

## Coursework 2: MATH 3018/6141 - Numerical methods

**Due: 13 December 2017**

In this coursework you will implement given numerical algorithms. The assessment will be based on

- correct implementation of the algorithms to solve the problems (task 1) – **5 marks**
- check simple case (task 2) and more complex case (task 3) in 2d and in 3d (task 4) – **2, 1, and 2 marks**
- choice and justification of algorithms – **2 marks**
- documentation of code, unit testing, robustness and error checking of code – **3 marks**.

The deadline is noon, Wednesday 13 December 2017 (week 11). For late submissions there is a penalty of 10% of the total marks for the assignment per day after the assignment is due, for up to 5 days. No marks will be obtained for submissions that are later than 5 days.

Your work must be submitted electronically via Blackboard. Only the Python files needed to produce the output specified in the tasks below is required.

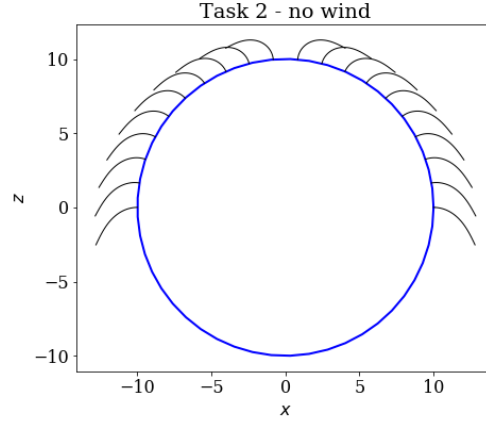


Figure 1: Locations of 20 hairs in the  $x, z$  plane under the force of gravity alone. The boundary value problem for each hair is solved, fixing the location where the hair meets the head.

## 1 Modelling hair

A major use of numerical methods is in the creation of computer graphics in films, games, and other media. The generation of “realistic” animations of, for example, smoke, or water, or explosions, are typical problems. One case that remains difficult is the simulation of hair, particularly under external forces such as wind and rain.

## 2 Model

The first reasonable model of human hair used the beam equation applied to each hair individually. This coursework follows that model. The model we use here makes a number of simplifying assumptions:

1. The human head is a sphere with radius 10 cm.
2. Each hair meets the head normal to the sphere.
3. The forces acting on each hair are gravity (in the  $-z$  direction) and a constant wind (in the  $+x$  direction).

We want to construct the location of each hair. We will use the parameter  $s$  along the length of the hair:  $s = 0$  where the hair meets the head, and  $s = L$  at the free end of the hair, where  $L = 4$  cm is the constant hair length.

The model works separately for each hair. It uses a mixture of coordinate systems. The location of the hair will be displayed in the  $(x, y, z)$  coordinate system. The location of the hair will depend on the parameter  $s$  along its length. The parameter  $s$  takes

the value  $L$  at the free end. The location where the hair meets the head, where  $s = 0$ , is given by

$$x(0) = R \cos(\theta_0) \cos(\phi_0), \quad (1a)$$

$$y(0) = -R \cos(\theta_0) \sin(\phi_0), \quad (1b)$$

$$z(0) = R \sin(\theta_0). \quad (1c)$$

Here  $\theta_0$  and  $\phi_0$  are coordinates on the head:  $\theta_0$  is the latitude (angle away from horizontal) and  $\phi_0$  is longitude (angle away from the  $x$ -direction).

By contrast  $\theta(s)$  and  $\phi(s)$  describe the local orientation in space of a given hair. The fact that the hair comes out of the head at right angles is given by the boundary conditions

$$\theta(0) = \theta_0, \quad (2a)$$

$$\phi(0) = \phi_0. \quad (2b)$$

The hair then bends under the influence of gravity and the wind force. Bending is described by the orientation  $(\theta(s), \phi(s))$  changing with  $s$ , as we go along the length of the hair. A completely rigid hair would have constant  $\theta(s)$  and  $\phi(s)$ . The simplified model leads to equations for each individual hair as the ODE system

