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DS 203 : Assignment 1

Exercise 1

1. Probability that the server is working is 0.8
if server is working, probability that each access attempt is successful = 0.9
P(first access attempt fails)
case 1 - server is not working
probability = 0.2
Case-2:- server working but access attempt fails
probability = $0.8 \times (1-0.9)$
$= 0.8 \times 0.1 = 0.08$

$\therefore$ $\boxed{\text{Total Probability that first access attempt fails} = 0.2 + 0.08 = 0.28}$

~~Ans~~

2. P(server is working | first access attempt fails)

$$= \frac{P(\text{server is working} \cap \text{first access attempt fails})}{P(\text{first access attempt fails})}$$

$$= \frac{0.8 \times (1-0.9)}{0.28}$$

0.8 ← P(server is working)

$(1-0.9)$ ← P(first access attempt fails)

0.28 ← (as calculated in first part)

$$= \frac{0.8 \times 0.1}{0.28} = \frac{\cancel{8} \times 1}{\cancel{28}\,7} \cdot 2 = \frac{2}{7}$$

$\therefore$ $\boxed{P(\text{server is working} \mid \text{first access attempt fails}) = 2/7 = 0.2857}$

3. P(second access attempt fails | first access attempt fails)

$$= \frac{P(\text{second access attempt fails} \cap \text{first access attempt fails})}{P(\text{first access attempt fails})}$$

$$= \frac{P[\text{first two attempts fail}]}{P[\text{first access attempt fails}]}$$

$$= \frac{0.2 + 0.8 \times 0.1 \times 0.1}{0.28}$$

case 1: web server not working

case 2: webserver working but access attempt fail $[1-0.9 = 0.1]$

0.28 → found in part 1.

$$= \frac{0.2 + 0.008}{0.28} = \frac{\cancel{0.2008}}{} \quad \frac{0.208}{0.28} = \frac{\cancel{208}\ \cancel{52}\ 26}{\cancel{280}\ \cancel{70}\ 35}$$

$$= \frac{26}{35} = 0.742857$$

$$\boxed{P(\text{second access attempt fails} \mid \text{first access attempt fails}) = \frac{26}{35} = 0.742857}$$

4. P[server is working | first and second attempt fails]

$$= \frac{P[\text{server is working} \cap \text{first and second attempt fails}]}{P[\text{first and second attempt fails}]}$$

$$= \frac{0.8 \times 0.1 \times 0.1}{0.2 + 0.8 \times 0.1 \times 0.1}$$

server working

first attempt fail

second attempt fail

same as numerator of 3rd part

$$= \frac{0.008}{0.2 + 0.008} = \frac{0.008}{0.208} = \frac{\cancel{8}\ 1}{\cancel{208}\ 26} = \frac{1}{26} = \cancel{0.03846}$$

$$\boxed{P[\text{server is working} \mid \text{first and second attempt fails}] = 1/26 = 0.03846}$$

Ans
1. 0.28
2. 2/7 = 0.2857
3. 26/35 = 0.742857
4. 1/26 = 0.03846

Exercise 2

$P_X(K) = 0.2$ if $3 \le K \le 7$
$P_X(K) = 0$ otherwise

X represents the lifetime rounded up to an integer number of years, of a car battery.

(i) Probability that a 3 year old battery is still working
$= 1 -$ probability that is doesn't work after 3 years
$= 1 - P_X(3)$
$= 1 - 0.2$
$= \boxed{0.8}$

(ii) $P(X>8 \mid X>5)$

~~For X>8~~ ~~Fore~~

For more than 8 years, least rounded up integer no. of years would be 8

and $P_X(8) = 0$
$P(X>8) = 0$
$\therefore \boxed{P(X>8 \mid X>5) = 0}$

Ans. i) 0.8
ii) 0

Exercise 3

shot is a success with probability p & miss with probability (1-p)
Y is the no. of shots required for first success.

