Multi-target Tracking via Mixed Integer Optimization

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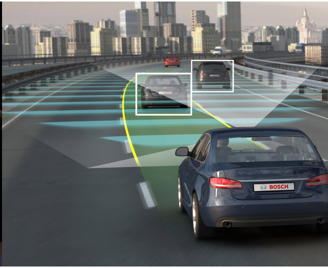
Overview

Section 1

Motivation

Multi-target Tracking (MTT) Applications





- Ballistic missile and aircraft defense
- Space applications
- Movement of ships and ground troops
- Autonomous vehicles and robotics
- Air traffic control

Background

- Predominantly statistically based approaches
- Rely on heavy probabilistic assumptions
  - radar detection process
  - underlying target dynamics/features
- Two most prevalent algorithms:
  - Multiple Hypothesis Tracker (MHT)
  - Joint Probability Data Association Filter (JPDAF)
- Little to no emphasis on optimization

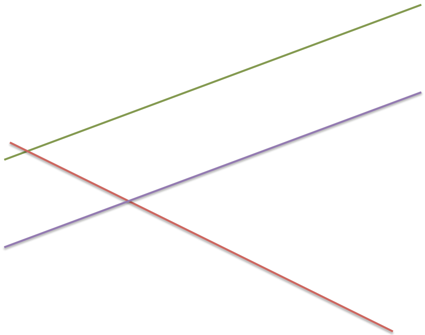
Our contributions

(i) Introduce novel approach using MIO models
(ii) Propose random local search heuristics
(iii) New measure of complexity and performance

