# Financial Engineering and Risk Management

## Modeling defaultable bonds

Martin Haugh Garud Iyengar

Columbia University
Industrial Engineering and Operations Research

# Defaultable bonds

Characterized by

- Coupon $c$
- Face value $F$
- Recovery value $R =$ random fraction of face value recovered on default

As before, will work directly with the risk neutral probability $\mathbb{Q}$

Will model the term structure of default using a $1$-step default probability

$$h(t) = \mathbb{Q}(\text{bond defaults in } [t, t+1) \mid \mathcal{F}_t).$$

Will calibrate $h(t)$ to market prices.

Will modify the binomial lattice to include defaults.

# Binomial lattice for short rates

Binomial lattice for the short rate

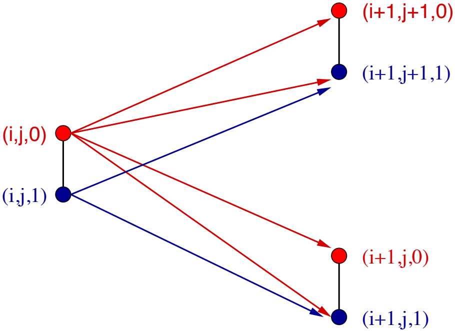
- nodes $(i, j)$: date $i = 0, \ldots, n$ and state $j = 0, \ldots, i$
- short rate $r_{ij}$
- state transition probability

$$\mathbb{Q}\big((i+1, s) \mid (i, j)\big) = \begin{cases} q_u & s = j+1 \\ q_d & s = j \\ 0 & \text{otherwise} \end{cases}$$

"Split" node $(i, j)$ by introducing a variable that encodes whether or not default has occurred before date $i$

- $(i, j, 0)$ = state $j$ on date $i$ with default time $\tau > i$
- $(i, j, 1)$ = state $j$ on date $i$ with default time $\tau \leq i$

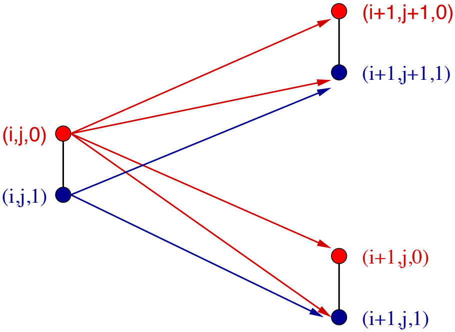
# Binomial lattice with defaults




Transitions from no-default state $(i, j, 0)$

$$\mathbb{Q}\big((i+1, s, \eta) \mid (i, j, 0)\big) = \begin{cases} q_u h_{ij} & s = j+1, \eta = 1 \text{ (default)} \\ q_u(1 - h_{ij}) & s = j+1, \eta = 0 \text{ (no default)} \\ q_d h_{ij} & s = j, \eta = 1 \text{ (default)} \\ q_d(1 - h_{ij}) & s = j, \eta = 0 \text{ (no default)} \\ 0 & \text{otherwise} \end{cases}$$

# Binomial lattice with defaults




Transitions from default state $(i,j,1)$: default state is an "absorbing" state

$$\mathbb{Q}\big((i+1,s,\eta) \mid (i,j,1)\big) = \begin{cases} q_u & s = j+1, \eta = 1 \text{ (default)} \\ q_d & s = j, \eta = 1 \text{ (default)} \\ 0 & \text{otherwise} \end{cases}$$

Conditional default probability $h_{ij}$ is state dependent.

# Default-free zero-coupon bonds (ZCBs)

Default-free zero-coupon bonds with expiration $T$

- pay \$1 in every state at the expiration date $T$
- no default possible

Pricing

- $Z^T_{i,j,\eta}$ = price of a bond maturing on date $T$ in node $(i,j,\eta)$
- Default events do not affect default-free bonds

$$Z^T_{i,j,1} = Z^T_{i,j,0} := Z^T_{i,j}$$

- Risk-neutral pricing:

$$Z^T_{i,j} = \frac{1}{1+r_{ij}}\left[q_u Z^T_{i+1,j+1} + q_d Z^T_{i+1,j}\right]$$

Calibrate the short-rate lattice using the prices of the default-free ZCBs and other default-free instruments.

# Defaultable ZCBs with no recovery

Defaultable zero-coupon bonds with expiration $T$ and no-recovery on default

- pay \$ 1 in every state at the expiration date $T$ provided default has not occurred at any date $t \leq T$.
- If default occurs at any date $t \leq T$, the bond pays $0$, i.e. there is no recovery.

