Unit-3
(Counting principles and relations)

The Principle of Inclusion-Exclusion

It defines that the number of elements in the union of set A and B is the sum of the numbers of elements in the sets minus the number of elements in their intersection

i.e $|A \cup B| = |A| + |B| - |A \cap B|$ ($|A|$: represent cardinality of A or no. of elements in A)

Example In a discrete mathematics class every student is a major in computer science or mathematics or both. The number of students having computer science as a major (possibly along ~~computer science~~ mathematics) is 25; the number of students having mathematics as a major (possibly along with computer science) is 13; and the number of students majoring in both computer science and mathematics is 8. How many students are in this class?

Sol^n

$|A|$: no. of students in the class majoring in C.S.

$|B|$: " " " " " in Maths.

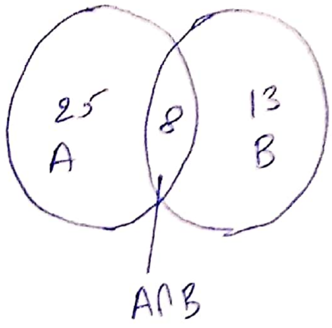
$|A \cap B|$: " " " " " in both C.S and maths.

$|A \cup B|$: no. of students " " majoring either C.S or math or both

Using principle of inclusion - exclusion.

$$|A \cup B| = |A| + |B| - |A \cap B| = 25 + 13 - 8$$

$$|A \cup B| = 30$$



<u>Example</u> How many positive integers not exceeding 1000 are divisible by 7 or 11?

<u>Sol^n</u>

let A : the set of positive integers not exceeding 1000 that are divisible by 7.

B : " " " " divisible 11.

A∩B : " " divisible by both 7 and 11

A∪B : " " divisible by either 7 or 11

By principle of inclusion and exclusion

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$= \left\lfloor \frac{1000}{7} \right\rfloor + \left\lfloor \frac{1000}{11} \right\rfloor - \left\lfloor \frac{1000}{7 \cdot 11} \right\rfloor$$

$$= 142 + 90 - 12 = 220$$

$\lfloor \; \rfloor$ floor function

$\lfloor a.a_1a_2 \rfloor = a$

Extended form of the principle of inclusion – exclusion

let $A_1, A_2, \dots A_n$ be finite sets, then

$$|A_1 \cup A_2 \cup \dots \cup A_n| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j|$$

$$+ \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| \dots + (-1)^{n+1} |A_1 \cap A_2 \cap \dots \cap A_n|$$

for $n = 3$

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3|$$
$$- |A_3 \cap A_1| + |A_1 \cap A_2 \cap A_3|$$

for $n = 4$

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3|$$
$$- |A_3 \cap A_4| - |A_4 \cap A_1| - |A_4 \cap A_2| - |A_1 \cap A_3|$$
$$+ |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4|$$
$$+ |A_2 \cap A_3 \cap A_4| - |A_1 \cap A_2 \cap A_3 \cap A_4|$$

Example Suppose that there are 1807 freshmen at your school. Of these, 453 are taking a course in Computer science, 567 are taking a course in mathematics, and 299 are taking courses in both Computer Science and mathematics. How many are not taking a course either in Computer Science or in mathematics?

Sol^n

let A: freshmen taking Computer science.

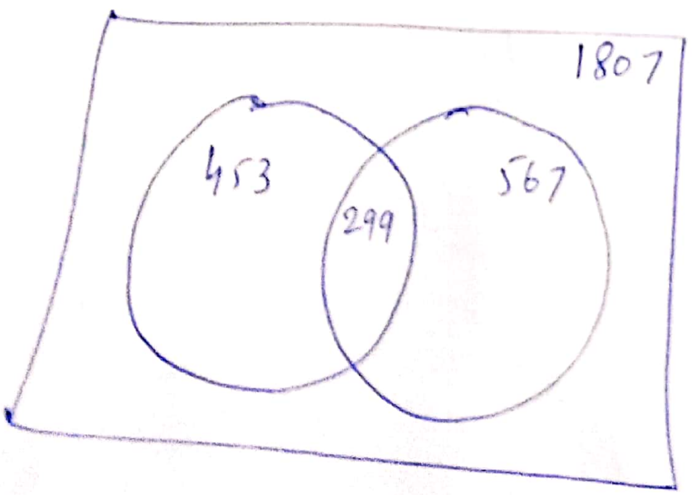
B: " " mathematics.