(i) $P(Y>3) = 1 - P(Y\le 3) = 1 - (P(Y=1) + P(Y=2) + P(Y=3))$

$$= 1 - (\underbrace{p}_{\text{1st shot success}} + \underbrace{(1-p)}_{\text{1st miss}}\underbrace{p}_{\text{2nd success}} + \underbrace{(1-p)(1-p)}_{\text{1st \& 2nd shot miss}}\underbrace{p}_{\text{3rd success}})$$

$= 1 - p(1 + (1-p) + (1-p)^2)$
$= 1 - p(1 + 1 - p + 1 + p^2 - 2p)$

$$P(Y>3) = 1 - p(3 - 3p + p^2)$$
$$= 1 - 3p + 3p^2 - p^3$$
$$\boxed{P(Y>3) = (1-p)^3}$$

For a general case $Y > n$,

$$P(Y>n) = 1 - (p + (1-p)p + \dots + (1-p)^{n-1}p)$$
$$= 1 - p(1 + (1-p) + (1-p)^2 \dots (1-p)^{n-1})$$
$$= 1 - p\left(\frac{1.(1-(1-p)^n)}{1-(1-p)}\right)$$
$$= 1 - \cancel{p}\left(\frac{1-(1-p)^n}{\cancel{p}}\right) = 1 - (1 - (1-p)^n)$$

$$P(Y>n) = (1-p)^n$$

(ii) $P(Y>8 \mid Y>5)$

$$= \frac{P(Y>8 \cap Y>5)}{P(Y>5)}$$
$$= \frac{P(Y>8)}{P(Y>5)}$$
$$= \frac{(1-p)^8}{(1-p)^5} = (1-p)^3$$

$$\boxed{P(Y>8 \mid Y>5) = (1-p)^3}$$

(iii) $Y = \{1, 2, 3, 4, \dots\}$

$P(Y=1) = p$ $\quad$ $P(Y=3) = (1-p)^2 p$

$P(Y=2) = (1-p)p$ $\quad$ $\therefore P(Y=i) = (1-p)^{i-1}p \quad (i \geq 1)$

$\therefore$ The type of probability distribution of Y is $\boxed{\text{Discrete RV of type Geometric}}$

Ans :- (i) $(1-p)^3$

(ii) $(1-p)^3$

(iii) Geometric type, (Discrete RV)

Exercise 4

(a) event E is independent of itself

$\therefore P(E \cap E) = P(E) \cdot P(E)$

$E \cap E \equiv E$

$P(E) = P(E) \cdot P(E)$

$P(E)(1 - P(E)) = 0$

$\therefore P(E) = 0$ or $1 - P(E) = 0$

i.e. $P(E) = 1$

$\therefore P(E) = 0$ or $P(E) = 1$

Hence shown.

(b) $P(A) = 0.3$

$P(B) = 0.4$

i) If A and B are independent

$P(A \cap B) = P(A) \cdot P(B)$

$P(A \cap B) = 0.3 \times 0.4 = 0.12$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.3 + 0.4 - 0.12$

$= 0.7 - 0.12$

$\boxed{P(A \cup B) = 0.58}$ → when A & B are independent

ii) If A and B are mutually exclusive.

$P(A \cap B) = 0$

$P(A \cup B) = P(A) + P(B)$

$= 0.3 + 0.4 = 0.7$

$\boxed{P(A \cup B) = 0.7}$ → when A and B are mutually exclusive.

(c) $P(A) = 0.6$, $P(B) = 0.8$

If the events are independent,

$P(A \cap B) = P(A) \cdot P(B) = 0.48$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.6 + 0.8 - 0.48$

$= 0.92$

$P(A \cup B) < 1$

$\boxed{\text{Hence the events could be independent.}}$

If the events are mutually exclusive,

$P(A \cup B) = P(A) + P(B)$

$= 1.4 > 1$

which is not possible.

