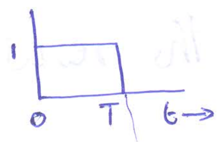
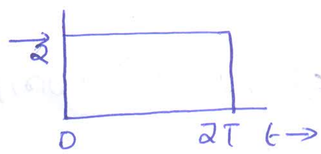
TUTORIAL-II — Solutions

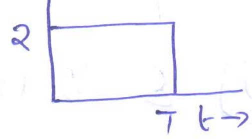
1 (a) $x_1(t) \longrightarrow$



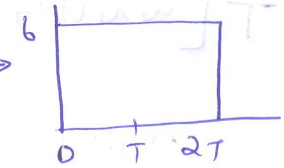
$x_2(t) \longrightarrow$



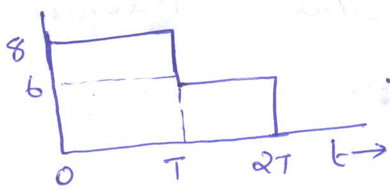
$\alpha x_1(t) \longrightarrow$
$[2x_1(t)]$



$\beta x_2(t)$
$3x_2(t) \longrightarrow$



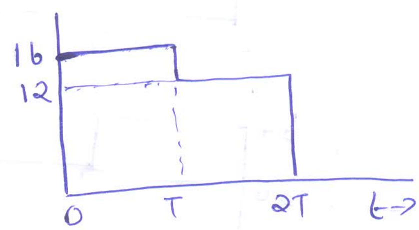
$\alpha x_1(t) + \beta x_2(t) \longrightarrow$



$\longrightarrow$ Let it be $x_3(t)$

The system given is $y(t) = 2x(t)$. Apply $x_3(t)$ as input the system. Then the output $y_3(t) = 2x_3(t)$

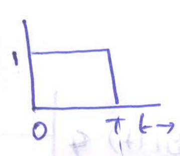
$y_3(t) = 2x_3(t) \longrightarrow$



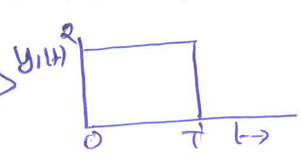
This is the response of the system to the input $\alpha x_1(t) + \beta x_2(t)$.

(b) $y(t) = 2x(t)$ is the system

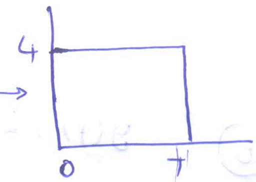
$x_1(t) \rightarrow$



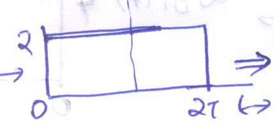
$\Rightarrow$



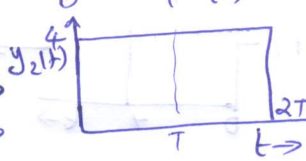
$\Rightarrow \alpha y_1(t) \rightarrow$



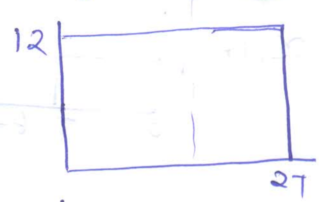
$x_2(t) \longrightarrow$



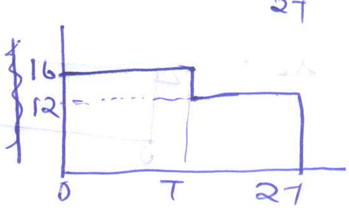
$\Rightarrow$



$\Rightarrow \beta y_2(t)$



$\Rightarrow \alpha y_1(t) + \beta y_2(t) \longrightarrow$



This is the response $\alpha y_1(t) + \beta y_2(t)$

(c) Now compare both the resultant wave forms

Then,

$$T[\alpha x_1(t) + \beta x_2(t)] = \alpha [y_1(t)] + \beta\, y_2(t)$$

where $y_1(t) = T[x_1(t)]$ & $y_2(t) = T[x_2(t)]$

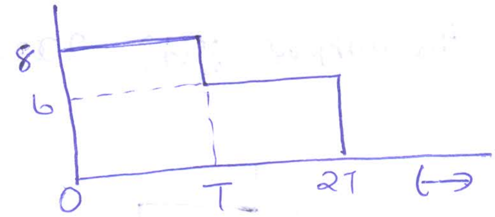
So the super position theorem is verified. And the system is linear.

