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Exact Results in Supersymmetric Gauge Theories

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Thesis submitted in partial fulfilment of the requirements
of the Degree of Doctor of Philosophy

September 2014

Abstract

In this thesis we discuss supersymmetric gauge theories, focusing on exact results achieved using methods of integrability. For the larger portion of the thesis we study the $\mathcal{N}=4$ super Yang-Mills theory in the planar limit, a recurring concept being the Konishi anomalous dimension, which is roughly the analogue for the mass of the proton in quantum chromodynamics. The $\mathcal{N}=4$ supersymmetric Yang-Mills theory is known to be integrable in the planar limit, which opens up a wealth of techniques one can employ in order to find results in this limit valid at any value of the coupling.

We begin with perturbation theory where the integrability of the theory first manifests itself. Here we are able to find the first exact result, the so-called slope function, which is the linear small spin expansion coefficient of the generalized Konishi anomalous dimension. We then move on to exact results mainly achieved using the novel quantum spectral curve approach, the method allowing one to find scaling dimensions of operators at arbitrary values of the coupling. As an example we find the second coefficient in the small spin expansion after the slope, which we call the curvature function. This allows us to extract non-trivial information about the Konishi operator.

Methods of integrability are also applicable to other supersymmetric gauge theories such as ABJM, which is also known to be integrable and in fact shares many similarities with $\mathcal{N}=4$ super Yang-Mills. We briefly review these parallel developments in the last chapter of the thesis.

Contents

1 Introduction

Everything is interesting if you go into it deeply enough.
– Richard Feynman

The title of this thesis is *Exact Results in Supersymmetric Gauge Theories*. A reasonable question to ask is – why would anyone care about that ? After all there is no evidence that supersymmetry is a true symmetry of nature and supersymmetric theories are mostly toy theories, we can not observe them in particle accelerators, as opposed to the Standard Model of particle physics. And indeed these are all valid points, however there are very good reasons for studying them.

Consider $\mathcal{N} = 4$ super Yang-Mills, from a pragmatic point of view it is the simplest non-trivial quantum field theory in four spacetime dimensions and since attempts at solving realistic QFTs such as the theory of strong interactions (QCD) have so far been futile, it seems like a good starting point – some go as far as calling it the harmonic oscillator of QFTs.

Another (and probably the main) reason why $\mathcal{N} = 4$ has been receiving so much attention in the last decades is the long list of mysterious and intriguing properties it seems to posses, making it almost an intellectual pursuit of understanding it. The theory has been surprising the theoretical physics community from the very beginning: it is a rare instance of a conformal theory in dimensions higher than two, it has a dual description in terms of a string theory and more recently it was discovered to be integrable in the planar limit. All of these properties give reasonable hope for actually solving the theory exactly, something that has never been achieved before for any four dimensional interacting QFT.

In the remainder of the section we give a proper introduction to the subject from a historic point of view focusing on $\mathcal{N} = 4$ SYM and its integrability aspect, for it is integrability that allows one to actually find exact results in the theory. We then give an overview of the thesis itself, emphasizing which parts of the text are reviews of known material and which parts constitute original work.

• Make sure you cite your referees!

1.1 Brief history of the subject

Quantum field theory has been at the spot light of theoretical physics since the beginning of the century when it was found that electromagnetism is described by the theory of quantum electrodynamics (QED). Since then people have been trying to fit other forces of nature into the QFT framework. Ultimately it worked: the theory of strong interactions, quantum chromodynamics or QCD for short, together with the electroweak theory, spontaneously broken down to QED, collectively make up the *Standard Model* of particle physics, which has been extensively tested in particle accelerators since then.

However nature did not give away her secrets without a fight. For some time it was though that strong interactions were described by a theory of vibrating strings, as it seemed to incorporate the so-called Regge trajectories observed in experiments [1]. Even after discovering QCD, which is a Yang-Mills gauge theory, stringy aspects of it were still evident and largely mysterious. Most notably lattice gauge theory calculations at strong coupling suggested that surfaces of color-electric fluxes between quarks could be given the interpretation of stretched strings [2], thus an idea of a gauge-string duality was starting to emerge. It was strongly re-enforced by t'Hooft, who showed that the perturbative expansion of $U(N)$ gauge theories in the large N limit could be rearranged into a genus expansion of surfaces triangulated by the double-line Feynman graphs, which strongly resembles string theory genus expansions [3].

However it was the work of Maldacena in the end of 1997 that sparked a true revolution in the field [4]. He formulated the first concrete conjecture, now universally referred to as *AdS/CFT*, for a duality between a gauge theory, the maximally supersymmetric $\mathcal{N}=4$ super Yang-Mills, and type IIB string theory on $AdS_5 \times S^5$. Polyakov had already shown that non-critical string theory in four-dimensions describing gauge fields should be complemented with an extra Liouville-like direction thus enriching the space to a curved five dimensional manifold [5]. Furthermore the gauge theory had to be defined on the boundary of this manifold. Maldacena's conjecture was consistent with this view, as the gauge theory was defined on the boundary of AdS_5, whereas the

S^5 was associated with the internal symmetries of the gauge fields. The idea of a higher dimensional theory being fully described by a theory living on the boundary was also considered before in the context of black hole physics [6, 7] and goes by the name of holography, thus AdS/CFT is also referred to as a holographic duality.

The duality can be motivated by considering a stack of N parallel D3 branes in type IIB string theory. Open strings moving on the branes can be described by $\mathcal{N}=4$ SYM with the gauge group $SU(N)$. Roughly the idea is that there are six extra dimensions transverse to the stack of branes, thus a string stretching between two of them can be viewed as a set of six scalar fields $(\Phi^i)^a{}_b$ defined in four dimensional spacetime carrying two extra indices denoting the branes it is attached to. These are precisely the indices of the adjoint representation of $SU(N)$. A similar argument can be put forward for other fields thus recovering the field content of $\mathcal{N}=4$ SYM. Far away from the branes we have closed strings propagating in empty space. In the low energy limit these systems decouple and far away from the branes we are left with ten dimensional supergravity.

Another way of looking at this system is considering the branes as a defect in spacetime, which from the point of view of supergravity is a source of curvature. The supergravity solution carrying D3 brane charge can be written down explicitly [8]. Far away from the branes it is obviously once again the usual flat space ten dimensional supergravity. However the near horizon the geometry of the brane system becomes $AdS_5 \times S^5$. Since both points of view end up with supergravity far away from the branes, one is tempted to identify the theories close to the branes, $\mathcal{N}=4$ SYM and type IIB string theory on $AdS_5 \times S^5$. This is exactly what Maldacena did in his seminal paper [4].

By studying the supergravity solution one can identify the parameters of the theories, namely $\mathcal{N}=4$ SYM is parametrized by the coupling constant g_{YM} and the number of colors N, whereas string theory has the string coupling constant g_s and the string length squared α'. These are identified in the following way

$$4\pi g_s = g_{YM}^2 \equiv \frac{\lambda}{N}, \qquad \frac{R^4}{\alpha'^2} = \lambda, \tag{1.1}$$

where λ is the t'Hooft coupling and R is the radius of both AdS_5 and S^5 which is fixed as only the ratio R^2/α' is measurable. A few things are to be noted here. First of all, the identification directly implements t'Hooft's idea of large N expansion of gauge theory, since $g_s \sim 1/N$. In fact in the large N limit only planar Feynman graphs survive and everything simplifies dramatically, a fact that we will take advantage of a lot in this thesis. In this limit the effective coupling constant of the gauge theory is λ.

for low lying states

The supergravity approximation is valid when $\alpha' \ll R^2$, which corresponds to strongly coupled gauge theory, thus the conjecture is of the weak-strong type. This fact is a blessing in disguise, since initially it seems very restrictive as one can not easily compare results of the theories. However it provides a possibility to access strongly coupled regimes of both theories, which was beyond reach before. Prescriptions for matching up observables on both sides of the correspondence were given in [9, 10]. However because of the weak-strong nature of the duality initial tests were performed only for BPS states, which are protected from quantum corrections. The first direct match was observed in [10] where it was shown that the spectrum of half-BPS single trace operators matches the Kaluza-Klein modes of type IIB supergravity. Another indirect confirmation of the conjecture was the formulation of type IIB string theory as a super-coset sigma model on the target space $PSU(2,2|4)/SO(2,4) \times SO(6)$, which has the same global symmetries as $\mathcal{N}=4$ SYM [11].

The situation changed dramatically in 2002 when Berenstein, Maldacena and Nastase devised a way to go beyond BPS checks [12]. The idea was to take an operator in gauge theory with large R-charge J and add some some impurities, effectively making it "near-BPS". The canonical example of such an operator is $\mathrm{Tr}(Z^J X^S)$, where Z and X are two complex scalar fields of $\mathcal{N}=4$ SYM, with X being the impurities ($S \ll J$). Since anomalous dimensions are suppressed like λ/J^2, perturbative gauge theory calculations are valid even at large λ, as long as $\lambda' \equiv \lambda/J^2 \ll 1$ and N is large. It is thus possible to compare gauge theory calculations with string theory results. From the string theory point of view this limit corresponds to excitations of point-like strings with angular momentum J moving at the speed of light around the great circle of S^5. The background seen by this string is the so-called pp-wave geometry and string theory in this background is tractable.

footnote! One should be carefull with limits order of limits issues. Ok only up to 3 loops

The discovery of the BMN limit was arguably the first time it was explicitly demonstrated how the world sheet theory of a string can be reconstructed by a physical picture of scalar fields dubbed as "impurities" propagating in a closed single trace operator of "background" scalar fields of the gauge theory. Shortly after this discovery Minahan and Zarembo revolutionized the subject once again by discovering Integrability at the end of 2002 [13]. They showed that in the large N limit single-trace operators of scalar fields can be identified with spin chains and their anomalous dimensions at one-loop in weak coupling are given by the energies of the corresponding spin chain states. These

spin-chain systems are known to be integrable, which in practice allows one to solve the problem exactly using techniques such as the Bethe ansatz [14]. This discovery sparked a very rapid development of integrability methods in AdS/CFT during the coming years.

Solving a quantum field theory in principle means finding all n-point correlation functions of all physical observables. Since $\mathcal{N}=4$ SYM is conformal it is enough to find all 2-point and 3-point correlators, as all higher point correlation functions can be decomposed in terms of these basic constituents. Due to conformal symmetry the two-point functions only depend on the scaling dimensions of operators, whereas for three point functions one also needs the so called structure constants C_{ijk} in addtion to the scaling dimensions. Integrability methods from the very beginning mainly focused on solving the spectral problem in the large N limit, that is finding the spectrum of operators with definite anomalous dimensions and their exact numeric values. The initial discovery of [13] was that the spectral problem was analogous to diagonalizing a spin chain Hamiltonian, which was identified with the dilatation operator of the superconformal symmetry of the theory. The eigenstates correspond to operators in the gauge theory and the eigenvalues are their anomalous dimensions.

Soon after the initial discovery of integrability a spin chain formulation at one-loop was found for the full $PSU(2,2|4)$ theory, not only the scalar sector [15]. The result was also extended to two and three loops [16]. Integrability was also discovered at strong coupling as it was shown that the Metsaev-Tseytlin sigma model is classically integrable [17]. With integrability methods now being available at both weak and strong coupling it was possible to compare results in the BMN limit. As expected, comparisons in the first two orders of the BMN coupling constant λ' showed promising agreement [18, 19, 20], however an order of limits problem emerged at three loops [16].

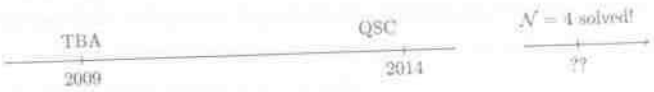
logic stops it seems

All of these results seemed to suggest that integrability may be an all loop phenomenon, only the surface of it being scratched so far. This notion was strongly reinforced when classical string integrability was reformulated in the elegant language of algebraic curves by Kazakov, Marshakov, Minahan and Zarembo (KMMZ), which made the connection with weak coupling more manifest [21]. The algebraic curve was

say, "which puzzled the community for some time"

interpreted as the continuum limit of Bethe equations, which made it possible to specu-
late about all loop equations. The first such attempt was made by Beisert, Dippel and
Staudacher (BDS), who conjectured a set of Bethe equations and a dispersion relation
which together successfully showcased some all-loop features [22]. This result was later
extended to all sectors of the theory [23]. The BDS result was quickly shown to be
incomplete as it was lacking a so-called dressing phase [24], a scalar function not con-
strained by symmetry of the problem. It was found to leading order at strong coupling
in [24] and later to one-loop in [25]. A crossing equation satisfied by the dressing phase
was soon found [26] and eventually solved by Beisert, Hernandez and Lopez (BHL) [27].
Collectively these results are often referred to as the asymptotic Bethe ansatz (ABA),
reminding that they are valid only for asymptotically long spin chains. When the states
are short so-called wrapping effects become relevant. At weak coupling they manifest
as long-range spin chain interactions wrapping around the chain, whereas at strong
coupling they are due to virtual particles self-interacting across the circumference of
the worldsheet [28, 29].

you can add that the dressing phase cured the 3 loop discrepancy



Once the asymptotic solution was found attention shifted to finite size corrections,
which once resolved would in principle complete the solution to the spectral problem
for single trace operators. Scattering corrections in finite volume for arbitrary QFTs
were first addressed by Lüscher [30], who derived a set of universal formulas. This
approach, while very general and not directly related to integrability, was employed to
calculate four [31] and five loop anomalous dimension coefficients [32] of the simplest
non-BPS operator with length two, the Konishi operator. The results agreed with
available diagrammatic four-loop calculations [33] and gave a new prediction for five
loops.

it was generalized by Janik et al. for non-relativistic case in place.

An alternative approach more in line with integrability is the Thermodynamic Bete
Ansatz (TBA). Its origins can be traced back to Yang and Yang [34], however it was
the work of Alexey Zamolodchikov [35, 36] that brought it to the mainstream. The
idea is to consider the partition function of a two dimensional integrable CFT and it's
"mirror" image found after exchanging length and time with a modular transformation.
At large imaginary times the partition function will be dominated by the ground state

1.1 Brief history of the subject 1 INTRODUCTION

energy, whereas in the mirror theory large time means asymptotic length, which is
under control using the asymptotic Bethe ansatz techniques. Thus one can evaluate the
partition function using the saddle point method and after rotating back to the original
theory compute the exact ground state energy. Excited states can then be reached
using analytic continuation. This approach was already proposed as an option for the
AdS/CFT system in [29] and was first discussed in depth in [37]. The TBA approach
crystallized in 2009 with multiple groups publishing results almost simultaneously [38,
39, 40, 41]. The Konishi anomalous dimension was initially checked at four [40] and five
[42, 43] loops by linearising the TBA equations, showing precise agreement with results
obtained using the Lüscher method. Ultimately the Konishi anomalous dimension was
calculated numerically for a wide range of values of the t'Hooft coupling constant [44].

And so the spectral problem seemed to be solved, at least in the case of Konishi an
exact and complete result was finally found, even if only numerically. However it was
increasingly becoming clear that the solution was not in its final and most elegant form.
Indeed the TBA equations are an infinite set of coupled integral equations, obviously
one has to employ various numerical tricks to actually solve them and this mostly works
in a case-by-case basis. Cases such as the $\mathfrak{sl}(2)$ sector of the theory, containing the
Konishi operator [40] and cusped Wilson lines [45, 46] have been worked out explicitly,
however it still remains a hard problem in general. From the very beginning alternative
formulations of the solution were being proposed. An infinite set of non-linear functional
equations, the so called Y-system was proposed already in [40], later completed with
analytical constraints coming from the TBA equations [47]. Connections of the Y-
system with the Hirota bilinear relation were later explored in [48] and the Y-system was
reduced to a finite set of non-linear integral equations (FiNLIE). The long sought beauty
of the solution to the spectral problem was arguably uncovered with the formulation of
the quantum spectral curve (QSC) approach [49, 50], also referred to as the $\mathbf{P}\mu$-system.
The whole TBA construction was ultimately reduced to a Riemann-Hilbert problem for
eight Q functions, which can be though of as the quantum analogues of quasimomenta
found in the algebraic curve construction. The QSC quickly showed its potential as
previously known results were rederived almost without any effort and new results were
being rapidly discovered [51, 52].

Thus one can safely say that the spectral problem in $\mathcal{N}=4$ SYM is by now very
well understood with numerical results readily available and deeper understanding of
the structure being within reach. Integrability methods have also been useful in other

areas such as three point functions [53, 54] and scattering amplitudes [55, 56], however the situation there is still not as complete. Having witnessed the successful resolution of the spectral problem it appears that $\mathcal{N}=4$ SYM is within reach of being solved completely. If this programme were to be successfully carried out it would be the first example of a four dimensional interacting quantum field theory being solved exactly. Undoubtedly this would provide a huge boost to our understanding of QFTs in general and hopefully bring us closed to solving QCD.

It turns out that $\mathcal{N}=4$ SYM is not the only example of an integrable supersymmetric gauge theory having a dual string description. Probably the most famous example involves the so called ABJM theory, proposed by Aharony, Bergman, Jafferis and Maldacena [57], following [58, 59, 60, 61]. It is a three-dimensional superconformal Chern-Simons gauge theory with $\mathcal{N}=6$ supersymmetry. This theory was conjectured to be the effective theory for a stack of M2 branes at a Z_k orbifold point. In the large N limit its gravitational dual turns out to be M-theory on $AdS_4 \times S^7/Z_k$. For large k and N with $\lambda = N/k$ fixed, the dual theory becomes type IIA superstring theory in $AdS_4 \times CP^3$. This duality is also integrable [62] and all of the developments outlined above have been reworked for it almost in parallel.

do you cite review?
there are several! Big review, also PhD thesises
may be useful?

1.2 Thesis overview

The thesis consists of four core chapters, the first three of which cover $\mathcal{N}=4$ super Yang-Mills. We start by introducing the theory in chapter 2 where we give its field content, Lagrangian and talk briefly about symmetries and their representations. We introduce the dual string theory in section 2.4 and talk briefly about its formulation as a super-coset sigma model.

Chapter 3 covers integrable structures found in $\mathcal{N}=4$ super Yang-Mills at strong and weak coupling, namely we discuss the spin chain picture at weak coupling in section 3.1 and the classical spectral curve picture found at strong coupling in section 3.3. A lot of focus in this chapter is put on the folded string solution described in section 3.3.3. It is the strong coupling dual to operators in the $\mathfrak{sl}(2)$ sector of $\mathcal{N}=4$ super Yang-Mills, which are described in section 3.1.2. A key result of the chapter is the calculation of the Konishi anomalous dimension up to two loops at strong coupling achieved by boosting the one-loop result with the help of the exact slope function found in section 3.2.1.

In chapter 4 we move away from the perturbative regime and introduce exact solution methods for the spectral problem of $\mathcal{N}=4$ super Yang-Mills – the thermodynamic Bethe ansatz and Y/T/Q-systems in section 4.1 and the novel quantum spectral curve construction in section 4.2. We then discuss exact solutions found using the quantum spectral curve starting with the slope function. We rederive it in section 4.3 and the calculation is then extended one order further to find the curvature function in 4.4. The Konishi anomalous dimension is revisited in section 4.5 where using the curvature function we boost the previously obtained two-loop strong coupling result to three loops. The chapter concludes with finding the anomalous dimension of a cusped Wilson line in the near-BPS limit in section 4.6 and addressing its classical limit.

Chapter 5 switches over from $\mathcal{N}=4$ super Yang-Mills to the ABJM theory and roughly follows the same path, however as most of the methods are very similar in spirit we move on much quicker. We introduce the theory in section 5.1 and discuss integrability in section 5.2. Section 5.3 describes the analogue of the folded string solution in ABJM, in particular the semi-classical quantization procedure of the solution. We finish with a short overview of exact results in section 5.4.

We end with conclusions and appendices containing some of the more technical details left out from the main text for brevity. The interdependencies between the chapters and sections of the text are shown in figure 1.

1.3 Original work

The thesis contains original work by the author from five papers published in collaboration with fellow colleagues while working towards the PhD degree. Section 3.4 is based on [63], where the two-loop strong coupling Konishi anomalous dimension was first calculated. The calculation relied on semi-classical quantization of the folded string solution in $\mathcal{N}=4$ super Yang-Mills, the exact analogous calculation was then performed by the author in [64, 65] for the ABJM theory, which is the basis for section 5.3. The subsections of 4.6 describing the classical limit of the cusped Wilson line are based on [66]. The remainder of chapter 4 concerning the slope and curvature functions and their use to find the three-loop Konishi anomalous dimension at strong coupling are based on the work done in [52].

Naturally in order to achieve a uniform flow throughout the text we introduced some filler sections outlining the basics of techniques we utilize later. These sections are kept short and are thoroughly filled with references to original work and/or reviews of the subject matter. We hope the reader is not offended or annoyed by the inhomogeneous level of detail in various sections of the text and enjoys the thesis in its present form!

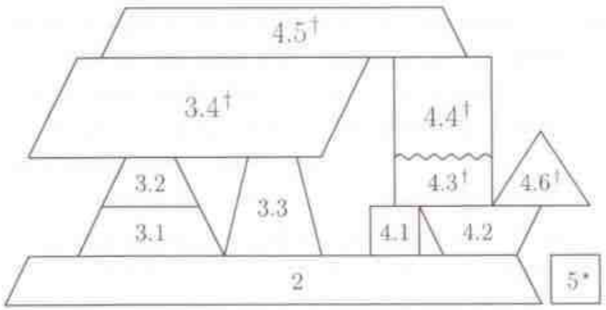


Figure 1: The interdependencies of the chapters and sections in the thesis. Sections containing mostly original work are marked with daggers. Section 5 parallels the main text with original work on the folded string quantization.

2 $\mathcal{N}=4$ super Yang-Mills

The devil is in the details.

- German Proverb

For the most part of this thesis we will be dealing with $\mathcal{N}=4$ super Yang-Mills theory. In this chapter we start off by defining it via its action and discussing its symmetries and observables. We also give an alternative formulation of the theory as a string theory, which is the core idea of the AdS/CFT correspondence. This formulation will later prove to be incredibly useful when discussing integrability and exact solutions.

2.1 Action

$\mathcal{N}=4$ super Yang-Mills theory is a quantum field theory much like the Standard Model of particle physics with a certain field content and interaction pattern. It was first discovered by considering $\mathcal{N}=1$ super Yang-Mills theory in $9+1$ spacetime dimensions [67], its action is given by

$$S=\int d^{10}x\,\mathrm{Tr}\left(-\frac{1}{4}F_{MN}F^{MN}+\frac{1}{2}\bar{\Psi}\Gamma^M\mathcal{D}_M\Psi\right),\quad M=1\ldots10,\tag{2.1}$$

where Ψ is a Majorana-Weyl spinor in $9+1$ dimensions with 16 real components and Γ^M are the appropriate gamma matrices. The covariant derivative $\mathcal{D}_M$ is defined as

$$\mathcal{D}_M=\partial_M-ig_{YM}\,[A_M,\,],\tag{2.2}$$

where g_{YM} is the Yang-Mills coupling constant. The gauge group is in principle arbitrary, but we choose $SU(N)$ in anticipation of the AdS/CFT correspondence. By dimensionally reducing this theory on a flat torus T^6 one recovers the maximally supersymmetric $\mathcal{N}=4$ Yang-Mills gauge theory in $3+1$ spacetime dimensions. The reduced action reads

$$\begin{aligned}S=\int d^4x\,\mathrm{Tr}\Big(&-\frac{1}{2}\mathcal{D}_\mu\Phi_I\mathcal{D}^\mu\Phi^I+\frac{g_{YM}^2}{4}[\Phi_I,\Phi_J][\Phi^I,\Phi^J]-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\\&-\bar{\psi}^a\bar{\sigma}^\mu\mathcal{D}_\mu\psi_a+\frac{ig_{YM}}{2}\sigma_I^{ab}\psi_a[\Phi^I,\psi_b]+\frac{ig_{YM}}{2}\sigma^I_{ab}\bar{\psi}^a[\Phi_I,\bar{\psi}^b]\Big).\end{aligned}\tag{2.3}$$

After dimensional reduction the gauge field A_M decomposes to the four dimensional gauge field A_μ and to six real scalar fields Φ_I whereas the Majorana-Weyl spinor Ψ_A

breaks up into four copies of the left and right Weyl spinors in four dimensions

$$\Psi_A \ (A=1,\dots,16) \ \to \ \bar{\psi}^a_{\dot{\alpha}},\ \psi_{\alpha a} \ (\alpha,\dot{\alpha}=1,2,\ \ a=1,\dots,4). \tag{2.4}$$

It also gives rise to the $SO(6)\simeq SU(4)$ symmetry called *R-symmetry*, which originally was part of the ten dimensional Poincare group, but now acts as an internal symmetry of the supercharges. It permutes the scalars, which live in the fundamental **6** of $SO(6)$ and the spinors, which live in the fundamental of $SU(4)$, namely the lower index a in $\psi_{\alpha a}$ transforms in **4**, while $\bar{\psi}^a_{\dot{\alpha}}$ transforms in $\bar{\mathbf{4}}$. From this it follows that we can combine the six real scalars Φ^I into three complex scalars Φ^{ab}, often denoted as X, Y and Z, which then transform under the second rank antisymmetric **6** of $SU(4)$. The gauge field is a singlet under R-symmetry.

