EE5609: Matrix Theory Assignment-5

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***Abstract*—This document finds the coordinates of foot of perpendicular using Singular Value Decomposition.**

Download the latex code from

https://github.com/saurabh13002/EE5609/tree/master/Assignment5_SVD

1 Problem

Perform SVD and find the foot of the perpendicular from any point outside the plane. $\begin{pmatrix}14 & -3 & -18\end{pmatrix}\mathbf{x}$ = -1

2 Solution

Let the point be $\begin{pmatrix}2\\3\\-5\end{pmatrix}$, First we find orthogonal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the given normal vector $\mathbf{n}$. Let, $\mathbf{m} = \begin{pmatrix}a\\b\\c\end{pmatrix}$, then

$$\mathbf{m^T n} = 0 \tag{2.0.1}$$

$$\implies \begin{pmatrix}a & b & c\end{pmatrix}\begin{pmatrix}14\\-3\\18\end{pmatrix} = 0 \tag{2.0.2}$$

$$\implies 14a - 3b + 18c = 0 \tag{2.0.3}$$

Putting a=1 and b=0 we get,

$$\mathbf{m_1} = \begin{pmatrix}1\\0\\\frac{-7}{9}\end{pmatrix} \tag{2.0.4}$$

Putting a=0 and b=1 we get,

$$\mathbf{m_2} = \begin{pmatrix}0\\1\\\frac{1}{6}\end{pmatrix} \tag{2.0.5}$$

Now we solve the equation,

$$\mathbf{Mx} = \mathbf{b} \tag{2.0.6}$$

Putting values in (2.0.6),

$$\begin{pmatrix}1 & 0\\0 & 1\\\frac{-7}{9} & \frac{1}{6}\end{pmatrix}\mathbf{x} = \begin{pmatrix}2\\3\\-5\end{pmatrix} \tag{2.0.7}$$

In order to solve (2.0.7), perform Singular Value Decomposition on $\mathbf{M}$ as follows,

$$\mathbf{M} = \mathbf{USV}^T \tag{2.0.8}$$

Where the columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{MM}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix}\frac{130}{81} & \frac{-7}{54}\\\frac{-7}{54} & \frac{37}{36}\end{pmatrix} \tag{2.0.9}$$

$$\mathbf{MM}^T = \begin{pmatrix}1 & 0 & \frac{-7}{9}\\0 & 1 & \frac{1}{6}\\\frac{-7}{9} & \frac{1}{6} & \frac{205}{324}\end{pmatrix} \tag{2.0.10}$$

From (2.0.6) putting (2.0.8) we get,

$$\mathbf{USV}^T\mathbf{x} = \mathbf{b} \tag{2.0.11}$$

$$\implies \mathbf{x} = \mathbf{VS_+U^Tb} \tag{2.0.12}$$

Where $\mathbf{S_+}$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$.Now, calculating eigen value of $\mathbf{MM}^T$,

$$\left|\mathbf{MM}^T - \lambda\mathbf{I}\right| = 0 \tag{2.0.13}$$

$$\implies \begin{pmatrix}1-\lambda & 0 & \frac{1}{2}\\0 & 1-\lambda & 1\\\frac{1}{2} & 1 & \frac{205}{324}-\lambda\end{pmatrix} = 0 \tag{2.0.14}$$

$$\implies -\lambda^3 + \frac{853}{324}\lambda^2 - \frac{529}{324}\lambda = 0 \tag{2.0.15}$$

Hence eigen values of $\mathbf{M}\mathbf{M}^T$ are,

$$\lambda_1 = \frac{529}{324} \tag{2.0.16}$$
$$\lambda_2 = 1 \tag{2.0.17}$$
$$\lambda_3 = 0 \tag{2.0.18}$$

Hence the eigen vectors of $\mathbf{M}\mathbf{M}^T$ are,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{-252}{205} \\ \frac{54}{205} \\ 1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} \frac{3}{4} \\ 1 \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} \frac{7}{9} \\ \frac{-1}{6} \\ 1 \end{pmatrix} \tag{2.0.19}$$

Normalizing the eigen vectors we get,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{-252}{23\sqrt{205}} \\ \frac{54}{23\sqrt{205}} \\ \frac{\sqrt{205}}{23} \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} \frac{3}{\sqrt{205}} \\ \frac{14}{\sqrt{205}} \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} \frac{14}{23} \\ \frac{-3}{23} \\ \frac{18}{23} \end{pmatrix} \tag{2.0.20}$$

Hence we obtain $\mathbf{U}$ of (2.0.8) as follows, $\mathbf{U}$

$$\begin{pmatrix} \frac{-252}{23\sqrt{205}} & \frac{3}{\sqrt{205}} & \frac{14}{23} \\ \frac{54}{23\sqrt{205}} & \frac{14}{\sqrt{205}} & -\frac{3}{23} \\ \frac{\sqrt{205}}{23} & 0 & \frac{18}{23} \end{pmatrix} \tag{2.0.21}$$

