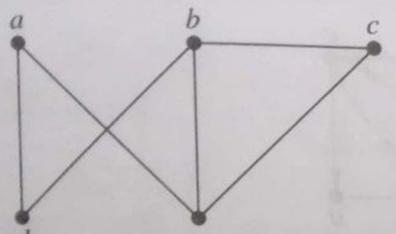
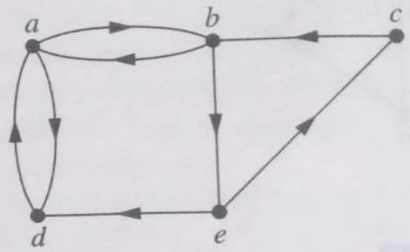
Theorem 2 can be used to find the length of the shortest path between two vertices of a graph (see Exercise 46), nd it can also be used to determine whether a graph is connected (see Exercises 51 and 52).

Exercises

1. Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? What are the lengths of those that are paths?
   a) a, e, b, c, b
   b) a, e, a, d, b, c, a
   c) e, b, a, d, b, e
   d) c, b, d, a, e, c

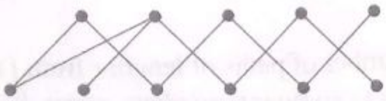


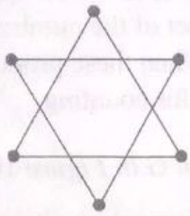
2. Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? What are the lengths of those that are paths?
   a) a, b, e, c, b
   b) a, d, a, d, a
   c) a, d, b, e, a
   d) a, b, e, c, b, d, a



In Exercises 3–5 determine whether the given graph is connected.

3.

4. 

5. 

6. How many connected components does each of the graphs in Exercises 3–5 have? For each graph find each of its connected components.

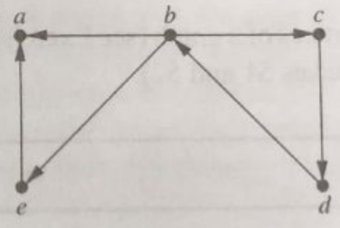
7. What do the connected components of acquaintanceship graphs represent?

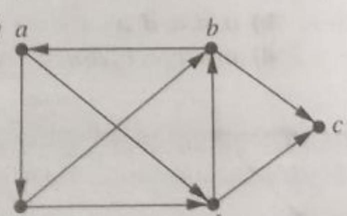
8. What do the connected components of a collaboration graph represent?

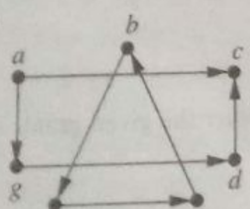
9. Explain why in the collaboration graph of mathematicians a vertex representing a mathematician is in the same connected component as the vertex representing Paul Erdős if and only if that mathematician has a finite Erdős number.

10. In the Hollywood graph (see Example 4 in Section 8.1), when is the vertex representing an actor in the same connected component as the vertex representing Kevin Bacon?

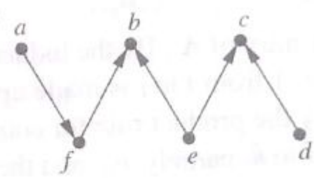
11. Determine whether each of these graphs is strongly connected and if not, whether it is weakly connected.

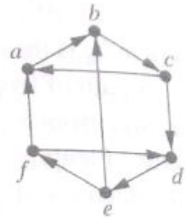
a) 

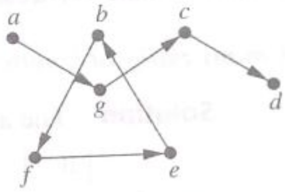
b) 

c) 

12. Determine whether each of these graphs is strongly connected and if not, whether it is weakly connected.

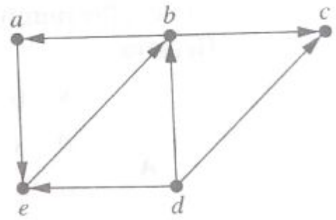
a) 

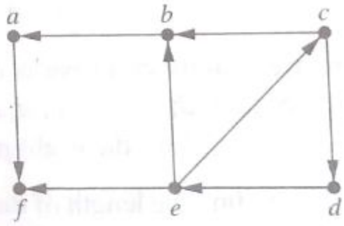
b) 