It is now a straightforward but rather tedious task to calculate the beta function for this theory. For any $SU(N)$ gauge theory at one loop level it is given by [68]

$$\beta(g) = -\frac{g^3_{YM}}{16\pi^2}\left(\frac{11}{3}N - \frac{1}{6}\sum_s C_s - \frac{1}{3}\sum_f \tilde{C}_f\right) \tag{2.5}$$

where the first sum is over the real scalars and the second one over the fermions. C_s and $\tilde{C}_f$ are the quadratic Casimirs, which in our case are equal to N since all fields are in the adjoint representation of the group. It is then easy to see that at least at one loop level the theory is conformally invariant. In fact the β function was shown to be identically zero to all orders in perturbation theory [69, 70], hence $\mathcal{N}=4$ super Yang-Mills is fully conformally invariant even after quantization. After discussing the full symmetry algebra of the theory and its representations we will give an elegant argument why this is true.

P. West is cited?

2.2 Observables

The theory has 16 on-shell degrees of freedom which make up the gauge multiplet of $\mathcal{N}=4$ supersymmetry, namely $(\Phi_I, \psi_\alpha, A_\mu)$. Gauge invariant operators are then formed by taking traces over the gauge group. An important class of operators are the *local operators*, which are traces of fields all evaluated at the same spacetime point. They have the general form

$$\begin{aligned}\mathcal{O}_{i_1\mu i_2\alpha\dots i_n\cdots j_1\nu\beta\dots j_n}(x) \ &= \mathrm{Tr}\left[\Phi_{i_1}(x)\mathcal{D}_\mu\Phi_{i_2}(x)\psi_\alpha(x)\dots\Phi_{i_n}(x)\right]\times\dots\\ &\cdots\times\mathrm{Tr}\left[\Phi_{j_1}(x)\mathcal{D}_\nu\psi_\beta(x)\dots\Phi_{j_n}(x)\right].\end{aligned} \tag{2.6}$$

In this thesis we will be exclusively focusing on the planar limit, which is the limit when the number of colors N is sent to infinity. Diagrams involving multi-trace operators are non-planar, hence suppressed in the large N limit and therefore we will only be considering single trace operators. An example of a non-local operator is the Wilson loop, given by

$$W_L = \mathrm{Tr}\left(\mathcal{P}\exp\oint_C dt\left(iA\cdot\dot{x} + \vec{\Phi}\cdot\vec{n}\,|\dot{x}|\right)\right), \tag{2.7}$$

which depends on the path $x^\mu(t)$ in spacetime, hence it is known as a *line operator*. It also depends on the coupling to the scalar fields, which is encoded in the six-dimensional unit vector $\vec{n}(t)$. The scalar field term can also be understood by recalling that the scalar fields are a result of dimensional reduction from $9+1$ dimensions, thus the coupling vector $\vec{n}(t)$ together with the curve $x^\mu(t)$ make up a path $x^M(t)$ in $9+1$ dimensional spacetime. In later sections of the text we will be considering cusped Wilson lines with other operators inserted at the cusp. We will be mostly working in these two classes of operators, however in principle one could go on and define surface operators, etc.

2.3 Symmetry

Conformal symmetry, supersymmetry and R-symmetry are a part of a bigger group $PSU(2,2|4)$, which is also known as the $\mathcal{N}=4$ *superconformal group*. It is the full symmetry ~~group of $\mathcal{N}=4$ super Yang-Mills and~~ is unbroken by quantum corrections. It is an example of a *supergroup*, i.e. a graded group containing bosonic and fermionic generators. The theory of supergroups is highly developed (see [71]) and much of the techniques from studying bosonic groups carry over to supergroups with some additional complications, i.e. Dynkin diagrams, root spaces, weights etc.

$PSU(2,2|4)$ has the bosonic subgroup of $SU(2,2)\times SU(4)$, where $SU(2,2)\simeq SO(2,4)$ is the conformal group in four dimensions and $SU(4)\simeq SO(6)$ is the R-symmetry. The conformal group has the Poincaré group as a subgroup, which has a total of 10 generators including four translations P_μ and six Lorentz transformations $M_{\mu\nu}$, in addition there is the generator for dilatations D and four special conformal generators K_μ. Their commutation relations read

$$\begin{gathered}[D, M_{\mu\nu}] = 0 \quad [D, P_\mu] = -iP_\mu \quad [D, K_\mu] = +iK_\mu,\\ [M_{\mu\nu}, P_\lambda] = -i(\eta_{\mu\nu}P_\nu - \eta_{\lambda\nu}P_\mu) \quad [M_{\mu\nu}, K_\lambda] = -i(\eta_{\mu\lambda}K_\nu - \eta_{\lambda\nu}K_\rho),\\ [P_\mu, K_\nu] = 2i(M_{\mu\nu} - \eta_{\mu\nu}D).\end{gathered} \tag{2.8}$$

$\mathcal{N}=4$ supersymmetry has 16 supercharges $Q_{\alpha a}$ and $\bar{Q}^a_{\dot{\alpha}}$ where $\alpha, \dot{\alpha}=1,2$ are the Weyl spinor indices and $a=1,\ldots,4$ are the R-symmetry indices. These generators have the usual commutation and anti-commutation relations with the Poincaré generators given by

$$\begin{gathered}
\{Q_{\alpha a}, \bar{Q}^b_{\dot{\alpha}}\} = \gamma^{\mu}_{\alpha\dot{\alpha}}\delta^b_a P_\mu \quad \{Q_{\alpha a}, Q_{\alpha b}\} = \{\bar{Q}^a_{\dot{\alpha}}, \bar{Q}^b_{\dot{\alpha}}\} = 0, \\
[M^{\mu\nu}, Q_{\alpha a}] = i\gamma^{\mu\nu}_{\alpha\beta}\epsilon^{\beta\gamma}Q_{\gamma a} \quad [M^{\mu\nu}, \bar{Q}^a_{\dot{\alpha}}] = i\gamma^{\mu\nu}_{\dot{\alpha}\dot{\beta}}\epsilon^{\dot{\beta}\dot{\gamma}}\bar{Q}^a_{\dot{\gamma}}, \\
[P_\mu, Q_{\alpha a}] = [P_\mu, \bar{Q}^b_{\dot{\alpha}}] = 0,
\end{gathered} \tag{2.9}$$

where $\gamma^{\mu\nu}_{\alpha\beta} = \gamma^{[\mu}_{\alpha\dot{\alpha}}\gamma^{\nu]}_{\beta\dot{\beta}}\epsilon^{\dot{\alpha}\dot{\beta}}$. Commutators between supercharges and the conformal generators are also non trivial and introduce new supercharges,

$$\begin{gathered}
[D, Q_{\alpha a}] = -\tfrac{1}{2}Q_{\alpha a} \quad [D, \bar{Q}^a_{\dot{\alpha}}] = -\tfrac{1}{2}\bar{Q}^a_{\dot{\alpha}}, \\
[K^\mu, Q_{\alpha a}] = \gamma^{\mu}_{\alpha\dot{\alpha}}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{S}_{\dot{\beta}a} \quad [K^\mu, \bar{Q}^a_{\dot{\alpha}}] = \gamma^{\mu}_{\alpha\dot{\alpha}}\epsilon^{\alpha\beta}S^a_{\beta},
\end{gathered} \tag{2.10}$$

where $\bar{S}_{\dot{\alpha}a}$ and S^a_α are the *special conformal supercharges*. They have opposite R-symmetry representations compared to the usual supercharges. The special supercharges bring the total of supercharges to 32. The commutation and anti-commutation relations for the special conformal supercharges are very much like the ones for normal supercharges,

$$\begin{gathered}
\{S^a_\alpha, \bar{S}_{\dot{\alpha}b}\} = \gamma^{\mu}_{\alpha\dot{\alpha}}\delta^a_b K_\mu \quad \{S^a_\alpha, S^b_\alpha\} = \{\bar{S}_{\dot{\alpha}a}, \bar{S}_{\dot{\alpha}b}\} = 0, \\
[M^{\mu\nu}, S^a_\alpha] = i\gamma^{\mu\nu}_{\alpha\beta}\epsilon^{\beta\gamma}S^a_\gamma \quad [M^{\mu\nu}, \bar{S}_{\dot{\alpha}a}] = i\gamma^{\mu\nu}_{\dot{\alpha}\dot{\beta}}\epsilon^{\dot{\beta}\dot{\gamma}}\bar{S}_{\dot{\gamma}a}, \\
[K_\mu, S^a_\alpha] = [K_\mu, \bar{S}_{\dot{\alpha}a}] = 0.
\end{gathered} \tag{2.11}$$

Finally the anti-commutation relations between the special conformal and usual supercharges close the algebra,

$$\begin{aligned}
\{Q_{\alpha a}, S^b_\beta\} &= -i\epsilon_{\alpha\beta}\sigma^{IJ}{}_a{}^b R_{IJ} + \gamma^{\mu\nu}_{\alpha\beta}\delta_a{}^b M_{\mu\nu} - \frac{1}{2}\epsilon_{\alpha\beta}\delta_a{}^b D \\
\{\bar{Q}^a_{\dot{\alpha}}, \bar{S}_{\dot{\beta}b}\} &= +i\epsilon_{\dot{\alpha}\dot{\beta}}\sigma^{IJa}{}_b R_{IJ} + \gamma^{\mu\nu}_{\dot{\alpha}\dot{\beta}}\delta^a{}_b M_{\mu\nu} - \frac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\delta^a{}_b D \\
\{Q_{\alpha a}, \bar{S}_{\dot{\beta}b}\} &= \{\bar{Q}^a_{\dot{\alpha}}, S^b_\beta\} = 0
\end{aligned} \tag{2.12}$$

where R_{IJ} are the generators of R-symmetry with $I, J=1,\ldots,6$. All supercharges transform under the two spinor representations of the R-symmetry group and all other generators commute with it. All of the generators can be organized as follows

$$\left(\begin{array}{c|c} K^\mu, P^\mu, M^{\mu\nu}, D & Q_{\alpha a}, \bar{S}_{\dot{\alpha}a} \\ \hline S^a_\alpha, \bar{Q}^a_{\dot{\alpha}} & R_{IJ} \end{array}\right) \tag{2.13}$$

where the generators in the diagonal blocks are bosonic and the ones in the anti-diagonal blocks are fermionic. They have a definite dimensions, which are not modified by radiative corrections

$$[D]=[L]=[\bar{L}]=[R]=0\,, \quad [P]=1\,,\ [K]=-1\,, \quad [Q]=1/2\,,\ [S]=-1/2\,. \tag{2.14}$$

In contrast, the classical dimensions of fields

$$[\Phi^I]=[A_\mu]=1\,, \quad [\psi_a]=\frac{3}{2}, \tag{2.15}$$

do receive radiative corrections and acquire *anomalous dimensions*, which together with the bare dimension make up the conformal dimension

$$\Delta = \Delta_0 + \gamma(g_{YM}). \tag{2.16}$$

The name is justified by the fact that in conformal field theories all two point functions are determined by the scaling dimensions of the fields. More than that, together with the knowledge of all three point functions they are enough to determine any n-point function. This is why finding conformal dimensions of all operators, i.e. the spectrum of the theory is a very important step in solving it.

2.3.1 Superconformal multiplets

Fields of the theory can be organized in unitary representations of the superconformal symmetry group, which are labeled by quantum numbers of the bosonic subgroup

$$\begin{array}{ccc} SO(1,3) & \times\ SO(1,1) & \times\ SU(4) \\ (s_+, s_-) & \Delta & [r_1, r_2, r_3] \end{array} \tag{2.17}$$

where (s_+, s_-) are the usual positive half-integer spin labels of the Lorentz group, Δ is the positive conformal dimension that can depend on the coupling and $[r_1, r_2, r_3]$ are Dynkin labels of the R-symmetry. All unitary representations of the superconformal group have been classified into four families [72, 73], here we give a short description of the classification.

Looking at the commutation relations of the conformal subgroup (2.8), we see that the operators P_μ and K_μ act as raising and lowering operators for the dilatation operator D – this gives a hint as to how we could construct representations of the group. The dilatation operator D is the generator of scalings, i.e. upon a rescaling $x \to \lambda x$ a local operator in a field theory scales as

$$\mathcal{O}(x) \to \lambda^{-\Delta}\mathcal{O}(\lambda x) \tag{2.18}$$

where Δ is the conformal dimension of the operator $\mathcal{O}(x)$. Restricting to the point $x=0$, which is a fixed point of scalings, we see that the conformal dimension is the eigenvalue of the dilatation operator.

$$[D, \mathcal{O}(0)] = -i\Delta\mathcal{O}(0). \tag{2.19}$$

It is now clear that acting on a field with K_μ should lower the dimension by one and acting with P_μ – raise it by one. We can show this explicitly using the Jacobi identity as

$$[D, [K_\mu, \mathcal{O}(0)]] = [[D, K_\mu], \mathcal{O}(0)] + [K_\mu, [D, \mathcal{O}(0)]] = -i(\Delta-1)\,[K_\mu, \mathcal{O}(0)]. \tag{2.20}$$

Since operators in a unitary quantum field theory should have positive dimensions (aside from the identity operator), we should not be able to keep lowering the dimension indefinitely, i.e. there should always be an operator that satisfies

$$[K_\mu, \tilde{\mathcal{O}}(0)] = 0. \tag{2.21}$$

We call such operators *conformal primary operators*. Acting on these with P_μ keeps producing operators with a dimension one higher – we call these the *descendants* of $\tilde{\mathcal{O}}(0)$. We can also act with the supercharges and looking at the commutators in (2.10) we see that they raise the dimension by 1/2, while the special conformal supercharges lower it by 1/2. Operators annihilated by special conformal supercharges are called *superconformal primaries*, which is a stronger condition that being a conformal primary.

(Super-)conformal primaries and their descendants make up multiplets that constitute the three families of discrete representations in the classification. They are further distinguished by the number of supercharges the primary commutes with. One example is a class of operators that satisfy the condition

$$\Delta = r_1 + r_2 + r_3, \tag{2.22}$$

a canonical representative would be a single-trace symmetrized scalar field operator such as

$$\mathcal{O}^{IJ\dots K}(x) = \mathrm{Tr}\left(\Phi(x)^{(I}\Phi(x)^J \dots \Phi(x)^{K)}\right). \tag{2.23}$$

These operators commute with half of the supercharges; thus they are referred to as half-BPS. A key fact is that operators in the same representation must have the same anomalous dimension, because the generators of the group can only change it by half integer steps and there's only a finite number of generators. What is more, operators

in the discrete BPS representations are protected from quantum corrections, because at any coupling the total dimension is always algebraically related to the Dynkin labels of the R-symmetry, e.g. as in (2.22). Since charges of compact groups are quantized it must meant that the dimension can't continuously depend on the coupling and hence the anomalous dimension must vanish. This is however not true for the fourth continuous non-BPS family of representations, hence operators from these multiplets do acquire anomalous dimensions.

Let us conclude the section with an elegant argument for why the beta function of $\mathcal{N}=4$ super Yang-Mills is zero. One can use the algebra and shown that the operators $\mathrm{Tr}\, F_+F_+$ and $\mathrm{Tr}\, F_-F_-$, where $F_\pm$ are the (anti-)self-dual field strengths, belong to the same multiplet as a superconformal primary [74], meaning that the $\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}$ term in the Lagrangian is protected from quantum corrections, hence so is the coupling constant g_{YM}. This argument is valid to all orders in perturbation theory, which means that $\mathcal{N}=4$ super Yang-Mills is conformally invariant to all orders in perturbation theory.

2.4 String description at strong coupling

As already briefly explained in the introduction, the AdS/CFT conjecture states that $\mathcal{N}=4$ super Yang-Mills is exactly dual to type IIB string theory on $AdS_5\times S^5$, [4, 9, 10]. To be more precise, the gauge group of the Yang-Mills theory is taken to be $SU(N)$ and the coupling constant g_{YM}. The string theory is defined on $AdS_5\times S^5$ where both AdS_5 and S^5 have radius R. The self-dual five-form field F_5^+ has integer flux through the sphere

$$\int_{S^5} F_5^+ = N, \tag{2.24}$$

and N is identified with the number of colours in the gauge theory. The string theory is further parametrized by the string coupling g_s and the string length squared α'. The following relations are conjectured to hold

$$4\pi g_s = g_{YM}^2 \equiv \frac{\lambda}{N}, \qquad \frac{R^4}{\alpha'^2} = \lambda, \tag{2.25}$$

where λ is the t'Hooft coupling. We will be working in the planar limit $N\to\infty$ with λ fixed. It is easy to see that in this limit $g_s\to 0$ and we are left with freely propagating strings. Furthermore, the regime of strongly coupled gauge theory when $\lambda\to\infty$ corresponds to the regime of string theory where the supegravity approximation is valid, namely $\alpha' \ll R^2$. The takeaway here is that one can formulate strongly coupled planar $\mathcal{N}=4$ super Yang-Mills as a classical theory of free strings on $AdS_5 \times S^5$.

2.4.1 Sigma model formulation

A very useful formulation of string theory on $AdS_5 \times S^5$ is the coset space sigma model [11] with the target superspace of

$$\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}. \tag{2.26}$$

The bosonic part of the supercoset where the string moves is given by

$$\frac{SO(2,4)\times SO(6)}{SO(1,4)\times SO(5)} = AdS_5 \times S^5, \tag{2.27}$$

which is constructed as the coset between the isometry and isotropy groups of $AdS_5 \times S^5$. The action is then written in terms of the algebra of $PSU(2,2|4)$.

The superalgebra $\mathfrak{psu}(2,2|4)$ has no realization in terms of matrices, instead it is the quotient of $\mathfrak{su}(2,2|4)$ by matrices proportional to the identity. On the other hand $\mathfrak{su}(2,2|4)$ is a matrix superalgebra spanned by 8×8 supertraceless matrices

$$M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right), \tag{2.28}$$

where the supertrace is defined as

$$\mathrm{STr}\, M = \mathrm{Tr} A - \mathrm{Tr} D. \tag{2.29}$$

A and D are elements of $\mathfrak{su}(2,2)$ and $\mathfrak{su}(4)$ respectively, whereas the fermionic components are related by

$$C = \left(\begin{array}{c|c} +\mathbb{1}_{2\times 2} & 0 \\ \hline 0 & -\mathbb{1}_{2\times 2} \end{array}\right) B^{\dagger}. \tag{2.30}$$

An important feature of this algebra is the following automorphism

$$\Omega \circ M = \left(\begin{array}{c|c} EA^T E & -EC^T E \\ \hline EB^T E & ED^T E \end{array}\right), \quad E = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{2.31}$$

which endows the algebra with a $\mathbb{Z}_4$ grading, since one can easily check that $\Omega^4 = 1$. This in turn means that any element of the algebra can be decomposed under this grading as

$$M = \sum_{i=0}^{3} M^{(i)}, \tag{2.32}$$

where

$$\begin{aligned} M^{(0,2)} &= \tfrac{1}{2}\left(\begin{array}{c|c} A \pm EA^T E & 0 \\ \hline 0 & D \pm ED^T E \end{array}\right) \\ M^{(1,3)} &= \tfrac{1}{2}\left(\begin{array}{c|c} 0 & B \pm iEC^T E \\ \hline C \mp iEB^T E & 0 \end{array}\right) \end{aligned} \tag{2.33}$$

and the morphism then acts on the elements of the decomposition as

$$\Omega \circ M^{(n)} = i^n M^{(n)}. \tag{2.34}$$

The Metsaev-Tseytlin action for the Green-Schwarz superstring is then given by

$$S = \frac{\sqrt{\lambda}}{4\pi} \int \mathrm{STr}\left(J^{(2)} \wedge * J^{(2)} - J^{(1)} \wedge J^{(3)} + \Lambda \wedge J^{(2)}\right), \tag{2.35}$$

which is written down in terms of the graded elements of the algebra current

$$J = -g^{-1} \mathrm{d} g \tag{2.36}$$

where $g(\sigma,\tau) \in PSU(2,2|4)$ is a map from the string worldsheet to the supergroup $PSU(2,2|4)$. The last term contains a Lagrange multiplier Λ, which ensures that $J^{(2)}$ is supertraceless, whereas all other components are manifestly traceless as seen from (2.33). Since the target space is the coset of $PSU(2,2|4)$ by $SO(1,4)\times SO(5)$, the map g has an extra gauge symmetry

$$g \to gH, \quad H \in SO(1,4)\times SO(5) \tag{2.37}$$

under which the components of the supercurrent transform as

$$J^{(0)} \to H^{-1} J^{(0)} H - H^{-1} \mathrm{d} H \tag{2.38}$$

$$J^{(i)} \to H^{-1} J^{(i)} H, \quad i = 1,2,3 \tag{2.39}$$

The equations of motion read

$$d * k = 0, \tag{2.40}$$

where $k = gKg^{-1}$ and

$$K = J^{(2)} + \frac{1}{2} * J^{(1)} - \frac{1}{2} * J^{(3)} - \frac{1}{2} * \Lambda. \tag{2.41}$$

They are equivalent to the conservation of the Noether current associated to the global left $PSU(2,2|4)$ multiplication symmetry.

Finally let us briefly remark on how the action reduces to the usual sigma model action if one restricts to bosonic fields. A purely bosonic representative of $PSU(2,2|4)$ has the form

$$g=\left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array}\right), \tag{2.42}$$

where $A \in SO(6) \simeq SU(4)$ and $D \in SO(2,4) \simeq SU(2,2)$. Then we see that AEA^T is a good parametrization of

$$\frac{SO(6)}{SO(5)} \simeq \frac{SU(4)}{SP(4)} = S^5, \tag{2.43}$$

since it is invariant under $A \to AH$ with $H \in SP(4)$. Similarly DED^T parametrizes AdS_5. If we now define the coordinates u^i and v^i in the following way

$$u^i\Gamma^S_i = AEA^T, \qquad v^i\Gamma^A_i = DED^T, \tag{2.44}$$

with Γ^S and Γ^A being the gamma matrices of $SO(6)$ and $SO(2,4)$ respectively, then by construction they will satisfy the following constraints

$$\begin{aligned} 1 &= u\cdot u \equiv +u_1^2+u_2^2+u_3^2+u_4^2+u_5^2+u_6^2 \\ 1 &= v\cdot v \equiv -v_1^2-v_2^2-v_3^2-v_4^2+v_5^2+v_6^2. \end{aligned} \tag{2.45}$$

and the action (2.35) will read

$$S_b = \frac{\sqrt{\lambda}}{4\pi}\int_0^{2\pi} d\sigma \int d\tau\, \sqrt{h}\,(h^{\mu\nu}\partial_\mu u\cdot\partial_\nu u + \lambda_u\,(u\cdot u-1) - (u\to v)), \tag{2.46}$$

which is just the usual non-linear sigma model for a string moving in $AdS_5\times S^5$.

3 Perturbative results

It is better to take many small steps in the right direction than to make a great leap forward only to stumble backward.

– Old Chinese Proverb

In this section we start attacking the problem of finding the spectrum and as expected we begin with perturbation theory. Starting at weak coupling we quickly stumble upon an amazing feature of the theory, so-called integrability, which allows one to apply numerous techniques that greatly simplify the problem. We demonstrate integrability from the string theoretic perspective at strong coupling as well, which suggests a unified picture of the integrable structure embedded in the theory persisting to all loops. After discussing results achievable via integrability in the perturbative regime we finish off with our first exact result, the slope function, which in turn allows one to extract novel information about the spectrum.

