqrt{S}}\,\frac{1}{\sqrt[4]{\lambda}} + \mathcal{O}\left(\frac{1}{\lambda^{\frac{3}{4}}}\right). \tag{4.208}$$

Thus we reformulate the problem entirely in terms of the coefficients A_i, B_i, C_i, etc. For example, the next coefficient in the series, namely the two-loop term is given by

$$\Delta^{(2)} = -\frac{\left(2A_2 + 4B_1 + J^2\right)^2 - 16A_1(A_3 + 2B_2 + 4C_1)}{16\sqrt{2}A_2^{3/2}}. \tag{4.209}$$

Further coefficients become more and more complicated, however a very clear pattern can be noticed after looking at these expressions: we see that the term $\Delta^{(n)}$ only contains coefficients with indices up to $n+1$, e.g. the tree level term $\Delta^{(0)}$ only depends on A_1, the one-loop term depends on A_1, A_2, B_1, etc. Thus we can associate the index of these coefficients with the loop level. Conversely, from the last section we learned that the letter of A_i, B_i, etc. can be associated with the order in S, i.e. the slope function fixed all A_i coefficients and the curvature function in principle fixes all B_i coefficients.

Looking at (4.208) we see that knowing A_i and B_i only takes us to one loop, in order to proceed we need to know some coefficients in the C_i and D_i series. This is where the next

(S,J)	$\lambda^{-5/4}$ prediction		$\lambda^{-5/4}$ fit	error	fit order
$(2,2)$	$\frac{15\,\zeta_5}{2}+6\,\zeta_3-\frac{1}{2}$	$=14.48929958$	14.12099034	2.61%	6
$(2,3)$	$\frac{15\,\zeta_5}{2}+\frac{63\,\zeta_3}{8}-\frac{1131}{512}$	$=15.03417190$	14.88260078	1.02%	5
$(2,4)$	$\frac{21\,\zeta_3}{2}+\frac{15\,\zeta_5}{2}-\frac{25}{8}$	$=17.27355565$	16.46106336	4.94%	7

Table 2: Comparisons of strong coupling expansion coefficients for $\lambda^{-5/4}$ obtained from fits to TBA data versus our predictions for various operators. The fit order is the order of polynomials used for the rational fit function (see [?] for details).

ingredient in this construction comes in, which is the knowledge of the classical energy and its semiclassical correction in the Frolov-Tseytlin limit, i.e. when $\mathcal{S}\equiv S/\sqrt{\lambda}$ and $\mathcal{J}\equiv J/\sqrt{\lambda}$ remain fixed, while $S,\,J,\,\lambda\to\infty$. Additionally we will also be taking the limit $\mathcal{S}\to 0$ in all of the expressions that follow. In particular, the square of the classical energy has a very nice form in these limits and is given by [?, ?]

$$\mathcal{D}^2_{\text{classical}} \;=\; \mathcal{J}^2+2\,\mathcal{S}\,\sqrt{\mathcal{J}^2+1}+\mathcal{S}^2\,\frac{2\mathcal{J}^2+3}{2\mathcal{J}^2+2}-\mathcal{S}^3\,\frac{\mathcal{J}^2+3}{8\left(\mathcal{J}^2+1\right)^{5/2}}+\mathcal{O}\left(\mathcal{S}^4\right)\;, \tag{4.210}$$

where $\mathcal{D}_{\text{classical}}\equiv\Delta_{classical}/\sqrt{\lambda}$. The 1-loop correction to the classical energy is given by

$$\Delta_{sc}\simeq\frac{-\mathcal{S}}{2\left(\mathcal{J}^3+\mathcal{J}\right)}+\mathcal{S}^2\left[\frac{3\mathcal{J}^4+11\mathcal{J}^2+17}{16\mathcal{J}^3\left(\mathcal{J}^2+1\right)^{5/2}}-\sum_{\substack{m>0\\ m\neq n}}\frac{n^3m^2\left(2m^2+n^2\mathcal{J}^2-n^2\right)}{\mathcal{J}^3\left(m^2-n^2\right)^2\left(m^2+n^2\mathcal{J}^2\right)^{3/2}}\right] \tag{4.211}$$

If the parameters $\mathcal{S}$ and $\mathcal{J}$ are fixed to some values then the sum can be evaluated explicitly in terms of zeta functions. We now add up the classical and the 1-loop contributions[19], take S and J fixed at strong coupling and compare the result to (4.200). By requiring consistency we are able to extract the following coefficients,

$$\begin{array}{llllll} A_1 & = & 2, & A_2 & = & -\;1\\ B_1 & = & 3/2, & B_2 & = & -\;3\,\zeta_3+\frac{3}{8}\\ C_1 & = & -\;3/8, & C_2 & = & \frac{1}{16}\left(60\,\zeta_3+60\,\zeta_5-17\right)\\ D_1 & = & 31/64, & D_2 & = & \frac{1}{512}(-5520\,\zeta_3-5120\,\zeta_5-3640\,\zeta_7+901)\end{array}$$

As discussed in the previous section, we can in principle extract all coefficients with indices 1 and 2. In order to find e.g. B_3 we would need to extend the quantization of the classical solution to the next order. Note that the coefficients A_1, A_2 and B_1, B_2 have the same exact values that we extracted from the slope and curvature functions.

The key observation in [?] was that once written in terms of the coefficients A_i, B_i, C_i, the two-loop term $\Delta^{(2)}$ only depends on $A_{1,2,3}$, $B_{1,2}$, C_1 as can be seen in (4.209). As discussed in the last section, the one-loop result fixes all of these constants except A_3, which in principle is a contribution from a true two-loop calculation. However we already fixed it from the slope function and thus we are able to find

$$\Delta^{(2)}=\frac{-21\,S^4+(24-96\,\zeta_3)S^3+4\left(5J^2-3\right)S^2+8J^2S-4J^4}{64\sqrt{2}\,S^{3/2}}. \tag{4.212}$$

[19]Note that they mix various orders of the coupling.

Now that we know the strong coupling expansion of the curvature function and thus all the coefficients B_i, we can do the same trick and find the three loop strong coupling scaling dimension coefficient $\Delta^{(3)}$, which now depends on $A_{1;2;3;4}$, $B_{1,2,3}$, $C_{1,2}$, D_1. We find it to be

$$\begin{aligned}\Delta^{(3)} &= \frac{187\, S^6 + 2\,(624\,\zeta_3 + 480\,\zeta_5 - 193)\, S^5 + \big(-146\, J^2 - 4\,(336\,\zeta_3 - 41)\big)\, S^4}{512\sqrt{2}\, S^{5/2}} + \\ &+ \frac{\big(32\,(6\,\zeta_3 + 7)\, J^2 - 88\big)\, S^3 + \big(-28\, J^4 + 40\, J^2\big)\, S^2 - 24\, J^4 S + 8\, J^6}{512\sqrt{2}\, S^{5/2}},\end{aligned} \tag{4.213}$$

for $S = 2$ it simplifies to

$$\Delta^{(3)}_{S=2} = \frac{1}{512}\left(J^6 - 20J^4 + 48J^2(4\zeta_3 - 1) + 192(12\,\zeta_3 + 20\,\zeta_5 + 1)\right) \tag{4.214}$$

and finally for the Konishi operator, which has $S = 2$ and $J = 2$ we get[20]

$$\Delta^{(3)}_{S=2,J=2} = \frac{15\,\zeta_5}{2} + 6\,\zeta_3 - \frac{1}{2}. \tag{4.215}$$

In order to compare our predictions with data available from TBA calculations [**?**], we employed Padé type fits as explained in [**?**]. The fit results are shown in table 2, we see that our predictions are within 5% error bounds, which is a rather good agreement. However we must be honest that for the $J = 3$ and especially $J = 4$ states we did not have as many data points as for the $J = 2$ state and the fit is somewhat shaky.

[20] The ζ_3 and ζ_5 terms are coming from semi-classics and were already known before [**?**] and match our result.

5 ABJM

5.1 Algebraic curve quantization in $AdS_4 \times \mathbb{CP}^3$

We give in this section a compact self-contained summary of the results of [?] using the language of off-shell fluctuation energies [?]. We shall work in the algebraic curve regularization and write all equations in terms of the σ-model coupling g. For large g, it is related to the 't Hooft coupling by

$$\lambda = N/k = 8\,g^2, \tag{5.1}$$

but, contrary to the $AdS_5 \times S^5$ case, this relation will get corrections at finite g. The classical algebraic curve for $AdS_4 \times \mathbb{CP}^3$ is a 10-sheeted Riemann surface. The spectral parameter moves on it and we shall consider 10 symmetric quasi momenta $q_i(x)$

$$(q_1, q_2, q_3, q_4, q_5) = (-q_{10}, -q_9, -q_8, -q_7, -q_6). \tag{5.2}$$

They can have branch cuts connecting the sheets with

$$q_i^+ - q_j^- = 2\,\pi\,n_{ij}. \tag{5.3}$$

In the terminology of [?], the physical polarizations (ij) can be split into *heavy* and *light* ones and are summarized in the following table:

	AdS_4	Fermions	$\mathbb{CP}^3$
heavy	$(1,10)(2,9)(1,9)$	$(1,7)(1,8)(2,7)(2,8)$	$(3,7)$
light		$(1,5)(1,6)(2,5)(2,6)$	$(3,5)(3,6)(4,5)(4,6)$

Virasoro constraints require that the poles of the quasi-momenta $q_i(x)$ at $x = \pm 1$ are synchronized according to

$$(q_1, q_2, q_3, q_4, q_5) = \frac{\alpha_+}{x-1}\,(1,1,1,1,0) + \cdots = \frac{\alpha_-}{x+1}\,(1,1,1,1,0) + \cdots . \tag{5.4}$$

Inversion symmetry reads

$$q_1(x) = -q_2(1/x), \qquad q_3(x) = 2\,\pi\,m - q_4(1/x), \qquad q_5(x) = q_5(1/x), \tag{5.5}$$

where $m \in \mathbb{Z}$ is a winding number. The asymptotic values of the quasi-momenta for a length L state with energy and spin E, S are

$$\begin{pmatrix} q_1(x) \\ q_2(x) \\ q_3(x) \\ q_4(x) \\ q_5(x) \end{pmatrix} = \frac{1}{2\,g\,x} \begin{pmatrix} L+E+S \\ L+E-S \\ L-M_r+M_s \\ L+M_r-M_u-M_v \\ M_v-M_u \end{pmatrix} + \cdots, \tag{5.6}$$

where $M_{r,u,v}$ are related to the $SU(4)$ representation of the state

$$[d_1, d_2, d_3] = [L-2M_u+M_r, M_u+M_v-2M_r+M_s, L-2M_v+M_r]. \tag{5.7}$$

5.2 Semiclassical quantization

Semiclassical quantization is achieved by perturbing quasi-momenta introducing extra poles that shift the quasi-momenta $q_i \to q_i + \delta q_i$. Virasoro constraints and inversion properties of the variations δq_i follow from those of the q_i's. In order to find the asymptotic expression of δq_i in terms of the number N_{ij} of extra fluctuations we can look at the details of polarized states and obtain

$$\begin{pmatrix} \delta q_1(x) \\ \delta q_2(x) \\ \delta q_3(x) \\ \delta q_4(x) \\ \delta q_5(x) \end{pmatrix} = \frac{1}{2\,g\,x} \begin{pmatrix} \delta E + N_{19} + 2\,N_{1,10} & +N_{15} + N_{16} + N_{17} + N_{18} & \\ \delta E + 2\,N_{29} + N_{19} & +N_{25} + N_{26} + N_{27} + N_{28} & \\ & -N_{18} - N_{28} & -N_{35} - N_{36} - N_{37} \\ & -N_{17} - N_{27} & -N_{45} - N_{46} - N_{37} \\ & +N_{15} - N_{16} + N_{25} - N_{26} & +N_{35} - N_{36} + N_{45} - N_{46} \end{pmatrix}. \tag{5.8}$$

The off-shell frequencies $\Omega^{ij}(x)$ are defined in order to have

$$\delta E = \sum_{n,ij} N_n^{ij}\,\Omega^{ij}(x_n^{ij}), \tag{5.9}$$

where the sum is over all pairs $(ij) \equiv (ji)$ of physical polarizations and integer values of n with

$$q_i(x_n^{ij}) - q_j(x_n^{ij}) = 2\,\pi\,n. \tag{5.10}$$

Also, the residues at the extra poles are

$$\delta q_i(x) = k_{ij}\,N_n^{ij}\,\frac{\alpha(x_n^{ij})}{x - x_n^{ij}}, \quad \text{with} \quad \alpha(x) = \frac{1}{2\,g}\,\frac{x^2}{x^2-1}, \tag{5.11}$$

and $k_{ij} = 0, \pm 1, \pm 2$ are the coefficients of N_{ij} in (5.8). By linear combination of frequencies and inversion (as in the $\mathrm{AdS}_5/\mathrm{CFT}_4$ case), we can derive all the off-shell frequencies in terms of two fundamental ones

$$\Omega_A(x) = \Omega^{15}(x), \qquad \Omega_B(x) = \Omega^{45}(x). \tag{5.12}$$

Their explicit expressions turns out to be

$$
\begin{aligned}
\Omega^{29} &= 2\left[-\Omega_A(1/x)+\Omega_A(0)\right], \\
\Omega^{1,10} &= 2\,\Omega_A(x), \\
\Omega^{19} &= \Omega_A(x)-\Omega_A(1/x)+\Omega_A(0), \\
\Omega^{37} &= \Omega_B(x)-\Omega_B(1/x)+\Omega_B(0), \\
\Omega^{35}=\Omega^{36} &= -\Omega_B(1/x)+\Omega_B(0), \\
\Omega^{45}=\Omega^{46} &= \Omega_B(x), \\
\Omega^{17} &= \Omega_A(x)+\Omega_B(x), \\
\Omega^{18} &= \Omega_A(x)-\Omega_B(1/x)+\Omega_B(0), \\
\Omega^{27} &= \Omega_B(x)-\Omega_A(1/x)+\Omega_A(0), \\
\Omega^{28} &= -\Omega_A(1/x)+\Omega_A(0)-\Omega_B(1/x)+\Omega_B(0), \\
\Omega^{15}=\Omega^{16} &= \Omega_A(x), \\
\Omega^{25}=\Omega^{26} &= -\Omega_A(1/x)+\Omega_A(0).
\end{aligned}
\tag{5.13}
$$

5.3 The folded string in $AdS_4 \times \mathbb{CP}^3$

We present the algebraic curve for the folded string in $AdS_4 \times \mathbb{CP}^3$ closely following the notation of [?]. In terms of the semiclassical variables

$$
\mathcal{S}=\frac{S}{4\,\pi\,g}, \qquad \mathcal{J}=\frac{J}{4\,\pi\,g}, \tag{5.14}
$$

the energy of the folded string can be expanded according to

$$
E = 4\,\pi\,g\;\mathcal{E}_0(\mathcal{J},\mathcal{S})+E_1(\mathcal{J},\mathcal{S})+\mathcal{O}\left(\frac{1}{g}\right), \tag{5.15}
$$

where the small $\mathcal{S}$ expansion of the classical contribution $\mathcal{E}_0$ reads

$$
\mathcal{E}_0=\mathcal{J}+\frac{\sqrt{\mathcal{J}^2+1}}{\mathcal{J}}\,\mathcal{S}-\frac{\mathcal{J}^2+2}{4\,\mathcal{J}^3(\mathcal{J}^2+1)}\,\mathcal{S}^2+\frac{3\,\mathcal{J}^6+13\,\mathcal{J}^4+20\,\mathcal{J}^2+8}{16\,\mathcal{J}^5\,(\mathcal{J}^2+1)^{5/2}}\,\mathcal{S}^3+\cdots. \tag{5.16}
$$

5.4 Quasi-momenta

The quasi momenta are closely related to those of the $AdS_5 \times S^5$ folded string since motion is still in $AdS_3 \times S^1$ and the $\mathbb{CP}^3$ part of the background plays almost no role. The only non trivial case is

$$
\begin{aligned}
q_1(x) &= \pi\,f(x)\left\{-\frac{J}{4\,\pi\,g}\left(\frac{1}{f(1)\,(1-x)}-\frac{1}{f(-1)(1+x)}\right)+\right. \\
&\quad -\frac{4}{\pi\,(a+b)(a-x)(a+x)}\left[(x-a)\,\mathbb{K}\left(\frac{(b-a)^2}{(b+a)^2}\right)+\right. \\
&\quad \left.\left.+2\,a\,\Pi\left(\frac{(b-a)(a+x)}{(a+b)(x-a)}\middle|\,\frac{(b-a)^2}{(b+a)^2}\right)\right]\right\}-\pi.
\end{aligned}
\tag{5.17}
$$

where the branch points obey $1 < a < b$ and

$$f(x) = \sqrt{x-a}\,\sqrt{x+a}\,\sqrt{x-b}\,\sqrt{x+b}, \tag{5.18}$$

$$\begin{aligned} S &= 2\,g\,\frac{ab+1}{ab}\left[b\,\mathbb{E}\left(1-\frac{a^2}{b^2}\right)-a\,\mathbb{K}\left(1-\frac{a^2}{b^2}\right)\right], \\ J &= \frac{4\,g}{b}\,\sqrt{(a^2-1)(b^2-1)}\,\mathbb{K}\left(1-\frac{a^2}{b^2}\right). \\ E &= 2\,g\,\frac{ab-1}{ab}\left[b\,\mathbb{E}\left(1-\frac{a^2}{b^2}\right)+a\,\mathbb{K}\left(1-\frac{a^2}{b^2}\right)\right]. \end{aligned} \tag{5.19}$$

The other quasi-momenta are

$$q_2(x) = -q_1(1/x), \tag{5.20}$$

$$q_3(x) = q_4(x) = \frac{J}{2\,g}\,\frac{x}{x^2-1}. \tag{5.21}$$

$$q_5(x) = 0. \tag{5.22}$$

The above expressions are valid for a folded string with minimal winding. Adding winding is trivial at the classical level, but requires non trivial changes at the one-loop level (see for instance [?] for a detailed analysis of the $AdS_5 \times S^5$ case).

