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Some assignment problems arising from multiple target tracking

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Abstract

Multiple target tracking is a subject devoted to the estimation of targets' or objects' states, e.g., position and velocity, over time using a single or multiple sensors. The development of modern tracking systems requires a wide variety of algorithms ranging from gating (preprocessing), state and bias estimation, and development of likelihood ratios to data association. The central problem is the data association problem of partitioning sensor reports into tracks and false alarms. From a data association perspective, multiple target tracking methods divide into two basic classes, single and multiple frame processing. The advantage of multiple frame methods is that current decisions are improved by the ability to change past decisions, making multiple frame methods the choice for difficult tracking problems. The classical multiple frame method that has been well developed is called multiple hypothesis tracking (MHT). In the last ten to fifteen years, a new method, called multiple frame assignments (MFA) has been developed by formulating MHT as a multi-dimensional assignment problem for which modern optimization methods can be utilized in the development of near-optimal solutions for real-time applications. This work reviews a number of the problem formulations, including two-dimensional asymmetric single and multi-assignment problems, the corresponding multi-dimensional versions, and the newer group assignment problems. Some of the current and future needs are also discussed.

Keywords: Target tracking; Multiple hypothesis tracking; Multi-dimensional assignment; Group assignment; Multiple hypothesis correlation

1. Introduction to data association

Multiple target tracking is a subject devoted to the estimation of the trajectory of objects using one or more sensors. The central problem is the data association problem of determining which sensor reports emanate from which object and which correspond to false reports. If one knows which reports emanate from which object, then estimation techniques such Kalman or IMM filtering [1] can be utilized to determine a good approximation to the state. In DOD applications, the objects of interest may be airplanes, ground vehicles, or missiles. The sensors might be radar or IR sensors.

From the point of view of data association, multiple target tracking methods divide into two broad classes, namely single frame and multiple frame methods. The single frame methods include nearest neighbor, global nearest neighbor (posed as a two-dimensional assignment problem) and JPDA (joint probabilistic data association) [2]. The most

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successful of the multiple frame methods are multiple hypothesis tracking (MHT) [3], which is based on efficient enumeration and pruning schemes, and multiple frame assignments (MFA) [4,5], based on Lagrangian relaxation algorithms for multi-dimensional assignment problems. The performance advantage of the multiple frame methods over the single frame methods follows from the ability to hold difficult decisions in abeyance until more information is available or, said in an equivalent way, the opportunity to change past decisions to improve current decisions.

Multiple frame data association methods are better able to handle closely spaced objects caused by dense threats with associated objects, false signals and clutter, radar multi-path, residual sensor registration biases, countermeasures, unresolved closely spaced objects, and data from 2-D sensors. Multiple frame data association methods offer improved performance in, for example, accuracy of the target tracks, discriminants, and covariance consistency and also reduced track switches, track breaks, and missed targets. Local sensor corruption can also be moderated in a composite tracker that uses multiple frame data association. Thus, for difficult tracking problems, the performance improvements of multiple frame methods over single frame methods are very significant.

MHT with an N-scan-back approximation in which all decisions are fixed N frames back is equivalently formulated as a multi-dimensional assignment problem [6]. Thus, developing efficient and real-time algorithms for the multi-dimensional assignment problem can significantly improve the quality of data association decisions over existing methods.

Multiple frame methods have been well-developed for a single sensor. The use of multiple sensors distributed across multiple platforms provides geometric and sensor diversity that can significantly enhance tracking capability. Thus, multiple frame methods have also been extensively developed for this problem using a centralized architecture in which all sensor reports are sent from each sensor to a central processing center. The tracking problem is “solved” at the central node and track states are sent back to the platforms of interest. On the other hand, a distributed or network-centric architecture is often preferable for estimation/fusion and data association for several reasons, namely, communication overload and delays and single-point-failure. Such architectures that use both single and multiple frame processing have recently been developed [7] and include the ability to achieve a consistent air picture. Such architectures use the same basic multi-dimensional assignment problem for data association.

While the above discussion focuses on measurement-to-track association, there is an equally important problem of associating (sometimes correlating) tracks to tracks from multiple sources. The same assignment problems presented in this work are applicable to this problem, as discussed in Section 6.2.

Given the central importance of data association, this work reviews the mathematical statement of the two- and multi-dimensional assignment problem and the current algorithms and future needs. In addition, a major part of this work focuses on the newly derived group assignment problems arising from cluster tracking.

2. Other issues in the design of modern tracking systems

Although data association is the central problem in multiple target tracking, there are many additional issues that must be addressed in the development of modern tracking systems. The books of Blackman [8], Blackman and Popoli [3], and Bar-Shalom [9–11] are all excellent references. Here is a brief summary focused mostly on network-centric tracking.

The objectives of network-centric tracking are to achieve enhanced estimation of tracks using geometric diversity and sensor variety not available in single platform tracking. To achieve this objective, one generally must address a host of problems or topics including (1) distributed data association and estimation [3]; (2) single integrated air picture [7]; (3) management of communication loading using such techniques as data pruning, data compression (e.g., tracklets [5]), push/request schemes, and target prioritization; (4) network topology of the communication architecture including the design of new communication architectures as well as the incorporation of legacy systems; (5) the types of information (e.g., measurements, tracks, tracklets) sent across the network; (6) sensor location and registration errors (sometimes called gridlock); (7) pedigree problems [12]; and (8) out-of-order, latent, and missing data due to both sensor and communication problems. In the Joint Composite Tracking Network (JCTN) effort [13], a candidate architecture has emerged in which each local platform is responsible for assigning its own measurements. These associated measurement reports (AMRs) are broadcast across the network with the corresponding track number. Such an architecture gives a consistent air picture across the platforms and shows a good reduction in communication loads [7]. This basic approach has been extended to multiple frame processing [7].

A prerequisite for formulating the association problem is the partitioning of the data into maximal proper frames. A proper frame is one in which the target is seen at most once. A maximal proper frame is a proper one that is maximal in the sense that, if an additional measurement is added, it is no longer proper. Achieving a pure maximal proper frame is rarely possible. The simplest example is that of a rotating radar in which a frame is a 360° sweep of the surveillance region. For agile sensors, the problem is more difficult due to sensor scheduling and beam overlaps, both at the sensor and network levels; however, good approximations seem to work reasonably well, especially in the presence of multiple frame processing.

Almost all data association problems are sparse due to preprocessing so that the number of reports that can be assigned to a track is generally limited. The value of this preprocessing cannot be over-estimated and is responsible, in large part, for the real-time performance of the resulting tracking system. A number of these are reviewed in the book by Blackman and Popoli [3]. Surprisingly, the solution of the data association problem requires only about 2%–5% of the total processing time in a tracking system. The majority of the time is expended in computation of costs for the assignment problem due to building the tree of possible extensions and filtering the resulting sequences of measurements. Thus, there is room for improved association algorithms that require additional time.

3. The two-dimensional assignment problem

The two-dimensional assignment problem that occurs most often in tracking is one that matches m objects from one set, e.g., tracks, to n objects from a second set, e.g., measurements, with the additional feature that each track need not be assigned, i.e., there may be a missed detection, and that each measurement need not be assigned, i.e., the measurement may be false.

The formulation of an assignment problem requires the specification of arcs a_{ij}. An arc a_{ij} represents an assignment between object i from the first set and object j from the second set. An arc that contains a zero index such as a_{0j} represents an assignment between an object on one frame and nothing on the other frame (e.g., missed detection or false alarm). To each arc a_{ij} corresponds a cost c_{ij} that represents the cost of the assignment. The computation of the cost coefficient typically represents the computationally most expensive step in the formulation and solution of the assignment problem. Thus it is important to compute cost coefficients only for assignments that are likely to be part of the solution to the assignment problem. These *dynamically feasible* arcs are identified in the tracking system by *coarse gating* algorithms that provide a "cheap" test for the feasibility of the assignment between objects i and j. Due to this preprocessing, the assignment problem is generally *sparse*. To explain the sparse formulation, let $I = \{0, 1, \ldots, m\}$ and $J = \{0, 1, \ldots, n\}$ and $A \subset \{(i, j) \mid (i, j) \in I \times J\}$ denote the collection of feasible arcs, each with a cost c_{ij}. Also, define $A(i) = \{j \mid (i, j) \in A\}$ and $B(j) = \{i \mid (i, j) \in A\}$ and require $0 \in A(i)$ for all $i \in I$, $0 \in B(j)$ for all $j \in J$, i.e., $A(0) = J$, and $B(0) = I$. $A(i)$ represents those elements in the second set that correspond to feasible arcs that originate from element i in the first set, and similarly for $B(j)$. The remaining requirements ensure that each element can be assigned to the zero index.

