EE5609: Matrix Theory Assignment-4

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***Abstract*—This document explains how to factorize a matrix using QR decomposition.**

Download the latex-tikz codes from

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https://github.com/saurabh13002/EE5609/tree/master/Assignment4/QR_V
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1 Problem

Perform QR decomposition of $\begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$

2 Explanation

Let $\mathbf{a}$ and $\mathbf{b}$ be columns of a $\mathbf{A}$. Then, the matrix $\mathbf{A}$ can be decomposed in the form as:

$$\mathbf{A} = \mathbf{QR} \tag{2.0.1}$$

such that

$$\mathbf{QQ}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I} \tag{2.0.2}$$

$$\mathbf{Q} = \begin{pmatrix} \mathbf{u_1} & \mathbf{u_2} \end{pmatrix} \tag{2.0.3}$$

$$\mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{2.0.4}$$

where

$$k_1 = \|\mathbf{a}\| \tag{2.0.5}$$

$$\mathbf{u_1} = \frac{\mathbf{a}}{k_1} \tag{2.0.6}$$

$$r_1 = \frac{\mathbf{u_1}^T\mathbf{b}}{\|\mathbf{u_1}\|^2} \tag{2.0.7}$$

$$\mathbf{u_2} = \frac{\mathbf{b} - r_1\mathbf{u_1}}{\|\mathbf{b} - r_1\mathbf{u_1}\|} \tag{2.0.8}$$

$$k_2 = \mathbf{u_2}^T\mathbf{b} \tag{2.0.9}$$

Then, the matrix can be represented as

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} = \begin{pmatrix} \mathbf{u_1} & \mathbf{u_2} \end{pmatrix} \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{2.0.10}$$

3 Solution

Let $\mathbf{A}$ be the given matrix. Then $\mathbf{A} = \begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$ and columns of $\mathbf{A}$ are $\mathbf{a}$ and $\mathbf{b}$, where

$$\mathbf{a} = \begin{pmatrix} 3 \\ -\sqrt{3} \end{pmatrix} \tag{3.0.1}$$

$$\mathbf{b} = \begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix} \tag{3.0.2}$$

Now, for given matrix from (2.0.5) and (2.0.6), we have

$$k_1 = \|\mathbf{a}\| = \sqrt{12} = 2\sqrt{3} \tag{3.0.3}$$

$$\mathbf{u_1} = \frac{1}{2\sqrt{3}} \begin{pmatrix} 3 \\ -\sqrt{3} \end{pmatrix} \tag{3.0.4}$$

By, (2.0.7), we find

$$r_1 = \frac{\frac{1}{2\sqrt{3}} \begin{pmatrix} 3 & -\sqrt{3} \end{pmatrix} \begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix}}{1} = -2 \tag{3.0.5}$$

Now, by (2.0.8)

$$\mathbf{u_2} = \frac{\begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix} - \frac{-2}{2\sqrt{3}} \begin{pmatrix} 3 \\ -\sqrt{3} \end{pmatrix}}{\left\| \begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix} - \frac{-2}{2\sqrt{3}} \begin{pmatrix} 3 \\ -\sqrt{3} \end{pmatrix} \right\|} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.0.6}$$

From (2.0.9),

$$k_2 = \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix} = 0 \tag{3.0.7}$$

Verification for $\mathbf{Q}^T\mathbf{Q} = I$

Here we observed that the second vector in $\mathbf{Q}$ is zero since the column vectors in $\mathbf{A}$ are dependent. Thus, $\mathbf{Q}$ is effectively written as:

$$\mathbf{Q} = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix} \tag{3.0.8}$$

We observe that $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$

$$\mathbf{Q}^T\mathbf{Q} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix} = 1 \tag{3.0.9}$$

Hence, Verified.

Now,for a 2x2 matrix $\mathbf{A}$ we can write matrix $\mathbf{QR}$ as product of respective coloumn and row vector as,

$$\begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 2\sqrt{3} & -2 \end{pmatrix} \tag{3.0.10}$$

which is the required $\mathbf{QR}$ decomposition of $\mathbf{A}$.