EE5609: Matrix Theory Assignment-9

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***Abstract*—This document uses properties of vector spaces and subspaces.**

Download the latex code from

https://github.com/saurabh13002/EE5609/tree/master/Assignment9

1 Problem

Show that the vectors

$$\alpha_{\mathbf{1}} = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \qquad \alpha_{\mathbf{2}} = \begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \tag{1.0.1}$$

$$\alpha_{\mathbf{3}} = \begin{pmatrix} 0 & -3 & 2 \end{pmatrix} \tag{1.0.2}$$

form a basis for $\mathbb{R}^3$. Express each of the standard basis vectors as linear combinations of $\begin{pmatrix} \alpha_{\mathbf{1}} & \alpha_{\mathbf{2}} & \alpha_{\mathbf{3}} \end{pmatrix}$

2 Theorem

Theorem 2.1. *Let* $\mathbf{V}$ *be an n-dimensional vector space over the field* $\mathbf{F}$*, and let* β *and* β' *be two ordered basis of* $\mathbf{V}$*. Then, there is a unique, necessarily invertible,* $n \times n$ *matrix* $\mathbf{P}$ *with entries in* $\mathbf{F}$ *such that*

1) $\left[\alpha\right]_{\beta} = \mathbf{P}\left[\alpha\right]_{\beta'}$
2) $\left[\alpha\right]_{\beta'} = \mathbf{P^{-1}}\left[\alpha\right]_{\beta}$

for every vector α *in* $\mathbf{V}$*. The columns of* $\mathbf{P}$ *are given by*

$$\mathbf{P_j} = \left[\alpha_{\mathbf{j}}\right]_{\beta} \qquad j = 1, 2, ..., n \tag{2.0.1}$$

3 Solution

In order to show that the set of vectors α_1, α_2,and α_3 are basis for $\mathbb{R}^3$. We first show that α_1, α_2,and α_3 a are linearly independent in $\mathbb{R}^3$ and also they span $\mathbb{R}^3$. Consider,

$$\mathbf{A} = \begin{pmatrix} \alpha_{\mathbf{1}}{}^T & \alpha_{\mathbf{2}}{}^T & \alpha_{\mathbf{3}}{}^T \end{pmatrix} \tag{3.0.1}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ -1 & 1 & 2 \end{pmatrix} \tag{3.0.2}$$

Now,by row reduction

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ -1 & 1 & 2 \end{pmatrix} \xleftrightarrow{R_3 = R_3 + R_1} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ 0 & 2 & 2 \end{pmatrix} \tag{3.0.3}$$

$$\xleftrightarrow{R_3 = R_3 - R_2} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ 0 & 0 & 5 \end{pmatrix} \tag{3.0.4}$$

$$\xleftrightarrow{R_2 = \frac{R_2}{2}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & \frac{-3}{2} \\ 0 & 0 & 5 \end{pmatrix} \tag{3.0.5}$$

$$\xleftrightarrow{R_1 = R_1 - R_2} \begin{pmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & \frac{-3}{2} \\ 0 & 0 & 5 \end{pmatrix} \tag{3.0.6}$$

$$\xleftrightarrow{R_3 = \frac{R_3}{5}} \begin{pmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & \frac{-3}{2} \\ 0 & 0 & 1 \end{pmatrix} \tag{3.0.7}$$

$$\xleftrightarrow{R_1 = R_1 - \frac{3}{2}R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{-3}{2} \\ 0 & 0 & 1 \end{pmatrix} \tag{3.0.8}$$

$$\xleftrightarrow{R_2 = R_2 + \frac{3}{2}R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.0.9}$$

(3.0.9) is the row reduced echelon form of $\mathbf{A}$ and since it is identity matrix of order 3, we say that vectors $\alpha_{\mathbf{1}}$, $\alpha_{\mathbf{2}}$, and $\alpha_{\mathbf{3}}$ are linearly independent and their column space is $\mathbb{R}^3$ which means vectors $\alpha_{\mathbf{1}}$, $\alpha_{\mathbf{2}}$, and $\alpha_{\mathbf{3}}$ span $\mathbb{R}^3$. Hence, vectors $\alpha_{\mathbf{1}}$, $\alpha_{\mathbf{2}}$, and $\alpha_{\mathbf{3}}$ form a basis for $\mathbb{R}^3$.

