D Proofs of propositions

Definition 4 is recalled below:

An enforcer for a timed property $\varphi \in tw(\Sigma)$ is a function

$$E^{\varphi} : tw(\Sigma) \rightarrow tw(\Sigma),$$

satisfying the following constraints:

- **Soundness**

$$\forall \sigma \in tw(\Sigma) : E^{\varphi}(\sigma) \models \varphi \vee E^{\varphi}(\sigma) = \epsilon \qquad \textbf{(Snd)}$$

- **Monotonicity**

$$\forall \sigma, \sigma' \in tw(\Sigma) : \sigma \preccurlyeq \sigma' \implies E^{\varphi}(\sigma) \preccurlyeq E^{\varphi}(\sigma') \qquad \textbf{(Mo)}$$

- **Transparency**

$$Tr_1 : \forall \sigma \in tw(\Sigma), \neg\exists \sigma' \in tw(\Sigma) : (\sigma' \preccurlyeq \sigma \wedge \text{delayable}_{\varphi}(\epsilon, \sigma') \neq \emptyset) \implies E^{\varphi}(\sigma) \triangleleft_d \sigma \quad \textbf{(Tr1)}$$

$$Tr_2 : \forall \sigma \in tw(\Sigma), \exists \sigma' \in tw(\Sigma) : (\sigma' \preccurlyeq \sigma \wedge \text{delayable}_{\varphi}(\epsilon, \sigma') \neq \emptyset) \implies E^{\varphi}(\sigma) \preccurlyeq_d \sigma \quad \textbf{(Tr2)}$$

where:

- $\text{delayable}_{\varphi}(\sigma_1, \sigma_2) = \{\sigma_2' \in tw(\Sigma) : (\sigma_2' =_d \sigma_2) \wedge (\sigma_1 \cdot \sigma_2' \in \text{pref}(\varphi)) \wedge (start(\sigma_2') \geq end(\sigma_2))\}$

Definition 5 is recalled below:

The enforcement function for a property φ is the function $E^{\varphi} : tw(\Sigma) \rightarrow tw(\Sigma)$ defined as: $\forall \sigma \in tw(\Sigma), \forall t \in \mathbb{R}_{\geq 0}, \forall a \in \Sigma, \mathcal{L} \subseteq tw(\Sigma)$,

$$E^{\varphi}(\sigma) = \Pi_1(\text{store}^{\varphi}(\sigma)), \text{ where}$$

$\text{store}^{\varphi} : tw(\Sigma) \rightarrow tw(\Sigma) \times tw(\Sigma)$ is defined as:

- $\text{store}^{\varphi}(\epsilon) = (\epsilon, \epsilon)$
- $\text{store}^{\varphi}(\sigma \cdot (t, a)) =$

$$\begin{cases} (\sigma_s \cdot min_{\preceq_{lex}, end}(k^{\varphi}(\sigma_s, \sigma_{ca})), \epsilon) & \text{if } k^{\varphi}(\sigma_s, \sigma_{ca}) \neq \emptyset, \\ (\sigma_s, \sigma_c) & \text{if } k^{pref(\varphi)}(\sigma_s, \sigma_{ca}) = \emptyset, \\ (\sigma_s, \sigma_{ca}) & \text{otherwise} \end{cases}$$

with

- $(\sigma_s, \sigma_c) = \text{store}^{\varphi}(\sigma)$,
- $\sigma_{ca} = \sigma_c \cdot (t, a)$

Definition 11 is recalled below:

A bounded enforcement function is $E^{\varphi,k} : tw(\Sigma) \rightarrow tw(\Sigma) \times \{\top, \bot\}$, and is defined as: $\forall \sigma \in tw(\Sigma), \forall t \in \mathbb{R}_{\geq 0}, \forall a \in \Sigma, \mathcal{L} \subseteq tw(\Sigma)$,

$$E^{\varphi,k}(\sigma) = (\Pi_1(\text{store}^{\varphi,k}(\sigma)), \Pi_3(\text{store}^{\varphi,k}(\sigma))), \text{ where:}$$