(2) The system is $y(t) = 2x^2(t)$

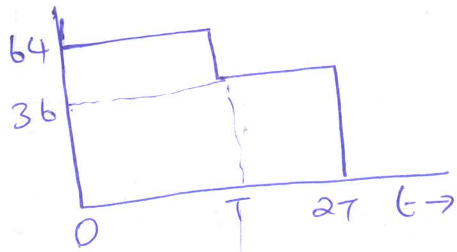
(a)

$\alpha x_1(t) + \beta x_2(t) \longrightarrow$

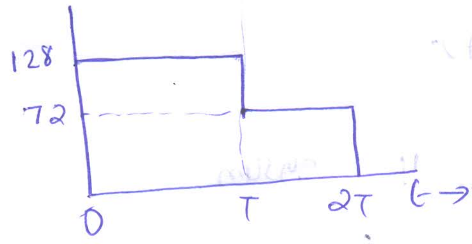
$[x_3(t)]$



$x_3^2(t) \longrightarrow$

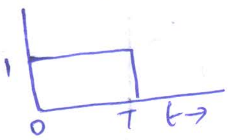


$y_3(t) = 2x_3^2(t) \longrightarrow$

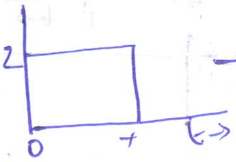


(b) $y(t) = 2x^2(t)$

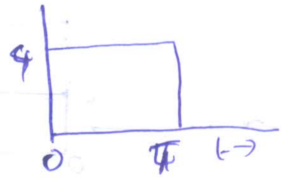
$x_1(t)$



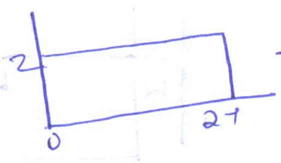
$\longrightarrow$ $y_1(t)$



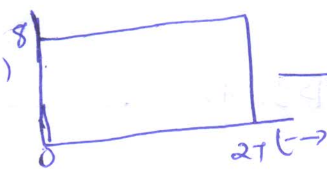
$\longrightarrow$ $\alpha y_1(t)$



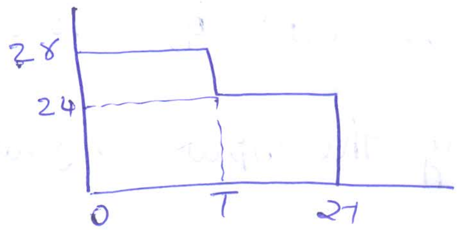
$x_2(t)$



$\longrightarrow$ $y_2(t)$



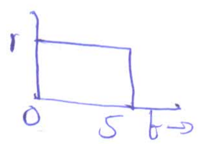
$\longrightarrow$ $\beta y_2(t)$

$\alpha y_1(t) + \beta y_2(t) \longrightarrow$ 

(c) On comparing both the waveforms we can conclude that they are not same. So the superposition theorem

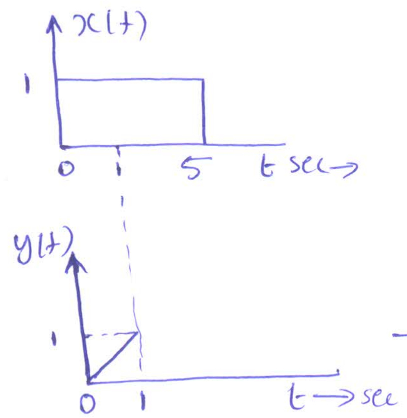
$T[\alpha x_1(t) + \beta x_2(t)] \neq \alpha T[x_1(t)] + \beta T[x_2(t)]$ is not valid.

So the system is not linear.

