Homework 3 for Introduction to topological data analysis

Winter 2024/25

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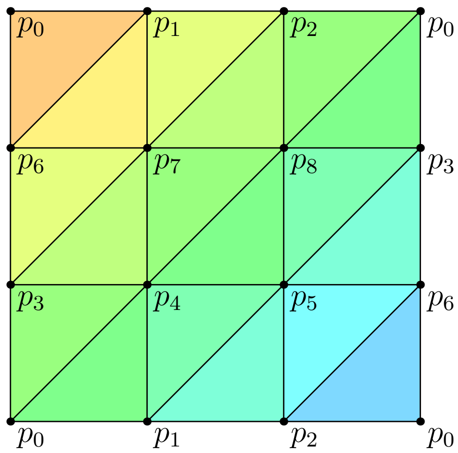
Exercises have to be completed online respectively submitted in written form. Every exercise is worth 10 points. *Problems* will be discussed in the tutorial. You can prepare them, but are not obliged to do so.

Exercise 9 — Complete the online test "Finite fields".

Exercise 10 — Complete the online test "Chains, cycles, boundaries".

Exercise 11 — On the set $P = \{p_0, p_1, \dots, p_8\}$ with fixed ordering $p_0 < p_1 < \cdots < p_8$, we consider the following abstract simplicial complex K:



Note that some points like p_3 appear multiple times in the picture, but all appearances represent the same point. For example, the 2-simplices $\{p_0, p_1, p_6\}$ and $\{p_0, p_2, p_6\}$ meet along their common face $\{p_0, p_6\}$. (In particular, the picture above is *not* a geometric realisation, it is just an easy way to describe K.)

(a) Determine the Euler characteristic $\chi(K)$.

Solve the following questions over the field $F = \mathbb{F}_2$ as well as over the field $F = \mathbb{Q}$ and compare your results.

(b) Compute the rank of the boundary maps

$$\partial_1 \colon C_1(K, F) \to C_0(K, F)$$

and

$$\partial_2 \colon C_2(K, F) \to C_1(K, F)\ .$$

(c) Compute the Betti numbers $\beta_n(K, F)$ for $n \in \{0, 1, 2\}$.

(d) For $n \in \{0, 1, 2\}$, construct a basis for the linear subspace $B_n(K, F) \subset C_n(K, F)$ of n-boundaries.

(e) For $n \in \{0, 1, 2\}$, extend your basis from (c) by $\beta_n(K, F)$ vectors in such a way that you get a basis for the linear subspace $Z_n(K, F) \subset C_n(K, F)$ of n-cycles.

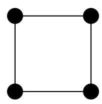
Exercise 12 — Let F be a field. Let K be an abstract simplicial complex on a finite set P. On the finite set $P' = P \cup \{\bullet\}$, where we added an additional point $\bullet$ to the set P, we consider the following abstract simplicial complex:

$$K' = K \cup \{\sigma \cup \{\bullet\} \mid \sigma \in K \text{ or } \sigma = \emptyset\} \ .$$

In other words, an n-simplex of K' is either an n-simplex of K, or $\sigma \cup \{\bullet\}$, where σ is an $(n-1)$-simplex of K (and for $n = 0$ we also have $\{\bullet\}$ itself).

(a) Verify that K' is indeed an abstract simplicial complex.

(b) Draw a picture of K' if K is the following complex:



(c) Show that $\chi(K') = 1$.

(d) Show that $\beta_0(K', F) = 1$.

(e) Show that $\beta_n(K', F) = 0$ for all $n > 0$.

Hand in: Wednesday, November 6, 12:15, online in Ilias