3.1 One loop at weak coupling

We begin with two point correlation functions of local operators. In any conformal field theory they are constrained by symmetry, namely for operators that are eigenvalues of dilatations they have the following form at all loop levels

$$\left\langle \mathcal{O}(x)\,\bar{\mathcal{O}}(y)\right\rangle \approx \frac{1}{|x-y|^{2\Delta}}, \tag{3.1}$$

where Δ is the scaling dimension of the operator and we ignore unphysical normalization factors. Classically $\Delta=\Delta_0$ is simply the mass dimension, but at the quantum level it receives radiative corrections and acquires an anomalous dimension γ, such that $\Delta(g_{YM})=\Delta_0+\gamma(g_{YM})$, where the anomalous dimension depends on the coupling. Usually the corrections are small and the correlator can be expanded perturbatively. Of course one has to be careful here, as expanding in γ would result in expressions like $\log|x-y|$, which do not make sense. To that end we introduce a scale μ and expand the following quantity instead

$$\mu^{-2\gamma}\left\langle \mathcal{O}(x)\,\bar{\mathcal{O}}(y)\right\rangle \approx \frac{1}{|x-y|^{2\Delta_0}}\left(1-\gamma\log\mu^2|x-y|^2\right), \tag{3.2}$$

however we will formally assume that the factor $\mu^{-\gamma}$ is absorbed into the field definition and thus we will ignore it from now on. We can now take some explicit local operator

$\mathcal{O}(x)$, calculate the correlator using perturbation theory and read off the anomalous dimension γ. Let us start with a very simple chiral primary operator

$$\Psi = \operatorname{Tr} Z^L = Z^a{}_b Z^b{}_c \ldots Z^l{}_a, \tag{3.3}$$

where the complex scalar field Z and its conjugate $\bar{Z}$ have the standard tree level correlators

$$\left\langle Z^a{}_b(x)\bar{Z}^{b'}{}_{a'}(y)\right\rangle_{\text{tree}} \approx \frac{\delta^a{}_{a'}\delta_b{}^{b'}}{|x-y|^2} \tag{3.4}$$

\rangle

In order to find the anomalous dimension of the operator Ψ we must calculate the correlator $< \Psi(x)\bar{\Psi}(x) >$. We do this by using Wick's theorem and plugging in the two-point correlator (3.4), which produces a lot of terms with delta function contractions between the adjoint indices. Some examples are

$$\ldots\delta^{a'}{}_a\,\delta^a{}_{a'}\,\delta^{b'}{}_b\,\delta^b{}_{b'}\,\delta^{c'}{}_c\,\delta^c{}_{c'}\,\ldots \tag{3.5a}$$

$$\ldots\delta^{a'}{}_c\,\delta^c{}_{a'}\,\delta^{b'}{}_a\,\delta^a{}_{b'}\,\delta^{c'}{}_b\,\delta^b{}_{c'}\,\ldots \tag{3.5b}$$

$$\ldots\delta^{a'}{}_a\,\delta^a{}_{b'}\,\delta^{c'}{}_b\,\delta^b{}_{a'}\,\delta^{b'}{}_c\,\delta^c{}_{c'}\,\ldots \tag{3.5c}$$

These contractions have a graphical interpretation. Consider the scalar field $Z^a{}_b$ as a dot and each contraction of the adjoint indices as a line connecting these dots, then the chiral primary operator Ψ is simply a circle due to the trace. Wick's theorem says that in order to find the correlator $< \Psi(x)\bar{\Psi}(x) >$ we must sum all possible ways we can connect the dots in the circle of Ψ to the dots in the circle of $\bar{\Psi}$. All the delta function contractions that we get after expanding the correlator represent precisely all the possible ways we can contract the dots in the circles. The three excerpts of contractions shown in (3.5) can be represented graphically as shown in figure 2. One can immediately notice that the first two are planar, while the third one is intersecting itself. Evaluating the three contractions we immediately see that planar ones produce a factor of N^3 while the non-planar one produces a factor of N, i.e. non-planar diagrams are suppressed and we can discard them once we take the planar limit $N \to \infty$. All that's left then are cyclic permutations of lines by shifting all of them as seen in figure 2 while going from (a) to (b). There are $L-1$ shifts that can be done in this way, since after making a full circle we return to the initial configuration. Thus finally for the chiral primary correlator at tree level we find

$$\left\langle \Psi(x)\bar{\Psi}(y)\right\rangle_{\text{tree}} \approx \frac{LN^L}{|x-y|^{2L}}, \tag{3.6}$$

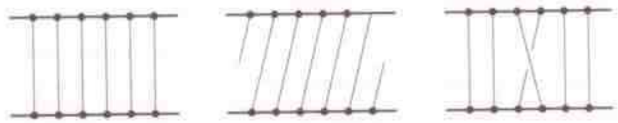


Figure 2: Possible types of Wick contractions (vertical lines) between single trace operators. The constituent scalar fields are represented by dots in the horizontal lines, which represent the successive index contractions due to the trace. First two figures are examples of planar contractions while the last one is an example of a non-planar contraction.

where N^L comes from the contractions and L from all the possible planar ways we can contract. This can easily be generalized for correlators of operators with arbitrary scalar fields $\Phi_{I_1I_2\ldots I_L}(x) = \operatorname{Tr}[\Phi_{I_1}(x)\Phi_{I_2}(x)\ldots\Phi_{I_L}(x)]$ to

$$\left\langle \Phi_{I_1I_2\ldots I_L}(x)\,\bar{\Phi}^{J_1J_2\ldots J_L}(y)\right\rangle_{\text{tree}} \approx \frac{1}{|x-y|^{2L}}\left(\delta_{I_1}^{J_1}\delta_{I_2}^{J_2}\ldots\delta_{I_L}^{J_L} + \text{cycles}\right), \tag{3.7}$$

where "cycles" refers to terms with the J indices pushed. I and J are flavor indices, the color indices are suppressed.

So far so good, but in order to calculate anomalous dimensions we have to go beyond tree level. This may seen like a highly non-trivial thing to do, since we expect not only scalar interactions, but also gluon exchanges and fermion loops appearing. Luckily the symmetry of the theory allows one to calculate all gluon and fermion effects in one go. First let's concentrate on the bosonic sector of the theory ignoring gluons. The action (2.3) contains a single scalar-only interaction term

$$\begin{aligned} S_\Phi &= \frac{g_{YM}^2}{4}\sum_{I,J}\int d^4x\,\operatorname{Tr}[\Phi_I,\Phi_J][\Phi_I,\Phi_J] \\ &= -\frac{g_{YM}^2}{2}\sum_{I,J}\int d^4x\,\left(\operatorname{Tr}\Phi_I\Phi_I\Phi_J\Phi_J - \operatorname{Tr}\Phi_I\Phi_J\Phi_I\Phi_J\right) \end{aligned} \tag{3.8}$$

In order to calculate the correlator (3.7) at one-loop level, one should insert this term and Wick contract. Just like in tree level, we only have to keep planar diagrams. For the interaction terms this means that only neighbouring fields can interact. This drastically reduces the number of terms we get after Wick contracting. Because of that it is enough to consider a length two operator $\Phi_{I_kI_{k+1}}$ and with a bit of work one can show that at one-loop level we get

$$\begin{aligned} &\left\langle \Phi_{I_kI_{k+1}}(x)\,\bar{\Phi}^{J_kJ_{k+1}}\right\rangle_{\text{one-loop}} = \frac{\lambda}{16\pi^2}\frac{\log(|\mu^2|x-y|^2|)}{|x-y|^{2L}} \times \\ &\times\left(2\delta_{I_k}{}^{J_{k+1}}\delta_{I_{k+1}}{}^{J_k} - \delta_{I_kI_{k+1}}\delta^{J_kJ_{k+1}} - \delta_{I_k}{}^{J_k}\delta_{I_{k+1}}{}^{J_{k+1}}\right), \end{aligned} \tag{3.9}$$

where $\lambda = g_{YM}^2 N$ is the t'Hooft coupling. Comparing this to (3.7) we see that effectively the interactions permute and contract the delta function indices. We can introduce exchange and trace operators to make this explicit. The permutation operator, also called the exchange operator, $\mathcal{P}_{l,l+1}$ is defined by it's action on a set of delta functions as

$$\mathcal{P}_{l,l+1}\, \delta_{I_1}{}^{J_1} \ldots \delta_{I_l}{}^{J_l} \delta_{I_{l+1}}{}^{J_{l+1}} \ldots \delta_{I_L}{}^{J_L} = \delta_{I_1}{}^{J_1} \ldots \delta_{I_l}{}^{J_{l+1}} \delta_{I_{l+1}}{}^{J_l} \ldots \delta_{I_L}{}^{J_L} \tag{3.10}$$

and the trace operator $\mathcal{K}_{l,l+1}$ is defined as

$$\mathcal{K}_{l,l+1}\, \delta_{I_1}{}^{J_1} \ldots \delta_{I_l}{}^{J_l} \delta_{I_{l+1}}{}^{J_{l+1}} \ldots \delta_{I_L}{}^{J_L} = \delta_{I_1}{}^{J_1} \ldots \delta_{I_l I_{l+1}} \delta^{J_l J_{l+1}} \ldots \delta_{I_L}{}^{J_L}. \tag{3.11}$$

Using these operators we can rewrite the correlator in (3.9) in a more compact notation

$$\begin{aligned}
&\left\langle \Phi_{I_k I_{k+1}}(x)\, \bar{\Phi}^{J_k J_{k+1}} \right\rangle_{\text{one-loop}} = \\
&\qquad = \frac{\lambda}{16\pi^2} \frac{\log(\mu^2 |x-y|^2)}{|x-y|^{2L}} \quad (2\,\mathcal{P}_{k,k+1} - \mathcal{K}_{k,k+1} - 1)\, \delta_{I_k}^{J_k} \delta_{I_{k+1}}^{J_{k+1}}.
\end{aligned} \tag{3.12}$$

This result includes only interactions with four scalars, however as mentioned before at one-loop level we can also have gluon interactions and fermion loops in scalar propagators. The nice thing about these is that such interactions don't alter the flavor index structure, i.e. there are no permutations or traces. Basically this happens because the gluon transforms trivially under R-symmetry and hence can't change the flavor index (which transforms under R-symmetry). Similarly, fermions can only appear in loops altering scalar self-energies, hence they also leave the flavor structure intact. Thus all of these interactions contribute a constant term C, which we can determine later. We can generalize our one-loop result with all interactions included for operators of arbitrary length,

$$\begin{aligned}
&\left\langle \Phi_{I_1 I_2 \ldots I_L}(x)\, \bar{\Phi}^{J_1 J_2 \ldots J_L}(y) \right\rangle_{\text{one-loop}} = \frac{\lambda}{16\pi^2} \frac{\log(\mu^2 |x-y|^2)}{|x-y|^{2L}} \times \\
&\times \sum_{l=1}^{L} (2\,\mathcal{P}_{l,l+1} - \mathcal{K}_{l,l+1} - 1 + C) \left(\delta_{I_1}^{J_1} \delta_{I_2}^{J_2} \ldots \delta_{I_L}^{J_L} + \text{cycles} \right).
\end{aligned}$$

Combining this with the tree level result (3.7) and comparing to the general expression of a two-point function at one-loop level (3.2) we can deduce the anomalous dimension γ, which now becomes an operator Γ because of the flavor mixing. It is given by

$$\Gamma = \frac{\lambda}{16\pi^2} \sum_{l=1}^{L} (-2\,\mathcal{P}_{l,l+1} + \mathcal{K}_{l,l+1} + 1 - C). \tag{3.13}$$

At first sight it may seem strange that what was supposed to be a number, i.e. a correction to the mass dimension of an operator has turned out to be an operator acting on the flavor space, i.e. a matrix. But this is very natural and in fact expected, since interactions can change the flavor of fields and we can't be sure that an operator at the quantum level has the same flavor indices as it does at the classical level. This line of thinking may lead to a natural question, why do we have mixing between the scalars only and not between all the fields in the theory including fermions, which miraculously do not appear. It turns out that this is a one-loop feature only and mixing becomes a problem at higher loop levels [13].

Now that we have acknowledged that the anomalous dimension is a matrix and found an expression for it, the next logical step would be diagonalizing it and finding the flavor eigenstates. One example of such an eigenstate is the chiral primary operator Ψ. Since it contains scalar fields of only one type, the permutation and trace operators act trivially on it. Thus we see that

$$\Gamma\,\Psi = \frac{\lambda}{16\pi^2} \sum_{l=1}^{L} (-2 + 1 - C)\,\Psi, \tag{3.14}$$

but we already saw that a chiral primary has an anomalous dimension of zero, which then fixes the constant C to -1. And finally we get

$$\Gamma = \frac{\lambda}{16\pi^2} \sum_{l=1}^{L} (2 - 2\,\mathcal{P}_{l,l+1} + \mathcal{K}_{l,l+1}). \tag{3.15}$$

A keen eye might already notice that this expression resembles a Hamiltonian of a spin chain. In fact, this is hardly surprising, since from the very beginning we were talking about fields as objects in some closed line, which indeed resembles a spin chain. Furthermore the correlators that we were calculating are nothing more that propagators from one state of the chain to another, hence no wonder that the operator describing this evolution looks like a Hamiltonian for a spin chain. This identification is very useful, because the spin chains that appear in AdS/CFT are integrable and can be solved exactly, which gives us hope that we can apply the same techniques here and solve the spectral problem in $\mathcal{N}=4$ super Yang-Mills exactly. The first steps towards this goal were outlined in the seminal paper [13], which launched the integrability program in AdS/CFT. However saying that the spectral problem can be solved exactly in this particular case is too strong, since we are only at one-loop level. Nevertheless one can apply the same techniques going beyond one-loop level, as we shall soon see in the coming sections.

3.1.1 The su(2) sector

In the previous section we considered single trace operators potentially containing all six scalar fields, we also mentioned that at higher loops the remaining fields of the theory start mixing in, i.e. the scalar sector is only closed at one-loop level. However it is easy to see that there exist sectors that are closed at all loops. The anomalous dimension matrix is simply the dilatation operator minus the bare dimension and from the algebra of the theory we know that dilatations commute with Lorentz and R-symmetries at any value of the coupling. We can thus conclude that only operators with the same bare dimensions, Lorentz charges and R-charges can mix when acting with the anomalous dimension matrix. Furthermore, since this is true at any value of the coupling it must follow that all the coefficients in

$$D = \sum_n \lambda^n D^{(2n)}, \tag{3.16}$$

commute with Lorentz and R-symmetry generators, here $D^{(2)} \equiv \Gamma$ is the one-loop dilatation operator found in the last section.

Arguably the simplest possible closed sector is the so-called $\mathfrak{su}(2)$ sector, containing only two scalar fields X and Z. An operator with M and $L-M$ scalars X and Z has the charges $(0,0,L;M,L-M,0)$, the only other operators with these charges are permutations of this operator, hence the sector is closed. The anomalous dimension operator in this sector is given by

$$\Gamma = \frac{\lambda}{8\pi^2}\sum_{l=1}^{L}(1-\mathcal{P}_{l,l+1}), \tag{3.17}$$

which lacks the contraction operator term compared to (3.15). We neglect it since operators in this sector do not contain both scalars and their conjugates, thus no contractions are possible. Up to a constant factor this is the same as the Hamiltonian for the Heisenberg spin chain (also called the XXX spin chain), which is a quantum description of a one dimensional magnet. The Hamiltonian is given by

$$\mathbf{H} = \sum_{l=1}^{L}(1-\mathcal{P}_{l,l-1}), \tag{3.18}$$

which can also be rewritten in terms of Pauli matrices as

$$\mathbf{H} = 2\sum_{l=1}^{L}\left(\frac{1}{4}-\vec{S}_l\cdot\vec{S}_{l+1}\right), \quad \vec{S}_l = \frac{1}{2}\vec{\sigma}_l. \tag{3.19}$$

Hence solving the spectral problem in this sector translates into solving the Schrödinger equation

$$\mathbf{H}\,|\psi\rangle = E\,|\psi\rangle, \tag{3.20}$$

where we now seek to find the energy eigenvalues for the Hamiltonian of the spin chain. If the chain is short, this is a trivial diagonalization problem that can be easily solved by a present day computer. However this problem was first solved analytically by Hans Bethe in a time when computers were still in their infancy. The original solution now goes by the name of *coordinate Bethe ansatz* and it is by far one of the most important and beautiful solutions in physics in the past century, which is still very widely used even to this day. The idea is to make an educated guess for the wave function $|\psi\rangle$, plug it in to the Schrödinger equation and determine when does it actually hold. This produces a set of algebraic Bethe ansatz equations for a set of variables unimaginatively called the Bethe roots. All observables can then be expressed in terms of these numbers as simple algebraic functions, thus transforming a diagonalization problem to an algebraic problem. This has an enormous advantage, since in the asymptotic limit, when the spin chain is very large, instead of diagonalizing an infinite matrix, the set of algebraic equations actually simplify and produce integral equations, which can be solved.

In the spin chain language the scalar fields can be treated as up and down spin states, i.e.

$$|\uparrow\rangle = Z = \begin{pmatrix}1\\0\end{pmatrix}, \quad |\downarrow\rangle = X = \begin{pmatrix}0\\1\end{pmatrix}, \tag{3.21}$$

thus local single trace operators can be treated as states of a spin chain, e.g.

$$\mathrm{Tr}(XXZXXZX) \equiv |\downarrow\downarrow\uparrow\downarrow\downarrow\uparrow\downarrow\rangle. \tag{3.22}$$

Due to the cyclicity of the trace all rotations of the chain are equivalent. We should also specify the periodicity boundary condition

$$\vec{S}_{L+1} = \vec{S}_1. \tag{3.23}$$

The operators $\vec{S}_l$ act as Pauli matrices on the l'th spin site and trivially on all the others. Since a spin "chain" with a single site would have a state space $\mathbb{C}^2$, a spin chain of length L has a state space $(\mathbb{C}^2)^{\otimes L}$, which has 2^L basis vectors and the Hamiltonian is then a $2^L\times 2^L$ matrix, which we need to diagonalize. Of course, technically the state space is smaller due to the cyclicity of the chain, however as is common in physics we stick with the redundant description for simplicity. Working directly with Pauli matrices one can find some simple results directly, e.g. it is trivial to show that the chiral primary operator

$$|\Psi\rangle = \mathrm{Tr}\, Z^L = |\uparrow\uparrow \ldots \uparrow\rangle \tag{3.24}$$

is an eigenstate of the Hamiltonian with zero energy, i.e. it is the ferromagnetic ground state of the spin chain, which we will denote as $|0\rangle$ from now on. This is expected, since we know that chiral primaries have zero anomalous dimensions. Another eigenstate of the Hamiltonian is the *single magnon* state, defined as

$$|p\rangle = \sum_{n=1}^{L} e^{ipn}|n\rangle, \tag{3.25}$$

where $|n\rangle$ is the ground state with the n'th spin flipped,

$$|n\rangle = S_n^- |0\rangle = |\uparrow\uparrow\uparrow \ldots \downarrow \ldots \uparrow\uparrow\uparrow\rangle, \tag{3.26}$$

here p is formally just a parameter, but it can be interpreted as the momentum of the excitation travelling in the spin chain. Due to the cyclicity of the chain the momentum is quantized,

$$p = \frac{2\pi}{L} n, \quad n \in \mathbb{Z}, \tag{3.27}$$

where n is the mode number. The energy of the excitation is given by the dispersion relation

$$E(p) = 4\sin^2\frac{p}{2}. \tag{3.28}$$

Now consider a two magnon state

$$|\psi\rangle = \sum_{n<m} \psi(n,m)\,|n,m\rangle, \quad |n,m\rangle = S_n^- S_m^- |0\rangle. \tag{3.29}$$

The situation is not so trivial this time, since the two magnons might scatter among themselves. We now plug this into (3.20) and find the conditions for $\psi(n,m)$, which are

$$\begin{aligned} E\,\psi(n,m) = 4\,\psi(n,m) - \psi(n+1,m) - \psi(n-1,m) \\ - \psi(n,m+1) - \psi(n,m-1) \end{aligned} \tag{3.30}$$

when $m > n+1$ and

$$E\,\psi(n,n+1) = 2\,\psi(n,n+1) - \psi(n-1,n+1) - \psi(n,n+2) \tag{3.31}$$

when $m = n+1$, i.e. when the two magnons scatter. The solution is now a superposition of single magnon states

$$\psi(n,m) = e^{ikn+ipm} + S(k,p)e^{ipn+ikm}, \tag{3.32}$$

where

$$S(p,k) = \frac{\frac{1}{2}\cot\frac{k}{2} - \frac{1}{2}\cot\frac{p}{2} - i}{\frac{1}{2}\cot\frac{k}{2} - \frac{1}{2}\cot\frac{p}{2} + i} \tag{3.33}$$

is the scattering matrix. As required, such a state is an eigenstate and the energy is given by

$$E = E(p) + E(k), \tag{3.34}$$

i.e. it is simply the sum of the single magnon energies. Finally the spin chain periodicity condition imposes the following equations

$$e^{ikL}\, S(p,k) = e^{ipL}\, S(k,p) = 1. \tag{3.35}$$

It is now straightforward to generalize this procedure, which is exactly what Bethe did. The wave function for M spins down can be written as

$$|\psi\rangle = \sum_{1\le l_1<l_2<\cdots<l_M\le L} \psi(l_1,l_2,\ldots,l_M)\, S_{l_1}^- S_{l_2}^- \ldots S_{l_M}^- |0\rangle. \tag{3.36}$$

The sum is chosen in a way so as not to over count states. The Bethe ansatz is the educated guess of the wave function

$$\psi(l_1,l_2,\ldots,l_M) = \sum_{\sigma \in perm(1,2,\ldots M)} A(p)\, e^{ip_{\sigma_1}l_1 + ip_{\sigma_2}l_2 + \cdots + ip_{\sigma_M}l_M}, \tag{3.37}$$

where the sum runs over all permutations of the down spin labels $1,2,\ldots,M$. p_i are the momenta of the down spins, which can be treated as excitations moving in the vacuum state of the spin chain. The ansatz then looks like a superposition of plane waves. As in the two magnon case, one should now plug in the ansatz and find the conditions that make it work. The result is a set of algebraic equations, called the *Bethe equations*

$$e^{ip_kL} = -\prod_{\substack{j=1\\ j\neq k}}^{M} \frac{e^{ip_j} - e^{ip_k} + 1}{e^{ip_k} - e^{ip_j} + 1} \quad \text{for } k = 1,2,\ldots,M \tag{3.38}$$

and the amplitude is given by

$$A(r) = \mathrm{sign}(\sigma) \prod_{j<k} \left(e^{ip_j} - e^{ip_k} + 1\right). \tag{3.39}$$

These equations can be interpreted physically once rewritten as

$$e^{ip_kL} \prod_{\substack{j=1\\ j\neq k}}^{M} S(p_j,p_k) = 1, \quad \text{where } S(p_j,p_k) = -\frac{e^{ip_k} - e^{ip_j} + 1}{e^{ip_j} - e^{ip_k} + 1} \tag{3.40}$$

This is simply saying that if we take a magnon, carry it around the spin chain, the total phase change which is a result of free propagation (represented by $e^{ip_k L}$) and scattering with other magnons (due to $S(p_j, p_k)$) must be trivial. Changing variables to

$$e^{ip_k} = \frac{u_k + i/2}{u_k - i/2}, \quad u_k = \frac{1}{2}\cot\frac{p_k}{2}, \tag{3.41}$$

brings the Bethe equations (3.38) to a more familiar form

$$\left(\frac{u_k + i/2}{u_k - i/2}\right)^L = \prod_{\substack{j=1\\ j\neq k}}^{M} \frac{u_k - u_j + i}{u_k - u_j - i}, \tag{3.42}$$

where now one solves for the Bethe roots u_k, also known as magnon rapidities. It is now straightforward to see that this general solution reproduces the two magnon scenario we discussed earlier. The energy of the M magnon state is given by

$$E = \sum_{k=1}^{M} \frac{1}{u_k^2 + 1/4}, \tag{3.43}$$

which also agrees with the single and two magnon examples.

They key thing worth noting in (3.40) is that the spin chain can be fully described in terms of the scattering matrix for just two particles, i.e. the full M particle scattering matrix factorizes. This is the defining property of integrability, since factorized scattering means that individual momenta are conserved in each two particle scattering producing a tower of conserved quantities – just the thing one would want in an integrable system.

3.1.2 The $\mathfrak{sl}(2)$ sector

The $\mathfrak{su}(2)$ sector has a finite dimensional state space for a given length L of the spin chain since we are dealing with finite dimensional representations of a compact group. The simplest non-compact closed sector is the $\mathfrak{sl}(2)$ sector, which consists of operators of the form [75]

$$\mathcal{O} = \mathrm{Tr}\left(Z^{J-1}\mathcal{D}_+^S Z\right) + \text{permutations}, \tag{3.44}$$

where $\mathcal{D}_+ = \mathcal{D}_1 + i\mathcal{D}_2$ is the lightcone covariant derivative with global charges given by $(\frac{1}{2}, \frac{1}{2}, 1; 0, 0, 0)$. Mixing simply redistributes the S covariant derivative applications among the J scalars. In this case we are dealing with infinite dimensional representations of $\mathfrak{sl}(2)$, namely the number of covariant derivatives is in principle unlimited.

