Financial Engineering & Risk Management

Introduction to Term Structure Lattice Models

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Fixed Income Markets

Fixed income markets are enormous and in fact bigger than equity markets.

According to *SIFMA*, in Q3 2012 the total outstanding amount of US bonds was $\$35.3$ trillion:

Government	\$10.7	30.4%
Municipal	\$3.7	10.5%
Mortgage	\$8.2	23.3%
Corporate	\$8.6	24.3%
Agency	\$2.4	6.7%
Asset-backed	\$1.7	4.8%
Total	\$35.3 tr	100%

– in comparison, size of US equity markets is approx $\$26$ trillion.

Fixed income derivatives markets are also enormous

– includes interest-rate and bond derivatives, credit derivatives, MBS and ABS

– will focus here on interest-rate and bond derivatives

– using binomial lattice models.

(The slides and Excel spreadsheet should be sufficient but Chapter 14 of Luenberger is an excellent reference for the material in this section.)

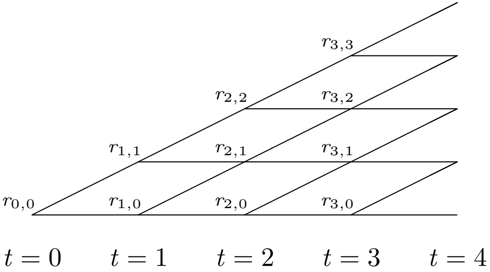
Binomial Models for the Short Rate

- Will use binomial lattice models as our vehicle for introducing:
  1. the mechanics of the most important fixed income derivative securities
     - bond futures (and forwards)
     - caplets and caps, floorlets and floors
     - swaps and swaptions
  2. the "philosophy" behind fixed income derivatives pricing
     – more on this soon.
- Fixed-income models are inherently more complex than security models
  - need to model evolution of entire **term-structure of interest rates**.
- The short-rate, r_t, is the variable of interest in many fixed income models
  – including binomial lattice models
  – r_t is the risk-free rate that applies between periods t and $t+1$
  – it is a **random process** but r_t is known by time t.

The “Philosophy” of Fixed Income Derivatives Pricing

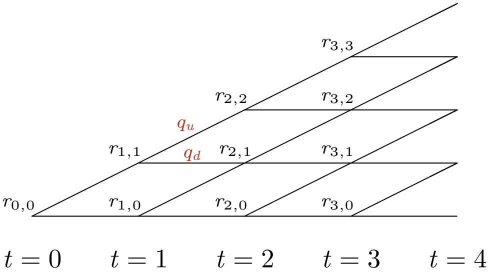
- We will simply specify risk-neutral probabilities for the short-rate, r_t
  - without any reference to the true probabilities of the short-rate
- This is in contrast to the binomial model for stocks where we specified p and $1 - p$
  - and then used replication arguments to obtain q and $1 - q$.
- We will price securities in such a way that **guarantees no-arbitrage**
- Will match market prices of liquid securities via a **calibration** procedure
  - often the most challenging part.
- Will see that derivatives pricing in practice is really about **extrapolating** from liquid security prices to illiquid security prices.

Binomial Models for the Short-Rate



- We will take zero-coupon bond (zcb) prices to be our basic securities
  - will use $Z_{i,j}^k$ for time i, state j price of a zcb that matures at time k
- Would like to specify binomial model by specifying all $Z_{i,j}^k$'s at all nodes
  - possible but awkward if we want to ensure no-arbitrage.
- Instead will specify the **short-rate**, $r_{i,j}$ at each node $N_{i,j}$
  - the risk-free rate that applies to the next period.

Binomial Models for the Short-Rate



- Let $Z_{i,j}$ be the date i, state j price of a non-coupon paying security.
- Will use risk-neutral pricing to price every security so that:

$$Z_{i,j} = \frac{1}{1 + r_{i,j}} \left[q_u \times Z_{i+1,j+1} + q_d \times Z_{i+1,j} \right] \tag{1}$$

  - where q_u and q_d are the risk-neutral probabilities of an up- and down-move
  - so $q_d + q_u = 1$ and must have $q_d > 0$ and $q_u > 0$.

- There can be **no arbitrage** when we price using (3). Why?

Binomial Models for the Short Rate

- If the security pays a "coupon", $C_{i+1,j}$, at date $i+1$ and state j then

$$Z_{i,j} = \frac{1}{1+r_{i,j}} \left[q_u \left(Z_{i+1,j+1} + C_{i+1,j+1} \right) + q_d \left(Z_{i+1,j} + C_{i+1,j} \right) \right] \quad (2)$$

  – where $Z_{i+1,\cdot}$ is now the **ex-coupon** price at date $i+1$.