2) The system given is $y(t) = t\,x(t)$ where $x(t) =$ 

a) To plot the response from $t = 0$ sec to $t = 1$ sec.. multiply each time instant (t) with corresponding amplitude value (1)

So

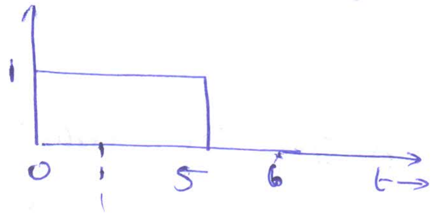


$\rightarrow$ response plotted from $t = 0$ sec to $t = 1$ sec

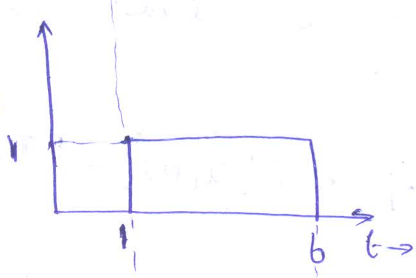
Delay the input $x(t)$ by 1sec

→ So shifting the input signal to right side by 1sec

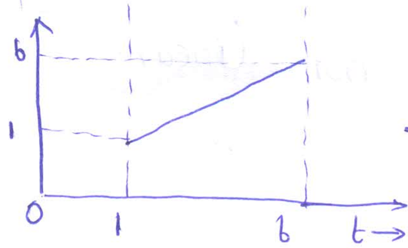
$x'(t)$ →



$x'(t) = x(t-1)$ →



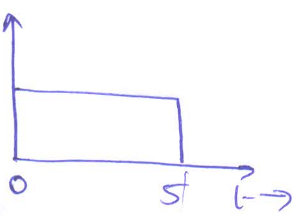
$y'(t)$



→ Response for delayed input

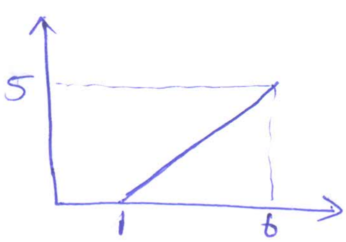
c) To verify ~~the~~ whether it is Time variant or Time invariant, plot the graph of the "response itself is getting delayed" by 1 sec".

$x(t)$



$y(t)$

$y(t-1)$



→ The response $y(t)$ itself is getting delayed by 1 sec.

On comparing both the graphs we can conclude that "the response due to the delayed input" is not the same as "the response itself is getting delayed". So the system is a Time variant system.

1) $y(t) = \int_{-\infty}^{t+3} x(t)\, dt$

$y(t) = \int_{-\infty}^{0} x(t)\, dt + \int_{0}^{t+3} x(t)\, dt.$

$\Downarrow$

The output at time t ie $y(t)$ requires an input at $(t+3)$ which is future input. So the system is not causal

2) $y(t) = x(-t)$

Take some sample time values around origin

$y(0) = x(0) \longrightarrow$ output at $t=0$ depends on input at $t=0$

$y(1) = x(-1) \longrightarrow$ " $t=1$ " " $t=-1$

$y(-2) = x(2) \longrightarrow$ " $t=-2$ " " $t=+2$

In the last case output at time $t=-2$ sec requires an input at time $t=2$ which is a future value of input. So here it anticipates a future input to produce output.

so the system is anticipatory and it is not causal.

3) $y(t) = (2t+3)\,x(t)$

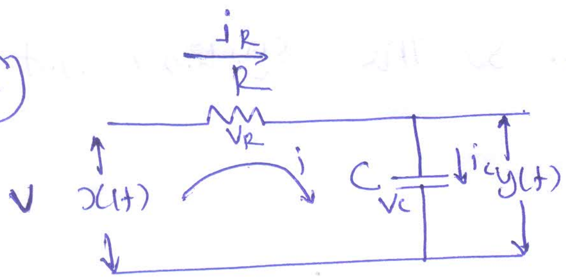
→ output at any time 't' depends on the input at the same time. It's not depending on any future values. So the system is causal.

4) $y(t) = |x(t)|$

→ Same reason as in 3^{rd} case

→ system is causal.

(4)



Let V_R and V_C be the voltages across R and C.

