Q.1 TUT4 - ANSWERS

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}, \forall t$$

Area under $f(x)$

$$= \int_{-\infty}^{+\infty} f(x).dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}.dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \left(\int_{-\infty}^{+\infty} e^{-x^2/2\sigma^2}.dx . \int_{-\infty}^{+\infty} e^{-x^2/2\sigma^2}.dx \right)^{1/2}$$

replacing x by y

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \left(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-\frac{1}{2\sigma^2}(x^2+y^2)}.dxdy \right)^{1/2}$$

taking $x = r\cos\theta$, $y = r\sin\theta$ $\quad dxdy = \sqrt{x^2+y^2}\,drd\theta$

$dxdy = rdrd\theta$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \left(\int_{0}^{2\pi} \int_{0}^{\infty} e^{-\frac{r^2}{2\sigma^2}}.r.dr.d\theta \right)^{1/2}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \left(2\pi \int_{0}^{\infty} e^{-\frac{r^2}{2\sigma^2}}.r.drd\theta \right)^{1/2}$$

taking $r^2 = m \Rightarrow 2rdr = dm \Rightarrow rdr = \frac{dm}{2}$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \left(\cancel{2}\pi \int_{0}^{\infty} e^{-\frac{m}{2\sigma^2}}.\frac{dm}{\cancel{2}} \right)^{1/2}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \left(\pi . \left[\frac{e^{-m/2\sigma^2}}{-1/2\sigma^2} \right]_0^{\infty} \right)^{1/2}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \left(2\pi\sigma^2 (0+1) \right)^{1/2} = \frac{\sqrt{2\pi\sigma^2}}{\sqrt{2\pi\sigma^2}} = \underline{\underline{1}}$$

Q.2

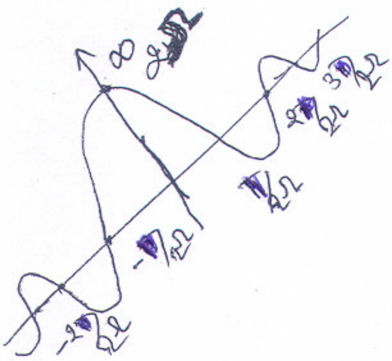
$$\int_{-\infty}^{+\infty} X(f).e^{j2\pi ft}.df$$

$$= \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} x(t').e^{-j2\pi ft'}.dt'\right).e^{j2\pi ft}.df$$

$$= \int_{-\infty}^{+\infty} x(t')\left(\underbrace{\int_{-\infty}^{+\infty} e^{+j(t-t').2\pi f}.df}\right).dt'$$

$$\Rightarrow \int_{-\infty}^{+\infty} e^{+j(t-t').2\pi f}.df = \lim_{\Omega\to\infty}\int_{-\Omega}^{+\Omega} e^{j(t-t').2\pi f}.df$$

$$= \lim_{\Omega\to\infty}\frac{1}{2\pi j(t-t')}\left(e^{j(t-t').2\pi f}\right]_{-\Omega}^{+\Omega}$$



$$= \lim_{\Omega\to\infty}\frac{\sin(2\pi\Omega(t-t'))(2\Omega)}{2\pi\Omega(t-t')}$$

$$= \lim_{\Omega\to\infty} \operatorname{sinc}(2\Omega(t-t')).(2\Omega)$$ ✓

$$= \delta(t-t')$$

Here

$$\int_{-\infty}^{+\infty} X(f).e^{j2\pi ft}.df = \int_{-\infty}^{+\infty} x(t').\delta(t-t').dt'$$

$$= \underline{\underline{x(t)}}$$

Same for Transform pair $H(f) \leftrightarrow h(t)$

a. Given

$x(t) = A\sin(2\pi f_1 t + \phi_1)$

Let the impulse response of the LTI system is $h(t)$ which is real

$\Rightarrow H^*(f) = H(-f)$

Let $\phi(t) = 2\pi f_1 t + \phi_1$

The output of the system is given by

$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} h(\gamma)\, x(t-\gamma)\, d\gamma$$

