Nepal Engineering College
Digital Signal Analysis & Processing
Answer of some selected questions

1. A digital communication link carries binary-coded words representing samples of an input signal,

$$x(t) = 3cos600\pi t + 2cos1800\pi t$$

The link is operated at 10,000 bits/s and each input sample is quantized into 1024 different voltage levels.

i. What is the sampling frequency & folding frequency?
ii. What is the Nyquist rate for the signal x(t) ?
iii. What are the frequencies in the resulting discrete time signal x[n] ?
iv. What is the resolution Δ ?

Solution: ➔

As the link is operated at 10,000 bits/s & each input sample is quantized into 1024 different voltage levels the each sampled value is represented by $\log_2 1024$ = 10 bits/sample

i. Then maximum sampling frequency, $F_s = \frac{10{,}000\ bits/sec}{10\ bits/sample} = 1000\ samples/sec$

Folding frequency (is the maximum frequency that can be represented uniquely by sampled signal), $\frac{F_s}{2} = 500\ samples/sec.$

ii. $x(t) = 3cos600\pi t + 2cos1800\pi t$

Here, F_1 = 300 Hz and F_2 = 900 Hz. Thus F_{max} = 900 Hz.

The Nyquist rate, F_N = 2Fmax = 1800 Hz.

iii. For sampling frequency, F_s = 1000 Hz,

$$x[n] \cong x(nT) = x\left(\frac{n}{F_s}\right) = 3cos2\pi\left(\frac{300}{1000}\right)n + 2cos2\pi\left(\frac{900}{1000}\right)n$$

$$= 3cos2\pi\left(\frac{3}{10}\right)n + 2cos2\pi\left(\frac{9}{10}\right)n$$

(*Here frequency f2 = 9/10 is greater than ½, so*)

$$= 3cos2\pi\left(\frac{3}{10}\right)n + 2cos2\pi\left(1 - \frac{1}{10}\right)n$$

$$= 3cos2\pi\left(\frac{3}{10}\right)n + 2cos2\pi\left(\frac{1}{10}\right)n$$

$$\therefore f_1 = \frac{3}{10}\ \&\ f_2 = \frac{1}{10}$$

$$Here\ both\ frequencies\ f_1 and\ f_2\ lies\ in\ the\ interval - \frac{1}{2} \le f \le \frac{1}{2}$$

iv. ADC resolution = 10 bits

Voltage resolution, $\Delta = \frac{x_{max}-x_{min}}{L-1} = \frac{5-(-5)}{1024-1} = 9.76\ mV$.

2. Check for linearity, time invariant and stability of following system:

$$\boldsymbol{y[n] = x[n^2-3]u[n] + x[n]u[n]}$$

Solution ➔

<u>For Linearity:</u>

$$y_1[n] = x_1[n^2-3]u[n] + x_1[n]u[n]$$

$$y_2[n] = x_2[n^2-3]u[n] + x_2[n]u[n]$$

$$let\ x_3[n] = a_1x_1[n] + a_2x_2[n]$$

$$\therefore y_3[n] = x_3[n^2-3]u[n] + x_3[n]u[n]$$

$$= \{a_1x_1[n^2-3] + a_2x_2[n^2-3]\}u[n] + \{a_1x_1[n] + a_2x_2[n]\}u[n]$$

$$= a_1\{x_1[n^2-3]u[n] + x_1[n]u[n]\} + a_2\{x_2[n^2-3]u[n] + x_2[n]u[n]\}$$

$$= a_1y_1[n] + a_2y_2[n]$$

Since $y_3[n] = a_1y_1[n] + a_2y_2[n]$, the system is linear.

<u>For Time invariant:</u>

Response of delayed input, $y(n,k) = x[n^2-3-k]u[n] + x_1[n-k]u[n]$

Delayed response, $y[n-k] = x[(n-k)^2-3]u[n-k] + x[n-k]u[n-k]$

Since $y(n,k) \neq y[n-k]$, the system is time variant.

<u>For Stabilty:</u>

Bounded Input condition: $|x[n]| \leq M_x < \infty$

For bounded output,

$$|y[n]| = |x[n^2-3]u[n] + x[n]u[n]|$$

$$\leq |x[n^2-3]u[n]| + |x[n]u[n]|$$

$$\leq M_x + M_x \leq M_y < \infty$$

Here, the output is bounded, therefore the system is stable.

3. **The impulse response h[n] of an LTI system is known to be zero, except in the interval $N_0 \leq n \leq N_1$. The input x[n] is known to be zero, except in the interval $N_2 \leq n \leq N_3$. As a result, the output is constrained to be zero, except in some interval $N_4 \leq n \leq N_5$. Determine N_4 and N_5 in terms of N_0, N_1, N_2, and N_3.**

Solution: -→

For an LTI system the output is obtained from the convolution of the input with impulse response of the system:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

Since $h[n] \neq 0$ for $N_0 \leq n \leq N_1$

$$y[n] = \sum_{k=N_0}^{N_1} h[k]x[n-k]$$

The input $x[n] \neq 0$ for $N_2 \leq n \leq N_3$, so

$x[n-k] \neq 0$ for $N_2 \leq (n-k) \leq N_3$

Note the minimum value of $(n-k)$ is N_2. Thus the lower bound on n, which occurs for $k = N_0$ is