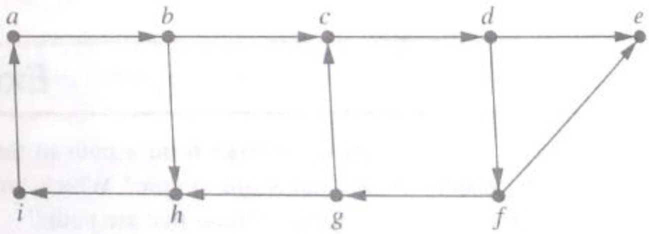
c) 

13. What do the strongly connected components of a telephone call graph represent?

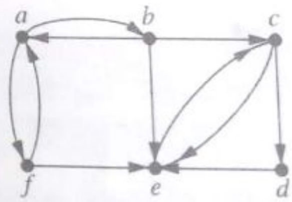
14. Find the strongly connected components of each of these graphs.

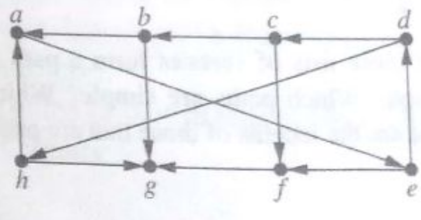
a) 

b) 

c) 

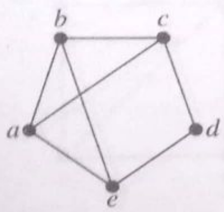
15. Find the strongly connected components of each of these graphs.

a) 

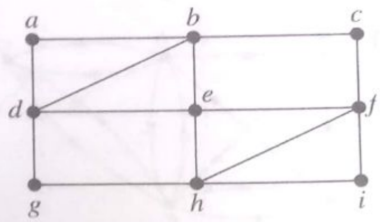
b) 

In Exercises 1–8 determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.

1.



2.



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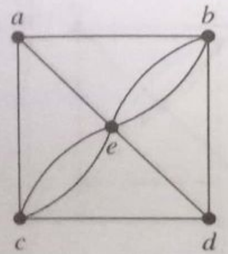

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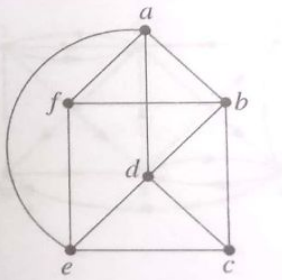

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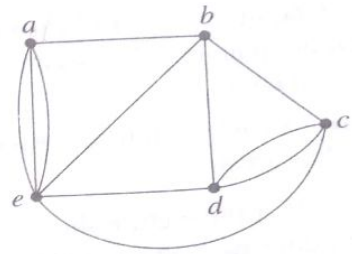

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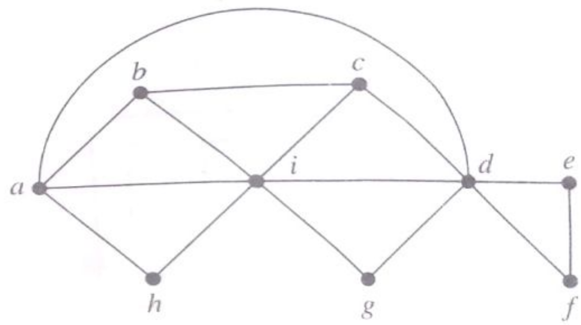
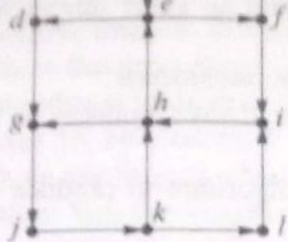




*24. Devise an algorithm for constructing Euler circuits in directed graphs.

25. Devise an algorithm for constructing Euler paths in directed graphs.

26. For which values of n do these graphs have an Euler circuit?

a) K_n b) C_n c) W_n d) Q_n

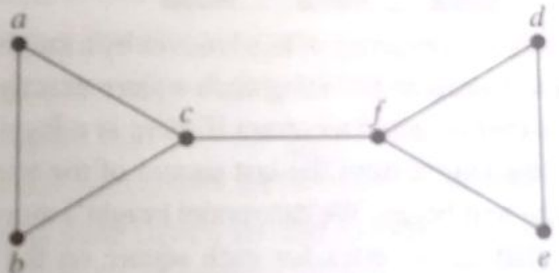
27. For which values of n do the graphs in Exercise 26 have an Euler path but no Euler circuit?