For generality, one can allow multi-assignment of the objects, i.e., object i can be assigned at most $m_i \geq 1$ times and j, at most $n_j \geq 1$ times. The resulting assignment problem is as follows:

$$
\begin{aligned}
\text{Minimize} \quad & \sum_{(i,j)\in A} c_{ij} x_{ij} \\
\text{Subject To} \quad & \sum_{j\in A(i)} x_{ij} = m_i \quad (i = 0, 1, \ldots, m), \\
& \sum_{i\in B(j)} x_{ij} = n_j \quad (j = 0, 1, \ldots, n), \\
& x_{ij} \in \begin{cases} \{0, 1\} & \text{for } i \neq 0 \text{ and } j \neq 0, \\ \{0, 1, \ldots, m_i\} & \text{for } j = 0 \text{ and } i = 1, \ldots, m, \\ \{0, 1, \ldots, n_j\} & \text{for } i = 0 \text{ and } j = 1, \ldots, n, \\ \{0, 1, \ldots, \text{ Min}\,\{m_0, n_0\}\} & \text{for } i = 0 \text{ and } j = 0, \end{cases}
\end{aligned}
\tag{1}
$$

where each $m_i \geq 1$, $n_j \geq 1$, $m_0 = \sum_{j=1}^{n} n_j$, $n_0 = \sum_{i=1}^{m} m_i$. Note that constraints on $i = 0$ and $j = 0$ have been added as well as the (0, 0) arc. Then, since $\sum_{i=0}^{m} m_i = \sum_{j=0}^{n} n_j$, this formulation is in the form of a general network flow problem. Since $0 \in A(i)$ for all $i \in I$ and $0 \in B(j)$ for all $j \in J$, the cardinalities of $A(i)$ and $B(j)$ should satisfy

$m_i + 1 \le |A(i)|$ for all $i \in I$ and $n_j + 1 \le |B(j)|$ for $j \in J$; otherwise, m_i should be reduced to $m_i = |A(i)| - 1$ for $i = 1, \ldots, m$ and n_j to $n_j = |B(j)| - 1$ for $j = 1, \ldots, n$.

3.1. The costs

In multiple target tracking applications and in track-to-track correlation problems, the costs c_{ij} are generally expressed in one of two forms. The first is $\tilde{c}_{ij} = -\ln(L_{ij})$, where L_{ij} is a likelihood ratio expressed as

$$L_{ij} = \frac{\Gamma_{ij}}{\Gamma_{i0}\Gamma_{0j}}.$$

Here, Γ_{ij} is the likelihood that i is associated with j, Γ_{i0} is the likelihood that i is not assigned, and Γ_{0j} is the likelihood that j is not assigned. In this expression, one generally assumes that $\Gamma_{00} = 1$ so that $L_{i0} = 1$, $L_{0j} = 1$, and $L_{00} = 1$ which translates into $\tilde{c}_{i0} = 0$, $\tilde{c}_{0j} = 0$, and $\tilde{c}_{00} = 0$. The second form of the cost is based on $c_{ij} = -\ln \Gamma_{ij}$ for all i and j, including zero. Clearly the equivalence between the two is that $\tilde{c}_{ij} = c_{ij} - c_{i0} - c_{0j} + c_{00}$. General expressions for these costs are established in the work of Poore [6] as well as Popoli and Blackman [3].

3.2. Cost shifting and score gating

Score gating or *fine gating* is a preprocessing technique that is often used in tracking and is based on the assignment costs [3]. While coarse gating allows one to remove an arc from the assignment problem before its assignment cost c_{ij} is computed, score gating allows one to remove the arc after its cost is computed but before the assignment problem is solved. To explain this, observe first that the assignment problem Eq. (1) can be transformed to the following one.

Theorem 1 (*Invariance Via Cost Shifting*). *The assignment problem* (1) *can be put into the following equivalent form where* $\tilde{c}_{ij} = c_{ij} - c_{i0} - c_{0j} + c_{00}$, *for which* $\tilde{c}_{i0} = 0$ *and* $\tilde{c}_{0j} = 0$, *i.e.,*

$$\begin{aligned}
&\text{Minimize} && \sum_{(i,j)\in A} \tilde{c}_{ij} x_{ij} + \sum_{i=1}^{m} m_i c_{i0} + \sum_{j=1}^{n} n_j c_{0j} \\
&\text{Subject To} && \sum_{j \in A(i)} x_{ij} = m_i \quad (i = 0, 1, \ldots, m), \\
& && \sum_{i \in B(j)} x_{ij} = n_j \quad (j = 0, 1, \ldots, n), \\
& && x_{ij} \in \begin{cases} \{0, 1\} & \text{for } i \neq 0 \text{ and } j \neq 0, \\ \{0, 1, \ldots, m_i\} & \text{for } j = 0 \text{ and } i = 1, \ldots, m, \\ \{0, 1, \ldots, n_j\} & \text{for } i = 0 \text{ and } j = 1, \ldots, n, \\ \{0, 1, \ldots, \operatorname{Min}\{m_0, n_0\}\} & \text{for } i = 0 \text{ and } j = 0 \end{cases}
\end{aligned} \tag{2}$$

where each $m_i \ge 1$, $n_j \ge 1$, $m_0 = \sum_{j=1}^{n} n_j$, $n_0 = \sum_{i=1}^{m} m_i$.

Fine gating for the assignment problem is based on the observation that any arc satisfying

$$\tilde{c}_{ij} \stackrel{def}{=} c_{ij} - c_{i0} - c_{0j} + c_{00} > 0$$

in Eq. (2) can be deleted from the problem since it is cheaper to assign i and j to 0.

3.3. Current algorithms

The most commonly used algorithm for tracking applications involving one-to-one assignments is probably the adaptation of the Jonker–Folgernant (JV) algorithm for the symmetric algorithm [14] to the asymmetric assignment problem Eq. (1). In particular, Drummond and Castañon [15,16] adapt the JV algorithm to a formulation that is mathematically equivalent to (1). One can view this algorithm as a naive forward and reverse auction [17] followed by a successive shortest path algorithm. Auction algorithms [17] are also popular due to their effectiveness for cases of highly rectangular problems.

While assignment algorithms give optimal solutions to the problem as posed, these solutions are not necessarily optimal for the tracking problem. An optimal tracking solution would probably be based on all data over all time, but this approach is computationally intractable for long time periods. (The multiple frame methods in the next section are a better approximation.) Also, the solutions are often *ambiguous* due to the stochastic nature of the tracking problem arising, for example, from sensor errors, biases, and inexact modeling of the motion of the object. This stochastic nature is reflected in the costs, often resulting in many solutions that are well within the noise level of the problem. Thus, the optimal solution to the assignment problem need not recover the truth in the tracking problem. This is made worse by the fact that many target identification systems assume perfect association of measurement to tracks and tracks to tracks. Given the importance of the identification problem, an assessment of the *ambiguity* of the optimal solution is needed.

4. The multi-dimensional assignment problem

The goal here is to explain some of the features of the multi-dimensional assignment problem most relevant to the data association problem in tracking. A derivation of the problem and the equivalence with MHT using an N-scan back approximation has been previously given [6] and will not be repeated here.

4.1. The multi-dimensional assignment problem

Consider a layered graph $\mathcal{G}$ with N distinct node sets (layers) $A_k = \{0, 1, \ldots, m_k\}$ indexed by $k \in K = \{1, \ldots, N\}$ and arcs $A \subset A_1 \times \cdots \times A_N$. (In tracking applications, a node set A_k is called a frame of data. For track initiation, all A_k's are sensor reports, while, for track maintenance, one set will be a list of tracks and a list of sensor reports not assigned to these tracks; the remaining frames contain sensor reports only.) A specific arc in A is denoted by the N-dimensional vector $a = (a(1), a(2), \ldots, a(N))$ where each $a(k) \in A_k$. Let $a_{k\ell}$ denote that $a \in A$ satisfying $a(k) = \ell$ and $a(j) = 0$ for all $j \neq k$. A is assumed to contain $a_{k\ell}$ for each $k = 1, \ldots, N$ and $\ell = 0, 1, \ldots, m_k$. Note that $a_{k0} = (0, 0, \ldots, 0) = 0$ for each $k = 1, \ldots, N$ and $b_{ka(k)} = (0, \ldots, 0, a(k), 0, \ldots, 0)$. Next, define a section $A_{k\ell} = \{a \in A \mid a(k) = \ell\}$. For each $k \in K$, associate to each section $A_{k\ell}$ a nonnegative demand $d_{k\ell} \in \Re$ for $\ell = 1, \ldots, m_k$. The variables $x_{a_{k\ell}}$ denote slack variables analogous to the variables x_{i0} and x_{0j} in Eq. (1) [17].

