

# EE5609: Matrix Theory

## Assignment-5

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**Abstract**—This document finds the coordinates of foot of perpendicular using Singular Value Decomposition.

Download the latex code from

[https://github.com/saurabh13002/EE5609/tree/master/Assignment5\\_SVD](https://github.com/saurabh13002/EE5609/tree/master/Assignment5_SVD)

### 1 PROBLEM

Perform SVD and find the foot of the perpendicular from any point outside the plane.  
 $(14 \ -3 \ -18)\mathbf{x} = -1$

### 2 SOLUTION

Let the point be  $\begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}$ , First we find orthogonal vectors  $\mathbf{m}_1$  and  $\mathbf{m}_2$  to the given normal vector  $\mathbf{n}$ .  
 Let,  $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ , then

$$\mathbf{m}^T \mathbf{n} = 0 \quad (2.0.1)$$

$$\Rightarrow (a \ b \ c) \begin{pmatrix} 14 \\ -3 \\ 18 \end{pmatrix} = 0 \quad (2.0.2)$$

$$\Rightarrow 14a - 3b + 18c = 0 \quad (2.0.3)$$

Putting  $a=1$  and  $b=0$  we get,

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ 0 \\ \frac{-7}{9} \end{pmatrix} \quad (2.0.4)$$

Putting  $a=0$  and  $b=1$  we get,

$$\mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{1}{6} \end{pmatrix} \quad (2.0.5)$$

Now we solve the equation,

$$\mathbf{M}\mathbf{x} = \mathbf{b} \quad (2.0.6)$$

Putting values in (2.0.6),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{-7}{9} & \frac{1}{6} \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix} \quad (2.0.7)$$

In order to solve (2.0.7), perform Singular Value Decomposition on  $\mathbf{M}$  as follows,

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \quad (2.0.8)$$

Where the columns of  $\mathbf{V}$  are the eigen vectors of  $\mathbf{M}^T \mathbf{M}$ , the columns of  $\mathbf{U}$  are the eigen vectors of  $\mathbf{M}\mathbf{M}^T$  and  $\mathbf{S}$  is diagonal matrix of singular value of eigenvalues of  $\mathbf{M}^T \mathbf{M}$ .

$$\mathbf{M}^T \mathbf{M} = \begin{pmatrix} \frac{130}{54} & \frac{-7}{36} \\ \frac{81}{54} & \frac{54}{36} \end{pmatrix} \quad (2.0.9)$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{-7}{9} & \frac{1}{6} \end{pmatrix} \quad (2.0.10)$$

From (2.0.6) putting (2.0.8) we get,

$$\mathbf{U}\mathbf{S}\mathbf{V}^T \mathbf{x} = \mathbf{b} \quad (2.0.11)$$

$$\Rightarrow \mathbf{x} = \mathbf{V}\mathbf{S}_+ \mathbf{U}^T \mathbf{b} \quad (2.0.12)$$

Where  $\mathbf{S}_+$  is Moore-Penrose Pseudo-Inverse of  $\mathbf{S}$ . Now, calculating eigen value of  $\mathbf{M}\mathbf{M}^T$ ,

$$|\mathbf{M}\mathbf{M}^T - \lambda \mathbf{I}| = 0 \quad (2.0.13)$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & \frac{1}{2} \\ 0 & 1-\lambda & 1 \\ \frac{1}{2} & 1 & \frac{205}{324} - \lambda \end{vmatrix} = 0 \quad (2.0.14)$$

$$\Rightarrow -\lambda^3 + \frac{853}{324}\lambda^2 - \frac{529}{324}\lambda = 0 \quad (2.0.15)$$

Hence eigen values of  $\mathbf{M}\mathbf{M}^T$  are,

$$\lambda_1 = \frac{529}{324} \quad (2.0.16)$$

$$\lambda_2 = 1 \quad (2.0.17)$$

$$\lambda_3 = 0 \quad (2.0.18)$$

Hence the eigen vectors of  $\mathbf{M}\mathbf{M}^T$  are,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{-252}{205} \\ \frac{54}{205} \\ 1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} \frac{3}{4} \\ 1 \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} \frac{7}{9} \\ \frac{-1}{6} \\ 1 \end{pmatrix} \quad (2.0.19)$$

Normalizing the eigen vectors we get,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{-252}{23\sqrt{205}} \\ \frac{54}{23\sqrt{205}} \\ \frac{\sqrt{205}}{23} \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} \frac{3}{\sqrt{205}} \\ \frac{14}{\sqrt{205}} \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} \frac{14}{23} \\ \frac{-3}{23} \\ \frac{18}{23} \end{pmatrix} \quad (2.0.20)$$

Hence we obtain  $\mathbf{U}$  of (2.0.8) as follows,  $\mathbf{U}$

$$\begin{pmatrix} \frac{-252}{23\sqrt{205}} & \frac{3}{\sqrt{205}} & \frac{14}{23} \\ \frac{54}{23\sqrt{205}} & \frac{14}{\sqrt{205}} & \frac{-3}{23} \\ \frac{\sqrt{205}}{23} & 0 & \frac{18}{23} \end{pmatrix} \quad (2.0.21)$$

After computing the singular values from eigen values  $\lambda_1, \lambda_2, \lambda_3$  we get  $\mathbf{S}$  of (2.0.8) as follows,

$$\mathbf{S} = \begin{pmatrix} \frac{23}{18} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (2.0.22)$$

