EE5609: Matrix Theory Assignment-4

Major Saurabh Joshi
MTech Artificial Intelligence
AI20MTECH13002

Abstract—This document explains how to factorize a matrix using QR decomposition.

Download the latex-tikz codes from

1 PROBLEM

Perform QR decomposition of $\begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$

2 EXPLANATION

Let **a** and **b** be columns of a **A**. Then, the matrix **A** can be decomposed in the form as:

$$\mathbf{A} = \mathbf{QR} \tag{2.0.1}$$

such that

$$\mathbf{QQ}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I} \tag{2.0.2}$$

$$\mathbf{Q} = \begin{pmatrix} \mathbf{u_1} & \mathbf{u_2} \end{pmatrix} \tag{2.0.3}$$

$$\mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{2.0.4}$$

where

$$k_1 = \|\mathbf{a}\| \tag{2.0.5}$$

$$\mathbf{u_1} = \frac{\mathbf{a}}{k_1} \tag{2.0.6}$$

$$r_1 = \frac{\mathbf{u_1}^T \mathbf{b}}{\|\mathbf{u_1}\|^2} \tag{2.0.7}$$

$$\mathbf{u_2} = \frac{\mathbf{b} - r_1\mathbf{u_1}}{\|\mathbf{b} - r_1\mathbf{u_1}\|} \tag{2.0.8}$$

$$k_2 = \mathbf{u_2}^T \mathbf{b} \tag{2.0.9}$$

Then, the matrix can be represented as

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} = \begin{pmatrix} \mathbf{u_1} & \mathbf{u_2} \end{pmatrix}\begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{2.0.10}$$

3 SOLUTION

Let **A** be the given matrix. Then $\mathbf{A} = \begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$ and columns of **A** are **a** and **b**, where

$$\mathbf{a} = \begin{pmatrix} 3 \\ -\sqrt{3} \end{pmatrix} \tag{3.0.1}$$

$$\mathbf{b} = \begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix} \tag{3.0.2}$$

Now, for given matrix from (2.0.5) and (2.0.6), we have

$$k_1 = \|\mathbf{a}\| = \sqrt{12} = 2\sqrt{3} \tag{3.0.3}$$

$$\mathbf{u_1} = \frac{1}{2\sqrt{3}}\begin{pmatrix} 3 \\ -\sqrt{3} \end{pmatrix} \tag{3.0.4}$$

By, (2.0.7), we find

$$r_1 = \frac{\frac{1}{2\sqrt{3}}\begin{pmatrix} 3 & -\sqrt{3} \end{pmatrix}\begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix}}{1} = -2 \tag{3.0.5}$$

Now, by (2.0.8)

$$\mathbf{u_2} = \frac{\begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix} - \frac{-2}{2\sqrt{3}}\begin{pmatrix} 3 \\ -\sqrt{3} \end{pmatrix}}{\left\|\begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix} - \frac{-2}{2\sqrt{3}}\begin{pmatrix} 3 \\ -\sqrt{3} \end{pmatrix}\right\|} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.0.6}$$

From (2.0.9),

$$k_2 = \begin{pmatrix} 0 & 0 \end{pmatrix}\begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix} = 0 \tag{3.0.7}$$

Now, by (2.0.10) we can write matrix **A** as

$$\begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 \\ -\frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} 2\sqrt{3} & -2 \\ 0 & 0 \end{pmatrix} \tag{3.0.8}$$

which is the required **QR** decomposition of **A**.

Verification for $\mathbf{Q}^T\mathbf{Q} = I$

Here we observed that the second vector in $\mathbf{Q}$ is zero since the column vectors in $\mathbf{A}$ are dependent. Thus, $\mathbf{Q}$ is effectively written as:

$$\mathbf{Q} = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix} \qquad (3.0.9)$$

We observe that $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$

$$\mathbf{Q}^T\mathbf{Q} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix} = 1 \qquad (3.0.10)$$

Hence, Verified.