The inequality constrained multi-index transportation problem converted to an equality problem using the slack variables $x_{a_{k\ell}}$ can be stated as

$$
\begin{aligned}
\text{Minimize} \quad & \sum_{a \in A} c_a x_a \\
\text{Subject To} \quad & \sum_{a \in A_{k\ell}} x_a = d_{k\ell} \quad \text{for } k \in K \text{ and } \ell \in A_k, \\
& x_a \geq 0,
\end{aligned}
$$

where each $d_{k\ell}$ is the given aforementioned demand for $\ell = 1, \ldots, m_k$. The value of d_{k0} is chosen to satisfy

$$
d_{k0} = \sum_{j=1;\, j \neq k}^{N} \sum_{\ell=1}^{m_j} d_{j\,\ell} + \Delta,
$$

where Δ is an additive positive number, normally taken to be zero. Since $\sum_{\ell=0}^{m_k} d_{k\ell} = \sum_{\ell=0}^{m_p} d_{p\ell}$ for all $k, p \in K$, there exists a solution to the above problem.

The N-dimensional multi-assignment problem of interest in tracking is a combinatorial version of the inequality constrained multi-index transportation problem and requires that $d_{k\ell} = n_{k\ell} \geq 1$ be an integer for each $k \in K$ and $\ell = 1, \ldots, m_k$. In addition, x_a is assumed to be a zero–one variable for each $a \in A$ with at least two nonzero indices. The remaining variables are assumed to be nonnegative integers as indicated below. The N-dimensional multi-assignment problem can then be expressed as

$$\begin{aligned}
\text{Minimize} \quad & \sum_{a \in A} c_a x_a \\
\text{Subject To} \quad & \sum_{a \in A_{k\ell}} x_a = n_{k\ell} \quad \text{for } k \in K \text{ and } \ell \in A_k, \\
& x_a \in \{0, 1\} \text{ for all indices } a \text{ containing at least two nonzero indices}, \\
& x_{a_{k\ell}} \in \{0, 1, \ldots, n_{k\ell}\}, \\
& x_0 \in \mathrm{Min}\{n_{10}, \ldots, n_{N0}\},
\end{aligned} \tag{3}$$

where each $n_{k\ell}$ denotes the given aforementioned demand for $\ell = 1, \ldots, m_k$. The value of n_{k0} for each k is given by

$$n_{k0} = \sum_{j=1;\, j \neq k}^{N} \sum_{\ell=1}^{m_j} n_{j\,\ell} + \Delta,$$

where Δ is an additive nonnegative integer, normally taken to be zero. Again, since $\sum_{\ell=0}^{m_k} n_{k\ell} = \sum_{\ell=0}^{m_p} n_{p\ell}$ for all $k, p \in K$, there exists a solution to the above problem. The problem most often solved in tracking is that in which $n_{k\ell} = 1$ for each $k \in K$ and $\ell = 1, \ldots, m_k$.

4.2. The costs

The costs $\tilde{c}_a$ for tracking are generally based on likelihood ratios L_a in the form

$$\begin{aligned}
L_a &= \frac{\Gamma_a}{\prod_{k=1}^{N} \Gamma_{b_{ka(k)}}} = \frac{\Gamma_a}{\prod_{k=1}^{N} \Gamma_{0\cdots 0a(k)0\cdots 0}}, \\
\tilde{c}_a &= -\ln(L_a).
\end{aligned}$$

Here Γ_a denotes the likelihood that the reports indexed by $a = (a(1), \ldots, a(N))$ go together, $\Gamma_{0\cdots 0a(k)0\cdots 0}$ represents the likelihood that $a(k)$ is not assigned to anything, and $\Gamma_0 = 1$. These likelihoods are based on the various sensor properties such as measurement noise, probabilities of detection and false alarms, density functions for track initiation, false alarms, and the likelihood that a measurement or report emanates from a particular target, and they are derived in reasonable generality in the article by Poore [6] and in the book by Blackman and Popoli [3]. Alternately, the costs are written directly in terms of the likelihoods themselves, i.e., $c_a = -\ln \Gamma_a$. The costs are related through $\tilde{c}_a = c_a - \sum_{k=1}^{N} c_{b_{ka(k)}} + (N-1)c_0$, where $c_0 = 0$ has been included for later analogy.

4.3. Preprocessing

In tracking applications, preprocessing techniques are generally called "gating". Many methods such as dynamic and filtering gating substantially reduce the number of arcs in $A \subset A_1 \times \cdots \times A_N$ present in the problem. A survey of these gating methods can be found in the book by Blackman and Popoli [3]. The following two techniques are based on decomposing the multi-dimensional assignment problem into independent problems and on removing arcs that cannot appear in an optimal solution.

4.3.1. Problem decomposition

Given the multi-dimensional assignment problem Eq. (3), one can attempt to decompose the problem into disjoint problems by determining the connected components of the associated layered graph. The following decomposition method, originally presented in the work of Poore et al. [18], uses graph theoretic methods. Define an undirected graph $\mathcal{G}(\mathcal{N}, \mathcal{A})$ with nodes $\mathcal{N}$ and arcs $\mathcal{A}$ by

$$\begin{aligned}
\mathcal{N} &= \tilde{A}_1 \times \cdots \times \tilde{A}_N, \\
\mathcal{A} &= \{a(k), a(\ell) \mid k \neq \ell, a(k) \neq 0, a(\ell) \neq 0, a \in A\},
\end{aligned}$$

where $\tilde{A}_k = \{1, 2, \ldots, m_k\} = A_k - \{0\}$. The nodes corresponding to zero index have been excluded from this graph since two variables that have only the zero index in common can be assigned independently. Connected components of the graph are then easily found by constructing a spanning forest via a depth first search [19]. Note that the decomposition algorithm depends only on the layered graph itself and not on the costs c_a.

As an aside, this decomposition often yields small problems that are best and more efficiently handled by a branch and bound or an explicit enumeration procedure [20]. For real-time applications, the remaining components are solved by a heuristic such as Lagrangian relaxation [20]. However, extensive decomposition can be a time sink and it is better to limit the number of components to, say, ten, unless one is using a parallel machine.

4.3.2. Fine gating based on costs

Arcs with at least two nonzero indices can be removed from the following if they are too large. This property follows as a result from the following theorem.

Theorem 2 (*Invariance Via Cost Shifting*). *Let* $\tilde{c}_a \stackrel{def}{=} c_a - \sum_{k=1}^{N} c_{b_{ka(k)}} + (N-1)c_0$. *Then, the minimizing solution and objective function value of the following problem is the same as that for the problem Eq. (3) :*

$$\begin{aligned}
\text{Minimize} \quad & \sum_{a \in A} \tilde{c}_a x_a + \sum_{k=1}^{N} \sum_{\ell=1}^{m_k} n_{k\ell}\, c_{a_{k\ell}} \\
\text{Subject To} \quad & \sum_{a \in A_{k\ell}} x_a = n_{k\ell} \quad \textit{for } k \in K \textit{ and } \ell \in A_k, \\
& x_a \in \{0,1\} \textit{ for all indices a containing at least two nonzero indices}, \\
& x_{a_{k\ell}} \in \{0,1,\ldots,n_{k\ell}\}, \\
& x_0 \in \text{Min}\{n_{10},\ldots,n_{N0}\}.
\end{aligned}$$

This equivalent problem has the feature that $\tilde{c}_{b_{ka(k)}} = \tilde{c}_{0\cdots 0a(k)0\cdots 0} = 0$, so that any arc $a \in A$ for which

$$\tilde{c}_a \stackrel{def}{=} c_a - \sum_{k=1}^{N} c_{b_{ka(k)}} + (N-1)c_0 > 0$$

can be deleted from the problem since it is cheaper to assign $x_{b_{ka(k)}} = 1$ for each $k = 1,\ldots,N$. This is generally called *fine or likelihood based gating* (as opposed to coarse or medium) in tracking.

4.4. Algorithms

The data association problem in multiple frame association is the multi-dimensional assignment problem Eq. (3) with $n_{k\ell} = 1$ for $k = 1,\ldots,N$ and $\ell = 1,\ldots,m_k$. Recall also that problem Eq. (3) is also the data association problem for the track oriented approach to MHT with an N-scan back approximation [6]. The problem is NP-hard even in the case $N = 3$ and a branch and bound algorithm is the only known algorithm for guaranteeing an optimal solution [21]. Thus, the general approach to obtaining good solutions is based on heuristics and considerable preprocessing. While there are a wide array of possibilities such as tabu search [22], GRASP, genetic algorithms, and Lagrangian relaxation, GRASP [23,24] and Lagrangian relaxation [25,4,5] have been reasonably well-developed.