$|A| = 453$, $|B| = 567$, $|A \cap B| = 299$

$|A \cup B| = |A| + |B| - |A \cap B| = 453 + 567 - 299$
$= 721$

Total freshmen $= 1807$

no. of freshmen not taking any course either C.S or maths $= 1807 - 721$
$= 1086$



Example A total of 1232 students have taken a course in spanish, 879 have taken a course in french, and 114 have taken a course in Russian. further, 103 have taken course in both Spanish and French, 23 have taken course in both spanish and Russian, and 14 have taken courses in both french and Russian. If 2092 students have taken at least one of spanish, French and Russian, how many students have taken a course in all three languages?

Sol^n

let S: students have taken spanish
F: " " " french
R: " " " Russian
$S\cap R$: " " " both spanish and Russian
$S\cap F$: " " " both spanish and french
$F\cap R$: " " " both French and Russian
$S\cap F\cap R$: " " " spanish, French and Russian
$S\cup F\cup R$: " " at least one of S, F OR R.

By principle of inclusion-exclusion

$$|S\cup F\cup R| = |S| + |F| + |R| - |S\cap F| - |S\cap R| - |F\cap R| + |F\cap R\cap S|$$

$$2092 = 1232 + 879 + 114 - 103 - 23 - 14 + |S\cap F\cap R|$$

$$|S\cap F\cap R| = 7$$

The pigeonhole Principle

If k is a positive integer (natural number) and k+1 or more objects are placed into 'k' boxes, then there is at least one box containing two or more of the objects.

$\left\{ \begin{array}{l} \text{box (Pigeonhole)} \\ \text{Objects (Pigeons).} \end{array} \right.$

Sol^n Proving by ~~Contraposition~~ Contradiction.

Suppose that none of the k boxes contains more than one object. Then the total number of objects would be at most k. This is a contradiction, because there are at least k+1 objects.

Corollary A function 'f' from a set with k+1 or more elements to a set with k elements is not one to one.

Example Among any group of 367 people, there must be at least two with the same birthday because there are only 366 possible birthday.

Example In any group of 27 English words, there must be at least two that begin with the same letter because there are 26 letters in the English alphabet.

Example How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points.?.

Sol^n

Points are

$\underbrace{0, 1, 2, 3, \ldots .99, 100}$

total 101 points in count

So, to have equal OR same points from 0 to 100

Such that at least two students receive the same score is 102.

The Generalized pigeonhole Principle

If N objects are placed into K boxes, then there is at least one box containing at least $\lceil N/K \rceil$ objects.

OR $\left\lfloor \frac{N+1}{K} \right\rfloor$

ceiling function.

2.1.+1

$\text{ceil}(x) = \lceil x \rceil = \min\{n \in \mathbb{Z} \mid n \geq x\}$

$\text{floor}(x) = \lfloor x \rfloor = \max\{m \in \mathbb{Z} \mid m \leq x\}$

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Example Among 100 people, how many (at least) born in the same month.

boxes 12

objects 100

by General Pigeonhole Principle

$$\left\lceil \frac{100}{12} \right\rceil = \lceil 8.33 \rceil = 9$$

(A) 7 (B) 8 (C) 9 (D) None

Example what is the minimum number of students required in a discrete mathematics class to be sure that at least 6 will receive the same grade, if there are five possible grades A, B, C, D and F?

Sol^n Given

$$\left\lceil \frac{N}{5} \right\rceil = 6$$

i.e

$$25+1 \left\{ \begin{array}{ccccc} A & B & C & D & F \\ A & B & C & D & F \\ A & B & C & D & F \\ A & B & C & D & F \\ A & B & C & D & F \end{array} \right.$$

A or B or C or D or F

(A) 24 (B) 25 (C) 26 ✓ (D) 27

Ex How many cards must be selected from a standard deck of 52 cards to ~~gua~~ guarantee that at least three cards of the same suit are chosen?