It is convenient to introduce a creation-annihilation operator algebra by defining

$$\left(\mathbf{a}^\dagger\right)^n |0\rangle \equiv \frac{1}{n!}(\mathcal{D}_+)^n Z, \tag{3.45}$$

where $|0\rangle$ is the state annihilated by $\mathbf{a}$. The canonical commutator is defined as usual with $[\mathbf{a}, \mathbf{a}^\dagger] = 1$. The sector is then invariant under the $\mathfrak{sl}(2)$ subalgebra of the full supercoformal algebra given by

$$J'_- = \mathbf{a}, \quad J'_3 = \frac{1}{2} + \mathbf{a}^\dagger\mathbf{a}, \quad J'_+ = \mathbf{a} + \mathbf{a}^\dagger\mathbf{a}^\dagger\mathbf{a}, \tag{3.46}$$

with the defining commutation relations among them

$$[J'_+, J'_-] = -2J'_3, \quad [J'_3, J'_\pm] = \pm J'_\pm. \tag{3.47}$$

Taking a trace of J operators with S covariant derivatives is then equivalent to an $\mathfrak{sl}(2)$ spin chain with $s = -1/2$ representations at each site. The Hamiltonian density is given by [75]

$$\mathbf{H}_{ij}\left(\mathbf{a}_i^\dagger\right)^k\left(\mathbf{a}_j^\dagger\right)^m|00\rangle = \sum_{k'=0}^{m+k}\left(\delta_{k=k'}\left(h(k)+h(m)\right) - \frac{\delta_{k\neq k'}}{|k-k'|}\right)\left(\mathbf{a}_i^\dagger\right)^{k'}\left(\mathbf{a}_j^\dagger\right)^{m+k-k'}|00\rangle, \tag{3.48}$$

where $h(k)$ is the k'th harmonic number given by $\sum_{i=1}^{k} 1/i$. The Hamiltonian is a sum of nearest neighbour interactions

$$\mathbf{H} = \sum_{i=1}^{J} \mathbf{H}_{i,i+1}. \tag{3.49}$$

The spin chain also admits a set of Bethe ansatz equations for the spectrum given by

$$\left(\frac{u_k + i/2}{u_k - i/2}\right)^J = \prod_{\substack{j=1\\ j\neq k}}^{S} \frac{u_k - u_j - i}{u_k - u_j + i}, \tag{3.50}$$

which are remarkably similar to the $\mathfrak{su}(2)$ equations (3.42). Once the Bethe roots are found the energy of the state can be found as

$$E = \sum_{k=1}^{S} \frac{1}{u_k^2 + 1/4}. \tag{3.51}$$

The most famous $\mathfrak{sl}(2)$ operator is the so called Konishi operator

$$\mathcal{O}_K \equiv \mathrm{Tr}\left(\mathcal{D}_+ Z\,\mathcal{D}_+ Z\right) - \mathrm{Tr}\left(Z\,\mathcal{D}_+^2 Z\right). \tag{3.52}$$

It has the classical dimension $\Delta_0 = 4$, which is obvious from dimensionality. A simple calculation shows that it is an eigenstate of the Hamiltonian (3.49) with eigenvalue 12. The same result can also be found from the Bethe ansatz equations (3.50), (3.51). It turns out that the Konishi operator is an eigenstate of the dilatation operator at all loops [76], thus it is a very convenient object to study. So far we can summarize our knowledge of its anomalous dimension as a weak coupling expansion

$$\Delta = 4 + 12g^2 + \mathcal{O}\left(g^4\right). \qquad (3.53)$$

Later sections of this thesis will be mostly concerned with the strong coupling expansion of this anomalous dimension.

3.1.3 Arbitrary sectors

The Bethe ansatz equations for the $\mathfrak{su}(2)$ sector (3.42) and for the $\mathfrak{sl}(2)$ sector (3.50) look remarkably similar, suggesting that there might be a generalization for arbitrary algebras and representations. And indeed such equations exist, they are given by [77]

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$$\left(\frac{u_{i,k} + \frac{i}{2}V_k}{u_{i,k} - \frac{i}{2}V_k}\right)^L = \prod_{l=1}^{r}\prod_{\substack{j=1 \\ j\neq i}}^{J_l} \frac{u_{i,k} - u_{j,l} + \frac{i}{2}M_{kl}}{u_{i,k} - u_{j,l} - \frac{i}{2}M_{kl}}, \qquad (3.54)$$

where M_{kl} is the Cartan matrix of the symmetry algebra and V_k is the vector of highest weights for the representation that the spin sites live in. This is a set of equations for the Bethe roots $u_{k,i}$, where $k = 1, \ldots, \mathrm{rank}(G)$ and $i = 1, \ldots, J_k$ with J_k being the number of excitations of type k (each type corresponds to a different node of the Dynkin diagram, hence k has $\mathrm{rank}(G)$ possible values). The total number of excitations is then $J = \sum J_k$. All of the conserved charges of the system can now be given in terms of the Bethe roots as

$$Q_r = \frac{i}{r-1}\sum_{l=1}^{r}\sum_{j=1}^{J_l}\left(\frac{1}{\left(u_{j,l} + \frac{i}{2}V_l\right)^{r-1}} - \frac{1}{\left(u_{j,l} - \frac{i}{2}V_l\right)^{r-1}}\right). \qquad (3.55)$$

In particular energy is simply the second conserved charge, $E = Q_2$. It is now trivial to check that these equations reproduce all of the Bethe equations discussed so far. It is also a matter of simple algebra to derive them for other closed sectors, such as $\mathfrak{su}(2|3)$ or even the full superconformal algebra $\mathfrak{psu}(2,2|4)$.

3.2 Higher loops and asymptotic length

The next step in solving the spectral problem is increasing the loop level. For the $\mathfrak{su}(2)$ sector this has first been done for two-loops by using symmetry constraints to fix the structure of the operator. The resulting dilatation operator is given by [16]

$$\Gamma_{2-loop} = \frac{\lambda}{8\pi^2}\sum_{l=1}^{L}\left(-4 + 6\,\mathcal{P}_{l,l+1} - \left(\mathcal{P}_{l,l+1}\mathcal{P}_{l+1,l+2} + \mathcal{P}_{l+1,l+2}\mathcal{P}_{l,l+1}\right)\right). \qquad (3.56)$$

In the spin chain picture this corresponds to a Hamiltonian for a long range spin chain with two nearest neighbour interactions. This is hardly surprising, in fact one can expect the range of the spin chain to increase together with the loop level, as can be easily seen from diagrammatic arguments. This long range spin chain is known to be integrable. In fact one can reverse the problem and ask what is the most general form of a long range spin chain Hamiltonian that is still integrable, given its nearest neighbour truncation. The result, up to unknown constant factors, has been worked out [78] and the structure of the Hamiltonian matches loop calculations that are currently available up to five loops. It is now widely believed that the dilatation operator is integrable to all loops.

The method of long range spin chain deformations also predicts how the Bethe ansatz equations get modified at higher loops. Surprisingly the only changes that have to be introduced are the rapidity map

$$u_k \pm \frac{i}{2}V_k \to x\left(u_k \pm \frac{i}{2}V_k\right), \quad u(x) = x + \sum_{k=3}^{\infty}\frac{\alpha_k}{x^{k-2}}, \qquad (3.57)$$

and the dressing phase for the scattering matrix

$$S(u_k - u_j) \to S(u_k - u_j)\exp\left(2i\theta(u_k, u_j)\right), \qquad (3.58)$$

with

$$\theta(u_k, u_j) = \sum_{s>r=2}^{\infty}\beta_{r,s}\left(q_r(u_k)q_s(u_j) - q_s(u_k)q_r(u_j)\right), \qquad (3.59)$$

where $q_r(u_k)$ is the eigenvalue of the conserved charge $\mathbf{Q}_r$ on the single magnon state $|u_k\rangle$. The constants α_k and $\beta_{r,s}$ contain dynamical information about the theory and should be determined by other means, such as loop calculations.

Going higher up in loops poses an additional complication, namely the fact that interactions get long ranged and can start wrapping around short operators, this is the so-called *wrapping problem*. For starters it is easiest to avoid it by considering asymptotically long operators. The rapidity map for $\mathcal{N} = 4$ SYM has been conjectured to be [22]

$$x + \frac{1}{x} = \frac{4\pi}{\sqrt{\lambda}}u, \qquad x^{\pm} + \frac{1}{x^{\pm}} = \frac{4\pi}{\sqrt{\lambda}}\left(u \pm \frac{i}{2}\right), \qquad (3.60)$$

whereas the dressing phase only appears at four loops [79]. The most general form of the Bethe ansatz equations (3.54) modified by the rapidity map and the dressing phase are referred to as the *asymptotic Bethe ansatz* equations. They have been extensively verified [80, 81] since their original proposal.

3.2.1 A glimpse ahead: the slope function

It is now a simple exercise to write down the asymptotic Bethe ansatz equations for the $\mathfrak{sl}(2)$ sector, which are [24]

$$\left(\frac{x_k^+}{x_k^-}\right)^J = \prod_{j\neq k}^{S} \frac{x_k^- - x_j^+}{x_k^+ - x_j^-} \frac{1-1/(x_k^+ x_j^-)}{1-1/(x_k^- x_j^+)} \sigma^2(u_k, u_j), \quad k = 1, \ldots S \tag{3.61}$$

where

$$\Delta = J + S + \gamma(g), \quad \gamma(g) = \frac{i\sqrt{\lambda}}{2\pi} \sum_{j=1}^{S} \left(\frac{1}{x_j^+} - \frac{1}{x_j^-}\right) \tag{3.62}$$

This asymptotic Bethe ansatz (3.61) is the first non-trivial exact result we encountered so far, even if only valid in the asymptotic limit. In this short paragraph we will demonstrate how it can be used to find the exact *slope function* $\gamma^{(1)}(g)$, which is defined as the linear term in the small S expansion of the anomalous dimension, namely

$$\gamma(g) = \gamma^{(1)}(g)\, S + \gamma^{(2)}(g)\, S^2 + \mathcal{O}\left(S^3\right). \tag{3.63}$$

The subleading coefficient $\gamma^{(2)}(g)$ is called the *curvature function* and it will be the main study object of section 4.4. We address the question of what it actually means to send an integer quantity S to zero in section 4.3.4.

The slope function was initially conjectured in [82] and later independently proved in [83] and [84], our derivation will follow the former reference. The starting point is the logarithm of the asymptotic Bethe ansatz (3.61), given by

$$\frac{J}{i} \log\left(\frac{x_k^+}{x_k^-}\right) - \sum_{j\neq k}^{S} \frac{1}{i} \log\left(\frac{x_k^- - x_j^+}{x_k^+ - x_j^-} \frac{1-1/(x_k^+ x_j^-)}{1-1/(x_k^- x_j^+)} \sigma^2(u_k, u_j)\right) = 2\pi n_k. \tag{3.64}$$

where n_k is the mode number of the k'th Bethe root. In the small S limit the number of Bethe roots also tends to zero and in this regime they stop interacting [82], thus we will consider the case when $n_k = n$ and the general result will simply be a linear combination of terms with different values of n_k. The key idea of the derivation is assuming that the result only depends on the combination $\Lambda \equiv n\sqrt{\lambda}$ and taking the small n limit. Obviously this limit is also the strong coupling limit, as $\lambda \sim 1/n^2 \to \infty$. This considerably simplifies the derivation, for starters we only need the strong coupling expansion of the dressing phase, which is given by [24]

$$\log \sigma(u_k, u_j) \simeq -\log\left(\frac{1-1/(x_k^+ x_j^-)}{1-1/(x_k^- x_j^+)}\right) + i\,(u_j - u_k) \log\left(\frac{x_j^- x_k^- - 1}{x_j^- x_k^+ - 1} \frac{x_j^+ x_k^+ - 1}{x_j^+ x_k^- - 1}\right) \tag{3.65}$$

Also, since $u_k \sim 1/n$ we can simplify the shifts in the rapidities u_k, namely

$$x_k^\pm = x\left(u_k \pm \frac{i}{2}\right) = x\left(\frac{1}{g}\left(x_k + \frac{1}{x_k}\right) \pm \frac{i}{2}\right) = x_k \pm \frac{i}{2g} \frac{x_k^2}{x_k^2 - 1} + \mathcal{O}\left(\frac{1}{g^2}\right). \tag{3.66}$$

Plugging in the leading order dressing phase expansion and getting rid of the shifts in the rapidities reduces the asymptotic Bethe ansatz equations (3.64) to

$$\sum_{j\neq k} \frac{2}{x_k - x_j} + \frac{1}{x_k}\left(J + \gamma + \frac{2}{1-x_k^2}\right) = \frac{\Lambda(x_k^2 - 1)}{2x_k^2}, \tag{3.67}$$

which are now starting to resemble equations found in matrix models. In anticipation of this we introduce the resolvent

$$G(x) = \sum_{j=1}^{S} \frac{1}{x - x_j}, \tag{3.68}$$

the anomalous dimension is then given by

$$\gamma = G(1) - G(-1). \tag{3.69}$$

We now multiply (3.67) by $(x - x_k)^{-1}$ and sum over k, which yields

$$G^2(x) + G'(x) + \left(\frac{J+\gamma+2}{x} - \frac{2x}{x^2-1} + \frac{\Lambda}{2}\frac{1-x^2}{x^2}\right) G(x) = F(x), \tag{3.70}$$

where

$$F(x) = \frac{\Lambda}{2}\frac{G(0) + G'(0)x}{x^2} + (J+\gamma+2)\frac{G(0)}{x} - \frac{G(1)}{x-1} - \frac{G(-1)}{x+1} \tag{3.71}$$

and we used the following well known identity from matrix model literature

$$\sum_{j\neq k} \frac{2}{(x-x_k)(x_k - x_j)} = G^2(x) + G'(x). \tag{3.72}$$

Next we expand (3.70) at large x, obviously is still has to be satisfied order by order. In this limit $G(x) \sim S/x$ and at second to leading order we find an equation for $G'(0)$,

$$\Lambda G'(0) = 2G(1) + 2G(-1) - 2G(0)(J+\gamma+2) - \Lambda S, \tag{3.73}$$

which we then stick back into $F(x)$ to produce

$$F(x) = \frac{\Lambda}{2}\left(\frac{G(0)}{x^2} - \frac{S}{x}\right) + \frac{G(-1)}{x(x+1)} - \frac{G(1)}{x(x-1)}, \tag{3.74}$$

thus introducing the parameter S, which in principle could now be non-integer. Finally we take the small S limit by noting that $G(x) \sim S$, keeping only the leading S term $\gamma^{(1)}$ in the anomalous dimension and dropping all sub-leading S terms. What remains is a first order linear differential equation, which after integrating gives

$$G(x) = \frac{x^2-1}{x^{J+2}} e^{\Lambda\frac{x^2+1}{2x}} \int_{x_0}^{x} dy\, F(y) \frac{y^{J+2}}{y^2-1} e^{-\Lambda\frac{y^2+1}{2y}}. \tag{3.75}$$

where x_0 is the integration constant. We can immediately set it to zero by requiring the resolvent to be finite at the origin. Furthermore we can fix the remaining unknown constants $G(0)$ and $G(\pm 1)$ by requiring analyticity of the resolvent, which is manifest in the definition (3.68). The integrand has poles at ± 1, which may lead to logarithmic singularities after integrating, unless the residues are zero. This provides the following constraints

$$\Lambda(G(0) - S) - G(+1)(2J + 1) + G(-1) = 0, \tag{3.76a}$$

$$\Lambda(G(0) + S) + G(-1)(2J + 1) - G(+1) = 0. \tag{3.76b}$$

Together with (3.69) we can solve this system of equations for the unknowns $G(0)$ and $G(\pm 1)$ in terms of S, J, Λ and $\gamma^{(1)}$. Plugging everything in and integrating by parts we get

$$G(x) = -\frac{\Lambda}{2}\frac{S}{J} - \frac{\gamma^{(1)}}{2x} - \frac{\Lambda}{4J}\frac{x^2-1}{x^{J+2}}e^{\Lambda\frac{x^2+1}{2x}}\int_0^x dy\left(\gamma^{(1)}Jy^{J-1} + \Lambda S y^J\right)e^{-\Lambda\frac{y^2+1}{2y}}. \tag{3.77}$$

We fix the final unknown $\gamma^{(1)}$ by requiring analyticity at the origin, since in general it can be a branch point. To that end we recall the integral representation of the modified Bessel function

$$I_\nu(\Lambda) = \frac{(-1)^{-\nu}}{2\pi i}\oint dy\, y^{-\nu-1}e^{\Lambda\frac{y^2+1}{2y}}, \tag{3.78}$$

where the integration contour starts at the origin, goes around counter-clockwise and returns to the origin. Since this integral has no branchpoints at the origin, we see that if we tune the integrand of (3.77) in the following way

$$\gamma^{(1)}J(-1)^J I_J(\Lambda) + \Lambda S(-1)^{J+1}I_{J+1}(\Lambda) = 0, \tag{3.79}$$

then $G(x)$ is also regular at the origin. This condition fixes the leading small S coefficient of the anomalous dimension, i.e. the slope function $\gamma^{(1)}$ to be

$$\gamma^{(1)}(\Lambda) = \frac{\Lambda}{J}\frac{I_{J+1}(\Lambda)}{I_J(\Lambda)}. \tag{3.80}$$

This result is exact in the sense that it is valid at any value of the coupling. We will be making use of it later in the text to extract non-trivial information about the Konishi anomalous dimension.

3.3 Strong coupling

Since integrability seems to be an all loop phenomenon as exemplified by the asymptotic Bethe ansatz, one would expect to find it at strong coupling as well where $\mathcal{N} = 4$ super

Yang-Mills admits a dual string theory formulation as discussed in section 2.4. In this section we showcase integrability of the classical $AdS_5 \times S^5$ string using the elegant language of classical spectral curves [21, 85].

3.3.1 The classical spectral curve

Recall that classical string theory on $AdS_5 \times S^5$ can be formulated as a super-coset sigma model, which is defined in terms of the algebra current $J = -g^{-1}dg$. This current has the property of being flat,

$$dJ - J \wedge J = 0, \tag{3.81}$$

what is more, one can find a one parameter family of connections [17]

$$\begin{aligned} L(x) = J^{(0)} + \frac{x^2+1}{x^2-1}J^{(2)} - \frac{2x}{x^2-1}\left(*J^{(0)} - \Lambda\right) \\ + \sqrt{\frac{x+1}{x-1}}J^{(1)} + \sqrt{\frac{x-1}{x+1}}J^{(3)}, \end{aligned} \tag{3.82}$$

which are flat for any x,

$$dL(x) - L(x) \wedge L(x) = 0. \tag{3.83}$$

Here $L(x)$ is the *Lax connection* and x is the spectral parameter. The existence of such a set of connections signals that the theory is at least classically integrable. This can be shown by constructing the monodromy matrix

$$\Omega(x) = \mathcal{P}\exp\oint_\gamma L(x), \tag{3.84}$$

where γ is any path wrapping the worldsheet cylinder. Since the connection is flat, by definition it is path independent and we can evaluate the integral along any constant τ loop. Furthermore, shifting the τ value corresponds to doing a similarity transformation on the monodromy matrix, meaning that the eigenvalues must be time independent. Thus we have an infinite tower of conserved charges, hinting that the theory is integrable. Technically to prove classical integrability one has to also show that the conserved charges are local and that they are in involution with each other [86, 87]. In order to find the eigenvalues of the monodromy matrix one has to solve the characteristic equation, which in this case is a polynomial of order eight. This equation in turn defines an eight-sheeted Riemann surface where the sheets can be connected with branch cuts of the square root type. We refer to this surface as the *classical spectral curve* or alternatively the algebraic curve. Denote the eigenvalues of the monodromy matrix as

$$\{e^{i\hat{p}_1(x)}, e^{i\hat{p}_2(x)}, e^{i\hat{p}_3(x)}, e^{i\hat{p}_4(x)} \,|\, e^{i\tilde{p}_1(x)}, e^{i\tilde{p}_2(x)}, e^{i\tilde{p}_3(x)}, e^{i\tilde{p}_4(x)}\}, \tag{3.85}$$

Figure 3: Examples of cuts connecting the eight sheets of the Riemann surface corresponding to the classical spectral curve for strings in $AdS_5 \times S^5$. The wavy line corresponds to the pole at $x = 1$.

where the quantities $p(x)$ are called *quasi-momenta* and we use the convention where hatted quantities correspond to AdS_5 variables and quantities with tildes correspond to S^5. The quasi-momenta $p(x)$ being logarithms of the eigenvalues live on the sheets of this Riemann surface. The key idea here is that the quasi-momenta provide an alternative representation of classical string solutions which lends itself better to generalization. Namely, this description of classical solutions is more natural in light of integrability, furthermore it is better suited for quantization.

The physical picture is that each cut between two sheets represents an excitation whose polarization is determined by the sheets it connects, an example is shown in figure 3. Four of the eight sheets correspond to the AdS_5 part of the string target space and the other four to the S^5 part. $AdS_5 \times S^5$ has $16 = 8_B + 8_F$ types of excitations. in the algebraic curve language this is implemented as a requirement that only sheets from the following sets be connected

$$i \in \{\hat{1}, \hat{2}, \tilde{1}, \tilde{2}\}, \quad j \in \{\hat{3}, \hat{4}, \tilde{3}, \tilde{4}\}, \tag{3.86}$$

furthermore bosonic excitations correspond to cuts between sheets of the same type (hat or tilde), whereas fermionic excitations connect sheets of different types. Obviously fermions do not exist at the classical level, thus cuts can only represent the 8 types of bosonic excitations. Fermionic excitations start appearing as microscopic cuts, i.e. poles during quantization. Solutions in closed sectors, e.g. strings moving in the $\mathbb{R} \times S^3$ submanifold of the target space will be limited to cuts between a subset of the eight sheets.

Denote the branch cut between sheets i and j as $\mathcal{C}^{ij}$, the quasi-momenta on these sheets have discontinuities when going through the cut given by

$$p_i(x + i\epsilon) - p_j(x - i\epsilon) = 2\pi n_{ij}, \tag{3.87}$$

where n_{ij} is an integer mode number arising due to the logarithm. For each cut we associate the so called *filling fraction* defined by

$$S_{ij} = \pm \frac{\lambda}{8\pi^2 i} \oint_{\mathcal{C}^{ij}} \left(1 - \frac{1}{x^2}\right) p_i(x) dx, \tag{3.88}$$

where the sign is $+1$ for $i = \hat{1}, \hat{2}$ and -1 for $i = \tilde{1}, \tilde{2}$. These are the action variables of the theory [88], roughly they measure the length of the cut and in the physical picture they correspond to the amplitude of the excitation. They can be shown to take on integer values, which is natural since we anticipate the classical cuts to be collections of large numbers of poles which condense in the classical limit. Thus we see that the algebraic curve construction acts like a Fourier decomposition – string solutions are described as collections of excitations each having definite polarizations, mode numbers and amplitudes.

Let us now review some of the analyticity properties of the quasi-momenta. Since the Lax connection has poles at $x = \pm 1$, so do the quasi-momenta (as shown in figure 3). Due to the Virasoro constraint, which comes about from the diffeomorphism invariance of the worldsheet, the residues of the quasi-momenta are constrained to

font

$$\{\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4 \,|\, \tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \tilde{p}_4\} = \frac{\{\alpha_\pm, \alpha_\pm, \beta_\pm, \beta_\pm \,|\, \alpha_\pm, \alpha_\pm, \beta_\pm, \beta_\pm\}}{x \pm 1}. \tag{3.89}$$

An additional constraint on the quasi-momenta comes from the fact that the algebra psu(2, 2|4) has an automorphism, which is the cause for the $\mathbb{Z}_4$ grading. The constraints are given by [89]

$$\begin{aligned} \tilde{p}_{1,2}(x) &= -\tilde{p}_{2,1}(1/x) - 2\pi m \\ \tilde{p}_{3,4}(x) &= -\tilde{p}_{4,3}(1/x) + 2\pi m \\ \hat{p}_{1,2,3,4}(x) &= -\hat{p}_{2,1,4,3}(1/x). \end{aligned} \tag{3.90}$$

These relations define an inversion symmetry $x \to 1/x$. Finally one can look at the asymptotics of the quasi-momenta as the spectral parameter becomes infinite. In this limit the Lax connection becomes related to the Noether currents of the theory and hence one can relate the quasi-momenta to the charges of the global symmetry algebra by [90, 85]

$$\left(\begin{array}{c} \hat{p}_1 \\ \hat{p}_2 \\ \hat{p}_3 \\ \hat{p}_4 \\ \hline \tilde{p}_1 \\ \tilde{p}_2 \\ \tilde{p}_3 \\ \tilde{p}_4 \end{array}\right) = \frac{2\pi}{x}\left(\begin{array}{c} +\mathcal{E} - \mathcal{S}_1 + \mathcal{S}_2 \\ +\mathcal{E} + \mathcal{S}_1 - \mathcal{S}_2 \\ -\mathcal{E} - \mathcal{S}_1 - \mathcal{S}_2 \\ -\mathcal{E} + \mathcal{S}_1 + \mathcal{S}_2 \\ \hline +\mathcal{J}_1 + \mathcal{J}_2 - \mathcal{J}_3 \\ +\mathcal{J}_1 - \mathcal{J}_2 + \mathcal{J}_3 \\ -\mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 \\ -\mathcal{J}_1 - \mathcal{J}_2 - \mathcal{J}_3 \end{array}\right), \tag{3.91}$$

where the charges are rescaled by $\mathcal{Q} = Q/\sqrt{\lambda}$. Thus we see that we can characterize the quasi-momenta by describing their behaviour at poles, under symmetries, by their asymptotics and their filling fractions.

