1 Simulating Brownian motion (BM) and geometric Brownian motion (GBM)

For an introduction to how one can construct BM, see the Appendix at the end of these notes.

A stochastic process $\mathbf{B} = \{B(t) : t \geq 0\}$ possessing (wp1) continuous sample paths is called standard Brownian motion (BM) if

1. $B(0) = 0$.
2. $\mathbf{B}$ has both stationary and independent increments.
3. $B(t) - B(s)$ has a normal distribution with mean 0 and variance $t - s,\ 0 \leq s < t$.

2) and *3)* together can be summarized by: If $t_0 = 0 < t_1 < t_2 < \cdots < t_k$, then the increment rvs $B(t_i) - B(t_{i-1}),\ i \in \{1, \ldots k\}$, are independent with $B(t_i) - B(t_{i-1}) \sim N(0, t_i - t_{i-1})$ (normal with mean 0 and variance $t_i - t_{i-1}$). In particular, $B(t_i) - B(t_{i-1})$ is independent of $B(t_{i-1}) = B(t_{i-1}) - B(0)$.

If we only wish to simulate $B(t)$ at one fixed value t, then we need only generate a unit normal $Z \sim N(0, 1)$ and set $B(t) = \sqrt{t}Z$. But typically, we will want to simulate (say) k values at times $t_0 = 0 < t_1 < t_2 < \cdots < t_k$ to get the entire vector (with correlated coordinates):

$$(B(t_1), \ldots, B(t_k)).$$

We can easily do so by using the fact that $B(s + t) = B(s) + (B(s + t) - B(s))$, and $B(s)$ and $B(s + t) - B(s)$ are independent normals.

If we generate k iid unit normals $Z_1, Z_2, \ldots, Z_k$, then we can construct the independent increments via $B(t_i) - B(t_{i-1}) = \sqrt{t_i - t_{i-1}}Z_i,\ i = 1, \ldots, k$.

Thus to simulate the values $B(t_1), \ldots, B(t_k)$, we can use the recursion
$B(t_{i+1}) = B(t_i) + (B(t_{i+1}) - B(t_i)) = B(t_i) + \sqrt{t_{i+1} - t_i}Z_{i+1},\ i \in \{0, \ldots k - 1\}$.

This yields:

Simulating Standard BM at times $0 = t_0 < t_1 < t_2 < \cdots < t_k$*:*

Sequentially generate unit normals $Z_1, Z_2, \ldots, Z_k$, and recursively define

$$\begin{aligned} B(t_1) &= \sqrt{t_1}Z_1 \\ B(t_2) &= B(t_1) + \sqrt{t_2 - t_1}Z_2 = \sqrt{t_1}Z_1 + \sqrt{t_2 - t_1}Z_2 \\ &\vdots \\ B(t_k) &= \sum_{i=1}^{k} \sqrt{t_i - t_{i-1}}Z_i. \end{aligned}$$

In the end, to simulate BM then, we need only generate unit normals, which we already learned how to do using (for example) the polar method or the acceptance rejection algorithm.

1.1 BM with drift

$X(t) = \sigma B(t) + \mu t$ will denote the BM with drift $\mu \in \mathbb{R}$ and variance term $\sigma > 0$. It has continuous sample paths and is defined by

1. $X(0) = 0$.
2. **X** has both stationary and independent increments.
3. $X(t) - X(s)$ has a normal distribution with mean $\mu(t-s)$ and variance $\sigma^2(t-s)$, $0 \le s < t$.

$X(t) - X(s)$ thus can be constructed (simulated) by generating a standard normal rv Z and setting $X(t) - X(s) = \sigma\sqrt{t-s}Z + \mu(t-s)$. Again, by the stationary and independent increments, we can simulate such a BM at times $0 = t_0 < t_1 < t_2 < \cdots < t_k$, by generating k iid unit normals $Z_1, Z_2, \ldots, Z_k$ and using the recursion
$X(t_{i+1}) = X(t_i) + (X(t_{i+1}) - X(t_i)) = X(t_i) + \sigma\sqrt{t_{i+1} - t_i}Z_{i+1} + \mu(t_{i+1} - t_i)$.

