

EE5609: Matrix Theory

Assignment-9

Major Saurabh Joshi
MTech Artificial Intelligence
AI20MTECH13002

Abstract—This document explains how to find the standard basis vector.

Download the latex code from

<https://github.com/saurabh13002/EE5609/tree/master/Assignment9>

1 PROBLEM

Show that the vectors

$$\alpha_1 = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \quad (1.0.1)$$

$$\alpha_3 = \begin{pmatrix} 0 & -3 & 2 \end{pmatrix} \quad (1.0.2)$$

form a basis for \mathbb{R}^3 . Express each of the standard basis vectors as linear combinations of $(\alpha_1 \ \alpha_2 \ \alpha_3)$

2 THEOREM

Theorem 2.1. Let \mathbf{V} be an n -dimensional vector space over the field \mathbf{F} , and let β and β' be two ordered basis of \mathbf{V} . Then, there is a unique, necessarily invertible, $n \times n$ matrix \mathbf{P} with entries in \mathbf{F} such that

$$1) \begin{bmatrix} \alpha \end{bmatrix}_{\beta} = \mathbf{P} \begin{bmatrix} \alpha \end{bmatrix}_{\beta'}$$

$$2) \begin{bmatrix} \alpha \end{bmatrix}_{\beta'} = \mathbf{P}^{-1} \begin{bmatrix} \alpha \end{bmatrix}_{\beta}$$

for every vector α in \mathbf{V} . The columns of \mathbf{P} are given by

$$\mathbf{P}_j = \begin{bmatrix} \alpha_j \end{bmatrix}_{\beta} \quad j = 1, 2, \dots, n \quad (2.0.1)$$

3 SOLUTION

In order to show that the set of vectors α_1, α_2 , and α_3 are basis for \mathbb{R}^3 . We first show that α_1, α_2 , and α_3 are linearly independent in \mathbb{R}^3 and also they span \mathbb{R}^3 . Consider,

$$\mathbf{A} = (\alpha_1^T \ \alpha_2^T \ \alpha_3^T) \quad (3.0.1)$$

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ -1 & 1 & 2 \end{pmatrix} \quad (3.0.2)$$

Now, by row reduction

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ -1 & 1 & 2 \end{pmatrix} \xleftrightarrow{R_3=R_3+R_1} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ 0 & 2 & 2 \end{pmatrix} \quad (3.0.3)$$

$$\xleftrightarrow{R_3=R_3-R_2} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ 0 & 0 & 5 \end{pmatrix} \quad (3.0.4)$$

$$\xleftrightarrow{R_2=\frac{R_2}{2}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 5 \end{pmatrix} \quad (3.0.5)$$

$$\xleftrightarrow{R_1=R_1-R_2} \begin{pmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 5 \end{pmatrix} \quad (3.0.6)$$

$$\xleftrightarrow{R_3=\frac{R_3}{5}} \begin{pmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 1 \end{pmatrix} \quad (3.0.7)$$

$$\xleftrightarrow{R_1=R_1-\frac{3}{2}R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 1 \end{pmatrix} \quad (3.0.8)$$

$$\xleftrightarrow{R_2=R_2+\frac{3}{2}R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.0.9)$$

(3.0.9) is the row reduced echelon form of \mathbf{A} and since it is identity matrix of order 3, we say that vectors α_1, α_2 , and α_3 are linearly independent and their column space is \mathbb{R}^3 which means vectors α_1, α_2 , and α_3 span \mathbb{R}^3 . Hence, vectors α_1, α_2 , and α_3 form a basis for \mathbb{R}^3 .

Now, use theorem (2.1), and calculate the inverse of (3.0.2) then the columns of \mathbf{A}^{-1} will give the coefficients to write the standard basis vectors in terms of α_i 's. We try to find the inverse of \mathbf{A} by

row-reducing the augmented matrix $\mathbf{A}|\mathbf{I}$

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ -1 & 1 & 2 \end{pmatrix} \quad (3.0.10)$$

Now, by row reducing $\mathbf{A}|\mathbf{I}$ as follows

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & -3 & 0 & 1 & 0 \\ -1 & 1 & 2 & 0 & 0 & 1 \end{pmatrix} \xleftrightarrow{R_3=R_3+R_1} \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & -3 & 0 & 1 & 0 \\ 0 & 2 & 2 & 1 & 0 & 1 \end{pmatrix} \quad (3.0.11)$$

$$\xleftrightarrow{R_3=R_3-R_2} \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & -3 & 0 & 1 & 0 \\ 0 & 0 & 5 & 1 & -1 & 1 \end{pmatrix} \quad (3.0.12)$$

$$\xleftrightarrow{R_2=\frac{R_2}{2}} \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -\frac{3}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 5 & 1 & -1 & 1 \end{pmatrix} \quad (3.0.13)$$

$$\xleftrightarrow{R_1=R_1-R_2} \begin{pmatrix} 1 & 0 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{3}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 5 & 1 & -1 & 1 \end{pmatrix} \quad (3.0.14)$$

$$\xleftrightarrow{R_3=\frac{R_3}{5}} \begin{pmatrix} 1 & 0 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{3}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} \end{pmatrix} \quad (3.0.15)$$

$$\xleftrightarrow{R_1=R_1-\frac{3R_3}{2}} \begin{pmatrix} 1 & 0 & 0 & \frac{7}{10} & \frac{-1}{5} & \frac{-3}{10} \\ 0 & 1 & -\frac{3}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} \end{pmatrix} \quad (3.0.16)$$

$$\xleftrightarrow{R_2=R_2+\frac{3R_3}{2}} \begin{pmatrix} 1 & 0 & 0 & \frac{7}{10} & \frac{-1}{5} & \frac{-3}{10} \\ 0 & 1 & 0 & \frac{3}{10} & \frac{1}{5} & \frac{3}{10} \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} \end{pmatrix} \quad (3.0.17)$$

Thus, by (3.0.17), we have

$$\mathbf{A}^{-1} = \begin{pmatrix} \frac{7}{10} & \frac{-1}{5} & \frac{-3}{10} \\ \frac{3}{10} & \frac{1}{5} & \frac{3}{10} \\ \frac{1}{5} & \frac{-1}{5} & \frac{1}{5} \end{pmatrix} \quad (3.0.18)$$

Now, let $\mathbf{e}_1 = (1 \ 0 \ 0)$, $\mathbf{e}_2 = (0 \ 1 \ 0)$, and $\mathbf{e}_3 = (0 \ 0 \ 1)$ be the standard basis for \mathbb{R}^3 . Hence, each of the standard basis vectors as linear combinations of $\alpha_1, \alpha_2, \alpha_3$ is as under

$$\mathbf{e}_1 = \frac{7}{10}\alpha_1 + \frac{3}{10}\alpha_2 + \frac{1}{5}\alpha_3 \quad (3.0.19)$$

$$\mathbf{e}_2 = -\frac{1}{5}\alpha_1 + \frac{1}{5}\alpha_2 - \frac{1}{5}\alpha_3 \quad (3.0.20)$$

$$\mathbf{e}_3 = \frac{-3}{10}\alpha_1 + \frac{3}{10}\alpha_2 + \frac{1}{5}\alpha_3 \quad (3.0.21)$$