The independent off-shell frequencies can be determined by the methods of [?]. The result is rather simple and reads [21]

$$\Omega_A(x) = \frac{1}{ab-1}\left(1-\frac{f(x)}{x^2-1}\right), \tag{5.23}$$

$$\Omega_B(x) = \frac{\sqrt{a^2-1}\,\sqrt{b^2-1}}{ab-1}\,\frac{1}{x^2-1}. \tag{5.24}$$

5.5 Integral representation for the one-loop correction to the energy

The one-loop shift of the energy is given in full generality by the following sum of zero point energies

$$E_1 = \frac{1}{2}\sum_{n=-\infty}^{\infty}\sum_{ij}(-1)^{F_{ij}}\,\omega_n^{ij}, \qquad \omega_n^{ij} = \Omega^{ij}(x_n^{ij}), \tag{5.25}$$

where the sum over ij is over the $8_B + 8_F$ physical polarizations and x_n^{ij} is the unique solution to the equation (5.10) under the condition $|x_n^{ij}| > 1$ [22].

In the same spirit as [?, ?], the infinite sum over on-shell frequencies can be evaluated by contour integration in the complex plane. The result is quite similar to the $AdS_5 \times S^5$ one and reads

$$E_1 = E_1^{\text{anomaly},1} + E_1^{\text{anomaly},2} + E_1^{\text{dressing}} + E_1^{\text{wrapping}}, \tag{5.26}$$

[21] Notice the important relation $\Omega_B(x) = -\Omega_B(1/x) + \Omega_B(0)$.

[22] If it happens that for some ij and n the above equation has no solution, then we shall say that the polarization (ij) has the missing mode n. Missing modes can be treated according to the procedure discussed in [?].

with [23]

$$
\begin{aligned}
E_1^{\text{anomaly},1} &= 2\int_a^b \frac{dx}{2\,\pi\,i}\left[\Omega^{1,10}(x)-\Omega^{1,10}(a)\right]\,\partial_x\,\log\sin q_1(x), && (5.27)\\
E_1^{\text{anomaly},2} &= -2\times 2\int_a^b \frac{dx}{2\,\pi\,i}\left[\Omega^{1,5}(x)-\Omega^{1,5}(a)\right]\,\partial_x\,\log\sin\frac{q_1(x)}{2}, && (5.28)\\
E_1^{\text{dressing}} &= \sum_{ij}(-1)^{F_{ij}}\int_{-1}^{1}\frac{dz}{2\,\pi\,i}\,\Omega^{ij}(z)\,\partial_z\frac{i\,[q_i(z)-q_j(z)]}{2}, && (5.29)\\
E_1^{\text{wrapping}} &= \sum_{ij}(-1)^{F_{ij}}\int_{-1}^{1}\frac{dz}{2\,\pi\,i}\,\Omega^{ij}(z)\,\partial_z\log(1-e^{-i\,(q_i(z)-q_j(z))}), && (5.30)
\end{aligned}
$$

As in $AdS_5\times S^5$, the labeling of the various contributions reminds their physical origin. In particular, dressing and wrapping contributions have been separated in order to split the asymptotic contribution from finite size effects. As in $AdS_5\times S^5$, the anomaly terms are special contributions arising from the deformation of contours and ultimately due to the presence of the algebraic curve cuts. The representation (5.25) is a compact formula for E_1 and can be evaluated numerically with minor effort. In order to understand it better, we shall now analyze the short and long string limit. In the former case, we shall evaluate the explicit sum over frequencies clarifying the relation with the contour integrals. In the latter, we shall extract the analytical expansion at large spin directly from (5.25).

5.6 Short string limit

The short string limit is generically $\mathcal{S}\to 0$. Regarding $\mathcal{J}$, we shall consider two cases. The first amounts to keeping $\mathcal{J}$ fixed, expanding in the end each coefficient of powers of $\mathcal{S}$ at small $\mathcal{J}$. This is precisely the procedure worked out in [?] in $AdS_5\times S^5$. In the second case, we shall keep the ratio $\rho=\mathcal{J}/\sqrt{\mathcal{S}}$ fixed as in [?]. The two expansions are related, but not equivalent and provide useful different information.

[23]Here, $x(z)=z+\sqrt{z^2-1}$. Also the anomaly contributions are computed integrating on the upper half complex plane.

5.7 Fixed $\mathcal{J}$ expansion

After a straightforward computation, our main result is

$$\begin{aligned} E_1 \;=\; & \Bigg(-\frac{1}{2\mathcal{J}^2} + \frac{\log(2)-\frac{1}{2}}{\mathcal{J}} + \frac{1}{4} + \mathcal{J}\left(-\frac{3\,\zeta(3)}{8} + \frac{1}{2} - \frac{\log(2)}{2}\right) - \frac{3\mathcal{J}^2}{16} + \\ & + \mathcal{J}^3\left(\frac{3\,\zeta(3)}{16} + \frac{45\,\zeta(5)}{128} - \frac{1}{2} + \frac{3\log(2)}{8}\right) + \cdots \Bigg)\mathcal{S} + \\ & + \Bigg(\frac{3}{4\mathcal{J}^4} + \frac{\frac{1}{2}-\log(2)}{\mathcal{J}^3} - \frac{1}{8\mathcal{J}^2} + \frac{\frac{1}{16}-\frac{3\,\zeta(3)}{4}}{\mathcal{J}} - \frac{1}{8} + \mathcal{J}\left(\frac{69\,\zeta(3)}{64} + \frac{165\,\zeta(5)}{128} - \frac{27}{32} + \frac{\log(2)}{2}\right) + \\ & + \frac{3\mathcal{J}^2}{8} + \mathcal{J}^3\left(-\frac{163\,\zeta(3)}{128} - \frac{405\,\zeta(5)}{256} - \frac{875\,\zeta(7)}{512} + \frac{235}{128} - \log(2)\right) + \cdots \Bigg)\mathcal{S}^2 + \\ & + \Bigg(-\frac{5}{4\mathcal{J}^6} + \frac{\frac{3\log(2)}{2} - \frac{3}{4}}{\mathcal{J}^5} + \frac{\frac{9\,\zeta(3)}{16} + \frac{1}{16}}{\mathcal{J}^3} + \frac{1}{16\mathcal{J}^2} + \frac{\frac{45\,\zeta(3)}{64} + \frac{75\,\zeta(5)}{256} - \frac{7}{32} + \frac{\log(2)}{8}}{\mathcal{J}} + \frac{11}{64} + \\ & + \mathcal{J}\left(-\frac{89\,\zeta(3)}{32} - \frac{745\,\zeta(5)}{256} - \frac{3815\,\zeta(7)}{2048} + 2 - \frac{33\log(2)}{32}\right) - \frac{465\mathcal{J}^2}{512} + \\ & + \mathcal{J}^3\left(\frac{5833\,\zeta(3)}{1024} + \frac{1585\,\zeta(5)}{256} + \frac{98035\,\zeta(7)}{16384} + \frac{259455\,\zeta(9)}{65536} - \frac{405}{64} + \frac{775\log(2)}{256}\right) + \cdots \Bigg)\mathcal{S}^3 + \cdots \end{aligned} \quad (5.31)$$

This expansion is rather similar to the one derived in [?] for $AdS_5 \times S^5$, but there are two remarkable differences:

1. The leading terms at small $\mathcal{J}$ are $\mathcal{O}(\mathcal{S}^n/\mathcal{J}^{2n})$. Instead, they were $\mathcal{O}(\mathcal{S}^n/\mathcal{J}^{2n-1})$ in $AdS_5 \times S^5$. Also, there are terms with all parities in $\mathcal{J}$ while in $AdS_5 \times S^5$, there appear only terms odd under $\mathcal{J} \to -\mathcal{J}$. The additional terms are important and we shall discuss them in more details later. Remarkably, they imply that if one scales $\mathcal{J} \sim \sqrt{\mathcal{S}}$ they give a constant contribution in the short string limit. This is different from $AdS_5 \times S^5$ where the energy correction vanishes like $\sqrt{\mathcal{S}}$ in this regime.
2. There are terms proportional to log(2). As we discuss in App. (**??**), these terms can be removed by expressing the energy correction in terms of the coupling in the world-sheet scheme. The scheme dependence is universal and agrees with that found in [?] for a circular string solution and in [?] for the giant magnon.

5.8 Fixed $\rho = \mathcal{J}/\sqrt{\mathcal{S}}$ expansion

The result in this limit is

$$E_1 = -\frac{1}{2}\,\mathcal{C}(\rho,\mathcal{S}) + a_{01}(\rho)\,\sqrt{\mathcal{S}} + a_{1,1}(\rho)\,\mathcal{S}^{3/2} + \mathcal{O}(\mathcal{S}^{5/2}), \quad (5.32)$$

where

$$a_{1,0}(\rho) \;=\; \frac{2\,\log(2)-1}{2\,\sqrt{\rho^2+2}}, \quad (5.33)$$

$$a_{1,1}(\rho) \;=\; -\frac{\log(2)\left(2\rho^4+6\rho^2+3\right)}{4\left(\rho^2+2\right)^{3/2}} + \frac{8\rho^4+25\rho^2+16}{16\left(\rho^2+2\right)^{3/2}} - \frac{3\left(\rho^2+3\right)\zeta(3)}{8\sqrt{\rho^2+2}}, \quad (5.34)$$

and $\mathcal{C}$ is related to the branch cut endpoints by the formula

$$\mathcal{C} = \frac{\sqrt{(a^2-1)\,(b^2-1)}}{1-a\,b} + 1. \tag{5.35}$$

Its expansion at small $\mathcal{S}$ with fixed $\rho = \mathcal{J}/\sqrt{\mathcal{S}}$ is

$$\mathcal{C} = 1 - \frac{\rho}{\sqrt{\rho^2+2}} - \frac{2\,\rho^3+5\,\rho}{4\,(\rho^2+2)^{3/2}}\,\mathcal{S} + \frac{\rho\,(12\,\rho^6+68\,\rho^4+126\,\rho^2+73)}{32\,(\rho^2+2)^{5/2}}\,\mathcal{S}^2 + \cdots \tag{5.36}$$

Expanding E_1 at large ρ we partially resum the calculation at fixed $\mathcal{J}$. Just to give an example, from the expansion

$$-\frac{1}{2}\left(1-\frac{\rho}{\sqrt{\rho^2+2}}\right) = -\frac{1}{2\rho^2}+\frac{3}{4\rho^4}-\frac{5}{4\rho^6}+\frac{35}{16\rho^8}-\frac{63}{16\rho^{10}}+\cdots\,, \tag{5.37}$$

we read the coefficients of **all** terms $\sim \mathcal{S}^n/\mathcal{J}^{2n}$. The first ones are of course in agreement with (5.31). As another non trivial example, the large ρ expansion of $a_{11}(\rho)$ is

$$\begin{aligned} a_{11}(\rho) \quad &= \quad \rho\left(-\frac{3\zeta(3)}{8}+\frac{1}{2}-\frac{\log(2)}{2}\right)+\frac{\frac{1}{16}-\frac{3\zeta(3)}{4}}{\rho}+\frac{\frac{9\zeta(3)}{16}+\frac{1}{16}}{\rho^3}+ \\ &\quad +\frac{-\frac{3\zeta(3)}{4}-\frac{1}{32}-\frac{\log(2)}{4}}{\rho^5}+\frac{\frac{75\zeta(3)}{64}-\frac{5}{32}+\frac{15\log(2)}{16}}{\rho^7}+\cdots\,, \end{aligned} \tag{5.38}$$

and allows to read the coefficients of all terms $\sim \mathcal{S}^n/\mathcal{J}^{2n-3}$.

5.9 Summation issues

The explicit sum over the infinite number of on-shell frequencies requires some care and a definite prescription since the sums are not absolutely convergent due to physically sensible cancellations between bosonic and fermionic contributions. As discussed in [**?**], the following summation prescription is natural from the point of view of the algebraic curve (see [**?**] for a different prescription) [24]

$$E_1 = \sum_{n=1}^{\infty} K_n, \tag{5.39}$$

where K_n is a particular grouping of heavy and light modes

$$K_n = \begin{cases} \omega_n^{\text{heavy}} + \omega_{n/2}^{\text{light}} & n \in 2\,\mathbb{Z} \\ \\ \omega_n^{\text{heavy}} & n \notin 2\,\mathbb{Z}, \end{cases} \tag{5.40}$$

with

$$\omega_n^{\text{heavy}} = \omega_n^{(AdS,1)} + \omega_n^{(AdS,2)} + \omega_n^{(AdS,3)} + \omega_n^{(\mathbb{CP},1)} - 2\,\omega_n^{(F,1)} - 2\,\omega_n^{(F,2)}, \tag{5.41}$$

$$\omega_n^{\text{light}} = 4\,\omega_n^{(\mathbb{CP},2)} - 2\,\omega_n^{(F,3)} - 2\,\omega_n^{(F,4)}. \tag{5.42}$$

[24] Notice that we exploit the $x \to -x$ symmetry of the classical algebraic curve as well as triviality of zero mode corrections.