Finally one may ask how this picture of Riemann surfaces with cuts emerges from the gauge theory perspective where the spectrum is described by the Bethe ansatz. In the scaling limit when lengths of operators become large the Bethe roots u_i start condensing in the complex plane and start looking like cuts. Naturally it is tempting to interpret the cuts of the algebraic curve as collections of very large numbers of poles. Ultimately a string Bethe ansatz was proposed describing the distribution of these poles [24],

$$\left(\frac{x_j^+}{x_j^-}\right)^L = \prod_{j\neq i}^{M} \frac{u_i - u_j + i}{u_i - u_j - i}\sigma^2_{AFS}(u_i, u_j), \tag{3.92}$$

where σ_{AFS} is the dressing phase. Compared to the asymptotic Bethe ansatz for the su(2) sector

$$\left(\frac{x_j^+}{x_j^-}\right)^L = \prod_{j\neq i}^{M} \frac{u_i - u_j + i}{u_i - u_j - i}, \tag{3.93}$$

it is natural to assume that there should be an interpolating Bethe ansatz valid to all orders of the coupling constant. Indeed the all-loop asymptotic Bethe ansatz for the full superconformal algebra was formulated [23] as we described briefly in the previous sections, which interpolated nicely between gauge theory and the algebraic curve. In particular the dressing phase σ_{AFS} is a limit of the full dressing phase (3.58) one finds when deforming to long range spin chains.

3.3.2 Quantization and semi-classics

Consider a classical string solution characterized by some conserved charges, expanding the superstring action around this solution produces a quadratic lagrangian whose quantization yields the semiclassical spectrum

$$E(\{N_{ij,n}\}) = E_{cl} + E_0 + \sum_{ij,n} N_{ij,n}\mathcal{E}_{ij,n}, \tag{3.94}$$

where $N_{ij,n}$ is the number of excited quanta with energy $\mathcal{E}_{ij,n}$. Here ij label the different polarizations and n the mode numbers of the excitations. The classical energy is E_{cl} whereas E_0 is the ground state energy coming from quantization, the last two terms in (3.94) are analogues of $\frac{1}{2}\omega$ and $N\omega$ for the harmonic oscillator. Just like in the case of the harmonic oscillator we can infer the ground state energy given the level spacings, it is simply

$$E_0 = \frac{1}{2}\sum_{ij,n}(-1)^{F_{ij}}\mathcal{E}_{ij,n}, \tag{3.95}$$

where $(-1)^{F_{ij}} = \pm 1$ for bosonic/fermionic excitations. In this section we will review the quantization procedure in the algebraic curve formalism [89], which is equivalent to the semi-classical computation of quadratic fluctuations in the sigma-model [91, 92, 93], yet is significantly more efficient.

Roughly the idea is that given a classical string solution represented by some cuts between sheets, as seen for example in figure 3, we perturb it by adding microscopic cuts, which can be treated as a finite number of poles. Just like before the indices ij denoting the connected sheets represent the polarization of the excitation, they can take on values given in (3.86), however unlike in the classical setting the excitations can be fermionic as well. The introduction of these fluctuations backreacts on the classical quasi-momenta $p_k(x)$ shifting them slightly to

$$p_k(x) \to p_k(x) + \delta_n^{ij} p_k(x). \tag{3.96}$$

The shifted quasi-momenta still have to satisfy (3.87), which determines the positions x_n^{ij} of the poles

$$p_i(x_n^{ij}) - p_j(x_n^{ij}) = 2\pi\, n_{ij}. \tag{3.97}$$

The fluctuation will add a pole to the quasi-momentum at this position

$$\delta_n^{ij} p_i = \epsilon_i \frac{\alpha(x_n^{ij})}{x - x_n^{ij}}. \tag{3.98}$$

where the signs are

$$\epsilon_{\hat{1}} = \epsilon_{\hat{2}} = -\epsilon_{\hat{3}} = -\epsilon_{\hat{4}} = -\epsilon_{\tilde{1}} = -\epsilon_{\tilde{2}} = \epsilon_{\tilde{3}} = \epsilon_{\tilde{4}} = 1 \tag{3.99}$$

and the residue is chosen such that the filling fraction (3.88) increases by one, namely

$$\alpha(x) = \frac{4\pi}{\sqrt{\lambda}} \frac{x^2}{x^2 - 1}. \tag{3.100}$$

The total shifted quasi-momentum is obtained by summing over all fluctuations

$$\delta p_i \sim \sum_{ij} \epsilon_i N_n^{ij} \frac{\alpha(x_n^{ij})}{x - x_n^{ij}} \tag{3.101}$$

It still has to satisfy all the analyticity properties outlined the the previous section, this in turn imposes a lot of constraints of the shifts themselves. The Virasoro constraint implies the synchronization of residues (3.89), which for the shifts translates to

$$\{\delta\hat{p}_1, \delta\hat{p}_2, \delta\hat{p}_3, \delta\hat{p}_4 \,|\, \delta\tilde{p}_1, \delta\tilde{p}_2, \delta\tilde{p}_3, \delta\tilde{p}_4\} = \frac{\{\delta\alpha_{\pm}, \delta\alpha_{\pm}, \delta\beta_{\pm}, \delta\beta_{\pm} \,|\, \delta\alpha_{\pm}, \delta\alpha_{\pm}, \delta\beta_{\pm}, \delta\beta_{\pm}\}}{x \pm 1}. \tag{3.102}$$

Similarly the asymptotics of the quasi-momenta encode the global charges as seen in (3.91), for the shifts this translates to

$$\begin{pmatrix} \delta\hat{p}_1 \\ \delta\hat{p}_2 \\ \delta\hat{p}_3 \\ \delta\hat{p}_4 \\ \hline \delta\tilde{p}_1 \\ \delta\tilde{p}_2 \\ \delta\tilde{p}_3 \\ \delta\tilde{p}_4 \end{pmatrix} = \frac{4\pi}{x\sqrt{\lambda}} \begin{pmatrix} +\delta\Delta/2 & +N_{\hat{1}\hat{4}} + N_{\hat{1}\hat{3}} + N_{\hat{1}\tilde{3}} + N_{\hat{1}\tilde{4}} \\ +\delta\Delta/2 & +N_{\hat{2}\hat{3}} + N_{\hat{2}\hat{4}} + N_{\hat{2}\tilde{4}} + N_{\hat{2}\tilde{3}} \\ -\delta\Delta/2 & -N_{\hat{2}\hat{3}} - N_{\hat{1}\hat{3}} - N_{\tilde{1}\hat{3}} - N_{\tilde{2}\hat{3}} \\ -\delta\Delta/2 & -N_{\hat{1}\hat{4}} - N_{\hat{2}\hat{4}} - N_{\tilde{2}\hat{4}} - N_{\tilde{1}\hat{4}} \\ \hline & -N_{\tilde{1}\tilde{4}} - N_{\tilde{1}\tilde{3}} - N_{\tilde{1}\hat{3}} - N_{\tilde{1}\hat{4}} \\ & -N_{\tilde{2}\tilde{3}} - N_{\tilde{2}\tilde{4}} - N_{\tilde{2}\hat{4}} - N_{\tilde{2}\hat{3}} \\ & +N_{\tilde{2}\tilde{3}} + N_{\tilde{1}\tilde{3}} + N_{\hat{1}\tilde{3}} + N_{\hat{2}\tilde{3}} \\ & +N_{\tilde{1}\tilde{4}} + N_{\tilde{2}\tilde{4}} + N_{\hat{2}\tilde{4}} + N_{\hat{1}\tilde{4}} \end{pmatrix} + \mathcal{O}\left(\frac{1}{x^2}\right), \tag{3.103}$$

where $\delta\Delta$ is the shift of the classical energy $\mathcal{E}$. From here one can read off the individual fluctuation frequencies

$$\Omega_n^{ij} = -2\delta_{i,\tilde{1}} + \frac{\lambda}{2\pi} \lim_{x\to\infty} x\, \delta_n^{ij} p_{\hat{1}}(x) \tag{3.104}$$

and the energy shift is then a sum over all frequencies

$$\delta\Delta = \sum_{ij,n} N_{ij}^n \Omega_n^{ij}. \tag{3.105}$$

The description outlined above is fully sufficient to calculate the semi-classical spectrum around a classical solution – one has to find the locations of poles, find shifts to the

quasi-momenta by utilizing their analyticity properties and finally calculate the 16 fluctuation frequencies and sum them up. This produces the energy shift

$$\delta\Delta = E(\{N_{ij,n}\}) - E(\{\}) = \sum_{ij,n} N_{ij}^n \Omega_n^{ij}. \tag{3.106}$$

Another quantity of interest is the one-loop shift

$$E_0 = \frac{1}{2} \sum_{ij,n} (-1)^{F_{ij}} \Omega_n^{ij} \tag{3.107}$$

appearing in the loop expansion of the energy of a string state as

$$E(\{\}) = E_{cl} + E_0 + \mathcal{O}\left(1/\sqrt{\lambda}\right), \tag{3.108}$$

where the classical energy E_{cl} is of order $\sqrt{\lambda}$ and E_0 is of order 1. One can of course proceed with semi-classical quantization, find all the fluctuation frequencies and sum them up by hand to find the one-loop shift, however it would be nicer to find the result in one go. To that end we introduce the off-shell fluctuations $\delta^{ij} p_k(x; y)$ which are defined by the same asymptotics as the on-shell fluctuations $\delta_n^{ij} p_k(x)$ but the position of the pole is left unspecified, namely

$$\delta_n^{ij} p_k(x) = \delta^{ij} p_k(x; y)\big|_{y = x_n^{ij}}. \tag{3.109}$$

Note that the off-shell quantity depends on the mode number n, which is a function of the pole position via (3.97), which we simply left unspecified as y in the off-shell quantity. Similarly we introduce off-shell fluctuation energies

$$\Omega_n^{ij} = \Omega^{ij}(y)\big|_{y = x_n^{ij}}, \tag{3.110}$$

which can easily be found if the on-shell frequencies are known by

$$\Omega^{ij}(y) = \Omega_n^{ij}\big|_{n \to \frac{p_i(y) - p_j(y)}{2\pi}}. \tag{3.111}$$

The main advantage of introducing the off-shell frequencies is that due to the $\mathbb{Z}_4$ grading of the $\mathfrak{psu}(2,2|4)$ algebra the quasi-momenta enjoy an inversion symmetry under $x \to 1/x$ as seen in (3.90). This constrains the off-shell frequencies as well. Consider symmetric classical configurations that have pairwise symmetric quasi-momenta

$$p_{\hat{1},\hat{2},\tilde{1},\tilde{2}} = -p_{\hat{4},\hat{3},\tilde{4},\tilde{3}}, \tag{3.112}$$

it is known to be the case for all rank one solutions [89], which in particular are dual to states in the $\mathfrak{su}(2)$ and $\mathfrak{sl}(2)$ sectors of $\mathcal{N} = 4$ super Yang-Mills. All of the off-shell frequencies can then be expressed in terms of two, namely $\Omega^{\tilde{2}\tilde{3}}$ and $\Omega^{\hat{2}\hat{3}}$, which we denote as the basis of frequencies. The rest of the fluctuations are given by

$$
\begin{aligned}
\Omega^{\tilde{1}\tilde{4}}(y) &= -\Omega^{\tilde{2}\tilde{3}}(1/y) + \Omega^{\tilde{2}\tilde{3}}(0)\\
\Omega^{\tilde{2}\tilde{4}}(y) = \Omega^{\tilde{1}\tilde{3}}(y) &= \frac{1}{2}\left(\Omega^{\tilde{2}\tilde{3}}(y) + \Omega^{\tilde{1}\tilde{4}}(y)\right) = \frac{1}{2}\left(\Omega^{\tilde{2}\tilde{3}}(y) - \Omega^{\tilde{2}\tilde{3}}(1/y) + \Omega^{\tilde{2}\tilde{3}}(0)\right)\\
\Omega^{\hat{1}\hat{4}}(y) &= -\Omega^{\hat{2}\hat{3}}(1/y) - 2\\
\Omega^{\hat{2}\hat{4}}(y) = \Omega^{\hat{1}\hat{3}}(y) &= \frac{1}{2}\left(\Omega^{\hat{2}\hat{3}}(y) + \Omega^{\hat{1}\hat{4}}(y)\right) = \frac{1}{2}\left(\Omega^{\hat{2}\hat{3}}(y) - \Omega^{\hat{2}\hat{3}}(1/y)\right) - 1\\
\Omega^{\tilde{2}\hat{4}}(y) = \Omega^{\tilde{1}\hat{3}}(y) &= \frac{1}{2}\left(\Omega^{\hat{2}\hat{3}}(y) + \Omega^{\tilde{1}\tilde{4}}(y)\right) = \frac{1}{2}\left(\Omega^{\hat{2}\hat{3}}(y) - \Omega^{\tilde{2}\tilde{3}}(1/y) + \Omega^{\tilde{2}\tilde{3}}(0)\right)\\
\Omega^{\hat{2}\tilde{4}}(y) = \Omega^{\hat{1}\tilde{3}}(y) &= \frac{1}{2}\left(\Omega^{\tilde{2}\tilde{3}}(y) + \Omega^{\hat{1}\hat{4}}(y)\right) = \frac{1}{2}\left(\Omega^{\tilde{2}\tilde{3}}(y) - \Omega^{\hat{2}\hat{3}}(1/y)\right) - 1\\
\Omega^{\tilde{1}\hat{4}}(y) = \Omega^{\hat{1}\tilde{4}}(y) &= \frac{1}{2}\left(\Omega^{\tilde{1}\tilde{4}}(y) + \Omega^{\hat{1}\hat{4}}(y)\right) = \frac{1}{2}\left(-\Omega^{\tilde{2}\tilde{3}}(1/y) - \Omega^{\hat{2}\hat{3}}(1/y) + \Omega^{\tilde{2}\tilde{3}}(0)\right) - 1\\
\Omega^{\tilde{2}\hat{3}}(y) = \Omega^{\hat{2}\tilde{3}}(y) &= \frac{1}{2}\left(\Omega^{\tilde{2}\tilde{3}}(y) + \Omega^{\hat{2}\hat{3}}(y)\right).
\end{aligned}
\tag{3.113}
$$

Knowing the off-shell frequencies and the quasi-momenta one can express the one-loop shift as a contour integral

$$
E_0 = \frac{1}{2}\sum_{ij}(-1)^{F_{ij}} \oint \frac{dx}{2\pi i}\left(\Omega^{ij}(x)\partial_x \log\sin\frac{p_i - p_j}{2}\right), \tag{3.114}
$$

where the integrand is chosen carefully such that it contains poles at each fluctuation insertion point x_n^{ij} with residues $\Omega^{ij}(x_n^{ij})$, so that the result is equivalent to (3.107). There are three contributions to one loop energy shift that are different by their nature. They can be separated into an "anomaly" contribution, a contribution from the dressing phase and a wrapping contribution, which is missing in the asymptotic Bethe ansatz.

$$
E_0 = \delta\Delta_{\text{anomaly}} + \delta\Delta_{\text{dressing}} + \delta\Delta_{\text{wrapping}}, \tag{3.115}
$$

where each of these contributions is simply an integral of some closed form expression,

$$
\delta\Delta_{\text{anomaly}} = -\frac{4}{ab-1}\int_a^b \frac{dx}{2\pi i}\frac{y(x)}{x^2-1}\partial_x \log\sin p_{\hat{2}}, \tag{3.116}
$$

$$
\delta\Delta_{\text{dressing}} = \sum_{ij}(-1)^{F_{ij}}\int_{-1}^{1}\frac{dz}{2\pi i}\left(\Omega^{ij}(z)\,\partial_z \frac{i(p_i - p_j)}{2}\right), \tag{3.117}
$$

$$
\delta\Delta_{\text{wrapping}} = \sum_{ij}(-1)^{F_{ij}}\int_{-1}^{1}\frac{dz}{2\pi i}\left(\Omega^{ij}(z)\,\partial_z \log(1 - e^{-i(p_i - p_j)})\right). \tag{3.118}
$$

As always i takes values $\hat{1}, \hat{2}, \tilde{1}, \tilde{2}$ whereas j runs over $\hat{3}, \hat{4}, \tilde{3}, \tilde{4}$.

3.3.3 Folded string

Operators (3.44) from the $\mathfrak{sl}(2)$ sector are known to be dual to folded rotating string solutions in $AdS_5 \times S^5$, these are closed strings rotating around their center of mass in an AdS_3 subspace of AdS_5 with spin S [91]. Additionally they orbit the big circle of S^5 with angular momentum J, also referred to as twist. These parameters as expected correspond to the number of scalars J and number of derivatives S in the gauge theory operators. In the classical regime these are assumed to scale as $\sqrt{\lambda}$, thus we use $\mathcal{S} = S/n\sqrt{\lambda}$ and $\mathcal{J} = J/n\sqrt{\lambda}$ when describing the classical solution, which corresponds to long operators in gauge theory. The number of spikes n corresponds to the mode number n in the language of Bethe states.

Given the explicit string solution (see [94] for details) one could follow the steps outlined in the classical spectral curve construction, namely calculate the monodromy matrix (3.84), diagonalize it and extract the quasi-momenta. While it is indeed possible to do, we will present an alternative method based on analyticity properties of the quasi-momenta when we discuss the classical limit of cusped Wilson lines in section 4.6.3. Here we present the result [95], which consists of two "basis" quasi-momenta

$$
\begin{aligned}
p_{\hat{2}} &= \pi n - 2\pi n\mathcal{J}\left(\frac{a}{a^2-1} - \frac{x}{x^2-1}\right)\sqrt{\frac{(a^2-1)(b^2-x^2)}{(b^2-1)(a^2-x^2)}}\\
&+ \frac{8\pi n ab\mathcal{S}F_1(x)}{(b-a)(ab+1)} + \frac{2\pi n\mathcal{J}(a-b)F_2(x)}{\sqrt{(a^2-1)(b^2-1)}},\\
p_{\tilde{2}} &= \frac{2\pi\mathcal{J}x}{x^2-1},
\end{aligned}
\tag{3.119}
$$

while the remaining functions can be determined by utilizing the $x \to 1/x$ inversion symmetry as shown in (3.90), resulting in

$$
p_{\hat{2}}(x) = -p_{\hat{3}}(x) = -p_{\hat{1}}(1/x) = p_{\hat{4}}(1/x), \tag{3.120}
$$

$$
p_{\tilde{2}}(x) = -p_{\tilde{3}}(x) = p_{\tilde{1}}(x) = -p_{\tilde{4}}(x). \tag{3.121}
$$

The functions $F_1(x)$ and $F_2(x)$ can be expressed in terms of the elliptic integrals:

$$
\begin{aligned}
F_1(x) &= iF\left(i\sinh^{-1}\sqrt{\frac{(b-a)(a-x)}{(b+a)(a+x)}}\,\middle|\,\frac{(a+b)^2}{(a-b)^2}\right),\\
F_2(x) &= iE\left(i\sinh^{-1}\sqrt{\frac{(b-a)(a-x)}{(b+a)(a+x)}}\,\middle|\,\frac{(a+b)^2}{(a-b)^2}\right).
\end{aligned}
$$

This is a two-cut solution with symmetric cuts on the real axis given by the branch points $a < b$ (and $-b < -a$). The classical energy Δ of the folded string is a function of the Lorentz spin $\mathcal{S}$, twist $\mathcal{J}$ and the mode number n. This function can be written in a parametric form in terms of the branch points a and b [95, 96, 21, 97]:

$$
\begin{aligned}
2\pi\mathcal{S} &= \frac{ab+1}{ab}\left[bE\left(1-\tfrac{a^2}{b^2}\right)-aK\left(1-\frac{a^2}{b^2}\right)\right], \\
2\pi\mathcal{J} &= \frac{2\sqrt{(a^2-1)(b^2-1)}}{b}K\left(1-\frac{a^2}{b^2}\right), \\
2\pi\mathcal{D} &= \frac{ab-1}{ab}\left[bE\left(1-\tfrac{a^2}{b^2}\right)+aK\left(1-\frac{a^2}{b^2}\right)\right],
\end{aligned}
\tag{3.122}
$$

where $\mathcal{D}=\Delta/n\sqrt{\lambda}$ and E, K are elliptic integrals of the first kind.

Next we proceed to the semi-classical quantization of the folded string solution, more precisely we are after the one-loop shift. Instead of calculating the 16 fluctuation frequencies Ω_n^{ij} we adopt the off-shell frequency formalism outlined above. Once again, the exact formulae for the off-shell frequencies can be found solely from analyticity constraints by first determining the off-shell shifts in quasi-momenta and then using the definitions (3.104) and (3.111). The answer turns out to be surprisingly simple and is given by the two basis frequencies

$$
\Omega^{23}(x) = \frac{2}{ab-1}\frac{\sqrt{a^2-1}\sqrt{b^2-1}}{x^2-1}, \tag{3.123}
$$

$$
\Omega^{23}(x) = \frac{2}{ab-1}\left(1-\frac{y(x)}{x^2-1}\right), \tag{3.124}
$$

where $y(x)=\sqrt{x-a}\sqrt{a+x}\sqrt{x-b}\sqrt{b+x}$. The remaining frequencies can be read off from the relations (3.113). What remains in order to find the one-loop shift is performing the sum (3.107) or alternatively evaluating the integral (3.114) numerically. The answer involves an infinite sum over all mode numbers hence it is not very illuminating at this point.

Let us now consider the $\mathcal{S}\to 0$ limit. The square of the classical energy has a very nice expansion in this limit

$$
\mathcal{D}^2 = \mathcal{J}^2+2\mathcal{S}\sqrt{\mathcal{J}^2+1}+\mathcal{S}^2\frac{2\mathcal{J}^2+3}{2\mathcal{J}^2+2}-\mathcal{S}^3\frac{\mathcal{J}^2+3}{8(\mathcal{J}^2+1)^{5/2}}+\mathcal{O}\left(\mathcal{S}^4\right). \tag{3.125}
$$

The one-loop shift expanded up to two orders in $\mathcal{S}$ reads

$$
\Delta \simeq \frac{-\mathcal{S}}{2(\mathcal{J}^3+\mathcal{J})}+\mathcal{S}^2\left[\frac{3\mathcal{J}^4+11\mathcal{J}^2+17}{16\mathcal{J}^3(\mathcal{J}^2+1)^{5/2}}-\sum_{\substack{m>0\\ m\neq n}}\frac{n^3m^2\left(2m^2+n^2\mathcal{J}^2-n^2\right)}{\mathcal{J}^3\left(m^2-n^2\right)^2\left(m^2+n^2\mathcal{J}^2\right)^{3/2}}\right] \tag{3.126}
$$

The sum is nothing but a sum over the fluctuation energies, whereas the remaining terms originate from the "zero"-modes $m=n$, which have to be treated separately. The sum can be very easily expanded for small $\mathcal{J}$. It is easy to see that the expansion

coefficients will be certain combinations of zeta-functions. It is also easy to see that the dependence on the mode number n is rather nontrivial. The expansion of the one loop energy first in small $\mathcal{S}$ up to a second order and then in small $\mathcal{J}$ reads

$$
\Delta_{1-\text{loop}} \simeq \begin{cases}
-\frac{\mathcal{S}}{2\mathcal{J}}+\mathcal{S}^2\left(+\frac{1}{2\mathcal{J}^3}-\frac{3\zeta_3}{2\mathcal{J}}-\frac{1}{16\mathcal{J}}\right) & \text{for } n=1, \\
-\frac{\mathcal{S}}{2\mathcal{J}}+\mathcal{S}^2\left(+\frac{1}{2\mathcal{J}^3}-\frac{12\zeta_3}{\mathcal{J}}-\frac{17}{16\mathcal{J}}\right) & \text{for } n=2, \\
-\frac{\mathcal{S}}{2\mathcal{J}}+\mathcal{S}^2\left(-\frac{5}{8\mathcal{J}^3}-\frac{81\zeta_3}{2\mathcal{J}}-\frac{7}{4\mathcal{J}}\right) & \text{for } n=3.
\end{cases} \tag{3.127}
$$

We note that the contributions $\mathcal{S}^2/\mathcal{J}^3$ are universal for $n=1$ and $n=2$, however starting from $n=3$ we get some nasty coefficient

3.4 Short strings

Even though semi-classical results are technically only valid for the charges scaling as $\sqrt{\lambda}$, there is evidence that re-expanding the classical energy plus the one-loop shift yields a reasonable result in terms of the unscaled charges J and S [98], thus providing one the possibility to probe so-called short string states, the Konishi operator being a prime example. Namely summing up the square root of (3.125) and (3.127), introducing the unscaled charges $S=\sqrt{\mu}\,\mathcal{S}$, $J=\sqrt{\mu}\,\mathcal{J}$ and re-expanding in terms of $\mu\equiv n^2\lambda$ yields the following one-loop energy for the folded string solution [95]

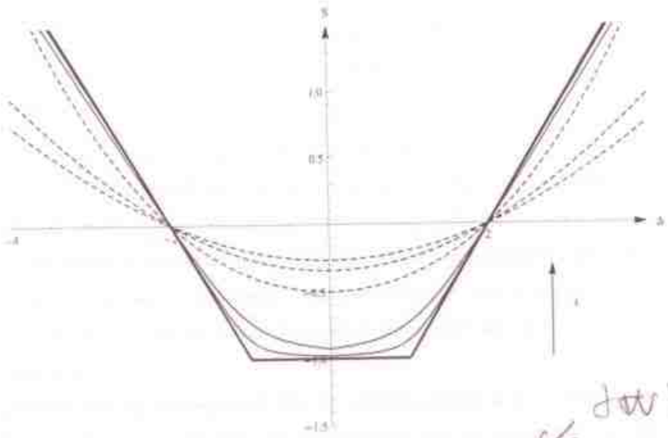
$$
\Delta_{S,J,n} = \sqrt{2S}\mu^{1/4}+\frac{2J^2+3S^2-2S}{4(2S)^{1/2}\mu^{1/4}}. \tag{3.128}
$$

For the Konishi operator we get

$$
\Delta_{2,2,1} = 2\lambda^{1/4}+\frac{2}{\lambda^{1/4}}+\mathcal{O}\left(\lambda^{-3/4}\right), \tag{3.129}
$$

where the first coefficient of the expansion was first derived in [99]. The sub-leading coefficient was first suggested to be 1 in [98], however a possible issue with the argument was soon pointed out in [100], where numerical calculations indeed confirmed the correct answer to be 2.