$\text{store}^{\varphi,k} : tw(\Sigma) \rightarrow tw(\Sigma) \times tw(\Sigma) \times \{\top, \bot\}$ is defined as:

- $\text{store}^{\varphi,k}(\epsilon) = (\epsilon, \epsilon, \top)$
- $\text{store}^{\varphi,k}(\sigma \cdot (t, a)) =$

$$\begin{cases} (\sigma_s \cdot min_{\preceq_{lex},end}(k^{\varphi}(\sigma_s, \sigma_{ca})), \epsilon, mode) & \text{if } k^{\varphi}(\sigma_s, \sigma_{ca}) \neq \emptyset, \\ (\sigma_s, \sigma_c, \bot) & \text{if } k^{pref(\varphi)}(\sigma_s, \sigma_{ca}) = \emptyset, \\ (\sigma_s, \sigma_{ca}, mode) & \text{if } k^{pref(\varphi)}(\sigma_s, \sigma_{ca}) \neq \emptyset \ \wedge |\sigma_{ca}| \leq k \\ (\sigma_s, \sigma_{ca \text{ } or \text{ } c}, stop) & \text{if } k^{pref(\varphi)}(\sigma_s, \sigma_{ca}) \neq \emptyset \ \wedge \ |\sigma_{ca}| > k \\ & \quad \wedge \text{ Get\_Subwords}^{\varphi,k}(\sigma_s, \sigma_{ca}) = \emptyset \\ (\sigma_s, \text{Clean}^{\varphi,k}(\sigma_s, \sigma_{ca}), \bot) & \text{if } k^{pref(\varphi)}(\sigma_s, \sigma_{ca}) \neq \emptyset \ \wedge |\sigma_{ca}| > k \\ & \quad \wedge \text{ Get\_Subwords}^{\varphi,k}(\sigma_s, \sigma_{ca}) \neq \emptyset \end{cases}$$

with

798 - $(\sigma_s, \sigma_c, mode) = \text{store}^{\varphi,k}(\sigma)$,
- $\sigma_{ca} = \sigma_c \cdot (t, a)$
- $E^{\varphi,k}_{\text{out}}(\sigma) = \Pi_1(E^{\varphi,k}(\sigma))$, and $E^{\varphi,k}_{\text{mode}}(\sigma) = \Pi_3(E^{\varphi,k}(\sigma))$
- $\text{buff}(E^{\varphi,k}(\sigma)) = \Pi_2(E^{\varphi,k}(\sigma)$

- $\text{Clean}^{\varphi,k} : tw(\Sigma) \times tw(\Sigma) \rightarrow tw(\Sigma)$
  $$\begin{aligned} \text{Clean}^{\varphi,k}(\sigma_s, \sigma_{ca}) = max_{\preceq_{lex}}(\sigma' \in \text{Get\_Subwords}^{\varphi,k}(\sigma_s, \sigma_{ca}) : \\ \forall \sigma'' \in \text{Get\_Subwords}^{\varphi,k}(\sigma_s, \sigma_{ca}),\ \sigma' \neq \sigma'' \wedge \\ |\sigma'| > |\sigma''| \wedge (\text{index}(\sigma', \sigma_{ca}) \leq \text{index}(\sigma'', \sigma_{ca}))) \end{aligned}$$

  where:
  $\text{index}(\sigma', \sigma_{ca}) = (i \in \mathbb{N} \mid i \in [1, |\sigma_{ca}|] : \sigma_{ca[i]} \neq \sigma'_{[i]})$