Soln

4 = boxes → pigeonhole

N = cards → pigeons

By generalised pigeonhole principle.

$$\left\lceil \frac{N}{4} \right\rceil \geq 3$$ ($\because$ at least 3 of the same suit)

find min. N for which it is satisfying.

ie $\left\lceil \frac{9}{4} \right\rceil \geq 3$ 9 is min. integer.

So, 9 cards need to be selected out of 52 cards.

Relations.

let A and B are two sets. then binary relation from A to B is a Subset of $A \times B$.

i.e $A = \{0, 1\}$, $B = \{a, b, c\}$

$A \times B = \{(0,a), (0,b), (0,c), (1,a), (1,b), (1,c)\}$

if R represent relation set,

then $R_1 = \{(0,a), (0,b)\} \subseteq A \times B$

$R_2 = \{(0,a)\} \subseteq A \times B$

etc.

Every possible Subset of $A \times B$ is a relation

So, Relation can be defined as

$R = \{(a,b) \mid aRb \text{ where } a \in A, b \in B\}$

aRb is Same as $(a,b) \in R$

$a \not R b$ is Same as $(a,b) \notin R$

☞ If relation on the set A to itself are of special interest.

Def^n A relation on the set A is a relation from A to A

i.e. A relation on a set A is a subset of $A \times A$.

Ex. Let $A = \{1, 2, 3, 4\}$, which ordered pairs are in the relation $R = \{(a,b) \mid a \text{ divides } b\}$

Sol^n

$R = \{(1,1), (2,2), (3,3), (4,4), (1,2), (1,3), (1,4), (2,4)\}$

~~Graph and Matrix of~~ relation

Types of Relation

Reflexive if $aRa \ \forall a \in A$

Symmetric if aRb, then $bRa \ \forall a, b \in A$

transitive if aRb and bRc, then $aRc \ \forall a, b, c \in A$

irreflexive if $a \not R a \ \forall a \in A$ (negation of reflexive)

Antisymmetric if aRb and bRa, then $a = b$

Not reflexive if $a \not R a$ for some 'a' $\in A$

asymmetric if $b \not R a$ when aRb for some $a, b \in$

So, ~~not~~ irreflexive implies not irreflexive

but Converse ~~~~ need not to be true,

Example $A = \{1, 2, 3\}$

$R_1 = \{(1,1), (2,2), (3,3), (1,2)\}$

$R_2 = \{(1,1), (2,2), (1,2)\}$

$R_3 = \{(1,2), (2,1), (3,1)\}$

R_1 is Reflexive

R_2 not reflexive

R_3 irreflexive.

Ex Consider these relations on the set of integers

(i) $R_1 = \{(a,b) \mid a \leq b\}$

(ii) $R_2 = \{(a,b) \mid a > b\}$

(iii) $R_3 = \{(a,b) \mid a + b \leq 3\}$

(iv) $R_4 = \{(a,b) \mid a = b \text{ or } a = -b\}$

check each for Reflexive, Symmetric and transitive.

(i)

$a R_1 a$ as $a = a$ ($a \leq b$ is given)

So, reflexive

if $a \leq b$ then it is necessary $b \nleq a$

So, not Symmetric it is asymmetric

i.e $a = 3, b = 4$

$3 < 4$ but $4 \nless 3$.

if $a \leq b$ and $b \leq c$ then $a \leq c$ for every $a, b, c \in \mathbb{Z}$

So, it is transitive.

(ii) not reflexive $\because a \ngtr a$

not Symmetric $a > b$ but $b \ngtr a$

transitive if $a > b$ and $b > c$, then $a > c$

(iii) $a + a = \overset{a}{2} + \overset{a}{2} = 4 \nleq 3$ not reflexive

~~not~~ Symmetric if $a + b \leq 3$ as aRb

then $b + a \leq 3$

So, Symmetric

if $a + b \leq 3$ and $b + c \leq 3$

then ~~$1 + 2 \leq 3$~~ ~~$2 + 2$~~

~~$a + c = a + b = b + c$~~

$a + 2b + c \leq 6$

$a + c \not\leq 6 - 2b$

$6 - 2b = 3$

So $a + c \not\leq 3$

($b = \frac{3}{2}$ which is not true as b is integer.)

=> Not transitive.

Ex. Is the "divides" relation on the set of positive integers reflexive?