In this section we will show how combining the knowledge of the semi-classical one-loop shift (3.126) and the slope function (3.80) can yield further information about the Konishi anomalous dimension. Later in the text when we find the next small spin coefficient after the slope, which we call the curvature function, we will revisit the techniques presented here in order to boost the obtain results one order further.



twist-2 op.

Figure 4: The BFKL trajectories $S(\Delta)$ for the ~~Konishi operator~~ at various values of the coupling. Solid black lines are obtained using the known two loop weak coupling expansion [101, 102] and dashed lines are obtained using the strong coupling expansion [103, 104, 105].

3.4.1 Structure of small spin expansions

Consider the classical string energy (3.125), the re-expansion of Δ^2 in the large $\mu \equiv \lambda n^2$ limit with S and J fixed has a particularly nice structure

$$\Delta^2 = J^2 + S\left(2\sqrt{\mu} + \frac{J^2}{\sqrt{\mu}} + \dots\right) + S^2\left(\frac{3}{2} - \frac{J^2}{2\sqrt{\mu}} + \dots\right) - S^3\left(\frac{3}{8\sqrt{\mu}} - \frac{13J^2}{16\sqrt[3]{\mu}} + \dots\right) + \mathcal{O}(S^4) \tag{3.130}$$

where each next term in S gets more and more suppressed for large μ. It was conjectured [82] that a generalization of this result holds for any operator at all values of the coupling constant, more precisely the statement was that making expansions of the scaling dimension squared first in $S \to 0$ and then in $\mu \to \infty$ should reveal the following structure

$$\Delta^2 = J^2 + S\left(A_1\sqrt{\mu} + A_2 + \dots\right) + S^2\left(B_1 + \frac{B_2}{\sqrt{\mu}} + \dots\right) + S^3\left(\frac{C_1}{\mu^{1/2}} + \frac{C_2}{\mu^{3/2}} + \dots\right) + \mathcal{O}(S^4), \tag{3.131}$$

where the coefficients A_i, B_i, C_i, etc. are some functions of J.

Indeed it is not hard to understand this constraint. A good entry point is considering the inverse relation $S(\Delta)$, frequently encountered in the context of BFKL [106], where one also usually sets $n = 1$. It satisfies a few basic properties, namely the curve $S(\Delta)$ goes through the points $(\pm J, 0)$ at any coupling, because at $S = 0$ the operator is

cite new paper?

BPS. At the same time for non-BPS states one should have $\Delta(\lambda) \propto \lambda^{1/4} \to \infty$ [9] which indicates that if Δ is fixed, S should go to zero, thus combining this with the knowledge of fixed points $(\pm J, 0)$ we conclude that at infinite coupling $S(\Delta)$ is simply the line $S = 0$. As the coupling becomes finite $S(\Delta)$ starts bending from the $S = 0$ line and starts looking like a parabola going through the points $\pm J$, as shown in figure 4. Based on this qualitative picture and the scaling $\Delta(\lambda) \propto \lambda^{1/4}$ at $\lambda \to \infty$ and fixed J and S, one can write down the following ansatz,

$$\begin{aligned} S(\Delta) &= (\Delta^2 - J^2)\Big(\alpha_1\frac{1}{\lambda^{1/2}} + \alpha_2\frac{1}{\lambda} + (\alpha_3 + \beta_3\Delta^2)\frac{1}{\lambda^{3/2}} + (\alpha_4 + \beta_4\Delta^2)\frac{1}{\lambda^2} \\ &+ (\alpha_5 + \beta_5\Delta^2 + \gamma_5\Delta^4)\frac{1}{\lambda^{5/2}} + (\alpha_6 + \beta_6\Delta^2 + \gamma_6\Delta^4)\frac{1}{\lambda^3} + \dots\Big). \end{aligned} \tag{3.132}$$

The reason for omitting odd powers of the scaling dimension from the ansatz will become clear later when we discuss the $\mathbf{P}\mu$-system, where we will see that only the square of Δ enters the equations. We can now invert the relation and express Δ in terms of S at strong coupling, which exactly reproduces (3.131). There exists a one-to-one mapping between the coefficients α_i, β_i, etc. and A_i, B_i etc, which is rather complicated but easy to find. The pattern in (3.131) continues to higher orders in S with further coefficients D_i, E_i, etc. and powers of μ suppressed incrementally. This structure is a non-trivial constraint on Δ itself as one easily finds from (3.131) that

$$\begin{aligned} \Delta &= J + \frac{S}{2J}\left(A_1\sqrt{\mu} + A_2 + \frac{A_3}{\sqrt{\mu}} + \dots\right) \\ &+ S^2\left(-\frac{A_1^2}{8J^3}\mu - \frac{A_1A_2}{4J^3}\sqrt{\mu} + \left[\frac{B_1}{2J} - \frac{A_2^2 + 2A_1A_3}{8J^3}\right] + \left[\frac{B_2}{2J} - \frac{A_2A_3 + A_1A_4}{4J^3}\right]\frac{1}{\sqrt{\mu}} + \dots\right), \end{aligned} \tag{3.133}$$

where we introduced the mode number by the naive replacement $\lambda \to \mu \equiv n^2\lambda$. By definition the coefficients of S and S^2 are the slope and curvature functions respectively as defined in (3.63), so now we have their expansions at strong coupling in terms of A_i, B_i, C_i, etc. Since the S coefficient only contains the constants A_i, we can find all of their values by simply expanding the slope function (3.80) that we found earlier at strong coupling. We get

$$A_1 = 2\ , \quad A_2 = -1\ , \quad A_3 = J^2 - \frac{1}{4}\ , \quad A_4 = J^2 - \frac{1}{4} \dots\ . \tag{3.134}$$

Note that in this series the power of J increases by two at every other member, which is a direct consequence of omitting odd powers of Δ from (3.132). We also expect the same pattern to hold for the coefficients B_i, C_i, etc. It is not hard to see that the sub-leading coefficient in the spin S only contains A_i and B_i coefficients, thus given the A's we could in principle fix all B's. Hence we conclude that the letters in A_i, B_i, C_i etc. are directly linked to the generalizations of the slope function $\gamma^{(n)}$.

3.4.2 Two-loop prediction

Let us now consider the case $n=1$. We are interested in the coefficients of the strong coupling expansion of Δ, namely

$$\Delta = \Delta^{(0)}\lambda^{\frac{1}{4}} + \Delta^{(1)}\lambda^{-\frac{1}{4}} + \Delta^{(2)}\lambda^{-\frac{3}{4}} + \Delta^{(3)}\lambda^{-\frac{5}{4}} + \dots \tag{3.135}$$

First, we utilize the structure (3.131) and by fixing S and J we re-expand the square root of Δ^2 at strong coupling to find

$$\Delta = \sqrt{A_1 S}\,\sqrt[4]{\lambda} + \frac{\sqrt{A_1}\,(J^2 + A_2 S + B_1 S^2)}{2A_1\sqrt{S}}\,\frac{1}{\sqrt[4]{\lambda}} + \mathcal{O}\left(\frac{1}{\lambda^{\frac{3}{4}}}\right). \tag{3.136}$$

Thus we reformulate the problem entirely in terms of the coefficients A_i, B_i, C_i, etc. For example, the next coefficient in the series, namely the two-loop term is given by

$$\Delta^{(2)} = -\frac{\left(2A_2 + 4B_1 + J^2\right)^2 - 16A_1(A_3 + 2B_2 + 4C_1)}{16\sqrt{2}A_2^{3/2}}. \tag{3.137}$$

Further coefficients become more and more complicated, however a very clear pattern can be noticed after looking at these expressions: we see that the term $\Delta^{(n)}$ only contains coefficients with indices up to $n+1$, e.g. the tree level term $\Delta^{(0)}$ only depends on A_1, the one-loop term depends on A_1, A_2, B_1, etc. Thus we can associate the index of these coefficients with the loop level. Conversely, from the last section we learned that the letter of A_i, B_i, etc. can be associated with the order in S, i.e. the slope function fixed all A_i coefficients and the curvature function in principle fixes all B_i coefficients.

Looking at (3.136) we see that knowing A_i and B_i only takes us to one loop, in order to proceed we need to know some coefficients in the C_i and D_i series. This is where the knowledge of the classical energy (3.125) and its semi-classical correction (3.126) come in handy. We add up the classical and the 1-loop contributions, take S and J fixed at strong coupling and compare the result to (3.131). By requiring consistency we are able to extract the following coefficients for $n=1$,

$$\begin{aligned}
A_1 &= 2, & A_2 &= -1 \\
B_1 &= 3/2, & B_2 &= -3\,\zeta_3 + \tfrac{3}{8} \\
C_1 &= -3/8, & C_2 &= \tfrac{1}{16}(60\,\zeta_3 + 60\,\zeta_5 - 17) \\
D_1 &= 31/64, & D_2 &= \tfrac{1}{512}(-5520\,\zeta_3 - 5120\,\zeta_5 - 3640\,\zeta_7 + 901).
\end{aligned} \tag{3.138}$$

As discussed in the previous section, we can in principle extract all coefficients with indices 1 and 2. In order to find e.g. B_3 we would need to extend the quantization of

the classical solution to the next order. For general mode numbers n one can extract the following values for B_i,

$$B_1 = \frac{3}{2}\,, \quad B_2 = \begin{cases} -3\,\zeta_3 + \frac{3}{8} & , \ n=1 \\ -24\,\zeta_3 - \frac{13}{8} & , \ n=2 \\ -81\,\zeta_3 - \frac{24}{8} & , \ n=3 \end{cases}. \tag{3.139}$$

Combining all of this information we find the following result for spin 2 operators

$$\Delta_{2,J,1} = 2\,\lambda^{1/4} + \frac{\frac{J^2}{4}+1}{\lambda^{1/4}} + \frac{-\frac{J^4}{64} + \frac{3J^2}{8} - 3\,\zeta(3) - \frac{3}{4}}{\lambda^{3/4}} + \mathcal{O}\left(\frac{1}{\lambda^{5/4}}\right), \tag{3.140}$$

which for Konishi reads

$$\Delta_{2,2,1} = 2\,\lambda^{1/4} + \frac{2}{\lambda^{1/4}} + \frac{\frac{1}{2} - 3\zeta_3}{\lambda^{3/4}} + \mathcal{O}\left(\frac{1}{\lambda^{5/4}}\right). \tag{3.141}$$

Generalizing the discussion above we conclude that this procedure yields the n-loop scaling dimension given the values of

$$A_{1,2,\dots,n+1}, \quad B_{1,2,\dots,n}, \quad C_{1,2,\dots,n-1}, \quad \dots, \quad A_1^{(n+1)}, \tag{3.142}$$

thus e.g. to go to three loops we would need to know $A_{1,2,3,4}$, $B_{1,2,3}$, $C_{1,2}$, D_1, where the only unknown at this point is B_3. In order to find it we would either need to find the curvature function $\gamma^{(2)}$ or quantize the classical string to one more order.

3.4.3 Inconsistencies for higher mode numbers

The analysis in the previous sections was done only up to second order in the small spin expansion, namely using the one-loop semi-classical energy given by (3.127). Extending our results to higher orders in the spin we found perfect agreement with the conjectured structure (3.131) for mode number $n=1$, yet for cases with $n>1$ there are inconsistencies. For $n=2$ the first inconsistency appears in the $\frac{S^3}{J^4}\mu$ term and for $n=3$ there are already inconsistencies at order S^2. We found that for $n>1$ one has to modify the structure in (3.131) by including negative coefficients in order for it to be consistent with our one-loop results. E.g. for $n=2$ the structure has to be modified starting with the S^3 term, which now becomes

$$\left(C_{-2}\,\mu + \frac{C_1}{\sqrt{\mu}} + \frac{C_2}{\mu} + \dots\right) S^3 \tag{3.143}$$

with $C_{-2} = 12/J^4$. To the next order in S we find

$$\left(D_{-3}\,\mu^{3/2} + D_{-2}\,\sqrt{\mu} + \frac{D_0}{\sqrt{\mu}} + \frac{D_1}{\mu} + \dots\right) S^4 \tag{3.144}$$

where $D_{-1} = -\frac{78}{J^6}$, $D_{-2} = -\frac{36}{J^4}$, $D_0 = \frac{21}{2J^2}$. For $n = 3$ the first modification already occurs at order S^2 and it can be resolved if the term $-\frac{9S^2\sqrt{\mu}}{4J^3}$ is added to (3.131). Thus effectively the conjectured structure (3.131) has to be modified as in the $n = 2$ case by including negative coefficients, which now depend on n in a non-trivial way. It is also worth noticing that since inconsistencies start appearing at orders of $\frac{S^2}{J^2}$ and $\frac{S^4}{J^4}$ for $n = 3$ and $n = 2$ respectively, one might guess that there should be an inconsistency at order $\frac{S^4}{J^6}$ for $n = 1$, however we found no such thing.

We will revisit this topic in section 4.5.1 when we try to perform this procedure again after having found the curvature function. As one might expect, similar issues appear in that case as well.

Y-system
was before TBA

4 Exact results

The less effort, the faster and more powerful you will be.
– Bruce Lee

In this chapter we leave the perturbative regime (weak and strong coupling) behind and move on towards exact results in planar $\mathcal{N} = 4$ super Yang-Mills, where by "exact" we mean available at any value of the coupling constant. In principle such results are the ultimate goal of the whole AdS/CFT programme and being able to extract them is a remarkable achievement.

One can argue that the spectral problem has been worked out exactly, at least conceptually, as we will review shortly. However in practice one immediately runs into technical difficulties with finding solutions, thus at the moment only certain calculations have been carried out explicitly. In this section we will present a few examples of exact results, most notably the slope and curvature functions, which are the two leading coefficients in the small spin expansion of the anomalous dimension of the folded string. As mentioned before they are indeed exact in the coupling constant, yet they are somewhat abstract and one might say of little use. We will argue the opposite by demonstrating how one can use them in order to find new information about physically relevant quantities such as the Konishi anomalous dimension.

We will start the chapter by reviewing the exact solution to the spectral problem of AdS/CFT, first discussing historic approaches and quickly moving on to the novel quantum spectral curve approach. We then devote the rest of the chapter to various exact results, mostly achieved using the quantum spectral curve construction.

4.1 Solution to the spectral problem

Historically the first solution to the spectral problem that fully incorporated finite length corrections was the thermodynamic Bethe ansatz (TBA). Being a set of infinite integral equations it was obviously very hard to use in practice and was thus soon reformulated as an infinite set of functional relations, the so called Y-system. Both of these formulations can now be seen as intermediary steps towards the more elegant quantum spectral curve construction, which we cover in more depth in the next section.

Figure 5: A high level illustration of the idea behind the TBA method: by double Wick rotating the theory one can exchange time with length and thus finite volume scattering with the ground state scattering at finite length.

4.1.1 Thermodynamic Bethe ansatz

The technical details of the thermodynamic Bethe ansatz (TBA) approach are overwhelmingly difficult, in fact the lack of elegance and simplicity in this approach led many to believe that it might be only an intermediary step towards a more satisfactory solution. That is why we will only illustrate the basic ideas behind the TBA at a conceptual level, the full gory technical details can be found in numerous literature reviews (see [109] for an overview).

In the spin chain picture we identify operators with states of the chain and the dilatation operator with the Hamiltonian. This mapping enables one to use the physical language of magnons with momenta propagating on a closed chain with the time evolution being determined by the Hamiltonian. The key idea of TBA is shown in figure 5: consider this system with time and space interchanged, more precisely we analytically continue in time by introducing $y \equiv it$ and consider a theory where y plays the role of space and τ defined via $x \equiv i\tau$ is time. We call this double Wick rotated theory the *mirror* of the original. Obviously it is a completely different theory, for example the dispersion gets inverted by switching from $(E, p) \to i(\tilde{p}, \tilde{E})$, thus if we start from the asymptotic dispersion $E(p)$ for the $\mathfrak{sl}(2)$ sector we end up with

$$E(p) = \sqrt{1 + 4g^2 \sin^2 \frac{p}{2}} \quad \Longleftrightarrow \quad \tilde{E}(\tilde{p}) = 2 \operatorname{arcsinh}\left(\frac{1}{2g}\sqrt{\tilde{p}^2 + 1}\right). \tag{4.1}$$

The scattering matrix also has a different pole structure, meaning that the theory has different bound states from the original ones. But the remarkable thing is that mirroring a theory preserves integrability, meaning that one can solve it with an asymptotic Bethe ansatz. One can use this fact, since the partition functions for these theories satisfy

the identity

$$Z(L, R) = \tilde{Z}(R, L), \tag{4.2}$$

where L is the length scale of the original theory and R is the time scale. At asymptotic time scales the partition function is dominated by contributions from the ground state, this applies to any length scale of the system, thus in the asymptotic time limit

$$Z(L, R) = \operatorname{Tr} e^{-RH(L)} \xrightarrow[R \to \infty]{} e^{-RE_0(L)}. \tag{4.3}$$

This limit corresponds to the infinite length limit for the mirror model where the spectrum is controlled by the asymptotic Bethe ansatz. Here we have

$$\tilde{Z}(R, L) = \operatorname{Tr} e^{-L\tilde{H}(R)} \xrightarrow[L \to \infty]{} \sum_n e^{-L\tilde{E}_n(R)}, \tag{4.4}$$

where $\tilde{H}$ is the Hamiltonian of the mirror theory. In the large R limit we introduce the density of particles in momentum space $\rho(\tilde{p}) = \Delta n/(R\Delta\tilde{p})$ and the logarithm of the mirror asymptotic Bethe equations reads

$$\tilde{p}(u) + \int du' \left(-i \log S(u, u')\right) \rho(u') = \frac{2\pi n}{R}, \tag{4.5}$$

where we introduce a parametrization for the momentum $\tilde{p}(u)$, just as in (3.41). Particles with momentum $\tilde{p}$ that are not excited but nevertheless satisfy the Bethe equations are called holes and their density is denoted by $\bar{\rho}$. The relation between the two densities is given by

$$\partial_u \tilde{p} - 2\pi (\rho + \bar{\rho}) = -\int du' K(u, u')\rho(u') \equiv -K \star \rho, \tag{4.6}$$

where the convolution kernel is

$$K(u, u') = -i\partial_u \log S(u, u'). \tag{4.7}$$

Finally the partition function can be written as

$$Z(L, R) = \int \mathcal{D}\rho\, e^{-L\tilde{E}[\rho] + S[\rho, \bar{\rho}]}, \tag{4.8}$$

where $S[\rho, \bar{\rho}]$ is the entropy of a given particle density configuration arising due to the fact that the densities do not uniquely specify a particle configuration. This functional integral can be evaluated in the saddle point approximation. The saddle point is found by solving the non-linear integral equation, which is called a thermodynamic Bethe ansatz equation,

$$\epsilon(u) - L\tilde{E}(u) = -\log(1 + e^{-\epsilon}) \star K \tag{4.9}$$

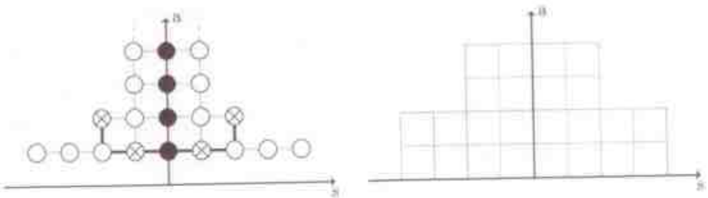


Figure 6: The domains of the $Y_{a,s}$ and $T_{a,s}$ functions (left and right respectively). The $Y_{a,s}$ functions are defined on the nodes, where the type of node signifies the type of excitation. The $T_{a,s}$ functions are defined on the latttice points of the grid.

in terms of the so-called pseudo-energy $\epsilon \equiv \log \bar{\rho}/\rho$. Once it is found the ground state energy is given by

$$E_0(L) = -\int \frac{du}{2\pi}(\partial_u \tilde{p}) \log\left(1 + e^{-\epsilon(u)}\right). \tag{4.10}$$

Excited states can also be reached in this way via analytic continuation in some parameter, for example the volume of the system [110]. Basically one can to introduce singularities in the integrand of (4.9) in such a way that the large L limit of the new solution coincides with the asymptotic Bethe ansatz solution. English?

The high level ideas outlined above have been successfully applied to the spectral problem in $\mathcal{N} = 4$ super Yang-Mills. The TBA equations schematically read [40, 41]

$$\log Y_{a,s}(u) = \delta_s^0 \, iL\, p_a(u) + \int dv\, K_{a,s}^{a',s'}(u,v) \log\left(1 + Y_{a',s'}(v)\right), \tag{4.11}$$

where the Y-functions are related to the pseudo-energies as $Y(u) = \exp(-\epsilon(u))$. A key difference from the example above is that here we can have different types of excitations which are labelled by the indices a, s. The hook diagram on the left of figure 6 indicates the ranges of values they can take on, here the type of node also distinguishes the type of particle. Finally the energy of the state is given by

$$E = \sum_j \epsilon_1(u_{4,j}) + \sum_{a=1}^{\infty} \int_{-\infty}^{\infty} \frac{du}{2\pi i} \frac{\partial \epsilon_a^*}{\partial u} \log\left(1 + Y_{a,0}^*(u)\right), \tag{4.12}$$

where the first term is basically the result of analytic continuation for excited states. It is highly non-trivial to actually find the correct analytic continuation that describes a specific excited state thus the applicability of this method is difficult and only a handful of operators can be studied, one particular example is the Konishi operator [44]. Another obvious difficulty is the fact that the TBA equations (4.11) are an infinite

set of coupled non-linear integral equations, which in practice limits their applicability to numeric calculations only.

4.1.2 Y/T/$\mathcal{Q}$-systems

The Y-functions introduced in (4.11) are technically defined on an infinite sheeted Riemann surface and have a complicated analytic structure, this is done so that one could rewrite the TBA equations as [40, 47]

$$\frac{Y_{a,s}^{+} Y_{a,s}^{-}}{Y_{a+1,s} Y_{a-1,s}} = \frac{(1 + Y_{a,s+1})(1 + Y_{a,s-1})}{(1 + Y_{a+1,s})(1 + Y_{a-1,s})}, \tag{4.13}$$

which are called the Y-system equations, here $Y_{a,s}^{\pm}(u) \equiv Y_{a,s}(u \pm i/2)$. The advantage of this form is that instead of an infinite set of integral equations we have a still infinite set of functional equations. Another notable feature of this system is that its form is universal for any excited state, which is now encoded in the analyticity properties of the Y-functions. It can also be cast into yet another form by introducing T-functions as

$$Y_{a,s} = \frac{T_{a,s+1} T_{a,s-1}}{T_{a+1,s} T_{a-1,s}}, \tag{4.14}$$

thus reducing the Y-system to the T-system

$$T_{a,s}^{+} T_{a,s}^{-} = T_{a+1,s} T_{a-1,s} + T_{a,s+1} T_{a,s-1}, \tag{4.15}$$

also known as the Hirota discrete bilinear equation, a well studied object in the context of classical integrability. The T-functions are defined on the lattice points of the T-hook shown on the right of figure 6. The advantage of this reformulation is that solutions of the T-system can be parametrized in terms of Wronskians built from eight independent $\mathcal{Q}$-functions, which have much simpler analytic properties than the Y-functions [111]. One can then build up a total of 2^8 $\mathcal{Q}$-functions denoted by $\mathcal{Q}_{A|J}(u)$ where $A, J \in \{1, 2, 3, 4\}$ are two ordered subsets of indices. These functions are defined through the QQ-relations

$$\mathcal{Q}_{A|I} \mathcal{Q}_{Aab|I} = \mathcal{Q}_{Aa|I}^{+} \mathcal{Q}_{Ab|I}^{-} - \mathcal{Q}_{Aa|I}^{-} \mathcal{Q}_{Ab|I}^{+}, \tag{4.16a}$$

$$\mathcal{Q}_{A|I} \mathcal{Q}_{A|Iij} = \mathcal{Q}_{A|Ii}^{+} \mathcal{Q}_{A|Ij}^{-} - \mathcal{Q}_{A|Ii}^{-} \mathcal{Q}_{A|Ij}^{+}, \tag{4.16b}$$

$$\mathcal{Q}_{Aa|I} \mathcal{Q}_{A|Ii} = \mathcal{Q}_{Aa|Ii}^{+} \mathcal{Q}_{A|I}^{-} - \mathcal{Q}_{A|I}^{+} \mathcal{Q}_{Aa|Ii}^{-}. \tag{4.16c}$$

In addition we impose the constraints $\mathcal{Q}_{\emptyset|\emptyset} = \mathcal{Q}_{1234|1234} = 1$ coming from normalization and the unimodularity of the superconformal group. A dual QQ system can then be introduced by $\mathcal{Q}^{\bar{A}|\bar{J}} \equiv (-1)^{|A||J|}\mathcal{Q}_{A|J}$ which satisfies the same QQ-relations. Here the bar over a subset means the subset complementary with respect to the full set $\{1,2,3,4\}$ and $|X|$ denotes the number of indexes in X. It was later shown that with a good choice of the eight basis $\mathcal{Q}$-functions it is possible to reduce the problem to a finite set of non-linear integral equations (FiNLIE) [48], which is better suited for numerical calculations.