- If we use (3) or (2) to price securities in the lattice model then arbitrage is not possible
  - Regardless of what probabilities we use! Why is this?

- In fact it is very common to simply set $q_u = q_d = 1/2$
  - and to calibrate other parameters to market prices.
- We will assume $q_u = q_d = 1/2$ in our examples.

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The Cash Account and Pricing Zero-Coupon Bonds

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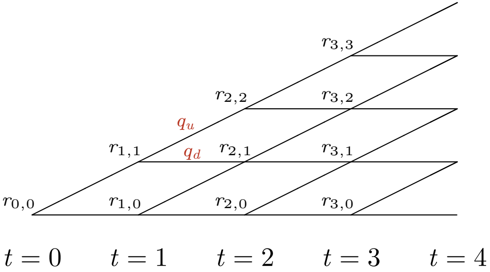
Binomial Models for the Short-Rate



- We use **risk-neutral** pricing to price every non-coupon paying security:

$$Z_{i,j} \;=\; \frac{1}{1 + r_{i,j}} \left[q_u \times Z_{i+1,j+1} \;+\; q_d \times Z_{i+1,j} \right] \tag{3}$$

  – $q_u > 0$ and $q_d > 0$ are the risk-neutral probabilities of an up- and down-move, respectively, of the short-rate.

- There can be **no arbitrage** when we price using (3). Why?

The Cash-Account

- The cash-account is a particular security that in each period earns interest at the short-rate
  - we use B_t to denote its value at time t and assume that $B_0 = 1$.
- The cash-account is **not** risk-free since B_{t+s} is not known at time t for any $s > 1$
  - it is **locally** risk-free since B_{t+1} is known at time t.
- Note that B_t satisfies $B_t \;=\; (1 + r_{0,0})(1 + r_1)\ldots(1 + r_{t-1})$
  - so that $B_t/B_{t+1} = 1/(1 + r_t)$.
- Risk-neutral pricing for a "non-coupon" paying security then takes the form:

$$\begin{aligned} Z_{t,j} &= \frac{1}{1 + r_{t,j}} \left[q_u \times Z_{t+1,j+1} \; + \; q_d \times Z_{t+1,j} \right] \\ &= \mathsf{E}_t^{\mathbb{Q}} \left[\frac{Z_{t+1}}{1 + r_{t,j}} \right] \\ &= \mathsf{E}_t^{\mathbb{Q}} \left[\frac{B_t}{B_{t+1}} Z_{t+1} \right] \end{aligned} \tag{4}$$

Risk-Neutral Pricing with the Cash-Account

- Therefore for a non-coupon paying security, (4) is equivalent to

$$\frac{Z_t}{B_t} = \mathsf{E}_t^{\mathbb{Q}}\left[\frac{Z_{t+1}}{B_{t+1}}\right] \tag{5}$$

- We can iterate (5) to obtain

$$\frac{Z_t}{B_t} = \mathsf{E}_t^{\mathbb{Q}}\left[\frac{Z_{t+s}}{B_{t+s}}\right] \tag{6}$$

for any non-coupon paying security and any $s > 0$.

Risk-Neutral Pricing with the Cash-Account

- **Risk-neutral** pricing for a "coupon" paying security takes the form:

$$
\begin{aligned}
Z_{t,j} &= \frac{1}{1+r_{t,j}} \left[q_u \left(Z_{t+1,j+1} + C_{t+1,j+1} \right) + q_d \left(Z_{t+1,j} + C_{t+1,j} \right) \right] \\
&= \mathsf{E}_t^{\mathbb{Q}} \left[\frac{Z_{t+1} + C_{t+1}}{1+r_{t,j}} \right] \qquad (7)
\end{aligned}
$$

- We can rewrite (7) as

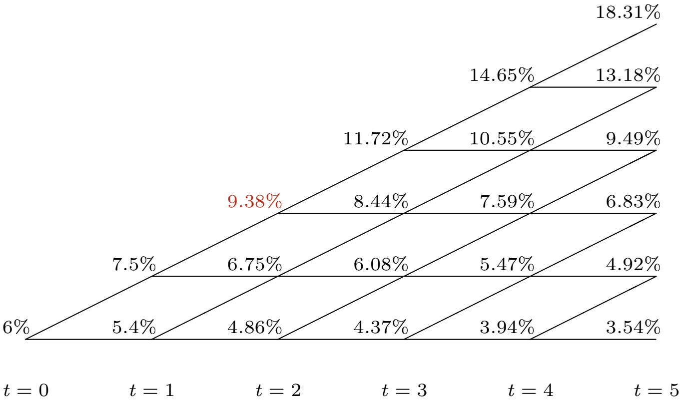
$$
\frac{Z_t}{B_t} = \mathsf{E}_t^{\mathbb{Q}} \left[\frac{C_{t+1}}{B_{t+1}} + \frac{Z_{t+1}}{B_{t+1}} \right] \qquad (8)
$$