- $\text{Get\_Subwords}^{\varphi,k} : tw(\Sigma) \times tw(\Sigma) \rightarrow 2^{tw(\Sigma)}$
  $$\begin{aligned} \text{Get\_Subwords}^{\varphi,k}(\sigma_s, \sigma_{ca}) = \{\sigma'' \in tw(\Sigma) \mid \\ \exists \sigma' \in \text{delayable}^{\varphi}(\sigma_s, \sigma_{ca}), |\sigma'| = k \wedge \\ \exists i, j, k \in \mathbb{N} \wedge 1 \leq i \leq j < k : \\ \sigma'' = \sigma'_{[1 \ldots i-1]} \cdot \sigma'_{[j+1 \ldots k]} \wedge \\ \sigma'_{[1 \ldots i-1]} \cdot \sigma'_{[i \ldots j]} \cdot \sigma'_{[j+1 \ldots k]} \sim_{\varphi} \sigma''\} \end{aligned}$$

799 Definition 10 is recalled below:

A bounded enforcer for a timed property $\varphi \in tw(\Sigma)$, equipped with a buffer of size k, is a function

$$E^{\varphi,k} : tw(\Sigma) \to tw(\Sigma) \times \{\bot, \top\}$$

satisfying the following constraints:

- **Soundness** :

$$\forall \sigma \in tw(\Sigma) : E^{\varphi,k}_{out}(\sigma) \models \varphi \vee E^{\varphi,k}_{out}(\sigma) = \epsilon \tag{SndB}$$

- **Monotonicity**:

$$Mo_1 : \forall \sigma, \sigma' \in tw(\Sigma) : \sigma \preccurlyeq \sigma' \implies E^{\varphi,k}_{out}(\sigma) \preccurlyeq E^{\varphi,k}_{out}(\sigma') \tag{Mo1B}$$

$$Mo_2 : \forall \sigma, \sigma' \in tw(\Sigma) : \sigma \preccurlyeq \sigma', (E^{\varphi,k}_{mode}(\sigma) = \bot \implies E^{\varphi,k}_{mode}(\sigma') = \bot) \tag{Mo2B}$$

- **Transparency**

$$Tr_1 : \forall \sigma \in tw(\Sigma), \neg\exists \sigma' \in tw(\Sigma) : (\sigma' \preccurlyeq \sigma \wedge delayable_\varphi(\epsilon, \sigma) \neq \emptyset) \vee degraded \implies E^{\varphi,k}_{out}(\sigma) \lhd_d \sigma \tag{Tr1B}$$

$$Tr_2 : \forall \sigma \in tw(\Sigma), \exists \sigma' \in tw(\Sigma) : (\sigma' \preccurlyeq \sigma \wedge delayable_\varphi(\epsilon, \sigma) \neq \emptyset) \wedge nominal \implies E^{\varphi,k}_{out}(\sigma) \preccurlyeq_d \sigma \tag{Tr2B}$$

where:

- $degraded = (E^{\varphi,k}_{mode}(\sigma) = (\bot \vee stop))$
- $nominal = \neg degraded$

D.1 Sketch of the proof (of Proposition 6)

Proposition: Given some property φ, its enforcement function E^φ as per Definition 5 satisfies the Soundness, Monotonicity, and Transparency constraints as per Definition 4.

Sketch of the proof of proposition 6:

The proof of **??** is straightforward by noticing that function store$^\varphi$ is monotonic on its first output $(\forall \sigma, \sigma' \in tw(\Sigma) : \sigma \preccurlyeq \sigma' \implies \Pi_1(\text{store}^\varphi(\sigma)) \preccurlyeq \Pi_1(\text{store}^\varphi(\sigma')))$.
Let us prove **??**, **??**, and **??** using induction on the input sequence σ.
Induction basis. If $\sigma = \epsilon$, store$^\varphi(\epsilon) = (\epsilon, \epsilon)$. The proposition holds trivially.
Induction step. We assume for $\sigma \in tw(\Sigma)$, if store$^\varphi(\sigma) = (\sigma_s, \sigma_c)$ and $\sigma_c \neq \epsilon$, then this proposition holds.
Let us prove, it holds for all $(t, a) \in (\mathbb{R}_{\geq 0} \times \Sigma)$. Following definition of store$^\varphi$, We distinguish two cases:

- Case 1: if $k^\varphi(\sigma_s, \sigma_{ca}) \neq \emptyset$
  In this case, we have $E^\varphi(\sigma \cdot (t, a)) = \sigma_s \cdot min_{\preceq_{lex}, end}(k^\varphi(\sigma_s, \sigma_{ca}))$ and from definition of k^φ, we have $k^\varphi(\sigma_s, \sigma_{ca}) \in \sigma_s^{-1} \cdot \varphi$. Thus, $E^\varphi(\sigma \cdot (t, a)) \models \varphi$ and **??** holds.

