

EE5609: Matrix Theory

Assignment-4

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Abstract—This document contains solution to determine the conic representing the given equation.

Download the latex-tikz codes from

<https://github.com/saurabh13002/EE5609/tree/master/Assignment4>

1 PROBLEM

What conic does the following equation represent.

$$y^2 - 2\sqrt{3}xy + 3x^2 + 6x - 4y + 5 = 0$$

Find the center and equation referred to centre.

2 SOLUTION

The general second degree equation can be expressed as follows,

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.0.1)$$

From the given second degree equation we get,

$$\mathbf{V} = \begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \quad (2.0.2)$$

$$\mathbf{u} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad (2.0.3)$$

$$f = 5 \quad (2.0.4)$$

Expanding the determinant of \mathbf{V} we observe,

$$\begin{vmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{vmatrix} = 0 \quad (2.0.5)$$

Also

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 3 & -\sqrt{3} & 3 \\ -\sqrt{3} & 1 & -2 \\ 3 & -2 & 5 \end{vmatrix} \neq 0 \quad (2.0.6)$$

Hence from (2.0.5) and (2.0.6) we conclude that given equation is a parabola. The characteristic equation of \mathbf{V} is given as follows,

$$|\mathbf{V} - \lambda \mathbf{I}| = \begin{vmatrix} 3 - \lambda & -\sqrt{3} \\ -\sqrt{3} & 1 - \lambda \end{vmatrix} = 0 \quad (2.0.7)$$

$$\implies \lambda^2 - 4\lambda = 0 \quad (2.0.8)$$

Hence the characteristic equation of \mathbf{V} is given by (2.0.8). The roots of (2.0.8) i.e the eigenvalues are given by

$$\lambda_1 = 0, \lambda_2 = 4 \quad (2.0.9)$$

The eigen vector \mathbf{p} is defined as,

$$\mathbf{V} \mathbf{p} = \lambda \mathbf{p} \quad (2.0.10)$$

$$\implies (\mathbf{V} - \lambda \mathbf{I}) \mathbf{p} = 0 \quad (2.0.11)$$

for $\lambda_1 = 0$,

$$(\mathbf{V} - \lambda_1 \mathbf{I}) = \begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \xrightarrow[R_1 = \frac{1}{\sqrt{3}} R_1]{R_2 = R_1 + R_2} \begin{pmatrix} \sqrt{3} & -1 \\ 0 & 0 \end{pmatrix} \quad (2.0.12)$$

Substituting equation 2.0.12 in equation 2.0.11 and upon normalizing we get we get

$$\implies \mathbf{p}_1 = \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix} \quad (2.0.13)$$

Again, for $\lambda_2 = 4$,

$$(\mathbf{V} - \lambda_2 \mathbf{I}) = \begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & -3 \end{pmatrix} \xrightarrow[R_1 = -\sqrt{3} R_1]{R_2 = -\sqrt{3} R_1 + R_2} \begin{pmatrix} 1 & \sqrt{3} \\ 0 & 0 \end{pmatrix} \quad (2.0.14)$$

Substituting equation 2.0.14 in equation 2.0.11 and upon normalizing we get

$$\mathbf{p}_2 = \begin{pmatrix} -\sqrt{3}/2 \\ 1/2 \end{pmatrix} \quad (2.0.15)$$

The matrix P,

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2) = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} \quad (2.0.16)$$

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \quad (2.0.17)$$

$$\eta = 2\mathbf{p}_1^T \mathbf{u} = 3 - 2\sqrt{3} \quad (2.0.18)$$

The focal length of the parabola is given by:

$$\left| \frac{\eta}{\lambda_2} \right| = \left| \frac{3 - 2\sqrt{3}}{4} \right| = 0.116 \quad (2.0.19)$$

When $|\mathbf{V}| = 0$, (2.0.1) can be written as

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = -\eta \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{y} \quad (2.0.20)$$

And the vertex \mathbf{c} is given by

$$\begin{pmatrix} \mathbf{u}^T + \frac{\eta}{2} \mathbf{p}_1^T \\ \mathbf{v}^T \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \frac{\eta}{2} \mathbf{p}_1 - \mathbf{u} \end{pmatrix} \quad (2.0.21)$$

Substituting the found values

$$\mathbf{u}^T + \frac{\eta}{2} \mathbf{p}_1^T = \begin{pmatrix} 3 & -2 \end{pmatrix} + \frac{3 - 2\sqrt{3}}{2} \begin{pmatrix} 1/2 & \sqrt{3}/2 \end{pmatrix} \quad (2.0.22)$$

$$\Rightarrow \mathbf{u}^T + \frac{\eta}{2} \mathbf{p}_1^T = \begin{pmatrix} \frac{15-2\sqrt{3}}{4} & \frac{-14+3\sqrt{3}}{4} \end{pmatrix} \quad (2.0.23)$$

$$\frac{\eta}{2} \mathbf{p}_1 - \mathbf{u} = \begin{pmatrix} \frac{-9-2\sqrt{3}}{4} \\ \frac{2+3\sqrt{3}}{4} \end{pmatrix} \quad (2.0.24)$$

using equations (2.0.3),(2.0.4),(2.0.13),(2.0.23),(2.0.24) and (2.0.13) in (2.0.21)

$$\begin{pmatrix} \frac{15-2\sqrt{3}}{4} & \frac{-14+3\sqrt{3}}{4} \\ 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -5 \\ \frac{-9-2\sqrt{3}}{4} \\ \frac{2+3\sqrt{3}}{4} \end{pmatrix} \quad (2.0.25)$$

