Tutorial-9 Answers ①

Tut-9 ANSWERS

① (a) $g(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn2\pi f_0 t}$

$n = -\infty$ to $+\infty$

$$C_n = \frac{1}{T}\int_0^T g(t)\, e^{-j2\pi n f_0 t}\, dt$$

Here $g(t) = 1 \qquad 0 \le t \le T/4$

$= -1 \qquad T/4 \le t \le 3T/4$

$= 1 \qquad 3T/4 \le t \le T$

$$C_n = \frac{1}{T}\left[\int_0^{T/4} 1\cdot e^{-j2\pi n f_0 t}\, dt + \int_{T/4}^{3T/4} -1\cdot e^{-j2\pi n f_0 t}\, dt + \int_{3T/4}^{T} 1\cdot e^{-j2\pi n f_0 t}\, dt\right]$$

$$= \frac{1}{T}\left[\left(\frac{e^{-j2\pi n f_0 t}}{-j2\pi n f_0}\right)_0^{T/4} + \left(\frac{e^{-j2\pi n f_0 t}}{-j2\pi n f_0}\right)_{T/4}^{3T/4} + \left(\frac{e^{-j2\pi n f_0 t}}{-j2\pi n f_0}\right)_{3T/4}^{T}\right]$$

$$= \frac{1}{T} * \frac{1}{j2\pi n f_0}\left[\left(1 - e^{-j2\pi n f_0 * \frac{T}{4}}\right) + \left(e^{-j2\pi n f_0 \frac{3T}{4}} - e^{-j2\pi n f_0 T/4}\right) + e^{-j2\pi n 3T/4} - e^{-j2\pi n f_0 T/4}\right]$$

But $f_0 T = 1$

$$= \frac{1}{j2\pi n}\left[1 - e^{-jn\pi/2} + e^{-j3n\pi/2} - e^{-jn\pi/2} + e^{-j3\pi n/2} + \ldots\right\} + e^{-j2\pi n}\Big]$$

$$= \frac{1}{\cancel{2}j\pi n}\left[\cancel{2}\left(e^{-j3\pi n/2} - e^{-jn\pi/2}\right)\right]$$

$$C_n = \frac{1}{j\pi n}\left[e^{-j3\pi n/2} - e^{-jn\pi/2}\right]$$

For n = even ie $n = \pm 2, \pm 4 \ldots$

$\underline{C_n = 0}$

For $n = 2k$

$$C_n = \frac{1}{j\pi n}\left[e^{-j3\pi\frac{2k}{2}} - e^{-j\pi/2 \times 2k}\right] \qquad k = \pm 1, \pm 2 -$$

$$= \frac{1}{j\pi n}\left[e^{-j3k\pi} - e^{-jk\pi}\right] = \underline{0}$$

For n is odd. $n = 2k+1$

$$C_n = \frac{1}{j\pi(2k+1)}\left[e^{-j3\pi\frac{(2k+1)}{2}} - e^{-j(2k+1)\pi/2}\right]$$

Let $k = 0$

$$C_1 = \frac{1}{j\pi 1}\left[e^{-\frac{j3\pi}{2}\times 1} - e^{-j\pi/2}\right]$$

$$= \frac{1}{j\pi \times 1}\left[\cos 3\pi/2 - j\sin 3\pi/2 - \cos\pi/2 + j\sin\pi/2\right]$$

$$= \frac{1}{j\pi}\left[0 + j - 0 + j\right] = \frac{2j}{j\pi} = \underline{2/\pi \times 1}$$

$$C_3 = \frac{1}{j\pi 3}\left[\cos 3\pi/2 \times 3 - j\sin 3\pi/2 \times 3 - \cos 3\pi/2 + j\sin 3\pi/2\right]$$

$$= \frac{1}{j\pi \times 3}\left[\cos - j\sin(4\pi + \pi/2) - 0 + j\sin(\pi + \pi/2)\right]$$

$$= \frac{1}{j\pi \times 3}(-j - j) = -2/\pi 3$$

$$C_n = \frac{1}{j\pi}\left[-j\sin(2k+1)3\pi/2 + j\sin(2k+1)\pi/2\right]$$

$$C_n = \frac{2}{|n|\pi} \qquad n = \pm 1, \pm 5 \cdots$$

$$= \frac{-2}{|n|\pi} \qquad n = \pm 3, \pm 7 \cdots$$

$$= 0 \qquad \text{even.}$$

$$g(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn2\pi f_0 t}$$

$n = \pm 1, \pm 3, \pm 5 \ldots$

$$\left.\begin{array}{ll} n=\pm 1 \longrightarrow & \frac{2}{\pi} e^{j2\pi f_0 t} \\ \& & \\ n=-1 & \frac{2}{\pi} e^{-j2\pi f_0 t} \end{array}\right\} \frac{2}{\pi}\left[e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right]$$