Lagrangian relaxation

An algorithm that has been particularly successful is Lagrangian relaxation in which an N-dimensional problem is relaxed to a two-dimensional one, the dual problem is approximately maximized, and the restoration of feasibility is posed as an $(N-1)$-dimensional problem. Here is a brief technical explanation. Multiply each of the constraint sets in Eq. (3) corresponding to $k = 3,\ldots,N$ ($\sum_{a \in A_{k\ell}} x_a - n_{k\ell} = 0$ for $k \in K$ and $\ell \in A_k$) by $u_{k\ell}$, add this to the objective function, and remove the corresponding constraints to obtain

$$\begin{aligned}
\Phi(u_3,\ldots,u_N) \quad & = \\
\text{Minimize} \quad & \sum_{a \in A} \left(c_a + \sum_{k=3}^{N} u_{ka(k)} \right) x_a - \sum_{k=3}^{N} u_{k\ell} n_{k\ell} \\
\text{Subject To} \quad & \sum_{a \in A_{k\ell}} x_a = n_{k\ell} \quad \ell \in A_k \text{ for } k = 1,2 \\
& x_a \in \{0,1\} \text{ for all indices } a \text{ containing at least two nonzero indices}, \\
& x_{a_{k\ell}} \in \{0,1,\ldots,n_{k\ell}\}, \\
& x_0 \in \text{Min}\{n_{10},\ldots,n_{N0}\}.
\end{aligned}$$

The dual problem then is to maximize $\Phi(u_3, \ldots, u_N)$. The solution of this maximization problem yields an alignment between the first two indices in a, namely $a(1)$ and $a(2)$. This essentially collapses the first two indices into a single index. Also, the constraints corresponding to $k = 3, \ldots, N$ need not be satisfied. Thus, to restore feasibility of the original problem, one formulates an $(N - 1)$-dimensional assignment problem of a form similar to Eq. (3), but with N replaced by $N - 1$. (This scheme is present in considerable detail in the paper by Poore and Robertson [4].) The procedure is repeated until one reaches the final two-dimensional problem. The optimization of the partial dual is performed using nonsmooth optimization methods [26,27]. These Lagrangian relaxation algorithms have performed exceptionally well, principally due to the base problem being a two-dimensional assignment problem and the structure of the costs. For example, in the case of three dimensions, the costs for track extension using filtering for estimation of the state have the form $c_{ijk} = c_{ij} + e_{jk}(i)$.

An alternate method for solving the N-dimensional assignment method arising from tracking is what is called "sequential processing". In this method, one solves the two-dimensional assignment problem for the first two frames using the costs c_{ij}. Then, given a solution to this problem, one next enumerates the indices (i, j) for which $x_{ij} = 1$ by an index, say ℓ. Then, fixing the assignment on the first two frames and denoting the corresponding assignment of i to j by the (i_ℓ, j_ℓ), one could match this to the problem on the third frame by solving the problem in the two indices (ℓ, k) and costs $\overline{c}_{\ell k} = c_{i_\ell, j_\ell, k}$. This "sequential processing" procedure also works for the more general multi-dimensional case. Obviously, once a decision is made for two frames, it cannot be changed, and this is precisely the weakness of the "sequential processing" method in that it does not use the additional information available for the full N-frames. A properly constructed Lagrangian relaxation algorithm can indeed produce solutions at least as good as this base "sequential processing" algorithm. Other algorithms based on for example, GRASP, tabu search, or genetic algorithms [28], must always do at least as well as the "sequential processing" algorithm.

Finally, the suggestion of this algorithm is due to Frieze and Yadegar [29] who proposed the basic Lagrangian relaxation algorithm for a different three-dimensional assignment problem.

Improvement algorithms

The Lagrangian relaxation algorithm generally produces very good solutions, well within the noise level of the problem. Lagrangian relaxation combined with a partial branch and bound [30] and Murty's algorithm [31] for the two-dimensional assignment problem can significantly improve the suboptimal solution. In spite of the success of these methods, additional improvement methods based on local search might also provide good solutions.

Ambiguity in assignments

As discussed in the previous section on the two-dimensional assignment problem, the assignments arising under the approximate algorithms are often *ambiguous* due to the stochastic nature of the tracking problem. Multiple frame data association methods such as MHT and MFA often treat the problem as a maximum likelihood estimation problem, in which each assignment solution has a likelihood, and an optimization algorithm tries to find the solution with the highest likelihood. However, in order to assess the ambiguity of the assignment problem, the normalized probability distribution on all assignment solutions must be considered. The uncertainty can be quantitatively measured as a single scalar quantity, the entropy h of the probability distribution $h(P) = \sum_x P(x)\log(P(x))$.

The entropy can be measured in bits, and represents the amount of uncertainty information a MLE tracker discards by making a firm assignment decision. For assignment problems with only one feasible solution, the entropy is zero, and for assignment problems with roughly uniform probability over K solutions, the entropy is $\log(K)$. The entropy has the nice property that it is additive over independent components of the assignment problem and should prove to be useful in assessing ambiguity in the assignments.

5. The group–cluster assignment problem

The assignment problems presented so far in this paper were concerned with the association of individual objects between frames. More recently a new class of assignment problems has been introduced that is concerned with the association of *groups* or *clusters* of objects between frames [32]. The goal of this section is to give a formulation of the *group–cluster assignment problem* for group–cluster tracking association and for the merged measurement problem. While the ideas apply equally well to single and multiple frame association, the technical development will be restricted to clusterings and matchings between two frames of data. The multiple frame analogue is reasonably straightforward and the three-dimensional version is given as an example to illustrate the generalization.

The principal idea in the formulation of the problem is to consider several clustering (or grouping) hypotheses for each frame of data. We distinguish between two approaches toward the solution of the group assignment problem. In the first approach we match *complete clustering hypotheses* of the frames of data. This approach is straightforward in its implementation since it does not require a new class of assignment solvers. Its solution requires the solution of multiple N-dimensional (one-to-one or multi-) assignment problems. In the second approach, the general *group assignment*, the distinct subclusters from all the cluster hypotheses are listed on each frame. The subclusters are then assigned across multiple frames of data subject to the set packing constraint on each frame. When the subclusters are each composed of a single individual object, then the resulting cluster assignment problem reduces to the usual multi-assignment or one-to-one assignment problem classically used in tracking.

5.1. Multiple clustering hypotheses

Given a data set (e.g., a set of measurements) $\mathcal{Z} = \{\mathbf{z}_1, \ldots, \mathbf{z}_N\}$ from an input space $Z \subset \mathbb{R}^d$, a clustering algorithm allows one to partition Z into natural groups or clusters based on some measure of similarity. See Ref. [33] for an excellent review of cluster algorithms. One can distinguish between hard and soft cluster algorithms [33].

Definition 1 (*Hard M-clustering*). A hard M-clustering of a data set $\mathcal{Z} \subset Z$ denotes the partitioning of $\mathcal{Z}$ into M sets (clusters, groups) $\{\mathcal{C}_1, \ldots \mathcal{C}_M\}$, such that (a) $\mathcal{C}_i \neq \emptyset$, $i = 1, \ldots, M$, (b) $\mathcal{C}_i \cap \mathcal{C}_j = \emptyset$, $i \neq j$, $i, j = 1, \ldots, M$, and (c) $\cup_{i=1}^{M} \mathcal{C}_i = \mathcal{Z}$.

In this paper we do not make a formal distinction between groups and clusters. A hard clustering assigns a data point to exactly one cluster. A famous hard clustering algorithm is the k-means or Isodata algorithm [34]. Soft (or fuzzy) clustering, on the other hand, allows the assignment of a data point to multiple classes through a membership function which, for the Bayesian approach, represents a probability that the data vector belongs to a given class.

Definition 2 (*Soft M-clustering*). A soft M-clustering of $\mathcal{Z}$ is characterized by M membership functions $u_i : \mathcal{Z} \to [0, 1]$ $(i = 1, \ldots, M)$ such that $\sum_{i=1}^{M} u_i(\mathbf{z}) = 1$ for all $\mathbf{z} \in \mathcal{Z}$ and $0 < \sum_{j=1}^{N} u_i(\mathbf{z}_j) < N$ $(i = 1, \ldots, M)$.