Since, $k^{\varphi}(\sigma_s, \sigma_{ca}) \neq \emptyset$, delayable$_{\varphi}(\sigma_s, \sigma_{ca}) \neq \emptyset$, Thus, **??** holds trivially.
Since, delayable$_{\varphi}(\sigma_s, \sigma_{ca}) \neq \emptyset$ and $E^{\varphi}(\sigma \cdot (t,a)) = \sigma_s \cdot min_{\preceq_{lex},end}(k^{\varphi}(\sigma_s, \sigma_{ca})) \preccurlyeq_d \sigma$ (from Definition of k^{φ}), **??** holds.

- Case 1: if $k^{\varphi}(\sigma_s, \sigma_{ca}) = \emptyset$
  In this case, we have $E^{\varphi}(\sigma \cdot (t,a)) = \sigma_s$. Since, from induction hypothesis, we assume that the proposition holds for σ, meaning when $E^{\varphi}(\sigma) = \sigma_s$, **??**, **??**, and **??** holds; Thus, when $E^{\varphi}(\sigma \cdot (t,a)) = \sigma_s$ **??**, **??**, and **??** will hold too.

D.2 Sketch of the proof (of Proposition 7)

Proposition [Optimal Suppression]: Given some property φ, its enforcement function E^{φ} as per Definition 5 satisfies the following constraint:

$$\begin{aligned}&\forall \sigma \in tw(\Sigma),\\ &\quad \exists \sigma_s, \sigma_c \in tw(\Sigma) : \mathrm{store}^{\varphi}(\sigma) = (\sigma_s, \sigma_c) \wedge\\ &\quad \forall (t,a) \in (\mathbb{R}_{\geq 0} \times \Sigma),\ t \geq end(\sigma_c) :\\ &\quad (\mathrm{delayable}_{\varphi}(\sigma_s, \sigma_c \cdot (t,a)) = \emptyset \implies \forall \sigma_{\mathrm{con}} \in tw(\Sigma) : start(\sigma_{\mathrm{con}}) \geq t,\\ &\qquad\qquad E^{\varphi}(\sigma \cdot (t,a) \cdot \sigma_{\mathrm{con}}) = E^{\varphi}(\sigma \cdot \sigma_{\mathrm{con}}))\end{aligned} \tag{\textbf{Opts}}$$

Sketch of the proof of Proposition 7:

Let us prove this using induction on the input sequence σ.
Induction basis. If $\sigma = \epsilon$ and $t = \epsilon$, store$^{\varphi}(\epsilon) = (\epsilon, \epsilon)$, the proposition holds trivially.
Induction step. We assume for $\sigma \in tw(\Sigma)$, store$^{\varphi}(\sigma) = (\sigma_s, \sigma_c)$.
Let us proove, it holds for all $(t,a) \in (\mathbb{R}_{\geq 0} \times \Sigma)$. Following definition of store$^{\varphi}$, we have three cases:

- Case 1: if $k^{\varphi}(\sigma_s, \sigma_{ca}) \neq \emptyset$
  In this case, we will have a set of delayed words w of σ_{ca} s.t. $\sigma_s \cdot w \in \varphi$. Thus, delayable$_{\varphi}$ will also be $\neq \emptyset$. Thus, proposition holds trivially.
- Case 2: if $k^{pref}(\sigma_s, \sigma_{ca}) = \emptyset$
  This is the case, where E^{φ} will suppress the event. Since $k^{pref}(\sigma_s, \sigma_{ca}) = \emptyset$, delayable$_{\varphi}$ will also be $= \emptyset$, leading to suppression of (t,a). Thus, proposition holds.
- Case 3: if $k^{\varphi}(\sigma_s, \sigma_{ca}) = \emptyset$ and $k^{pref}(\sigma_s, \sigma_{ca}) \neq \emptyset$
  Since, $k^{pref}(\sigma_s, \sigma_{ca}) \neq \emptyset$, thus, delayable$_{\varphi}$ will also be $\neq \emptyset$. Thus, proposition holds trivially.