$N_4 = N_0 + N_2$

Using similar argument

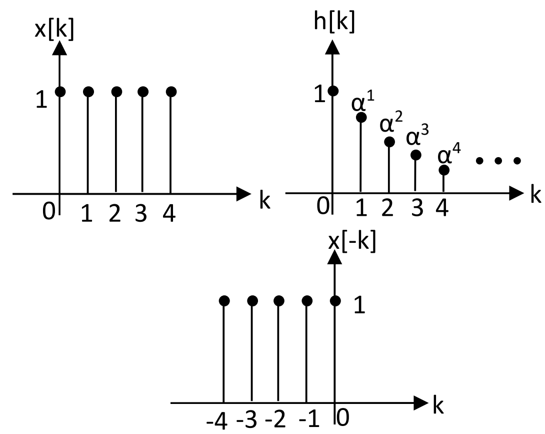
$N_5 = N_1 + N_3$ $N_0 + N_2 \leq n \leq N_1 + N_3$

4. **Find the response of the system to the input signal $x[n] = u[n] - u[n-5]$. The impulse response of the system is given by $x[n] = \alpha^n u[n]$.**

Solution: From figure

$$y[n] = 0 \ for \ n < 0$$
$$y[0] = h[k]x[-k] = 1$$
$$y[1] = h[k]x[1-k] = 1 + \alpha$$
$$y[2] = h[k]x[2-k] = 1 + \alpha + \alpha^2$$
$$y[3] = h[k]x[3-k] = 1 + \alpha + \alpha^2 + \alpha^3$$
$$y[4] = h[k]x[4-k] = 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4$$
$$y[5] = h[k]x[5-k] = \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5$$
$$y[6] = h[k]x[6-k] = \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6$$

And so on..



Therefore,

$$y[n] = \begin{cases} 0, & for\ n < 0 \\ \sum_{k=0}^{n} \alpha^k, & 0 \le n \le 4 \\ \sum_{k=n-4}^{n} \alpha^k, & for\ \ n > 4 \end{cases}$$

5. **Find z-transform of $x[n] = a^n(cos\omega_0 n)u[n]$.**
   Solution ➔

$$x[n] = a^n(cos\omega_0 n)u[n] = \frac{1}{2}a^n e^{j\omega_0 n}u[n] + \frac{1}{2}a^n e^{-j\omega_0 n}u[n]$$

We can write the above expression as $x[n] = v[n] + v^*[n]$

$$where\ v[n] = \frac{1}{2}\alpha^n u[n]\ \ with\ \alpha = ae^{j\omega_0}$$

The z-transform V(z) of v[n] by linearity and scaling in z-domain properties:

$$V(z) = \frac{1}{2}.\frac{1}{1-\alpha z^{-1}} = \frac{1}{2}.\frac{1}{1-ae^{j\omega_0}z^{-1}}, |z| > |\alpha| = a$$

From conjugate of complex sequence property,

$$V^*(z^*) = \frac{1}{2}.\frac{1}{1-\alpha^* z^{-1}} = \frac{1}{2}.\frac{1}{1-ae^{-j\omega_0}z^{-1}}, |z| > |\alpha| = a$$

By linearity property of the z-transform, we obtain

$$X(z) = V(z) + V^*(z^*)$$

$$= \frac{1}{2}\left(\frac{1}{1-ae^{j\omega_0}z^{-1}} + \frac{1}{1-ae^{-j\omega_0}z^{-1}}\right) = \frac{1-(acos\omega_0)z^{-1}}{1-(2acos\omega_0)z^{-1}+a^2z^{-2}} \quad |z| > a$$

6. **Given the system y[n] = 3y[n-1] – 0.1y[n-2] + x[n]**
   a. **Find the system function, H(z).**
   b. **Is the system IIR or FIR?**
   c. **Plot the pole-zero diagram.**
   d. **Plot the magnitude response. [PU Fall 2012 5(a)]**

Solution: →

a. Taking z-transform of both sides, we get

$$Y(z) = 3z^{-1}Y(z) - 0.1z^{-2}Y(z) + X(z)$$

$$=> \ Y(z)\{1 - 3z^{-1} + 0.1z^{-2}\} = X(z)$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - 3z^{-1} + 0.1z^{-2}}$$

This is the required System function of given system.

b. As the system contains poles the given system is IIR. (If the system contain poles then system is IIR)

c. Also,

$$H(z) = \frac{1}{1 - 3z^{-1} + 0.1z^{-2}} = \frac{1}{(1 - 2.9663z^{-1})(1 - 0.0337z^{-1})}$$

The system contains poles at $p_1 = 2.966$ and $p_2 = 0.034$, and zeros are at $z_1 = z_2 = 0$, the pole-zero diagram is shown below:

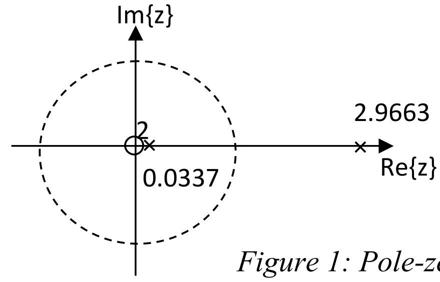


Figure 1: Pole-zero plot

d. For the frequency response, we will take $z = e^{j\omega}$

$$\therefore H(e^{j\omega}) = \frac{1}{(1 - 2.9663e^{-j\omega})(1 - 0.0337e^{-j\omega})}$$