The short string expansion of K_n takes the form

$$K_p \quad = \quad (-1)^p\,\mathcal{C} + \widehat{K}_p \tag{5.43}$$

where $\mathcal{C}$, given in (5.35), is independent on p and the sum of $\widehat{K}_p$ (which start at $\mathcal{O}(\mathcal{S})$) is convergent. The alternating constant $\mathcal{C}$ poses some problems because we have to give a meaning to

$$-\,\mathcal{C} + \mathcal{C} - \mathcal{C} + \mathcal{C} + \cdots\,. \tag{5.44}$$

An analysis of the integral representation shows that it automatically selects the choice

$$-\,\mathcal{C} + \mathcal{C} - \mathcal{C} + \mathcal{C} + \cdots \equiv -\frac{1}{2}\,\mathcal{C} \tag{5.45}$$

Later, we shall provide various consistency checks of this prescription. In particular, we shall see that it is necessary in order to match the asymptotic Bethe Ansatz equations when wrapping effects are subtracted. Notice also that the expansion of $\mathcal{C}$ at fixed $\mathcal{J}$ is

$$\mathcal{C} = \frac{\mathcal{S}}{\mathcal{J}^2\sqrt{\mathcal{J}^2+1}} - \frac{\left(3\mathcal{J}^4+11\mathcal{J}^2+6\right)\mathcal{S}^2}{4\mathcal{J}^4\left(\mathcal{J}^2+1\right)^2} + \frac{12\,\mathcal{J}^8+75\,\mathcal{J}^6+173\,\mathcal{J}^4+140\,\mathcal{J}^2+40}{16\,\mathcal{J}^6\left(\mathcal{J}^2+1\right)^{7/2}}\,\mathcal{S}^3 + \cdots\,, \tag{5.46}$$

so, upon expanding at small $\mathcal{J}$, it provides precisely the terms with even/odd $\mathcal{J}$ exponents in the coefficients of the odd/even powers of $\mathcal{S}$ in (5.31).

Apart from the $\mathcal{C}$ term, the integral representation implements the Gromov-Mikhailov (GM) prescription. The reason is that the singularities at $|x| = 1$ are avoided by implicitly encircling them by a small circumference. This cut-off on $|x-1|$ translates in a bound on the highest mode n that correlates heavy/light polarizations according to GM. In other words the highest mode for light polarizations is asymptotically half the highest mode for heavy polarizations.

As a numerical check of the agreement between the integral representation and the series expansion, we fix $\rho = 1$ in table (3) and show the value of E_1 from our analytical resummation and result from the integral. The agreement is very good already at moderately small $\mathcal{S}$.

$\mathcal{S}$	E_1 from (5.32)	E_1
1/10	-0.18790	-0.17987
1/50	-0.19461	-0.19443
1/100	-0.19934	-0.19930
1/300	-0.20449	-0.20448
1/500	-0.206075	-0.206075

Table 3: Comparison between resummation at fixed ratio $\rho = 1$ and integral representation. The asymptotic value for $\mathcal{S} \to 0$ is $(\sqrt{3}-3)/6 \simeq -0.211$, but already at $\mathcal{S} = 1/500$ we have 6 digits agreement.

A similar check at fixed $\mathcal{J}$ is shown in Fig. (**??**) where we plot the asymptotic expansion (5.31) and the exact numerical E_1 as functions of $\mathcal{S}$ at $\mathcal{J} = 1/5$.

5.10 The slope function

The one-loop correction E_1 tends to zero linearly with $\mathcal{S}$ when $\mathcal{S} \to 0$ at fixed $\mathcal{J}$. The slope ratio

$$\sigma(\mathcal{J}) = \lim_{\mathcal{S}\to 0} \frac{E_1(\mathcal{S},\mathcal{J})}{\mathcal{S}}, \tag{5.47}$$

is an important quantity related to the conjectures in [68] [25] . It is known that it does not receive dressing corrections both in $AdS_5 \times S^5$ and in $AdS_4 \times \mathbb{CP}^3$ since such contributions start at order $\mathcal{S}^2$ [68]. It also does not receive wrapping corrections in AdS_5. Instead, in the case of AdS_4 the slope has a non vanishing wrapping contribution. For instance, a rough evaluation at $\mathcal{J} = 1$ gives a definitely non zero value around -0.042.

Indeed, an analytical calculation shows that the wrapping contribution to the slope in $AdS_4 \times \mathbb{CP}^3$ is exactly

$$\sigma^{\rm wrap}(\mathcal{J}) = \sum_{n=-\infty}^{\infty} \sigma_n = -\frac{1}{2\,\mathcal{J}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\sqrt{\mathcal{J}^4 + (n^2+1)\,\mathcal{J}^2 + n^2}}. \tag{5.48}$$

This formula is in perfect agreement with numerics since for instance

$$\sigma^{\rm wrap}(\mathcal{J} = 1) = -0.04177765487955882481 4\dots. \tag{5.49}$$

The large $\mathcal{J}$ limit of this expression is exponentially suppressed as it should

$$\sigma^{\rm wrap}(\mathcal{J}) = -\frac{\sqrt{2}}{\mathcal{J}^{5/2}}\, e^{-\pi\,\mathcal{J}} + \cdots \tag{5.50}$$

To analyze the small $\mathcal{J}$ limit it is convenient to split this contribution into the $n = 0$ term plus the rest. The result is very intriguing. For the $n = 0$ term, we find

$$\sigma^{\rm wrap}_{n=0} = -\frac{1}{2\,\mathcal{J}^2\sqrt{\mathcal{J}^2+1}} = -\frac{1}{2\mathcal{J}^2} + \frac{1}{4} - \frac{3\mathcal{J}^2}{16} + \frac{5\mathcal{J}^4}{32} - \frac{35\mathcal{J}^6}{256} + \frac{63\mathcal{J}^8}{512} + \cdots. \tag{5.51}$$

This is precisely the set of terms even under $\mathcal{J} \to -\mathcal{J}$ in the full slope which is the first term of (5.31). Similarly, we can consider the rest of $\sigma^{\rm wrap}$ and expand at small $\mathcal{J}$. We find

$$\begin{aligned}\sum_{n\neq 0} \sigma^{\rm wrap}_n \;=\;& \frac{\log(2)}{\mathcal{J}} + \mathcal{J}\left(-\frac{3\zeta(3)}{8} - \frac{\log(2)}{2}\right) + \mathcal{J}^3\left(\frac{3\zeta(3)}{16} + \frac{45\zeta(5)}{128} + \frac{3\log(2)}{8}\right) + \\ & + \mathcal{J}^5\left(-\frac{9\zeta(3)}{64} - \frac{45\zeta(5)}{256} - \frac{315\zeta(7)}{1024} - \frac{5\log(2)}{16}\right) + O\left(\mathcal{J}^6\right).\end{aligned} \tag{5.52}$$

Comparing again with (5.31), we see that we are reproducing all the irrational terms of the slope, involving zeta functions or log(2). The remaining terms are the same as in $AdS_5 \times S^5$ [26],

$$\sigma(\mathcal{J}) - \sigma^{\rm wrap}(\mathcal{J}) = -\frac{1}{2\,\mathcal{J}} + \frac{\mathcal{J}}{2} - \frac{\mathcal{J}^3}{2} + \cdots. \tag{5.53}$$

[25] The exact slope mentioned in [68] is the coefficient of S in the expansion of E^2. This line of analysis is suggested by the simplicity of the marginality condition in $AdS_5 \times S^5$ (see [?] for a general discussion). Here, it is simpler to discuss the quantity $\sigma(\mathcal{J})$.

[26] This is due to the fact that the BAE are essentially the same as for $\mathfrak{sl}(2)$ sector in $AdS_5 \times S^5$. This is however a nontrivial test that all is done correctly.

Thus, we are led to the following expression for the one-loop full slope

$$\sigma(\mathcal{J}) = -\frac{1}{2\,\mathcal{J}}\left[\frac{1}{\mathcal{J}^2+1} + \sum_{n=-\infty}^{\infty}\frac{(-1)^n}{\sqrt{\mathcal{J}^4+(n^2+1)\,\mathcal{J}^2+n^2}}\right]. \tag{5.54}$$

The above analysis of the slope is a confirmation that the various terms in (5.31) are organized in the expected way. The asymptotic contribution is precisely the same as in $AdS_5\times S^5$, while wrapping is different and is exponentially suppressed for large operators. This is a property of the integral representation and a confirm that the prescription (5.45) is correct.

5.11 Weak coupling

It is interesting to evaluate the slope at weak coupling. In principle, this requires the knowledge of the anomalous dimensions of short $\mathfrak{sl}(2)$ operators in closed form as a function of the spin at a certain length (*i.e.* twist, in the gauge theory dictionary). This information is available for the asymptotic contribution, but not for the wrapping, which is only known as a series expansion at large spin and low twist [**?**, **?**]. Nevertheless, if we are interested in the correction to the slope only (so, just the first term at small spin), then the Lüscher form of the wrapping correction presented in [**?**] is enough [27]. At twist-1, and following the notation of [**?**], the wrapping correction enters at four loops and is expressed by the following function of the integer spin N of the gauge theory operator

$$\gamma_4^{\text{wrapping}}(N) = \gamma_2(N)\,\mathcal{W}(N), \qquad \gamma_2(N) = 4\,[S_1(N) - S_{-1}(N)]. \tag{5.55}$$

Here, $S_a(N)$ are generalized harmonic sums while $\mathcal{W}(N)$ is a complicated expression depending on the Baxter polynomial $Q_N(u)$ associated with the Bethe roots. The first factor $\gamma_2(N)$ is nothing but the two-loop anomalous dimension of the twist-1 operators. In the small N limit, it starts at $\mathcal{O}(N)$. Thus, the factor $\mathcal{W}(N)$ can be evaluated at $N=0$ where the Baxter polynomial trivializes $Q_0(u)=1$. After a straightforward calculation, one finds that (on the even N branch),

$$\gamma_4^{\text{wrapping}}(N) = -\frac{\pi^4}{3}\,N + \mathcal{O}(N^2). \tag{5.56}$$

So, even at weak coupling, we find a correction to the slope coming from the wrapping terms [28].

5.12 Long string limit

The large S behaviour of the one-loop energy E_1 can be computed starting from the integral representation. Let us first summarize the result valid for $AdS_5\times S^5$ from [**?**]. We scale $\mathcal{J}$

[27] We kindly thank B. Basso for this important remark.

[28] Notice that the reason why such a contribution is absent in $AdS_5\times S^5$ is simply that the factor analogous to $\gamma_2(N)$ is squared in the wrapping contribution. This leads immediately to a contribution to the slope of order $\mathcal{O}(N^2)$.

with $\mathcal{S}$ for $\mathcal{S} \gg 1$ according to

$$\mathcal{J} = \frac{\ell}{\pi} \log\left(\frac{8\,\pi\,\mathcal{S}}{\sqrt{\ell^2+1}}\right), \tag{5.57}$$

where we assume $\ell > 0$ [29]. Then, the one-loop energy correction can be written

$$E_1^{AdS_5} = f_{10}^{AdS_5}(\ell)\,\log\left(\frac{8\,\pi\,\mathcal{S}}{\sqrt{\ell^2+1}}\right) + f_{11}^{AdS_5}(\ell) + \frac{c^{AdS_5}}{\log\left(\frac{8\,\pi\,\mathcal{S}}{\sqrt{\ell^2+1}}\right)} + \cdots\,, \tag{5.58}$$

with

$$\begin{aligned} f_{10}^{AdS_5}(\ell) &= \frac{\sqrt{\ell^2+1} + 2\left(\ell^2+1\right)\log\left(\frac{1}{\ell^2}+1\right) - \left(\ell^2+2\right)\log\left(\frac{\sqrt{\ell^2+2}}{\sqrt{\ell^2+1}-1}\right) - 1}{\pi\sqrt{\ell^2+1}}, \\ f_{11}^{AdS_5}(\ell) &= \frac{2\left(\log\left(1-\frac{1}{(\ell^2+1)^2}\right) + 2\sqrt{\ell^2+1}\cot^{-1}\left(\sqrt{\ell^2+1}\right) + 2\coth^{-1}\left(\sqrt{\ell^2+1}\right) - 2\ell\cot^{-1}(\ell)\right)}{\pi\sqrt{\ell^2+1}}., \\ c^{AdS_5}(\ell) &= -\frac{\pi}{12\left(\ell^2+1\right)}. \end{aligned} \tag{5.59}$$

The expansion in $AdS_4 \times \mathbb{CP}^3$ can be derived in the same way as in [?] and the result is simply

$$E_1^{AdS_4} = f_{10}^{AdS_4}(\ell)\,\log\left(\frac{8\,\pi\,\mathcal{S}}{\sqrt{\ell^2+1}}\right) + f_{11}^{AdS_4}(\ell) + \frac{c^{\mathrm{AdS}_4}}{\log\left(\frac{8\,\pi\,\mathcal{S}}{\sqrt{\ell^2+1}}\right)} + \cdots\,, \tag{5.60}$$

with

$$\begin{aligned} f_{10}^{AdS_4}(\ell) &= \frac{1}{2}\,f_{10}^{AdS_5}(\ell), \\ f_{11}^{AdS_4}(\ell) &= \frac{1}{2}\,f_{11}^{AdS_5}(\ell), \\ c^{AdS_4}(\ell) &= 2\,c^{AdS_5}(\ell) = -\frac{\pi}{6\left(\ell^2+1\right)}. \end{aligned} \tag{5.61}$$

This formula can be easily checked numerically from the explicit evaluation of the integral representation. Notice that the simple $\frac{1}{2}$ rule for the leading two terms is in agreement with the result of [?]. The correction $\sim 1/\log\mathcal{S}$ comes from the anomaly terms. It is twice bigger than in SYM.

The explanation of this fact is as follows [30]. The low energy effective theory of the Gubser-Klebanov-Polyakov (GKP) string in $AdS_4 \times \mathbb{CP}^3$ has two massless modes at finite chemical potential ℓ. Namely, one massless Dirac Fermion and one massless boson that gives a central charge 2 (in $AdS_5 \times S^5$ one has only one massless boson giving central charge 1). Also, concerning the $\ell \to 0$ limit, the other low-energy modes acquire a mass proportional to ℓ at small ℓ and their contribution is exponentially suppressed with the effective length $\log\mathcal{S}$ at fixed ℓ. When $\ell \to 0$ they become massless and contribute at leading order to 5 units of central charge (there are actually 4 bosons with mass ℓ and one with mass $\ell/2$ while there were only four with mass ℓ in $AdS_5 \times S^5$). In other words it should be true that in the small ℓ limit the $1/\log\mathcal{S}$ gets corrected by 5 extra units of central charge giving a total $-(2+5)\frac{\pi}{12\,\log\mathcal{S}}$ for the energy of the vacuum state (i.e. the twist 1 state of the theory). Indeed, $2+5=7$ is the correct

[29] This means that the case $\ell = 0$, or $\mathcal{J} = 0$ has to be treated separately as discussed in [?].

[30] We thank B. Basso for clarifying this point as well as the $\ell \to 0$ limit.

central charge of the low-energy effective theory on the GKP background [?, ?]. Instead, in $AdS_5 \times S^5$ the final result for $\ell \to 0$ (i.e. for twist 2) was coming with $1+4=5$ units of central charge, which is the correct central charge of the $O(6)$ model.

5.13 Relation with marginality condition

Let us define $\Lambda \equiv \lambda$ in $AdS_5 \times S^5$, and $\Lambda = 16\,\pi^2\,g^2$ in $AdS_4 \times \mathbb{CP}^3$. The role of Λ is to emphasize the close analogy between the expressions in the two cases. For the folded string in $AdS_5 \times S^5$, the energy admits the following expansion

$$\begin{aligned} E^2 &= J^2 + \left(A_1\sqrt{\Lambda} + A_2 + \frac{A_3}{\sqrt{\Lambda}} + \cdots\right) S + \left(B_1 + \frac{B_2}{\sqrt{\Lambda}} + \frac{B_3}{\Lambda} + \cdots\right) S^2 + \\ &+ \left(\frac{C_1}{\sqrt{\Lambda}} + \frac{C_2}{\Lambda} + \frac{C_3}{\Lambda^{3/2}} + \cdots\right) S^3 + \cdots , \end{aligned} \tag{5.62}$$

where the following exact formula for the constants A_i has been conjectured in [68]:

$$A_1\sqrt{\Lambda} + A_2 + \frac{A_3}{\sqrt{\Lambda}} + \cdots = 2\sqrt{\Lambda}\,Y_J(\sqrt{\Lambda}), \qquad Y_J(x) = \frac{d}{dx}\log I_J(x). \tag{5.63}$$

Expanding at large λ, we find the first values

$$\begin{aligned} A_1 &= 2, \\ A_2 &= -1, \\ A_3 &= J^2 - \tfrac{1}{4}, \\ A_4 &= J^2 - \tfrac{1}{4}, \end{aligned} \qquad \begin{aligned} A_5 &= -\tfrac{1}{4}\,J^4 + \tfrac{13}{8}\,J^2 - \tfrac{25}{64}, \\ A_6 &= -J^4 + \tfrac{7}{2}\,J^2 - \tfrac{13}{16}, \\ A_7 &= \tfrac{J^6}{8} - \tfrac{115 J^4}{32} + \tfrac{1187 J^2}{128} - \tfrac{1073}{512}. \end{aligned} \tag{5.64}$$

Also, it is know that $B_1 = \frac{3}{2}$ and $B_2 = \frac{3}{8} - 3\,\zeta(3)$ [?] .