$\boxed{\text{Hence the events cannot be mutually exclusive.}}$

Exercise 5

1. $F(x) = \begin{cases} \dfrac{e^{-x^2}}{4} & \text{if } x < 0 \\ 1 - \dfrac{e^{-x^2}}{4} & \text{if } x \geq 0 \end{cases}$

This is a $\boxed{\text{valid CDF}}$.

$$\begin{aligned} P(X^2 > 5) &= 1 - P(X^2 \leq 5) \\ &= 1 - P(-\sqrt{5} \leq X \leq \sqrt{5}) \\ &= 1 - (F(\sqrt{5}) - F(-\sqrt{5})) \\ &= 1 - \left(1 - \frac{e^{-5}}{4} - \frac{e^{-5}}{4}\right) \\ &= \frac{2e^{-5}}{4} = \frac{e^{-5}}{2} \end{aligned}$$

$$\boxed{P(X^2 > 5) = \frac{e^{-5}}{2} = 0.003369}$$

2. $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 0.5 + e^{-x} & \text{if } 0 \le x < 3 \\ 1 & \text{if } x \geq 3 \end{cases}$

This is $\boxed{\text{not a valid CDF.}}$ CDF $F_X(x)$ should be non-decreasing in x but this function is a decreasing function in the interval $0 \le x \le 3$.

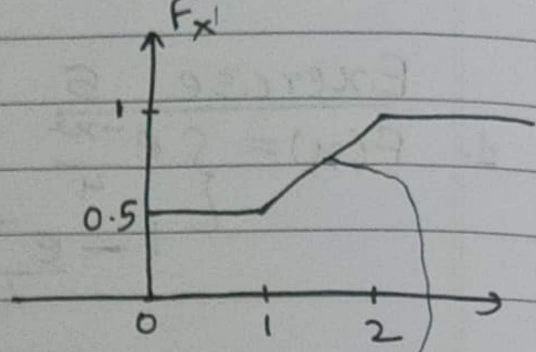
3. $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 0.5 + x/20 & \text{if } 0 \le x \le 10 \\ 1 & \text{if } x \geq 10 \end{cases}$

This is a $\boxed{\text{valid CDF.}}$

$$P(X^2 > 5) = 1 - P(X^2 \le 5)$$
$$= 1 - P(-\sqrt{5} \le x \le \sqrt{5})$$
$$= 1 - (F(\sqrt{5}) - F(-\sqrt{5}))$$
$$= 1 - \left(0.5 + \frac{\sqrt{5}}{20}\right)$$
$$= 0.5 - \frac{\sqrt{5}}{20} = \frac{1}{2} - \frac{\sqrt{5}}{20}$$
$$= \frac{10 - \sqrt{5}}{20}$$

$$\boxed{P(X^2 > 5) = \frac{10 - \sqrt{5}}{20} = 0.3882}$$

Exercise 6

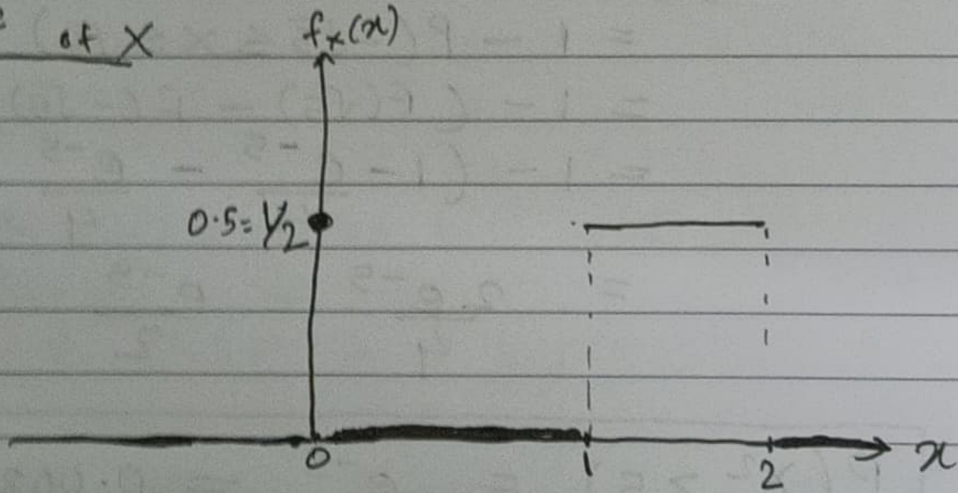


equation: $y - 1 = 0.5(x-2)$
$y = 0.5x$
slope $= 0.5$

1. $P(X \le 0.8)$
$= F_X(0.8)$
$= 0.5$ (using graph)