Magnitude is given by

$$|H(e^{j\omega})| = \left|\frac{1}{(1 - 2.9663e^{-j\omega})(1 - 0.0337e^{-j\omega})}\right| = \frac{1}{|1 - 2.9663e^{-j\omega}||1 - 0.0337e^{-j\omega}|}$$

$$= \frac{1}{\sqrt{(1 - 2.9663cos\omega)^2 + (2.9663sin\omega)^2}.\sqrt{(1 - 0.0337cos\omega)^2 + (0.0337sin\omega)^2}}$$

For the magnitude response, we can find the ω Vs. $|H(e^{j\omega})|$ as,

ω	$\|H(e^{j\omega})\|$
0	0.5263
π/6	0.4770
2π/6	0.3890
3π/6	0.3193
4π/6	0.2751
5π/6	0.2514
π	0.2493

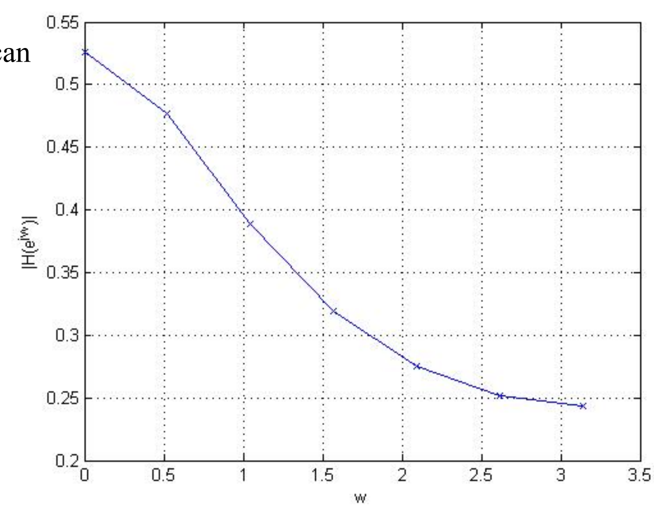


Figure 2: Magnitude Response

7. Determine the inverse z-transform of

$$\boldsymbol{H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 + 4z^{-1} + 4z^{-2}}}$$

For all possible ROCs. [PU 2009 Spring, 3(a)].

Solution: →

We have,

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 + 4z^{-1} + 4z^{-2}} = \frac{1 + 2z^{-1} + z^{-2}}{(1 + 2z^{-1})^2}$$

$$= \frac{1}{(1 + 2z^{-1})^2} + \frac{2z^{-1}}{(1 + 2z^{-1})^2} + \frac{z^{-2}}{(1 + 2z^{-1})^2}$$

$$= -0.5z \times \frac{-2z^{-1}}{(1 - (-2)z^{-1})^2} + (-1) \times \frac{-2z^{-1}}{(1 - (-2)z^{-1})^2} + (-0.5)z^{-1} \times \frac{-2z^{-1}}{(1 - (-2)z^{-1})^2}$$

The system have poles at $p_1 = -2$ and $p_2 = -2$. Therefore the possible ROCs are $|z|>2$ and $|z|<2$.

a. For ROC $|z| > 2$ (Causal Signal)

$$\therefore h[n] = -0.5(n+1)(-2)^{n+1}u[n+1] - n(-2)^n u[n] - 0.5(n-1)(-2)^{n-1}u[n-1]$$

b. For ROC $|z| < 2$ (Anticausal signal)

$$\therefore h[n] = 0.5(n+1)(-2)^{n+1}u[-n-2] + n(-2)^n u[-n-1] + 0.5(n-1)(-2)^{n-1}u[-n]$$

Note: We have used shifting property of z-transform and following z-transform table:

$$na^n u[n] \stackrel{z}{\longleftrightarrow} \frac{az^{-1}}{(1 - az^{-1})^2}, \qquad |z| > |a|$$

$$-na^n u[-n-1] \stackrel{z}{\longleftrightarrow} \frac{az^{-1}}{(1 - az^{-1})^2}, \qquad |z| < |a|$$

After using shifting property,

$$(n-k)a^{n-k}u[n-k] \stackrel{z}{\longleftrightarrow} z^{-k} \times \frac{az^{-1}}{(1 - az^{-1})^2}, \qquad |z| > |a|$$

$$-(n-k)a^{n-k}u[-(n-k)-1] \stackrel{z}{\longleftrightarrow} z^{-k} \times \frac{az^{-1}}{(1 - az^{-1})^2}, \qquad |z| < |a|$$

8. If y[n] = {1, 1, 2, -1, 3} is output and impulse response is h[n] = {1, 2, 3} find discrete-time signal x[n].

Solution: →

Taking z-transform,

$$Y(z) = 1 + z^{-1} + 2z^{-2} - z^{-3} + 3z^{-4}$$
$$H(z) = 1 + 2z^{-1} + 3z^{-2}$$
$$\therefore X(z) = \frac{Y(z)}{H(z)} = \frac{1 + z^{-1} + 2z^{-2} - z^{-3} + 3z^{-4}}{1 + 2z^{-1} + 3z^{-2}}$$

By division method:

$$1+2z^{-1}+3z^{-2}\,|\;1+z^{-1}+2z^{-2}-z^{-3}+3z^{-4}\,|\,1-z^{-1}+z^{-2}$$
$$|1+2z^{-1}+3z^{-2}\qquad\qquad |$$
$$|\,-z^{-1}-z^{-2}-z^{-3}+3z^{-4}|$$
$$|-z^{-1}-2z^{-2}-3z^{-3}\qquad |$$
$$|\,z^{-2}+\;2z^{-3}+3z^{-4}|$$
$$|z^{-2}+\;2z^{-3}+3z^{-4}|$$