The expansion (5.62) is very convenient since all powers of S have a coefficient with an expansion at large Λ starting with a more and more suppressed term. The simplicity of (5.62) is a special feature of the folded string with two cusps. If winding is allowed, it is known that such structure is lost as discussed in [?] (see also the results of [?]).

For the folded string in $AdS_4 \times \mathbb{CP}^3$, the expansion with fixed $\mathcal{J}$ [31] has the general form (see the Appendices of [?])

$$E = \sqrt{\Lambda}\;\mathcal{E}_0 + \sum_{\ell=0}^{\infty}\frac{1}{(\sqrt{\Lambda})^\ell}\sum_{p=1}^{\infty}\sum_{q=-2p}^{\infty} v_{pq}^{(\ell)}\,\mathcal{J}^q\,\mathcal{S}^p, \tag{5.65}$$

where the classical energy is [32]

$$\mathcal{E}_0 = \mathcal{J} + \frac{\sqrt{\mathcal{J}^2+1}}{\mathcal{J}}\,\mathcal{S} - \frac{\mathcal{J}^2+2}{4\,\mathcal{J}^3(\mathcal{J}^2+1)}\,\mathcal{S}^2 + \frac{3\,\mathcal{J}^6 + 13\,\mathcal{J}^4 + 20\,\mathcal{J}^2 + 8}{16\,\mathcal{J}^5\,(\mathcal{J}^2+1)^{5/2}}\,\mathcal{S}^3 + \cdots \tag{5.66}$$

and the semiclassical computation provides $v_{pq}^{(0)}$ according to the results in (5.31).

[31] Actually when we speak about fixed $\mathcal{J}$ we mean small $\mathcal{S}$ followed by small $\mathcal{J}$.

[32] Note that there is a typo in the $\mathcal{S}^3$ term in the introduction to [?]

Expanding E^2, we find that (5.62) takes the following form

$$
\begin{aligned}
E^2 - J^2 = \\
&+\Bigg[\left(2-\frac{1}{J}\right)\sqrt{\Lambda}+\left(\frac{2v^{(1)}_{1,-2}}{J}-1+2\log(2)\right)+\sqrt{\frac{1}{\Lambda}}\left(\frac{2v^{(2)}_{1,-2}}{J}+2v^{(1)}_{1,-1}+J^2+\frac{J}{2}\right)+\cdots\Bigg]\,S+ \\
&+\Bigg[\left(\frac{1}{4J^4}+\frac{1}{2J^3}\right)\Lambda+\sqrt{\Lambda}\left(-\frac{v^{(1)}_{1,-2}}{J^4}+\frac{2v^{(1)}_{1,-2}}{J^3}+\frac{2v^{(1)}_{2,-4}}{J^3}+\frac{1}{2J^3}-\frac{\log(2)}{J^3}\right)+\cdots\Bigg]\,S^2+ \\
&+\Bigg[\left(-\frac{3}{4J^6}-\frac{1}{2J^5}\right)\Lambda^{3/2}+\cdots\Bigg]\,S^3+\cdots\,. \qquad (5.67)
\end{aligned}
$$

This structure is different from (5.62) since higher powers of S are not associated with terms that are more and more suppressed at large Λ. This is possible since the new terms not present in (5.62) are associated with suitable inverse powers of J. The same phenomenon is discussed in [?] for the folded string in $AdS_5\times S^5$ with non-trivial winding. As we discussed in Sec. (5.10), wrapping corrections are responsible for these terms.

5.14 Prediction for short states

We can provide a prediction for the strong coupling expansion of the energy of short states that in principle could be tested by TBA calculations. To this aim, we can start from our results at fixed $\rho=\mathcal{J}/\sqrt{\mathcal{S}}$, and re-expand at large Λ the sum of the (scaled) classical energy

$$\mathcal{E}_0 \;=\; \sqrt{(\rho^2+2)\,\mathcal{S}}\left[1+\frac{2\,\rho^2+3}{4\,(\rho^2+2)}\,\mathcal{S}-\frac{4\,\rho^6+20\,\rho^4+34\,\rho^2+21}{32\,(\rho^2+2)^2}\,\mathcal{S}^2+\cdots\right] \qquad (5.68)$$

and the one-loop contribution (5.32). The result is

$$E=(4\,\pi\,g)^{1/2}\,\sqrt{2\,S}-\frac{1}{2}+\frac{\sqrt{2\,S}}{(4\,\pi\,g)^{1/2}}\left(\frac{J\,(J+1)}{4\,S}+\frac{3\,S}{8}-\frac{1}{4}+\frac{1}{2}\,\log(2)\right)+\cdots\,. \qquad (5.69)$$

The same expansion where we remark that one of the effect of the $\mathcal{C}$ term is the constant $\mathcal{O}(\widetilde{\Lambda}^0)$ contribution.

The same expansion can be written in terms of the coupling $g_{\rm WS}$ in the world-sheet regularization whose relation with g is [?, ?]

$$g=g_{\rm WS}-\frac{\log(2)}{4\pi}+\cdots\,. \qquad (5.70)$$

After this replacement, eq.(5.71) reads

$$E=(4\,\pi\,g_{\rm WS})^{1/2}\,\sqrt{2\,S}-\frac{1}{2}+\frac{\sqrt{2\,S}}{(4\,\pi\,g_{\rm WS})^{1/2}}\left(\frac{J\,(J+1)}{4\,S}+\frac{3\,S}{8}-\frac{1}{4}\right)+\cdots\,, \qquad (5.71)$$

without log(2) term. This is correct since in world-sheet regularization all modes are treated with uniform cutoff.

6 Conclusions

Conclude with a tearful and heroic description about the journey of Konishi through the land of integrability - from weak to strong coupling.

A Summary of notation and definitions

In this appendix we summarize some notation used throughout the paper.

Laurent expansions in x

We often represent functions of the spectral parameter u as a series in x

$$f(u) = \sum_{n=-\infty}^{\infty} f_n x^n \tag{A.1}$$

with

$$u = g(x + 1/x). \tag{A.2}$$

We denote by $[f]_+$ and $[f]_-$ part of the series with positive and negative powers of x:

$$[f]_+ = \sum_{n=1}^{\infty} f_n x^n, \tag{A.3}$$

$$[f]_- = \sum_{n=1}^{\infty} f_{-n} x^{-n}. \tag{A.4}$$

As a function of u, $x(u)$ has a cut from $-2g$ to $2g$. For any function $f(u)$ with such a cut we denote another branch of $f(u)$ obtained by analytic continuation (from $\Im\, u > 0$) around the branch point $u = 2g$ by $\tilde{f}(u)$. In particular, $\tilde{x} = 1/x$.

Functions $\sinh_\pm$ and $\cosh_\pm$

We define $I_k = I_k(4\pi g)$, where $I_k(u)$ is the modified Bessel function of the first kind. Then

$$\sinh_+ = [\sinh(2\pi u)]_+ = \sum_{k=1}^{\infty} I_{2k-1} x^{2k-1}, \tag{A.5}$$

$$\sinh_- = [\sinh(2\pi u)]_- = \sum_{k=1}^{\infty} I_{2k-1} x^{-2k+1}, \tag{A.6}$$

$$\cosh_+ = [\cosh(2\pi u)]_+ = \sum_{k=1}^{\infty} I_{2k} x^{2k}, \tag{A.7}$$

$$\cosh_- = [\cosh(2\pi u)]_- = \sum_{k=1}^{\infty} I_{2k} x^{-2k}. \tag{A.8}$$

In some cases we denote for brevity

$$\mathrm{sh}_-^x = \sinh_-(x), \quad \mathrm{ch}_-^x = \cosh_-(x). \tag{A.9}$$

Integral kernels

In order to solve for $\mathbf{P}_a^{(1)}$ in section 4.6.3 we introduce integral operators H and K with kernels

$$H(u,v) = -\frac{1}{4\pi i}\frac{\sqrt{u-2g}\sqrt{u+2g}}{\sqrt{v-2g}\sqrt{v+2g}}\frac{1}{u-v}dv, \tag{A.10}$$

$$K(u,v) = +\frac{1}{4\pi i}\frac{1}{u-v}dv, \tag{A.11}$$

which satisfy

$$\tilde{f}+f=h \ , \ f=H\cdot h \quad \text{and} \quad \tilde{f}-f=h \ , \ f=K\cdot h. \tag{A.12}$$

Since the purpose of H and K is to solve equations of the type A.12, H usually acts on functions h such that $\tilde{h}=h$, whereas K acts on h such that $\tilde{h}=-h$. On the corresponding classes of functions H and K can be represented by kernels which are equal up to a sign?

$$H(u,v) = -\frac{1}{2\pi i}\frac{1}{x_u-x_v}dx_v\Big|_{\tilde{h}=h}, \tag{A.13}$$

$$K(u,v) = \frac{1}{2\pi i}\frac{1}{x_u-x_v}dx_v\Big|_{\tilde{h}=-h}. \tag{A.14}$$

In order to be able to deal with series in half-integer powers of x in section 4.6.5 we introduce modified kernels:

$$H^*\cdot f \equiv \frac{x+1}{\sqrt{x}}H\cdot\frac{\sqrt{x}}{x+1}f, \tag{A.15}$$

$$K^*\cdot f \equiv \frac{x+1}{\sqrt{x}}K\cdot\frac{\sqrt{x}}{x+1}f. \tag{A.16}$$

Finally, to write the solution to equations of the type (4.126), we introduce the operator Γ' and its more symmetric version Γ

$$\left(\Gamma'\cdot h\right)(u) \equiv \oint_{-2g}^{2g}\frac{dv}{4\pi i}\partial_u\log\frac{\Gamma[i(u-v)+1]}{\Gamma[-i(u-v)]}h(v), \tag{A.17}$$

$$\left(\Gamma\cdot h\right)(u) \equiv \oint_{-2g}^{2g}\frac{dv}{4\pi i}\partial_u\log\frac{\Gamma[i(u-v)+1]}{\Gamma[-i(u-v)+1]}h(v). \tag{A.18}$$

Periodized Chebyshev polynomials

Periodized Chebyshev polynomials appearing in $\mu_{ab}^{(1)}$ are defined as

$$p'_a(u) = \Sigma\cdot[x^a+1/x^a] = 2\Sigma\cdot\left[T_a\left(\frac{u}{2g}\right)\right], \tag{A.19}$$

$$p_a(u) = p'_a(u)+\frac{1}{2}\left(x^a(u)+x^{-a}(u)\right), \tag{A.20}$$

where $T_a(u)$ are Chebyshev polynomials of the first kind. Here is the explicit form for the first five of them:

$$p_0' = -i(u - i/2), \tag{A.21}$$
$$p_1' = -i\frac{u(u-i)}{4g}, \tag{A.22}$$
$$p_2' = -i\frac{(u-i/2)(-6g^2+u^2-iu)}{6g^2}, \tag{A.23}$$
$$p_3' = -i\frac{u(u-i)\left(-6g^2+u(u-i)\right)}{8g^3}, \tag{A.24}$$
$$p_4' = -i\frac{\left(u-\frac{i}{2}\right)\left(30g^4-20g^2u^2+20ig^2u+3u^4-6iu^3-2u^2-iu\right)}{30g^4}. \tag{A.25}$$

B The slope function for odd J

Here we give details on solving the $\mathbf{P}\mu$-system for odd J at leading order in the spin. First, the parity of the μ_{ab} functions is different from the even J case, which can be seen from the asymptotics (4.22). Following arguments similar to the discussion for even J in section **??**, we obtain

$$\mu_{12} = 1,\ \mu_{13} = 0,\ \mu_{14} = 0,\ \mu_{24} = \cosh(2\pi u),\ \mu_{34} = 1. \tag{B.1}$$

Plugging these μ_{ab} into (4.20) we get a system of equations for $\mathbf{P}_a$

$$\tilde{\mathbf{P}}_1 = -\mathbf{P}_3, \tag{B.2}$$
$$\tilde{\mathbf{P}}_2 = -\mathbf{P}_4 - \mathbf{P}_1\cosh(2\pi u), \tag{B.3}$$
$$\tilde{\mathbf{P}}_3 = -\mathbf{P}_1, \tag{B.4}$$
$$\tilde{\mathbf{P}}_4 = -\mathbf{P}_2 + \mathbf{P}_3\cosh(2\pi u). \tag{B.5}$$

This system can be solved in a similar way to the even J case. The only important difference is that due to asymptotics (4.21) the $\mathbf{P}_a$ acquire an extra branch point at $u = \infty$.

Let us first rewrite the equations for $\mathbf{P}_1, \mathbf{P}_3$ as

$$\tilde{\mathbf{P}}_1 + \tilde{\mathbf{P}}_3 \quad = \quad -(\mathbf{P}_1 + \mathbf{P}_3) \tag{B.6}$$
$$\tilde{\mathbf{P}}_1 - \tilde{\mathbf{P}}_3 \quad = \quad \mathbf{P}_1 - \mathbf{P}_3. \tag{B.7}$$

This, together with the asymptotics (4.21) implies $\mathbf{P}_1 = \epsilon x^{-J/2}$, $\mathbf{P}_3 = -\epsilon x^{J/2}$ where ϵ is a constant. Let us note that these $\mathbf{P}_1, \mathbf{P}_3$ contain half-integer powers of x, and the analytic continuation around the branch points at $\pm 2g$ replaces $\sqrt{x} \to 1/\sqrt{x}$. Now, taking the sum and difference of the equations for $\mathbf{P}_2,\ \mathbf{P}_4$ we get

$$\tilde{\mathbf{P}}_2 + \tilde{\mathbf{P}}_4 + \mathbf{P}_2 + \mathbf{P}_4 = -a_1\left(x^{J/2} + x^{-J/2}\right)\cosh 2\pi u \tag{B.8}$$
$$\tilde{\mathbf{P}}_2 - \tilde{\mathbf{P}}_4 - (\mathbf{P}_2 - \mathbf{P}_4) = a_1\left(x^{J/2} - x^{-J/2}\right)\cosh 2\pi u \tag{B.9}$$

We can split the expansion

$$\cosh 2\pi u = \sum_{k=-\infty}^{\infty} I_{2k} x^{2k} \tag{B.10}$$

into the positive and negative parts according to

$$\cosh 2\pi u = \cosh_- + \cosh_+ + I_0 \tag{B.11}$$

where

$$\cosh_+ = \sum_{k=1}^{\infty} I_{2k} x^{2k}, \qquad \cosh_- = \sum_{k=1}^{\infty} I_{2k} x^{-2k}. \tag{B.12}$$

Then we can write

$$\mathbf{P}_2 + \mathbf{P}_4 = -a_1(x^{J/2} + x^{-J/2}) \cosh_- - a_1 I_0 x^{-J/2} + Q, \tag{B.13}$$

$$\mathbf{P}_2 - \mathbf{P}_4 = -a_1(x^{J/2} - x^{-J/2}) \cosh_- + a_1 I_0 x^{-J/2} + P, \tag{B.14}$$

where Q and P are some polynomials in $\sqrt{x}, 1/\sqrt{x}$ satisfying

$$\tilde{Q} = -Q, \ \tilde{P} = P. \tag{B.15}$$

We get

$$\mathbf{P}_2 = -a_1 x^{J/2} \cosh_- + \frac{Q+P}{2}, \tag{B.16}$$

$$\mathbf{P}_4 = a_1 x^{-J/2} \cosh_- - a_1 I_0 x^{-J/2} + \frac{Q-P}{2}. \tag{B.17}$$

Now imposing the correct asymptotics of $\mathbf{P}_2$ we find

$$\frac{P+Q}{2} = a_1 x^{J/2} \sum_{k=1}^{\frac{J-1}{2}} I_{2k} x^{-2k} \tag{B.18}$$

Due to (B.15) this relation fixes Q and P completely, and we obtain the solution given in section 2.1,

$$\mu_{12} = 1, \ \mu_{13} = 0, \ \mu_{14} = 0, \ \mu_{24} = \cosh(2\pi u), \ \mu_{34} = 1, \tag{B.19}$$

$$\mathbf{P}_1 = a_1 x^{-J/2}, \tag{B.20}$$

$$\mathbf{P}_2 = -a_1 x^{J/2} \sum_{k=-\infty}^{-\frac{J+1}{2}} I_{2k} x^{2k}, \tag{B.21}$$

$$\mathbf{P}_3 = -a_1 x^{J/2}, \tag{B.22}$$

$$\mathbf{P}_4 = a_1 x^{-J/2} \cosh_- - a_1 x^{-J/2} \sum_{k=1}^{\frac{J-1}{2}} I_{2k} x^{2k} - a_1 I_0 x^{-J/2}. \tag{B.23}$$

Notice that the branch point at infinity is absent from the product of any two $\mathbf{P}$'s, as it should be [?], [?]. One can check that this solution gives again the correct result (4.120) for the slope function.