Now, calculating eigen value of  $\mathbf{M}^T\mathbf{M}$ ,

$$|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}| = 0 \quad (2.0.23)$$

$$\Rightarrow \begin{pmatrix} \frac{130}{81} - \lambda & \frac{-7}{54} \\ \frac{-7}{54} & \frac{37}{36} - \lambda \end{pmatrix} = 0 \quad (2.0.24)$$

$$\Rightarrow \lambda^2 - \frac{853}{324}\lambda + \frac{529}{324} = 0 \quad (2.0.25)$$

Hence eigen values of  $\mathbf{M}^T\mathbf{M}$  are,

$$\lambda_4 = \frac{529}{324} \quad (2.0.26)$$

$$\lambda_5 = 1 \quad (2.0.27)$$

Hence the eigen vectors of  $\mathbf{M}^T\mathbf{M}$  are,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{-14}{3} \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} \frac{3}{14} \\ 1 \end{pmatrix} \quad (2.0.28)$$

Normalizing the eigen vectors we get,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{-14}{\sqrt{205}} \\ \frac{3}{\sqrt{205}} \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} \frac{3}{\sqrt{205}} \\ \frac{14}{\sqrt{205}} \end{pmatrix} \quad (2.0.29)$$

Hence we obtain  $\mathbf{V}$  of (2.0.8) as follows,

$$\mathbf{V} = \begin{pmatrix} \frac{-14}{\sqrt{205}} & \frac{3}{\sqrt{205}} \\ \frac{3}{\sqrt{205}} & \frac{14}{\sqrt{205}} \end{pmatrix} \quad (2.0.30)$$

From (2.0.8) we get the Singular Value Decomposition of  $\mathbf{M}$ ,

$$\mathbf{M} = \begin{pmatrix} \frac{-252}{23\sqrt{205}} & \frac{3}{\sqrt{205}} & \frac{14}{23} \\ \frac{54}{23\sqrt{205}} & \frac{14}{\sqrt{205}} & \frac{-3}{23} \\ \frac{\sqrt{205}}{23} & 0 & \frac{18}{23} \end{pmatrix} \begin{pmatrix} \frac{23}{18} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{-14}{\sqrt{205}} & \frac{3}{\sqrt{205}} \\ \frac{3}{\sqrt{205}} & \frac{14}{\sqrt{205}} \end{pmatrix}^T \quad (2.0.31)$$

Moore-Penrose Pseudo inverse of  $\mathbf{S}$  is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{18}{23} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (2.0.32)$$

From (2.0.11) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{415107}{100000} \\ \frac{167623}{50000} \\ -\frac{154347}{50000} \end{pmatrix} \quad (2.0.33)$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{1624313691}{500000000} \\ \frac{167623}{50000} \end{pmatrix} \quad (2.0.34)$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{969580693}{250000000} \\ \frac{32467791}{12500000} \end{pmatrix} = \begin{pmatrix} 3.87 \\ 2.59 \end{pmatrix} \quad (2.0.35)$$

Verifying the solution of (2.0.35) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \quad (2.0.36)$$

Evaluating the R.H.S in (2.0.36) we get,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} \frac{53}{9} \\ \frac{13}{6} \end{pmatrix} \quad (2.0.37)$$

$$\Rightarrow \begin{pmatrix} \frac{130}{81} & \frac{-7}{54} \\ \frac{-7}{54} & \frac{37}{36} \end{pmatrix} \mathbf{x} = \begin{pmatrix} \frac{53}{9} \\ \frac{13}{6} \end{pmatrix} \quad (2.0.38)$$

Solving the augmented matrix of (2.0.38) we get,

$$\left( \begin{array}{cc|c} \frac{130}{81} & \frac{-7}{54} & \frac{53}{9} \\ \frac{-7}{54} & \frac{37}{36} & \frac{13}{6} \end{array} \right) \xrightarrow{R_1 = \frac{81}{130}R_1} \left( \begin{array}{cc|c} 1 & \frac{-21}{260} & \frac{477}{130} \\ \frac{-7}{54} & \frac{37}{36} & \frac{13}{6} \end{array} \right) \quad (2.0.39)$$

$$\xrightarrow{R_2 = R_2 + \frac{7}{54}R_1} \left( \begin{array}{cc|c} 1 & \frac{-21}{260} & \frac{477}{130} \\ 0 & \frac{260}{520} & \frac{130}{687} \end{array} \right) \quad (2.0.40)$$

$$\xrightarrow{R_2 = \frac{520}{260}R_2} \left( \begin{array}{cc|c} 1 & \frac{-21}{260} & \frac{477}{130} \\ 0 & 1 & \frac{130}{529} \end{array} \right) \quad (2.0.41)$$

$$\xrightarrow{R_1 = R_1 + \frac{21}{260}R_2} \left( \begin{array}{cc|c} 1 & 0 & \frac{2052}{529} \\ 0 & 1 & \frac{130}{529} \end{array} \right) \quad (2.0.42)$$

From equation (2.0.42), solution is given by,

$$\mathbf{x} = \begin{pmatrix} \frac{2052}{529} \\ \frac{1374}{529} \end{pmatrix} = \begin{pmatrix} 3.87 \\ 2.59 \end{pmatrix} \quad (2.0.43)$$

Comparing results of  $\mathbf{x}$  from (2.0.35) and (2.0.43), we can say that the solution is verified.