Soln yes, $a | a \quad \forall a \in \mathbb{Z}$

Ex Is the "divides" relation on the set of integers reflexive?

Soln NO, $\because \exists\ 0$ such that $0 \nmid 0$

So, not reflexive.

Ex if $A = \{1, 2, 3, 4\}$

then $R = \{(1,1), (1,2), (2,1)\}$

Is it reflexive, symmetric, antisymmetric, irreflexive, not reflexive: ∵ not for every a, aRa

· Symmetric: if $(1,2) \in R$ then $(2,1) \in R$

not antisymmetric: ∵ there is no $b \not R a$ when aRb.

not irreflexive; not for every a, $a \not R a$

it is not reflexive.

H.W If $A = \{1, 2, 3, 4\}$

$R_1 = \{(3,4)\}$, $R_2 = \{(1,1), (1,2), (1,3), (1,4),$
$(2,2), (2,3), (2,4), (3,3),$
$(3,4), (4,4)\}$

Check Reflexive,
symmetric
transitive
irreflexive
asymmetric
antisymmetric
for R_1 and R_2?

Counting number of relations.

* No. of relations on set A with 'n' elements
$$= 2^{n^2}$$

* No. of reflexive relation $= 2^{n(n-1)}$

* " " Symmetric relation $= 2^{\frac{n(n+1)}{2}}$

* " " Antisymmetric $= 2^n \cdot 3^{\frac{n(n-1)}{2}}$

* " " asymmetric $= 3^{\frac{n(n-1)}{2}}$

* " " irreflexive $= 2^{n(n-1)}$

if $|A| = m$, $|B| = n$

then total no. of relation from A to B is
$$= 2^{m \cdot n}$$

Ex How many binary relations are there on a set S with 9 distinct elements,

(a) 2^{90} (b) 2^{100} (c) 2^{81} ✓ (d) 2^{60}

Ex

______ number of reflexive relations are there on a set of 12 distinct elements.

(a) 2^{132} ✓ (b) 2^{131} (c) 2^{130} (d) 2^{129}

$(2^{n(n-1)})$

Combining Relations.

Since relations behaves like sets only, So, we can find their union, intersection, difference etc.

for example

let $A = \{1,2,3\}$, $B = \{1,2,3,4\}$.

let $R_1 = \{(1,1),(2,2),(3,3)\}$, $R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$

$R_1 \cup R_2 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(3,3)\}$

$R_1 \cap R_2 = \{(1,1)\}$

$R_1 - R_2 = \{(2,2),(3,3)\}$

$R_2 - R_1 = \{(1,2),(1,3),(1,4)\}$

$R_1 \oplus R_2 = (R_1 \cup R_2) - (R_1 \cap R_2)$

$= \{(1,2),(1,3),(1,4),(2,2),(3,3)\}$

Since
$$\left.\begin{aligned} R_1 \cup R_2 &\subseteq A \times B \\ R_1 \cap R_2 &\subseteq A \times B \\ R_1 - R_2 &\subseteq A \times B \\ R_2 - R_1 &\subseteq A \times B \end{aligned}\right\}$$
union, intersection, direct sum difference of two relations are also a relation.

Example

If $R_1 = \{(x,y) \mid x < y\}$

$R_2 = \{(x,y) \mid x > y\}$ where $x, y \in \mathbb{R}$.

What are $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_2 - R_1$ and $R_1 \oplus R_2$

(i) $R_1 \cup R_2 = \{(x,y) \mid x < y$ ~~and~~ OR $x > y\}$

$= \{(x,y) \mid x \neq y\}$

(ii) $R_1 \cap R_2 = \{\phi\}$ there is no x, y s.t $x < y$ and $y < x$ togeather.

(iii) $R_1 - R_2 = R_1$

(iv) $R_2 - R_1 = R_2$

(v) $R_1 \oplus R_2 = (R_1 \cup R_2) - (R_1 \cap R_2)$

$= \{(x,y) \mid x \neq y\}$

Composition ~~Combining Relation with itself~~

Composition of relation is also a kind of combining two relations.

<u>Def^n</u> let R be a relation from a set A to a set B and S be relation from B to C then composite of R and S is the relation consist of (a, c) where $a \in A$, $c \in C$ for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$.