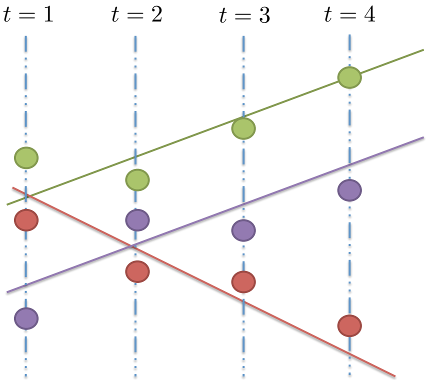
Section 2

Problem Description

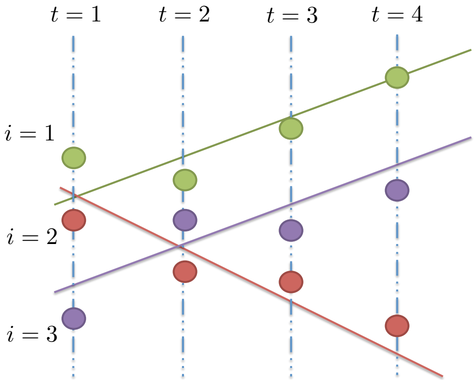
The MTT Problem



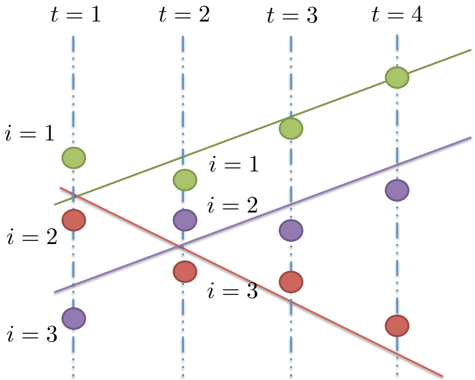
The MTT Problem



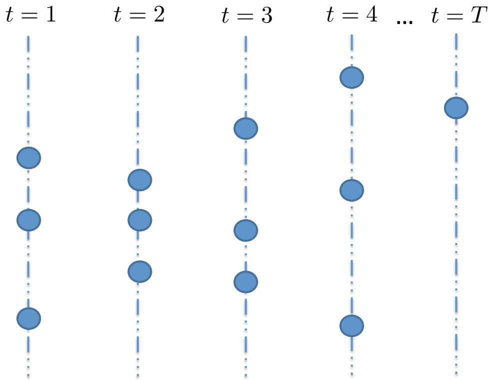
The MTT Problem



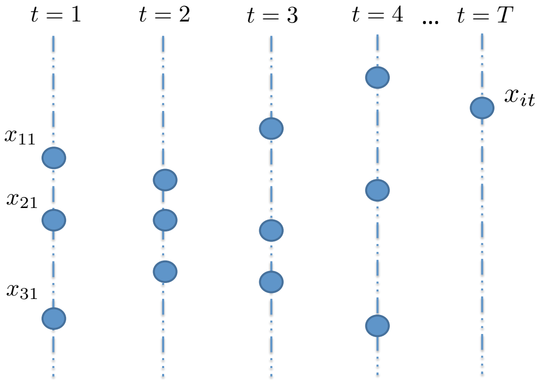
The MTT Problem



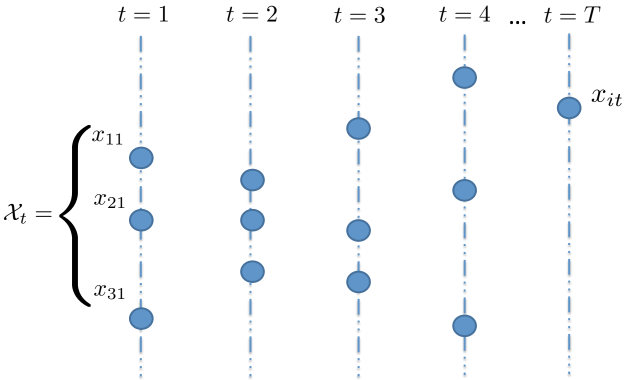
Notation



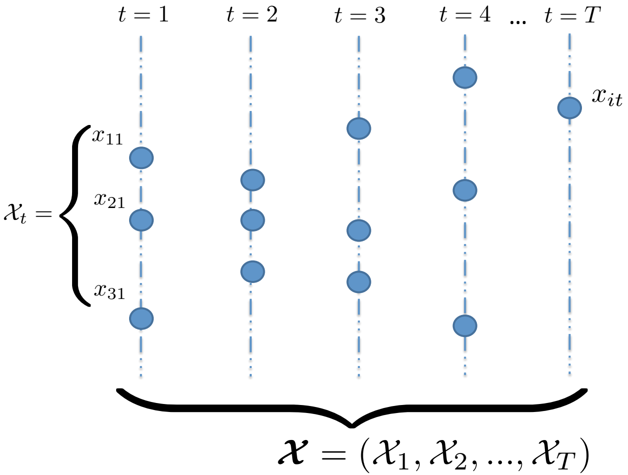
Notation



Notation



Notation


$t = 1$
$t = 2$
$t = 3$
$t = 4$
...
$t = T$
x_{it}
x_{11}
x_{21}
x_{31}
$\mathcal{X}_t =$
$\boldsymbol{\mathcal{X}} = (\mathcal{X}_1, \mathcal{X}_2, ..., \mathcal{X}_T)$

General Assumptions

Assumption

(i) *All targets have constant velocity*

(ii) *Each target's dynamics are independent*

(iii) *The number of targets remains constant*

(iv) *The detection errors are independent*

Section 3

MIO Models

Assumptions for case of no detection ambiguity

Assumption

(i) *The sensor generates exactly one detection for each target in each scan i.e., no missed detections.*

(ii) *The sensor does not generate any spurious detections i.e., no false alarms.*

Corollary

The number of targets is known/fixed, denoted by P

Decision Variables

Assignment variables:

$$y_{itj} = \begin{cases} 1, & \text{if detection } x_{it} \text{ is assigned to trajectory } j, \\ 0, & \text{otherwise.} \end{cases}$$

Trajectory estimation variables:

Estimated initial position of trajectory j: $\alpha_j \in \mathbb{R}^n$
Estimated velocity of trajectory j: $\beta_j \in \mathbb{R}^n$

Objective Function

$$\underset{y_{itj},\alpha_j,\beta_j}{\text{minimize:}} \quad \sum_{j=1}^{P} \sum_{t=1}^{T} \left\| \sum_{i=1}^{P} y_{itj} x_{it} - (\alpha_j + \beta_j t) \right\|$$

- $\sum_{i=1}^{P} y_{itj} x_{it}$ = detection associated with target j at time t
- Measures cost of assignment
- ℓ_1 vs. ℓ_2

Constraints

For each target and each scan, each detection must be assigned to exactly one target:

$$\sum_{j=1}^{P} y_{itj} = 1 \qquad \forall i, t.$$

Similarly, for each scan, each target must be assigned exactly one detection:

$$\sum_{i=1}^{P} y_{itj} = 1 \qquad \forall j, t.$$

Overall Formulation

$$
\begin{aligned}
\underset{\psi_{jt}}{\text{minimize:}} \quad & \sum_{j=1}^{P} \sum_{t=1}^{T} \psi_{jt} \\
\text{subject to:} \quad & \sum_{j=1}^{P} y_{itj} = 1 \qquad \forall i, t \\
& \sum_{i=1}^{P} y_{itj} = 1 \qquad \forall j, t \\
& \sum_{i=1}^{P} y_{itj} x_{it} - \alpha_j - \beta_j t \leq \psi_{jt} \qquad \forall j, t \\
& -\left(\sum_{i=1}^{P} y_{itj} x_{it} - \alpha_j - \beta_j t \right) \geq \psi_{jt} \qquad \forall j, t \\
& y_{itj} \in \{0, 1\} \quad \forall i, t, j \\
& \alpha_j \in \mathbb{R}^n \quad \forall j, \quad \beta_j \in \mathbb{R}^n \quad \forall j
\end{aligned}
$$

A need to generalize

When extending to detection ambiguity, there is no one to one assignment, so

$$\sum_{i=1}^{P} y_{itj} x_{it} \neq \text{detection associated with target } j \text{ at time } t.$$

Alternate Approach

$$z_{jt} = \begin{cases} x_{it}, & \text{if } y_{itj} = 1, \\ \mathit{free}, & \text{otherwise.} \end{cases}$$

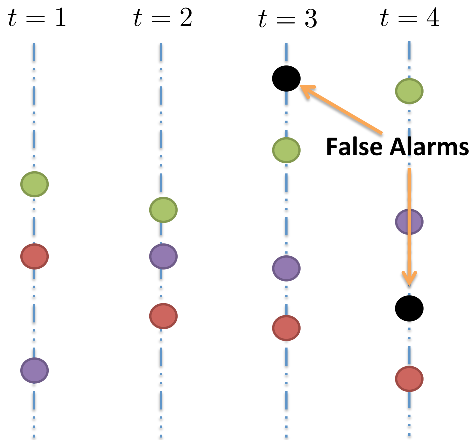
$$\underset{z_{jt},\alpha_j,\beta_j}{\text{minimize:}} \sum_{j=1}^{P} \sum_{t=1}^{T} \|z_{jt} - \alpha_j - \beta_j t\|$$