Simulating BM with drift μ and variance term σ at times $0 = t_0 < t_1 < t_2 < \cdots < t_k$:

Sequentially generate unit normals $Z_1, Z_2, \ldots, Z_k$, and recursively define

$$\begin{aligned} X(t_1) &= \sigma\sqrt{t_1}Z_1 + \mu t_1 \\ X(t_2) &= X(t_1) + \sigma\sqrt{t_2 - t_1}Z_2 + \mu(t_2 - t_1) = \sigma\sqrt{t_1}Z_1 + \mu t_1 + \sigma\sqrt{t_2 - t_1}Z_2 + \mu(t_2 - t_1) \\ &\vdots \\ X(t_k) &= \sum_{i=1}^{k} (\sigma\sqrt{t_i - t_{i-1}}Z_i + \mu(t_i - t_{i-1})). \end{aligned}$$

1.2 Geometric BM

Geometric Brownian motion (GBM) is given by

$$S(t) = S(0)e^{X(t)},\ t \ge 0,$$

where $X(t) = \sigma B(t) + \mu t,\ t \ge 0$, is a BM. $e^{X(t)}$ has a lognormal distribution for each fixed $t > 0$. In general if $Y = e^X$ is lognormal with $X \sim N(\mu, \sigma^2)$, then we can easily simulate Y via setting $Y = e^{\sigma Z + \mu}$, with $Z \sim N(0, 1)$.

Moreover, for any $0 \le s < t$ it holds that

$$S(t) = S(0)\frac{S(s)}{S(0)} \times \frac{S(t)}{S(s)} = S(0)e^{X(s)} \times e^{X(t)-X(s)},$$

and since the increment $X(s)$ is independent of the increment $X(t) - X(s)$, we conclude that the consecutive ratios $\frac{S(s)}{S(0)}$ and $\frac{S(t)}{S(s)}$ are independent lognormals. We can thus simulate the pair $(S(s), S(t))$ by generating two iid $N(0, 1)$ rvs, Z_1, Z_2 and setting $S(s) = S(0)e^{\sigma\sqrt{s}Z_1 + \mu s},\ S(t) = S(s)e^{\sigma\sqrt{t-s}Z_2 + \mu(t-s)} = S(0)e^{\sigma\sqrt{s}Z_1 + \mu s} \times e^{\sigma\sqrt{t-s}Z_2 + \mu(t-s)}$.

More generally, for $0 = t_0 < t_1 < t_2 < \cdots < t_k$, define $Y_i = S(t_i)/S(t_{i-1}),\ i \in \{1, 2, \ldots, k\}$. Then we can write

$$\begin{aligned} S(t_1) &= S(0)Y_1 \\ S(t_2) &= S(t_1)Y_2 = S_0Y_1 \times Y_2 \\ &\vdots \\ S(t_k) &= S(t_{k-1})Y_k = S_0Y_1 \times Y_2 \times \cdots \times Y_k. \end{aligned}$$

The Y_i are independent lognormal rvs and can be constructed by generating k iid $N(0,1)$ rvs, $Z_1, Z_2, \ldots Z_k$ and setting

$$Y_i = e^{\sigma\sqrt{t_i - t_{i-1}}Z_i + \mu(t_i - t_{i-1})},\ i \in \{1, 2, \ldots, k\}. \tag{1}$$

Simulating paths of GBM is thus an easy consequence of our algorithm for simulating BM since for $0 = t_0 < t_1 < t_2 < \cdots < t_k$, the following recursion holds

$$S(t_{i+1}) = S(t_i)e^{X(t_{i+1}) - X(t_i)},\ i \in \{0, 1, \ldots, k-1\}.$$

Simulating Geometric BM (with drift μ and variance term σ) at times $0 = t_0 < t_1 < t_2 < \cdots < t_k$:

Sequentially generate unit normals $Z_1, Z_2, \ldots, Z_k$, and set the Y_i as in (1). Then recursively define

$$\begin{aligned} S(t_1) &= S(0)Y_1 \\ S(t_2) &= S(t_1)Y_2 = S_0Y_1 \times Y_2 \\ &\vdots \\ S(t_k) &= S(t_{k-1})Y_k = S_0Y_1 \times Y_2 \times \cdots \times Y_k. \end{aligned}$$

1.3 Applications in Financial Engineering

Here we let $S(0) = S_0$ denote the price per share of a risky asset (stock) initially, and $S(t) = S_0e^{X(t)}$ as the price at time t. The classic European call option with expiration date T and strike price K has payoff at time T of $C_T = (S(T) - K)^+$. The famous Black-Scholes-Merton option pricing theory/formula makes this option's price known explicitly, but other options(derivatives of the stock) are typically impossible to compute exactly; Monte Carlo simulation is thus commonly used to do estimate the prices.

Asian call option

A variation on a European call option (that is cheaper) is to average the price of the stock over the time interval $[0, T]$ and use that instead of $S(T)$ yielding payoff

$$\Big(\frac{1}{T}\int_0^T S(t)dt - K\Big)^+.$$

In practice one can't compute (or simulate exactly) such an average as $\frac{1}{T}\int_0^T S(t)dt$ to offer such an option, and instead one samples the price at a sequence of times such as the beginning of

each day. So let's assume that time is in days, that T is an integer and thus we will consider the payoff

$$C_T = \Big(\frac{1}{T}\sum_{n=1}^{T} S(n) - K\Big)^+, \tag{2}$$

and our objective is to estimate the expected payoff, $E(C_T)$. We thus need to simulate iid copies of C_T and then average.

Using the Y_i rvs from the previous Section, they are now iid since $t_i - t_{i-1} = 1$, and thus each $X(t_i) - X(t_{i-1)}$ has a $N(\mu, \sigma^2)$ distribution.

Using the recursion

$$S(n+1) = S(n)e^{X(n+1)-X(n)},\ n \in \{0, \dots T-1\},$$

we conclude that we can represent the stock prices by introducing T iid unit normals $Z_1, \dots, Z_T$, and rewriting

$$S(n+1) = S(n)e^{\sigma Z_{n+1}+\mu}.$$

We thus can construct a copy of C_T.

Up-and-out call option

Another variation on a European call option (that is cheaper) is the introduction of a upper barrier $b > 0$ which, in order that the holder receive payoff $(S(T) - K)^+$ at time T, the stock price must remain below the barrier at pre-specified times $0 < t_1 < t_2 < \cdots < t_k < T$. Thus the payoff is

$$C_T = (S(T) - K)^+ I\{S(t_1) < b, \dots, S(t_k) < b\};$$

if $S(t_i) \geq b$ at any one of the k check times $t_1, \dots, t_k$, then $C_T = 0$.

Using $S(t_{i+1}) = S(t_i)e^{X(t_{i+1})-X(t_i)}$, and noting that $X(t_{i+1})-X(t_i) \sim N(\mu(t_{i+1}-t_i), \sigma^2(t_{i+1}-t_i))$, we can construct sequentially the $S(t_i)$ and check if $S(t_i) \geq b$; if so then we stop and set $C_T = 0$; otherwise we continue to the next check point. If all k check points are passed without violating the barrier constraint, then we finally construct $S(T) = S(t_k)e^{X(T)-X(t_k)}$ with $X(T) - X(t_k) \sim N(\mu(T-t_k), \sigma^2(T-t_k))$, and set $C_T = (S(T) - K)^+$. This then gives us our first sample of C_T.

We illustrate this below:

Algorithm for generating one copy of C_T:

Input k and the k times $0 < t_1 < t_2 < \cdots < t_k < T$.
Initialize $t = t_1,\ S = S_0,\ i = 1$.