- More generally, we can iterate (8) we obtain

$$
\frac{Z_t}{B_t} = \mathsf{E}_t^{\mathbb{Q}} \left[\sum_{j=t+1}^{t+s} \frac{C_j}{B_j} + \frac{Z_{t+s}}{B_{t+s}} \right] \qquad (9)
$$

- Pricing using (9) ensures **no-arbitrage**
  - note that (6) is a special case of (9).

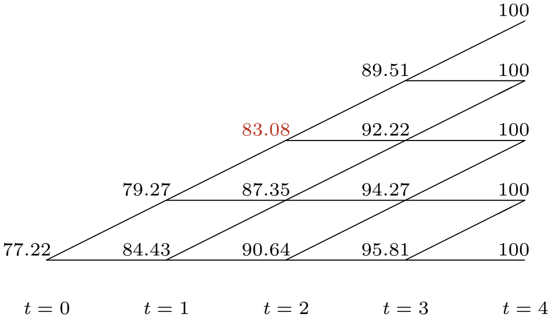
A Sample Short-Rate lattice



The short-rate, r, grows by a factor of $u = 1.25$ or $d = .9$ in each period

– not very realistic but more than sufficient for our purposes.

Pricing a ZCB that Matures at Time t=4

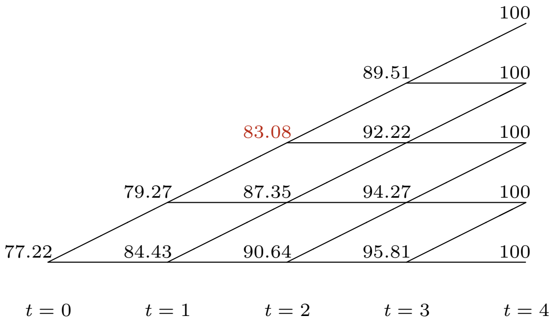


$$\text{e.g.} \quad 83.08 = \frac{1}{1 + .0938}\left[\frac{1}{2} \times 89.51 + \frac{1}{2} \times 92.22\right].$$

Can compute the term-structure by pricing ZCB's of every maturity and then backing out the spot-rates for those maturities

- so $s_4 = 6.68\%$ assuming per-period compounding, i.e., $77.22(1 + s_4)^4 = 100$.

Pricing a ZCB that Matures at Time t=4



Therefore can compute compute Z_0^1, Z_0^2, Z_0^3 and Z_0^4

– and then compute s_1, s_2, s_3 and s_4 to obtain the **term-structure of interest rates** at time $t = 0$.

At $t = 1$ we will compute new ZCB prices and obtain a new term-structure

– model for the short-rate, r_t, therefore defines a model for the term-structure!

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Fixed Income Derivatives: Options on Bonds

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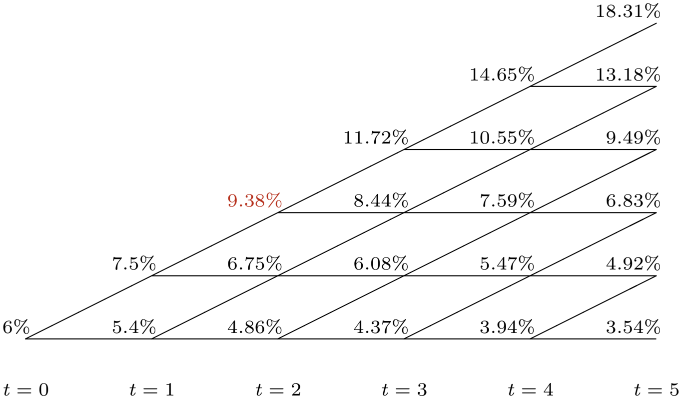
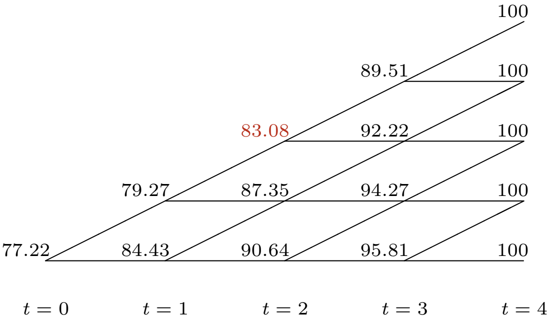
Our Sample Short-Rate lattice