$$M_t(1 - y_{itj}) \geq |z_{jt} - x_{it} y_{itj}| \qquad \forall i, t, j$$

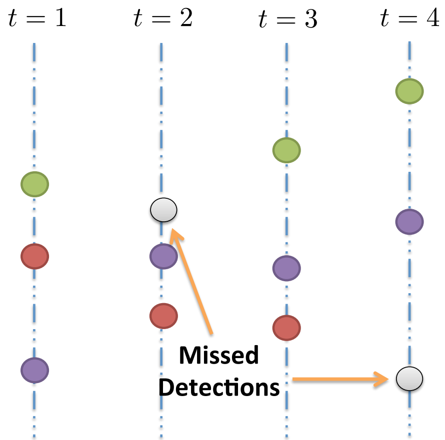
Generalized Formulation

$$
\begin{aligned}
\underset{\psi_{jt}}{\text{minimize:}} \quad & \sum_{j=1}^{P} \sum_{t=1}^{T} \psi_{jt} \\
\text{subject to:} \quad & \sum_{j=1}^{P} y_{itj} = 1 \qquad \forall i, t \\
& \sum_{i=1}^{P} y_{itj} = 1 \qquad \forall j, t \\
& x_{it} y_{itj} + M_t(1 - y_{itj}) \geq z_{jt} \qquad \forall i, t, j \\
& x_{it} y_{itj} - M_t(1 - y_{itj}) \leq z_{jt} \qquad \forall i, t, j \\
& z_{jt} - \alpha_j - \beta_j t \leq \psi_{jt} \qquad \forall i, j, t \\
& -(z_{jt} - \alpha_j - \beta_j t) \geq \psi_{jt} \qquad \forall i, j, t \\
& y_{itj} \in \{0, 1\} \quad \forall i, t, j \\
& \alpha_j \in \mathbb{R}^n \quad \forall j, \quad \beta_j \in \mathbb{R}^n \quad \forall j, \quad z_{jt} \in \mathbb{R}^n \quad \forall j, t
\end{aligned}
$$

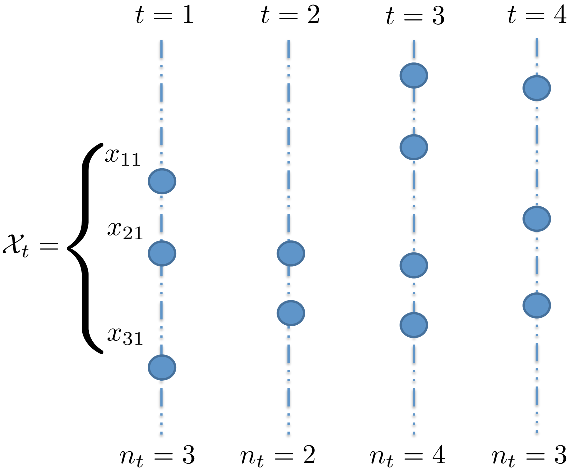
Detection Ambiguity - False Alarms



Detection Ambiguity - Missed Detections



Detection Ambiguity Notation



Assumptions for Case of Detection Ambiguity

$$N_0 = \min_t \ n_t$$

$$N_1 = \max_t \ n_t$$

Assumption

(i) *The sensor generates at most one detection for each target in each scan i.e., there can be missed detections.*

(ii) *The sensor can generate detections that do not originate from any target i.e., there can be false alarms.*

(iii) *The number of true targets P satisfies $N_0 \leq P \leq N_1$.*

How to handle unknown number of targets P?

Two methodologies:

1. Find optimal P using additional decision variables
2. Find optimal P by:
    - For each $N_0 \leq P \leq N_1$ solve problem using MIO for known P
    - Choose P with best objective function value
    - Parallelizable

Decision Variables

False alarms:

$$F_{it} = \begin{cases} 1, & \text{if detection } i \text{ at time } t \text{ is a false alarm,} \\ 0, & \text{otherwise.} \end{cases}$$

Missed Detections:

$$M_{jt} = \begin{cases} 1, & \text{if detection for trajectory } j \text{ at time } t \text{ is} \\ & \text{a missed detection,} \\ 0, & \text{otherwise.} \end{cases}$$

Generalized Formulation

$$
\begin{aligned}
\underset{\psi_{jt}}{\text{minimize:}} \quad & \sum_{j=1}^{P} \sum_{t=1}^{T} \psi_{jt} \\
\text{subject to:} \quad & \sum_{j=1}^{P} y_{itj} = 1 \qquad \forall i, t \\
& \sum_{i=1}^{P} y_{itj} = 1 \qquad \forall j, t \\
& x_{it} y_{itj} + M_t(1 - y_{itj}) \geq z_{jt} \qquad \forall i, t, j \\
& x_{it} y_{itj} - M_t(1 - y_{itj}) \leq z_{jt} \qquad \forall i, t, j \\
& z_{jt} - \alpha_j - \beta_j t \leq \psi_{jt} \qquad \forall i, j, t \\
& -(z_{jt} - \alpha_j - \beta_j t) \geq \psi_{jt} \qquad \forall i, j, t \\
& y_{itj} \in \{0, 1\} \quad \forall i, t, j \\
& \alpha_j \in \mathbb{R}^n \quad \forall j, \quad \beta_j \in \mathbb{R}^n \quad \forall j, \quad z_{jt} \in \mathbb{R}^n \quad \forall j, t
\end{aligned}
$$

Modification of Constraints

$$
\begin{aligned}
\underset{\psi_{jt}}{\text{minimize:}} \quad & \sum_{j=1}^{P}\sum_{t=1}^{T}\psi_{jt} \\
\text{subject to:} \quad & \sum_{j=1}^{P} y_{itj} + F_{it} = 1 \qquad \forall i, t \\
& \sum_{i=1}^{n_t} y_{itj} + M_{jt} = 1 \qquad \forall j, t \\
& x_{it} y_{itj} + M_t(1 - y_{itj}) \geq z_{jt} \qquad \forall i, t, j \\
& x_{it} y_{itj} - M_t(1 - y_{itj}) \leq z_{jt} \qquad \forall i, t, j \\
& z_{jt} - \alpha_j - \beta_j t \leq \psi_{jt} \qquad \forall i, j, t \\
& -(z_{jt} - \alpha_j - \beta_j t) \geq \psi_{jt} \qquad \forall i, j, t \\
& y_{itj} \in \{0, 1\} \quad \forall i, t, j, \quad F_{it} \in \{0, 1\} \quad \forall i, t, \quad M_{jt} \in \{0, 1\} \quad \forall j, t \\
& \alpha_j \in \mathbb{R}^n \quad \forall j, \quad \beta_j \in \mathbb{R}^n \quad \forall j, \quad z_{jt} \in \mathbb{R}^n \quad \forall j, t,
\end{aligned}
$$