By performing row reductions on augmented matrix

$$\begin{pmatrix} \frac{15-2\sqrt{3}}{4} & \frac{-14+3\sqrt{3}}{4} & -5 \\ 3 & -\sqrt{3} & \frac{(-9-2\sqrt{3})}{4} \\ -\sqrt{3} & 1 & \frac{2+3\sqrt{3}}{4} \end{pmatrix} R_2 \leftrightarrow R_1 \quad (2.0.26)$$

$$\begin{pmatrix} 3 & -\sqrt{3} & \frac{(-9-2\sqrt{3})}{4} \\ \frac{15-2\sqrt{3}}{4} & \frac{-14+3\sqrt{3}}{4} & -5 \\ -\sqrt{3} & 1 & \frac{2+3\sqrt{3}}{4} \end{pmatrix}$$

$$\begin{pmatrix} 3 & -\sqrt{3} & \frac{(-9-2\sqrt{3})}{4} \\ \frac{15-2\sqrt{3}}{4} & \frac{-14+3\sqrt{3}}{4} & -5 \\ -\sqrt{3} & 1 & \frac{2+3\sqrt{3}}{4} \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - \frac{15-2\sqrt{3}}{12} R_1} \begin{pmatrix} 3 & -\sqrt{3} & \frac{(-9-2\sqrt{3})}{4} \\ 0 & 2(\sqrt{3}-2) & \frac{(4\sqrt{3}-39)}{16} \\ \sqrt{3} & 1 & \frac{2+3\sqrt{3}}{4} \end{pmatrix} \quad (2.0.27)$$

Therefore,

$$\begin{pmatrix} 3 & -\sqrt{3} & \frac{(-9-2\sqrt{3})}{4} \\ 0 & 2(\sqrt{3}-2) & \frac{(4\sqrt{3}-39)}{16} \\ -\sqrt{3} & 1 & \frac{(2+3\sqrt{3})}{4} \end{pmatrix} \xrightarrow{R_3 \leftarrow R_3 + \frac{1}{\sqrt{3}} R_1} \begin{pmatrix} 3 & -\frac{433}{250} & -\frac{311}{100} \\ 0 & -\frac{107}{200} & -2 \\ 0 & 0 & 0 \end{pmatrix} \quad (2.0.28)$$

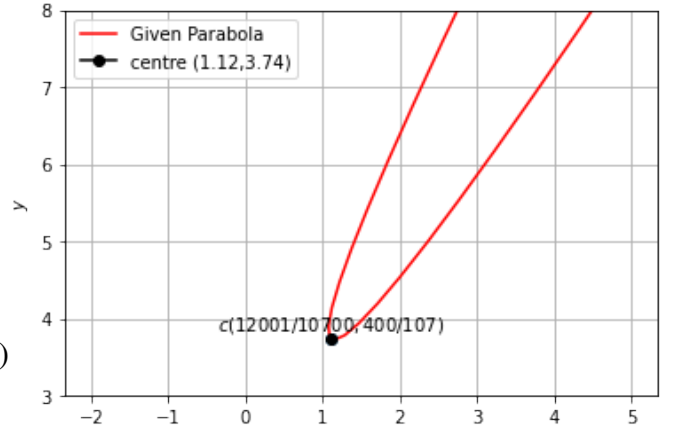


Fig. 1: Parabola with the center c

$$\begin{pmatrix} 3 & -\frac{433}{250} & -\frac{311}{100} \\ 0 & -\frac{107}{200} & -2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 \leftarrow \frac{R_1}{3}} \begin{pmatrix} 1 & -\frac{433}{750} & -\frac{311}{300} \\ 0 & -\frac{107}{200} & -2 \\ 0 & 0 & 0 \end{pmatrix} \quad (2.0.29)$$

$$\begin{pmatrix} 1 & -\frac{433}{750} & -\frac{311}{300} \\ 0 & -\frac{107}{200} & -2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 \leftarrow -\frac{200}{107} R_2} \begin{pmatrix} 1 & -\frac{433}{750} & -\frac{311}{300} \\ 0 & 1 & \frac{400}{107} \\ 0 & 0 & 0 \end{pmatrix} \quad (2.0.30)$$

$$\begin{pmatrix} 1 & -\frac{433}{750} & -\frac{311}{300} \\ 0 & 1 & \frac{400}{107} \\ 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 + \frac{433}{750}R_2} \begin{pmatrix} 1 & 0 & \frac{12001}{10700} \\ 0 & 1 & \frac{400}{107} \\ 0 & 0 & 0 \end{pmatrix} \quad (2.0.31)$$

On solving for values of \mathbf{c} from (2.0.31) The vertex of parabola is $\mathbf{c} = \begin{pmatrix} \frac{12001}{10700} \\ \frac{400}{107} \end{pmatrix}$.