4.2 Quantum spectral curve

The so far discussed gradually improving formulations of the solution to the spectral problem seem to be pointing to some ultimate simplification. In this section we discuss the quantum spectral curve (QSC) approach which many believe finally unveils the long anticipated beauty of the spectral problem.

4.2.1 Emergence from the $\mathcal{Q}$-system

The $\mathcal{Q}$-functions defined in the previous section have a disadvantage of having a rather complicated analytic structure, namely they are defined on infinite sheeted Riemann surfaces with square root type cuts parallel to the real axis between the branch points $\pm 2g + in$ with $n \in \mathbb{Z}$ going either through infinity (long cuts) or through the imaginary axis (short cuts). Apart from the branch cuts the $\mathcal{Q}$ functions do not have other singularities and are otherwise regular functions. However in order to fully define them one has to specify their analytic continuations through all of these cuts. Let us now introduce the notation

$$\mathbf{P}_a \equiv \mathcal{Q}_{a|\emptyset}, \quad \mathbf{P}^a \equiv \mathcal{Q}^{a|\emptyset}, \tag{4.17}$$

$$\mathbf{Q}_j \equiv \mathcal{Q}_{\emptyset|j}, \quad \mathbf{Q}^j \equiv \mathcal{Q}^{\emptyset|j}, \tag{4.18}$$

where $a, j = 1,2,3,4$. The nice thing about these is that $\mathbf{P}_i$ and $\mathbf{P}^i$ have only a single short cut $u \in [-2g, 2g]$ on their main sheet of the Riemann surface whereas $\mathbf{Q}_i$ and $\mathbf{Q}^i$ only have long cuts $u \in (-\infty, -2g] \cup [2g, \infty)$. Using the QQ-relations one can always use either the eight $\mathbf{P}$ or the eight $\mathbf{Q}$ functions to find all of the remaining ones. Take the $\mathbf{P}$ functions, we define their analytic continuation through the short cuts as

$$\tilde{\mathbf{P}}_a = \mu_{ab}(u)\,\mathbf{P}^b \ , \quad \tilde{\mathbf{P}}^a = \mu^{ab}(u)\,\mathbf{P}_b \tag{4.19}$$

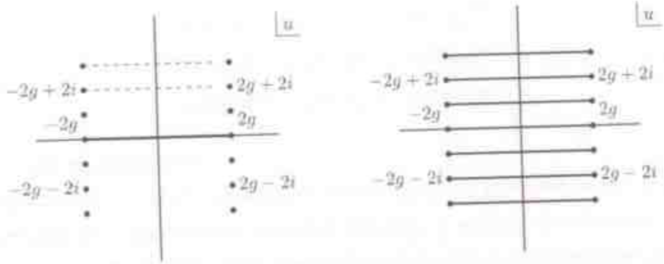


Figure 7: The location of branch cuts in u for the functions $\mathbf{P}_a(u)$ (left) and $\tilde{\mu}_{ab}(u)$ (right). The infinitely many cuts of $\tilde{\mathbf{P}}_a$ are shown on the left picture with dashed lines.

where $\mu_{ab}(u)$ is an antisymmetric matrix with components having infinitely many branch points at $u \in \pm 2g + i\mathbb{Z}$. The matrix itself has a unit Pfaffian which translates to the constraints on the components

$$\mu_{12}\,\mu_{34} - \mu_{13}\,\mu_{24} + \mu_{14}^2 \ = \ 1\,, \tag{4.20}$$

$$\mu_{14} = \mu_{23} \tag{4.21}$$

and the inverse matrix is defined by $\mu^{ab} = -\frac{1}{2}\epsilon^{abcd}\mu_{cd}$. These functions can also be shown to be periodic

$$\tilde{\mu}_{ab}(u+i) = \tilde{\mu}_{ab}(u), \tag{4.22}$$

where we distinguish between the short/long cut version of the same function by adding a hat/check over the symbol. For the short cut versions $\mu_{ab}(u) \equiv \hat{\mu}_{ab}(u)$ of the functions this condition reads

$$\tilde{\mu}_{ab}(u) = \mu_{ab}(u+i). \tag{4.23}$$

The magic of the whole quantum spectral curve construction is that the analytic continuations of the monodromies μ_{ab} through the cuts are again given by the same functions [50]

$$\tilde{\mu}_{ab} - \mu_{ab} = \mathbf{P}_a\tilde{\mathbf{P}}_b - \mathbf{P}_b\tilde{\mathbf{P}}_a \tag{4.24}$$

and the $\mathbf{P}\mu$-system of eight $\mathbf{P}$ and five μ functions closes! The cut structure is shown in figure 7. An analogous closure can be implemented for the $\mathbf{Q}$ functions by denoting the analytic continuations

$$\tilde{\mathbf{Q}}_i = \omega_{ij}\mathbf{Q}^j, \tag{4.25}$$

where the monodromy ω is related to μ by

$$\omega_{ij} = \tilde{Q}^-_{a|i}\tilde{Q}^-_{b|j}\,\mu^{ab}. \tag{4.26}$$

and itself has the following analytic continuation

$$\tilde{\omega}_{ij} - \omega_{ij} = \mathbf{Q}_i \tilde{\mathbf{Q}}_j - \mathbf{Q}_j \tilde{\mathbf{Q}}_i \tag{4.27}$$

thus closing the $\mathbf{Q}\omega$-system. Both the $\mathbf{P}\mu$ and the $\mathbf{Q}\omega$-systems are complete in the sense they describe all solutions in the theory. Since they are related it is a matter of convenience to use one or the other and for the remainder of the thesis we will stick with the $\mathbf{P}\mu$-system.

4.2.2 Asymptotics

Tracing back from the Q-system to the Y-system one can find the following relation

$$Y_{11}Y_{22} = 1 + \frac{\mathbf{P}_1\tilde{\mathbf{P}}_2 - \mathbf{P}_2\tilde{\mathbf{P}}_1}{\mu_{12}} = \frac{\mu_{12}(u+i)}{\mu_{12}(u)}, \tag{4.28}$$

and from the T-system one can find

$$T_{1,s}(u) = \mathbf{P}_1\left(u + \frac{is}{2}\right)\mathbf{P}_2\left(u - \frac{is}{2}\right) - \mathbf{P}_2\left(u + \frac{is}{2}\right)\mathbf{P}_1\left(u - \frac{is}{2}\right), \tag{4.29}$$

thus given a solution to the $\mathbf{P}\mu$-system one can fully reconstruct all of the T-functions using the T-system equations (4.15) and then reconstruct the Y functions using (4.14). The Y-functions encode the global charges in their asymptotics, for example in the $\mathfrak{sl}(2)$ sector one has [48]

$$\log Y_{11}Y_{22} \simeq i\frac{\Delta - J}{u}, \tag{4.30}$$

implying through (4.28) that the asymptotics of the $\mathbf{P}$ and μ functions also encode the global charges of the state/operator being described.

The picture described so far strongly resembles the classical spectral curve construction outlined in section 3.3.1. Quite expectedly the classical spectral curve turns out to be the strong coupling limit of the quantum spectral curve as implied by the naming [50]. More precisely one can think of the classical spectral curve as an WKB approximation to the quantum case, namely the quasi-momenta are related to the $\mathbf{P}$ and $\mathbf{Q}$ functions as

$$\mathbf{P}_i \simeq e^{i\int^u \tilde{p}_i(u)\,du}, \quad \mathbf{Q}_i \simeq e^{i\int^u \hat{p}_i(u)\,du} \tag{4.31}$$

where the four $\mathbf{P}_i$ functions correspond to the S^5 quasi-momenta and the four $\mathbf{Q}_i$ functions to the AdS_5 part. The emergence of the cuts can be seen by recalling that the quasi-momenta are the eigenvalues of the monodromy matrix (3.82), which develops the cuts after changing variables from x to u via the Zhukovsky map

$$x + \frac{1}{x} = \frac{u}{g}. \tag{4.32}$$

which is the classical analogue of the rapidity map (3.60) from the asymptotic Bethe ansatz. Similarly how the classical quasi-momenta encoded the global charges in their asymptotics as shown in (3.91), so do their quantum analogues $\mathbf{P}$ [50]

$$\begin{pmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \\ \mathbf{P}_4 \end{pmatrix} \simeq \begin{pmatrix} A_1\, u^{\frac{-J_1-J_2+J_3-2}{2}} \\ A_2\, u^{\frac{-J_1+J_2-J_3}{2}} \\ A_3\, u^{\frac{+J_1-J_2-J_3-2}{2}} \\ A_4\, u^{\frac{+J_1+J_2+J_3}{2}} \end{pmatrix}, \quad \begin{pmatrix} \mathbf{P}^1 \\ \mathbf{P}^2 \\ \mathbf{P}^3 \\ \mathbf{P}^4 \end{pmatrix} \simeq \begin{pmatrix} A^1\, u^{\frac{+J_1+J_2-J_3}{2}} \\ A^2\, u^{\frac{+J_1-J_2+J_3-2}{2}} \\ A^3\, u^{\frac{-J_1+J_2+J_3}{2}} \\ A^4\, u^{\frac{-J_1-J_2-J_3-2}{2}} \end{pmatrix} \tag{4.33}$$

and similarly for $\mathbf{Q}$

$$\begin{pmatrix} \mathbf{Q}_1 \\ \mathbf{Q}_2 \\ \mathbf{Q}_3 \\ \mathbf{Q}_4 \end{pmatrix} \simeq \begin{pmatrix} B_1\, u^{\frac{+\Delta-S_1-S_2}{2}} \\ B_2\, u^{\frac{+\Delta+S_1+S_2-2}{2}} \\ B_3\, u^{\frac{-\Delta-S_1+S_2}{2}} \\ B_4\, u^{\frac{-\Delta+S_1-S_2-2}{2}} \end{pmatrix}, \quad \begin{pmatrix} \mathbf{Q}^1 \\ \mathbf{Q}^2 \\ \mathbf{Q}^3 \\ \mathbf{Q}^4 \end{pmatrix} \simeq \begin{pmatrix} B^1\, u^{\frac{-\Delta+S_1+S_2-2}{2}} \\ B^2\, u^{\frac{-\Delta-S_1-S_2}{2}} \\ B^3\, u^{\frac{+\Delta+S_1-S_2-2}{2}} \\ B^4\, u^{\frac{+\Delta-S_1+S_2}{2}} \end{pmatrix}. \tag{4.34}$$

Since the $\mathbf{P}\mu$ and $\mathbf{Q}\omega$-systems are coupled via the QQ-relations the above asymptotics have to be compatible thus producing algebraic relations between the coefficients A and the global charges. We will be using simplified relations for the solutions we will consider later, the most general case can be found in [50].

4.3 Revisiting the slope function

In section 3.2.1 we derived the slope function which is the leading small spin expansion coefficient of the anomalous dimension for the generalized Konishi operator in the $\mathfrak{sl}(2)$ sector. We used the asymptotic Bethe ansatz as the starting point of the derivation which is justified by the fact that finite size effects are irrelevant for the slope function. In this section we will derive the slope function (3.80) using the $\mathbf{P}\mu$-system, which is not only more concise but also generalizes to higher orders as we shall demonstrate in the next section when we derive the next coefficient in the small spin expansion, the curvature function.

4.3.1 $\mathbf{P}\mu$-system for the $\mathfrak{sl}(2)$ sector

The first simplification we can employ is the fact that solutions in the $\mathfrak{sl}(2)$ sector are symmetric under the left-right exchange of the Y-functions $Y_{a,s} = Y_{a,-s}$, which implies the following relations for the $\mathbf{P}$ and $\mathbf{Q}$ functions

$$\mathbf{P}^a = \chi^{ac}\mathbf{P}_c, \qquad \mathbf{Q}^i = \chi^{ij}\mathbf{Q}_j \tag{4.35}$$

and thus the analytic continuation for the $\mathbf{P}\mu$-system reads

$$\tilde{\mathbf{P}}_a = -\mu_{ab}\chi^{bc}\mathbf{P}_c, \quad \text{with} \quad \chi^{ab} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{4.36}$$

and

$$\tilde{\mu}_{ab} - \mu_{ab} = \mathbf{P}_a\tilde{\mathbf{P}}_b - \mathbf{P}_b\tilde{\mathbf{P}}_a\,. \tag{4.37}$$

Explicitly the $\mathbf{P}\mu$-system equations are now given by

$$\tilde{\mathbf{P}}_1 = -\mathbf{P}_3\mu_{12} + \mathbf{P}_2\mu_{13} - \mathbf{P}_1\mu_{14} \tag{4.38}$$

$$\tilde{\mathbf{P}}_2 = -\mathbf{P}_4\mu_{12} \qquad + \mathbf{P}_2\mu_{14} - \mathbf{P}_1\mu_{24} \tag{4.39}$$

$$\tilde{\mathbf{P}}_3 = \qquad -\mathbf{P}_4\mu_{13} + \mathbf{P}_3\mu_{14} \qquad - \mathbf{P}_1\mu_{34} \tag{4.40}$$

$$\tilde{\mathbf{P}}_4 = \qquad -\mathbf{P}_4\mu_{14} + \mathbf{P}_3\mu_{24} - \mathbf{P}_2\mu_{34}\,. \tag{4.41}$$

The above equations ensure that the branch points of $\mathbf{P}_a$ and μ_{ab} are of the square root type, i.e. $\tilde{\tilde{\mathbf{P}}}_a = \mathbf{P}_a$ and $\tilde{\tilde{\mu}}_{ab} = \mu_{ab}$. Finally, we require that $\mathbf{P}_a$ and μ_{ab} do not have any singularities except these branch points.

The large u asymptotics for the $\mathbf{P}_a$ functions are given by (4.33) and can be uniquely fixed for the μ_{ab} functions using the $\mathbf{P}\mu$-system (4.36)-(4.37). For the $\mathfrak{sl}(2)$ sector they are given by [49]

$$\mathbf{P}_a \sim \left(A_1 u^{-J/2}, A_2 u^{-J/2-1}, A_3 u^{J/2}, A_4 u^{J/2-1}\right) \tag{4.42}$$

$$(\mu_{12}, \mu_{13}, \mu_{14}, \mu_{24}, \mu_{34}) \sim \left(u^{\Delta-J}, u^{\Delta+1}, u^{\Delta}, u^{\Delta-1}, u^{\Delta+J}\right) \tag{4.43}$$

where J is the twist of the gauge theory operator, and Δ is its conformal dimension. With these asymptotics, the equations (4.36)-(4.37) form a closed system which fixes $\mathbf{P}_a$ and μ_{ab}. Lastly, the spin S of the operator is related [49] to the leading coefficients A_a of the $\mathbf{P}_a$ functions

$$A_1A_4 = \frac{\left((J+S-2)^2-\Delta^2\right)\left((J-S)^2-\Delta^2\right)}{16iJ(J-1)} \tag{4.44}$$

$$A_2A_3 = \frac{\left((J-S+2)^2-\Delta^2\right)\left((J+S)^2-\Delta^2\right)}{16iJ(J+1)} \tag{4.45}$$

mention that this is related to the asymptotics of Q_i's

which as discussed in the last section comes from the compatibility condition of the asymptotics for the $\mathbf{P}\mu$ and $\mathbf{Q}\omega$-systems.

Additionally the simplified $\mathbf{P}\mu$-system enjoys a symmetry preserving all of its essential features. It has the form of a linear transformation of $\mathbf{P}_a$ and μ_{ab} which leaves the system (4.36)-(4.37) and the asymptotics (4.42), (4.43) invariant. Indeed, consider a general linear transformation $\mathbf{P}'_a = R_a{}^b\mathbf{P}_b$ with a non-degenerate constant matrix R. In order to preserve the $\mathbf{P}\mu$-system equations, μ should at the same time transform as

$$\mu' = -R\mu\chi R^{-1}\chi. \tag{4.46}$$

here is small problem

Such a transformation also preserves the form of (4.37) if

$$R^T\chi R\chi = -1\,, \tag{4.47}$$

notations changed

which also automatically ensures antisymmetry of μ_{ab} and the constraints (4.20), (4.21). However in general this transformation will spoil the asymptotics of $\mathbf{P}_a$. These asymptotics are ordered as $|\mathbf{P}_2| < |\mathbf{P}_1| < |\mathbf{P}_4| < |\mathbf{P}_3|$, which implies that the matrix R must have the following structure

$P_1 \leftrightarrow P_2 \quad P_3 \leftrightarrow P_4$

wrt. 4.33

$$R = \begin{pmatrix} * & * & 0 & 0 \\ 0 & * & 0 & 0 \\ * & * & * & * \\ * & * & 0 & * \end{pmatrix}. \tag{4.48}$$

This matrix would of course be lower triangular if we ordered $\mathbf{P}_a$ by their asymptotics. The general form of R which satisfies (4.47) and does not spoil the asymptotics generates a 6-parametric transformation, which we will call a γ-transformation. The simplest γ-transformation is the following rescaling:

$$\mathbf{P}_1 \to \alpha\,\mathbf{P}_1, \quad \mathbf{P}_2 \to \beta\,\mathbf{P}_2, \quad \mathbf{P}_3 \to 1/\beta\,\mathbf{P}_3, \quad \mathbf{P}_4 \to 1/\alpha\,\mathbf{P}_4. \tag{4.49}$$

$$\mu_{12} \to \alpha\beta\,\mu_{12}, \quad \mu_{13} \to \frac{\alpha}{\beta}\mu_{13}, \quad \mu_{14} \to \mu_{14}\,, \quad \mu_{24} \to \frac{\beta}{\alpha}\mu_{24}, \quad \mu_{34} \to \frac{1}{\alpha\beta}\mu_{34}. \tag{4.50}$$

with α, β being constants. In all the solutions that we consider all functions $\mathbf{P}_a$ turn out to be functions of definite parity, so it makes sense to consider γ-transformations which preserve parity. $\mathbf{P}_1$ and $\mathbf{P}_2$ always have opposite parity as one can see from from (4.42) and thus should not mix under such transformations, the same is true about $\mathbf{P}_3$ and $\mathbf{P}_4$. Thus depending on parity of J the parity-preserving γ-transformations are where we used the resolvent (4.225). Using the relation (4.224) between the resolvent and the quasi-momentum we find

$$G(0) = \frac{g}{L}\left(p''(0)/4 - \theta\right), \tag{4.243}$$

so the final expression for the cusp anomalous dimension in terms of the quasi-momentum is given by

$$\Gamma_L(g) = -\frac{g^2}{2}\partial_L p''_L(0), \tag{4.244}$$

where $p(x)$ is given in terms of the parameters of the branch points r and ψ in (4.235). They are implicitly defined through L/g and θ by the equations (4.238) and (4.239). In order to get Γ_L we express ∂_L though ∂_r and ∂_ψ and then apply (4.244) to (4.235). Finally we obtain a very simple result in terms of r and ψ

$$\Gamma_L(g) = g(\phi - \theta)\,(r - 1/r)\cos\psi. \tag{4.245}$$

We can now check our formula (4.245) in the limit $\phi = 0$ and $\theta \to 0$ considered in section E.2 of [46]. As the angles go to zero, the branch points approach the unit circle: $r \to 1$, thus the formula (4.245) gives

$$\Gamma_L(g) = 2\,g\,\theta(r-1)\cos\psi. \tag{4.246}$$

In this limit $r - 1 \propto \theta$, and the coefficient of proportionality can be found by expanding the equation (4.239) for θ around $r = 1$,

$$2(1-r)\frac{\mathbb{E}\left(\sin^2\psi\right)}{\cos\psi} = \theta/2. \tag{4.247}$$

Plugging it into the formula above we get

$$\Gamma_L(g) = g\,\theta^2\,\frac{\cos^2\psi}{2\mathbb{E}\left(\sin^2\psi\right)} \tag{4.248}$$

which perfectly agrees with (190) of [46]. We should note that the equation (4.239) is written in the approximation $\phi \approx \theta$ and now on the top of it we want to take a limit $\theta \to 0$. Since before we have neglected the terms $\mathcal{O}(\theta - \phi)^2$, the result, which is now of the order $\mathcal{O}(\theta)^2$ will not generally be reproduced. However, we found that here and in several other formulas correct small angle limit is reproduced if before taking θ, ϕ to zero we replace θ and ϕ by the middle angle $\phi_0 = (\phi + \theta)/2$, which is in our case equal to $\theta/2$.

4.6.4 Matching the string solution

As we have mentioned before, in the classical $L \sim \sqrt{\lambda} \to \infty$ limit $\Gamma_L(\lambda)$ can be matched with the energy of an open string. The class of string solutions we are interested in was introduced in [45] and generalized in [46]. It is a string in $AdS_3 \times S^3$ governed by the parameters θ, ϕ, AdS_3 charge E and S^3 charge L, furthermore the four parameters are restricted by the Virasoro constraint. The ansatz for the embedding coordinates of AdS^3 and S^3 is

$$y_1 + iy_2 = e^{i\kappa\tau}\sqrt{1 + r^2(\sigma)},\;\; y_3 + iy_4 = r(\sigma)e^{i\phi(\sigma)}, \tag{4.249}$$

$$x_1 + ix_2 = e^{i\gamma\tau}\sqrt{1 + \rho^2(\sigma)},\;\; x_3 + ix_4 = r(\sigma)e^{if(\sigma)}. \tag{4.250}$$

The range of the worldsheet coordinate is $-s/2 < \sigma < s/2$, where s is to be found dynamically. The angles θ and ϕ parametrizing the cusp enter the string solution through the boundary conditions $\phi(\pm s/2) = \pm(\pi - \phi)/2$ and $f(\pm s/2) = \pm\theta/2$. The equations of motion and Virasoro constraints lead to the following system of equations (see Appendix E of [46] for more details, also [148]):

$$f(\gamma, l_\theta) = f(\kappa, l_\phi), \tag{4.251}$$

$$h(\gamma, l_\theta) = \theta,\;\; h(\kappa, l_\phi) = \phi, \tag{4.252}$$

$$g(\gamma, l_\theta) = L,\;\; g(\kappa, l_\phi) = E, \tag{4.253}$$

where

$$f(\gamma, l) = \frac{2\sqrt{2}}{\sqrt{\gamma^2 + k^2 + 1}}\,\mathbb{K}\left(\frac{-k^2 + \gamma^2 + 1}{k^2 + \gamma^2 + 1}\right), \tag{4.254}$$

$$h(\gamma, l) = \frac{2l}{k(1 + k^2 - \gamma^2)}\left[(1 + \gamma^2 + k^2)\,\Pi\left(\frac{k^2 - 2l^2 - \gamma^2 + 1}{2k^2}\middle|\frac{k^2 - \gamma^2 - 1}{2k^2}\right) - 2\gamma^2\,\mathbb{K}\left(\frac{k^2 - \gamma^2 - 1}{2k^2}\right)\right], \tag{4.255}$$

$$g(\gamma, l) = -2\sqrt{2}\,\frac{\sqrt{\gamma^2 + k^2 + 1}}{\gamma}\left[\mathbb{E}\left(\frac{-k^2 + \gamma^2 + 1}{k^2 + \gamma^2 + 1}\right) - \mathbb{K}\left(\frac{-k^2 + \gamma^2 + 1}{k^2 + \gamma^2 + 1}\right)\right], \tag{4.256}$$

$$k^4 = \gamma^4 - 2\gamma^2 + 4\gamma^2 l^2 + 1.$$

One can see that the variables θ, l_θ, γ and L are responsible for the S^3 part of the solution, while ϕ, l_ϕ, κ and E are their analogues for AdS_3. The two parts of the solution are connected only by the Virasoro condition which leads to (4.251). We are interested in the limit when $\theta \approx \phi$. In this limit the two groups of variables responsible for S^3 and AdS_3 parts of the solution become close to each other, namely $l_\theta \approx l_\phi$ and

$E \approx L$. The cusp anomalous dimension should be compared with the difference $E - L$, because L is the classical part of the dimension of the observable W_L. To find $E - L$ we linearise the system (4.254),(4.255),(4.256) around $\phi \approx \theta$, which yields

$$E - L = (\phi - \theta) \left| \frac{\partial(g, f)}{\partial(l, \kappa)} \right| / \left| \frac{\partial(h, f)}{\partial(l, \kappa)} \right| . \tag{4.257}$$

Plugging in here the explicit form of g, f and h one gets as a result an extremely complicated expression with a lot of elliptic functions. However, there exists a parametrization in which the result looks surprisingly simple, it comes from comparison of the string conserved charges with the corresponding quantities of the algebraic curve. One can notice that the equations for θ and L/g in the end of the last subsection have the same structure as the equations (4.252) and (4.253). Indeed, it is possible to match them precisely if one chooses the correct identification of parameters of the string solution l_θ, γ with the parameters of the algebraic curve r, ψ. We used various elliptic function identities to bring the equations to identical form after the following identifications

$$\gamma = \frac{2r}{\sqrt{r^4 - 2r^2 \cos 2\psi + 1}}, \; l_\theta = \frac{(r^2 - 1)\cos\psi}{\sqrt{r^4 - 2r^2 \cos 2\psi + 1}}. \tag{4.258}$$

As another confirmation of correctness of this identification, after plugging it into (4.257) the complicated expression reduces to the following simple formula for the classical energy

$$E - L = g(\phi - \theta)(r - 1/r)\cos\psi, \tag{4.259}$$

which exactly coincides with the matrix model result (4.245). Notice that this can be rewritten as a sum over the branch points of the algebraic curve

$$E - L = \frac{g}{2}(\phi - \theta) \sum_i a_i, \tag{4.260}$$

where $a_i = \{r\, e^{i\psi}, r\, e^{-i\psi}, -1/r\, e^{i\psi}, -1/r\, e^{-i\psi}\}$.