$$= \frac{4}{\pi} \cos 2\pi f_0 t$$

$$\left.\begin{array}{ll} n=+3 \rightarrow & \frac{-2}{3\pi} e^{j2\pi 3 f_0 t} \\ n=-3 \rightarrow & -2/3\pi \, e^{-j2\pi 3 f_0 t} \end{array}\right\} = \cancel{\frac{-4}{3\pi} \cos 2\pi \times 0}$$

$$= -2/3\pi \left[e^{j2\pi 3 f_0 t} + e^{-j2\pi 3 f_0 t}\right]$$

$$= \frac{-2}{3\pi} 2\cos 2\pi 3 f_0 t$$

$$= \frac{-4}{3\pi} \cos 2\pi (3 f_0) t$$

So on.. So

$$g(t) = \frac{4}{\pi} \cos 2\pi f_0 t - \frac{4}{3\pi} \cos 2\pi (3f_0) t + \frac{4}{5\pi} \cos 2\pi (5 f_0) t \ldots$$

where $f_0 = 10^5$ Hz i.e $T = 10^{-5}$ sec.

(4) The mean square error - ($\mathcal{E}$)

$$\mathcal{E} = \frac{1}{T}\int_0^T \left| x(t) - \sum_{k=-K}^{K} X_k e^{j2\pi k F_1 t} \right|^2 dt$$

$$= \frac{1}{T}\int_0^T \left(x(t) - \sum_{k=-K}^{K} X_k e^{j2\pi k F_1 t}\right)\left(x(t) - \sum_{k=-K}^{K} X_k e^{j2\pi k F_1 t}\right)^* dt$$

changing $k \rightleftharpoons l$ (in second term)

$$= \frac{1}{T}\int_0^T \left(x(t) - \sum_{k=-K}^{K} X_k e^{j2\pi k F_1 t}\right)\left(x(t) - \sum_{l=-K}^{K} X_l e^{j2\pi l F_1 t}\right)^* dt$$

$$= \frac{1}{T}\left[\int_0^T \left[|x(t)|^2 dt\right] - \int_0^T \left[x(t)\sum_{l=-K}^{K} X_l^* e^{-j2\pi l F_1 t} + x^*(t)\sum_{k=-K}^{K} X_k e^{j2\pi k F_1 t}\right] dt + \int_0^T \left(\sum_{k=-K}^{K} X_k e^{j2\pi k F_1 t} \sum_{l=-K}^{K} X_l^* e^{-j2\pi l F_1 t}\right) dt\right.$$

$\frac{1}{T}\int_0^T |x(t)|^2 dt$ = Avg: power = P and by using $\left[\begin{aligned} & x^* y + y^* x \\ & = x^* y + (x^* y)^* \\ & = 2R[x^* y] \end{aligned}\right]$

$$\int_0^T e^{j2\pi (k-l)F_1 t} dt = \begin{cases} T & k = l \\ 0 & k \neq l \end{cases}$$

$$= P - 2R\left\{\int_0^T \sum_{l=-K}^{K} X_l^* \frac{1}{T} x(t) e^{-j2\pi l F_1 t} dt\right\} + \sum_{l=-K}^{K} |X_l|^2$$

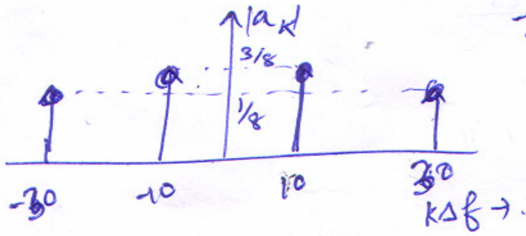
(3) (1) $\cos^3 20\pi t$.