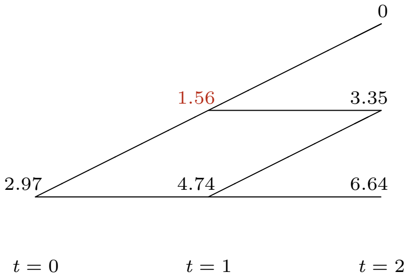


Pricing a ZCB that Matures at Time t=4



$$\text{e.g. } 83.08 = \frac{1}{1 + .0938}\left[\frac{1}{2} \times 89.51 + \frac{1}{2} \times 92.22\right].$$

Pricing a European Call Option on the ZCB



Strike $= \$84$

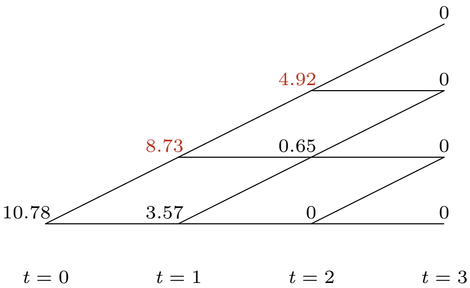
Option Expiration at $t = 2$

Option Payoff $= \max\left(0,\ Z^4_{2,\cdot} - 84\right)$

Underlying ZCB Matures at $t = 4$

$$\text{e.g.} \quad 1.56 \; = \; \frac{1}{1 + .075} \left[\frac{1}{2} \times 0 \; + \; \frac{1}{2} \times 3.35\right].$$

Pricing an American Put Option on a ZCB



Strike $= \$88$
Expiration at $t = 3$
Payoff at $t = 3$ is $\max(0,\ 88 - Z_{3,\cdot}^4)$
Underlying ZCB Matures at $t = 4$

$$\text{e.g.} \quad 4.92 \;=\; \max\left\{88 - 83.08\,,\ \frac{1}{1 + .0938}\ \left[\frac{1}{2} \times 0 \;+\; \frac{1}{2} \times 0\right]\right\}.$$

$$\text{e.g.} \quad 8.73 \;=\; \max\left\{88 - 79.27\,,\ \frac{1}{1 + .075}\ \left[\frac{1}{2} \times 4.92 \;+\; \frac{1}{2} \times 0.65\right]\right\}.$$

Turns out it's optimal early-exercise everywhere

– not a very realistic example.

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Fixed Income Derivatives: Bond Forwards

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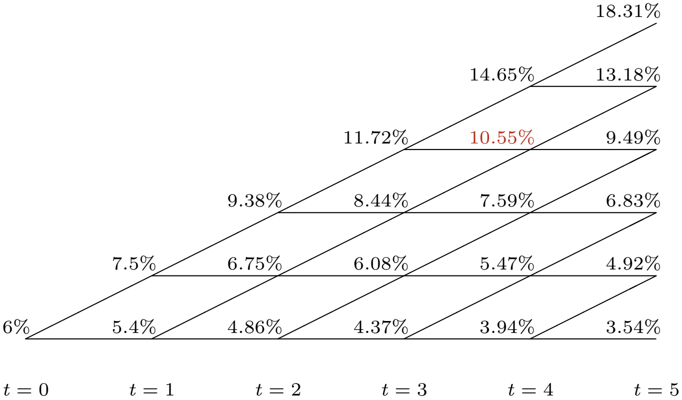
Our Sample Short-Rate lattice



Pricing a Forward on a Coupon-Bearing Bond

- Delivery at $t = 4$ of a 2-year 10% coupon-bearing bond.
- We assume delivery takes place just after a coupon has been paid.
- In the pricing lattice we use backwards induction to compute the $t = 4$ ex-coupon price of the bond.
- Let G_0 be the forward price at $t = 0$ and let Z_4^6 be the ex-coupon bond price at $t = 4$. Then risk-neutral pricing implies

$$0 \quad = \quad \mathsf{E}_0^{\mathbb{Q}} \left[\frac{Z_4^6 - G_0}{B_4} \right]$$

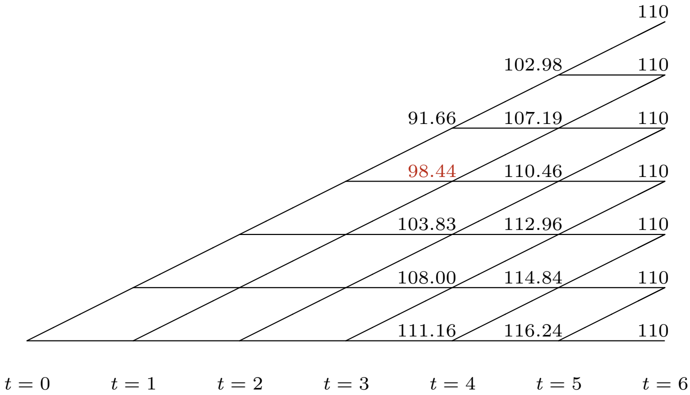
where B_4 is the value of the cash-account at $t = 4$.

