Financial Engineering and Risk Management

Interest rates and fixed income instruments

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Simple and compound interest

Definition. An amount A invested for n periods at a simple interest rate of r per period is worth $A(1 + n \cdot r)$ at maturity.

Definition. An amount A invested for n periods at a compound interest rate of r per period is worth $A(1 + r)^n$ at maturity.

Interest rates are typically quoted on annual basis, even if the compounding period is less than 1 year.

- n compounding periods in each year
- rate of interest r
- A invested for y years yields $A\left(1 + \frac{r}{n}\right)^{y \cdot n}$

Definition. Continuous compounding corresponds to the situation where the length of the compounding period goes to zero. Therefore, an amount A invested for y years is worth $\lim_{n\to\infty} A(1 + r/n)^{yn} = Ae^{ry}$ at maturity.

Present value

Price p of a contract that pays $\mathbf{c} = (c_0, c_1, c_2, \ldots, c_N)$

- $c_k > 0 \equiv$ cash inflow, and $c_k < 0 \equiv$ cash outflow

Present Value (PV) assuming interest rate r per period

$$PV(\mathbf{c}; r) = c_0 + \frac{c_1}{(1+r)} + \frac{c_2}{(1+r)^2} + \ldots \frac{c_N}{(1+r)^N} = \sum_{k=0}^{N} \frac{c_k}{(1+r)^k}.$$

No-arbitrage argument: Suppose one can borrow and lend at rate r

Cash flows	$t=0$	$t=1$	$t=2$	$t=k$	$t=T$
Buy contract	$-p + c_0$	c_1	c_2	c_k	c_T
Borrow $c_1/(1+r)$ up to time 1	$c_1/(1+r)$	$-c_1$			
Borrow $c_2/(1+r)^2$ up to time 2	$c_2/(1+r)^2$		$-c_2$		
Borrow $c_k/(1+r)^k$ up to time k	$c_k/(1+r)^k$			$-c_k$	
Borrow $c_T/(1+r)^T$ up to time T	$c_T/(1+r)^T$				$-c_T$

- Portfolio cash flows $= 0$ for times $k \geq 1$
- Price of portfolio: $p - \sum_{k=0}^{T} c_k/(1+r)^k \geq 0 \Rightarrow p \geq \sum_{k=0}^{T} c_k/(1+r)^k$

Present value (contd.)

To obtain the upper bound: **reverse** the portfolio.

Cash flows	$t = 0$	$t = 1$	$t = 2$	$t = k$	$t = T$
Sell contract	$p - c_0$	$-c_1$	$-c_2$	$-c_k$	$-c_T$
Lend $c_1/(1+r)$ up to time 1	$-c_1/(1+r)$	c_1			
Lend $c_2/(1+r)^2$ up to time 2	$-c_2/(1+r)^2$		c_2		
Lend $c_k/(1+r)^k$ up to time k	$-c_k/(1+r)^k$			c_k	
Lend $c_T/(1+r)^T$ up to time T	$-c_T/(1+r)^T$				c_T

- Portfolio cash flows $= 0$ for times $k \geq 1$
- Price of portfolio: $\sum_{k=0}^{T} c_k/(1+r)^k - p \geq 0 \Rightarrow p \leq \sum_{k=0}^{T} c_k/(1+r)^k$

The two bounds together imply: $p = PV(\mathbf{c}; r)$

Important we could both lend and borrow at rate r

- What if the lending rate is different from borrowing rate?

Different lending and borrowing rates

Can lend at rate r_L and borrow rate at rate r_B: $r_L \leq r_B$

Portfolio: buy contract, and borrow $\frac{c_k}{(1+r_B)^k}$ for k years, $k = 1, \ldots, N$

- Cash flow in year k: $c_k - \frac{c_k}{(1+r_B)^k}(1 + r_B)^k = 0$ for $k \geq 1$
- No-arbitrage: price $= p - c_0 - \sum_{k=1}^{N} \frac{c_k}{(1+r_B)^k} \geq 0$
- Lower bound on price $p \geq PV(\mathbf{c}; r_B)$

Portfolio: sell contract, and lend $\frac{c_k}{(1+r_L)^k}$ for k years, $k = 1, \ldots, N$

- Cash flow in year k: $-c_k + \frac{c_k}{(1+r_L)^k}(1 + r_L)^k = 0$ for $k \geq 1$
- No-arbitrage: price $= -p + c_0 + \sum_{k=1}^{N} \frac{c_k}{(1+r_L)^k} \geq 0$
- Upper bound on price $p \leq PV(\mathbf{c}; r_L)$

Bounds on the price $PV(\mathbf{c}; r_B) \leq p \leq PV(\mathbf{c}; r_L)$

How is the price set?

Fixed income securities

Fixed income securities "guarantee" a fixed cash flow. Are these risk-free?

- Default risk
- Inflation risk
- Market risk

Perpetuity: $c_k = A$ for all $k \geq 1$

$$p = \sum_{k=1}^{\infty} \frac{A}{(1+r)^k} = \frac{A}{r}$$

Annuity: $c_k = A$ for all $k = 1, \ldots, n$

$$\begin{aligned} \text{Annuity} \quad &= \quad \text{Perpetuity} - \text{Perpetuity starting in year } n+1 \\ \text{Price } p \quad &= \quad \frac{A}{r} - \frac{1}{(1+r)^n} \cdot \frac{A}{r} = \frac{A}{r}\left(1 - \frac{1}{(1+r)^n}\right) \end{aligned}$$

Bonds

Features of bonds

- Face value F: usually 100 or 1000
- Coupon rate α: pays $c = \alpha F/2$ every six months
- Maturity T: Date of the payment of the face value and the last coupon
- Price P
- Quality rating: S&P Ratings AAA, AA, BBB, BB, CCC, CC

Bonds differ in many dimensions ... hard to compare bonds

Yield to maturity λ

$$P = \sum_{k=1}^{2T} \frac{c}{(1+\lambda/2)^k} + \frac{F}{(1+\lambda/2)^{2T}}$$

Annual interest rate at which price $P =$ present value of coupon payments

Yield to maturity

Yield to maturity λ

$$P = \sum_{k=1}^{2T} \frac{c}{(1+\lambda/2)^k} + \frac{F}{(1+\lambda/2)^{2T}}$$

Why do we think in terms of yields?

- Summarizes face value, coupon, maturity, and quality
- Relates to quality: lower quality $\rightarrow$ lower price $\rightarrow$ higher yield to maturity
- Relates to interest rate movements

But ... yield to maturity is a crude measure. Does not capture everything.