Homework 4 for Topological Data Analysis

Winter 2024/25

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Exercise 13 — Let $V = \mathbb{R}^3$ and $U = \{(\lambda, \lambda, \lambda) \mid \lambda \in \mathbb{R}\}$. Let $\pi\colon V \to V/U$ be the canonical projection. We consider the vectors

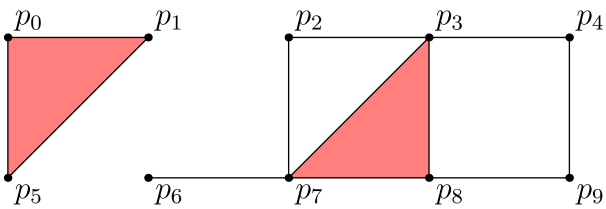
$$v_1 = (1,2,3),\ v_2 = (0,1,1),\ v_3 = (1,3,5)\ \in V$$

and their images $[v_i] = \pi(v_i) = v_i + U \in V/U$.

(a) Are $[v_1]$ and $[v_2]$ linearly independent? Are $[v_1]$ and $[v_3]$ linearly independent?

(b) Compute a basis for $W = \mathsf{span}([v_1], [v_2], [v_3])$.

Exercise 14 — In this exercise we take all coefficients in the field $\mathbb{F}_2$. We consider the following abstract simplicial complex K:



We also define the following 1-chains in K:

$$\begin{aligned}
\sigma_0 &= \{p_0, p_1\} + \{p_1, p_5\} + \{p_5, p_0\} \\
\sigma_1 &= \{p_2, p_3\} + \{p_3, p_7\} + \{p_7, p_2\} \\
\sigma_2 &= \{p_3, p_4\} + \{p_4, p_9\} + \{p_9, p_8\} + \{p_8, p_3\} \\
\sigma_3 &= \{p_2, p_3\} + \{p_3, p_4\} + \{p_4, p_9\} + \{p_9, p_8\} + \{p_8, p_7\} + \{p_7, p_2\}
\end{aligned}$$

(a) Verify that the σ_i are cycles and thus define homology classes $[\sigma_i] \in H_1(K, \mathbb{F}_2)$.

(b) Which of the following equations are true in $H_0(K, \mathbb{F}_2)$? If possible, provide a 1-chain whose boundary explains why the equation is true in homology.

$$[p_7] = [p_8] \qquad [p_1] = [p_3] \qquad [p_2] = [p_9]$$

(c) Which of the following equations are true in $H_1(K, \mathbb{F}_2)$? If possible, provide a 2-chain whose boundary explains why the equation is true in homology.

$$[\sigma_0] = 0 \qquad [\sigma_1] = [\sigma_2] \qquad [\sigma_1] + [\sigma_2] = [\sigma_3]$$

Exercise 15 — Let K and L be abstract simplicial complexes on a finite set P.

(a) Prove that $K \cup L$ and $K \cap L$ are abstract simplicial complexes as well.

(b) Show that we have $\chi(K \cup L) = \chi(K) + \chi(L) - \chi(K \cap L)$.

(c) Give an example where $\beta_n(K \cup L) \neq \beta_n(K) + \beta_n(L) - \beta_n(K \cap L)$.

(d) Show that if $K \cap L = \emptyset$, then we have $\beta_n(K \cup L) = \beta_n(K) + \beta_n(L)$.

Programming Challenge (10 bonus points) In this challenge, you analyse a big data set with the help of a computer. All persons who send the correct solution via email to `matthias.paulsen@uni-marburg.de`, along with a short description how they obtained their answer (e. g. source code of a program), receive 10 bonus points. Furthermore, the first person who submits the correct solution is rewarded with a small prize. You can use whatever software or programming language you like, as long as you are able to obtain the correct answer with it.

On Ilias, you can download a data set $P \subset \mathbb{R}^3$ of 1000 points. Your task is the following:

(a) Find the smallest value r such that $\mathrm{VR}_r(P)$ is connected.

(b) For this r, determine all Betti numbers of $\mathrm{VR}_r(P)$.

Hand in: Wednesday, November 13, 12:15, online in Ilias