Changes to Objective Function

$$
\begin{aligned}
\underset{\psi_{jt}}{\text{minimize:}} \quad & \sum_{j=1}^{P}\sum_{t=1}^{T}\psi_{jt} + \theta \cdot TF + \phi \cdot TM \\
\text{subject to:} \quad & \sum_{j=1}^{P} y_{itj} + F_{it} = 1 \qquad \forall i, t \\
& \sum_{i=1}^{n_t} y_{itj} + M_{jt} = 1 \qquad \forall j, t \\
& \sum_{i=1}^{n_t}\sum_{t=1}^{T} F_{it} = TF \\
& \sum_{j=1}^{P}\sum_{t=1}^{T} M_{jt} = TM \\
& x_{it}y_{itj} + M_t(1 - y_{itj}) \geq z_{jt} \qquad \forall i, t, j \\
& x_{it}y_{itj} - M_t(1 - y_{itj}) \leq z_{jt} \qquad \forall i, t, j \\
& z_{jt} - \alpha_j - \beta_j t \leq \psi_{jt} \qquad \forall i, j, t \\
& -(z_{jt} - \alpha_j - \beta_j t) \geq \psi_{jt} \qquad \forall i, j, t \\
& y_{itj} \in \{0,1\} \quad \forall i, t, j, \quad F_{it} \in \{0,1\} \quad \forall i, t, \quad M_{jt} \in \{0,1\} \quad \forall j, t \\
& \alpha_j \in \mathbb{R}^n \quad \forall j, \quad \beta_j \in \mathbb{R}^n \quad \forall j, \quad z_{jt} \in \mathbb{R}^n \quad \forall j, t, \quad TF \in \mathbb{R}^n, TM \in \mathbb{R}^n
\end{aligned}
$$

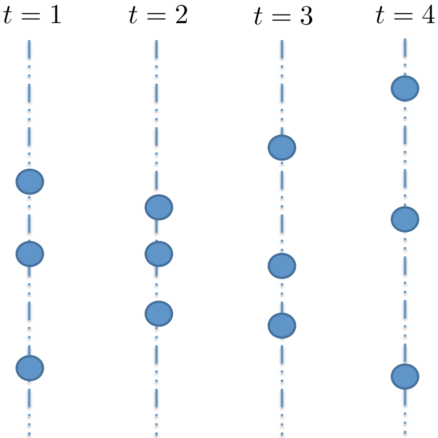
Robust MIO Model

$$
\begin{aligned}
\underset{\psi_{jt}}{\text{minimize:}}\quad & \sum_{j=1}^{P}\sum_{t=1}^{T}\psi_{jt} + \theta\cdot TF + \phi\cdot TM \\
\text{subject to:}\quad & \sum_{j=1}^{P} y_{itj} + F_{it} = 1 \qquad \forall i,t \\
& \sum_{i=1}^{n_t} y_{itj} + M_{jt} = 1 \qquad \forall j,t \\
& \sum_{i=1}^{n_t}\sum_{t=1}^{T} F_{it} = TF \\
& \sum_{j=1}^{P}\sum_{t=1}^{T} M_{jt} = TM \\
& x_{it}y_{itj} + M_t(1-y_{itj}) \geq z_{jt} \qquad \forall i,t,j \\
& x_{it}y_{itj} - M_t(1-y_{itj}) \leq z_{jt} \qquad \forall i,t,j \\
& z_{jt} - \alpha_j - \beta_j t \leq \psi_{jt} \qquad \forall j,t \\
& -(z_{jt} - \alpha_j - \beta_j t) \leq \psi_{jt} \qquad \forall j,t \\
& y_{itj} \in \{0,1\} \quad \forall i,t,j, \quad F_{it} \in \{0,1\} \quad \forall i,t, \quad M_{jt} \in \{0,1\} \quad \forall j,t \\
& \alpha_j \in \mathbb{R}^n \quad \forall j, \quad \beta_j \in \mathbb{R}^n \quad \forall j, \quad z_{jt} \in \mathbb{R}^n \quad \forall j,t, \quad TF \in \mathbb{R}^n, TM \in \mathbb{R}^n
\end{aligned}
$$