Composite of R and S is denoted by $S \circ R$.

<u>Example</u> let $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4\}$, $C = \{0, 1, 2\}$

$R = \{(1,1), (1,4), (2,3), (3,1), (3,4)\}$ from A to B

$S = \{(1,2), (2,0), (3,1), (3,2), (4,1)\}$ from B to C

$$S \circ R = \{(1,0), (1,1), (2,1), (2,2), (3,0), (3,1)\}$$

for $S \circ R$

$$\underset{\in R}{(a,b)} \quad \underset{\in S}{(b,c)}$$

$(a,c) \in S \circ R$

for $R \circ S$

$$\underset{\in S}{(a,b)} \quad \underset{\in R}{(b,c)}$$

$(a,c) \in R \circ S$.

<u>Def^n</u> let R be relation on the set A. The powers R^n, $n = 1, 2, 3 \ldots$ are defined

as $R^{n+1} = R^n \circ R$

$n = 1$, $R^2 = R \circ R$

$n = 2$, $R^3 = R^2 \circ R$

$R^4 = R^3 \circ R$

$\vdots$

<u>Example</u> let $R = \{(1,1), (2,1), (3,2), (4,3)\}$

find R^n where $n = 2, 3, 4 \ldots$

$R^2 = R \circ R = \{(1,1), (2,1), (3,1), (4,2)\}$

$R^3 = R^2 \circ R = \{(1,1), (2,1), (3,1), (4,1)\}$

$R^4 = R^3 \circ R = \{(1,1), (2,1), (3,1), (4,1)\} = R^3$

$\therefore R^n = R^3 \quad n = 5, 6, 7, \ldots$

Inverse, empty, full relation.

If $R = \{(a,b) \mid a \in A, b \in B\} \subseteq A \times B$

$R^{-1} = \{(b,a) \mid a \in A, b \in B\} \subseteq B \times A$

(Inverse relation.)

if $R = A \times B$, then R is full relation.

if $\phi \subseteq A \times B$, then ϕ is empty relation.

Matrix and graph of relation.

let $A = \{a_1, a_2, \ldots a_m\}$

$B = \{b_1, b_2, \ldots b_n\}$

$M_R = [m_{ij}]_{m \times n}$ Matrix of relation $R \subseteq A \times B$

where $m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$

if $m = n$ then M_R will be Square matrix.

<u>Example</u> if $A = \{1, 2, 3\}$

$B = \{1, 2\}$

$R = \{(a, b) \mid a > b\}$

i.e $R = \{(2,1), (3,1), (3,2)\}$

$$M_R = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \end{array}$$

3×2

- <u>Directed graph for Relation</u>

Each element of relation set is represented by a point and each orderd pair is represent by an arc indicated by an arrow.

<u>Directed graph</u>

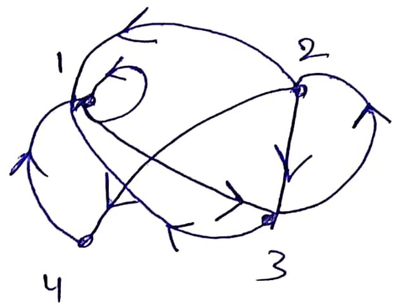
it Consist of vertices V (nodes) and edges E (ordered pairs) if $a \, R \, b$

initial vertex, terminal vertex.

Ex Draw a digraph or directed graph for the following relation

$R = \{(1,1), (1,3), (2,1), (2,3), (2,4), (3,1), (3,2), (4,1)\}$ on a set $\{1,2,3,4\}$

Mark all nodes from the set



Transitive closures (R^*)

$R = R \cup R^2 \cup R^3 \cup \dots \cup R^n$

~~Ex~~

Ex Suppose that the relation R on a set is represented by the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Is R reflexive, symmetric, antisymmetric iii Since all diagonal is 1, so reflexive

$M_R = M_R^T \Rightarrow$ Symmetric

Not antisymmetric

$\because$ $M_{ij} = 0$ and $M_{ji} = 0$
for $i \neq j$

Properties of relation and matrix

* A relation R is reflexive if diagonal elements are 1.

* " " irreflexive if diagonal are 0.

