EE5609: Matrix Theory Assignment-4

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***Abstract*—This document contains solution to determine the conic representing the given equation.**

Download the latex-tikz codes from

https://github.com/saurabh13002/EE5609/tree/master/Assignment4

1 Problem

What conic does the following equation represent.

$$y^2 - 2\sqrt{3}xy + 3x^2 + 6x - 4y + 5 = 0$$

Find the center and equation refereed to centre.

2 Solution

The general second degree equation can be expressed as follows,

$$\mathbf{x^T V x} + 2\mathbf{u^T x} + f = 0 \tag{2.0.1}$$

From the given second degree equation we get,

$$\mathbf{V} = \begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \tag{2.0.2}$$

$$\mathbf{u} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{2.0.3}$$

$$f = 5 \tag{2.0.4}$$

Expanding the determinant of $\mathbf{V}$ we observe,

$$\begin{vmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{vmatrix} = 0 \tag{2.0.5}$$

Also

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 3 & -\sqrt{3} & 3 \\ -\sqrt{3} & 1 & -2 \\ 3 & -2 & 5 \end{vmatrix} \neq 0 \tag{2.0.6}$$

Hence from (2.0.5) and (2.0.6) we conclude that given equation is a parabola. The characteristic equation of $\mathbf{V}$ is given as follows,

$$\left|\mathbf{V} - \lambda\mathbf{I}\right| = \begin{vmatrix} 3-\lambda & -\sqrt{3} \\ -\sqrt{3} & 1-\lambda \end{vmatrix} = 0 \tag{2.0.7}$$

$$\implies \lambda^2 - 4\lambda = 0 \tag{2.0.8}$$

Hence the characteristic equation of $\mathbf{V}$ is given by (2.0.8). The roots of (2.0.8) i.e the eigenvalues are given by

$$\lambda_1 = 0, \lambda_2 = 4 \tag{2.0.9}$$

The eigen vector $\mathbf{p}$ is defined as,

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{2.0.10}$$

$$\implies (\mathbf{V} - \lambda\mathbf{I})\,\mathbf{p} = 0 \tag{2.0.11}$$

for $\lambda_1 = 0$,

$$(\mathbf{V} - \lambda_1\mathbf{I}) = \begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \xleftrightarrow[R_1 = \frac{1}{\sqrt{3}}R_1]{R_2 = R_1 + R_2} \begin{pmatrix} \sqrt{3} & -1 \\ 0 & 0 \end{pmatrix} \tag{2.0.12}$$

Substiuting equation 2.0.12 in equation 2.0.11 and upon normalizing we get we get

$$\implies \mathbf{p_1} = \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix} \tag{2.0.13}$$

Again, for $\lambda_2 = 4$,

$$(\mathbf{V} - \lambda_2\mathbf{I}) = \begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & -3 \end{pmatrix} \xleftrightarrow[R_1 = -\sqrt{3}R_1]{R_2 = -\sqrt{3}R_1 + R_2} \begin{pmatrix} 1 & \sqrt{3} \\ 0 & 0 \end{pmatrix} \tag{2.0.14}$$

Substiuting equation 2.0.14 in equation 2.0.11 and upon normalizing we get

$$\mathbf{p_2} = \begin{pmatrix} -\sqrt{3}/2 \\ 1/2 \end{pmatrix} \tag{2.0.15}$$

The matrix P,

$$\mathbf{P} = \begin{pmatrix}\mathbf{p_1} & \mathbf{p_2}\end{pmatrix} = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} \tag{2.0.16}$$

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \tag{2.0.17}$$

$$\eta = 2\mathbf{p_1}^T\mathbf{u} = 3 - 2\sqrt{3} \tag{2.0.18}$$

The focal length of the parabola is given by:

$$\left|\frac{\eta}{\lambda_2}\right| = \left|\frac{3-2\sqrt{3}}{4}\right| = 0.116 \tag{2.0.19}$$

When $\left|\mathbf{V}\right| = 0$, (2.0.1) can be written as

$$\mathbf{y^T D y} = -\eta\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{y} \tag{2.0.20}$$

And the vertex $\mathbf{c}$ is given by

$$\begin{pmatrix} \mathbf{u^T} + \frac{\eta}{2}\mathbf{p_1^T} \\ \mathbf{V} \end{pmatrix}\mathbf{c} = \begin{pmatrix} -f \\ \frac{\eta}{2}\mathbf{p_1} - \mathbf{u} \end{pmatrix} \tag{2.0.21}$$