$\boxed{\cos 3A = 4\cos^3 A - 3\cos A}$

$$= \frac{1}{4}\left[\cos 60\pi t + 3\cos 20\pi t\right].$$

$$= \frac{1}{4}\left\{\frac{e^{60\pi t} + e^{-60\pi t}}{2} + \frac{3e^{20\pi t} + 3e^{-20\pi t}}{2}\right\}.$$

$e^{jn\omega_0 t}$



fourier series Coeff
$3/8$ at -10 and 10 Hz
$1/8$ at -30 and 30 Hz

$\boxed{\sin 3A = 3\sin A - 4\sin^3 A}$

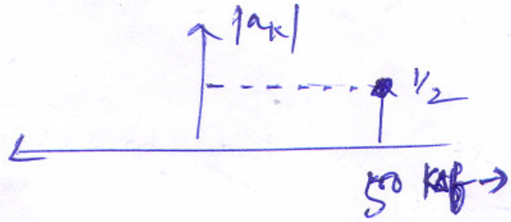
(2) $\sin^3 6\pi t$.

$$= \frac{3}{4}\sin 6\pi t - \frac{1}{4}\sin 18\pi t.$$

$$= \frac{3}{4}\left[\frac{e^{6\pi t} + e^{-6\pi t}}{2j}\right] - \frac{1}{4}\left[\frac{e^{18\pi t} + e^{-18\pi t}}{2j}\right].$$

fourier series Coeff $\rightarrow$ $\frac{3}{8j}$ at 3 Hz and -3 Hz
$\rightarrow$ $-\frac{1}{8j}$ at 9 Hz and -9 Hz.

(3) $\frac{1}{2}e^{j100\pi t}$.



fourier coeff $\rightarrow$ $1/2$ at 50 Hz.

(2) The trigonometric Fourier Series of a function $x(t)$ is defined as,

$$x(t) = a_0 + \sum_{K=1}^{\infty} a_K \cos(2\pi K f_0 t) + b_K \sin(2\pi K f_0 t)$$

where $f_0 = \frac{1}{T_0}$ and T_0 is the fundamental period of $x(t)$

In this Case, $T_0 = \pi$

$\therefore$ DC Coefficient,

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt = \frac{1}{\pi}\int_0^{\pi} e^{-t/2}\,dt \quad = \frac{1-e^{-\pi/2}}{1/2}$$

$$= \underline{\underline{0.504}}$$

Coeff of Cosines,

$$a_K = \frac{2}{T_0}\int_0^{T_0} x(t)\cos 2\pi f_0 K t\,dt = \frac{2}{\pi}\int_0^{\pi} e^{-t/2}\cos 2Kt\,dt$$

$$= \underline{\underline{0.504\left(\frac{2}{1+16K^2}\right)}}$$

Coeff of Sines,

$$b_K = \frac{2}{T_0}\int_0^{T_0} x(t)\sin 2\pi f_0 K t\,dt = \frac{2}{\pi}\int_0^{\pi} e^{-t/2}\sin 2Kt\,dt$$

$$= \underline{\underline{0.504\left(\frac{8K}{1+16K^2}\right)}}$$

So $x(t)$ Can be represented as,

$$x(t) = 0.504\left[1 + \sum_{K=1}^{\infty}\frac{2}{1+16K^2}\left(\cos 2Kt + 4K\sin 2Kt\right)\right]$$

$$\mathcal{E} = P - 2\,\mathrm{Re}\left\{\sum_{l=-k}^{k} X_l^* \frac{1}{T}\int_0^T x(t)\, e^{-j2\pi l F_1 t}\, dt\right\} + \sum_{l=-k}^{k} |X_l|^2$$

where p is avg power in $x(t)$ and X_l is a Complex Constant, $X_l = |X_l|\, e^{j\phi_l}$ where $\phi_l = \angle X_l$

The integral is also a complex Constant,

$$\frac{1}{T}\int_0^T x(t)\, e^{-j2\pi l F_1 t}\, dt = B_l\, e^{j\theta_1} \qquad \mathrm{Re}[e^{j\theta}] = \cos\theta$$

Substituting this into the expression for $\mathcal{E}$

~~$\mathcal{E} = P - 2\mathrm{Re}$~~ $\{$

$$\mathcal{E} = P - 2\sum_{l=-k}^{k} B_l |X_l| \cos(\theta_l - \phi_l) + \sum_{l=-k}^{k} |X_l|^2.$$

$\mathcal{E}$ will be minimum when $\theta_l = \phi_l$ ie $\cos(\theta_l - \phi_l) = 1$

$\therefore$

$$\mathcal{E}_K = P - 2\sum_{l=-k}^{k} B_l |X_l| + \sum_{l=-k}^{k} |X_l|^2.$$

In order to find the X_K that minimize the error, differentiate,

$$\frac{d\mathcal{E}_K}{d|X_K|} = 0.$$

ie $-2B_K + 2|X_K| = 0 \qquad -k \le K \le k$

ie $|X_K| = B_K$

So $X_K = B_K\, e^{j\theta_K} = \dfrac{1}{T}\displaystyle\int_0^T x(t)\, e^{-j2\pi K f_1 t}\, dt$

This is the formula for K^{th} complex Fourier Series Coefficient.