After computing the singular values from eigen values $\lambda_1, \lambda_2, \lambda_3$ we get $\mathbf{S}$ of (2.0.8) as follows,

$$\mathbf{S} = \begin{pmatrix} \frac{23}{18} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.0.22}$$

Now, calculating eigen value of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \tag{2.0.23}$$
$$\implies \begin{pmatrix} \frac{130}{81} - \lambda & \frac{-7}{54} \\ \frac{-7}{54} & \frac{37}{36} - \lambda \end{pmatrix} = 0 \tag{2.0.24}$$
$$\implies \lambda^2 - \frac{853}{324}\lambda + \frac{529}{324} = 0 \tag{2.0.25}$$

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_4 = \frac{529}{324} \tag{2.0.26}$$
$$\lambda_5 = 1 \tag{2.0.27}$$

Hence the eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{-14}{3} \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} \frac{3}{14} \\ 1 \end{pmatrix} \tag{2.0.28}$$

Normalizing the eigen vectors we get,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{-14}{\sqrt{205}} \\ \frac{3}{\sqrt{205}} \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} \frac{3}{\sqrt{205}} \\ \frac{14}{\sqrt{205}} \end{pmatrix} \tag{2.0.29}$$

Hence we obtain $\mathbf{V}$ of (2.0.8) as follows,

$$\mathbf{V} = \begin{pmatrix} \frac{-14}{\sqrt{205}} & \frac{3}{\sqrt{205}} \\ \frac{3}{\sqrt{205}} & \frac{14}{\sqrt{205}} \end{pmatrix} \tag{2.0.30}$$

From (2.0.8) we get the Singular Value Decomposition of $\mathbf{M}$,

$$\mathbf{M} = \begin{pmatrix} \frac{-252}{23\sqrt{205}} & \frac{3}{\sqrt{205}} & \frac{14}{23} \\ \frac{54}{23\sqrt{205}} & \frac{14}{\sqrt{205}} & -\frac{3}{23} \\ \frac{\sqrt{205}}{23} & 0 & \frac{18}{23} \end{pmatrix} \begin{pmatrix} \frac{23}{18} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{-14}{\sqrt{205}} & \frac{3}{\sqrt{205}} \\ \frac{3}{\sqrt{205}} & \frac{14}{\sqrt{205}} \end{pmatrix}^T \tag{2.0.31}$$

Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{18}{23} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.0.32}$$

From (2.0.11) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{1367}{23\sqrt{205}} \\ \frac{48}{\sqrt{205}} \\ -\frac{71}{23} \end{pmatrix} \tag{2.0.33}$$
$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{24606}{529\sqrt{205}} \\ \frac{48}{\sqrt{205}} \end{pmatrix} \tag{2.0.34}$$
$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{2052}{529} \\ \frac{1374}{529} \end{pmatrix} \tag{2.0.35}$$

Verifying the solution of (2.0.35) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{2.0.36}$$

Evaluating the R.H.S in (2.0.36) we get,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} \frac{53}{9} \\ \frac{13}{6} \end{pmatrix} \tag{2.0.37}$$
$$\implies \begin{pmatrix} \frac{130}{81} & \frac{-7}{54} \\ \frac{-7}{54} & \frac{37}{36} \end{pmatrix}\mathbf{x} = \begin{pmatrix} \frac{53}{9} \\ \frac{13}{6} \end{pmatrix} \tag{2.0.38}$$

Solving the augmented matrix of (2.0.38) we get,

$$\begin{pmatrix} \frac{130}{81} & \frac{-7}{54} & \frac{53}{9} \\ \frac{-7}{54} & \frac{37}{36} & \frac{13}{6} \end{pmatrix} \xleftrightarrow{R_1 = \frac{81}{130}R_1} \begin{pmatrix} 1 & \frac{-21}{260} & \frac{477}{130} \\ \frac{-7}{54} & \frac{37}{36} & \frac{13}{6} \end{pmatrix} \tag{2.0.39}$$
$$\xleftrightarrow{R_2 = R_2 + \frac{7}{54}R_1} \begin{pmatrix} 1 & \frac{-21}{260} & \frac{477}{130} \\ 0 & \frac{529}{520} & \frac{687}{260} \end{pmatrix} \tag{2.0.40}$$
$$\xleftrightarrow{R_2 = \frac{520}{529}R_2} \begin{pmatrix} 1 & \frac{-21}{260} & \frac{477}{130} \\ 0 & 1 & \frac{1374}{529} \end{pmatrix} \tag{2.0.41}$$
$$\xleftrightarrow{R_1 = R_1 + \frac{21}{260}R_2} \begin{pmatrix} 1 & 0 & \frac{2052}{529} \\ 0 & 1 & \frac{1374}{529} \end{pmatrix} \tag{2.0.42}$$

From equation (2.0.42), solution is given by,

$$\mathbf{x} = \begin{pmatrix} \frac{2052}{529} \\ \frac{1374}{529} \end{pmatrix} \tag{2.0.43}$$

Comparing results of $\mathbf{x}$ from (2.0.35) and (2.0.43), we can say that the solution is verified.