By using KVL.

$V = V_R + V_C.$ where

$V_R = i_R R$

$V_C = \frac{1}{C}\int i_c\,dt \Rightarrow i_c = C\frac{dV_C}{dt}$

But R and C are in series. So,

$i_R = i_c = i$

So

$$V = V_R + V_C$$

$$= i_R R + V_C$$

$$V = RC\frac{dV_C}{dt} + V_C$$

The input and ~~out~~ response given in the figure are $x(t)$ and $y(t)$ respectively. where

$y(t) \rightarrow$ voltage across C

$x(t) \rightarrow$ voltage given as input.

On substituting in this equation

$$x(t) = RC\frac{dy(t)}{dt} + y(t) \Rightarrow \tau\frac{dy(t)}{dt} = x(t) - y(t)$$

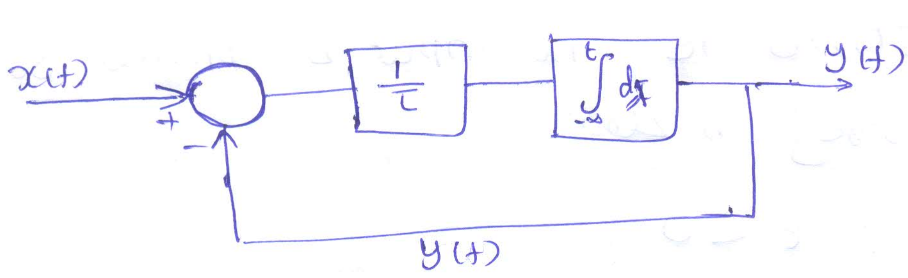
$$\frac{dy(t)}{dt} = \frac{1}{RC}\left[x(t) - y(t)\right]$$ where $RC = \tau$ is the time constant of RC circuit.

$$\frac{dy(t)}{dt} = \frac{1}{\tau}\left[x(t) - y(t)\right]$$

On integrating both sides from $-\infty$ to t

$$y(t) = \frac{1}{\tau}\int_{-\infty}^{t}\left[x(t) - y(t)\right]$$

This can be modeled by the following block diagram



In the above differential equation $x(t)$ is the input and $y(t)$ is the output.

Since we are interested in impulse response replace ~~$\delta(t)$ to~~ $x(t) = \delta(t)$ and

$y(t) = h(t)$

Now the differential equation will become

$$\tau \frac{d}{dt} h(t) + h(t) = \delta(t)$$

This is a linear differential equation. So the ~~stm~~ system characterized by this differential equation is a linear system.

For a linear system the output is zero as long as the input is zero. So the system is causal also.

$\delta(t)$ is existing only at $t=0$.

~~For time $t<0$; $\delta(t)=0$ ie no input is applied.~~

$\delta(t) = 0$ for $t \neq 0$

So consider each time ranges one by one

a) For $t<0$

For $t<0$; $\delta(t)=0$ ie no input is applied. So $h(t)$ [the response] is zero.

$h(t) = 0 \quad \because t<0$

b) For $t=0$

The input $\delta(t)$ is applied exactly at $t=0$ sec.

Here we consider an infinitesimally small time after $t=0$

ie $t=0^+$

The RC circuit we are using is a real system. Every real time system will be having ~~an~~ inertia (a property of the system) associated with that system.

So ~~within~~ Due to this inertial property the system will not be able to respond at the same time when the input like impulse (acting for small duration almost zero) is applied. ~~This~~ In this time interval the system will not have any response but still the input will be present.

Now the time $t=0^+$ is selected such that the time interval from $t=0$ to $t=0^+$ is less than this dead time (the time interval for which the system will not produce any response even if input is present).