$$= \int_{-\infty}^{\infty} h(\gamma)\, A\sin(2\pi f_1(t-\gamma) + \phi_1)\, d\gamma$$

$$= \frac{A}{2j}\int_{-\infty}^{\infty} h(\gamma)\left[e^{j(\phi(t) - 2\pi f_1\gamma)} - e^{-j(\phi(t) - 2\pi f_1\gamma)}\right] d\gamma$$

$$= \frac{A}{2j}\int_{-\infty}^{\infty} h(\gamma)\left[e^{j\phi(t)}\cdot e^{-j2\pi f_1\gamma} - e^{-j\phi(t)}\cdot e^{j2\pi f_1\gamma}\right] d\gamma$$

$$= \frac{A}{2j}\left[e^{j\phi(t)}\int_{-\infty}^{\infty} h(\gamma)\cdot e^{-j2\pi f_1\gamma}\, d\gamma - e^{-j\phi(t)}\int_{-\infty}^{\infty} h(\gamma)\, e^{j2\pi f_1\gamma}\, d\gamma\right]$$

$$= \frac{A}{2j}\left[e^{j\phi(t)} H(f_1) - e^{-j\phi(t)} H(-f_1)\right]$$

$$= \frac{A}{2j}\left[e^{j\phi(t)} H(f_1) - e^{-j\phi(t)} H^*(f_1)\right]$$

As $h(t)$ is real $\Rightarrow H^*(f) = H(-f)$

$$= \frac{A}{2j}\left[e^{j\phi(t)} |H(f_1)| e^{j\theta(f_1)} - e^{-j\phi(t)} |H(f_1)| e^{+j\theta(-f_1)}\right]$$

Let $\theta(f_0)$ is the angle of $H(f)$ at $f = f_0$

$$= \frac{A}{2j}\left[|H(f_1)|\left[e^{j[\phi(t)+\theta(f_1)]} - e^{-j(\phi(t)+\theta(f_1))}\right]\right]$$

$$= \frac{A}{\cancel{2j}} |H(f_1)| \left[\sin(\phi(t) + \theta(f_1))\right]$$

$$= A\,|H(f_1)|\,\sin(2\pi f_1 t + \phi_1 + \theta(f_1))$$ //

input

$x(t) = A\sin(2\pi f_1 t + \phi_1)$

<u>input frequency is f_1 Hz</u>

c) output

$y(t) = A\,|H(f)|\,\angle H(f)\,\sin(2\pi f_1 t + \phi_1)$

There is change only in amplitude & phase hence

<u>output frequency is f_1 Hz</u>

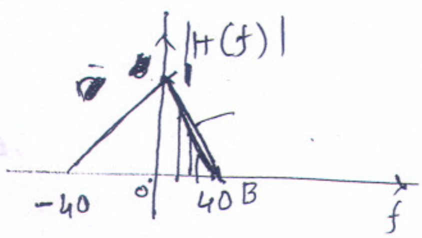
d) Amplitude : $A\,|H(f_1)|$

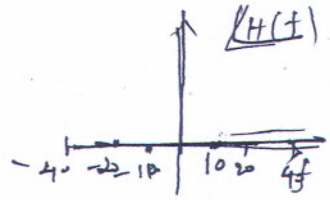
phase : $\phi_1 + \angle H(f_1)$ $\tan^{-1}\left(\frac{H_I(f_1)}{H_R(f_1)}\right)$

When a sinusoidal signal is passed through an LTI system, there will be only change in Amplitude and phase of the input signal

Given

$$x(t) = 10\sin(2\pi 10t + 45^\circ) + 20\sin(2\pi 20t + 90^\circ)$$





$y(t)$

$$10 \times \underbrace{|H(10)|}\ \sin\left(2\pi 10t + 45^\circ + \angle H(10)\right)$$

$$+\ 20 \times |H(20)|\ \sin\left(2\pi 20t + 90^\circ + \angle H(20)\right)$$

$|H(10)|$ = magnitude of $H(f)$ at $f = 10$ Hz

from the given graph

$H(10) = 3/4$

Consider two points on the line.