C NLO solution of $P\mu$ system: details

In this appendix we will provide more details on the solution of the $\mathbf{P}\mu$-system and calculation of curvature function for $J = 2, 3, 4$ which was presented in the main text in sections 4.6.1, 4.6.5, C.4.

C.1 NLO corrections to μ_{ab} for $J = 2$

Here we present some details of calculation of NLO corrections to μ_{ab} for $J = 2$ omitted in the main text. As described in section 4.6.2, $\mu_{ab}^{(1)}$ are found as solutions of (4.126) with appropriate asymptotics. The general solution of this equation consists of a general solution of the corresponding homogeneous equation (which can be reduced to one-parametric form (4.147)) and a particular solution of the inhomogeneous one. The latter can be taken to be

$$\mu_{ab}^{disc} = \Sigma \cdot \left(\mathbf{P}_a^{(1)}\tilde{\mathbf{P}}_b^{(1)} - \mathbf{P}_b^{(1)}\tilde{\mathbf{P}}_a^{(1)}\right). \tag{C.1}$$

One can get rid of the operation Σ, expressing μ_{ab}^{disc} in terms of Γ' and p'_a. This procedure is based on two facts: the definition (4.132) of p'_a and the statement that on functions decaying at infinity Σ coincides with Γ' defined by (4.130). After a straightforward but long calculation we find

$$\mu_{31}^{disc} = \epsilon^2\Sigma\left(\frac{1}{x^2} - x^2\right) = -\epsilon^2 \quad \left(\Gamma \cdot x^2 + p_2\right), \tag{C.2}$$

$$\mu_{41}^{disc} = \epsilon^2\left[-2I_1p_1 - 4I_1\Gamma\cdot x + \sinh(2\pi u)\left(\Gamma\cdot x^2 + p_0\right) + \Gamma\cdot\sinh_-\left(x - \frac{1}{x}\right)^2\right], \tag{C.3}$$

$$\mu_{43}^{disc} = -2\epsilon^2\left[-2I_1p_1 - 4I_1\Gamma\cdot x + \sinh(2\pi u)(p_2 - p_0) + \Gamma\cdot\sinh_-\left(x - \frac{1}{x}\right)^2\right], \tag{C.4}$$

$$\mu_{21}^{disc} = \epsilon^2\left[2I_1\Gamma\cdot x - \sinh(2\pi u)\,\Gamma\cdot x^2 - \Gamma\cdot\sinh_-\left(x^2 + \frac{1}{x^2}\right)\right], \tag{C.5}$$

$$\mu_{24}^{disc} = \epsilon^2\left[2I_1\Gamma\cdot\sinh_-\left(x + \frac{1}{x}\right) + I_1^2p_0 + \right. \tag{C.6}$$

$$\left. + \sinh(2\pi u)\Gamma\cdot\sinh_-\left(x^2 - \frac{1}{x^2}\right) - \Gamma\cdot\sinh_-^2\left(x^2 - \frac{1}{x^2}\right)\right]. \tag{C.7}$$

Here we write Γ and p_a instead of Γ' and p'_a taking into account the discussion between equations (4.152) - (4.157).

C.2 NLO solution of the $\mathbf{P}\mu$-system at $J = 3$

In this appendix we present some intermediate formulas for the calculation of curvature function for $J = 3$ in section 4.6.5 omitted in the main text.

- The particular solution of the inhomogeneous equation (4.126) which we construct as $\mu_{31}^{disc} = \Sigma \cdot \left(\mathbf{P}_a^{(1)}\tilde{\mathbf{P}}_b^{(1)} - \mathbf{P}_b^{(1)}\tilde{\mathbf{P}}_a^{(1)}\right)$ can be written using the operation Γ and p_a defined by (4.157) and (4.155)[33]

$$\mu_{31}^{disc} = \Sigma \cdot (\mathbf{P}_3\tilde{\mathbf{P}}_1 - \mathbf{P}_1\tilde{\mathbf{P}}_3) = -2\epsilon^2 \left[\Gamma x^3 + p_3\right], \tag{C.8}$$

$$\mu_{41}^{disc} = -\epsilon^2 \left[2p_2 I_2 + 2I_2\Gamma x^2 + 2\Gamma \cdot \cosh_- + (I_0 - \cosh(2\pi u))p_0\right], \tag{C.9}$$

$$\mu_{34}^{disc} = \epsilon^2 \left[2I_2\Gamma x + I_0\Gamma x^3 - \Gamma \cdot (x^3 + x^{-3})\cosh_- + \cosh(2\pi u)(2p_3 + \Gamma x^3)\right], \tag{C.10}$$

$$\mu_{21}^{disc} = \epsilon^2 \left[2I_2\Gamma x + (I_0 - \cosh(2\pi u))\Gamma x^3 - \Gamma\left((x^3 + x^{-3})\cosh(2\pi u)\right)\right], \tag{C.11}$$

$$\mu_{24}^{disc} = -2\epsilon^2 \left[-\frac{1}{2}\Gamma \cdot \cosh_-^2 \left(x^3 - x^{-3}\right) + \left(\frac{\cosh(2\pi u)}{2} - I_0\right)\Gamma \cdot \frac{\cosh_-}{x^3}\right. \tag{C.12}$$

$$-I_2\Gamma \cdot \left(x + \frac{1}{x}\right)\cosh_- - \frac{1}{2}\cosh(2\pi u)\Gamma \cdot x^3\cosh_- + \tag{C.13}$$

$$\left. + \frac{I_0}{2}\left(I_0 - \cosh(2\pi u)\right)\Gamma \cdot x^3 + \frac{I_1 I_2}{2\pi g}\Gamma x - I_2^2 p_1\right]. \tag{C.14}$$

- The zero mode of the system (**??**)-(**??**), which we added to the solution in Eqs. (**??**)-(**??**) to ensure correct asymptotics, is

$$\begin{aligned}
\mathbf{P}_1^{\rm zm} &= L_1 x^{-1/2} + L_3 x^{1/2}, \qquad\qquad (C.15)\\
\mathbf{P}_2^{\rm zm} &= -L_1 x^{1/2}\mathrm{ch}_- + L_2 x^{-1/2} - L_3 x^{-1/2}\left(\mathrm{ch}_- + \frac{1}{2}I_0\right) + L_4\left(x^{1/2} - x^{-1/2}\right),\\
\mathbf{P}_3^{\rm zm} &= -L_1 x^{1/2} - L_3 x^{1/2},\\
\mathbf{P}_4^{\rm zm} &= -L_1\left(I_0 x^{-1/2} + x^{-1/2}\cosh_-\right) - L_2 x^{1/2} + L_4(x^{1/2} - x^{-1/2})\\
&- L_3 x^{1/2}\left(\mathrm{ch}_- + \frac{1}{2}I_0\right).
\end{aligned}$$

C.3 NLO solution of the $\mathbf{P}\mu$-system at $J = 4$

Solution of the $\mathbf{P}\mu$ system at NLO for $J = 4$ is completely analogous to the case of $J = 2$. The starting point is the LO solution (4.104)-(4.107). As described in section C.1, from LO $\mathbf{P}_a$ we

[33] Alternatively one can use p'_a and Γ' instead of p_a and Γ- see the discussion between the equations (4.152) - (4.157)

can find μ_{ab} at NLO. Its discontinuous part is

$$\mu_{31}^{disc} = -\epsilon^2 \left(\Gamma \cdot x^4 + p_4\right), \tag{C.16}$$
$$\mu_{41}^{disc} = \frac{1}{2}\epsilon^2 \left(\sinh(2\pi u)\left(p_0 + \Gamma \cdot x^4\right) + 2\left(I_1 p_1 + I_3 p_3\right) + \right. \tag{C.17}$$
$$\left. +\Gamma \cdot \sinh_- \left(x^2 - \frac{1}{x^2}\right)^2 - 2\left(I_1 + I_3\right)\left(\Gamma \cdot x^3 + \Gamma \cdot x\right)\right), \tag{C.18}$$
$$\mu_{43}^{disc} = \epsilon^2 \left((p_4 - p_0)\sinh(2\pi u) + 2\left(I_1 p_1 + I_3 p_3\right) - \right. \tag{C.19}$$
$$\left. -\Gamma \cdot \sinh_- \left(x^2 - \frac{1}{x^2}\right)^2 + 2\left(I_1 + I_3\right)\left(\Gamma \cdot x^3 + \Gamma \cdot x\right)\right), \tag{C.20}$$
$$\mu_{21}^{disc} = \epsilon^2 \left(-\frac{1}{2}\sinh(2\pi u)\Gamma \cdot x^4 + I_1 p_3 + I_3 p_1 - \right. \tag{C.21}$$
$$\left. -\frac{1}{2}\Gamma \cdot \sinh_- \left(x^4 + \frac{1}{x^4}\right) + I_1 \Gamma \cdot x^3 + I_3 \Gamma \cdot x\right), \tag{C.22}$$
$$\mu_{24}^{disc} = \epsilon^2 \left(\frac{1}{2}\sinh(2\pi u)\Gamma \cdot \sinh_- \left(x^4 - \frac{1}{x^4}\right) + I_3^2 p_2 + I_1 I_3 p_0 - \right. \tag{C.23}$$
$$-\frac{1}{2}\Gamma \cdot \sinh_-^2 \left(x^4 - \frac{1}{x^4}\right) + I_1 \Gamma \cdot \sinh_- \left(x^3 + \frac{1}{x^3}\right) + \tag{C.24}$$
$$\left. +I_3 \Gamma \cdot \sinh_- \left(x + \frac{1}{x}\right) + \left(I_3^2 - I_1^2\right)\Gamma \cdot x^2\right), \tag{C.25}$$

and as discussed for $J = 2$ the zero mode can be brought to the form

$$\pi_{12} = 0,\ \pi_{13} = 0,\ \pi_{14} = 0, \tag{C.26}$$
$$\pi_{24} = c_{1,24}\cosh 2\pi u,\ \pi_{34} = 0. \tag{C.27}$$

After that, we calculate r_a by formula (4.152) and solve the expanded to NLO $\mathbf{P}\mu$ system for $\mathbf{P}_a^{(1)}$ as

$$\mathbf{P}_3^{(1)} = H \cdot r_3, \tag{C.28}$$
$$\mathbf{P}_1^{(1)} = \frac{1}{2}\mathbf{P}_3^{(1)} + K \cdot \left(r_1 - \frac{1}{2}r_3\right), \tag{C.29}$$
$$\mathbf{P}_4^{(1)} = K \cdot \left[(H \cdot r_3)\sinh(2\pi u) + r_4 - \frac{1}{2}r_3 \sinh(2\pi u)\right] - C(x + 1/x), \tag{C.30}$$
$$\mathbf{P}_2^{(1)} = H \cdot \left[-\mathbf{P}_4^{(1)} - \mathbf{P}_1^{(1)}\sinh(2\pi u) + r_2\right] + C/x, \tag{C.31}$$

where C is a constant which is fixed by requiring correct asymptotics of $\mathbf{P}_2$. Finally we find leading coefficients A_a of $\P_a^{(1)}$ and use expanded up to $\mathcal{O}(S^2)$ formulas (4.23), (4.24) in the same way as in section 4.6.4 to obtain the result (C.32).

C.4 Result for $J = 4$

The final result for the curvature function at $J = 4$ reads

$$
\begin{aligned}
\gamma^{(2)}_{J=4} = & \oint \frac{du_x}{2\pi i} \oint \frac{du_y}{2\pi i} \frac{1}{ig^2(I_3 - I_5)^3} \Bigg[\qquad \text{(C.32)} \\
& \frac{2\left(\mathrm{sh}_-^x\right)^2 y^4 \left(I_3\left(x^{10}+1\right) - I_5 x^2\left(x^6+1\right)\right)}{x^4\left(x^2-1\right)} - \frac{2\left(\mathrm{sh}_-^y\right)^2 x^4\left(y^8-1\right)\left(I_3 x^2 - I_5\right)}{\left(x^2-1\right)y^4} + \\
& + \frac{4\mathrm{sh}_-^x \mathrm{sh}_-^y \left(x^4 y^4 - 1\right)\left(I_3 + I_3 x^6 y^4 - I_5 x^2\left(x^2 y^4 + 1\right)\right)}{x^4\left(x^2-1\right)y^4} \\
& + \mathrm{sh}_-^y \Bigg(\left(y^4 + y^{-4}\right)x^{-1}\left(\left(I_1 I_5 - I_3^2\right)\left(3x^4+1\right) - 2I_1 I_3 x^6\right) + \\
& + \frac{2I_3 x^2\left(I_5\left(x^2+1\right)x^2 + I_1\left(1-x^2\right)\right) - I_1 I_5\left(x^2-1\right)^2 + I_3^2\left(-2x^6+x^4+1\right)}{x(x^2-1)} + \\
& + 2\left(y^3 + y^{-3}\right)\frac{I_1 I_3 x^6 - I_1 I_5 x^4 - I_3^2\left(x^2-1\right)}{x^2-1} - \\
& - 2I_3\left(y + y^{-1}\right)\frac{I_1\left(x^2-1\right) - I_3\left(x^6 - x^2 + 1\right) + I_5\left(x^4 - x^2 + 1\right)}{x^2-1}\Bigg) + \\
& + \frac{4x^6 y^2 I_3\left(I_3^2 - I_1^2\right)}{x^2-1} + \frac{4xyI_1\left(I_3 y^2 + I_1\right)\left(I_3 + I_5\right)}{x^2-1} + \\
& \frac{2y^4\left(I_1 + I_3\right)\left(I_1 I_5 - I_3^2\right)}{x^2-1} - \frac{2y\left(y^2+1\right)\left(I_1 + I_3\right)\left(I_1 I_5 - I_3^2\right)}{x\left(x^2-1\right)} - \\
& - \frac{2x^3 y\left(I_1 + I_3\right)\left(I_1\left(2I_3 + \left(3\,y^2+1\right)I_5\right) - I_3\left(2I_5 y^2 + \left(y^2+3\right)I_3\right)\right)}{x^2-1} \\
& + \frac{2x^2 y^4\left(-I_3^3 - I_1\left(3I_3 + I_5\right)I_3 + I_1^2 I_5\right)}{x^2-1} + \frac{2x^4 y\left(I_1^2\left(2yI_5 - 2y^3 I_3\right) - 2y\left(y^2+1\right)I_3^2 I_5\right)}{x^2-1} + \\
& + \frac{4x^5 y I_3\left(2I_1^2 y^2 + I_3\left(I_5 - I_3\right)y^2 + I_1\left(I_3 + I_5\right)\right)}{x^2-1}\Bigg] \frac{1}{4\pi i}\partial_u \log\frac{\Gamma(iu_x - iu_y + 1)}{\Gamma(1 - iu_x + iu_y)}
\end{aligned}
$$

where, similarly to $J = 2, 3$, the integrals go around the branch between $-2g$ and $2g$.

