EE5609: Matrix Theory Assignment-10

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***Abstract*—This document solves problem on linear independence .**

Download the latex code from

https://github.com/saurabh13002/EE5609/tree/master/Assignment10

1 Problem

Let $\mathbf{V}$ be the vector space over the complex numbers of all functions from $\mathbb{R}$ into $\mathbb{C}$, i.e., the space of all complex-valued functions on the real line. Let $f_1(x) = 1,\ f_2(x) = e^{ix},\ f_3(x) = e^{-ix}$.

(a) Prove that f_1, f_2, and f_3 are linearly independent.
(b) Let $g_1(x) = 1,\ g_2(x) = \cos x,\ g_3(x) = \sin x$. Find an invertible 3×3 matrix $\mathbf{P}$ such that

$$g_j = \sum_{i=1}^{3} \mathbf{P}_{ij} f_i \tag{1.0.1}$$

2 Solution

2.1 a

Given,

$$f_1(x) = 1 \tag{2.1.1}$$
$$f_2(x) = e^{ix} \tag{2.1.2}$$
$$f_3(x) = e^{-ix} \tag{2.1.3}$$

For f_1, f_2, and f_3 to be linearly independent, the following condition must satisfy.

$$\alpha_1 + \alpha_2 f_2 + \alpha_3 f_3 = 0 \tag{2.1.4}$$

$\forall \alpha_i = 0$ and i =1,2,3 Substitute (2.1.1),(2.1.2), (2.1.3) in (2.1.4), we get

$$\alpha_1 + \alpha_2 e^{ix} + \alpha_3 e^{-ix} = 0 \tag{2.1.5}$$

By Eulers Formula in (2.1.5) i.e

$$e^{ix} = \cos x + i \sin x \tag{2.1.6}$$
$$\alpha_1 + \alpha_2 \cos x + i\alpha_2 \sin x + \alpha_3 \cos x - i\alpha_3 \sin x = 0 \tag{2.1.7}$$

equating real and imaginary parts

$$\alpha_1 + \alpha_2 \cos x + \alpha_3 \cos x = 0 \tag{2.1.8}$$
$$\alpha_2 \sin x - \alpha_3 \sin x = 0 \tag{2.1.9}$$

Therefore from (2.1.9)

$$\alpha_2 = \alpha_3 \tag{2.1.10}$$

substituting (2.1.10) in (2.1.8)

$$\alpha_1 + \alpha_2 \cos x + \alpha_3 \cos x = 0 \tag{2.1.11}$$
$$\implies \alpha_1 + 2\alpha_3 \cos x = 0 \tag{2.1.12}$$

differentiating (2.1.12) wrt x

$$-2\alpha_3 \sin x = 0 \tag{2.1.13}$$

Therefore

$$\alpha_3 = 0 \tag{2.1.14}$$

and from (2.1.10)

$$\alpha_3 = \alpha_2 = 0 \tag{2.1.15}$$

Substituting, (2.1.15) in (2.1.11)

$$\alpha_1 = 0 \tag{2.1.16}$$

Therefore $\alpha_1 = 0$. Thus, from (2.1.5)

$$\alpha_1 + \alpha_2 e^{ix} + \alpha_3 e^{-ix} = 0 \tag{2.1.17}$$
$$\implies \alpha_1 = \alpha_2 = \alpha_3 = 0 \tag{2.1.18}$$

and

$$\alpha_1 + \alpha_2 e^{ix} + \alpha_3 e^{-ix} = 0 \iff \alpha_1 = \alpha_2 = \alpha_3 = 0 \tag{2.1.19}$$

Hence, $f_1, f_2, and f_3$ are linearly independent

2.2 b

Given,

$$g_1(x) = 1 = f_1 \tag{2.2.1}$$

$$g_2(x) = \cos x = \frac{e^{ix} + e{-}ix}{2} = \frac{f_2}{2} + \frac{f_3}{2} \tag{2.2.2}$$

$$g_3(x) = \sin x = \frac{e^{ix} - e{-}ix}{2i} = \frac{f_2}{2i} - \frac{f_3}{2i} = -\frac{i}{2}f_2 + \frac{i}{2}f_3 \tag{2.2.3}$$

Now (2.2.1), (2.2.2), (2.2.3) can be converted to matrix form as below.

$$\begin{pmatrix} g_1 & g_2 & g_3 \end{pmatrix} = \begin{pmatrix} f_1 & f_2 & f_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{i}{2} \\ 0 & \frac{1}{2} & \frac{i}{2} \end{pmatrix} \tag{2.2.4}$$

Therefore, on comparing with (1.0.1) we get

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{i}{2} \\ 0 & \frac{1}{2} & \frac{i}{2} \end{pmatrix} \tag{2.2.5}$$

Now to verify invertibility of $\mathbf{P}$ we use row reduction.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{i}{2} \\ 0 & \frac{1}{2} & \frac{i}{2} \end{pmatrix} \xleftrightarrow{R_3 = R_3 - R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{i}{2} \\ 0 & 0 & i \end{pmatrix} \tag{2.2.6}$$

we got rank of matrix $\mathbf{P}$ = 3 and is a full rank matrix. Therefore, $\mathbf{P}$ is invertible. Hence verified.