Introduction to Markov Decision Processes

INDE 597

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  - $(X_1, \ldots, X_{20})$: outcomes of 20 (fair) coin tosses.
  - $(Y_1, \ldots, Y_{365})$: cars sold in Houston up to day t of 2019.
  - $(Z_1, \ldots, Z_T)$: length of queue at the Coffeehouse every hour.

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    If balls are not replaced, W_{t+1} depends on <u>all</u> previous draws; not Markovian.

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    - Goal: maximize total profit.

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where r is a terminal reward.

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- Goal: find π that maximizes this reward. Optimal value function

$$V^*(s_0) = \max_\pi V^\pi(s_0).$$

Examples: Queuing

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- Decision epochs – every hour.
- State = number of jobs waiting to be served.
- Action = number of jobs accepted for service in each period, $\mathcal{A} = 0, 1, \ldots, s_t\}$.
- Next state = $s_{t+1} = s_t - a_t +$ new arrivals (random).
- Reward for completing a job, cost for keeping jobs waiting.

Examples: Inventory management

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- Determine optimal policy for ordering inventory in the face of uncertain demand.
- Decision epochs – every week.
- Seller observes the current inventory (state) and decides how much more inventory (action) to acquire for the week.
- Random demand d_t during the week.
- $s_{t+1} = \max\{s_t + a_t - d_t, 0\}$.
- Reward = sale revenue - purchase cost.

Example: medical treatment planning

- Find optimal dosing policy for ‘best’ disease progression.
- Observe a ‘disease score’ (state) of a patient, and prescribe a dose (action) of medication (rheumatoid arthritis, radiation therapy).
- $s_{t+1} = f(s_t, a_t)$ clinically determined.
- Reward = improvement in patient’s condition.

Solving MDPs: Bellman's Principle of Optimality

Recall objective

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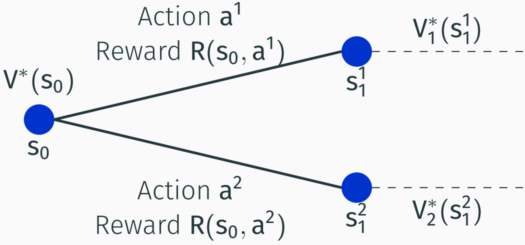
Bellman equations, solved by backward recursion.

Solving MDPs: Bellman's Principle of Optimality



$$V^*(s_0) = \max\left\{R(s_0, a^1) + \lambda V_1^*(s_1^1), R(s_0, a^2) + \lambda V_1^*(s_1^2)\right\}.$$

Infinite-horizon MDPs

- Decision epochs $t = 0, 1, \ldots$
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where $\lambda \in (0, 1)$ is a discount factor.

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No 'terminal' reward.

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- No backward recursion.
- Solution methods:
  - Value iteration
  - Policy iteration
  - Linear programming

Value iteration

- Define Bellman operator

$$\mathcal{L}(v)(s) = \max_{a \in \mathcal{A}} \; R(s,a) + \lambda \mathbf{E}[v(s')].$$

- V^* is the fixed point of this operator.
- Value iteration relies on the Contraction Mapping Theorem.
- Algorithm:
    - Initial guess $V^{(0)}$.
    - Compute $V^{(k)} = \mathcal{L}(V^{(k-1)})$.
    - Iterate until stopping condition.

Policy Iteration

- Now we iterate over policies to find a better one.
- Algorithm:
  - Initial guess $\pi^{(0)}$.
  - Evaluate policy using matrix inversion:

$$V^{\pi^{(k)}}(s) = R(s, \pi(s)) + \mathrm{E}[V^{(\pi^{(k)}}(s')].$$

  - Improve policy

$$\pi^{(k+1)}(s) = \operatorname*{argmax}_{a \in \mathcal{A}} \left\{ R(s, a) + \mathrm{E}[V^{(\pi^{(k)}}(s')] \right\}.$$

  - Iterate until stopping condition.

Linear Programming

- Note that

$$\begin{aligned} V^*(s) &= \max_{a \in \mathcal{A}} \left\{ R(s,a) + \mathbf{E}[V^*(s')] \right\} \; \forall \, s \\ &\geq R(s,a) + \mathbf{E}[V^*(s')] \; \forall \, s, a. \end{aligned}$$

- V^* is the solution to the following LP:

$$\begin{aligned} \min \; & \sum_{s \in \mathcal{S}} \beta(s) v(s) \\ \text{s.t. } & v(s) \geq R(s,a) + \mathbf{E}[v(s')] \; \forall \, s \in \mathcal{S}, a \in \mathcal{A}. \end{aligned}$$

Conclusion

- MDPs form a flexible modeling environment for sequential decision making problems.
- Several extensions – partially observable MDPs, reinforcement learning.
- Limitations:
  - 'Curse of dimensionality'
  - Data requirements.