D Weak coupling expansion – details

First, we give the expansion of our results for the slope-to slope functions $\gamma_J^{(2)}$ to 10 loops. We start with $J = 2$:

$$\begin{aligned}
\gamma^{(2)}_{J=2} &= -8g^2\zeta_3 + g^4\left(140\zeta_5 - \frac{32\pi^2\zeta_3}{3}\right) + g^6\left(200\pi^2\zeta_5 - 2016\zeta_7\right) \\
&+ g^8\left(-\frac{16\pi^6\zeta_3}{45} - \frac{88\pi^4\zeta_5}{9} - \frac{9296\pi^2\zeta_7}{3} + 27720\zeta_9\right) \\
&+ g^{10}\left(\frac{208\pi^8\zeta_3}{405} + \frac{160\pi^6\zeta_5}{27} + 144\pi^4\zeta_7 + 45440\pi^2\zeta_9 - 377520\zeta_{11}\right) \\
&+ g^{12}\left(-\frac{7904\pi^{10}\zeta_3}{14175} - \frac{17296\pi^8\zeta_5}{4725} - \frac{128\pi^6\zeta_7}{15} - \frac{6312\pi^4\zeta_9}{5}\right. \\
&\quad \left. -653400\pi^2\zeta_{11} + 5153148\zeta_{13}\right) \\
&+ g^{14}\left(\frac{1504\pi^{12}\zeta_3}{2835} + \frac{106576\pi^{10}\zeta_5}{42525} - \frac{18992\pi^8\zeta_7}{405} - \frac{16976\pi^6\zeta_9}{15}\right. \\
&\quad \left. + \frac{25696\pi^4\zeta_{11}}{9} + \frac{28003976\pi^2\zeta_{13}}{3} - 70790720\zeta_{15}\right) \\
&+ g^{16}\left(-\frac{178112\pi^{14}\zeta_3}{382725} - \frac{239488\pi^{12}\zeta_5}{127575} + \frac{2604416\pi^{10}\zeta_7}{42525} + \frac{8871152\pi^8\zeta_9}{4725}\right. \\
&\quad + \frac{30157072\pi^6\zeta_{11}}{945} + \frac{8224216\pi^4\zeta_{13}}{45} - 133253120\pi^2\zeta_{15} \\
&\quad \left. +979945824\zeta_{17}\right) \\
&+ g^{18}\left(\frac{147712\pi^{16}\zeta_3}{382725} + \frac{940672\pi^{14}\zeta_5}{637875} - \frac{490528\pi^{12}\zeta_7}{8505} - \frac{358016\pi^{10}\zeta_9}{189}\right. \\
&\quad - \frac{37441312\pi^8\zeta_{11}}{945} - \frac{9616256\pi^6\zeta_{13}}{15} - \frac{16988608\pi^4\zeta_{15}}{3} \\
&\quad \left. +1905790848\pi^2\zeta_{17} - 13671272160\zeta_{19}\right) \\
&+ g^{20}\left(-\frac{135748672\pi^{18}\zeta_3}{442047375} - \frac{103683872\pi^{16}\zeta_5}{88409475} + \frac{1408423616\pi^{14}\zeta_7}{29469825}\right. \\
&\quad + \frac{2288692288\pi^{12}\zeta_9}{1403325} + \frac{34713664\pi^{10}\zeta_{11}}{945} + \frac{73329568\pi^8\zeta_{13}}{105} \\
&\quad + \frac{305679296\pi^6\zeta_{15}}{27} + 121666688\pi^4\zeta_{17} - 27342544320\pi^2\zeta_{19} \\
&\quad \left. +192157325360\zeta_{21}\right)
\end{aligned} \tag{D.1}$$

Next, for $J = 3$,

$$
\begin{aligned}
\gamma^{(2)}_{J=3} &= -2g^2\zeta_3 + g^4\left(12\zeta_5 - \frac{4\pi^2\zeta_3}{3}\right) + g^6\left(\frac{2\pi^4\zeta_3}{45} + 8\pi^2\zeta_5 - 28\zeta_7\right) \\
&+ g^8\left(-\frac{4\pi^6\zeta_3}{45} - \frac{4\pi^4\zeta_5}{15} - 528\zeta_9\right) \\
&+ g^{10}\left(\frac{934\pi^8\zeta_3}{14175} + \frac{8\pi^6\zeta_5}{9} - \frac{82\pi^4\zeta_7}{9} - 900\pi^2\zeta_9 + 12870\zeta_{11}\right) \\
&+ g^{12}\left(-\frac{572\pi^{10}\zeta_3}{14175} - \frac{104\pi^8\zeta_5}{175} - \frac{256\pi^6\zeta_7}{45} + \frac{2476\pi^4\zeta_9}{9}\right. \\
&\qquad \left.+\frac{57860\pi^2\zeta_{11}}{3} - 208208\zeta_{13}\right) \\
&+ g^{14}\left(\frac{2878\pi^{12}\zeta_3}{127575} + \frac{404\pi^{10}\zeta_5}{1215} + \frac{326\pi^8\zeta_7}{75} + \frac{3352\pi^6\zeta_9}{135}\right. \\
&\qquad \left.- \frac{80806\pi^4\zeta_{11}}{15} - 316316\pi^2\zeta_{13} + 2994992\zeta_{15}\right) \\
&+ g^{16}\left(-\frac{159604\pi^{14}\zeta_3}{13395375} - \frac{257204\pi^{12}\zeta_5}{1488375} - \frac{14836\pi^{10}\zeta_7}{6075} - \frac{71552\pi^8\zeta_9}{2025}\right. \\
&\qquad \left.+\frac{4948\pi^6\zeta_{11}}{189} + \frac{4163068\pi^4\zeta_{13}}{45} + \frac{14129024\pi^2\zeta_{15}}{3} - 41116608\zeta_{17}\right) \\
&+ g^{18}\left(\frac{494954\pi^{16}\zeta_3}{81860625} + \frac{156368\pi^{14}\zeta_5}{1819125} + \frac{6796474\pi^{12}\zeta_7}{5457375} + \frac{332\pi^{10}\zeta_9}{15}\right. \\
&\qquad +\frac{1745318\pi^8\zeta_{11}}{4725} - \frac{868088\pi^6\zeta_{13}}{315} - \frac{22594208\pi^4\zeta_{15}}{15} \\
&\qquad \left.-67084992\pi^2\zeta_{17} + 553361016\zeta_{19}\right) \\
&+ g^{20}\left(-\frac{940132\pi^{18}\zeta_3}{315748125} - \frac{244456\pi^{16}\zeta_5}{5893965} - \frac{29637008\pi^{14}\zeta_7}{49116375} - \frac{11808196\pi^{12}\zeta_9}{1002375}\right. \\
&\qquad - \frac{2265364\pi^{10}\zeta_{11}}{8505} - \frac{68767984\pi^8\zeta_{13}}{14175} + \frac{480208\pi^6\zeta_{15}}{9} \\
&\qquad \left.+ \frac{71785288\pi^4\zeta_{17}}{3} + 934787840\pi^2\zeta_{19} - 7390666360\zeta_{21}\right)
\end{aligned} \tag{D.2}
$$

Finally, for $J = 4$,

$$\begin{aligned}
\gamma^{(2)}_{J=4} &= g^2\left(-\frac{14\zeta_3}{5}+\frac{48\zeta_5}{\pi^2}-\frac{252\zeta_7}{\pi^4}\right) \qquad \text{(D.3)}\\
&+ g^4\left(-\frac{22\pi^2\zeta_3}{25}+\frac{474\zeta_5}{5}-\frac{8568\zeta_7}{5\pi^2}+\frac{8316\zeta_9}{\pi^4}\right)\\
&+ g^6\left(\frac{32\pi^4\zeta_3}{875}+\frac{3656\pi^2\zeta_5}{175}-\frac{56568\zeta_7}{25}+\frac{196128\zeta_9}{5\pi^2}-\frac{185328\zeta_{11}}{\pi^4}\right)\\
&+ g^8\Big(-\frac{4\pi^6\zeta_3}{175}-\frac{68\pi^4\zeta_5}{75}-\frac{55312\pi^2\zeta_7}{125}+\frac{1113396\zeta_9}{25}-\frac{3763188\zeta_{11}}{5\pi^2}\\
&\quad +\frac{3513510\zeta_{13}}{\pi^4}\Big)\\
&+ g^{10}\Big(\frac{176\pi^8\zeta_3}{16875}+\frac{2488\pi^6\zeta_5}{7875}+\frac{2448\pi^4\zeta_7}{125}+\frac{209532\pi^2\zeta_9}{25}-\frac{3969878\zeta_{11}}{5}\\
&\quad +\frac{13213200\zeta_{13}}{\pi^2}-\frac{61261200\zeta_{15}}{\pi^4}\Big)\\
&+ g^{12}\Big(-\frac{88072\pi^{10}\zeta_3}{20671875}-\frac{449816\pi^8\zeta_5}{4134375}-\frac{327212\pi^6\zeta_7}{65625}-\frac{338536\pi^4\zeta_9}{875}\\
&\quad -\frac{129520798\pi^2\zeta_{11}}{875}+\frac{66969474\zeta_{13}}{5}-\frac{220540320\zeta_{15}}{\pi^2}\\
&\quad +\frac{1017636048\zeta_{17}}{\pi^4}\Big)\\
&+ g^{14}\Big(\frac{795136\pi^{12}\zeta_3}{487265625}+\frac{522784\pi^{10}\zeta_5}{13921875}+\frac{4021288\pi^8\zeta_7}{2953125}+\frac{1869152\pi^6\zeta_9}{21875}\\
&\quad +\frac{18573952\pi^4\zeta_{11}}{2625}+\frac{62633272\pi^2\zeta_{13}}{25}-\frac{1092799344\zeta_{15}}{5}\\
&\quad +\frac{17844607872\zeta_{17}}{5\pi^2}-\frac{16405526592\zeta_{19}}{\pi^4}\Big)\\
&+ g^{16}\Big(-\frac{30581888\pi^{14}\zeta_3}{51162890625}-\frac{43988768\pi^{12}\zeta_5}{3410859375}-\frac{446380184\pi^{10}\zeta_7}{1136953125}\\
&\quad -\frac{20108936\pi^8\zeta_9}{984375}-\frac{31755036\pi^6\zeta_{11}}{21875}-\frac{321449336\pi^4\zeta_{13}}{2625}\\
&\quad -\frac{1031925232\pi^2\zeta_{15}}{25}+\frac{87296960712\zeta_{17}}{25}-\frac{283092985656\zeta_{19}}{5\pi^2}\\
&\quad +\frac{259412389236\zeta_{21}}{\pi^4}\Big)\\
&+ g^{18}\Big(\frac{6706432\pi^{16}\zeta_3}{31672265625}+\frac{816838192\pi^{14}\zeta_5}{186232921875}+\frac{2004636572\pi^{12}\zeta_7}{17054296875}\\
&\quad +\frac{1950592976\pi^{10}\zeta_9}{378984375}+\frac{2220222512\pi^8\zeta_{11}}{6890625}+\frac{20963856\pi^6\zeta_{13}}{875}\\
&\quad +\frac{254959316\pi^4\zeta_{15}}{125}+\frac{584553371616\pi^2\zeta_{17}}{875}\\
&\quad -\frac{1375388084412\zeta_{19}}{25}+\frac{4432313039616\zeta_{21}}{5\pi^2}-\frac{4049650420200\zeta_{23}}{\pi^4}\Big)
\end{aligned}$$

$$\begin{aligned}
&+ g^{20}\Big(-\frac{15308976272\pi^{18}\zeta_3}{209512037109375}-\frac{1764947984\pi^{16}\zeta_5}{1197211640625}-\frac{18667123736\pi^{14}\zeta_7}{517313671875}\\
&\quad -\frac{538293689008\pi^{12}\zeta_9}{399070546875}-\frac{657466372\pi^{10}\zeta_{11}}{8859375}-\frac{119709052\pi^8\zeta_{13}}{23625}\\
&\quad -\frac{9095498848\pi^6\zeta_{15}}{23625}-\frac{260407748416\pi^4\zeta_{17}}{7875}-\frac{1869110789976\pi^2\zeta_{19}}{175}\\
&\quad +\frac{4293062840352\zeta_{21}}{5}-\frac{13755955395600\zeta_{23}}{\pi^2}+\frac{62673161265000\zeta_{25}}{\pi^4}\Big)
\end{aligned}$$

For future reference we have also computed[34] the weak coupling expansion of the anomalous dimensions at order S^3, using the known predictions from ABA which are available for any spin at $J=2$ and $J=3$. For $J=2$ we have computed the expansion to three loops[35]:

$$\begin{aligned}\gamma^{(3)}_{J=2} &= g^2\frac{4}{45}\pi^4 + g^4\left(40\zeta_3^2 - \frac{28\pi^6}{405}\right) \\ &+ g^6\left(\frac{192}{5}\zeta_{5,3} - \frac{6992\zeta_3\zeta_5}{5} + \frac{280\pi^2\zeta_3^2}{3} + \frac{6962\pi^8}{212625}\right) + \mathcal{O}(g^8)\end{aligned} \tag{D.4}$$

Compared to the S^2 part, a new feature is the appearance of multiple zeta values – here we have $\zeta_{5,3}$, which is defined by

$$\zeta_{a_1,a_2,\ldots,a_k} = \sum_{0<n_1<n_2<\ldots<n_k<\infty} \frac{1}{n_1^{a_1}n_2^{a_2}\ldots n_k^{a_k}} \tag{D.5}$$

and cannot be reduced to simple zeta values ζ_n.

For $J=3$ we have obtained the expansion to four loops:

$$\begin{aligned}\gamma^{(3)}_{J=3} &= \frac{1}{90}\pi^4 g^2 + g^4\left(4\zeta_3^2 + \frac{\pi^6}{1890}\right) + g^6\left(4\zeta_{5,3} + 4\pi^2\zeta_3^2 - 72\zeta_3\zeta_5 - \frac{2\pi^8}{675}\right) \\ &+ g^8\left(-112\zeta_{2,8} + \frac{20}{3}\pi^2\zeta_{5,3} + 728\zeta_3\zeta_7 + 448\zeta_5^2 - \frac{224}{3}\pi^2\zeta_3\zeta_5\right. \\ &\left.+\frac{4\pi^4\zeta_3^2}{5} - \frac{41\pi^{10}}{133650}\right) + \mathcal{O}(g^{10})\end{aligned} \tag{D.6}$$

E Higher mode numbers

E.1 Slope function for generic filling fractions and mode numbers

Let us extend the discussion of section **??** by considering the state corresponding to a solution of the asymptotic Bethe equations with arbitrary mode numbers and filling fractions[36]. We expect that in the ¶μ system this should correspond to[37]

$$\mu_{24} = \sum_{n=-\infty}^{\infty} C_n e^{2\pi n u}. \tag{E.1}$$

As an example, for the ground state twist operator we have $\mu_{24} = \sinh(2\pi u)$, which is reproduced by choosing $C_{-1} = -1/2, C_1 = 1/2$ and all other C's set to 0.