The last requirement assures that the soft clustering does not produce a hard clustering. Given any soft clustering, a hard partitioning can be obtained by assigning a data point only to its most likely group. A widely used soft-clustering algorithm is the Expectation-Maximization (EM) algorithm [35]. We say that a clustering is a hard (soft) *complete* clustering if any data point can be assigned to exactly (at least one) cluster.

Next we assume that we start with two lists of objects (e.g., measurements, features, or tracks). In a first step we hypothesize a set of complete candidate clusterings of the two data lists. Here is a formal definition.

Definition 3. Let P and Q denote two lists of objects and let $\mathcal{H}(P) = \{H_i(P)\}_{i \in I_H}$ and $\mathcal{H}(Q) = \{H_j(Q)\}_{j \in J_H}$ denote collections of complete clusterings of P and Q, respectively. In addition, let $\mathcal{P} = \{P_i\}_{i \in I}$, and $\mathcal{Q} = \{Q_j\}_{j \in J}$ denote the collection of all distinct clusters from the hypotheses $\mathcal{H}(P)$ and $\mathcal{H}(Q)$, respectively.

The first formulation of the cluster assignment problem will be based on explicit enumeration while the second and third formulation formulate the problem as a single assignment problem in which the distinct subclusters in $\mathcal{P}$ are matched to subclusters in $\mathcal{Q}$ in such a way that (1) the set packing property is maintained for both sets and (2) multiple assignments between the subclusters in the different frames are allowed. We distinguish between a *hard set packing property* and a *soft set packing property*.

Definition 4 (*Hard Clusterings Set Packing Property*). Find a subcollection $\{P_{i_1}, \ldots, P_{i_M}\}$ $(M \leq I)$ of $\mathcal{P}$ that is matched to a subcollection $\{Q_{j_1}, \ldots, Q_{j_N}\}$ $(N \leq J)$ of $\mathcal{Q}$ with the requirements that $\{P_{i_p}\}_{p=1}^{M}$ and $\{Q_{j_q}\}_{q=1}^{N}$ are set packings of P and Q, respectively, i.e.,

$$\text{(a)}\, P_{i_p} \neq \emptyset; \qquad \text{(b)}\, \cup_{p=1}^{M} P_{i_p} \subseteq P; \qquad \text{(c)}\, P_{i_p} \cap P_{i_q} = \emptyset$$

and similarly for $\mathcal{Q}$. In addition, each P_i should be allowed to be multiply assigned to a Q_j, and vice-versa.

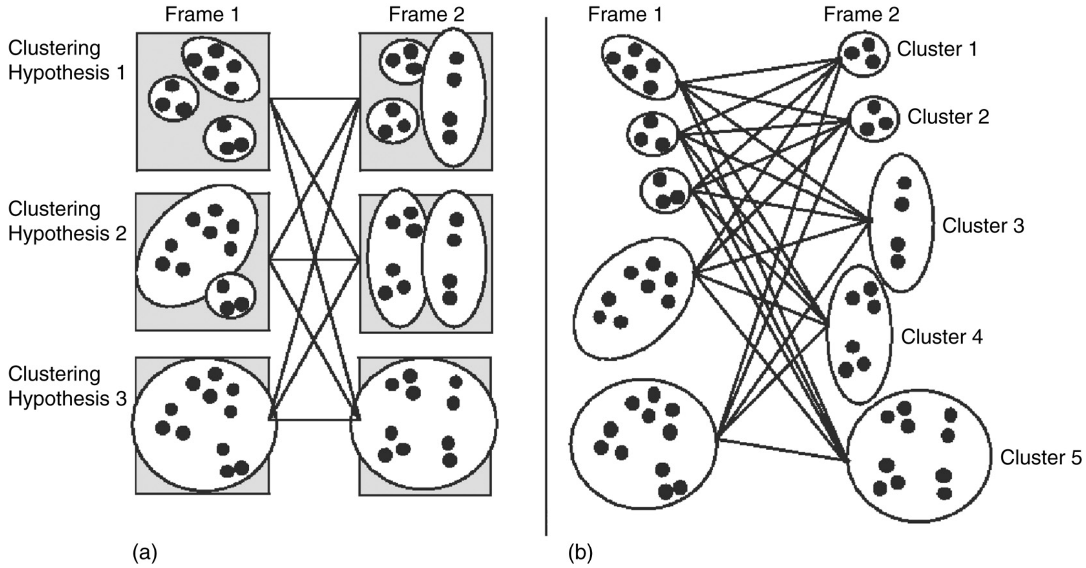

Fig. 1. Illustration of two formulations of the group–cluster assignment problem. (a) Matching complete clusterings between frames, and (b) matching clusters between frames.

Soft clusterings that allow overlap between the clusters cannot satisfy the partitioning property $P_{i_p} \cap P_{i_q} = \emptyset$ of the hard cluster. A number of schemes attempt to approximate this requirement and a discussion can be found in the book *Pattern Recognition* by Theodoridis and Koutroumbas [33], which is also an excellent reference for clustering methods. The soft clustering conditions similar to (a), (b), and (c) of the hard set packing property are as follows.

Definition 5 (*Soft Clusterings Set Packing Property*). Out of all the class memberships functions, find $u_{i_1}, \ldots, u_{i_M}$ satisfying

$$\text{(a)} \sum_{z \in \mathcal{Z}} u_{i_p}(z) > 0 \; (p = 1, \ldots, M), \qquad \text{(b)} \sum_{p=1}^{M} u_{i_p}(z) \leq 1, \text{ for all } z \in \mathcal{Z}, \text{ and}$$

$$\text{(c)} \frac{1}{N} \leq PC = \frac{1}{M} \sum_{p=1}^{M} \sum_{j=1}^{N} u_{i_p}^2(z_j) \leq 1.$$

If the partition coefficient $PC \sim 1$, then the cluster is almost hard. For the remainder of the work we will restrict development to the use of hard clusterings.

Fig. 1 illustrates the two classes of group–cluster tracking assignment problems that are formulated in this paper. The illustration shows two frames of data in which Frame 1 consists of 11 observations, and Frame 2 of 10 observations. Fig. 1(a) illustrates the approach that matches complete clustering hypotheses between frames and shows three candidate group–clusterings of the respective frames. Fig. 1(b) shows the unique clusters from all clustering hypotheses and illustrates the resulting assignment problem that matches group–clusters between frames. This second and general class of the group assignment problem is discussed in Section 5.2.

Solution through explicit enumeration of the assignments

One possible solution of the cluster tracking assignment problem is to determine the best score among the different complete clusterings in $\mathcal{H}(P)$ and a complete clustering in $\mathcal{H}(Q)$. Let $H_i(P) = \{P_{ik}\}_{k=1}^{P_i}$ and $H_j(Q) = \{Q_{jl}\}_{l=1}^{Q_j}$ denote the ith and jth complete clusterings of P and Q respectively. If the subclusters P_{ik} and Q_{jl} are to be assigned m_k^{ij} and n_l^{ij} times, respectively, then the assignment problem between $\mathcal{H}_i(P)$ and $\mathcal{H}_j(Q)$ is (using notation as in Section 3):

$$
\begin{aligned}
\text{Minimize} \quad & \sum_{(k,l)\in A} c_{kl}^{ij} x_{kl}, \\
\text{Subject To} \quad & \sum_{l\in A(k)} x_{kl} \le m_k^{ij} \quad (k = 1, \ldots, P_i), \\
& \sum_{k\in B(l)} x_{kl} \le n_l^{ij} \quad (l = 1, \ldots, Q_j), \\
& x_{ij} \in \{0, 1\},
\end{aligned} \tag{4}
$$

where each $m_k^{ij} \ge 1$ and $n_l^{ij} \ge 1$. Having computed the optimal score for each pairing (i, j), one chooses the one with the best score from this list. Note that if the number of complete clusterings in $\mathcal{H}(P)$ is M and the number in $\mathcal{H}(Q)$ is N, then one must solve $M \times N$ assignment problems. This number grows substantially over multiple frames of data.