1. Generate Z. Reset $S = Se^{\sigma\sqrt{t}Z+\mu t}$.
2. If $S \geq b$ and $i \leq k$, then set $C_T = 0$ and stop.
3. Otherwise if $S < b$ and $i \leq k$, reset $i = i + 1$, $t = t_i - t_{i-1}$ and go back to *(1)*.
4. Otherwise if $i = k + 1$, then reset $t = T - t_k$, generate Z, reset $S = Se^{\sigma\sqrt{t}Z+\mu t}$, and set $C_T = (S - K)^+$ and stop.

1.4 Other Applications in Finance

Monte Carlo simulation can also be used to estimate other quantities of interest in finance that do not involve derivatives. For example, suppose you invest in two different stocks, $S_1(t)$ and $S_2(t)$, buying N_1 shares of the first and N_2 of the second. At time $t = 0$ you pay $V(0) = N_1S_1(0) + N_2S_2(0)$ for this portfolio, and at time t it will be worth $V(t) = N_1S_1(t) + N_2S_2(t)$. Although each stock price on its own has a lognormal distribution, the sum of the two does not; computations involving this sum can be intractable. The stocks are typically correlated too. Suppose you wish to compute the probability that at time $t = T$, the value of your investment has increased by at least 10%. That is, you wish to compute

$$P(V(T) \geq (1.1)V(0)).$$

This is equivalent to taking expected values of the indicator function

$$X = I\{N_1S_1(T) + N_2S_2(T) \geq (1.1)V(0)\}.$$

For illustration, let's suppose the two stocks are independent in which case by letting Z_1 and Z_2 denote two generated iid unit normals, we can construct X via

$$X = I\{N_1S_1(0)e^{\sigma_1\sqrt{T}Z_1+\mu_1T} + N_2S_2(0)e^{\sigma_2\sqrt{T}Z_2+\mu_2T} \geq (1.1)V(0)\}.$$

In general, some correlation exists between the two stocks, and this can be incorporated by generating the unit normals with a desired correlation:

> One can construct *correlated* unit normals Z_1 and Z_2 with any desired correlation coefficient $-1 < \rho < 1$ by first generating Y_1 and Y_2 iid unit ones and setting $Z_1 = Y_1$ and $Z_2 = \rho Y_1 + \sqrt{1-\rho^2}Y_2$.

Finally, simulating n iid copies $X_1, X_2, \ldots, X_n$ of X yields our estimator

$$P(V(T) \geq (1.1)V(0)) \approx \frac{1}{n}\sum_{i=1}^{n} X_i.$$

Note: Taking two independent standard Brownian motions, $W_1(t),\ W_2(t)$ we can construct a correlated two-dimensional Brownian motion via defining
$B_1(t) = W_1(t),\ B_2(t) = \rho W_1(t) + \sqrt{1-\rho^2}W_2(t)$.

1.5 APPENDIX: Construction of Brownian motion from the simple symmetric random walk

Recall the simple *symmetric* random walk, $R_0 = 0$,

$$R_n = \Delta_1 + \cdots + \Delta_n = \sum_{i=1}^{n} \Delta_i,\ n \geq 1,$$

where the Δ_i are iid with $P(\Delta = -1) = P(\Delta = 1) = 0.5$. thus $E(\Delta) = 0$ and $Var(\Delta) = E(\Delta^2) = 1$.

We view time n in minutes, and R_n as the position at time n of a particle, moving on $\mathbb{R}$, which every minute takes a step, of size 1, equally likley to be forwards or backwards. Because $E(\Delta) = 0$ and $Var(\Delta) = 1$, it follows that $E(R_n) = 0$ and $Var(R_n) = n,\ n \geq 0$.