It is straightforward to solve the ¶μ system in the same way as in section **??**, and we find the energy

$$\gamma = \frac{\sqrt{\lambda}}{J}\frac{\sum_n C_n I_{J+1}(n\sqrt{\lambda})}{\sum_n C_n I_J(n\sqrt{\lambda})/n} S, \tag{E.2}$$

[34] As described in the main text (see section 5), in the calculations we used several Mathematica packages for dealing with harmonic sums.

[35] We remind that in our notation the anomalous dimension is written as $\gamma = \gamma^{(1)}S + \gamma^{(2)}S^2 + \gamma^{(3)}S^3 + \ldots$

[36] For simplicity we also consider even J here.

[37] We no longer expect μ_{24} to be either even or odd, since in the Bethe ansatz description of the state with generic mode numbers and filling fractions the Bethe roots are not distributed symmetrically.

which can also be written in a more familiar form as

$$\gamma = \sum_n \alpha_n \frac{n\sqrt{\lambda}}{J} \frac{I_{J+1}(n\sqrt{\lambda})}{I_J(n\sqrt{\lambda})} S, \tag{E.3}$$

where

$$\alpha_n = \frac{C_n I_J(n\sqrt{\lambda})/n}{\sum_m C_m I_J(m\sqrt{\lambda})/m} \tag{E.4}$$

are the filling fractions.

The coefficients C_n are additionally constrained by

$$\sum_n C_n I_J(n\sqrt{\lambda}) = 0, \tag{E.5}$$

which ensures that the $\mathbf{P}_a$ functions have correct asymptotics. This constraint implies a relation between the filling fractions,

$$\sum_n n\,\alpha_n = 0, \tag{E.6}$$

which is also familiar from the asymptotic Bethe ansatz.

E.2 Curvature function and higher mode numbers

In the main text we discussed the NLO solutions to the $\mathbf{P}\mu$ system which are based on the leading order solutions (4.104)-(4.107) or (4.109)-(4.112). One of the assumptions for constructing the leading order solution was to allow μ_{ab} to have only $e^{\pm 2\pi u}$ in asymptotics at infinity (we recall that this led to all μ's being constatnt except μ_{24} which is equal to $\sinh(2\pi u)$ or $\cosh(2\pi u)$), while in principle requiring μ_{ab} to be periodic one could also allow to have $e^{2n\pi u}$ with any integer n. Thus a natural generalization of the leading order solution is to consider $\mu_{24} = \sinh(2\pi n u)$ or $\mu_{24} = \cosh(2\pi n u)$, where n is an arbitrary integer. As discussed above (see the end of section 4.5 and appendix E.1), we believe that at the leading order in S such solutions correspond to states with mode numbers equal to n, and they reproduce the slope function for this case.

Proceeding to order S^2, the calculation of the curvature function $\gamma^{(2)}(g)$ with $\mu_{24} = \sinh(2\pi n u)$ or $\mu_{24} = \cosh(2\pi n u)$ can be done following the same steps as for $n = 1$. The final results for $J = 2, 3$ and 4 are given by exactly the same formulas as for $n = 1$ ((4.167), (4.184) and (C.32) respectively) – the only difference is that now one should set in those expressions

$$I_k = I_k(4\pi n g), \tag{E.7}$$

$$\mathrm{sh}^x_- = [\sinh(2\pi n u_x)]_-\,, \tag{E.8}$$

$$\mathrm{sh}^y_- = [\sinh(2\pi n u_y)]_-\,, \tag{E.9}$$

$$\mathrm{ch}^x_- = [\cosh(2\pi n u_x)]_-\,, \tag{E.10}$$

$$\mathrm{ch}^y_- = [\cosh(2\pi n u_y)]_-\,. \tag{E.11}$$

It would be natural to assume that this solution of the $\mathbf{P}\mu$ system describes anomalous dimensions for states with mode number n at order S^2. However we found some peculiarities in the strong coupling expansion of the result. The strong coupling data available for comparison in the literature for states with $n > 1$ also relies on some conjectures (see [68], [?]), so the interpretation of this solution is not fully clear to us.

The weak coupling expansion for this case turns out to be related in a simple way to the $n = 1$ case. One should just replace $\pi \to n\pi$ in the expansions for $n = 1$ which were given in (D.1), (D.2), (D.3). For example,

$$\gamma^{(2)}_{J=2} \quad = \quad -8g^2\zeta_3 + g^4\left(140\zeta_5 - \frac{32n^2\pi^2\zeta_3}{3}\right) + g^6\left(200n^2\pi^2\zeta_5 - 2016\zeta_7\right) + \ldots \tag{E.12}$$

It would be interesting to compare these weak coupling predictions to results obtained from the asymptotic Bethe ansatz (or by other means) as it was done for $n = 1$ in section **??**.

Let us now discuss the strong coupling expansion. According to Basso's conjecture [68] (see also [?]), the structure of the expansion may be obtained from

$$\Delta^2 = J^2 + S\left(A_1\sqrt{\mu} + A_2 + \ldots\right) + S^2\left(B_1 + \frac{B_2}{\sqrt{\mu}} + \ldots\right) + S^3\left(\frac{C_1}{\mu^{1/2}} + \frac{C_2}{\mu^{3/2}} + \ldots\right) + \mathcal{O}(S^4)\,, \tag{E.13}$$

where $\mu = n^2\lambda$. This gives

$$\begin{aligned}\Delta \quad &= \quad J + \frac{S}{2J}\left(A_1 n\sqrt{\lambda} + A_2 + \frac{A_3}{n\sqrt{\lambda}} + \ldots\right) \\ &+ \quad S^2\left(-\frac{A_1^2}{8J^3}n^2\lambda - \frac{A_1A_2}{4J^3}n\sqrt{\lambda} + \left[\frac{B_1}{2J} - \frac{A_2^2 + 2A_1A_3}{8J^3}\right] + \left[\frac{B_2}{2J} - \frac{A_2A_3 + A_1A_4}{4J^3}\right]\frac{1}{n\sqrt{\lambda}} + \ldots\right) + \mathcal{O}(S^3)\,.\end{aligned} \tag{E.14}$$

where A_i are known from Basso's slope function. Substituting them, we find

$$\gamma^{(2)}_J(g) = -\frac{8\pi^2g^2n^2}{J^3} + \frac{2\pi gn}{J^3} + \frac{B_1 - 1}{2J} + \frac{8B_2J^2 - 4J^2 + 1}{64\pi gJ^3n} + \ldots \tag{E.15}$$

However, already in [?] some inconsistencies were found if one assumes this structure for $n > 1$. Let us extend that analysis by comparing the prediction (E.15) to our results from the $\mathbf{P}\mu$-system. To compute the expansion of our results, similarly to the $n = 1$ case, we evaluated $\gamma^{(2)}_J(g)$ numerically for many values of g, and then fitted the result by powers of g. As for $n = 1$ we found with high precision (about $\pm 10^{-16}$) that the first several coefficients involve only rational numbers and powers of π. Our results for $n = 2, 3$ and $J = 2, 3, 4$ are summarized below:

$$\gamma^{(2)}_{J=2,n=2}(g) = -4\pi^2g^2 + \frac{\pi g}{2} + \frac{17}{8} - \frac{0.29584877037648771(2)}{g} + \ldots \tag{E.16}$$

$$\gamma^{(2)}_{J=3,n=2}(g) = -\frac{32}{27}\pi^2g^2 + \frac{4\pi g}{27} + \frac{17}{12} - \frac{0.2928304112866493(9)}{g} + \ldots \tag{E.17}$$

$$\gamma^{(2)}_{J=4,n=2}(g) = -\frac{1}{2}\pi^2g^2 + \frac{\pi g}{16} + \frac{17}{16} - \frac{0.319909936615448(9)}{g} + \ldots \tag{E.18}$$

$$\gamma^{(2)}_{J=2,n=3}(g) = -9\pi^2g^2 + \frac{3\pi g}{4} + \frac{23}{4} - \frac{0.8137483(9)}{g} + \ldots \tag{E.19}$$

$$\gamma^{(2)}_{J=3,n=3}(g) = -\frac{8}{3}\pi^2 g^2 + \frac{2\pi g}{9} + \frac{23}{6} - \frac{0.892016609(2)}{g} + \ldots \tag{E.20}$$

$$\gamma^{(2)}_{J=4,n=3}(g) = -\frac{9}{8}\pi^2 g^2 + \frac{3\pi g}{32} + \frac{23}{8} - \frac{1.035945580(6)}{g} + \ldots \tag{E.21}$$

Here in the coefficient of $\frac{1}{g}$ the digit in brackets is the last known one within our precision[38].

Comparing to (E.15) we find full agreement in the first two terms (of order g^2 and of order g). The next term in (E.15) (of order g^0) is determined by B_1, which in [?] was found to be

$$B_1 = \frac{3}{2} \tag{E.22}$$

for all n, J, based on consistency with the classical energy. However, comparing our results with (E.15) we find a different value:

$$\begin{aligned} B_1 &= \frac{19}{2} \quad \text{for} \quad n = 2\,, \\ B_1 &= 23 \quad \text{for} \quad n = 3\,. \end{aligned} \tag{E.23}$$

For both $n = 2$ and $n = 3$ this prediction for B_1 is independent of J.

The next term is of order $\frac{1}{g}$ and is determined by B_2, which in [?] was fixed to

$$B_2 = \begin{cases} -3\,\zeta_3 + \frac{3}{8} & ,\ n = 1 \\ -24\,\zeta_3 - \frac{13}{8} & ,\ n = 2 \\ -81\,\zeta_3 - \frac{24}{8} & ,\ n = 3 \end{cases} . \tag{E.24}$$

However, this does not agree with our numerical predictions for $n = 2$ and 3. Furthermore, for $n = 2$ we extracted the coefficient of $\frac{1}{g}$ with high precision (about 10^{-17}, see (E.16)) but were unable to fit it as a combination of simple zeta values using the EZ-Face calculator [?].

Thus our results appear to disagree with the values of B_1 and B_2 obtained in [?], but how to interpret this is not clear to us. Although our solution of the $\mathbf{P}\mu$-system for $n > 1$ looks fine at order S, it may be that to capture anomalous dimensions at order S^2 some other solution should be used. Another option is that the ansatz for the structure of anomalous dimensions at strong coupling may need to be modified when $n > 1$ (as already suspected in [?]), and our results may help provide some guidance in this case.

F Exact formulae for one-loop correction

F.1 Main formula for one-loop correction and notations

In [?] a general formula was derived describing the one loop correction to the energy of the generic (S, J, n) folded string solution. There are three contributions to one loop energy shift that are different by their nature. They can be separated into an "anomaly" contribution, a

[38] We did not seek to achieve high precision in this coefficient for $n = 3$.

contribution from the dressing phase and a wrapping contribution, which is missing in the ABA approach, but present in the Y-system

$$\Delta_{1-\text{loop}} = \delta\Delta_{\text{anomaly}} + \delta\Delta_{\text{dressing}} + \delta\Delta_{\text{wrapping}} \,, \tag{F.1}$$

where each of these contributions is simply an integral of some closed form expression,

$$\delta\Delta_{\text{anomaly}} = -\frac{4}{ab-1}\int_a^b \frac{dx}{2\pi i}\frac{y(x)}{x^2-1}\partial_x \log\sin p_{\hat{2}} \,, \tag{F.2}$$

$$\delta\Delta_{\text{dressing}} = \sum_{ij}(-1)^{F_{ij}}\int_{-1}^{1}\frac{dz}{2\pi i}\left(\Omega^{ij}(z)\,\partial_z\frac{i(p_i-p_j)}{2}\right) \,, \tag{F.3}$$

$$\delta\Delta_{\text{wrapping}} = \sum_{ij}(-1)^{F_{ij}}\int_{-1}^{1}\frac{dz}{2\pi i}\left(\Omega^{ij}(z)\,\partial_z\log(1-e^{-i(p_i-p_j)})\right) \,. \tag{F.4}$$

in this sum i takes values $\hat{1},\hat{2},\tilde{1},\tilde{2}$ whereas j runs over $\hat{3},\hat{4},\tilde{3},\tilde{4}$.

Let us explain the notations. The quasi-momenta:

$$\begin{aligned} p_{\hat{2}} &= \pi n - 2\pi n\mathcal{J}\left(\frac{a}{a^2-1}-\frac{x}{x^2-1}\right)\sqrt{\frac{(a^2-1)(b^2-x^2)}{(b^2-1)(a^2-x^2)}} \\ &+ \frac{8\pi nab\mathcal{S}F_1(x)}{(b-a)(ab+1)} + \frac{2\pi n\mathcal{J}(a-b)F_2(x)}{\sqrt{(a^2-1)(b^2-1)}}, \\ p_{\tilde{2}} &= \frac{2\pi\mathcal{J}x}{x^2-1}. \end{aligned} \tag{F.5}$$

The integer n (the mode number) is related to the number of spikes. All the other quasi-momenta can be found from

$$p_{\hat{2}}(x) = -p_{\hat{3}}(x) = -p_{\hat{1}}(1/x) = p_{\hat{4}}(1/x) \,, \tag{F.6}$$

$$p_{\tilde{2}}(x) = -p_{\tilde{3}}(x) = p_{\tilde{1}}(x) = -p_{\tilde{4}}(x) \,. \tag{F.7}$$

The functions $F_1(x)$ and $F_2(x)$ can be expressed in terms of the elliptic integrals:

$$\begin{aligned} F_1(x) &= iF\left(i\sinh^{-1}\sqrt{\frac{(b-a)(a-x)}{(b+a)(a+x)}}\,\Big|\,\frac{(a+b)^2}{(a-b)^2}\right) \,, \\ F_2(x) &= iE\left(i\sinh^{-1}\sqrt{\frac{(b-a)(a-x)}{(b+a)(a+x)}}\,\Big|\,\frac{(a+b)^2}{(a-b)^2}\right) \,. \end{aligned}$$

Finally the off-shell fluctuation energies are

$$\begin{aligned} \Omega^{\hat{1}\hat{4}}(x) &= -\Omega^{\hat{2}\hat{3}}(1/x) - 2 \,, \\ \Omega^{\hat{1}\hat{3}}(x) &= \Omega^{\hat{2}\hat{4}}(x) = \frac{1}{2}\Omega^{\hat{1}\hat{4}}(x) + \frac{1}{2}\Omega^{\hat{2}\hat{3}}(x) \,, \\ \Omega^{\hat{1}\tilde{3}}(x) &= \Omega^{\hat{1}\tilde{4}}(x) = \Omega^{\hat{4}\tilde{1}}(x) = \Omega^{\hat{4}\tilde{2}}(x) = \frac{1}{2}\,\Omega^{\tilde{2}\tilde{3}}(x) + \frac{1}{2}\,\Omega^{\hat{1}\hat{4}}(x), \\ \Omega^{\hat{2}\tilde{3}}(x) &= \Omega^{\hat{2}\tilde{4}}(x) = \Omega^{\tilde{1}\hat{3}}(x) = \Omega^{\tilde{2}\hat{3}}(x) = \frac{1}{2}\,\Omega^{\tilde{2}\tilde{3}}(x) + \frac{1}{2}\,\Omega^{\hat{2}\hat{3}}(x), \\ \Omega^{\tilde{2}\tilde{3}}(x) &= \Omega^{\tilde{2}\tilde{4}}(x) = \Omega^{\tilde{1}\tilde{3}}(x) = \Omega^{\tilde{1}\tilde{4}}(x) \,, \end{aligned} \tag{F.8}$$

where

$$\Omega^{\tilde{2}\tilde{3}}(x) = \frac{2}{ab-1}\frac{\sqrt{a^2-1}\sqrt{b^2-1}}{x^2-1}\,, \tag{F.9}$$

$$\Omega^{\hat{2}\hat{3}}(x) = \frac{2}{ab-1}\left(1-\frac{y(x)}{x^2-1}\right)\,. \tag{F.10}$$

and $y(x)=\sqrt{x-a}\sqrt{a+x}\sqrt{x-b}\sqrt{b+x}$.