So for $t=0^+$ sec

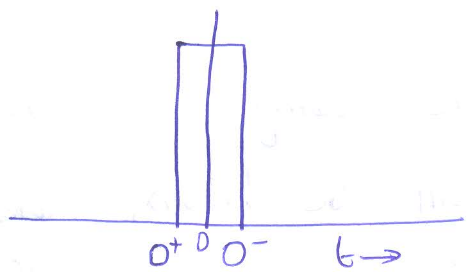
$$h(t) = \frac{1}{\tau}\int_{-\infty}^{0^+} [\delta(t) - \underbrace{h(t)}_{\text{This will be zero}}]\, dt = \frac{1}{\tau}\int_{-\infty}^{0^+} \delta(t)\, dt + 0$$

$$= \frac{1}{\tau}$$

So $h(0^+) = \frac{1}{\tau}$

Now to find the response at $t=0$

For this consider the approximation for the impulse as



we got $h(0^+) = \frac{1}{\tau}$

Now we are considering up to $t=0$, up to this time ie ~~up to 0~~ only half of the area of impulse is covered.

The system will respond to the signal energy (signal strength) which is the area in the case of impulse. For an area of 1 we got response $1/\tau$. So for ½ the area we will get ½ of $1/\tau$ ie because it is an LTI s/m.

$$h(0) = 1/\tau \times 1/2 = \frac{1}{2\tau}$$

c) For $t>0$

For time $t>0$, the ~~respon~~ input $\delta(t)=0$.

But the differential equation should be valid for $t>0$ also.

So

$$\tau \frac{d\,h(t)}{dt} + h(t) = 0$$

Now find the solution of the differential equation i.e $h(t)$

The general solution is of the form

$$h(t) = Ke^{st}$$

substitute in the equation

$$\tau \times K \times e^{st} \times s + Ke^{st} = 0$$

$$Ke^{st}(\tau s + 1) = 0$$

Here we consider $t > 0$. So to find the K and s we can use $h(0^+)$.

Note: → Here we cannot use $h(0)$ because we are considering $t > 0$. So we can't use $h(0)$ to find K or s.

$h(0^+) = \frac{1}{\tau}$ at $t = 0^+ \simeq 0$. substitute in $h(t) = Ke^{st}$

~~K (1 + τ~~

$$\frac{1}{\tau} = K \times e^0 \Rightarrow K = 1/\tau$$

$$\frac{1}{\tau} \times e^{st}(1 + \tau s) = 0$$

$$e^{st} \neq 0 \Rightarrow 1 + \tau s = 0$$

$$\Rightarrow s = -1/\tau$$

So $h(t) = \frac{1}{\tau} e^{-st/\tau}$ for $t > 0$

So

$$h(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{2\tau} & t = 0 \\ \frac{1}{\tau} & t > 0 \end{cases}$$

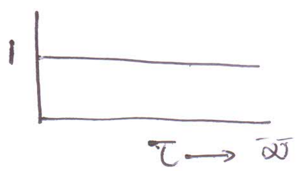
OR - we can use variable seperable method to find Ke^{st}.

5 a)

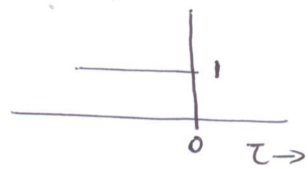
$$x(t) = u(t)$$
$$h(t) = u(t)$$
$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = x(t) * h(t)$$

$$y(t) = \int_{-\infty}^{\infty} u(\tau)\, u(t-\tau)\, d\tau$$

$u(\tau) \longrightarrow$

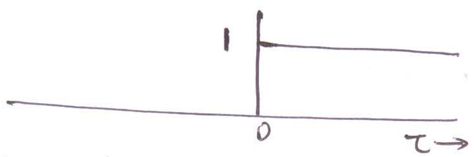


$u(-\tau) =$

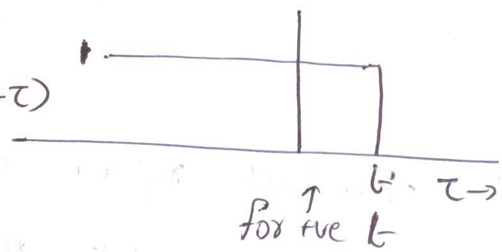


$u(t-\tau)$ is obtained by right shifting $u(-\tau)$ by t units.