5 Developments in ABJM

Insanity: doing the same thing over and over again and expecting different results.

a Bit too tought — Albert Einstein

In this last chapter of the thesis we leave $\mathcal{N} = 4$ super Yang-Mills behind and switch to the so-called ABJM theory, originally proposed by (and thus named after) Aharony, Bergman, Jafferis and Maldacena in [57] following [58, 150, 151, 59, 60]. This theory also has a string theory dual and exhibits integrability, as we shall review shortly. Many of these features are very similar to the ones we presented in the case of $\mathcal{N} = 4$ super Yang-Mills. In order to distinguish the two models we will often refer to ABJM and its string dual as AdS_4/CFT_3 and to the $\mathcal{N} = 4$ SYM and its dual as AdS_5/CFT_4. We will keep this chapter short, going into more detail only when discussing the classical spectral curve.

5.1 Short introduction

The ABJM theory is a three-dimensional superconformal Chern-Simons gauge theory with $\mathcal{N} = 6$ supersymmetry. The gauge group is $U(N) \times \hat{U}(N)$ at levels $\pm k$, where A_μ and $\hat{A}_\mu$ are the associated gauge fields. The field content consists of four complex scalars ϕ^I, four Dirac fermions ψ_I, their adjoints $\phi_I = \phi^{I\dagger}$, $\psi^I = \psi_I^\dagger$, and two gauge fields, A_μ and $\hat{A}_\mu$. The index I is an $SU(4)$ R-symmetry index. The fields ϕ^I and ψ_I transform in the $(\bar{N}, N)$ representation of the gauge group and their adjoints transform in the $(N, \bar{N})$ representation. The global symmetry group is the orhthosymplectic supergroup $OSp(6|4)$ whose bosonic part is the direct product of the R-symmetry group $SO(6) \simeq SU(4)$ and the conformal group $SO(2,3)$. As before, we are interested in gauge invariant single trace operators. The Chern-Simons level k acts like the coupling constant with large k corresponding to weak coupling. The planar limit is defined by $k, N \to \infty$ keeping $\lambda \equiv N/k$ fixed.

This theory was conjectured [57] to be dual to M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$ with four-form flux $F^{(4)} \sim N$ through AdS_4. Alternatively one can say that it is the effective theory for a stack of N M2 branes moving on a $\mathbb{C}^4/\mathbb{Z}_k$ orbifold point. In the limit $k^5 \gg N$ M-theory is well approximated by weakly coupled type IIA string theory on $AdS_4 \times \mathbb{CP}^3$. Here one also has RR four-form flux $F^{(4)} \sim N$ through AdS_4 and RR where K_n is a particular grouping of heavy and light modes

$$K_n = \begin{cases} \omega_n^{\text{heavy}} + \omega_{n/2}^{\text{light}} & n \in 2\mathbb{Z} \\ \omega_n^{\text{heavy}} & n \notin 2\mathbb{Z}. \end{cases} \tag{5.46}$$

with

$$\omega_n^{\text{heavy}} = \omega_n^{(AdS,1)} + \omega_n^{(AdS,2)} + \omega_n^{(AdS,3)} + \omega_n^{(\mathrm{CP},1)} - 2\,\omega_n^{(F,1)} - 2\,\omega_n^{(F,2)}, \tag{5.47}$$
$$\omega_n^{\text{light}} = 4\,\omega_n^{(\mathrm{CP},2)} - 2\,\omega_n^{(F,3)} - 2\,\omega_n^{(F,4)}. \tag{5.48}$$

Notice that we exploit the $x \to -x$ symmetry of the classical algebraic curve as well as triviality of zero mode corrections. The short string expansion of K_n takes the form

$$K_p = (-1)^p\,\mathcal{C} + \widetilde{K}_p \tag{5.49}$$

where $\mathcal{C}$, given in (5.41), is independent on p and the sum of $\widetilde{K}_p$ (which start at $\mathcal{O}(\mathcal{S})$) is convergent. The alternating constant $\mathcal{C}$ poses some problems because we have to give a meaning to

$$-\mathcal{C} + \mathcal{C} - \mathcal{C} + \mathcal{C} + \cdots. \tag{5.50}$$

An analysis of the integral representation shows that it automatically selects the choice

$$-\mathcal{C} + \mathcal{C} - \mathcal{C} + \mathcal{C} + \cdots \equiv -\frac{1}{2}\mathcal{C} \tag{5.51}$$

Later, we shall provide various consistency checks of this prescription. In particular, we shall see that it is necessary in order to match the asymptotic Bethe Ansatz equations when wrapping effects are subtracted. Notice also that the expansion of $\mathcal{C}$ at fixed $\mathcal{J}$ is

$$\mathcal{C} \simeq \frac{\mathcal{S}}{\mathcal{J}^2\sqrt{\mathcal{J}^2+1}} - \frac{(3\mathcal{J}^4 + 11\mathcal{J}^2 + 6)\,\mathcal{S}^2}{4\mathcal{J}^4\,(\mathcal{J}^2+1)^2} + \frac{12\,\mathcal{J}^8 + 75\,\mathcal{J}^6 + 173\,\mathcal{J}^4 + 140\,\mathcal{J}^2 + 40}{16\,\mathcal{J}^6\,(\mathcal{J}^2+1)^{7/2}}\,\mathcal{S}^3, \tag{5.52}$$

so, upon expanding at small $\mathcal{J}$, it provides precisely the terms with even/odd $\mathcal{J}$ exponents in the coefficients of the odd/even powers of $\mathcal{S}$ in (5.37).

Apart from the $\mathcal{C}$ term, the integral representation implements the Gromov-Mikhailov (GM) prescription. The reason is that the singularities at $|x| = 1$ are avoided by implicitly encircling them by a small circumference. This cut-off on $|x-1|$ translates in a bound on the highest mode n that correlates heavy/light polarizations according to GM. In other words the highest mode for light polarizations is asymptotically half the highest mode for heavy polarizations.

As a numerical check of the agreement between the integral representation and the series expansion, we fix $\rho = 1$ in table 2 and show the value of E_1 from our analytical resummation and result from the integral. The agreement is very good already at moderately small $\mathcal{S}$.

$\mathcal{S}$	E_1 from (5.38)	E_1
1/10	-0.18790	-0.17987
1/50	-0.19461	-0.19443
1/100	-0.19934	-0.19930
1/300	-0.20449	-0.20448
1/500	-0.206075	-0.206075

Table 2: Comparison between resummation at fixed ratio $\rho = 1$ and integral representation. The asymptotic value for $\mathcal{S} \to 0$ is $(\sqrt{3}-3)/6 \simeq -0.211$, but already at $\mathcal{S} = 1/500$ we have 6 digit agreement.

5.3.3 The slope function

The one-loop correction E_1 tends to zero linearly with $\mathcal{S}$ when $\mathcal{S} \to 0$ at fixed $\mathcal{J}$. The slope ratio

$$\sigma(\mathcal{J}) = \lim_{\mathcal{S}\to 0} \frac{E_1(\mathcal{S},\mathcal{J})}{\mathcal{S}}, \tag{5.53}$$

is the analogue of the slope function we encountered in $AdS_5 \times S^5$. It is known that it does not receive dressing corrections both in $AdS_5 \times S^5$ and in $AdS_4 \times \mathrm{CP}^3$ since such contributions start at order $\mathcal{S}^2$ [82]. It also does not receive wrapping corrections in AdS_5. Instead, in the case of AdS_4 the slope has a non vanishing wrapping contribution. For instance, a rough evaluation at $\mathcal{J} = 1$ gives a definitely non zero value around -0.042. Indeed, an analytical calculation shows that the wrapping contribution to the slope in $AdS_4 \times \mathrm{CP}^3$ is exactly

$$\sigma^{\text{wrap}}(\mathcal{J}) = \sum_{n=-\infty}^{\infty} \sigma_n = -\frac{1}{2\,\mathcal{J}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\sqrt{\mathcal{J}^4 + (n^2+1)\,\mathcal{J}^2 + n^2}}. \tag{5.54}$$

This formula is in perfect agreement with numerics since for instance

$$\sigma^{\text{wrap}}(\mathcal{J} = 1) = -0.041777654879558824814\ldots. \tag{5.55}$$

The large $\mathcal{J}$ limit of this expression is exponentially suppressed as it should

$$\sigma^{\text{wrap}}(\mathcal{J}) = -\frac{\sqrt{2}}{\mathcal{J}^{5/2}}\,e^{-\pi\mathcal{J}} + \cdots \tag{5.56}$$

To analyze the small $\mathcal{J}$ limit it is convenient to split this contribution into the $n = 0$ term plus the rest. The result is very intriguing. For the $n = 0$ term, we find

$$\sigma^{\text{wrap}}_{n=0} = -\frac{1}{2\,\mathcal{J}^2\sqrt{\mathcal{J}^2+1}} = -\frac{1}{2\mathcal{J}^2} + \frac{1}{4} - \frac{3\mathcal{J}^2}{16} + \frac{5\mathcal{J}^4}{32} - \frac{35\mathcal{J}^6}{256} + \frac{63\mathcal{J}^8}{512} + \cdots. \quad (5.57)$$

This is precisely the set of terms even under $\mathcal{J} \to -\mathcal{J}$ in the full slope which is the first term of (5.37). Similarly, we can consider the rest of σ^{wrap} and expand at small $\mathcal{J}$. We find

$$\begin{aligned}\sum_{n\neq 0}\sigma^{\text{wrap}}_n \;=\;& \frac{\log(2)}{\mathcal{J}} + \mathcal{J}\left(-\frac{3\zeta(3)}{8} - \frac{\log(2)}{2}\right) + \mathcal{J}^3\left(\frac{3\zeta(3)}{16} + \frac{45\zeta(5)}{128} + \frac{3\log(2)}{8}\right) + \\ & + \mathcal{J}^5\left(-\frac{9\zeta(3)}{64} - \frac{45\zeta(5)}{256} - \frac{315\zeta(7)}{1024} - \frac{5\log(2)}{16}\right) + O\left(\mathcal{J}^6\right). \quad (5.58)\end{aligned}$$

Comparing again with (5.37), we see that we are reproducing all the irrational terms of the slope, involving zeta functions or $\log(2)$. The remaining terms are the same as in $AdS_5 \times S^5$

$$\sigma(\mathcal{J}) - \sigma^{\text{wrap}}(\mathcal{J}) = -\frac{1}{2\,\mathcal{J}} + \frac{\mathcal{J}}{2} - \frac{\mathcal{J}^3}{2} + \cdots. \quad (5.59)$$

This is due to the fact that the BAE are essentially the same as for $\mathfrak{sl}(2)$ sector in $AdS_5 \times S^5$. This is however a nontrivial test that all is done correctly. Thus, we are led to the following expression for the one-loop full slope

$$\sigma(\mathcal{J}) = -\frac{1}{2\,\mathcal{J}}\left[\frac{1}{\mathcal{J}^2+1} + \sum_{n=-\infty}^{\infty}\frac{(-1)^n}{\sqrt{\mathcal{J}^4 + (n^2+1)\,\mathcal{J}^2 + n^2}}\right]. \quad (5.60)$$

The above analysis of the slope is a confirmation that the various terms in (5.37) are organized in the expected way. The asymptotic contribution is precisely the same as in $AdS_5 \times S^5$, while wrapping is different and is exponentially suppressed for large operators. This is a property of the integral representation and a confirm that the prescription (5.51) is correct.

5.3.4 Slope at weak coupling

It is interesting to evaluate the slope at weak coupling. In principle, this requires the knowledge of the anomalous dimensions of short $\mathfrak{sl}(2)$ operators in closed form as a function of the spin at a certain length (*i.e.* twist, in the gauge theory dictionary). This information is available for the asymptotic contribution, but not for the wrapping, which is only known as a series expansion at large spin and low twist [161, 162]. Nevertheless, if we are interested in the correction to the slope only (so, just the first term at small spin), then the Lüscher form of the wrapping correction presented in

check that this is same as in the paper with Grisha. We had a bit different ansatz which incorporates some extra symmetries

[161] is enough. At twist-1, and following the notation of [161], the wrapping correction enters at four loops and is expressed by the following function of the integer spin N of the gauge theory operator

$$\gamma_4^{\text{wrapping}}(N) = \gamma_2(N)\,\mathcal{W}(N), \qquad \gamma_2(N) = 4\,[S_1(N) - S_{-1}(N)]. \quad (5.61)$$

Here, $S_a(N)$ are generalized harmonic sums while $\mathcal{W}(N)$ is a complicated expression depending on the Baxter polynomial $Q_N(u)$ associated with the Bethe roots. The first factor $\gamma_2(N)$ is nothing but the two-loop anomalous dimension of the twist-1 operators. In the small N limit, it starts at $\mathcal{O}(N)$. Thus, the factor $\mathcal{W}(N)$ can be evaluated at $N = 0$ where the Baxter polynomial trivializes $Q_0(u) = 1$. After a straightforward calculation, one finds that (on the even N branch).

$$\gamma_4^{\text{wrapping}}(N) = -\frac{\pi^4}{3}\,N + \mathcal{O}(N^2). \quad (5.62)$$

So, even at weak coupling, we find a correction to the slope coming from the wrapping terms. Notice that the reason why such a contribution is absent in $AdS_5 \times S^5$ is simply that the factor analogous to $\gamma_2(N)$ is squared in the wrapping contribution. This leads immediately to a contribution to the slope of order $\mathcal{O}(N^2)$.

5.3.5 Prediction for short states

We can provide a prediction for the strong coupling expansion of the energy of short states that in principle could be tested by TBA calculations. To this aim, we can start from our results at fixed $\rho = \mathcal{J}/\sqrt{\mathcal{S}}$, and re-expand at large Λ the sum of the (scaled) classical energy

$$\mathcal{E}_0 \;=\; \sqrt{(\rho^2+2)\,\mathcal{S}}\left[1 + \frac{2\,\rho^2+3}{4\,(\rho^2+2)}\,\mathcal{S} - \frac{1\,\rho^6 + 20\,\rho^4 + 34\,\rho^2 + 21}{32\,(\rho^2+2)^2}\,\mathcal{S}^2 + \cdots\right] \quad (5.63)$$

and the one-loop contribution (5.38). The result is

$$E = (4\,\pi\,g)^{1/2}\sqrt{2\,S} - \frac{1}{2} + \frac{\sqrt{2\,S}}{(4\,\pi\,g)^{1/2}}\left(\frac{J\,(J+1)}{4\,S} + \frac{3\,S}{8} - \frac{1}{4} + \frac{1}{2}\,\log(2)\right) + \cdots. \quad (5.64)$$

The same expansion where we remark that one of the effect of the C term is the constant $\mathcal{O}(\widetilde{\Lambda}^0)$ contribution. The same expansion can be written in terms of the coupling g_{WS} in the world-sheet regularization whose relation with g is [158, 159]

$$g = g_{\text{WS}} - \frac{\log(2)}{4\pi} + \cdots. \quad (5.65)$$

After this replacement, eq.(5.64) reads

$$E = (4\,\pi\,g_{\text{WS}})^{1/2}\sqrt{2\,S} - \frac{1}{2} + \frac{\sqrt{2\,S}}{(4\,\pi\,g_{\text{WS}})^{1/2}}\left(\frac{J\,(J+1)}{4\,S} + \frac{3\,S}{8} - \frac{1}{4}\right) + \cdots. \quad (5.66)$$

the quantum spectral curve, a second quantization of sorts. Obviously at the moment there are just wild speculations.

In conclusion, AdS/CFT dualities and integrability are immensely powerful tools when dealing with gauge theories, so powerful that they enable one to find results exact in the coupling constant. Obviously the ultimate goal of this research programme is to learn something that could be applied to real world theories such as QCD. If we knew as much about QCD as we do about $\mathcal{N}=4$ super Yang-Mills by now we could analytically find the mass of the proton. Unfortunately we are not there yet, but one can dream.

Do you refer to the all appendixes in the main text?

A Summary of notation and definitions

In this appendix we summarize some notation used ubiquitously throughout the thesis.

A.1 Laurent expansions in x

We often represent functions of the spectral parameter u as a series in x

$$f(u)=\sum_{n=-\infty}^{\infty} f_n x^n \tag{A.1}$$

with

$$u=g(x+1/x). \tag{A.2}$$

We denote by $[f]_+$ and $[f]_-$ part of the series with positive and negative powers of x:

$$[f]_+=\sum_{n=1}^{\infty} f_n x^n, \tag{A.3}$$

$$[f]_-=\sum_{n=1}^{\infty} f_{-n} x^{-n}. \tag{A.4}$$

As a function of u, $x(u)$ has a cut from $-2g$ to $2g$. For any function $f(u)$ with such a cut we denote another branch of $f(u)$ obtained by analytic continuation (from $\Im u>0$) around the branch point $u=2g$ by $\tilde{f}(u)$. In particular, $\tilde{x}=1/x$.

A.2 Functions $\sinh_\pm$ and $\cosh_\pm$

We define $I_k=I_k(4\pi g)$, where $I_k(u)$ is the modified Bessel function of the first kind. Then

$$\sinh_+=[\sinh(2\pi u)]_+=\sum_{k=1}^{\infty} I_{2k-1}x^{2k-1}, \tag{A.5}$$

$$\sinh_-=[\sinh(2\pi u)]_-=\sum_{k=1}^{\infty} I_{2k-1}x^{-2k+1}, \tag{A.6}$$

$$\cosh_+=[\cosh(2\pi u)]_+=\sum_{k=1}^{\infty} I_{2k}x^{2k}, \tag{A.7}$$

$$\cosh_-=[\cosh(2\pi u)]_-=\sum_{k=1}^{\infty} I_{2k}x^{-2k}. \tag{A.8}$$

In some cases we denote for brevity

$$\mathrm{sh}_-^x=\sinh_-(x), \quad \mathrm{ch}_-^x=\cosh_-(x). \tag{A.9}$$

A.3 Integral kernels

In order to solve for $\mathbf{P}_a^{(1)}$ in section 4.4.2 we introduce integral operators H and K with kernels

$$H(u,v) = -\frac{1}{4\pi i}\frac{\sqrt{u-2g}\sqrt{u+2g}}{\sqrt{v-2g}\sqrt{v+2g}}\frac{1}{u-v}dv, \tag{A.10}$$

$$K(u,v) = +\frac{1}{4\pi i}\frac{1}{u-v}dv, \tag{A.11}$$

which satisfy

$$\tilde{f}+f=h \ , \ f=H\cdot h \quad \text{and} \quad \tilde{f}-f=h \ , \ f=K\cdot h. \tag{A.12}$$

Since the purpose of H and K is to solve equations of the type (A.12), H usually acts on functions h such that $\tilde{h}=h$, whereas K acts on h such that $\tilde{h}=-h$. On the corresponding classes of functions H and K can be represented by kernels which are equal up to a sign

$$H(u,v) = -\frac{1}{2\pi i}\frac{1}{x_u-x_v}dx_v\Big|_{\tilde{h}=h}, \tag{A.13}$$

$$K(u,v) = \frac{1}{2\pi i}\frac{1}{x_u-x_v}dx_v\Big|_{\tilde{h}=-h}. \tag{A.14}$$

In order to be able to deal with series in half-integer powers of x in section 4.4.4 we introduce modified kernels:

$$H^*\cdot f \equiv \frac{x+1}{\sqrt{x}}H\cdot\frac{\sqrt{x}}{x+1}f, \tag{A.15}$$

$$K^*\cdot f \equiv \frac{x+1}{\sqrt{x}}K\cdot\frac{\sqrt{x}}{x+1}f. \tag{A.16}$$

Finally, to write the solution to equations of the type (4.102), we introduce the operator Γ' and its more symmetric version Γ

$$(\Gamma'\cdot h)(u) \equiv \oint_{-2g}^{2g}\frac{dv}{4\pi i}\partial_u \log\frac{\Gamma[i(u-v)+1]}{\Gamma[-i(u-v)]}h(v), \tag{A.17}$$

$$(\Gamma\cdot h)(u) \equiv \oint_{-2g}^{2g}\frac{dv}{4\pi i}\partial_u \log\frac{\Gamma[i(u-v)+1]}{\Gamma[-i(u-v)+1]}h(v). \tag{A.18}$$

A.4 Periodized Chebyshev polynomials

Periodized Chebyshev polynomials appearing in $\mu_{ab}^{(1)}$ are defined as

$$p'_a(u) = \Sigma\cdot[x^a + 1/x^a] = 2\Sigma\cdot\left[T_a\left(\frac{u}{2g}\right)\right], \tag{A.19}$$

$$p_a(u) = p'_a(u) + \frac{1}{2}\left(x^a(u)+x^{-a}(u)\right), \tag{A.20}$$

where $T_a(u)$ are Chebyshev polynomials of the first kind. Here is the explicit form for the first five of them:

$$p'_0 = -i(u-i/2), \tag{A.21}$$

$$p'_1 = -i\frac{u(u-i)}{4g}, \tag{A.22}$$

$$p'_2 = -i\frac{(u-i/2)(-6g^2+u^2-iu)}{6g^2}, \tag{A.23}$$

$$p'_3 = -i\frac{u(u-i)\left(-6g^2+u(u-i)\right)}{8g^3}, \tag{A.24}$$

$$p'_4 = -i\frac{\left(u-\frac{i}{2}\right)\left(30g^4-20g^2u^2+20ig^2u+3u^4-6iu^3-2u^2-iu\right)}{30g^4}. \tag{A.25}$$

B Slope function: details

Here we fill in some of the details and provide generalizations for the $\mathbf{P}\mu$-system solution of the slope function discussed in section 4.3.

B.1 Solution for odd J

Here we give details on solving the $\mathbf{P}\mu$-system for odd J at leading order in the spin. First, the parity of the μ_{ab} functions is different from the even J case, which can be seen from the asymptotics (4.43). Following arguments similar to the discussion for even J in section 4.3.2, we obtain

$$\mu_{12}=1,\ \mu_{13}=0,\ \mu_{14}=0,\ \mu_{24}=\cosh(2\pi u),\ \mu_{34}=1. \tag{B.1}$$

Plugging these μ_{ab} into (4.41) we get a system of equations for $\mathbf{P}_a$

$$\tilde{\mathbf{P}}_1 = -\mathbf{P}_3, \tag{B.2}$$

$$\tilde{\mathbf{P}}_2 = -\mathbf{P}_4 - \mathbf{P}_1\cosh(2\pi u), \tag{B.3}$$

$$\tilde{\mathbf{P}}_3 = -\mathbf{P}_1, \tag{B.4}$$

$$\tilde{\mathbf{P}}_4 = -\mathbf{P}_2 + \mathbf{P}_3\cosh(2\pi u). \tag{B.5}$$

This system can be solved in a similar way to the even J case. The only important difference is that due to asymptotics (4.42) the $\mathbf{P}_a$ acquire an extra branch point at