* " " Symmetric if $M_R = M_R^T$

* " " antisymmetric if either $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$

* $M_1 \vee M_2$ is Join matrix represented for $R_1 \cup R_2$

* $M_1 \wedge M_2$ is meet matrix " " $R_1 \cap R_2$

Properties of relation and graph

* A relation R is reflexive if there is loop at every node of directed graph.

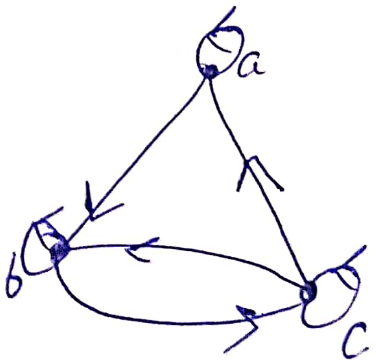
* " " reflexive if there is no loop on any node.

* " Symmetric if every edge between two distinct nodes, ~~has~~ an edge is always present in opposite direction.

* " asymmetric if there ~~is~~ are never two edges in opposite direction between distinct nodes.

* " antisymmetric if there ~~exist~~ is an edge from a to b and b to c then there exist edge a to c also.

Ex if $R = \{(a,a), (a,b), (b,b), (b,c), (c,c), (c,b), (c,a)\}$ is represented as



reflexive (All loops)
Not Symmetric.
aRb but $b \not R a$
Not transitive
aRb, bRc
but $a \not R c$

Equivalence Relation

A relation on a set A is called an equivalence relation if it is
reflexive
Symmetric
transitive.

Ex. $R = \{ (a,b) \in R \mid a = b$ or $a = -b$
is an equivalence relation.

Ex. $R = \{(a,b) \in R \mid$ ~~a - b is an integer~~ where $a, b \in$

$(a,b) \in R$ iff $a-b$ is an integer. where $a, b \in \mathbb{R}$
equivalence relation.

Ex. 'divides' relation on the set of positive integers is not an equivalence relation.

Pa~~rtial~~ ~~ordering~~

Partial ordering

Def^n A relation R on a set S is called a partial ordering or partial order if it is reflexive, antisymmetric and transitive.

Note A set S together with a partial ordering R is called a partially ordered set or poset denoted by (S, R)

Ex Show that "greater than or equal" relation ($\geq$) is a partial ordering on the set of integer.

$a \geq a$ reflexive

$a \geq b$ & $b \leq a \Rightarrow a = b$ antisymmetric

$a \geq b$ & $b \geq c \Rightarrow a \geq c$ transitive.

Ex Is $(Z^+, |)$ a poset?

$a | a$ reflexive

$a | b$ & $b | a \Rightarrow a = b$ antisymmetric

$a | b$ & $b | c \Rightarrow a | c$ transitive.

Yes, $(Z^+, |)$ is a poset.

Ex If A is any finite set and $P(A)$ is power set of A, then show that $(P(A), \subseteq)$ is Poset.

$A \subseteq A \quad \forall A \in P(A)$ (Reflexive)

$A \subseteq B$ & $B \subseteq A \Rightarrow A = B$ (Anti symmetric)

$A \subseteq B$ & $B \subseteq C \Rightarrow A \subseteq B$ (transitive)

Comparable and non-Comparable OR in Comparable elements

If $(S, \preceq)$ is a poset

and $a, b \in S$

then a, b are said to be Comparable if either $a \preceq b$ or $b \preceq a$

and

in Comparable if neither $a \preceq b$ nor $b \preceq a$

Ex In Poset $(Z^+, |)$, 3 and 9 Comparable

yes, $3 | 9$

But 5 and 7 are not as $5 \nmid 7$ and $7 \nmid 5$

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Total orderd set

If $(S, \preceq)$ is a poset and every two elements of S are comparable, then S is called a totally ordered OR linearly ordered set.

and '$\preceq$' is called total order OR linear order

Ex. $(\mathbb{Z}, \leq)$ is totally ordered

Every pair of integer are comparable.

Ex $(\mathbb{Z}, |)$ is not totally ordered

$5 \nmid 7$ not comparable.

Masse Diagrams

Masse diagram can be constructed only for poset $(S, \preceq)$.

steps

(i) Remove all loop from the digraph

(ii) Remove all edges which comes under transitivity relation.