11.a) Number of vertices

Count the unique vertices $P_0, P_1, \dots P_8$. Looking at the di[agram] we see that there are 9 unique vertices.

$$f_0 = 9$$

Number of edges

· ~~Horizontal edges~~: There are ~~12~~ ~~horizontal edges~~

~~Number of~~ Horizontal Edges $= \{P_0, P_1\}, \{P_1, P_2\}, \{P_2, P_0\}, \{P_6, P_7\}, \{P_7, P_8\}$
$\{P_8, P_3\}, \{P_3, P_4\}, \{P_4, P_5\}, \{P_5, P_6\}$.

Vertical Edges $= \{P_0, P_6\}, \{P_1, P_7\}, \{P_2, P_8\}, \{P_0, P_3\}, \{P_6, P_3\},$
$\{P_7, P_4\}, \{P_8, P_5\}, \{P_3, P_6\},$

Diagonal Edges $= \{P_6, P_1\}, \{P_3, P_7\}, \{P_7, P_2\}, \{P_0, P_4\}, \{P_4, P_8\},$
$\{P_0, P_8\}, \{P_1, P_5\}, \{P_3, P_5\}, \{P_6, P_2\}$.

Horizontal Edges $= 9$

Vertical edges $= 9$

Diagonal edges $= 9$

Edges $= 27$

No. of faces $= \{P_0, P_6, P_1\}, \{P_1, P_6, P_7\} \dots \dots$
$\{P_5, P_2, P_6\}, \{P_2, P_6, P_0\}$

$= \underline{18}$

Calculating Euler characteristic

$$\chi = V - E + F = 9 - 27 + 18 = 0 //$$

b) Rank of $\partial_1 : C_1(K,F) \to C_0(K,F)$

- The dimension of $C_1(K,F)$ is the number of edges $E = 27$.
- The dimension of $C_0(K,F)$ is the number of vertices $V = 9$
- The first Betti number $\beta_0 = 1$, indicating there is one connected component.

By rank-nullity theorem, we have:

$$\dim(\operatorname{Im} \partial_1) = \dim(C_0(K,F)) - \beta_0 = 9 - 1 = 8$$

Therefore, the rank of δ_1 is 8.

Rank of $\delta_2 : C_2(K,F) \to C_1(K,F)$

- The dimension of $C_2(K,F)$ is the number of faces $F = 18$
- The dimension of $C_1(K,F)$ is the number of edges $E = 27$
- The second Betti number $\beta_1 = 2$, indicating two independent loops

Using the rank-nullity theorem for δ_2:

$$\dim(\operatorname{Im} \delta_2) = \dim(C_1(K,F) - \beta_1 = 18 - 2 = 16$$

Therefore the rank of δ_2 is 16.

11 (c) Betti numbers $\beta_n(K,F)$ $n \in \{0,1,2\}$

Betti numbers (β_n) are defined as the ranked of homology groups.

$$H_n(K,F) = \frac{Z_n(K,F)}{B_n(K,F)}$$

[$Z_n(K,F)$: is the group of n-cycles of $C_n(K,F)$ that map to zero under ∂_n

$B_n(K,F)$: is the group of n-boundaries

β_0 : The no. of connected components

β_1 : The no. of 1-dim. holes on loops

β_2 : The no. of 2-dim. voids.

$\dim(C_0(K,F)) = 9$

$\dim(C_1(K,F)) = 27$

$\dim(C_2(K,F)) = 18$

$\text{rank}(\partial_1) = 8$

$\text{rank}(\partial_2) = 16$

$\underline{\beta_0:}$

Using the rank - nullity theorem of

$$\partial_1 : C_1(K, F) \rightarrow C_0(K, F)$$

$$\beta_0 = \dim(C_0(K, F)) - \text{rank}(\partial_1) = 9 - 8 = 1$$

$\therefore$ there is 1 connected component in K.

$\underline{\beta_1:}$

Using rank - nullity theorem for

$$\partial_2 : C_2(K, F) \rightarrow C_1(K, F)$$

$$\beta_1 = \dim(C_1(K, F)) - \text{rank}(\partial_2) \cancel{= 27 - 8 - 16}$$
$$- \text{rank}(\partial_1)$$

$$= 27 - 8 - 16 = 3$$

$\therefore$ there are 3 independent 1-dimensional loops in K.

$\underline{\beta_2}$:

Since $C_3(K, F) = 0$ (no 3-simplices in the complex)

Therefore β_2 is the nullity of ∂_2;

$\beta_2 = \dim(C_2(K,F)) - \text{rank}(\partial_2) = 18-16 = 2$

$\therefore$ there are 2 independent 2-dimensional voids.

$\beta_0 = 1$

$\beta_1 = 3$

$\beta_2 = 2$

$\underline{\text{ii) d)}}$ For $n \in \{0, 1, 2\}$, the basis of an linear subspace $B_n(K,F) \subset C_n(K,F)$ of n-boundaries can be examined by the computed Betti numbers:

$\beta_0 = 1$, $\beta_1 = 3$, $\beta_2 = 2$

1. $\underline{H_0(K,F)}$:-

Since $\beta_0 = 1$, there is only one connected component for simplical complex K.

* Any single vertex can represent the connected component as basis element.