- Rearranging terms and using the fact that G_0 is known at date $t = 0$ we obtain

$$G_0 \;=\; \frac{\mathsf{E}_0^{\mathbb{Q}} \left[Z_4^6 / B_4 \right]}{\mathsf{E}_0^{\mathbb{Q}} \left[1 / B_4 \right]}. \tag{10}$$

- Recall that $\mathsf{E}_0^{\mathbb{Q}} \left[1/B_4 \right]$ is time $t = 0$ price of a ZCB maturing at $t = 4$ but with a face value $\$1$
  - have already calculated this to be $.7722$.

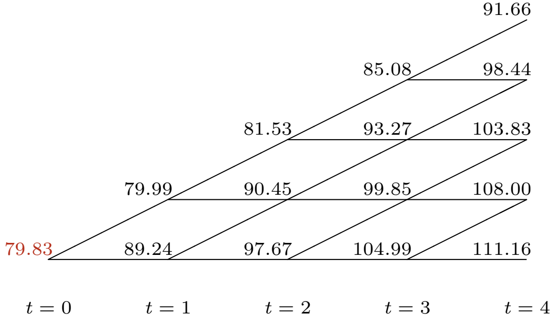
Pricing a Forward on a Coupon-Bearing Bond



First find ex-coupon price, Z_4^6, of the bond at time $t = 4$:

$$\text{e.g. } 98.44 = \frac{1}{1 + .1055}\left[\frac{1}{2} \times 107.19 + \frac{1}{2} \times 110.46\right].$$

Pricing a Forward on a Coupon-Bearing Bond



Now work backwards in lattice to compute $\mathsf{E}_0^{\mathbb{Q}}\left[Z_4^6/B_4\right] = 79.83$.

Can now use (13) to obtain

$$G_0 = \frac{79.83}{0.7722} = 103.38.$$

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Fixed Income Derivatives: Bond Futures

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Pricing Futures Contracts

- Let F_k be the date k price of a futures contract that expires after n periods.
- Let S_k denote the time k price of the security underlying the futures contract.
- Then $F_n = S_n$, i.e., at expiration the futures price and the underlying security price must coincide.
- Can compute the futures price at $t = n - 1$ by recalling that anytime we enter a futures contract, the initial value of the contract is 0.
- Therefore the futures price, F_{n-1}, at date $t = n - 1$ must satisfy (why?)

$$\frac{0}{B_{n-1}} = \mathsf{E}^{\mathbb{Q}}_{n-1}\left[\frac{F_n - F_{n-1}}{B_n}\right].$$

Pricing Futures Contracts

- Since B_n and F_{n-1} are both known at date $t = n-1$, we therefore have

$$F_{n-1} = \mathsf{E}^{\mathbb{Q}}_{n-1}[F_n].$$

- By the same argument, we obtain

$$F_k = \mathsf{E}^{\mathbb{Q}}_k[F_{k+1}] \quad \text{for } 0 \leq k < n.$$

- Can then use the law of iterated expectations to obtain

$$F_0 \;=\; \mathsf{E}^{\mathbb{Q}}_0\left[F_n\right].$$

- Since $F_n = S_n$ we have

$$F_0 \;=\; \mathsf{E}^{\mathbb{Q}}_0\left[S_n\right] \tag{11}$$

  – holds regardless of whether or not underlying security pays coupons etc.

- In contrast corresponding forward price, G_0, satisfies

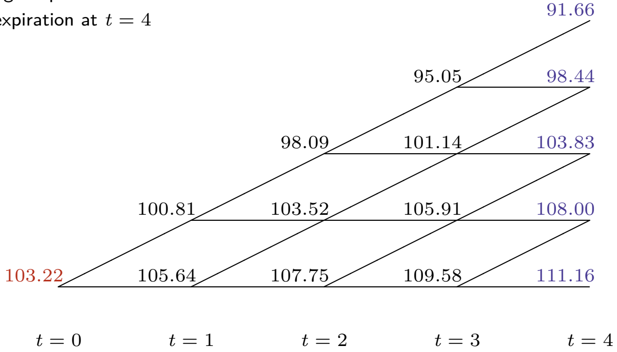
$$G_0 \;=\; \frac{\mathsf{E}^{\mathbb{Q}}_0\left[S_n/B_n\right]}{\mathsf{E}^{\mathbb{Q}}_0\left[1/B_n\right]}. \tag{12}$$

A Futures Contract on a Coupon-Bearing Bond

Futures contract written on same coupon bond as earlier forward contract

Underlying coupon bond matures at time $t = 6$

Futures expiration at $t = 4$



Note that the forward price, 103.38, and futures price, 103.22, are close

– but not equal!