In the small $\mathcal{S}$ limit these expressions can be expanded,

$$\begin{aligned}
\delta\Delta_{anomaly} &= \frac{-1}{2(\mathcal{J}^3+\mathcal{J})}\,\mathcal{S} + \left[\frac{2\,\mathcal{J}^4+15\,\mathcal{J}^2+4}{16\,\mathcal{J}^3(\mathcal{J}^2+1)^{5/2}} - \frac{\pi^2 n^2}{12\,\mathcal{J}^3\sqrt{\mathcal{J}^2+1}}\right]\mathcal{S}^2 \\
&+ \left[\frac{3\,\mathcal{J}^8-32\,\mathcal{J}^6-146\,\mathcal{J}^4-68\,\mathcal{J}^2-16}{64\,\mathcal{J}^5(1+\mathcal{J}^2)^4} + \frac{\pi^2n^2(\mathcal{J}^4+4\,\mathcal{J}^2+2)}{24\,\mathcal{J}^5(1+\mathcal{J}^2)^2} + \frac{\pi^4 n^4}{180\,\mathcal{J}^5}\right]\mathcal{S}^3 + \mathcal{O}(\mathcal{S}^4) \\
\delta\Delta_{dressing} &= \left[\frac{n\,(\mathcal{J}^2+2)\coth^{-1}(\sqrt{\mathcal{J}^2+1}+\mathcal{J})}{\mathcal{J}^3(\mathcal{J}^2+1)^{3/2}} - \frac{n}{2\,\mathcal{J}^3(\mathcal{J}^2+1)}\right]\mathcal{S}^2 \\
&+ \left[-\frac{n(3\,\mathcal{J}^6+13\,\mathcal{J}^4+22\,\mathcal{J}^2+8)\coth^{-1}(\sqrt{\mathcal{J}^2+1}+\mathcal{J})}{2\,\mathcal{J}^5(1+\mathcal{J}^2)^3} + \frac{n(9\,\mathcal{J}^4+31\,\mathcal{J}^2+10)}{12\,\mathcal{J}^5(1+\mathcal{J}^2)^{5/2}}\right]\mathcal{S}^3 + \mathcal{O}(\mathcal{S}^4)
\end{aligned} \tag{F.11}$$

the expansion of the third integral $\delta\Delta_{wrapping}$ is more complicated, and we advice the reader to use the equation (4.211) instead which includes all contributions. What we can, however, say is that $\delta\Delta_{wrapping} = \mathcal{O}(e^{-2\pi\mathcal{J}})$ and thus this term is irrelevant for the large $\mathcal{J}$ expansion. This makes the expressions (F.11) particularly convenient for small $\mathcal{S}$ followed by large $\mathcal{J}$ expansions, where as the exact $\mathcal{J}$ expression in (4.211) is not very convenient since the sum of the expansion does not converge.

F.2 One Loop (S, J) Folded String *Mathematica* Code

In order to fix all our conventions as well as for the convenience of the reader we include a simplified *Mathematica* code we used to check our results numerically

```
GS=((2*(a*b+1))*(b*EllipticE[1-a^2/b^2]-a*EllipticK[1-a^2/b^2]))/(4*Pi*a*b);
GJ=((4*Sqrt[(a^2-1)*(b^2-1)])*EllipticK[1-a^2/b^2])/(4*Pi*b);
y=Sqrt[x-a]*Sqrt[x+a]*Sqrt[x-b]*Sqrt[x+b];
F1[x_] =I*EllipticF[I*ArcSinh[Sqrt[-(((a-b)*(a-x))/((a+b)*(a+x)))]], (a+b)^2/(a-b)^2];
F2[x_] =I*EllipticE[I*ArcSinh[Sqrt[-(((a-b)*(a-x))/((a+b)*(a+x)))]], (a+b)^2/(a-b)^2];
pA[x_] =n*Pi-2*Pi*n*j*(a/(a^2-1)-x/(x^2-1))*Sqrt[((a^2-1)*(b^2-x^2))/((b^2-1)*(a^2-x^2))] +
       (8*a*b*s*Pi*n*F1[x])/((b-a)*(a*b+1))+(2*Pi*n*j*(a-b)*F2[x])/Sqrt[(a^2-1)*(b^2-1)];
pS[x_]=(2*Pi*n*j*x)/(x^2-1);
X[z_]=z+Sqrt[z^2-1];
OA[x_]=(2*(1-y/(x^2-1)))/(a*b-1);
OS[x_]=(2*(-(y /. x->1)))/((a*b-1)*(x^2-1));
ab[j_, s_] :=ab[j, s]=Chop[FindRoot[{s==GS, j==GJ}, {{b, Sqrt[j^2+1]+j+Sqrt[s]/10}
                                                   ,{a, Sqrt[j^2+1]+j-Sqrt[s]/10}}]];
OneLoop[jj_, ss_, nn_] := Block[{sb0=Join[ab[jj, ss], {j->jj, s->ss, n->nn}]},
tn0=(2*Im[pA[X[z]]-pS[X[z]]]*Im[D[OA[X[z]]-OS[X[z]], z]])/Pi /. sb0;
Edressing=NIntegrate[tn0, {z, 0, 1}];
tn1=(2*D[OS[X[z]], z]*Log[((1-Exp[(-I)*pS[X[z]]-I*pA[X[z]]])*(1-Exp[(-I)*pS[X[z]]+I*pA[1/X[z]]]))/
                                                          (1-Exp[-2*I*pS[X[z]]])^2])/Pi /. sb0;
tn2=-((2*D[OA[X[z]], z]*Log[((1-Exp[-2*I*pA[X[z]]])*(1-Exp[(-I)*pA[X[z]]+I*pA[1/X[z]]]))/
                                                 (1-Exp[(-I)*pS[X[z]]-I*pA[X[z]]])^2])/Pi) /. sb0;
Ewrapping=NIntegrate[Im[tn1+tn2], {z, 0, 1}];
tn=-((4*y*D[Log[Sin[pA[x]]], x])/((a*b-1)*(2*Pi*I)*(x^2-1))) /. sb0;
Eanomaly=Re[NIntegrate[tn, {x, a /. sb0, ((a+b)*(1+I))/(2*10) /. sb0, b /. sb0}]];
Edressing+Ewrapping+Eanomaly];
```

To compute $\Delta_{1-\text{loop}}$ simply run `OneLoop[`$\mathcal{J}, \mathcal{S}, n$`]` in *Mathematica*.

G S^3 and S^4 order

The $\mathcal{S}^3$ order term in the expression of (4.211) is given by

$$\delta\Delta^{(3)}_{1-loop} = -\frac{6\,\mathcal{J}^8+48\,\mathcal{J}^6+138\,\mathcal{J}^4+352\,\mathcal{J}^2+117}{64\,\mathcal{J}^5(\,\mathcal{J}^2+1)^4} + \sum_{m>0,m\neq n}\frac{P_3(n,m,\mathcal{J})}{2\,\mathcal{J}^5(\,\mathcal{J}^2+1)^{3/2}(m^2-n^2)^4(\mathcal{J}^2n^2+m^2)^{5/2}} \tag{G.1}$$

and the $\mathcal{S}^4$ order term is given by

$$\delta\Delta^{(4)}_{1-loop} = \frac{45\mathcal{J}^{12}+717\mathcal{J}^{10}+3429\mathcal{J}^8+11205\mathcal{J}^6+27601\mathcal{J}^4+15789\mathcal{J}^2+3305}{1024\mathcal{J}^7\left(\mathcal{J}^2+1\right)^{11/2}} - \sum_{m>0,m\neq n}\frac{P_4(n,m,\mathcal{J})}{16\mathcal{J}^7\left(\mathcal{J}^2+1\right)^3(m^2-n^2)^6\left(m^2+n^2\mathcal{J}^2\right)^{7/2}} \tag{G.2}$$

where

$$\begin{aligned} P_3(n,m,\mathcal{J}) = &+ m^{10}n^3\left(4\mathcal{J}^4+11\mathcal{J}^2+6\right)+2m^8n^5\left(3\mathcal{J}^6+5\mathcal{J}^4-6\mathcal{J}^2-6\right) \\ &+ 2m^6n^7\left(\mathcal{J}^8-4\mathcal{J}^6-11\mathcal{J}^4+6\mathcal{J}^2+9\right)+2m^4n^9\left(-2\mathcal{J}^8+9\mathcal{J}^6+29\mathcal{J}^4+14\mathcal{J}^2-2\right) \\ &+ m^2n^{11}\mathcal{J}^2\left(10\mathcal{J}^6+16\mathcal{J}^4-2\mathcal{J}^2-7\right)\ , \end{aligned} \tag{G.3}$$

$$\begin{aligned} P_4(n,m,\mathcal{J}) = &+ 4m^{16}n^3\left(8\mathcal{J}^8+42\mathcal{J}^6+85\mathcal{J}^4+68\mathcal{J}^2+20\right) \\ &+ m^{14}n^5\left(80\mathcal{J}^{10}+302\mathcal{J}^8+199\mathcal{J}^6-703\mathcal{J}^4-936\mathcal{J}^2-340\right) \\ &+ m^{12}n^7\left(64\mathcal{J}^{12}-12\mathcal{J}^{10}-893\mathcal{J}^8-1765\mathcal{J}^6+151\mathcal{J}^4+1587\mathcal{J}^2+740\right) \\ &+ m^{10}n^9\left(16\mathcal{J}^{14}-222\mathcal{J}^{12}-587\mathcal{J}^{10}+1209\mathcal{J}^8+5444\mathcal{J}^6+4374\mathcal{J}^4+388\mathcal{J}^2-520\right) \\ &+ 2m^8n^{11}\left(-38\mathcal{J}^{14}+200\mathcal{J}^{12}+1446\mathcal{J}^{10}+2505\mathcal{J}^8+769\mathcal{J}^6-511\mathcal{J}^4+17\mathcal{J}^2+210\right) \\ &+ m^6n^{13}\left(200\mathcal{J}^{14}+572\mathcal{J}^{12}+206\mathcal{J}^{10}-176\mathcal{J}^8+2199\mathcal{J}^6+3085\mathcal{J}^4+1068\mathcal{J}^2-60\right) \\ &+ m^4n^{15}\mathcal{J}^2\left(-76\mathcal{J}^{12}+464\mathcal{J}^{10}+2920\mathcal{J}^8+5315\mathcal{J}^6+3667\mathcal{J}^4+643\mathcal{J}^2-173\right) \\ &+ m^2n^{17}\mathcal{J}^4\left(256\mathcal{J}^{10}+962\mathcal{J}^8+1221\mathcal{J}^6+401\mathcal{J}^4-250\mathcal{J}^2-148\right)\ . \end{aligned} \tag{G.4}$$

The expansion (3.88) can also be written in higher orders of $\mathcal{S}$ and $\mathcal{J}$, for $n=1$ we get

$$\begin{aligned} \Delta_{1-loop} = &\left(\frac{-1}{2\,\mathcal{J}}+\frac{\mathcal{J}}{2}\right)\mathcal{S}+\left(\frac{1}{2\,\mathcal{J}^3}-\left[\frac{3\,\zeta_3}{2}+\frac{1}{16}\right]\frac{1}{\mathcal{J}}+\left[\frac{3\,\zeta_3}{2}+\frac{15\,\zeta_5}{8}-\frac{21}{32}\right]\mathcal{J}\right)\mathcal{S}^2 \\ + &\left(\frac{-3}{4\,\mathcal{J}^5}+\left[\frac{3\,\zeta_3}{2}+\frac{3}{16}\right]\frac{1}{\mathcal{J}^3}+\left[\frac{9\,\zeta_3}{8}-\frac{1}{32}\right]\frac{1}{\mathcal{J}}+\left[\frac{5}{4}-\frac{17\,\zeta_3}{4}-\frac{65\,\zeta_5}{16}-\frac{35\,\zeta_7}{16}\right]\mathcal{J}\right)\mathcal{S}^3 \\ + &\left(\frac{5}{4\mathcal{J}^7}-\left[\frac{7}{32}+\frac{9\zeta_3}{4}\right]\frac{1}{\mathcal{J}^5}-\left[\frac{3\zeta_3}{4}-\frac{15\zeta_5}{16}+\frac{5}{32}\right]\frac{1}{\mathcal{J}^3}-\left[\frac{145\zeta_3}{64}+\frac{45\zeta_5}{32}+\frac{175\zeta_7}{128}+\frac{27}{1024}\right]\frac{1}{\mathcal{J}}\right)\mathcal{S}^4+\mathcal{O}(\mathcal{S}^5)\ , \end{aligned} \tag{G.5}$$

for $n=2$,

$$\begin{aligned} \Delta_{1-loop} = &\left(\frac{-1}{2\,\mathcal{J}}+\frac{\mathcal{J}}{2}\right)\mathcal{S}+\left(\frac{1}{2\,\mathcal{J}^3}-\left[12\,\zeta_3+\frac{17}{16}\right]\frac{1}{\mathcal{J}}+\left[12\,\zeta_3+60\,\zeta_5+\frac{27}{32}\right]\mathcal{J}\right)\mathcal{S}^2 \\ + &\left(\frac{21}{4\,\mathcal{J}^5}+\left[12\,\zeta_3+\frac{19}{16}\right]\frac{1}{\mathcal{J}^3}+\left[9\,\zeta_3+\frac{47}{32}\right]\frac{1}{\mathcal{J}}-\left[\frac{19}{4}+34\,\zeta_3+130\,\zeta_5+280\,\zeta_7\right]\mathcal{J}\right)\mathcal{S}^3 \\ + &\left(-\frac{175}{4\mathcal{J}^7}-\left[\frac{727}{32}+18\zeta_3\right]\frac{1}{\mathcal{J}^5}-\left[6\zeta_3-30\zeta_5-\frac{155}{32}\right]\frac{1}{\mathcal{J}^3}-\left[\frac{145\zeta_3}{8}+45\zeta_5+175\zeta_7+\frac{7419}{1024}\right]\frac{1}{\mathcal{J}}\right)\mathcal{S}^4+\mathcal{O}(\mathcal{S}^5)\ , \end{aligned} \tag{G.6}$$

and finally for $n = 3$,

$$\begin{aligned}\Delta_{1-loop} = & \left(\frac{-1}{2\,\mathcal{J}} + \frac{\mathcal{J}}{2}\right)\mathcal{S} + \left(\frac{-5}{8\,\mathcal{J}^3} - \left[\frac{81\,\zeta_3}{2} + \frac{7}{4}\right]\frac{1}{\mathcal{J}} + \left[\frac{81\,\zeta_3}{2} + \frac{3645\,\zeta_5}{8} - \frac{147}{64}\right]\mathcal{J}\right)\mathcal{S}^2 \\
+ & \left(\frac{1245}{32\,\mathcal{J}^5} + \left[\frac{81\,\zeta_3}{2} + \frac{39}{16}\right]\frac{1}{\mathcal{J}^3} + \left[\frac{243\,\zeta_3}{8} + \frac{89}{32}\right]\frac{1}{\mathcal{J}} - \left[\frac{89}{8} + \frac{459\,\zeta_3}{4} + \frac{15795\,\zeta_5}{16} + \frac{76545\,\zeta_7}{16}\right]\mathcal{J}\right)\mathcal{S}^3 \\
- & \left(\frac{258785}{512\,\mathcal{J}^7} + \left[\frac{243\,\zeta_3}{4} + \frac{251423}{1024}\right]\frac{1}{\mathcal{J}^5} - \left[\frac{3645\,\zeta_5}{16} - \frac{81\,\zeta_3}{4} + \frac{256229}{4096}\right]\frac{1}{\mathcal{J}^3}\right. \\
+ & \left.\frac{27\,(907200\,\zeta_7 + 103680\,\zeta_5 + 18560\,\zeta_3 + 13457)}{8192\,\mathcal{J}}\right)\mathcal{S}^4 + \mathcal{O}(\mathcal{S}^5).\end{aligned} \tag{G.7}$$

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