5.2. *The group–cluster assignment problem*

In the previous section, we considered a formulation of the cluster tracking problem wherein the objective was to find the best matching between a complete clustering on one frame to one on the next, chosen from multiple possible complete clusterings on each frame. This approach essentially enumerates the assignment problems to find the best matching of a complete clustering on two distinct frames of data. In this section, the collection of all of these problems is collapsed into a single assignment problem. While this formulation is not guaranteed to solve the same problem as in the previous section, it is guaranteed to produce a matching that has an overall score which is at least as good as, if not better than, that found by matching complete clusterings to complete clusterings. As before, let $\mathcal{H}(P)$ and $\mathcal{H}(Q)$ denote collections of complete clusterings of P and Q, respectively. Next, let $\mathcal{P} = \{P_i\}_{i\in I}$ and $\mathcal{Q} = \{Q_j\}_{j\in J}$ denote the collection of all unique clusters from the hypotheses $\mathcal{H}(P) = \{H_i(P)\}_{i\in I_H}$ and $\mathcal{H}(Q) = \{H_j(Q)\}_{j\in J_H}$, respectively. The second formulation of the cluster tracking assignment problem attempts to match the clusters in $\mathcal{P}$ to clusters in $\mathcal{Q}$ while maintaining the set packing property for each. Note that the set packing property discussed in the previous subsection doesn't require that all the data be used. If not, then the remaining objects in P can be put into an additional set and combined with those actually assigned to form a set partitioning as used in the definition in the clustering. Thus, the problem formulated in this section is in a sense more general than that formulated in the previous section. We next present several formulations of the group–cluster assignment problem.

5.2.1. *First formulation: constraints enumerated by individual objects in P and Q*

The case in which each group in one list is assigned to at most one group in the other list has a particularly attractive form. While this appears to restrict this approach to one-to-one assignments, the use of *subgroups* within a particular group adds additional flexibility for multi-assignment as explained later. To preserve the selection of subsets of $\mathcal{P}$ and $\mathcal{Q}$, we introduce the following definitions.

Definition 6. Let P and Q denote two lists of objects and let $\mathcal{P} = \{P_i\}_{i\in I}$ and $\mathcal{Q} = \{Q_j\}_{j\in J}$ denote collections of subsets of P and Q, respectively. Define the indicator functions

$$
m_{ki} = \begin{cases} 1 & \text{if object } k \in P \text{ is in } P_i, \\ 0 & \text{otherwise} \end{cases}, \quad \text{and} \quad n_{lj} = \begin{cases} 1 & \text{if object } l \in Q \text{ is in } Q_j, \\ 0 & \text{otherwise.} \end{cases}
$$

Given this definition, the problem formulation is

$$
\begin{aligned}
\text{Minimize} \quad & \sum_{(i,j)\in A} c_{ij} x_{ij}, \\
\text{Subject To} \quad & \sum_{(i,j)\in A} m_{ki} x_{ij} \le 1 \quad (k \in P), \\
& \sum_{(i,j)\in A} n_{lj} x_{ij} \le 1 \quad (l \in Q), \\
& x_{ij} \in \{0, 1\}.
\end{aligned} \tag{5}
$$

The key new component of this formulation in Eq. (5) is the use of the constraints $\sum_{(i,j)\in A} m_{ki} x_{ij} \leq 1 (k \in P)$ which says that an object $k \in P$ can be present in at most one pairing $(i, j) \in A$ and that any group i can be assigned to at most one group j. A similar statement holds for objects $l \in Q$. Thus the groups that end up actually being assigned have the properties of a set packing. This particular formulation incorporates a set packing formulation for a single data set commonly used, e.g., in auctions. Also, the constraints in this formulation are posed in terms of the individual objects themselves rather than groups and thus may contain many redundant ones.

Multiple assignments via subgroups

The formulation Eq. (5) admits multi-assignment in a very structured manner if one allows subgroups within a group. Here is an example of how this might be used. Suppose a group (cluster) P_i on the first frame is to be allowed to be assigned to two groups Q_r, Q_s on the second frame. One way to accomplish this within the current formulation is to form another group, say $Q_{n+1} = \{Q_r, Q_s\}$ composed of subgroups Q_r and Q_s, and add this group to $\mathcal{Q}$. In fact, this formulation may be the preferred one for controlling multiple assignments between groups in one data set to those in another, especially for many-to-one assignments.

The case of singleton groups

The usual one-to-one assignment problem can be seen as a special case of Eq. (5) with the following identification. Let $P = \{1, \ldots, m\}$, $Q = \{1, \ldots, n\}$, $P_i = \{i\}$ for $i = 1, \ldots, m$, $(I = \{1, \ldots, m\})$, $Q_j = \{j\}$ for $j = 1, \ldots, n$, $(J = \{1, \ldots, n\})$, $m_i = 1$, and $n_j = 1$. Then the $m_{ik} = \delta_{ik}$ and $n_{lj} = \delta_{lj}$, so that the above assignment problem Eq. (5) reduces to the usual *one-to-one assignment* problem. The δ_{ik} are defined to be one if $i = k$ and zero otherwise. Here is the resulting problem:

$$
\begin{aligned}
\text{Minimize} \quad & \sum_{(i,j)\in A} c_{ij} x_{ij}, \\
\text{Subject To} \quad & \sum_{(i,j)\in A} \delta_{ki} x_{ij} \leq 1 \quad (k \in P) \Rightarrow \sum_{j\in A(i)} x_{ij} \leq 1 \quad \text{for } i = 1, \ldots, m \\
& \sum_{(i,j)\in A} \delta_{lj} x_{ij} \leq 1 \quad (l \in Q) \Rightarrow \sum_{i\in B(j)} x_{ij} \leq 1 \quad \text{for } j = 1, \ldots, n \\
& x_{ij} \in \{0, 1\}.
\end{aligned} \tag{6}
$$

5.2.2. Second formulation: constraints enumerated by subclusters in $\mathcal{P}$ and $\mathcal{Q}$

A more general formulation of the cluster tracking multi-assignment problem allows the multi-assignment between groups P_i of $\mathcal{P}$ and Q_j of $\mathcal{Q}$ directly and then adds the set packing as an additional constraint. Using the hard set packing constraint this problem can be expressed as:

$$
\begin{aligned}
\text{Minimize} \quad & \sum_{(i,j)\in A} c_{ij} x_{ij}, \\
\text{Subject To} \quad & \sum_{j\in A(i)} x_{ij} \leq m_i \quad (i \in I), \\
& \sum_{i\in B(j)} x_{ij} \leq n_j \quad (j \in J), \\
\text{(HSP)} \quad & x_{i_1 j_1} + x_{i_2 j_2} \leq 1 \text{ for all } (i_1, j_1) \text{ and } (i_2, j_2) \in A \\
& \text{for which } i_1 \neq i_2 \text{ and } P_{i_1} \cap P_{i_2} \neq \emptyset \\
& \text{or } j_1 \neq j_2 \text{ and } Q_{j_1} \cap Q_{j_2} \neq \emptyset, \\
& x_{ij} \in \{0, 1\}.
\end{aligned} \tag{7}
$$

The constraint (HSP) of Eq. (7) is the aforementioned constraint on the (hard) set packing requirement for the final assignment.

A special case for one-to-one assignments

In the above formulation, when $m_i = 1$ and $n_j = 1$ for all (i, j), the constraints $x_{i_1 j_1} + x_{i_2 j_2} \leq 1$ for all (i_1, j_1) and $(i_2, j_2) \in A$ for which $i_1 \neq i_2$ and $P_{i_1} \cap P_{i_2} \neq \emptyset$ or $j_1 \neq j_2$ and $Q_{j_1} \cap Q_{j_2} \neq \emptyset$ can be replaced by sums. The resulting problem for this special case is

$$
\begin{aligned}
\text{Minimize} \quad & \sum_{(i,j)\in A} c_{ij} x_{ij}, \\
\text{Subject To} \quad & \sum_{j\in A(i)} x_{ij} \le 1 \quad (i \in I), \\
& \sum_{i\in B(j)} x_{ij} \le 1 \quad (j \in J), \\
& \sum_{j\in A(i_1)} x_{i_1 j} + \sum_{j\in A(i_2)} x_{i_2 j} \le 1 \quad \text{for which } i_1 \ne i_2 \text{ and } P_{i_1} \cap P_{i_2} \ne \emptyset, \\
& \sum_{i\in B(j_1)} x_{i j_1} + \sum_{i\in B(j_2)} x_{i j_2} \le 1 \quad \text{for which } j_1 \ne j_2 \text{ and } Q_{j_1} \cap Q_{j_2} \ne \emptyset, \\
& x_{ij} \in \{0, 1\}.
\end{aligned} \tag{8}
$$

A Lagrangian relaxation algorithm

This second formulation of the cluster assignment problem is particularly well-suited to a Lagrangian relaxation algorithm in that the set packing constraint can be Lagrangian relaxed to the base problem consisting of either the usual one-to-one assignment problem or the multi-assignment problem. The nonsmooth optimization of the resulting problem is relatively straightforward. The final step in restoring the set packing constraint is that which remains to be solved.