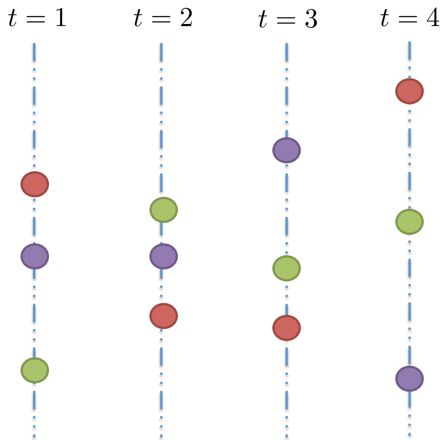
Section 4

Randomized Local Search Heuristics

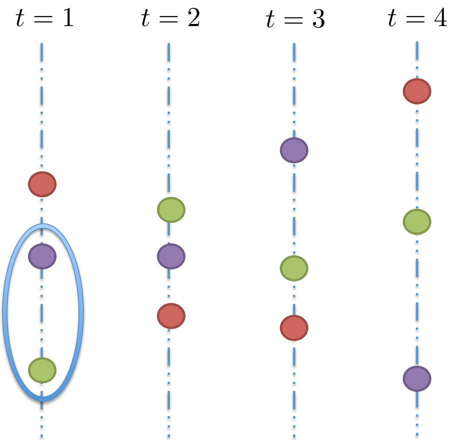
Original Problem Data



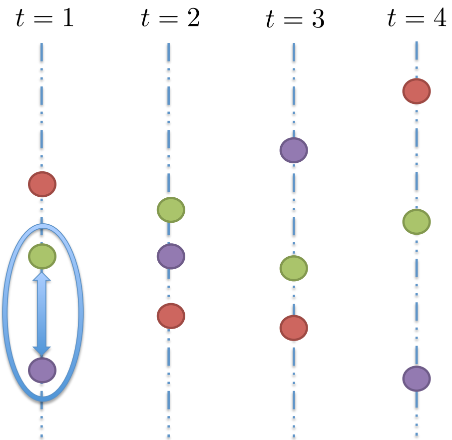
Randomize Initial Starting Point



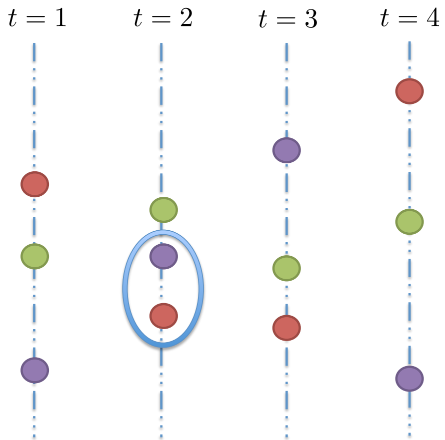
Swapping Mechanism



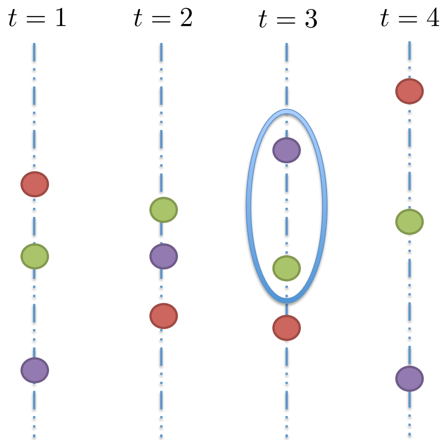
Swapping Mechanism



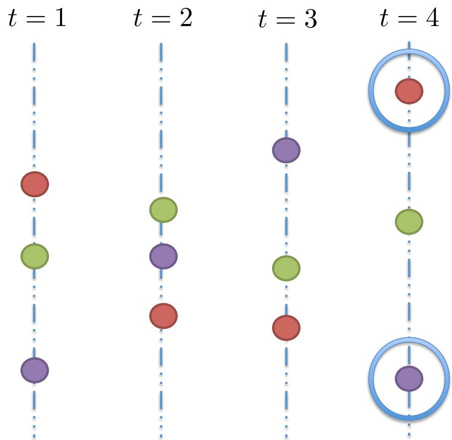
Swapping Mechanism



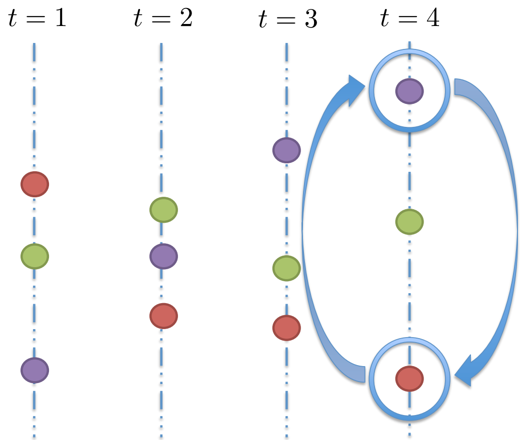
Swapping Mechanism



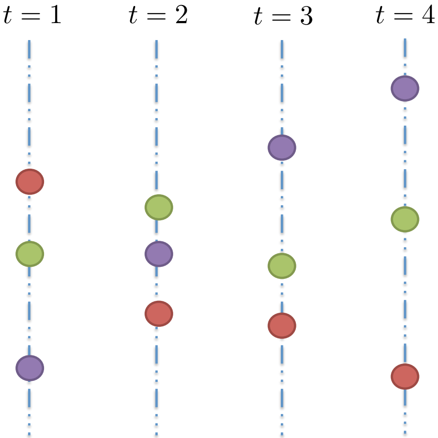
Swapping Mechanism



Swapping Mechanism



Termination



Section 5

Scenario Complexity and Performance Metrics

Data Association Complexity

Complexity Metric:

$$\rho = \frac{\sum_{t=1}^{T} \sum_{i<j} c_{ijt}}{\binom{P}{2} T},$$

where

$$c_{ijt} = \begin{cases} 1, & \text{if } D_{ijt} > h(\sigma), \\ 0, & \text{otherwise.} \end{cases}$$

and D_{ijt} is the distance between two targets i and j at time t:

$$D_{ijt} = \|\alpha_i^{\text{true}} + \beta_i^{\text{true}} t - \alpha_j^{\text{true}} + \beta_j^{\text{true}} t\|$$

Data Association Performance Metric

No detection ambiguity:

$$Accuracy = \frac{\#\text{ correct assignments}}{\text{Total }\#\text{ of detections}} = \frac{\#\text{ correct assignments}}{PT}$$

Detection ambiguity:

$$Accuracy = \frac{\#\text{ correct assignments}}{PT + \#\text{ False Alarms}}$$

Trajectory Estimation

Complexity Metric: σ
Performance Metric:

$$\delta = \frac{\sum_{t=1}^{T}\sum_{j=1}^{P} \|\bar{x}_{jt} - \hat{x}_{jt}\|}{PT}$$

where:

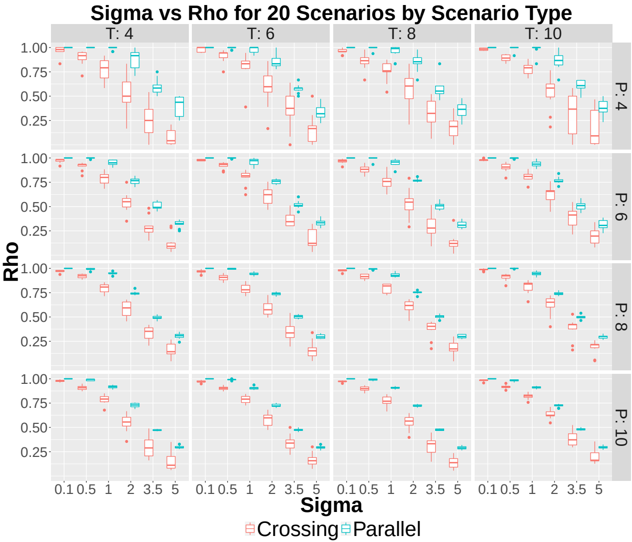
$$\bar{x}_{jt} = \alpha_j^{\text{true}} + \beta_j^{\text{true}} t.$$

$$\hat{x}_{jt} = \alpha_j + \beta_j t.$$

Section 6

Experimental Simulations and Computational Results

Sigma vs.Rho



Sigma vs Rho for 20 Scenarios by Scenario Type
T: 4
T: 6
T: 8
T: 10
P: 4
P: 6
P: 8
P: 10
Rho
1.00
0.75
0.50
0.25
Sigma
0.1
0.5
1
2
3.5
5
Crossing
Parallel

Heuristic Run Times

		Basic Heuristic Run Times (in milliseconds)		
P	T	Min	Mean	Max
4	4	0.07	0.10	0.18
4	6	0.18	0.24	0.38
4	8	0.34	0.45	0.62
4	10	0.58	0.76	1.02
6	4	0.11	0.15	0.25
6	6	0.31	0.39	0.58
6	8	0.64	0.81	1.05
6	10	1.24	1.56	2.02
8	4	0.14	0.19	0.30
8	6	0.46	0.57	0.86
8	8	0.95	1.24	1.58
8	10	2.07	2.53	3.37
10	4	0.19	0.25	0.41
10	6	0.63	0.80	1.03
10	8	1.44	1.84	2.44
10	10	2.96	3.73	4.56

Solution Types

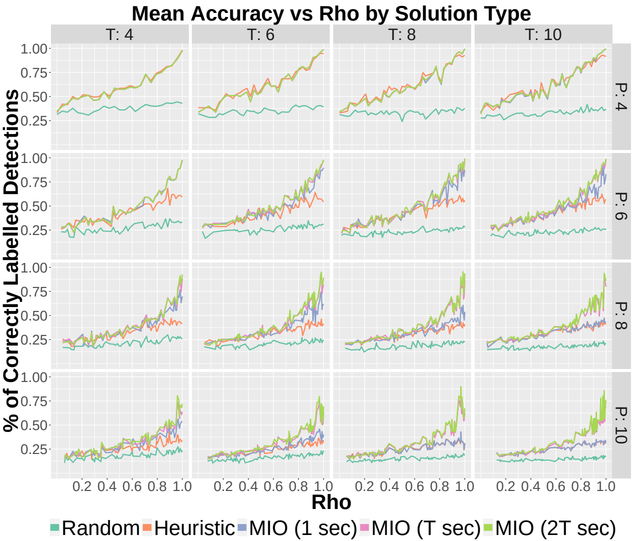
Two Benchmark Solutions:

1. *Random* := randomly generate detection assignments
2. *Ideal* := detection assignments are exactly correct

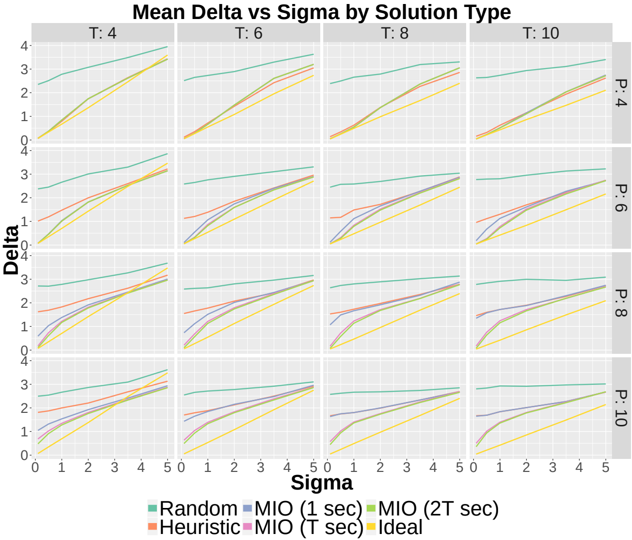
Two Algorithmic Solutions:

1. *Heuristic* := initialized with 1,000 starting points
2. *MIO Model* := heuristic solution provided as a warm start

Data Association Accuracy



Trajectory Estimation Error



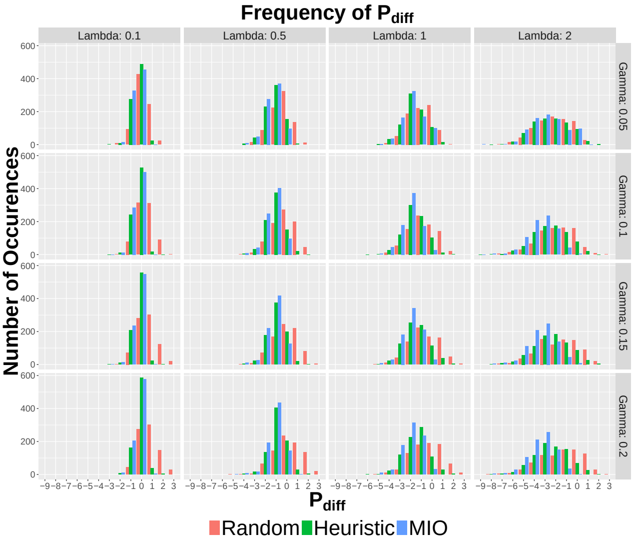
Summary of cases without detection ambiguity

- Heuristic
  - Runs in milliseconds for single starting point
  - Scalability maintained through parallelization
- MIO
  - Scalable with respect to increases in both P and T
  - Warm start crucial for scalability
  - High quality solutions after T or fewer seconds.
  - Tradeoff between correct data associations good trajectory estimation
  - Increasing $T \Rightarrow$ improved solution quality
    $\Rightarrow$ increased computational complexity