$$\boxed{P(X \le 0.8) = 0.5}$$

2. PDF of X



$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$= \int_1^2 x \cdot \tfrac{1}{2}\, dx = \left.\frac{x^2}{4}\right|_1^2 = \frac{3}{4}$$

$$\boxed{E(X) = \frac{3}{4} = 0.75}$$

3. $Var(X) = E(X^2) - (E(X))^2$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \int_1^2 x^2 \cdot \frac{1}{2}\, dx$$

$$= \left. x^3/6 \right|_1^2 = 7/6$$

$$E(X^2) - (E(X))^2 = 7/6 - (3/4)^2 = 7/6 - 9/16$$

$$= \frac{56 - 27}{48} = 29/48$$

$$\boxed{Var(X) = 29/48 = 0.60416667}$$

<u>Exercise 7</u>

$$f(x) = \begin{cases} ce^{-2x} & \text{if } 0 \leq x < \infty \\ 0 & \text{if } x < 0 \end{cases}$$

$$\int_{-\infty}^{\infty} f_x(x)\,dx = 1$$

$$\int_0^{\infty} ce^{-2x}\,dx = 1$$

$$c\frac{e^{-2x}}{-2}\Big|_0^{\infty} = 1$$

$$0 - \left(-\frac{c}{2}\right) = 1$$

$$\frac{c}{2} = 1$$

$$\boxed{c = 2}$$

$$P(X>2) = \int_2^{\infty} ce^{-2x}\,dx$$

$$= c\frac{e^{-2x}}{-2}\Big|_2^{\infty}$$

$$= 0 - \left(\frac{-ce^{-4}}{2}\right)$$

$$= \frac{ce^{-4}}{2} = \frac{2 \times e^{-4}}{2} = e^{-4}$$

$$\boxed{\cancel{P(X>2) = e^{-4}}}$$

$$\boxed{P(X>2) = e^{-4} \approx 0.0183}$$

Exercise 8

Probability of heads = 0.7
tossed 3 times.

X = no. of heads that appear in the three tosses

$P(X=0) = (0.3)^3 = 0.027$

$P(X=1) = {}^3C_1 \times 0.7 \times (0.3)^2 = 0.189$

$P(X=2) = {}^3C_2 \times (0.7)^2 \times 0.3 = 0.441$

$P(X=3) = (0.7)^3 = 0.343$

Ans :- $\boxed{\begin{array}{ll} P(0) = 0.027 & P(1) = 0.189 \\ P(2) = 0.441 & P(3) = 0.343 \end{array}}$

Exercise 9

$$f_X(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$P(X \ge 0.4 \mid X \le 0.8) = \frac{P(X \ge 0.4 \cap X \le 0.8)}{P(X \le 0.8)}$$

$$= \frac{P(0.4 \le X \le 0.8)}{P(X \le 0.8)}$$

$$= \frac{\int_{0.4}^{0.8} 2x\,dx}{\int_{0}^{0.8} 2x\,dx} = \frac{x^2 \Big|_{0.4}^{0.8}}{x^2 \Big|_{0}^{0.8}}$$

$$= \frac{(0.8)^2 - (0.4)^2}{(0.8)^2}$$

$$= 1 - (1/2)^2$$

$$= 1 - 1/4 = 3/4$$

Ans :- $\boxed{P(X \ge 0.4 \mid X \le 0.8) = 3/4 = 0.75}$

Exercise 10

X is exponentially distributed random variable with parameter λ.