Therefore X(z) is given by,

$$X(z)=1-z^{-1}+z^{-2}$$

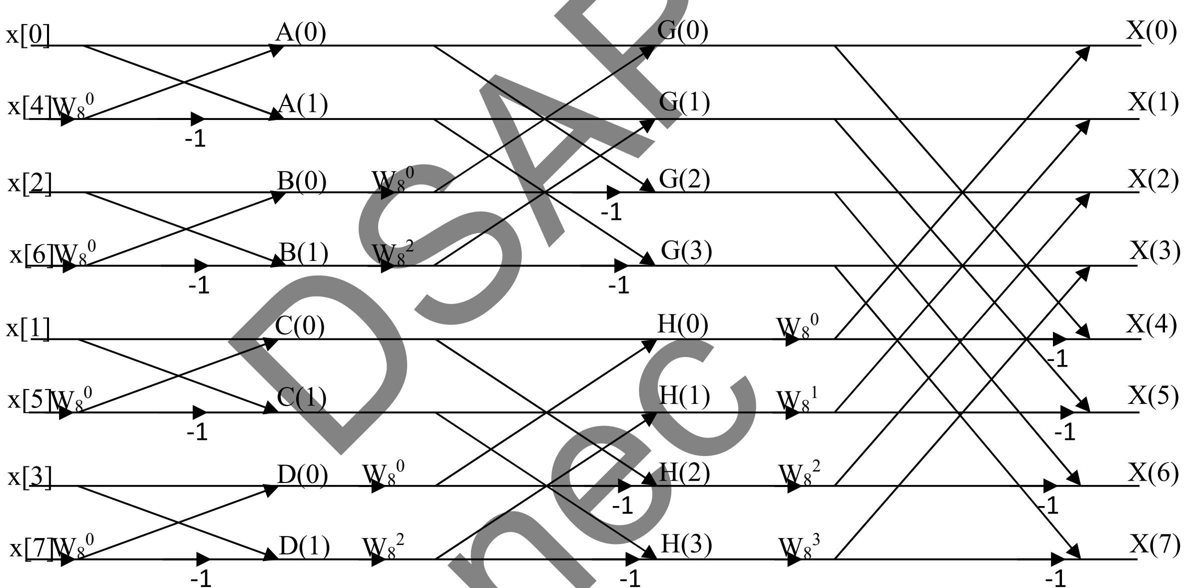
Taking inverse z-transfrom, we get,

$$x[n]=\{1,-1,1\}.$$

9. Find fft of the signal x[n] = {1, 2, 5, 4, -3, 0, 4, 1}.

Solution: ➔

$$Here, W_8^0=1$$



$$W_8^1=e^{-j\frac{2\pi}{8}}=\cos\frac{2\pi}{8}-jsin\frac{2\pi}{8}=0.707-j0.707$$

$$W_8^2=e^{-\left(\frac{j2\pi}{8}\right)\times 2}=\cos\frac{\pi}{2}-jsin\frac{\pi}{2}=-j$$

$$W_8^3=e^{-j\left(\frac{2\pi}{8}\right)\times 3}=\cos\frac{3\pi}{4}-jsin\frac{3\pi}{4}=0.707-j0.707$$

Output of first Stage:

$A(0)=x[0]+W_8^0x[4]=1+1\times -3=-2,\qquad A(1)=x[0]-W_8^0.x[4]=1-1\times -3=4$

$B(0)=x[2]+W_8^0.x[6]=5+1\times 4=9,\qquad B(1)=x[2]-W_8^0.x[6]=5-1\times 4=1$

$C(0)=x[1]+W_8^0.x[5]=2+1\times 0=2,\qquad C(1)=x[1]-W_8^0.x[5]=2-1\times 0=2$

$D(0) = x[3] + W_8^0.x[7] = 4 + 1 \times 1 = 5, \qquad D(1) = x[3] - W_8^0.x[7] = 4 - 1 \times 1 = 3$

Output of second stage:

$G(0) = A(0) + W_8^0.B(0) = -2 + 1 \times 9 = 7, \qquad H(0) = C(0) + W_8^0.D(0) = 2 + 1 \times 5 = 7$

$G(1) = A(1) + W_8^2.B(1) = 4 + (-j) \times 1 = 4 - j,$
$H(1) = C(1) + W_8^2.D(1) = 2 + (-j) \times 3 = 2 - j3$

$G(2) = A(0) - W_8^0.B(0) = -2 - 1 \times 9 = -11, \qquad H(2) = A(0) - W_8^0.B(0) = 2 - 1 \times 5 = -3$

$G(3) = A(1) - W_8^2.B(1) = 4 - (-j) \times 1 = 4 + j,$
$H(3) = C(1) - W_8^2.D(1) = 2 - (-j) \times 3 = 2 + j3$

Output of third stage:

$X(0) = G(0) + W_8^0.H(0) = 7 + 1 \times 7 = 14$

$X(1) = G(1) + W_8^1.H(1) = (4 - j) + (0.707 - j0.707) \times (2 - j3) = 3.293 - j4.535$

$X(2) = G(2) + W_8^2.H(2) = -11 + (-j) \times (-3) = -11 + j3$

$X(3) = G(3) + W_8^2.H(3) = (4 + j) + (-0.707 - j0.707) \times (2 + j3) = 4.707 - j2.535$

$X(4) = G(0) - W_8^0.H(0) = 7 - 1 \times 7 = 0$

$X(5) = G(1) - W_8^1.H(1) = (4 - j) - (0.707 - j0.707) \times (2 - j3) = 4.707 + j2.535$

