16) Given that,

$K_0 \subset K_1 \subset K_2 \subset K_3 \subset K_4$

of simplical complexes:

a) Betti Numbers,

- $\beta_0(K_i)$ counts the no. of connected components
- $\beta_1(K_i)$ counts the no. of loops (holes).

<u>H_0 :-</u>

- At K_0, there are three connected components ($\beta_0 = 3$).
- At K_1, the components merge to form two connected components ($\beta_0 = 2$).
- At K_2, one more merge happens, resulting in one connected components ($\beta_0 = 1$).
- K_3 and K_4 remain connected ($\beta_0 = 1$).

<u>H_1 :-</u>

- At K_0 and K_1, there are no loops ($\beta_1 = 0$).
- At K_2, a loop forms ($\beta_1 = 1$).

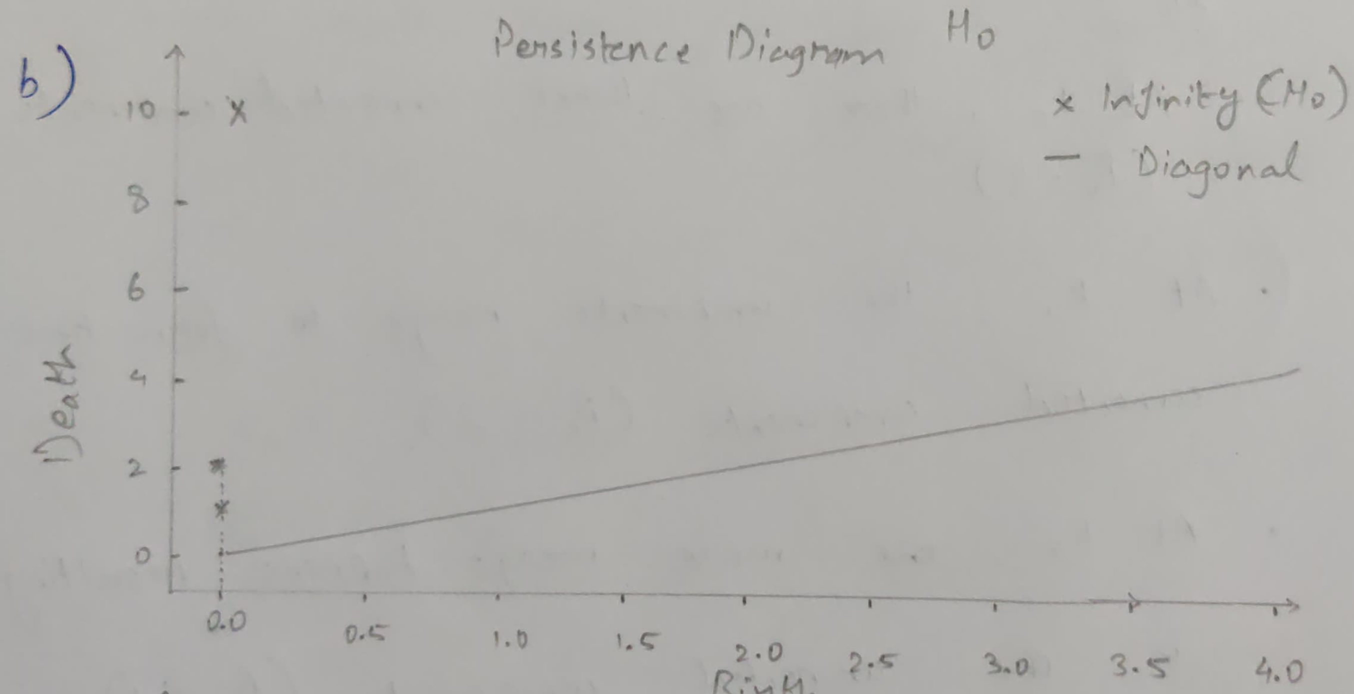
- At K_3, another loop appears ($\beta_1 = 2$).