$A : (0,1)$

$B : (40,0)$

$$y - 1 = \frac{1}{-40}(x - 0)$$

$$\Rightarrow y - 1 = \frac{1}{40}(-x)$$

$$\Rightarrow 40y - 40 = -x$$

$$\Rightarrow x + 40y = 40$$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$y - 1 = \frac{1 - 0}{0 - 40}(x - 0)$$

$40 \to 1$

$10 \to 1$

Now when $x = 10$ Hz

$$\Rightarrow y = \frac{30}{40} = \frac{3}{4} = |H(10)|$$

Now when $x = 20$ Hz

$$\Rightarrow y = \frac{20}{40} = \frac{1}{2} = |H(20)|$$

$$\boxed{y(t) = A|H(f)|\ \angle H(f)\ \sin\theta}$$

$$\therefore y(t) = 10\left(\frac{3}{4}\right)\sin(2\pi 10t + 45^\circ) + 20\left(\frac{1}{2}\right)\sin(2\pi 20t + 90^\circ)$$

[As $\angle H(f) = 0$ for all frequencies]

Given

$$x(t) = 10\cos(2\pi 50 t)$$

We know that

$$\boxed{y(t) = A\,|H(f)|\,\angle H(f)\,\sin\theta}$$

System passes 50 Hz signal

$$|H(50)| = |H(f)|_{f=50} = 1 \quad \text{and} \quad \angle H(f)\big|_{f=50} = \frac{-\pi}{2} = \angle H(50)$$

$$\Rightarrow y(t) = 10(1)\cos\left(2\pi 50 t - \frac{\pi}{2}\right) = 10\cos 2\pi 50\left(t - \frac{\cancel{\pi}}{2}\cdot\frac{1}{2\cancel{\pi}50}\right)$$

$$= 10\cos 2\pi 50\left(t - 5\times10^{-3}\right)$$

$$\Rightarrow \omega t_d = \frac{\pi}{2}$$

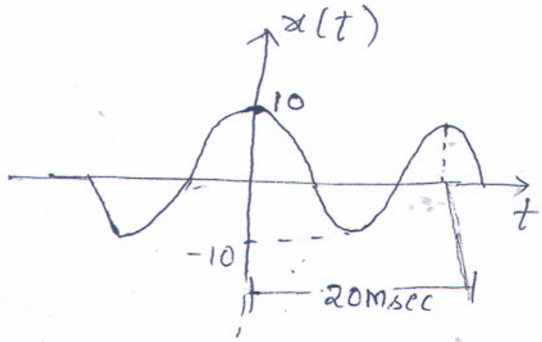
$$\Rightarrow t_d = \frac{\pi}{2\times 2\pi(50)}$$

$$= \frac{1}{200}$$

$$= 5\text{ msec.}$$

a) delay introduced by the system on 50 Hz signal is 5 msec.

b)

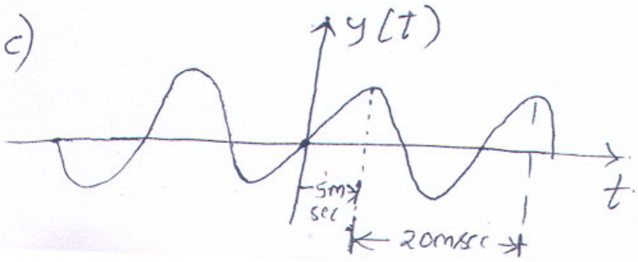


$f = 50$

Timeperiod (T)

$$T = \frac{1}{50} = \frac{1000}{50}\text{ msec} = 20\text{ msec.}$$

c)



The signal is just shifted right by 5 msec