Financial Engineering and Risk Management

Introduction to no-arbitrage

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Contracts, prices and no-arbitrage

Consider the following contract

- Pay price p at time $t = 0$
- Receive c_k at time $t = k$, $k = 1, \ldots, T$

Note that the cash flow c_k could be negative!

The no-arbitrage condition bounds the price p for this contract.

- Weak No-Arbitrage: $c_k \geq 0$ for all $k \geq 1 \Rightarrow p \geq 0$
- Strong No-Arbitrage: $c_k \geq 0$ for all $k \geq 1$ and $c_\ell > 0$ for some $\ell \Rightarrow p > 0$

Essentially eliminate the possibility of a **free-lunch**!

Rationale for the weak no-arbitrage condition: Suppose $p < 0$

- Since $c_k \geq 0$ for all $k \geq 1$, the buyer receives $-p > 0$ at time 0, and then does not lose money thereafter. Free lunch!
- Seller can increase price as long as $p \leq 0$, and still have buyers available.
- Buyers will be willing to pay a higher price in order to compete.

Assumptions underlying no-arbitrage

Rationale for the strong no-arbitrage condition

- Suppose $p \leq 0$.
- Recall that $c_\ell > 0$ for some $\ell \geq 1$. Therefore, a free lunch as long as $p \leq 0$.
- We can only guarantee that $p > 0$ but not the precise value!

Implicit assumptions underlying the no-arbitrage condition

- Markets are liquid: sufficient number of buyers and sellers
- Price information is available to all buyers and sellers
- Competition in supply and demand will correct any deviation from no-arbitrage prices

Pricing a simple bond

What is the price p of a contract that pays A dollars in 1 year?

Suppose one is able to borrow and lend unlimited amounts at an interest rate of r per year.

Construct the following portfolio

- **Buy** the contract at price p
- **Borrow** $A/(1+r)$ at interest rate r

Cash flows associated with this portfolio

Price of portfolio	Cashflow in 1 year
$z = p - \frac{A}{1+r}$	$A - A = 0$

Weak No-arbitrage: $c_1 \geq 0$ implies price $z \geq 0$, i.e. $p \geq \frac{A}{1+r}$.

Pricing a simple bond (contd.)

Next, construct the following portfolio

- **Sell** the contract at price p
- **Lend** $A/(1+r)$ at interest rate r

Cash flows associated with this portfolio

Price of portfolio	Cashflow in 1 year
$z = \frac{A}{1+r} - p$	$-A + A = 0$

Weak No-arbitrage: $c_1 \geq 0$ implies price $z \geq 0$, i.e. $p \leq \frac{A}{1+r}$.

Two results together imply: $p = \frac{A}{1+r}$. Surprise?

The result relied on the ability to borrow **and** lend at rate r.

- What if borrowing and lending rates are different?
- What if the borrowing and lending markets are elastic?