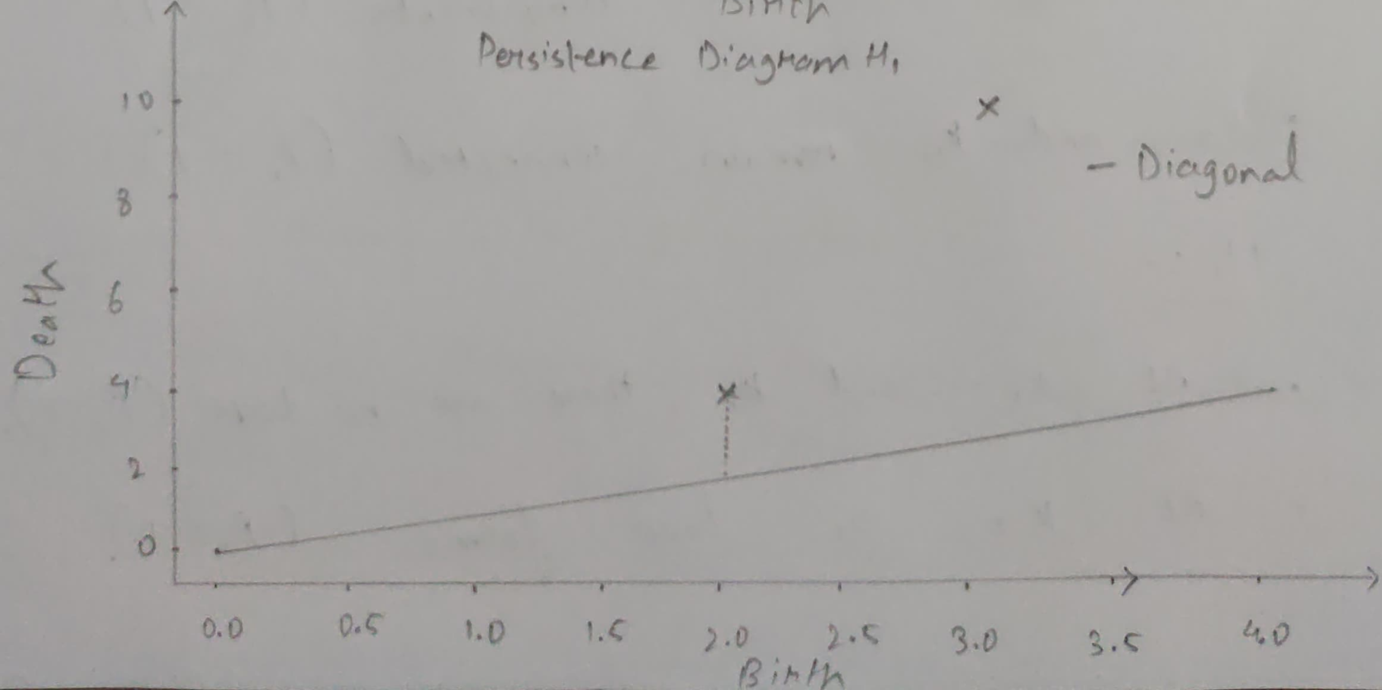
- At K_4, one loop persists, and another disappears ($\beta_1 = 1$).

Connected complexes:-

- A simplical complex is connected if $\beta_0 = 1$. Thus, K_2, K_3, and K_4 are connected.

b)





Persistence Diagram for H_0 (0-dimensional):

birth-death pairs for connected components H_0:-

- Component 1: Born at 0, dies at 2 (0,2).
- Component 2: Born at 0, dies at 1 (0,1).
- Component 3: Born at 0, ~~dies at~~ persists $(0,\infty)$

Persistence Diagram for H_1 (1-dimensional):

birth-death pairs for loops from H_1:-

- Loop 1: Born at 2, dies at 4 (2,4).
- Loop 2: Born at 3, persists $(3,\infty)$.

c) Betti Numbers $\beta_0^{i,j}$ and $\beta_1^{i,j}$ for all $i \leq j$.

$\beta_0^{i,j}$ (0-dimensional features):

- $\beta_0^{0,0} = 3$ (three components in K_0).
- $\beta_0^{0,1} = 3$, $\beta_0^{1,1} = 2$, and so on, reducing to 1 after K_2.
- General behavior: decreases as component merge.

$\beta_1^{i,j}$ (1- dimensional features):

- $\beta_1^{2,3} = 1$, $\beta_1^{3,4} = 2$, etc., representing loop persistence across stages.

- General behavior: increases where loops appears and decreases where they merge.

d) $H_0(K_i) \to H_0(K_j)$ (Connected components):

- Injective: The map is injective if no new connected components are created between K_i and K_j.

  - for all $i < j$, the map is injective because components merge, no split.

- Surjective: The map is surjective if all connected connected components in K_j are from K_i.

  - Surjective for all $i < j$, since no new components appear.

$H_1(K_i) \longrightarrow H_1(K_j)$ (loops):

- Injective : The map is injective if no new loops are connected between K_i and K_j.
  - Not injective between $K_2 \rightarrow K_3$ (new loops).
- Surjective : The map is surjective if all loops in K_j exists in K_i.
  - Surjective for $K_3 \rightarrow K_4$ (loop persists)

e) Example for a filtration of simplical complexes

1. Points :
   - Start with three points.
2. Edges :
   - Add edges between points at filtration steps K_1 and K_2, gradually merging components.
3. Loops :-
   - Add triangles at K_2 and K_3 to form loops.

This setup mirrors the barcode diagrams by capturing the births and deaths of connected components and loops.