Now $u(\tau)$ is



$u(t-\tau)$



~~For t < 0~~

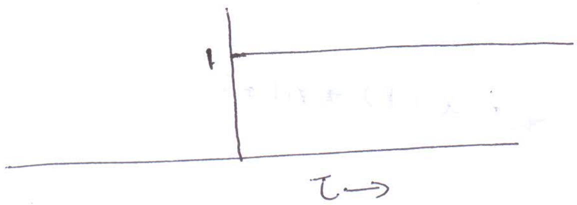
For $t < 0$, $u(t-\tau)$ is

For t from $-\infty$ to 0, there is no overlap of $u(\tau)$ and $(t-\tau)$. So the product $[u(\tau)\, u(t-\tau) = 0]$ is zero. So

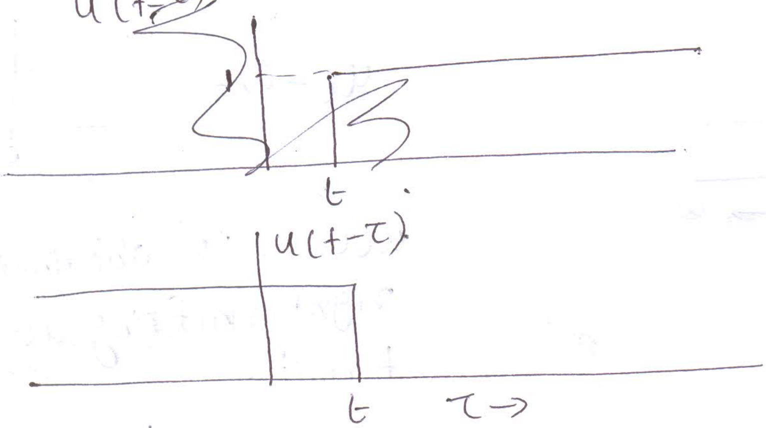
$y(t) = 0$ for $-\infty < t < 0$

For $0 < t < \infty$

$u(\tau)$

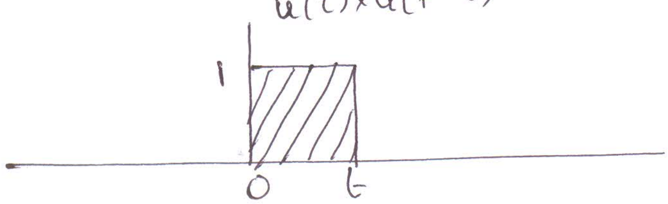


$u(t-\tau)$



product region is now as shown

$u(\tau) \times u(t-\tau)$



$$y(t) = \int_0^t 1 \, d\tau = \underline{t} \quad \text{for } 0 < t < \infty$$

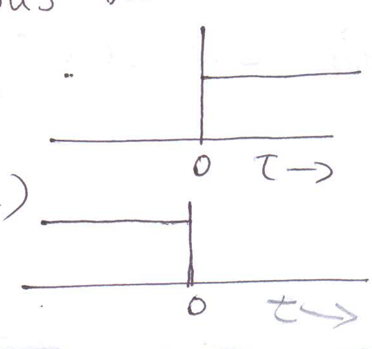
So this is a ramp function

Now the issue comes for $t = 0$

at $t = 0$

$u(\tau)$

$u(t-\tau)$



we are defining the unit step function as 15

$$u(t) = 0 \quad t<0$$
$$= 1/2 \quad t=0$$
$$= 1 \quad t>0$$

At $t=0$ both are over lapping so $1/2 \times 1/2 = 1/4$ will be the product. But the ~~range~~ limits of integration will be 0 and 0 (since we are considering at $\tau = '0'$ i.e the integration variable): so the result will be zero so $y(t) = 0$ for $t=0$

So

$$y(t) = 0 \quad t \leq 0$$
$$y(t) = t \quad t>0$$

→ Mathematical explanation for taking $\int_{0^-}^{0^+} y(t)dt = 0$

we have

$$\tau_c \frac{dh(t)}{dt} + h(t) = \delta(t)$$

or

$$\tau_c \frac{d}{dt}y(t) + y(t) = \cancel{x(t)}\ \delta(t) \quad : \text{where } y(t) = h(t)$$

Integrate both sides within the limit 0^- to 0^+

$$\tau_c \int_{0^-}^{0^+} \frac{d}{dt}y(t) + \underbrace{\int_{0^-}^{0^+} y(t)dt}_{\downarrow} = \int_{0^-}^{0^+} \delta(t)dt$$

$$\tau_c \left[y(0^+) - y(0^-) \right] + 0 = 1$$

Here we have taken $\int_{0^-}^{0^+} y(t)dt = 0$ ∴ The following ~~reason~~ is the explanation for this.