Now,use theorem (2.1), and calculate the inverse of (3.0.2) then the columns of $\mathbf{A^{-1}}$ will give the coefficients to write the standard basis vectors in terms of $\alpha_i's$. We try to find the inverse of $\mathbf{A}$ by row-reducing the augumented matrix.**A|I**

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ -1 & 1 & 2 \end{pmatrix} \tag{3.0.10}$$

Now, by row reducing **A|I** as follows

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & -3 & 0 & 1 & 0 \\ -1 & 1 & 2 & 0 & 0 & 1 \end{pmatrix}$$

$$\xleftrightarrow{R_3=R_3+R_1} \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & -3 & 0 & 1 & 0 \\ 0 & 2 & 2 & 1 & 0 & 1 \end{pmatrix} \tag{3.0.11}$$

$$\xleftrightarrow{R_3=R_3-R_2} \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & -3 & 0 & 1 & 0 \\ 0 & 0 & 5 & 1 & -1 & 1 \end{pmatrix} \tag{3.0.12}$$

$$\xleftrightarrow{R_2=\frac{R_2}{2}} \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & \frac{-3}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 5 & 1 & -1 & 1 \end{pmatrix} \tag{3.0.13}$$

$$\xleftrightarrow{R_1=R_1-R_2} \begin{pmatrix} 1 & 0 & \frac{3}{2} & 1 & \frac{-1}{2} & 0 \\ 0 & 1 & \frac{-3}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 5 & 1 & -1 & 1 \end{pmatrix} \tag{3.0.14}$$

$$\xleftrightarrow{R_3=\frac{R_3}{5}} \begin{pmatrix} 1 & 0 & \frac{3}{2} & 1 & \frac{-1}{2} & 0 \\ 0 & 1 & \frac{-3}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{5} & \frac{-1}{5} & \frac{1}{5} \end{pmatrix} \tag{3.0.15}$$

$$\xleftrightarrow{R_1=R_1-\frac{3R_3}{2}} \begin{pmatrix} 1 & 0 & 0 & \frac{7}{10} & \frac{-1}{5} & \frac{-3}{10} \\ 0 & 1 & \frac{-3}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{5} & \frac{-1}{5} & \frac{1}{5} \end{pmatrix} \tag{3.0.16}$$

$$\xleftrightarrow{R_2=R_2+\frac{3R_3}{2}} \begin{pmatrix} 1 & 0 & 0 & \frac{7}{10} & \frac{-1}{5} & \frac{-3}{10} \\ 0 & 1 & 0 & \frac{3}{10} & \frac{1}{5} & \frac{3}{10} \\ 0 & 0 & 1 & \frac{1}{5} & \frac{-1}{5} & \frac{1}{5} \end{pmatrix} \tag{3.0.17}$$

Thus, by (3.0.17), we have

$$\mathbf{A}^{-1} = \begin{pmatrix} \frac{7}{10} & \frac{-1}{5} & \frac{-3}{10} \\ \frac{3}{10} & \frac{1}{5} & \frac{3}{10} \\ \frac{1}{5} & \frac{-1}{5} & \frac{1}{5} \end{pmatrix} \tag{3.0.18}$$

Now, let $\mathbf{e_1} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$, $\mathbf{e_2} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$, and $\mathbf{e_3} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$ be the standard basis for $\mathbb{R}^3$. Hence,each of the standard basis vectors as linear combinations of $\alpha_1, \alpha_2, \alpha_3$ is as under

$$\mathbf{e_1} = \frac{7}{10}\alpha_1 + \frac{3}{10}\alpha_2 + \frac{1}{5}\alpha_3 \tag{3.0.19}$$

$$\mathbf{e_2} = -\frac{1}{5}\alpha_1 + \frac{1}{5}\alpha_2 - \frac{1}{5}\alpha_3 \tag{3.0.20}$$

$$\mathbf{e_3} = \frac{-3}{10}\alpha_1 + \frac{3}{10}\alpha_2 + \frac{1}{5}\alpha_3 \tag{3.0.21}$$