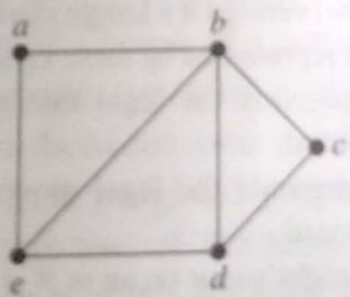
28. For which values of m and n does the complete bipartite graph $K_{m,n}$ have an

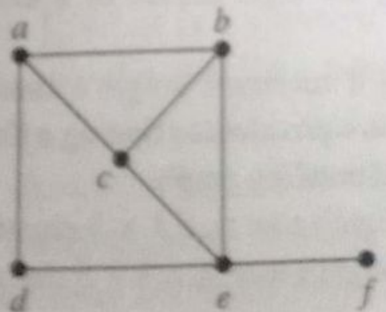
a) Euler circuit? b) Euler path?

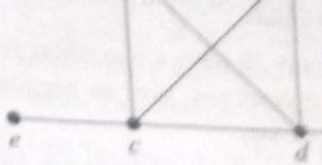
29. Find the least number of times it is necessary to lift a pencil from the paper when drawing each of the graphs in Exercises 1–7 without retracing an y part of the graph.

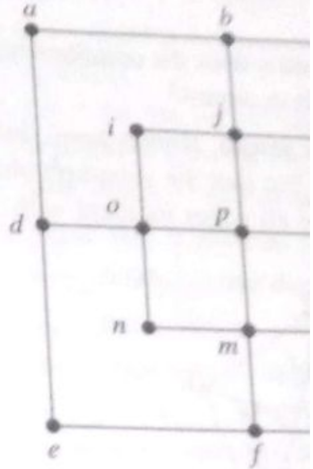
In Exercises 30–36 determine whether the given graph has a Hamilton circuit. If it does, find such a circuit. If it does not, give an argument to show why no such circuit exists.

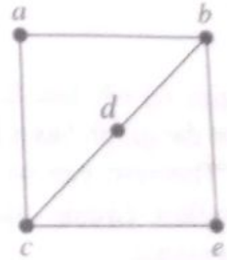
30. 

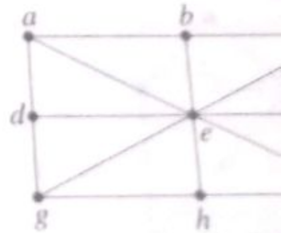
31. 

32. 



34. 

35. 

36. 

37. Does the graph in E so, find such a path show why no such

38. Does the graph in so, find such a pat show why no such

39. Does the graph in so, find such a pa show why no suc

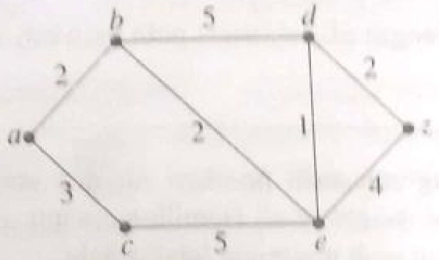
40. Does the graph i so, find such a p show why no su

*41. Does the graph so, find such a show why no s

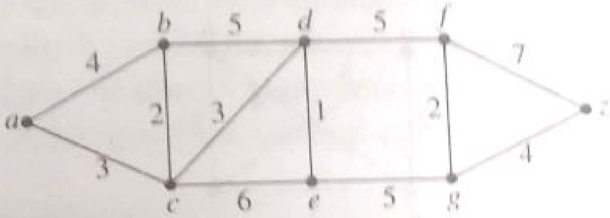
1. For each of these problems about a subway system, describe a weighted graph model that can be used to solve the problem.
   a) What is the least amount of time required to travel between two stops?
   b) What is the minimum distance that can be traveled to reach a stop from another stop?
   c) What is the least fare required to travel between two stops if fares between stops are added to give the total fare?