Pricing

- $\bar{Z}^T_{i,j,\eta}$ = price of a bond maturing on date $T$ in node $(i, j, \eta)$
- No recovery implies that $\bar{Z}^T_{i,j,1} \equiv 0$ in all default nodes $(i, j, 1)$
- Risk-neutral pricing:

$$\begin{aligned}\bar{Z}^T_{i,j,0} &= \frac{1}{1+r_{ij}}\Big[q_u(1-h_{ij})\bar{Z}^T_{i+1,j+1,0} + q_d(1-h_{ij})\bar{Z}^T_{i+1,j,0}\Big] \\ &\quad + \frac{1}{1+r_{ij}}\Big[q_u h_{ij}\underbrace{\bar{Z}^T_{i+1,j+1,1}}_{\equiv 0} + q_d h_{ij}\underbrace{\bar{Z}^T_{i+1,j,1}}_{\equiv 0}\Big]\end{aligned}$$

Calibrate $h_{ij}$ using the prices of the defaultable ZCBs.

# Defaultable ZCBs with no-recovery (contd)

Risk-neutral prices

$$\begin{aligned}\bar{Z}^T_{i,j,0} &= \frac{1-h_{ij}}{1+r_{ij}}\left[q_u \bar{Z}^T_{i+1,j+1,0} + q_d \bar{Z}^T_{i+1,j,0}\right] \\ &\approx e^{-(r_{ij}+h_{ij})}\mathbb{E}^{\bar{\mathbb{Q}}}_i[\bar{Z}^T_{i+1,\cdot,\cdot}]\end{aligned}$$

where $\bar{\mathbb{Q}}$ is the default-free risk-neutral probability.

Price of a defaultable ZCB is set by discounting the expected value by $(r_{ij} + h_{ij})$

- $h_{ij}$ is the 1-period **credit spread**
- conditional probability of default $h_{ij}$ also called the **hazard rate**.

# ZCBs with recovery

**Assumption**: Random recovery $\tilde{R}$ is independent of the default and interest dynamics. Let $R = \mathbb{E}[\tilde{R}]$

$\bar{Z}^T_{i,j,\eta}$ = price of a bond maturing on date $T$ in node $(i, j, \eta)$ after recovery

- $\bar{Z}^T_{i,j,1} \equiv 0$ in all default nodes $(i, j, 1)$

Risk-neutral pricing

$$\begin{aligned} \bar{Z}^T_{i,j,0} &= \frac{1}{1 + r_{ij}} \Big[ q_u (1 - h_{ij}) \bar{Z}^T_{i+1,j+1,0} + q_d (1 - h_{ij}) \bar{Z}^T_{i+1,j,0} \Big] \\ &\quad + \frac{1}{1 + r_{ij}} \Big[ q_u h_{ij} R + q_d h_{ij} R \Big] \end{aligned}$$

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## Pricing defaultable bonds

**Martin Haugh** **Garud Iyengar**

Columbia University
Industrial Engineering and Operations Research

# State-independent hazard rates

**Assumption**: hazard rates $h_{ij}$ are state-independent, i.e. $h_{ij} = h_i$

- Ensures that the default event is independent of interest rate dynamics
- Let $q(t) =$ risk-neutral probability that the bond survives until date $t$
- Then $q(t+1) = (1-h_t)q(t) = \prod_{k=0}^{t}(1-h_k)$

Let $I(t)$ denote the indicator variable that bond survives up to time $t$, i.e.

$$I(t) = \begin{cases} 1 & \text{Bond is not in default at time } t \\ 0 & \text{Otherwise} \end{cases}$$

Then the indicator variable for default at time $t$ is $I(t-1) - I(t)$.

From the definition of $I(t)$ is follows that

$$\mathbb{E}_0^Q[I(t)] = q(t)$$

# Pricing bonds with recovery

**Assumption**: Random recovery rate $\tilde{R}$ is independent of interest rate dynamics under $\mathbb{Q}$. Let $R = \mathbb{E}_0^Q[\tilde{R}]$

- $\tilde{R}$ is the fraction of the face value $F$ paid on default

Pricing details:

- $t_0 = 0$ is the current date
- $\{t_1, \ldots, t_n\}$ are the future dates at which coupons are paid out.
- The coupon is paid on date $t_k$ only if $I(t_k) = 1$. Therefore the random cash flow associated with the coupon payment on date $t_k$ is $cI(t_k)$.
- The face value $F$ is paid on date $t_n$ only if $I(t_n) = 1$. Therefore, the random cash flow associated with the face value payment on date $t_n$ is $FI(t_n)$.
- The recovery $\tilde{R}(t_k)F$ is paid on date $t_k$ is the bond defaults on date $t_k$. Therefore, the random cash flow associated with recovery on date $t_k$ is $\tilde{R}(t_k)F\big(I(t_{k-1}) - I(t_k)\big)$.