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Fixed Income Derivatives: Caplets and Floorlets

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Pricing a Caplet

A caplet is similar to a European call option on the interest rate, r_t.

- Usually settled in arrears but they may also be settled in advance.
- If maturity is τ and strike is c, then payoff of a caplet (settled in arrears) at time τ is

$$(r_{\tau-1} - c)^+$$

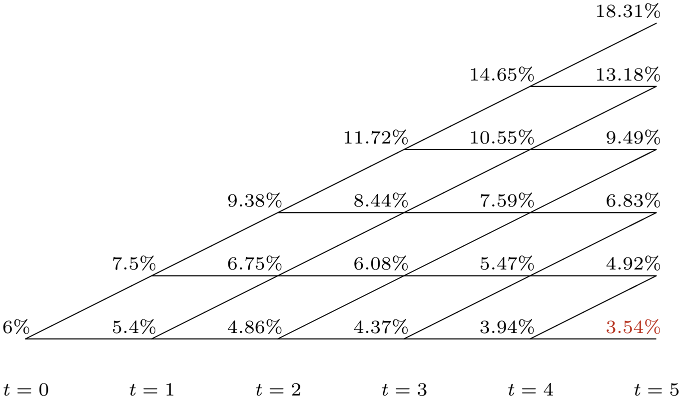
  – so the caplet is a call option on the short rate prevailing at time $\tau - 1$, settled at time τ.

A floorlet is the same as a caplet except the payoff is $(c - r_{\tau-1})^+$.

A cap consists of a sequence of caplets all of which have the same strike.

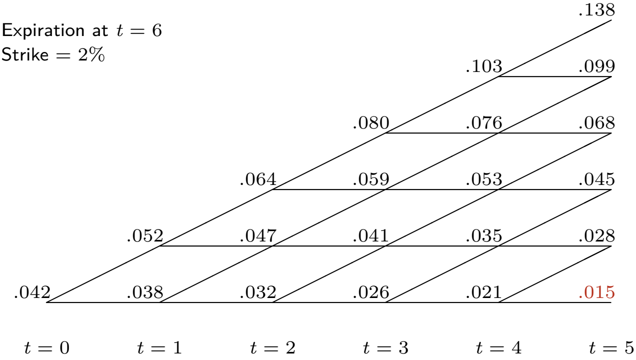
A floor consists of a sequence of floorlets all of which have the same strike.

Our Short-Rate lattice



Pricing a Caplet

Expiration at $t = 6$
Strike $= 2\%$



Note that it is easier to record the time $t = 6$ cash flows at their time 5 predecessor nodes, and then discount them appropriately:

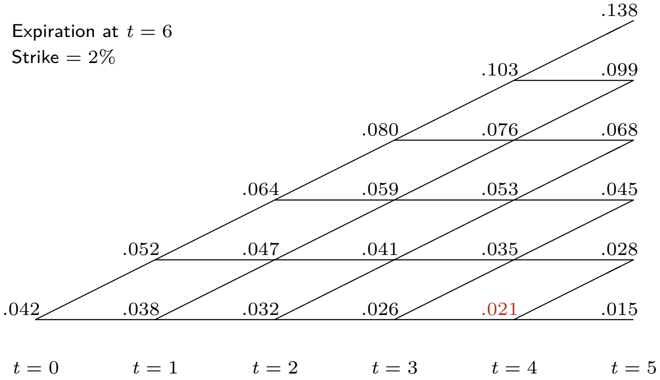
– so $(r_5 - c)^+$ at $t = 6$ is worth $(r_5 - c)^+/(1 + r_5)$ at $t = 5$.

A sample calculation:

$$0.015 \quad = \quad \frac{\max(0,\ .0354 - .02)}{1 + .0354}$$

Pricing a Caplet

Expiration at $t = 6$
Strike $= 2\%$



Now work backwards in the lattice to find the price at $t = 0$.