$X(6) = G(2) - W_8^2.H(2) = -11 - (-j) \times (-3) = -11 - j3$

$X(7) = G(3) - W_8^2.H(3) = (4 + j) - (-0.707 - j0.707) \times (2 + j3) = 3.293 + j4.535$

$\therefore X(k) = \{14, 3.293 - j4.535, -11 + j3, 4.707 - j2.535, 0, 4.707 + j2.535, -11 - j3, 3.293 + j4.535\}$

10. Draw the lattice structure of the following system:

$$H(z) = \frac{1 + 3z^{-1} + 3z^{-2} + z^{-3}}{2 - 1.8z^{-1} + 1.28z^{-2} - 1.152z^{-3}}. Is\ this\ system\ stable?$$

Solution: ➔

$$Given, H(z) = \frac{C_3(z)}{2A_3(z)} = \frac{1 + 3z^{-1} + 3z^{-2} + z^{-3}}{2(1 - 0.9z^{-1} + 0.64z^{-2} - 0.576z^{-3})}$$

Note: We have to make $\alpha_m(0) = 1$ for mathematical convenience.

First consider, $A_3(z) = 1 - 0.9z^{-1} + 0.64z^{-2} - 0.576z^{-3}$

$$B_3(z) = -0.576 + 0.64z^{-1} - 0.9z^{-2} + z^{-3}. (reverse\ polynomial\ of\ A_3(z))$$

$$We\ know, k_m = \alpha_m(m), therefore\ k_3 = -0.576$$

$$Also, A_{m-1}(z) = \frac{A_m(z) - k_m B_m(z)}{1 - k_m^2}$$

$$therefore, A_2(z) = \frac{A_3(z) - k_3 B_3(z)}{(1 - k_3^2)}$$

$$= \frac{1 - 0.9\,z^{-1} + 0.64z^{-2} - 0.576z^{-3} — 0.576(-0.576 + 0.64z^{-1} - 0.9z^{-2} + z^{-3})}{1 — 0.5762}$$

$$= 1 - 0.795z^{-1} + 0.182z^{-2}$$

$$\therefore k_2 = 0.182$$

Similarly,

$$A_1(z) = \frac{A_2(z) - k_2 B_2(z)}{(1 - k_2^2)} = 1 - 0.672z^{-1}$$

$$\therefore k_2 = 0.182$$

Now consider the numerator part $C_3(z)$

$$C_3(z) = c_3(0) + c_3(1)z^{-1} + c_3(2)z^{-2} + c_3(3)z^{-3} = 1 + 3z^{-1} + 3z^{-2} + z^{-3}$$

$$We\ know, v_m = c_m(m). Therefore\ v_3 = c_3(3) = 1$$

$$Now\ we\ have, C_{m-1}(z) = C_m(z) - v_m B_m(z)$$

$$\therefore C_2(z) = C_3(z) - v_3 B_3(z)\ \ (Note: B_3(z)\ is\ from\ lattice\ part)$$

$$C_3(z) = 1 + 3z^{-1} + 3z^{-2} + z^{-3} - 1(-0.576 + 0.64z^{-1} - 0.9z^{-2} + z^{-3})$$

$$= 1.576 + 2.36z^{-1} + 3.9z^{-2}$$

$$\therefore v_2 = c_2(2) = 3.9$$

Similarly, $C_1(z) = C_2(z) - v_2 B_2(z) = 0.866 + 5.46z^{-1} ==> \therefore v_1 = c_1(1) = 5.46$

And, $C_0(z) = C_1(z) - v_1 B_1(z) = 4.535 ==> \therefore v_0 = c_0(0) = C_0(Z) = 4.535$

For Stability, $|k_m| < 1$. Therefore the system is stable.

The lattice ladder structure is shown below:

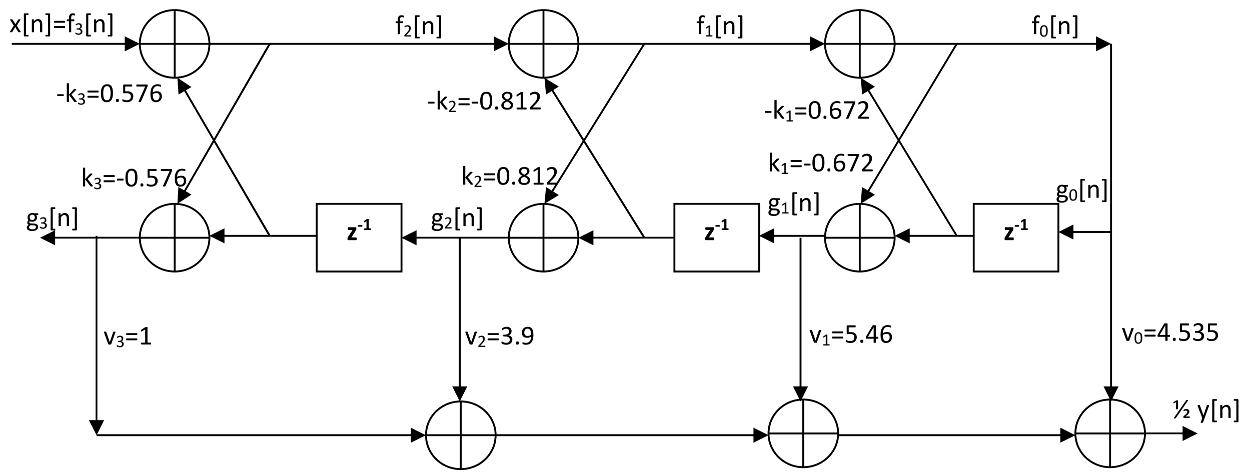


Fig: Lattice-ladder structure of given question.

