

Financial Engineering and Risk Management

Options

Martin Haugh

Garud Iyengar

Columbia University

Industrial Engineering and Operations Research

Options

Definition. A **European Call** Option gives the buyer the right but not the obligation to purchase 1 unit of the underlying at specified price K (strike price) at a specified time T (expiration).

Definition. An **American Call** Option gives the buyer the right but not the obligation to purchase 1 unit of the underlying at specified price K (strike price) at any time until a specified time T (expiration).

Definition. A **European Put** Option gives the buyer the right but not the obligation to sell 1 unit of the underlying at specified price K (strike price) at a specified time T (expiration).

Definition. An **American Put** Option gives the buyer the right but not the obligation to sell 1 unit of the underlying at specified price K (strike price) at any time until a specified time T (expiration).

Payoff and intrinsic value of a call option

Payoff of a European call option at expiration T

- price $S_T < K$: do not exercise the option, payoff = 0
- price $S_T > K$: exercise option and sell in the spot market, payoff = $S_T - K$

Payoff = $\max\{S_T - K, 0\}$... nonlinear in the price S_T

Intrinsic value of a call option at time $t \leq T = \max\{S_t - K, 0\}$

- In the money: $S_t > K$
- At the money: $S_t = K$
- Out of the money: $S_t < K$

Everything works the other way for **put** options.

Payoff and intrinsic value of a put option

Payoff of a European put option at expiration T

- price $S_T < K$: exercise the option and see in spot market, payoff = $K - S_T$
- price $S_T > K$: do not exercise option, payoff = 0

Payoff = $\max\{K - S_T, 0\}$... nonlinear in the price S_T

Intrinsic value of a put option at time $t \leq T = \max\{K - S_t, 0\}$

- In the money: $S_t < K$
- At the money: $S_t = K$
- Out of the money: $S_t > K$

Pricing options

Nonlinear payoff ... cannot price without a model for the underlying

Prices of options

- European put/call with strike K and expiration T : $p_E(t; K, T)$, $c_E(t; K, T)$
- American put/call with strike K and expiration T : $p_A(t; K, T)$, $c_A(t; K, T)$

European put-call parity at time t for non-dividend paying stock:

$$p_E(t; K, T) + S_t = c_E(t; K, T) + Kd(t, T)$$

Trading strategy

- At time t buy European call with strike K and expiration T
- At time t sell European put with strike K and expiration T
- At time t (short) sell 1 unit of underlying and buy at time T
- Lend $K \cdot d(t, T)$ dollars up to time T

No-arbitrage argument

- Cash flow at time T : $\max\{S_T - K, 0\} - \max\{K - S_T, 0\} - S_T + K = 0$
- Cash flow at time t : $-c_E(t; K, T) + p_E(t; K, T) + S_t - Kd(t, T) = 0$

Bounds on prices of European options

Price of American option \geq Price of European option

- $c_A(t; K, T) \geq c_E(t; K, T)$, and $p_A(t; K, T) \geq p_E(t; K, T)$

Lower bound on European options as a function of stock price S_t

- $c_E(t; K, T) = \max\{S_t + p_E(t; K, T) - Kd(t, T), 0\} \geq \max\{S_t - Kd(t, T), 0\}$
- $p_E(t; K, T) = \max\{Kd(t, T) + c_E(t; K, T) - S_t, 0\} \geq \max\{Kd(t, T) - S_t, 0\}$

Upper bound on European options as a function of stock price S_t

- $\max\{S_T - K, 0\} \leq S_T$ implies $c_E(t; K, T) \leq S_t$
- $\max\{K - S_T, 0\} \leq K$ implies $p_E(t; K, T) \leq Kd(t, T)$

Effect of dividends $p_E(t; K, T) + S_t - D = c_E(t; K, T) + Kd(t, T)$

- D = present value of all dividends until maturity

Bounds on prices of American options

Price of American call as function of stock price S_t :

- $c_A(t, K, T) \geq c_E(t; K, T) \geq \max \{S_t - Kd(t, T), 0\} > \max \{S_t - K, 0\}$

Thus, the price of an American call is always strictly greater than the exercise value of the call option.

Never optimal to exercise an American call on a non-dividend paying stock early!
 $c_A(t; K, T) = c_E(t, K, T)$

Price of American put as function of stock price S_t :

- Bound $p_A(t, K, T) \geq p_E(t; K, T) \geq \max \{Kd(t, T) - S_t, 0\}$
- But the exercise value of a American put option is $\max \{K - S_t, 0\}$

Bounds do not tell us much!