A sample calculation:

$$0.021 \quad = \quad \frac{1}{1.0394}\left[\frac{1}{2} \times 0.028 \; + \; \frac{1}{2} \times 0.015\right]$$

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Fixed Income Derivatives: Swaps and Swaptions

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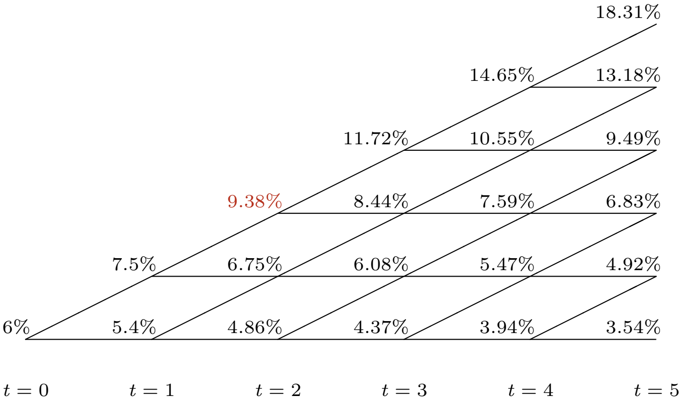
Our Short-Rate lattice



Want to price an interest-rate swap with fixed rate of 5% that expires at $t = 6$

- first payment at $t = 1$ and final payment at $t = 6$
- payment of $\pm(r_{i,j} - K)$ made at time $t = i + 1$ if in state j at time i.

Pricing Swaps

Expiration, i.e. last payment, at $t = 6$
First payment, at $t = 1$
Strike $=$ fixed rate $= 5\%$

$t=0$	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$
					.1125
				.1648	.0723
			.1793	.1014	.0410
		.1686	.1021	.0512	.0172
	.1403	.0829	.0400	.0122	−.0008
.0990	.0496	.0137	−.0085	−.0174	−.0141

Note that it is easier to record the time t cash flows at their time $t-1$ predecessor nodes, and then discount them appropriately:

– so $(r_{5,5} - K)$ at $t = 6$ is worth $\pm(r_{5,5} - K)/(1 + r_{5,5}) = .0723$ at $t = 5$.

A sample calculation:

$$0.1686 \quad = \quad \frac{1}{1.0938}\left[(.0938 - .05) \; + \; \frac{1}{2} \times 0.1793 \; + \; \frac{1}{2} \times 0.1021\right]$$

Pricing Swaptions

- A swaption is an option on a swap.
- Consider a swaption on the swap of the previous slide
  - will assume that the option strike is 0%
    - not to be confused with the strike, i.e. fixed rate, of underlying swap
  - and the swaption expiration is at $t = 3$.
- Swaption value at expiration is therefore $\max(0, \ S_3)$ where $S_3 \equiv$ underlying swap price at $t = 3$.
- Value at dates $0 \leq t < 3$ computed in usual manner by working backwards in the lattice
  - but underlying cash-flows of swap are not included at those times.

Pricing Swaptions

Fixed rate in Swap $= 5\%$
Underlying Swap Expiration at $t = 6$
Option Expiration at $t = 3$

$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
					.1125
				.1648	.0723
			.1793	.1014	.0410
		.1286	.1021	.0512	.0172
	.0908	.0665	.0400	.0122	$-.0008$
.0620	.0406	.0191	**0**	$-.0174$	$-.0141$

Swaption price is computed by determining payoff at maturity, i.e $t = 3$ and then working backwards in the lattice.

A sample calculation:

$$.0908 \;=\; \frac{1}{1 + .075}\left[\frac{1}{2} \times .1286 \;+\; \frac{1}{2} \times .0665\right]$$

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The Forward Equations

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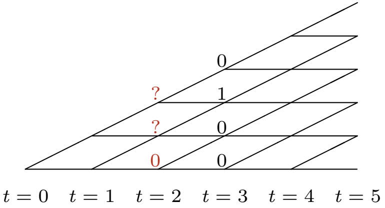
The Forward Equations

- $P^e_{i,j}$ denotes the time 0 price of a security that pays $\$1$ at time i, state j and 0 at every other time and state.
- Call such a security an elementary security and $P^e_{i,j}$ is its state price.
- Can see that elementary security prices satisfy the forward equations

$$
\begin{aligned}
P^e_{k+1,s} &= \frac{P^e_{k,s-1}}{2(1+r_{k,s-1})} + \frac{P^e_{k,s}}{2(1+r_{k,s})}, \quad 0 < s < k+1 \qquad (13) \\
P^e_{k+1,0} &= \frac{1}{2}\frac{P^e_{k,0}}{(1+r_{k,0})} \\
P^e_{k+1,k+1} &= \frac{1}{2}\frac{P^e_{k,k}}{(1+r_{k,k})}.
\end{aligned}
$$

with $P^e_{0,0} = 1$.