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$P(X > a+b \mid X > a)$ $\quad a, b > 0$

$= \dfrac{P(X > a+b \cap X > a)}{P(X > a)}$ $\quad a+b > a$

$= \dfrac{P(X > a+b)}{P(X > a)}$

$$= \frac{\int_{a+b}^{\infty} \lambda e^{-\lambda x}\, dx}{\int_{a}^{\infty} \lambda e^{-\lambda x}\, dx} = \frac{\frac{e^{-\lambda x}}{-\lambda}\Big|_{a+b}^{\infty}}{\frac{e^{-\lambda x}}{-\lambda}\Big|_{a}^{\infty}}$$

$$= \frac{0 - \left(\frac{e^{-\lambda(a+b)}}{-\lambda}\right)}{0 - \left(\frac{e^{-\lambda a}}{-\lambda}\right)} = \frac{e^{-\lambda(a+b)}}{e^{-\lambda a}} = e^{-\lambda b}$$

$$\boxed{P(X > a+b \mid X > a) = e^{-\lambda b}}$$

Exercise 11

Five coins tossed, $E \to$ all coin land heads.

$$I_E = \begin{cases} 1 & \text{if } E \text{ occurs} \\ 0 & \text{if } E^c \text{ occurs} \end{cases}$$

For $\{H, H, H, H, H\}$ that is 5 heads, I_E equals to 1 (since I_E is 1 when E occurs and E is the event in which all coin land heads).

$$P\{I_E = 1\} = P(\text{all 5 heads})$$
$$= \left(\tfrac{1}{2}\right)^5$$
$$= 1/32 = 0.03125$$

$$\boxed{P\{I_E = 1\} = 1/32 = 0.03125}$$

Exercise 12

$$F(b) = \begin{cases} 0 & b < 0 \\ 1/2 & 0 \le b < 1 \\ 1 & 1 \le b < \infty \end{cases}$$

$P(b=0) = 1/2 = 0.5$ ($\because$ a jump of 0.5 in cdf at $b=0$)

$P(b=1) = 1/2 = 0.5$ ($\because$ a further jump of 0.5 so total 1 in ~~at~~ cdf at $b=1$)

Ans :- $\boxed{\begin{array}{l} P(b=0) = 0.5 \\ P(b=1) = 0.5 \\ P=0 \text{ for rest other } b \end{array}}$

Exercise 13

urn → 3 white balls, 3 black balls.

P(to draw white ball) = 3/6 = 1/2

P(to draw black ball) = 3/6 = 1/2

After a ball is drawn, it is replaced so for the next draw the probability to draw respective balls remains same.

Out of first 4 balls, exactly two are white

~~WWHH~~ ~~WWBB~~, ~~WBW~~

There are $4C_2$ ways of drawing exactly 2 white

For each case probability to draw that 4 balls corresponding to that case $= (1/2)^4 = 1/16$

$4C_2$ cases

So total probability $= 4C_2 \times 1/16 = 6 \times 1/16 = 3/8$

$= \boxed{3/8}$

Ans $\boxed{\text{Probability that of the first 4 balls drawn, exactly 2 are white is } 3/8.}$

Exercise 14

Probability of heads $= p$, flipped until r^{th} head appears

X = number of flips required.

$P(X=n)$ = n flips required to get r heads and n^{th} flip is head.

So the last flip is head and rest $r-1$ heads are in the $n-1$ flips.

So $^{n-1}C_{r-1}$ ways of choosing which flips would be head.

Probability of r heads $= p^r$ and probability of remaining $n-r$ not being heads $= (1-p)^{n-r}$.

$$\therefore P(X=n) = {}^{n-1}C_{r-1} \cdot p^r (1-p)^{n-r}$$

$$\boxed{P(X=n) = \binom{n-1}{r-1} p^r (1-p)^{n-r}, \quad n \geq r}$$

Hence the given argument is true.