11. Design an FIR filter with 9 coefficients for the following specifications:

- Passband edge frequency = 0.5 KHz
- Sampling frequency = 2 KHz
- Use suitable window in design.

Solution:

Given,

$M = 9$ $\quad\quad$ $F_c = 0.5$ KHz $\quad\quad$ $F_s = 2$ KHz

We know,

$\omega_c = 2\pi f = 2\pi\, F_c/F_s = 0.5\pi$

The desired frequency response is given by

$$H_d(\omega) = \begin{cases} e^{-j\omega\tau} & for\ |\omega| \le \omega_c \\ 0 & elsewhere \end{cases}$$

Taking inverse discrete-time Fourier transform to obtain desired unit sample response, we get,

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-0.5\pi}^{-0.5\pi} e^{-j\omega\tau}\, e^{j\omega n} d\omega$$

Solving above equation we get,

$$h_d(n) = \begin{cases} \dfrac{\sin 0.5\pi(n-\tau)}{\pi(n-\tau)}, & for\ n \ne \tau \\ \dfrac{1}{2}, & for\ n = \tau \end{cases}$$

We know for symmetric FIR filter, the conditions are h[n] = h[M – 1 – n] & $\tau = \frac{M-1}{2}$

for M = 9, the value of τ = 4

$$\therefore h_d(n) = \begin{cases} \dfrac{\sin 0.5\pi(n-4)}{\pi(n-\tau)}, & for\ n \neq 4 \\ \dfrac{1}{2}, & for\ n = 4 \end{cases}$$

As minimum stopband attenuation condition is not given, We can take any window for the design. So, taking Hanning window for our design,

Then the required unit sample is given by,

h[n] = h_d[n].w[n]

Taking Hanning window in design, the required unit sample of given filter is given in table below:

N	h_d[n]	$w[n] = 0.5\left(1-\cos\frac{2\pi n}{M-1}\right)$	h[n] = h_d[n].w[n]
0	0	0	0
1	-0.106	0.146	0.015
2	0	0.5	0
3	0.318	0.854	0.269
4	0.5	1	0.5
5	0.318	0.854	0.269
6	0	0.5	0
7	-0.106	0.146	0.015
8	0	0	0

12. Design an FIR linear phase filter using Kaiser window to meet the following specifications.

$$0.99 \leq \left|H(e^{jw})\right| \leq 1.01, for\ 0 \leq |w| \leq 0.19\pi$$
$$\left|H(e^{jw})\right| \leq 0.01, \quad for\ 0.21\pi \leq |w| \leq \pi$$

Solution:

Step I: Given data

The given specifications may be rewritten as under

$$1-0.01 \leq |H(e^{jw})| \leq 1+0.01, \text{ for } 0 \leq \omega \leq 0.19\pi$$
$$|H(e^{jw})| \leq 0.01, \text{ for } 0.21\pi \leq \omega \leq \pi$$

On comparing the above given specifications with magnitude response of filter,

δ_1=0.01 δ_2=0.01

ω_p =0.19π ω_s =0.21π

Δω = ω_s - ω_p = 0.02π

δ = minimum of δ_1 and δ_2 = 0.01

Now, $A = -20\log_{10}\delta = 40$

Step 2: To determine cutoff frequency ω_c

Cutoff frequency is given by

$\omega_c = (\omega_p + \omega_s)/2 = 0.2\pi$

Step 3: To obtain β and M

Here A = 40, which lies in the range of 21 to 51

Hence, β can be obtained as,

$\beta = 0.5842(A-21)^{0.4} + 0.07886(A-21) = 3.395$

and $M = \frac{A-8}{2.285\Delta\omega} = 222.88 \cong 223$

Step 4: To obtain expression of Kaiser window

The window will be defined as under

$$w[n] = \frac{I_0\left[\beta\left(\sqrt{1-\left(\left(n-\frac{M}{2}\right)\Big/\frac{M}{2}\right)^2}\right)\right]}{I_0(\beta)}\ for\ 0 \le n \le M$$

The function $I_0()$ in the above expression may be calculated with the help of following equation

$$I_0(x) = 1 + \sum_{r=1}^{\infty}\left[\frac{\left(x/2\right)^r}{r!}\right]^2$$

Since, it is an infinite series, maximum possible terms must be taken to reduce the error.

The ideal desired frequency response is given by

$$H_d(\omega) = \begin{cases} e^{-j\omega\tau\left(\frac{M-1}{2}\right)} & for\ |\omega| \le \omega_c \\ 0\ elsewhere \end{cases}$$

Now, $h_d(n)$ may be obtained by taking inverse Fourier transform of above equation, i.e

$$h_d(n) = \frac{\sin\left[\omega_c\left(n-\frac{M}{2}\right)\right]}{\pi\left(n-\frac{M}{2}\right)},\ \ for\ 0 \le n \le M,$$

Substituting for $\omega_c = 0.2\pi$ and M = 223,

$$h_d(n) = \frac{\sin[0.2\pi(n-111.5)]}{\pi(n-111.5)}, \;\; for\; 0 \le n \le 223,$$

The unit sample response of FIR filter may be obtained by windowing, i.e.

$$h[n] = h_d[n]\omega[n]$$

$$h(n) = \frac{\sin[0.2\pi(n-111.5)]}{\pi(n-111.5)} \frac{I_0\left[3.395\left(\sqrt{1-\left({(n-115.5)}/{111.5}\right)^2}\right)\right]}{I_0(3.395)}, \;\; for\; 0 \le n \le 223,$$

This is the required relationship for unit sample response FIR filter using Kaiser Window.