# Pricing formula

Let $B(t)$ denote the value of the cash account at time $t$.

Then the price $\bar{P}(0)$ of defaultable fixed coupon bond is given by

$$
\begin{aligned}
\bar{P}(0) &= \mathbb{E}_0^{\mathbb{Q}}\left[\sum_{k=1}^{n}\frac{cI(t_k)}{B(t_k)} + FI(t_n)\frac{B(t)}{B(t_n)} + \sum_{k=1}^{n}\frac{R(t_k)F}{B(t_k)}\left(I(t_{k-1}) - I(t_k)\right)\right] \\
&= \sum_{k=1}^{n} c\mathbb{E}_0^{\mathbb{Q}}[I(t_k)]\cdot\mathbb{E}_0^{\mathbb{Q}}\left[\frac{1}{B(t_k)}\right] + F\mathbb{E}_0^{\mathbb{Q}}[I(t_n)]\cdot\mathbb{E}_0^{\mathbb{Q}}\left[\frac{1}{B(t_n)}\right] \\
&\quad + RF\sum_{k=1}^{n}\left(\mathbb{E}_0^{\mathbb{Q}}[I(t_{k-1})] - \mathbb{E}_0^{\mathbb{Q}}[I(t_k)]\right)\cdot\mathbb{E}_0^{\mathbb{Q}}\left[\frac{1}{B(t_k)}\right] \\
&= \sum_{k=1}^{n} cq(t_k)Z_0^{t_k} + Fq(t_n)Z_0^{t_n} + RF\sum_{k=1}^{n}\big(q(t_{k-1}) - q(t_k)\big)Z_0^{t_k} \\
&= \sum_{k=1}^{n} cq(t_k)d(0,t_k) + Fq(t_n)d(0,t_n) + RF\sum_{k=1}^{n}\big(q(t_{k-1}) - q(t_k)\big)d(0,t_k)
\end{aligned}
$$

# Calibrating hazard rates

Assume interest rate $r$ is deterministic and known.

Model price $P(\boldsymbol{h})$ of defaultable bond is a function $\boldsymbol{h} = (h_0, \ldots, h_{n-1})$.

Observe market prices for $m$ bonds

- $P_i^{\mathsf{mkt}} =$ market price for $i$-th bond with expected recovery $R_i$, $i = 1, \ldots, m$

Assumption: default of all bonds induced by the same "credit event".

Model calibration

- Model price of $i$-th bond: $P_i(\boldsymbol{h})$
- Pricing error: $f(h) = \sum_{i=1}^{m} (P_i^{\mathsf{mkt}} - P_i(\boldsymbol{h}))^2$
- Calibration problem: $\min_{\boldsymbol{h} \geq 0} f(\boldsymbol{h})$

Numerical example in spreadsheet `bonds_cds.xlsx`

# Financial Engineering and Risk Management

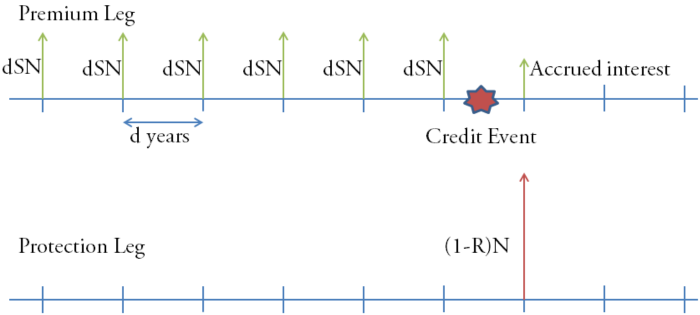
## Credit Default Swaps

Martin Haugh    Garud Iyengar

Columbia University
Industrial Engineering and Operations Research

# Credit Default Swaps

The seller of a credit default swap (CDS) agrees to compensate the buyer in the event of a loan default or some other credit event on a reference entity, e.g. a corporation, or sovereign, in return for periodic premium payments.