Deriving the Forward Equations



Consider the security that pays $\$1$ only at $t = 3$ and only in state 2

– value of this security is $P^e_{3,2}$ by definition.

But can also work backwards in lattice to price it. Its value at node $N_{2,2}$ is

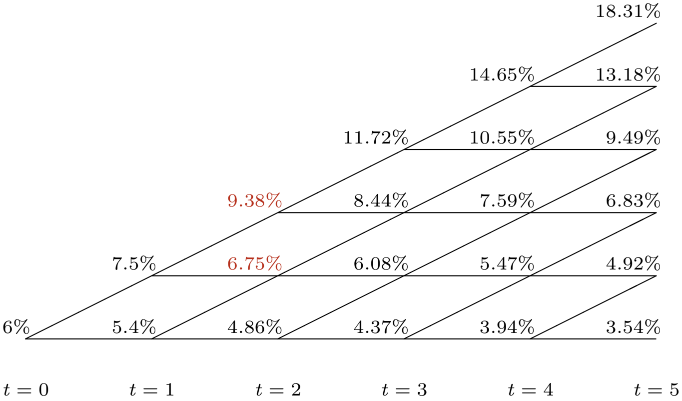
$$\frac{1}{1+r_{2,2}}\left[\frac{1}{2}\times 0 \;+\; \frac{1}{2}\times 1\right] \;=\; \frac{1}{2(1+r_{2,2})}$$

its value at node $N_{2,0}$ is 0, and its value at node $N_{2,1}$ is

$$\frac{1}{1+r_{2,1}}\left[\frac{1}{2}\times 1 \;+\; \frac{1}{2}\times 0\right] \;=\; \frac{1}{2(1+r_{2,1})}.$$

Therefore $P^e_{3,2} = \frac{1}{2(1+r_{2,2})} \times P^e_{2,2} \;+\; \frac{1}{2(1+r_{2,1})} \times P^e_{2,1} \;+\; 0 \times P^e_{2,0}$.

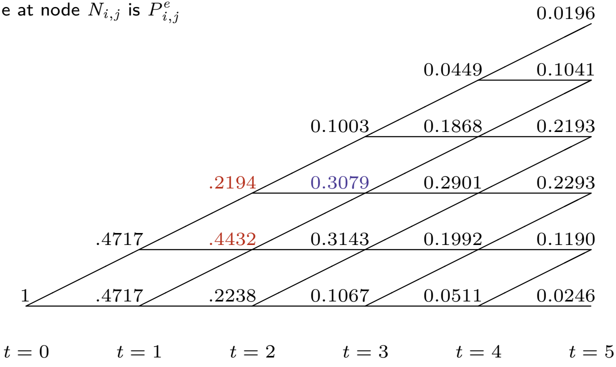
Our Short-Rate lattice



Now compute the forward prices by iterating the equations **forward** starting with $P^e_{0,0} = 1$.

… and the Corresponding Elementary Prices

Key: Value at node $N_{i,j}$ is $P^e_{i,j}$



Sample calculations:

$$
\begin{aligned}
.3079 &= \frac{P^e_{k,s-1}}{2(1 + r_{k,s-1})} + \frac{P^e_{k,s}}{2(1 + r_{k,s})} \\
&= \frac{.4432}{2(1 + .0675)} + \frac{.2194}{2(1 + .0938)}
\end{aligned}
$$

Derivative Prices Via Elementary Prices

Given the elementary prices the calculation of some security prices becomes very straightforward:

e.g. Can calculate Z_0^4 as

$$\begin{aligned} Z_0^4 &= 100 \times (.0449 + .1868 + .2901 + .1992 + .0511) \\ &= 77.22 \end{aligned}$$

– as calculated before.

Derivative Prices Via Elementary Prices

Consider a forward-start swap that begins at $t = 1$ and ends at $t = 3$

- notional principal is $\$1$ million
- fixed rate in the swap is 7%
- payments at $t = i$ for $i = 2, 3$ are based as usual on fixed rate minus floating rate that prevailed at $t = i - 1$

The "forward" feature of the swap is that it begins at $t = 1$

- first payment is then at $t = 2$ since payments are made in arrears.

Question: What is the value, V_0, of the forward swap today at $t = 0$?

Solution: The value is given by

$$\begin{aligned} V_0 &= \frac{(.07 - .0938)}{1.0938} \times .2194 + \frac{(.07 - .0675)}{1.0675} \times .4432 + \frac{(.07 - .0486)}{1.0486} \times .2238 \\ &\quad + \frac{(.07 - .075)}{1.075} \times .4717 + \frac{(.07 - .054)}{1.054} \times .4717 \\ &= \$5,800. \end{aligned}$$