The general group-assignment equation (7) contains a two-dimensional assignment with merged measurements as a special case. Recently, Briers et al. have introduced a solution approach to solve this two-dimensional assignment problem with merged measurements that uses the auction algorithm together with Lagrangian relaxation [36].

5.3. *The three-dimensional problem*

The multi-dimensional assignment versions of the above problems have been presented elsewhere [37]. Here is a brief summary. As before, let $\mathcal{H}(P)$, $\mathcal{H}(Q)$, and $\mathcal{H}(R)$ denote collections of complete clusterings of P, Q, and R, respectively. Next, let $\mathcal{P} = \{P_i\}_{i\in I}$, $\mathcal{Q} = \{Q_j\}_{j\in J}$, and $\mathcal{R} = \{R_k\}_{k\in K_c}$ denote the collection of all unique clusters from the hypotheses $\mathcal{H}(P) = \{H_i(P)\}_{i\in I_H}$, $\mathcal{H}(Q) = \{H_j(Q)\}_{j\in J_H}$, and $\mathcal{H}(R) = \{H_k(R)\}_{k\in K_H}$, respectively.

Definition 7. Let P, Q, and R denote three lists of objects and let $\mathcal{P} = \{P_i\}_{i\in I}$, $\mathcal{Q} = \{Q_j\}_{j\in J}$, and $\mathcal{R} = \{R_k\}_{k\in K}$ denote collections of subsets of P, Q, and R, respectively. Define the indicator functions

$$
m_{pi} = \begin{cases} 1 & \text{if object } p \in P \text{ is in } P_i, \\ 0 & \text{otherwise,} \end{cases} \qquad n_{qj} = \begin{cases} 1 & \text{if object } q \in Q \text{ is in } Q_j, \\ 0 & \text{otherwise,} \end{cases}
$$

$$
o_{rk} = \begin{cases} 1 & \text{if object } r \in R \text{ is in } R_k, \\ 0 & \text{otherwise.} \end{cases}
$$

Given this definition, the problem formulation is

$$
\begin{aligned}
\text{Minimize} \quad & \sum_{(i,j,k)\in A} c_{ijk} x_{ijk}, \\
\text{Subject To} \quad & \sum_{(i,j,k)\in A} m_{pi} x_{ijk} \le 1 \quad (p \in P), \\
& \sum_{(i,j,k)\in A} n_{qj} x_{ijk} \le 1 \quad (q \in Q), \\
& \sum_{(i,j,k)\in A} o_{rk} x_{ijk} \le 1 \quad (r \in R), \\
& x_{ijk} \in \{0, 1\}.
\end{aligned} \tag{9}
$$

The constraints are enumerated based on the objects in P, Q, and R. Analogous to the second formulation of the two-dimensional cluster assignment problem, we have

$$
\begin{aligned}
\text{Minimize} \quad & \sum_{(i,j,k)\in A} c_{ijk} x_{ijk}, \\
\text{Subject To} \quad & \sum_{(j,k)\in A(i)} x_{ijk} \le m_i \quad (i \in I), \\
& \sum_{(i,k)\in B(j)} x_{ijk} \le n_j \quad (j \in J), \\
& \sum_{(i,j)\in C(k)} x_{ijk} \le o_k \quad (k \in K), \\
& x_{i_1 j_1 k_1} + x_{i_2 j_2 k_2} \le 1 \text{ for all } (i_1, j_1, k_1) \text{ and } (i_2, j_2, k_2) \in A \\
& \text{ for which } i_1 \ne i_2 \text{ and } P_{i_1} \cap P_{i_2} \ne \emptyset \text{ or} \\
& \qquad\qquad j_1 \ne j_2 \text{ and } Q_{j_1} \cap Q_{j_2} \ne \emptyset \text{ or} \\
& \qquad\qquad k_1 \ne k_2 \text{ and } R_{k_1} \cap R_{k_2} \ne \emptyset, \\
& x_{ijk} \in \{0, 1\},
\end{aligned}
\tag{10}
$$

where $m_i \ge 1$, $n_j \ge 1$, and $o_k \ge 1$. If $m_i = 1$, $n_j = 1$, and $o_k = 1$, then the set packing constraints can be replaced by sums similar to that discussed above for the two-dimensional problem.

6. Other tracking problems formulated as a group assignment problem

The group–cluster assignment problem formulation in the previous section has significant other applications in tracking beyond that motivated by cluster tracking. Fig. 2 illustrates the approaches to solving the group assignment problem through a simple example that illustrates three frames of data that contain 11, 10, and 8 objects, respectively. Through a problem-dependent mechanism, a number of partitioning, grouping, or clustering hypotheses are formed for each frame. The group assignment, that represents the problem of associating groups of objects over multiple frames, can then be solved either through the approach that explicitly enumerates the assignments between complete data clusterings or by solving the general group assignment. This general formalism applies to various problems and this section motivates three important classes of problems that fit within the scope of the group assignment problem: (1) the merged measurement problem, (2) multiple hypothesis track-to-track correlation, and (3) centroiding of narrowband radar returns. Similarities of the group assignment to two new forms of auctions, *combinatorial auctions* and *cooperative bidding auctions*, were motivated in [38]. In combinatorial auctions a bidder is allowed to bid on bundles or groups of sale items [39]. These problems are instances of the set packing problem and similar in scope to the merged measurement problem discussed in Section 6.1. With the emergence of the electronic market place an even more general form of auction has appeared that combines coalition forming of bidders with combinatorial auction [40]. In this problem, several bidders may form a coalition to bid on bundles of items. For example, when items are offered in bundles at wholesale prices, several bidders may improve their payoff by buying bundles in a coalition compared to buying the items of interest separately. In the final assignment of such an auction both the items and the bidders need to satisfy a set packing constraint. While each of the aforementioned problems has its unique mechanism of forming the multiple frame of data clustering hypotheses and of scoring assignments between clusters, they may be cast into the framework of the group assignment.

6.1. The merged measurement assignment problem

Much of the motivation for the formulation of the cluster assignment problem presented in the previous sections has been based on forming multiple clustering hypotheses on each frame of data and deciding which clustering hypothesis is correct based on viewing a window of frames of data. The objective in this section is to explain how the merged measurement problem, originally formulated by Blair et al. [41] follows exactly this same approach.

We assume that we have a set of tracks P and a set of measurements Q and let $\mathcal{H}(P)$ denote a collection of hypotheses that two or more tracks can be associated with a merged measurement and let $\mathcal{H}(Q)$ denote collections of complete clusterings of P and Q, respectively. Next, let $\mathcal{P} = \{P_i\}_{i\in I}$ and $\mathcal{Q} = \{Q_j\}_{j\in J}$ where $Q_j = \{j\}$ denotes the jth measurement. In this notation, we hypothesize that $P_i = \{i\}$ denotes the individual tracks for $i = 1, \ldots, M$, and

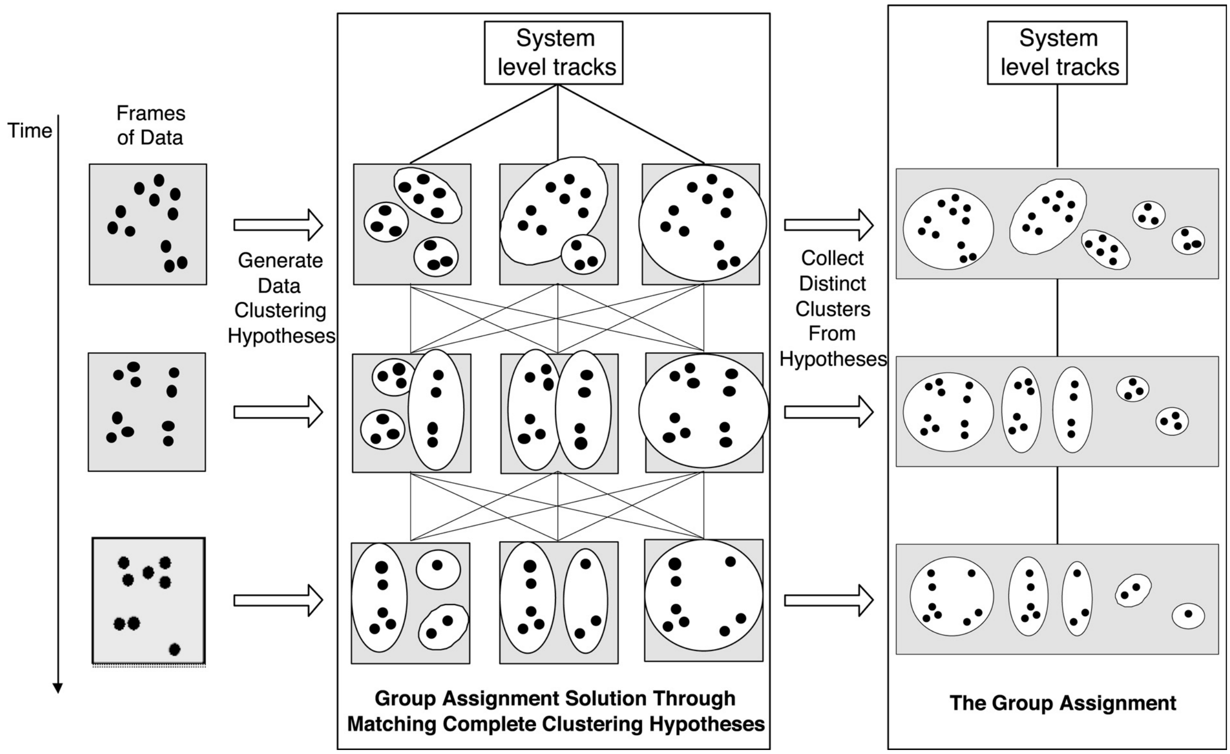

Fig. 2. Illustration of the solution approaches to solve the group assignment problem.