- $N$ = notional principal amount of credit protection
- Accrued interest = interest accumulated from last coupon to default
- $S$ = coupon or spread
- CDS seller has to compensate the buyer $(1 - R)N$ on default.

# Credit Default Swaps: Numerical Example

Consider a (hypothetical) $2$-year CDS

- Notional principal $N = \$1$ million
- Spread $S = 160$ bps
- Quarterly premium payments

Suppose a default event occurs in month 16 of the 24 month protection period and the recovery rate $R = 45\%$.

Payments of protection **buyer**

- Fixed premiums in months $3, 6, 9, 12, 15 = \frac{SN}{4} = \$4000$
- Accrued interest in month 18 $= \frac{SN}{12} = \$1333.33$

Payment of protection **seller**

- Default contingent payment in month $18 = (1 - R)N = \$550,000$

# Basic model for the CDS cash flows

- Let $\{t_k = \delta k : k = 1, \ldots, t_n\}$ denote times of coupon payments. Typically $\delta = \frac{1}{4}$, i.e. quarterly payments, and the dates of the payments are Mar. 20, Jun. 20, Sept. 20, Dec. 20.

- If reference entity **is not in default** at time $t_k$, the buyer pays the premium $\delta SN$ where $S$ is called the CDS spread or coupon.

- If the reference entity **defaults** at time $\tau \in (t_{k-1}, t_k]$, the contract terminates at time $t_k$. At time $t_k$
  - the buyer pays the accrued interest $(t_k - \tau)SN$, and
  - the buyer receives $(1 - R)N$ where $R$ denotes the recovery rate of the underlying.

# CDS contract details

Standardized by the International Swaps and Derivative Association in 1999.

Changes were made in 2003, and then again in 2009, may happen yet again if CDSs become exchange traded.

Many difficult issues

- How does one decide a credit event has occurred?
- How does one determine the recovery rate?

Many many details . . .

- How is the spread set? For junk bonds vs investment grade bonds? For sovereigns?
- When is coupon paid? In advance or in arrears?
- How is the spread quoted?

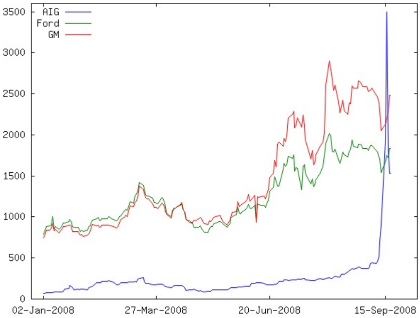
Will focus on the basic model to illustrate the details of pricing and sensitivity to hazard rates.

# CDS spreads measure of default risk

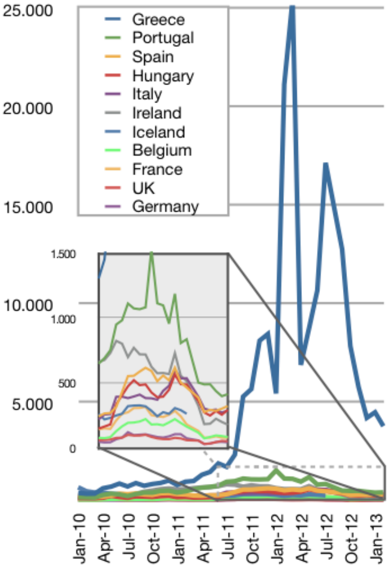
Show later that CDS spread $S \approx (1-R)h$ where $h$ is the hazard rate, or conditional probability of default.

For fixed $R$, the CDS spreads are directly proportional to the hazard rate $h$.

Plot of the 5yr CDS spreads for Ford, GM and AIG in the first 9 months of 2008.



## Sovereign Credit Default Swaps




Source: Bloomberg, CNBC

# Development and applications of CDS

Development credited to Blythe Masters from J. P. Morgan in 1994

- J. P. Morgan extended a $4.8 billion credit line to Exxon to cover the possible punitive damage from Exxon Valdez spill
- Bought protection from the European Bank for Reconstruction and Development using a CDS.

CDS market has grown tremendously

- By the end of 2007, the CDS market had notional value of $62 trillion.
- The Depository Trust and Clearing Corporation (DTCC) estimates **gross** notional amount in 2012 as $25 trillion.

Initially developed for hedging

- Hedge concentrations of credit risk privately - maintain good client relations.
- Hedge credit exposures where no publicly traded debt exists.