$$\frac{d^2\theta}{ds^2} = s f_g \cos(\theta) + s f_x \cos(\phi) \sin(\theta), \quad (3a)$$

$$\frac{d^2\phi}{ds^2} = -s f_x \sin(\phi) \sin(\theta). \quad (3b)$$

$f_g$  is proportional to the force from gravity. After including appropriate values for human hair stiffness,  $f_g = 0.1 \text{ cm}^{-3}$  is the value used here.  $f_x$  is proportional to the force from the wind. Finally, the hair can move freely at its end (at  $s = L$ ), giving the boundary conditions

$$\frac{d\theta}{ds}(L) = 0, \quad (4a)$$

$$\frac{d\phi}{ds}(L) = 0. \quad (4b)$$

We can then find  $\theta(s)$  and  $\phi(s)$ , by solving the BVP defined in equation (3) with the boundary conditions at the start from equation (2) and at the end from equation (4). From  $\theta(s)$  and  $\phi(s)$ , the  $(x, y, z)$  coordinates can be found as

$$\frac{dx}{ds} = \cos(\theta) \cos(\phi), \quad (5a)$$

$$\frac{dy}{ds} = -\cos(\theta) \sin(\phi), \quad (5b)$$

$$\frac{dz}{ds} = \sin(\theta), \quad (5c)$$

with initial conditions given by the locations on the head from equation (1).

**Note:** When the initial angle  $\phi_0 \equiv 0$  the system simplifies, as  $\phi(s) \equiv 0$ . In this case the boundary value problem to solve is equation (3a) with boundary conditions (2a) at the start and (4a) at the end.

### 3 Task

1. First assume that  $\phi_0 \equiv 0$ . Write a function that, given  $L, R, f_x, f_g$ , and a list of values of  $\theta_0$  which specify where the hairs meet the head, returns the  $(x, z)$  coordinates of the hairs.
2. Fix  $\phi = 0$  (work within the  $(x, z)$  plane) and  $f_x = 0$  (no force due to the wind). We fix  $f_g = 0.1$  throughout. Compute and plot the location of 20 hairs with  $L = 4$  where the locations where the hairs meet the head are given by  $\theta_0$  being evenly spaced on  $[0, \pi]$ . Plot the locations in the  $(x, z)$  plane.
3. Repeat the previous task with  $f_x = 0.1$ .
4. Now allow  $\phi_0$  to vary freely. Write a function that given  $L, R, f_x, f_g$ , and a list of values of  $\theta_0$  and  $\phi_0$  which specify where the hairs meet the head, returns the  $(x, y, z)$  coordinates of the hairs.
5. Compute the locations of 100 hairs, with the force due to the wind set to  $f_x = 0.05$ . These should meet the head at a  $10 \times 10$  grid of  $\theta, \phi$  locations, where  $\theta_0$  is evenly spaced on  $[0, 0.49\pi]$  and  $\phi_0$  is evenly spaced on  $[0, \pi]$ . Plot the results in the  $(x, z)$  plane, the  $(y, z)$  plane, and in 3 dimensions.

#### 3.1 Summary and assessment criteria

You should submit the code electronically as noted above.

The primary assessment criteria will be a general code that solves the hair-modelling problem. In top-level code comments the reasons for the choice of algorithm should be clearly given, stating the (dis)advantages of this and alternative methods.

The secondary assessment criteria will be the clarity and robustness of your code. Your functions and scripts must be well documented with appropriate internal comments describing inputs, outputs, what the function does and how it does it. The code should also be well structured to make it easy to follow. Input must be checked and sensible error messages given when problems are encountered. The clarity of the output (such as plots or results printed to the command window) is also important.

Code efficiency is not important unless the algorithm takes an exceptional amount of time to run.

The original paper that uses a “physical” approach to hair modelling is by Anjyo et al. from 1992. A survey of hair modelling approaches is given by Bartails et al. (2006). The particular model used here is a generalisation of Howison’s book on Practical Applied Mathematics. For a broader range of modelling techniques applied to computer graphics, Ron Fedkiw’s webpage is a good place to start.