

EE5609: Matrix Theory

Assignment-4

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Abstract—This document explains how to factorize a matrix using QR decomposition.

3 SOLUTION

Download the latex-tikz codes from

https://github.com/saurabh13002/EE5609/tree/master/Assignment4/QR_V

Let \mathbf{A} be the given matrix. Then $\mathbf{A} = \begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$ and columns of \mathbf{A} are \mathbf{a} and \mathbf{b} , where

1 PROBLEM

Perform QR decomposition of $\begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$

$$\mathbf{a} = \begin{pmatrix} 3 \\ -\sqrt{3} \end{pmatrix} \quad (3.0.1)$$

$$\mathbf{b} = \begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix} \quad (3.0.2)$$

2 EXPLANATION

Let \mathbf{a} and \mathbf{b} be columns of a \mathbf{A} . Then, the matrix \mathbf{A} can be decomposed in the form as:

$$\mathbf{A} = \mathbf{Q}\mathbf{R} \quad (2.0.1)$$

such that

$$\mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I} \quad (2.0.2)$$

$$\mathbf{Q} = (\mathbf{u}_1 \quad \mathbf{u}_2) \quad (2.0.3)$$

$$\mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \quad (2.0.4)$$

where

$$k_1 = \|\mathbf{a}\| \quad (2.0.5)$$

$$\mathbf{u}_1 = \frac{\mathbf{a}}{k_1} \quad (2.0.6)$$

$$r_1 = \frac{\mathbf{u}_1^T \mathbf{b}}{\|\mathbf{u}_1\|^2} \quad (2.0.7)$$

$$\mathbf{u}_2 = \frac{\mathbf{b} - r_1 \mathbf{u}_1}{\|\mathbf{b} - r_1 \mathbf{u}_1\|} \quad (2.0.8)$$

$$k_2 = \mathbf{u}_2^T \mathbf{b} \quad (2.0.9)$$

Then, the matrix can be represented as

$$(\mathbf{a} \quad \mathbf{b}) = (\mathbf{u}_1 \quad \mathbf{u}_2) \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \quad (2.0.10)$$

Now, for given matrix from (2.0.5) and (2.0.6), we have

$$k_1 = \|\mathbf{a}\| = \sqrt{12} = 2\sqrt{3} \quad (3.0.3)$$

$$\mathbf{u}_1 = \frac{1}{2\sqrt{3}} \begin{pmatrix} 3 \\ -\sqrt{3} \end{pmatrix} \quad (3.0.4)$$

By, (2.0.7), we find

$$r_1 = \frac{\frac{1}{2\sqrt{3}} (3 \quad -\sqrt{3}) \begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix}}{1} = -2 \quad (3.0.5)$$

Now, by (2.0.8)

$$\mathbf{u}_2 = \frac{\begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix} - \frac{-2}{2\sqrt{3}} \begin{pmatrix} 3 \\ -\sqrt{3} \end{pmatrix}}{\left\| \begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix} - \frac{-2}{2\sqrt{3}} \begin{pmatrix} 3 \\ -\sqrt{3} \end{pmatrix} \right\|} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.0.6)$$

Since the vector is zero, it is linearly dependent and we just skip it. From (2.0.9),

$$k_2 = (0 \quad 0) \begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix} = 0 \quad (3.0.7)$$

Now,

$$\mathbf{Q} = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{-1}{2} \end{pmatrix} \quad (3.0.8)$$

Now, we observe that $\mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I}$

$$\begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 \end{pmatrix} = \mathbf{I} \quad (3.0.9)$$

Now, by (2.0.10) we can write matrix \mathbf{A} as

$$\begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 2\sqrt{3} & -2 \end{pmatrix} \quad (3.0.10)$$

which is the required \mathbf{QR} decomposition of \mathbf{A} .