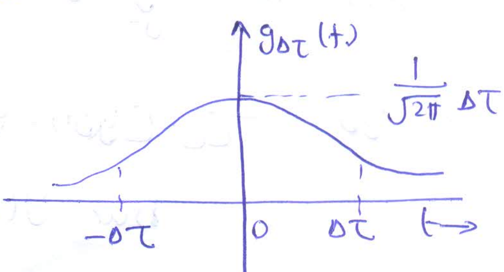
If $\int_{0^-}^{0^+} y(t)dt \neq 0$, then $y(t)$ has to be an impulse function.

Now we will find an approximation to this impulse. An impulse function can be written as the limit of Gaussian pulse. ie

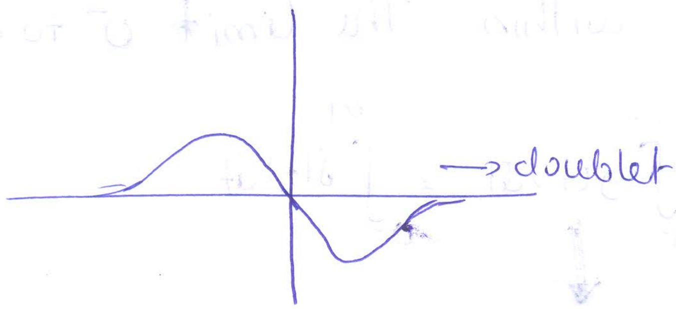
$$\delta(t) = \lim_{\Delta\tau \to 0} g_{\Delta\tau}(t)$$

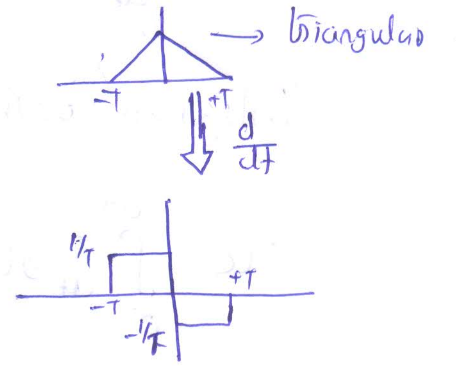
$$g_{\Delta\tau}(t) = \frac{1}{\sqrt{2\pi}\,\Delta\tau} e^{-\frac{1}{2}\frac{t^2}{(\Delta\tau)^2}}.$$

The gaussian pulse will look like



In this case ie if $y(t)$ was an impulse function, then $\frac{dy(t)}{dt}$ will be a doublet function. which will look like as follows:





So if $y(t)$ is an impulse function then $y'(t)$ should be indicating zero area.

As per the equation we must have

$$\tau(\overset{\text{doublet}}{\quad}) + y(t) = \delta(t)$$

But if we dont take $\int_{0^-}^{0^+} y(t)\,dt = 0$, then $y(t)$ will be an impulse fn and it means

$$\tau\left(\underset{+ve/-ve}{\overset{\text{doublet}}{\cancel{\text{something}}}}\right) + \underset{(+ve)}{\text{impulse}} = \underset{(+ve)}{\text{impulse}}.$$

So it will violate the equation. and its not been possible.

So the $\int_{0^-}^{0^+} y(t)\,dt = 0$ is to be taken.

So, $\tau_c\,[y(0^+) - y(0^-)] = 1$

but $y(0^-) = 0$ because no input applied for $t<0$

At $t=0^-$ there is no input so $y(0^-)=0$

$\Rightarrow$

$$\tau \times h(0^+) = 1$$

$$h(0^+) = 1/\tau$$