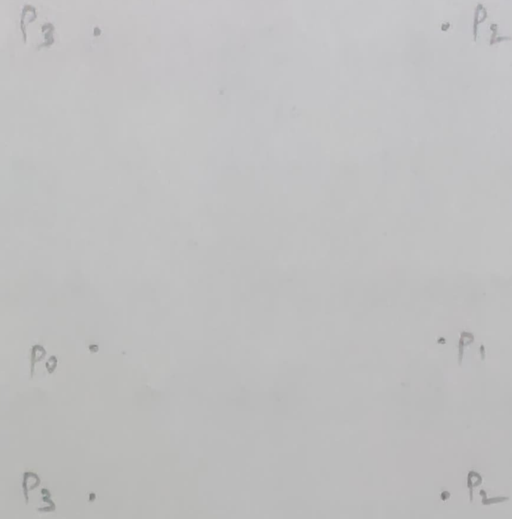
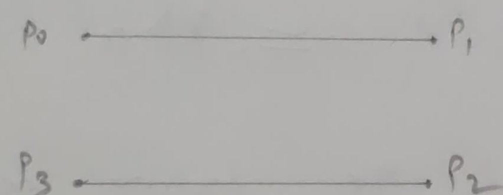
18) a)

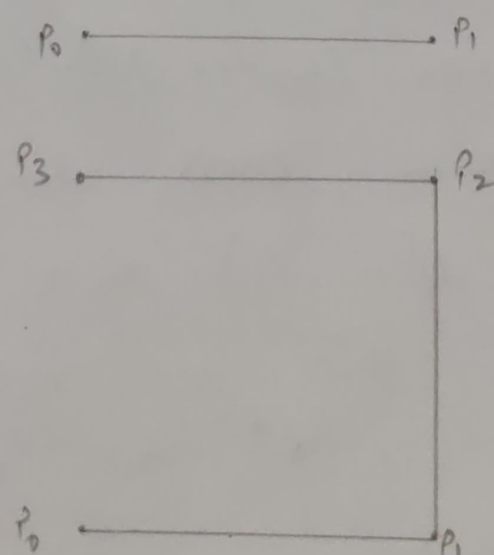
step 0:



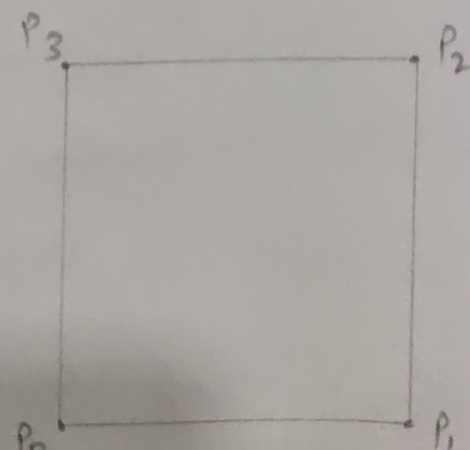
step 1:-



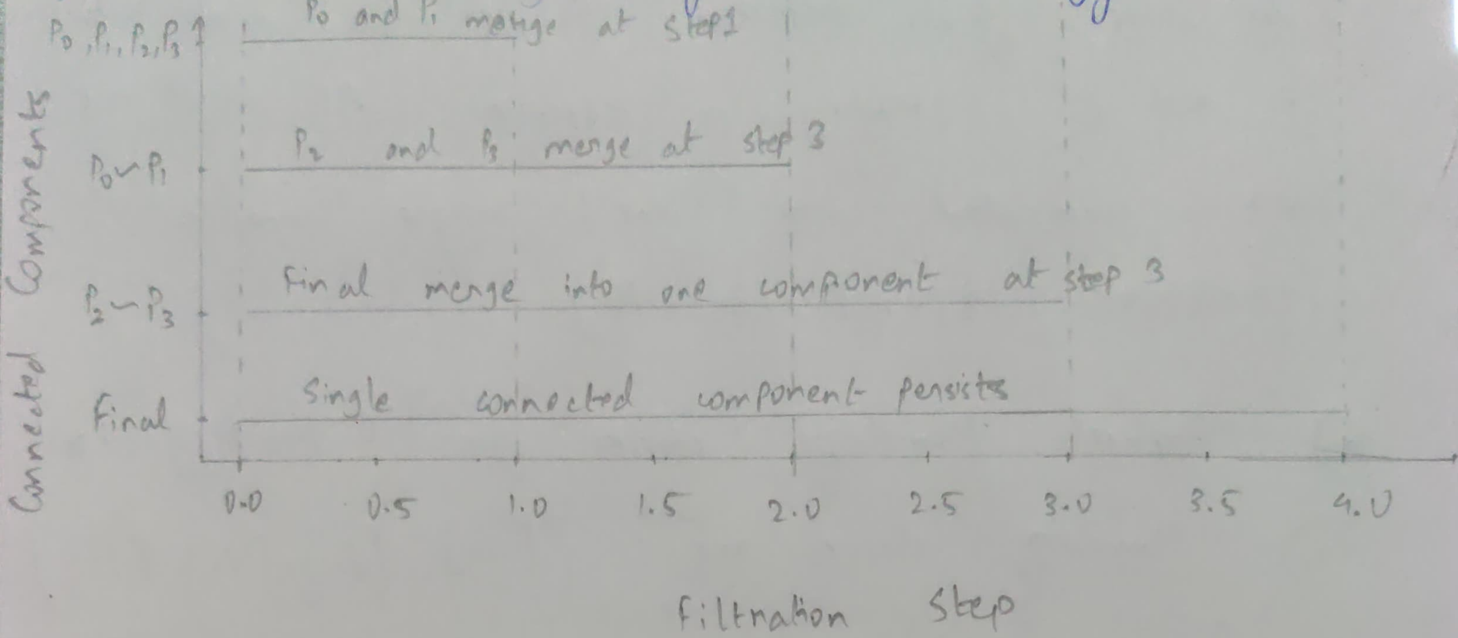
step 2:-



step 4:-



18) a) Barcode Diagram for 0- Homology H_0 :-



step 0 :-

- Initially, all points are disconnected.

  $\{P_0\}$, $\{P_1\}$, $\{P_2\}$, $\{P_3\}$

step 1 :-

- An edge forms between P_0 and P_1, merging these points into one connected component :

  $\{P_0, P_1\}$, $\{P_2\}$, $\{P_3\}$.

step 2 :-

- An edge forms between P_2 and P_3, merging these points into another connected component

  $\{P_0, P_1\}$, $\{P_2, P_3\}$.

Step 3:-

- An edge forms between p_1 and p_3, and between p_0 and p_2, creating a square. All points are now connected into one single connected component: $\{p_0, p_1, p_2, p_3\}$

b) Explicit Homology classes for 0-homology H_0:-

1.) Homology classes are : $\{[p_0], [p_1], [p_2], [p_3]\}$.

2.) $[p_0]$ and $[p_1]$ are identified, so the classes are reduced to : $\{[p_0 \sim p_1], [p_2], [p_3]\}$

3.) $[p_2]$ and $[p_3]$ are identified, so the classes are : $\{[p_0 \sim p_1], [p_2 \sim p_3]\}$.

4.) All points are identified into one class:

$\{[p_0 \sim p_1 \sim p_2 \sim p_3]\}$.

Lifespan of classes are:-

- $[p_0]$, $[p_1]$: Appear at step 0, merge at step 1.
- $[p_2]$, $[p_3]$: Appear at step 0, merge at step 2.
- Final Component : Appear at step 0, persists indefinitely.

19) A topological feature that persists in the Vietoris-Rips complex $VR_r(P)$ for a sufficiently long interval corresponds to a feature in the Cech complex $C_r(P)$. This is expressed as:

Any feature $VR_r(P)$ that persists in the interval $[r_1, r]$ satisfies:

$$C_{r_1}(P) \subseteq VR_{r_1}(P) \subseteq C_{r_2}(P),$$

implying that the feature exists in the Cech complex at some scalar $r \in [r_1, r_2]$.

i.) <u>Geometric Context</u>:-

- Vietoris-Rips Complex $VR_r(P)$:- Built using edges between points where the pairwise distance $d(p_i, p_j) \leq r$. It may include simplices that are not geometrical realizable in R^d but are combined consistent.

- Cech Complex $C_r(P)$: Built by covering each point in P with a ball of

$r/2$. A simplex exists if and only if ball intersect.

2.) Inclusion Relationship :

- For a given r, $VR_r(P)$ contains all simplices of $C_r(P)$ because any intersection of balls implies pairwise distances $\le r$. However, $VR_r(P)$ might have simplices that are not in $C_r(P)$, making $VR_r(P)$ a superset:

$$C_r(P) \subseteq VR_r(P)$$

3.) Persistence Stability:

- Features that persist for a long interval $[r_1, r_2]$ in $VR_r(P)$ must also be a part of $C_r(P)$, as persistent features are invariant under small pertubations of data. If a feature appears in $VR_r(P)$ for a sufficiently wide range of r, it is guaranteed to also be present in $CR_r(P)$ for at least some subset of range.

4.) Topological feature: Long-persisting feature are more likely to reflect true topological properties of P