D.3 Sketch of the proof (of Proposition 12)

Proposition: Given some property φ and the maximum buffer size k, let $n \in \mathbb{N}$ be the number of locations in $\mathcal{A}_{\varphi}$. If $k \geq n$, then the enforcement function $E^{\varphi,k}$ as per Definition 11 satisfies the Soundness, Monotonicity, and Transparency constraints as per Definition 10.

Sketch of the proof of proposition 12:

The proof of **??** is straightforward by noticing that function store$^{\varphi,k}$ is monotonic on its first output $(\forall \sigma, \sigma' \in tw(\Sigma) : \sigma \preccurlyeq \sigma' \implies \Pi_1(\mathrm{store}^{\varphi,k}(\sigma)) \preccurlyeq \Pi_1(\mathrm{store}^{\varphi,k}(\sigma')))$.
The proof of **??** is straightforward by noticing that function store$^{\varphi,k}$ does not bring the

nominal mode back in any of the cases once the mode has changed to $\bot$.
Let us now prove **??**, **??**, and **??** by an induction on the length of the input timed word σ.
Induction basis. If $\sigma = \epsilon$, $\text{store}^{\varphi,k}(\epsilon) = (\epsilon, \epsilon, \top)$. Since, $E^{\varphi,k}_{\text{out}}(\sigma) = \Pi_1(E^{\varphi,k}(\sigma)) = \Pi_1(\text{store}^{\varphi,k}(\sigma) = \epsilon$, thus **??**, **??**, and **??** holds trivially.
Induction step. We assume for $\sigma \in tw(\Sigma)$, if $\text{store}^{\varphi,k}(\sigma) = (\sigma_s, \sigma_c)$ and $\sigma_c \neq \epsilon$, then this proposition holds.
Let us prove, it holds for all $(t, a) \in (\mathbb{R}_{\geq 0} \times \Sigma)$. Following definition of $\text{store}^{\varphi,k}$, we have following cases to examine:

- Case 1: $k^{\varphi}(\sigma_s, \sigma_{ca}) \neq \emptyset$
  $E^{\varphi,k}_{\text{out}}(\sigma \cdot (t, a)) = \sigma_s \cdot min_{\preceq_{lex},end}(k^{\varphi}(\sigma_s, \sigma_{ca})$.
  Since, from definition of k^{φ}, $k^{\varphi}(\sigma_s, \sigma_{ca}) \in \sigma_s^{-1} \cdot \varphi$, thus, **??** holds.
  Since, $k^{\varphi}(\sigma_s, \sigma_{ca}) \neq \emptyset$, $\text{delayable}_{\varphi}(\sigma_s, \sigma_{ca})$ will also be $\neq \emptyset$ and $E^{\varphi,k}_{\text{mode}}(\sigma \cdot (t, a)) = mode = \top \neq$ degraded, Thus, **??** holds trivially.
  Since, $\text{delayable}_{\varphi}(\sigma_s, \sigma_{ca}) \neq \emptyset$ and $E^{\varphi,k}_{\text{mode}}(\sigma \cdot (t, a)) = mode = \top =$ nominal and $E^{\varphi,k}(\sigma \cdot (t, a)) = \sigma_s \cdot min_{\preceq_{lex},end}(k^{\varphi}(\sigma_s, \sigma_{ca})) \preccurlyeq_d \sigma \cdot (t, a)$ (from Definition of k^{φ}), **??** holds.