* Lets denoted this basis by (v_0), where v_0 is one of the vatices in the complex

2. $\underline{H_1(K, F)}$

$\beta_1 = 3$,

that means there are ~~3~~ three independent one dimensional loops in K.

* Identify three 1-dimensional cycles in K that do not bound any 2-dimensional faces.

* Lets denote these loops as z_1, z_2, z_3 where each z_i is closed path formed by edges in the complex.

These loops are representative of three independent cycles that forms a basis of $H_1(K, F)$.

3) $H_2(K, F)$:-

$\beta_2 = 2,$

This means that there are two - independent

2- dimensional voids in K.

* Identify two 2 - dimensional surfaces with K

that are not boundary of any 3 - dimension

solid K.

* Lets denote these surfaces as $s_1, s_2,$

Where each s_i represents a collection of

faces that enclose a void.

11) e) For $n \in \{0, 1, 2\}$ to form a basis

for the space of n- cycles

$$Z_n(K, F) \subset C_n(K, F) \quad n \in \{0, 1, 2\}$$

For $n = 0$:- $Z_0(K, F)$:-

Given,

* $\beta_0 = 1$

* Basis for $H_0(K, F) = V_0$

* Since $Z_0(K, F)$ consists of all vertices in the connected component, the basis for $Z_0(K, F)$ includes all vertices.

$$Z_0(K, F) = \{[v_1], [v_2], \dots [v_g]\}.$$

So, the basis of $Z_0(K, F)$ is set of all g vertices in K.

For <u>n = 1</u>

* $Z_1(K, F)$
* Given that $\beta_1 = 3$, meaning three independent 1 - dimensional cycle.
* Basis for $Z_1(K, F) = Z_1$
* To extend this basis $Z_1(K, F)$ we need to add additional cycles that captures all 1 - dimensional cycles.

$$Z_1(K, F) = \{[z_1], [z_2], [z_3] \dots \dots\}$$

Here, z_1, z_2, z_3 are part of a basis for 1 - dimensional cycle space.

For $\underline{n = 2}$

* Basis of $Z_2(K, F)$
* Given that $\beta_2 = 2$
* Basis for $Z_2(K, F) = s_1$
* To form a basis for $Z_2(K, F)$, we add any additional 2-cycles that form closed surface without boundaries of 3-dimensional simplices.

$$Z_2(K, F) = \{[s_1], [s_2] \ldots\ldots\}$$

- Basis for $Z_0(K, F)$: All vertices in the complex $[v_1], [v_2], [v_3] \ldots [v_9]$
- Basis for $Z_1(K, F)$: A combination of independent loops ~~$[v_1], [v_2], [v_3]$~~ ... $[z_1], [z_2], [z_3]$.
- Basis for $Z_2(K, F)$: Independent 2-dim. ~~basis~~ voids as $[s_1], [s_2]$.

12 a) $k' = k \cup \{\sigma \cup \{\cdot\} \; \sigma \in k \text{ or } \sigma = \phi\}$

Since k is given to be an abstract simplical complex on set P

$\forall \in k$, if $T' \subseteq T$ then $T' \in k$

then new simplices added to form k'

$\sigma \cup \{\cdot\}$ for each $\sigma \in k$ or $\sigma \in \phi$

let $\tau \subseteq \sigma \cup \{\cdot\}$ then, T can be 2 types - If $\cdot \in T$, then $T = \sigma' \cup \{\cdot\}$ for some $\sigma' \subseteq \sigma$

since $\sigma \in k$ and k is closed under subsets $\sigma \in k$, which implies $\sigma' \cup \{\cdot\} \in k'$

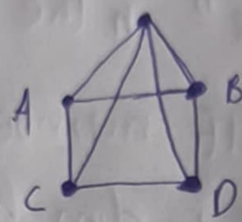
If $\cdot \notin T$, then $T \subseteq \sigma$, since $\sigma \in k$ and k is an abstract simplical complex, $T \in k$ and thus $T \in k'$

By the definition of an abstract simplical complex $\phi \in k$ since k is a simplical complex $\phi \in k'$

$\forall T \in k'$, if $T' \subseteq T$ the $T' \in k'$

~~Since~~ Hence k' is an abstract simplical complex

12 b)



12 c)

Total vertices $f_0(k') = 5$

Total edges: $f_1(k') = 8$

Total faces $f_2(k') = 5$

The entire structure forms a 3D tetrahedron so ~~we~~ we have 1 filled 2 complex

total 3 simpleces $f_3(k') = 1$

$$\chi(k') = f_0(k') - f_1(k') + f_2(k') - f_3(k')$$

$$\chi(k') = 5 - 8 + 5 - 1 = \underline{\underline{1}}$$

d) The Betti number β_0 represents the number of connected components in k'

Since k' is connected (all vertices are connecting through.

∴ there is only one connected component. Therefore $\beta_0(k'F) = 1$

e) The higher Betti numbers represents higher dimensional "holes" in the complex.

1) β_1: The Betti number β_1 represents the number of 1- dim holes (loops). In K, the square would form a single loop (1-cycle), But in K', this loop is filled by the addition of triangles (2 simplices), so there are no 1- dim holes. Thus $\beta_1 = 0$.

2) β_2 and higher: since K' is embedded by R^2 there are no higher - dim holes, so $\beta_n = 0$ for $n > 1$

Therefore:

$$\beta_n(K', F) = 0 \text{ for all } n > 0$$