Substituting the found values

$$\mathbf{u}^T + \frac{\eta}{2}\mathbf{p_1}^T = \begin{pmatrix}3 & -2\end{pmatrix} + \frac{3-2\sqrt{3}}{2}\begin{pmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix} \tag{2.0.22}$$

$$\implies \mathbf{u}^T + \frac{\eta}{2}\mathbf{p_1}^T = \begin{pmatrix}\frac{15-2\sqrt{3}}{4} & \frac{-14+3\sqrt{3}}{4}\end{pmatrix} \tag{2.0.23}$$

$$\frac{\eta}{2}\mathbf{p_1} - \mathbf{u} = \begin{pmatrix}\frac{-9-2\sqrt{3}}{4} \\ \frac{2+3\sqrt{3}}{4}\end{pmatrix} \tag{2.0.24}$$

using equations (2.0.3),(2.0.4),(2.0.13),(2.0.23),(2.0.24) and (2.0.13) in (2.0.21)

$$\begin{pmatrix} \frac{15-2\sqrt{3}}{4} & \frac{-14+3\sqrt{3}}{4} \\ 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}\mathbf{c} = \begin{pmatrix} -5 \\ \frac{-9-2\sqrt{3}}{4} \\ \frac{2+3\sqrt{3}}{4} \end{pmatrix} \tag{2.0.25}$$

By performing row reductions on augmented matrix

$$\begin{pmatrix} \frac{15-2\sqrt{3}}{4} & \frac{-14+3\sqrt{3}}{4} & -5 \\ 3 & -\sqrt{3} & \frac{(-9-2\sqrt{3})}{4} \\ -\sqrt{3} & 1 & \frac{2+3\sqrt{3}}{4} \end{pmatrix} R_2 \leftrightarrow R_1$$

$$\begin{pmatrix} 3 & -\sqrt{3} & \frac{(-9-2\sqrt{3})}{4} \\ \frac{15-2\sqrt{3}}{4} & \frac{-14+3\sqrt{3}}{4} & -5 \\ -\sqrt{3} & 1 & \frac{2+3\sqrt{3}}{4} \end{pmatrix} \tag{2.0.26}$$

$$\begin{pmatrix} 3 & -\sqrt{3} & \frac{(-9-2\sqrt{3})}{4} \\ \frac{15-2\sqrt{3}}{4} & \frac{-14+3\sqrt{3}}{4} & -5 \\ -\sqrt{3} & 1 & \frac{2+3\sqrt{3}}{4} \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - \frac{15-2\sqrt{3}}{12}R_1}$$

$$\begin{pmatrix} 3 & -\sqrt{3} & \frac{(-9-2\sqrt{3})}{4} \\ 0 & 2(\sqrt{3}-2) & \frac{(4\sqrt{3}-39)}{16} \\ \sqrt{3} & 1 & \frac{2+3\sqrt{3}}{4} \end{pmatrix} \tag{2.0.27}$$

Therefore,

$$\begin{pmatrix} 3 & -\sqrt{3} & \frac{(-9-2\sqrt{3})}{4} \\ 0 & 2(\sqrt{3}-2) & (4\sqrt{3}-39) \\ -\sqrt{3} & 1 & \frac{(2+3\sqrt{3})}{4} \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 + \frac{1}{\sqrt{3}}R_1}$$

$$\begin{pmatrix} 3 & -\sqrt{3} & \frac{(-9-2\sqrt{3})}{4} \\ 0 & 2(\sqrt{3}-2) & \frac{(4\sqrt{3}-39)}{16} \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.28}$$

On solving for values of $\mathbf{c}$ from the augmented matrix we get

$$\begin{pmatrix} 3 & -1.732 & -3.11 \\ 0 & -0.535 & -2 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.29}$$

Hence $\mathbf{c}=\begin{pmatrix}1.11 \\ 3.73\end{pmatrix}$.The vertex of parabola at (1.11, 3.73).

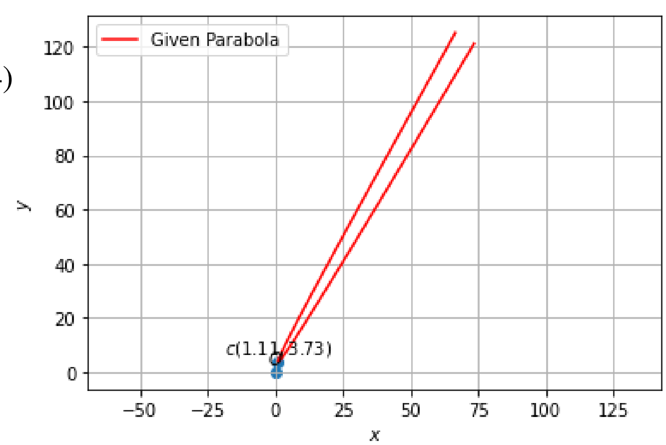

Fig. 1: Parabola with the center c