Choosing a large integer $k > 1$, if we instead make the particle take a step every $1/k$ minutes and make the step size $1/\sqrt{k}$, then by time t the particle will have taken a large number, $n = tk$, of steps and its position will be

$$B_k(t) = \frac{1}{\sqrt{k}} \sum_{i=1}^{tk} \Delta_i. \tag{3}$$

(By convention if tk is not an integer then we replace it by the largest integer less than or equal to it; denoted by $[tk]$.) This leads to the particle taking many many iid steps, but each of small magnitude, in any given interval of time. We expect that as $k \to \infty$, these small steps become a continuum and the process $\{B_k(t) : t \geq 0\}$ should converge to a process $\{B(t) : t \geq 0\}$ with continuous sample paths. We call this process Brownian motion (BM) after the Scottish botanist Robert Brown.[1] Its properties will be derived next.

Notice that for fixed k, any increment

$$B_k(t) - B_k(s) = \frac{1}{\sqrt{k}} \sum_{i=sk+1}^{tk} \Delta_i, \ 0 \leq s < t,$$

has a distribution that only depends on the length, $t - s$, of the time interval $(s, t]$ because it only depends on the number, $k(t - s)$, of iid Δ_i making up its construction. Thus we deduce that the limiting process (as $k \to \infty$) will possess *stationary increments*: *The distribution of any increment $B(t) - B(s)$ has a distribution that only depends on the length of the time interval $t - s$. In particular, $B(t) - B(s)$ has the same distribution as does $B(t - s)$.*

Notice further that given two non-overlapping time intervals, $(t_1, t_2]$ and $(t_3, t_4]$, $0 \leq t_1 < t_2 < t_3 < t_4$, the corresponding increments

$$B_k(t_4) - B_k(t_3) = \frac{1}{\sqrt{k}} \sum_{i=t_3k+1}^{t_4k} \Delta_i, \tag{4}$$

$$B_k(t_2) - B_k(t_1) = \frac{1}{\sqrt{k}} \sum_{i=t_1k+1}^{t_2k} \Delta_i, \tag{5}$$

are independent because they are constructed from different Δ_i. Thus we deduce that the limiting process (as $k \to \infty$) will also possess *independent increments*: *For any non-overlapping time intervals, $(t_1, t_2]$ and $(t_3, t_4]$, the increment rvs $I_1 = B(t_2) - B(t_1)$ and $I_2 = B(t_4) - B(t_3)$ are independent.*

Observing that $E(B_k(t)) = 0$ and $Var(B_k(t)) = [tk]/k \to t$, $k \to \infty$, we infer that the limiting process will satisfy $E(B(t)) = 0$, $Var(B(t)) = t$ just like the random walk $\{R_n\}$ does in discrete-time n ($E(R_n) = 0$, $Var(R_n) = n$).

Finally, a direct application of the central limit theorem CLT yields

$$B_k(t) = \sqrt{t}\Big(\frac{1}{\sqrt{kt}} \sum_{i=1}^{tk} \Delta_i\Big) \Longrightarrow N(0, t), \ k \to \infty, \ \textit{in distribution},$$

[1] Brown himself noticed in 1827, while carrying out some experiments, the unusual "motion" of particles within pollen grains suspended in water, under his microscope. The physical cause of such motion (bombardment of the particles by water molecules undergoing thermal motion) was not formalized via kinetic theory until Einstein in 1905. The rigorous mathematical construction of a stochastic process as a model for such motion is due to the mathematician Norbert Weiner; that is why it is sometimes called a Weiner process.

and we conclude that for each fixed $t > 0$, $B(t)$ has a normal distribution with mean 0 and variance t. Similarly, using the stationary and independent increments property, we conclude that $B(t) - B(s)$ has a normal distribution with mean 0 and variance $t - s$, and more generally:

> *the limiting BM process is a process with continuous sample paths that has both stationary and independent normally distributed (Gaussian) increments: If* $t_0 = 0 < t_1 < t_2 < \cdots < t_n$*, then the rvs.* $B(t_i) - B(t_{i-1}),\ i \in \{1, \ldots n\}$*, are independent with* $B(t_i) - B(t_{i-1}) \sim N(0, t_i - t_{i-1})$*.*