# Development and applications of CDS

Although a CDS can be used to protect against losses it is very different from an insurance policy.

- A CDS is a contract that can be written to cover anything.
- Can buy protection even when one does not hold the debt.
- CDS are easy to create and (until recently) completely unregulated.

Investing (speculation) quickly became the main application

- Unfunded way to take a credit risk – leverage possible.
- Tailor credit exposure to match precise requirements.
- Taking views on the credit quality of an reference credit.
- Buying protection is easier than shorting the asset.
- Arbitrage between the reference bond coupon and CDS spreads.

# CDS, the financial crisis and the debt crisis

CDS positions are not transparent.

- Riskiness of financial intermediaries cannot be accurately evaluated.
- Threatened trust in **all** counterparties – since no one knew who faced losses.

CDS trades are conducted on an OTC market.

- Impossible for any dealer to know the previous deals of a customer.
- **Result**: AIG was able to leverage its high credit rating to sell approximately $\$500$ billion worth of CDS.
- Allowed a small number of CDS dealers to take a huge amount of risk.
- The interconnected obligations of the dealers let to worries about contagion.

CDS can be adversely affect the cost of borrowing of a firm or a country.

- Speculators purchase CDS without holding underlying debt – naked CDS.
- This drives the spread higher, i.e. the firm starts to appear riskier.
- The cost of borrowing of the firm increases, and can lead to its collapse.

# Financial Engineering and Risk Management

## Pricing Credit Default Swaps

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# CDS pricing

Value of CDS to buyer $=$ Risk-neutral value of protection

$-$ Risk neutral value of premiums

**Assumption**: Default event uniformly distributed over the premium interval $\delta$

Risk neutral value of single premium payment on date $t_k$

$$\delta SN \cdot \mathbb{E}_0^{\mathbb{Q}}\left[\frac{I(t_k)}{B(t_k)}\right] = \delta SNq(t_k)Z_0^{t_k} = \delta SNq(t_k)d(0,t_k)$$

Risk neutral value of all premium payments

$$\sum_{k=1}^{n} \delta SNq(t_k)d(0,t_k)$$

# CDS pricing (contd)

Risk-neutral value of accrued interest if default $\tau \in (t_{k-1}, t_k]$

$$
\begin{aligned}
\frac{\delta}{2} SN \cdot \mathbb{E}_0^{\mathbb{Q}}\left[\frac{I(t_{k-1}) - I(t_k)}{B(t_k)}\right] &= \frac{\delta SN}{2}\big(q(t_{k-1}) - q(t_k)\big) Z_0^{t_k} \\
&= \frac{\delta SN}{2}\big(q(t_{k-1}) - q(t_k)\big) d(0, t_k)
\end{aligned}
$$

Risk neutral value of premium and accrued interest can approximated by

$$
\begin{aligned}
\delta SN \sum_{k=1}^{n} q(t_k) d(0, t_k) + \frac{\delta SN}{2} \sum_{k=1}^{n} \big(q(t_{k-1}) - q(t_k)\big) d(0, t_k) \\
= \frac{\delta SN}{2} \sum_{k=1}^{n} \big(q(t_{k-1}) + q(t_k)\big) d(0, t_k)
\end{aligned}
$$

# CDS pricing (contd)

Risk-neutral present value of the protection (contigent payment)

$$
\begin{aligned}
&(1-R)N\sum_{k=1}^{n}\mathbb{E}_0^Q\left[\frac{I(t_{k-1})-I(t_k)}{B(t_k)}\right] \\
&= \quad (1-R)N\sum_{k=1}^{n}\big(q(t_{k-1})-q(t_k)\big)\,d(0,t_k)
\end{aligned}
$$

par spread $S_{\text{par}}$ = spread that makes the value of the contract equal to zero.

$$
S_{\text{par}} \quad = \quad \frac{(1-R)\sum_{k=1}^{n}\big(q(t_{k-1})-q(t_k)\big)\,d(0,t_k)}{\frac{\delta}{2}\sum_{k=1}^{n}\big(q(t_{k-1})+q(t_k)\big)\,d(0,t_k)}
$$

Suppose $q(t_k)\approx(1-h)q(t_{k-1})$. Then

$$
S_{\text{par}} \approx \frac{(1-R)h}{(1-h/2)}
$$

Increasing in the hazard rate $h$ and decreasing in recovery rate $R$