13. Design a lowpass Butterworth filter to meet the following specifications.
Passband gain = 0.89
Passband frequency edge = 30 Hz
Attenuation = 0.20
Stopband edge = 75 Hz
(Use impulse invariance and Bilinear Transformation method to convert the filter from analog to digital).

Solution:→

We have $\Omega_p = 2\pi \times 30 = 60\pi$ rad/s (F_p = 30 Hz)

$\Omega_s = 2\pi \times 75 = 150\pi$ rad/s (F_s = 75 Hz)

$\alpha_{max} = -20\log_{10}(0.89) = 1.0122\; dB$

$\alpha_{min} = -20\log_{10}(0.20) = 13.9794\; dB$

$\varepsilon^2 = 10^{0.1\alpha_{max}} - 1 = 10^{0.10122} - 1 = 0.2625 ==> \varepsilon = 0.5123$

$\delta^2 = 10^{0.1\alpha_{min}} - 1 = 10^{1.39794} - 1 = 24 ==> \delta = 4.899$

$$\therefore N = \frac{\log_{10}\left(\frac{\delta}{\varepsilon}\right)}{\log_{10}\left(\frac{\Omega_s}{\Omega_p}\right)} = \frac{\log_{10}\left(\frac{4.899}{0.5123}\right)}{\log_{10}\left(\frac{150\pi}{60\pi}\right)} = \frac{0.9806}{0.3979} = 2.4644$$

Since the number of poles must be an integer, we round up to N = 3.

Matching the frequency response exactly at stopband produces,

$$\Omega_c = \frac{\Omega_s}{\delta^{\frac{1}{N}}} = \frac{150\pi}{4.899^{\frac{1}{3}}} = \;\; 277.4632$$

[*Note: If we were instead to match the frequency response at passband, we would obtain*

$$\Omega_c = \frac{\Omega_p}{\varepsilon^{\frac{1}{N}}} = \frac{60\pi}{0.5123^{\frac{1}{3}}} = 235.5734$$

In principle, any value of the critical frequency that satisfies $\frac{\Omega_p}{\varepsilon^{\frac{1}{N}}} \le \Omega_c \le \frac{\Omega_s}{\delta^{\frac{1}{N}}}$ *would be valid.]*

For N = 3 and $\Omega_c = 1$ (Normalized Transfer function)

$$H(s) = \frac{1}{(s+1)(s^2+s+1)}$$

(This can be obtained from formula as well as table given)

This transfer function is normalized for $\Omega_c = 1$. However $\Omega_c = 277.46$, we need to denormalize H(s) by $\Omega_c = 277.46$ rad/s

$$H(s)|\, s = {}^{s}/_{\Omega_c} = \frac{{\Omega_c}^3}{(s+\Omega_c)(s^2+\Omega_c s+{\Omega_c}^2)} = \frac{2.136\times 10^7}{(s+277.46)(s^2+277.46s+76984)}$$

i. Using Impulse invariance method:

$$H(s) = \frac{2.136\times 10^7}{(s+277.46)(s^2+277.46s+76984)} = \frac{A}{s+277.46} + \frac{Bs+c}{s^2+277.46s+76984}$$

Solving we get, A = 277.46, B = -277.46 and c = 0.

$$\therefore H(s) = \frac{277.46}{s+277.46} - \frac{277.46s}{s^2+277.46s+76984}$$

$$= \frac{277.46}{s+277.46} - \frac{277.46s}{s^2+2\times s\times 138.73s+138.73^2+230.29^2}$$

$$= \frac{277.46}{s+277.46} - \frac{277.46(s+138.73)-38492}{(s+138.73)^2+230.29^2}$$

$$= \frac{277.46}{s+277.46} - \frac{277.46(s+138.73)}{(s+138.73)^2+230.29^2} + \frac{\frac{38492}{230.29} * 230.29}{(s+138.73)^2+230.29^2}$$

Using mapping method:

$$H(z) = \frac{277.46}{1-e^{-277.46T}z^{-1}} - 277.46\times\frac{1-e^{-138.73T}\cos(230.29T)\,z^{-1}}{1-2e^{-138.73T}\cos(230.29T)\,z^{-1}+e^{-277.46T}z^{-2}}$$
$$+ 167.14\times\frac{e^{-138.73T}\sin(230.29T)\,z^{-1}}{1-2e^{-138.73T}\cos(230.29T)\,z^{-1}+e^{-277.46T}z^{-2}}$$

Analog Domain H(s)	**Digital Domain H(z)**
$\frac{1}{s-p_k}$	$\frac{1}{1-e^{p_kT}z^{-1}}$

$\dfrac{s+a}{(s+a)^2+b^2}$	$\dfrac{1-e^{-aT}cos(bT)z^{-1}}{1-2e^{-aT}cos(bT)z^{-1}+e^{-2aT}z^{-2}}$
$\dfrac{b}{(s+a)^2+b^2}$	$\dfrac{e^{-aT}sin(bT)z^{-1}}{1-2e^{-aT}cos(bT)z^{-1}+e^{-2aT}z^{-2}}$

ii. Using Bilinear Transformation Method:

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right).$$

$$H(s) = \frac{2.136\times 10^7}{(s+277.46)(s^2+277.46s+76984)}$$

$$\therefore H(z) = \frac{2.136\times 10^7}{\left(\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+277.46\right)\left(\left(\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right)^2+277.46\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+76984\right)}$$

Butterworth Polynomials in Factorized form	
Order(n)	H(s) = $H(s) = \frac{1}{A(s)}, Where\ A(s)$
1	(s+1)
2	$(s^2+1.4142s+1)$
3	$(s+1)(s^2+s+1)$
4	$(s^2+1.8478s+1)(s^2+0.7654s+1)$
5	$(s+1)(s^2+0.6180s+1)(s^2+1.6180s+1)$
6	$(s^2+1.414s+1)(s^2+0.5176s+1)(s^2+1.9319s+1)$
7	$(s+1)(s^2+0.445s+1)(s^2+1.247s+1)(s^2+1.802s+1)$

14. Design a digital lowpass Butterworth filter to meet the following specifications.

- **Passband cutoff frequency: $\omega_p = 0.15\pi$**
- **Stopband cutoff frequency: $\omega_s = 0.35\pi$**
- **Passband ripple: $-3dB \le \left|H(e^{j\omega})\right| \le 0dB,\ |\omega| \le \omega_p$**
- **Stopband ripple: $|H(e^{j\omega}) \le -20dB,\ \omega_s \le |\omega| \le \pi$**

Use impuse invariance method.

Solution:

Given,

$$\alpha_{max} = 3dB,\ \ \alpha_{min} = 20dB, \omega_p = 0.15\pi, \omega_s = 0.35\pi$$

$\varepsilon^2 = 10^{0.1\alpha_{max}} - 1 = 10^{0.3} - 1 = 0.995 ==> \varepsilon = 0.998$
$\delta^2 = 10^{0.1\alpha_{min}} - 1 = 10^2 - 1 = 99 ==> \delta = 9.95$
Using the impulse invariance design procedure, we have noted that the relation between frequency in the continuous-time and discrete-time domains is ω = ΩT, where T is merely a design parameter. Leaving T as an arbitrary constant for now, we obtain

$$\Omega_p = \frac{\omega_p}{T} = \frac{0.47124}{T},\ \ and\ \Omega_s = \frac{\omega_s}{T} = \frac{1.0996}{T}$$

$$\therefore N = \frac{\log_{10}\left(\frac{\delta}{\varepsilon}\right)}{\log_{10}\left(\frac{\Omega_s}{\Omega_p}\right)} = \frac{\log_{10}\left(\frac{9.95}{0.998}\right)}{\log_{10}\left(\frac{1.0996}{0.47124}\right)} = \frac{0.998}{0.369} = 2.712$$

Since the number of poles must be an integer, we round up to N = 3.

Matching the frequency response exactly at passband produces,

$$\Omega_c = \frac{\Omega_p}{\varepsilon^{\frac{1}{N}}} = \frac{0.4716}{T}$$

For N = 3 and $\Omega_c = 1$ (Normalized Transfer function)

$$H(s) = \frac{1}{(s+1)(s^2+s+1)}$$

(This can be obtained from formula as well as table given)

This transfer function is normalized for $\Omega_c = 1$. However $\Omega_c = 0.4716/T$, we need to denormalise H(s) by $\Omega_c = 0.4716/T$

$$H(s)|\ s = {}^{s}/_{\Omega_c} = \frac{{\Omega_c}^3}{(s+\Omega_c)(s^2+\Omega_c s+{\Omega_c}^2)} = \frac{\left(\frac{0.4716}{T}\right)^3}{\left(s+\frac{0.4716}{T}\right)\left(s^2+\frac{0.4716}{T}s+\left(\frac{0.4716}{T}\right)^2\right)}$$

Using partial fractions we can rewrite the system function of the continuous-time prototype filter as

$$H(s) = \sum_{k=1}^{N} \frac{A_k}{s-s_k},$$

$$where\ the\ parameters\ A_k are\ the\ continuous\ time\ residues\ of\ the\ poles\ s_k.$$

Using the MATLAB routine *residue* we find the residues for the three poles, producing the transfer function

$$H(s) = \frac{0.4719}{s+\frac{0.4716}{T}} \pm \frac{0.236-0.136j}{s-\frac{-0.236+0.408j}{T}} + \frac{-0.236+0.136j}{s-\frac{-0.236-0.408j}{T}}$$

The corresponding discrete-time filter has the transfer function

$$H(z) = \sum_{k=1}^{N} \frac{A_k}{1-e^{s_k T}z^{-1}}$$

Note that the poles of the continuous-time filter, s_k are all of the form of a (generally complex) constant divided by the parameter T. Since s_k is multiplied by T wherever it appears in the equation for H(z) above, the specific value chosen for T has no effect at all on the discrete-time filter that results from the design process. Hence we normally let T = 1 for simplicity. This produces the transfer function

$$H(z) = \frac{0.4716}{1 - e^{-0.4716}z^{-1}} + \frac{-0.236 - 0.136j}{1 - e^{(-0.236+0.408j)}z^{-1}} + \frac{-0.236 + 0.136j}{1 - e^{(-0.236-0.408j)}z^{-1}}$$

Combining we get,

$$H(z) = \frac{0.4716}{1 - 0.624z^{-1}} + \frac{-0.472 + 0.341z^{-1}}{1 - 1.45z^{-1} + 0.624z^{-2}}$$