- Case 2: $k^{pref(\varphi)}(\sigma_s, \sigma_{ca}) = \emptyset$
  Since, from induction hypothesis, for σ, $E^{\varphi,k}(\sigma) = \sigma_s$ we assumed that **??** holds; and in this case, we have $E^{\varphi,k}(\sigma \cdot (t, a)) = \sigma_s$, thus **??** holds here too.
  Since, $k^{pref(\varphi)}(\sigma_s, \sigma_{ca}) = \emptyset$ thus $\text{delayable}_{\varphi}(\sigma_s, \sigma_{ca}) = \emptyset$, thus, **??** holds trivially.
  Since, $\text{delayable}_{\varphi}(\sigma_s, \sigma_{ca}) = \emptyset$, (t, a) have been suppressed and $E^{\varphi,k}_{\text{out}}(\sigma \cdot (t, a)) = \sigma_s \triangleleft_d \sigma \cdot (t, a)$, thus **??** holds.

- Case 3: $k^{pref(\varphi)}(\sigma_s, \sigma_{ca}) \neq \emptyset \ \wedge |\sigma_{ca}| \leq k$
  Same as case 2, **??** holds here too.
  Since, $k^{pref(\varphi)}(\sigma_s, \sigma_{ca}) \neq \emptyset$, thus $\text{delayable}_{\varphi}(\sigma_s, \sigma_{ca}) \neq \emptyset$, and $E^{\varphi,k}_{\text{mode}}(\sigma \cdot (t, a)) = mode = \top =$ nominal. Thus, **??** holds trivially.
  Also, $\text{delayable}_{\varphi}(\sigma_s, \sigma_{ca}) \neq \emptyset$ and $E^{\varphi,k}_{\text{mode}}(\sigma \cdot (t, a)) = \top$. $E^{\varphi,k}(\sigma \cdot (t, a)) = \sigma_s \preccurlyeq_d \sigma \cdot (t, a)$ (from Definition of k^{φ}), thus, **??** holds.

- Case 4: $k^{pref(\varphi)}(\sigma_s, \sigma_{ca}) \neq \emptyset \ \wedge \ |\sigma_{ca}| > k \wedge \ \text{Get\_Subwords}^{\varphi,k}(\sigma_s, \sigma_{ca}) = \emptyset$
  Same as case 2, **??** holds here too.
  Since, $E^{\varphi,k}_{\text{mode}}(\sigma \cdot (t, a)) = stop = degraded$ and $E^{\varphi,k}_{\text{out}}(\sigma) = \sigma_s \triangleleft_d \sigma \cdot (t, a)$, thus **??** holds.
  Since, $E^{\varphi,k}_{\text{mode}}(\sigma \cdot (t, a)) = stop \neq nominal$, **??** holds trivially.

- Case 5: $k^{pref(\varphi)}(\sigma_s, \sigma_{ca}) \neq \emptyset \ \wedge |\sigma_{ca}| > k \wedge \ \text{Get\_Subwords}^{\varphi,k}(\sigma_s, \sigma_{ca}) \neq \emptyset$
  Same as case 2, **??** holds here too.
  Since, $E^{\varphi,k}_{\text{mode}}(\sigma \cdot (t, a)) = \bot = degraded$, **??** and **??** holds in this case too; the reasoning is similar to the one provided for the above case.

D.4 Informal proof (of Proposition 18)

Proposition: Given some property φ and the maximum buffer size k, its enforcement function $E^{\varphi,k}$ as per Definition 11 satisfies **Opt1B** and **Opt2B** properties.

Proof for **Opt2B** can be read from [10].

Informal proof of proposition 18 for **Opt2B**:

If the buffer has not reached its maximum capacity, $F^{\varphi,k}$ has to insert/add other events, in order to produce a longer output than $E^{\varphi,k}$, which Def. 10 prevents. Otherwise when the buffer is exhausted, $F^{\varphi,k}$ has to produce maximal output by removing the least number of events such that ∞-compatible($F^{\varphi,k}$) holds, which is already ensured by $\text{clean}^{\varphi,\text{k}}$ of Def. 11. Thus, the proposition holds.