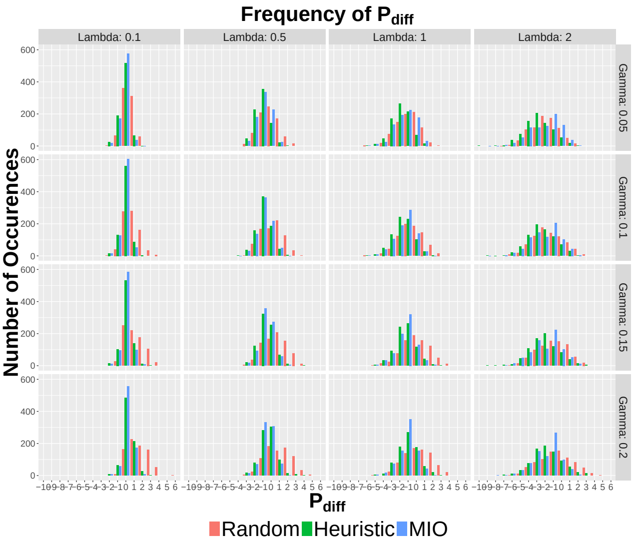
Robust Heuristic Run Times

		Robust Heuristic Run Times (in milliseconds)		
$P_{\text{estimated}}$	T	Min	Mean	Max
2	4	0.15	0.23	0.41
2	6	0.42	0.56	0.93
2	8	0.77	1.04	2.24
2	10	1.27	1.73	3.07
4	4	0.15	0.34	1.04
4	6	0.50	0.94	2.69
4	8	1.09	1.88	3.87
4	10	2.12	3.25	7.20
6	4	0.14	0.42	0.96
6	6	0.57	1.29	4.45
6	8	1.33	2.66	5.82
6	10	2.53	4.61	9.4
8	4	0.16	0.50	1.10
8	6	0.60	1.59	3.46
8	8	1.38	3.37	6.87
8	10	2.63	5.84	12.40
10	4	0.18	0.55	1.10
10	6	0.72	1.82	3.98
10	8	1.53	3.96	8.18
10	10	3.42	6.93	13.93
12	4	0.16	0.56	0.99
12	6	0.99	1.95	3.96
12	8	1.74	4.33	8.69
12	10	3.40	7.71	15.10

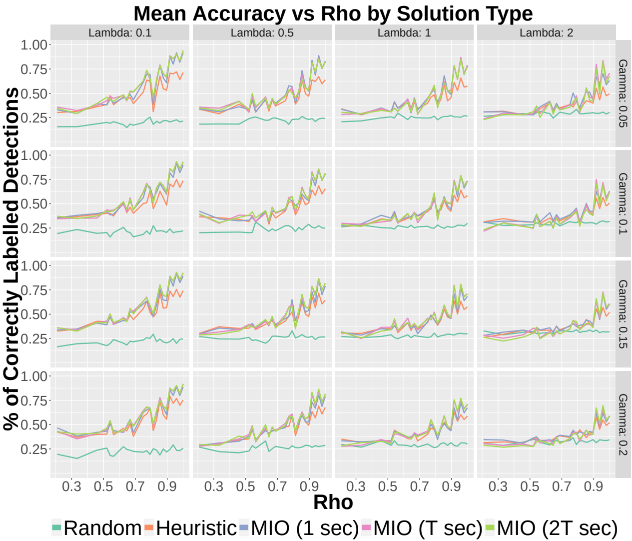
Estimating the Number of Targets ($P_{true} = 4$)



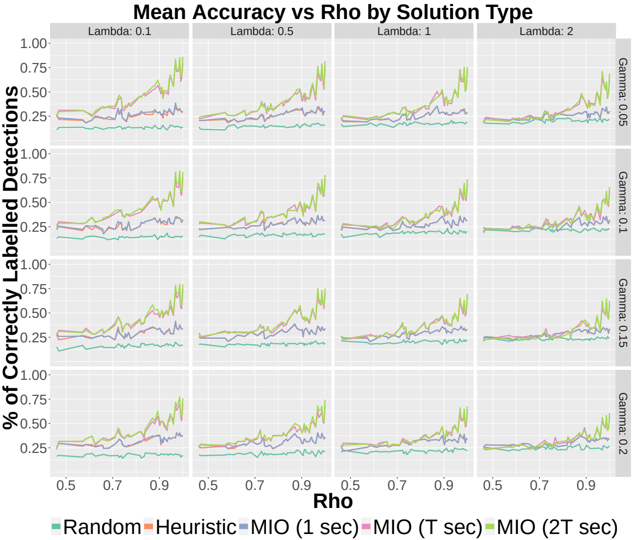
Estimating the Number of Targets ($P_{true} = 8$)



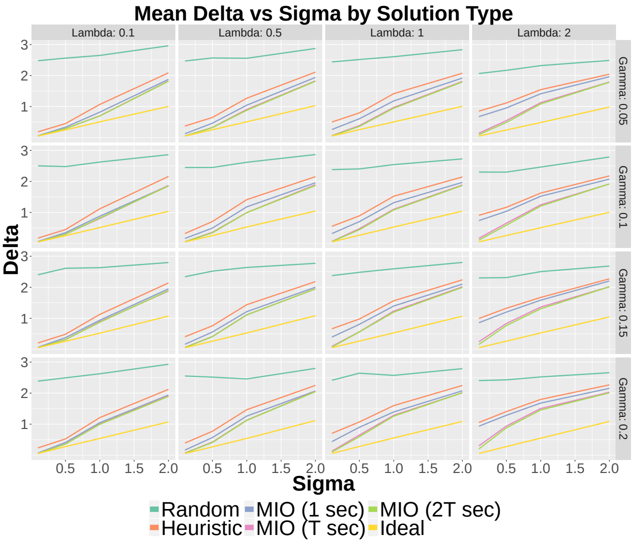
Data Association Accuracy ($P_{true} = 4$)



