Quantum Galton Board (Quantum walk)

Saurabh Mhatre

saurabh.mhatre1995@gmail.com

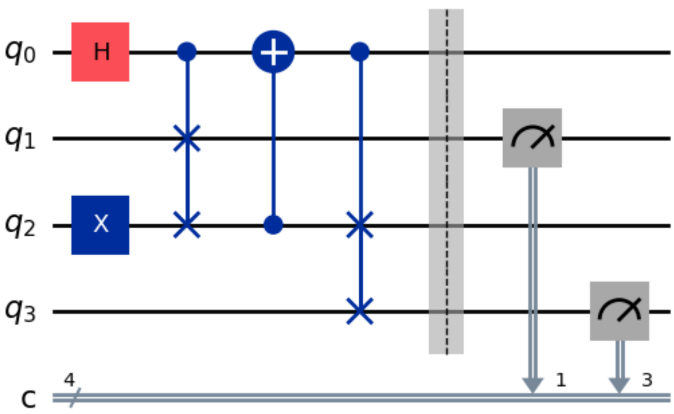
1 Motivation

Galton board is a fascinating toy model for showcasing the central limit theorem, which states that the probability distribution for a system that evolves with multiple independent random events always converges to the normal distribution. And the most peculiar thing is that this convergence is unaltered (albeit it might be shifted) even when the probability of each independent random event is biased. I got fascinated by the role of Galton Board in illustaring this idea due to a YouTube video posted by 3Blue1Brown a few years back, which lead me to choose this challenge.

Random walk problems help us to simulate random motions and to predict the probabilities of particular space occupancy. However, the two kinds of random walks, classical and quantum walk give drastically different probability densities. This attributes to the classical walk tending to center around particle's initial position with distribution spread proportional to the number of events ($\sigma \propto n$), while quantum walk tends to spread away from the initial position much faster ($\sigma \propto \sqrt{n}$) due wave function interference. In the paper given in the challenge, authors proposed a quantum circuit that mimics a Galton board and in general the classical walk. However the base of the circuit is flexible and can be modified to get quantum walk. First, I would like to give an overview of the work presented by Mark Carney and Ben Varcoe [1]

2 Quantum Galton Board (QGB)

The circuit design proposed by the authors simulates statistical distributions by linearly combining amplitude branchings derived from a hardware-inspired "peg" model. It encodes the independent paths in superposition and manipulates them with only three gate types and reset operations. Due to this simplicity the QGB achieves an exponential trajectory generation (2^n) using only $\big(O(n^2)\big)$ resources, and presents a compelling route to practical quantum sampling.



As shown in the above circuit diagram, the each peg consists 2 CSWAP or Friedkin gates and one CNOT gate. The system or ball is initialized using the X gate and the control qubit (coin), which introduces the superposition to the system through Hadamard gate. The role of quantum gates in sequence is as follows:

- Hadamard and CNOT use the superposition to place the "peg" amplitude into branching states,
- Controlled-SWAPs route the amplitude across possible paths,
- RESET gates reuse control ancilla qubits without retaining which-path information.

The circuit layers replicate rows of the classical board, building superposition over all possible descent paths. This leads to following state in case of one peg system,

$$|q_2 q_1 q_0\rangle = \frac{1}{\sqrt{2}}(|001\rangle + |100\rangle),$$

with states ($|001\rangle$)and ($|100\rangle$)being the two possible bins and amplitude square $\left(1/\sqrt{2}\right)^2$is the probability for finding the system in one of these states.

2.1 Biased QGB

Another fascinating thing about this circuit is it's universality (it's in the title!). By tuning the control ancilla qubit using a biased rotation ($R_x(\theta)$) rather than an equal-weight splitthe QGB allows amplitude weighting to reflect arbitrary probabilities at each branching.

$$|q_2 q_1 q_0\rangle = \sin(\frac{\theta}{2})|001\rangle + \cos(\frac{\theta}{2})|100\rangle,$$

In the continuous limit, this makes it a universal statistical simulator, capable of approximating Gaussian, binomial, and even exotic distributions by programming the rotation angles per layer.

References

1. Mark Carney and Ben Varcoe. Universal statistical simulator. pages –.