P_i $(i = M+1, \ldots, M+U)$ denotes combinations of these M tracks that might be associated with the unresolved measurements. Then, the problem

$$\begin{aligned} &\text{Minimize} && \sum_{(i,j)\in A} c_{ij} x_{ij}, \\ &\text{Subject To} && \sum_{(i,j)\in A} m_{ki} x_{ij} \le 1 \quad (k \in P), \\ &&& \sum_{i\in B(j)} x_{ij} \le 1 \quad (j = 1, \ldots, N) \\ &&& x_{ij} \in \{0, 1\} \end{aligned} \tag{11}$$

where

$$m_{ki} = \begin{cases} 1 & \text{if object } k \in P \text{ is in } P_i, \\ 0 & \text{otherwise,} \end{cases}$$

is equivalent to the formulation presented in the work of Blair et al., but perhaps a little different in notation in that we have used the indicator function m_{ki} instead of the double sum found in that paper. Thus, this formulation fits within the second cluster assignment formulation. A third formulation of the merged measurement assignment problem is given by Chen et al. [42], but this formulation is equivalent to the second cluster assignment formulation above wherein only one-to-one assignments are allowed in the association between groups.

A multiple-hypothesis, multiple-frame, approach to resolve merged measurements from an IR sensor (pixel-clusters on the focal plane) was recently introduced by Gadaleta et al. [43]. This pixel-cluster decomposition approach is similar to the multiple-hypothesis clustering approach discussed in Section 5.1 and can be generalized to fit the general group assignment.

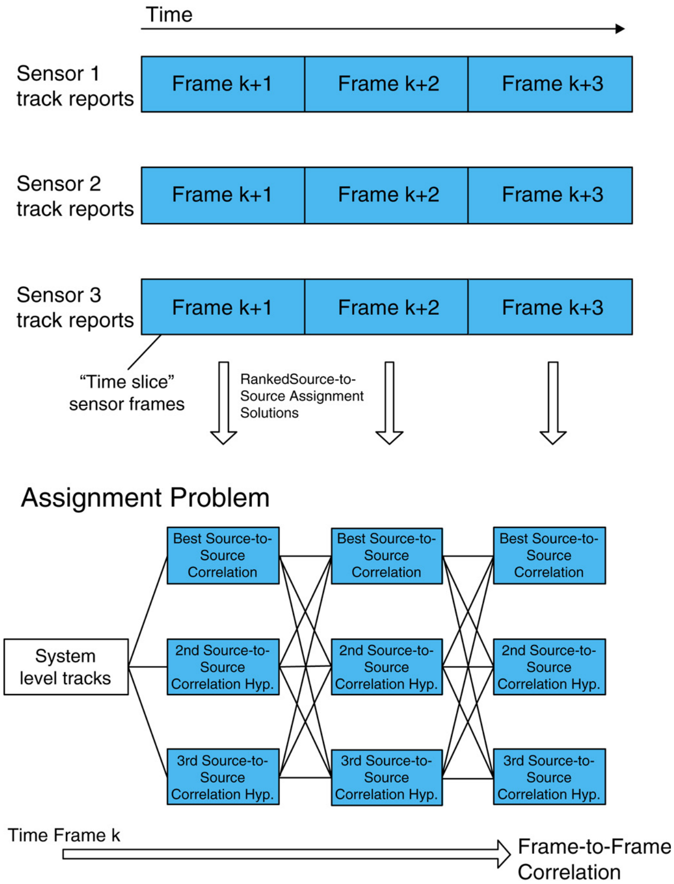

Fig. 3. Illustration of multiple hypothesis, multiple frame, correlation approach to frame-to-frame matching in source-to-source track fusion.

6.2. *Multiple hypothesis correlation in track-to-track fusion management*

In addition to measurement fusion, the problem of associating and fusing track states from multiple sources is an important problem. With respect to architectures for track-to-track correlation and fusion, one can distinguish between two fundamentally different architectures for track fusion: (1) *sensor-to-sensor track fusion* and (2) *sensor-to-system track fusion* [44]. The sensor-to-sensor architecture has the fundamental advantage that only the cross-correlations among the sensor-level tracks need to be addressed. However, since system level tracks are not directly maintained, the architecture requires a mechanism for matching and maintaining system-level tracks over time. Fig. 3 motivates a multiple hypothesis, multiple frame, correlation approach for system-level track management in sensor-to-sensor track fusion architectures that is similar to the multiple hypothesis clustering approach for group tracking discussed in Section 5.1.

In sensor-to-sensor track fusion, one first correlates tracks received from a number, say M, of sensors within a given time interval to obtain a set of *source-correlated tracks*. The source-correlation process represents an assignment problem over M frames of source track reports. The solution to the assignment problem allows one to form the source-correlated tracks. If the data is contentious, e.g., due to closely spaced objects, then the assignment may have many near likely solutions that can be exploited to form *multiple source-track correlation hypotheses* for a given set of sensor time frames. One efficient approach to generate the multiple source-to-source correlation hypotheses is to use Murty's algorithm [45] to find ranked assignment solutions of the source-to-source correlation assignment problem, or more generally, by solving a set of sequential two-dimensional assignment problems similar to the hypotheses generation in a multiple hypothesis tracking (MHT) system [46].

Given source-correlated tracks, the fundamental problem is to assign a set of source-correlated tracks to a set of system tracks that were formed on a previous frame. The multiple hypothesis, multiple frame, frame-to-frame matching approach uses the multiple generated source-track correlation hypotheses to form multiple system track extension hypotheses over multiple frames of data. The optimal system track extension is the *shortest path*, i.e., lowest cost path, through the *trellis* illustrated in the lower part of Fig. 3. Further details of this proposed approach, including a novel formulation for the assignment costs, can be found in [47].

6.3. Multiple hypothesis centroiding of radar returns

Slocumb and Blair introduced an approach to the maximum likelihood segmentation and centroid processing of narrowband radar data [48]. The goal of the algorithm is to form a single measurement report per object, given, potentially, multiple returns on an object. Multiple returns may be received when an object straddles two range cells. Given the primitive radar measurements, the approach forms multiple primitive measurement partition hypotheses and selects the optimal partition using a maximum likelihood approach.

An extension of the approach is possible that forms multiple partition hypotheses over multiple frames of data, potentially exploiting system tracker feedback for hypothesis generation and scoring [49]. In this general sense, the centroiding problem fits within the framework of the group assignment problem.

7. Summary

Data association is a central problem in the development of modern target tracking systems. This paper reviewed the mathematical statement of the two- and multi-dimensional assignment problem and the current algorithms and future needs. In addition, the newly derived group assignment problems arising from cluster tracking were presented. In cluster tracking the fundamental problem is to partition data into groups of data points and to assign correctly between groups of measurements on consecutive frames. The proposed approach requires the formation of multiple clustering hypotheses for each given frame of data, using either hard or soft clustering techniques. The optimal clustering for a frame of data can then be obtained from the solution of the group assignment problem which minimizes the cost of assigning clusters between frames of data. The formulated group assignment problem is of sufficient generality to deal with four major classes of problems, namely the (a) group–cluster tracking problem, (b) pixel-cluster tracking problem, (c) merged measurement problem, and (d) multiple hypothesis track-to-track correlation. In addition, the formulation accommodates one-to-one, many-to-one, and many-to-many assignments. Most importantly, these formulations represent generalized data association in the sense that the assignment problem reduces to the classical one if the groups do not overlap.

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