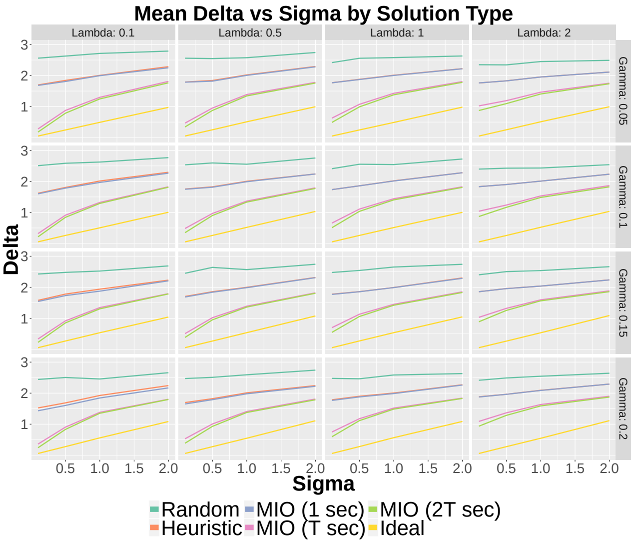
Data Association Accuracy ($P_{true} = 8$)



Trajectory Estimation Error ($P_{true} = 4$)



Trajectory Estimation Error ($P_{true} = 8$)



Summary of cases with detection ambiguity

- Heuristic
  - Slower and scales less efficiently than basic heuristic
  - Run times remain within milliseconds for single starting point
  - Scalability still maintained through parallelization
- MIO
  - Correct estimation of P through parameter tuning.
  - Data association accuracy deteriorates by 5%-10% because of detection ambiguity.
  - Overestimating P does not influence quality of trajectory estimation.
  - MIO solution more robust to γ than to λ.

Section 7

Summary and Future Work

Summary of Contributions

(i) Introduced MIO models for solving MTT
- Simple and interpretable
- Requires few assumptions
- Requires at most two parameters

(ii) Proposed heuristics
- Very efficient and scalable
- Useful as warm start for MIO model

(iii) New metrics provide additional insight

Future Work

- Extensions to non-linear trajectories
- More complex penalty functions
- Extensions to sliding window algorithm

Section 8

Penalties

False Alarm Penalty θ

λ	σ 0.1	0.5	1.0	2.0
0.1	1.7	2.6	3.1	3.5
0.5	1.1	1.9	2.3	2.5
1.0	0.9	1.2	1.6	1.8
2.0	0.5	0.9	0.9	1.0

Table: False alarm penalties (θ) as a function of λ and σ.

Missed Detection Penalty ϕ

λ	γ	σ 0.1	0.5	1	2
0.10	0.05	0.20	0.50	0.80	0.70
0.10	0.10	0.10	0.30	0.50	0.50
0.10	0.15	0.10	0.20	0.40	0.40
0.10	0.20	0.10	0.10	0.30	0.40
0.50	0.05	0.20	0.50	0.80	0.80
0.50	0.10	0.20	0.30	0.50	0.60
0.50	0.15	0.20	0.25	0.40	0.40
0.50	0.20	0.10	0.20	0.30	0.40
1.00	0.05	0.30	0.70	0.80	0.80
1.00	0.10	0.20	0.40	0.50	0.60
1.00	0.15	0.20	0.25	0.40	0.40
1.00	0.20	0.10	0.20	0.30	0.40
2.00	0.05	0.30	0.70	0.90	1.00
2.00	0.10	0.20	0.50	0.60	0.60
2.00	0.15	0.20	0.25	0.40	0.50
2.00	0.20	0.10	0.20	0.30	0.40

Table: Missed detection penalties (ϕ) as a function of λ, γ, and σ.

Section 9

Trajectory Assignment Pairing

$P_{true} = P_{est} = P$

$$\begin{aligned}
\underset{y_{ij}}{\text{minimize:}} \quad & \sum_{i=1}^{P} \sum_{j=1}^{P} c_{ij} y_{ij} \\
\text{subject to:} \quad & \sum_{i=1}^{P} y_{ij} = 1 \qquad \forall j = 1, ..., P \\
& \sum_{j=1}^{P} y_{ij} = 1 \qquad \forall i = 1, ..., P \\
& y_{ij} \in \{0, 1\} \quad \forall i = 1, ..., P, j = 1, ..., P
\end{aligned}$$

$P_{true} \leq P_{est}$

$$\begin{aligned}
\underset{y_{ij}}{\text{minimize:}} \quad & \sum_{i=1}^{P_{\text{true}}} \sum_{j=1}^{P_{\text{est}}} c_{ij} y_{ij} \\
\text{subject to:} \quad & \sum_{i=1}^{P_{\text{true}}} y_{ij} = 1 \qquad \forall j = 1, \ldots, P_{\text{est}} \\
& \sum_{j=1}^{P_{\text{est}}} y_{ij} \leq 1 \qquad \forall i = 1, \ldots, P_{\text{true}} \\
& y_{ij} \in \{0, 1\} \quad \forall i = 1, \ldots, P_{\text{true}}, j = 1, \ldots, P_{\text{est}}.
\end{aligned}$$

$P_{true} \geq P_{est}$

$$
\begin{aligned}
\underset{y_{ij}}{\text{minimize:}} \quad & \sum_{i=1}^{P_{\text{true}}} \sum_{j=1}^{P_{\text{est}}} c_{ij} y_{ij} \\
\text{subject to:} \quad & \sum_{i=1}^{P_{\text{true}}} y_{ij} \leq 1 \qquad \forall j = 1, \ldots, P_{\text{est}} \\
& \sum_{j=1}^{P_{\text{est}}} y_{ij} = 1 \qquad \forall i = 1, \ldots, P_{\text{true}} \\
& y_{ij} \in \{0, 1\} \quad \forall i = 1, \ldots, P_{\text{true}}, j = 1, \ldots, P_{\text{est}}.
\end{aligned}
$$

Section 10

Misc Backup Slides

Heuristic Performance by Number of Starting Points