In Exercises 2–4 find the length of a shortest path between a and z in the given weighted graph.

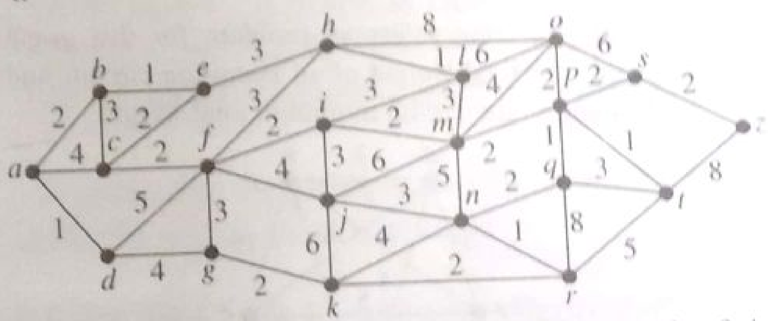
2.



3.



4.



5. Find a shortest path between a and z in each of the weighted graphs in Exercises 2–4.

6. Find the length of a shortest path between these pairs of vertices in the weighted graph in Exercise 3.
   a) a and d b) a and f
   c) c and f d) b and z

7. Find shortest paths in the weighted graph in Exercise 3 between the pairs of vertices in Exercise 6.

8. Find a shortest path (in mileage) between each of the following pairs of cities in the airline system shown in Figure 1.
   a) New York and Los Angeles
   b) Boston and San Francisco
   c) Miami and Denver
   d) Miami and Los Angeles

9. Find a combination of flights with the least total air t between the pairs of cities in Exercise 8, using the f times shown in Figure 1.

10. Find a least expensive combination of flights conne the pairs of cities in Exercise 8, using the fares sho Figure 1.

11. Find a shortest route (in distance) between com centers in each of these pairs of cities in the comm tions network shown in Figure 2.
    a) Boston and Los Angeles
    b) New York and San Francisco
    c) Dallas and San Francisco
    d) Denver and New York

12. Find a route with the shortest response time the pairs of computer centers in Exercise 11 u response times given in Figure 2.

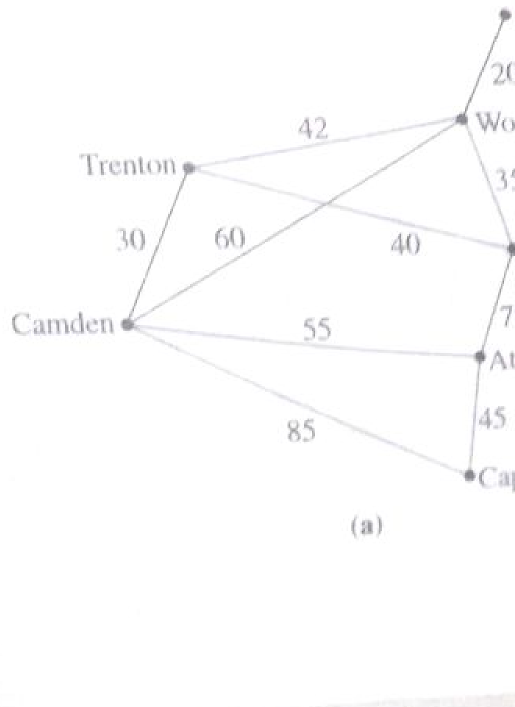
13. Find a least expensive route, in monthly lease between the pairs of computer centers in Ex using the lease charges given in Figure 2.

14. Explain how to find a path with the least numbe between two vertices in an undirected graph by ing it as a shortest path problem in a weighted

15. Extend Dijkstra's algorithm for finding the lengt est path between two vertices in a weighted nected graph so that the length of a shortest p the vertex a and every other vertex of the grap

16. Extend Dijkstra's algorithm for finding the shortest path between two vertices in a wei connected graph so that a shortest path b vertices is constructed.

17. The weighted graphs in the figures her major roads in New Jersey. Part (a) show between cities on these roads; part (b) sh



(a)

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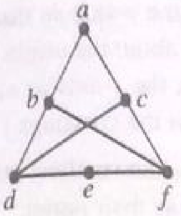
5.



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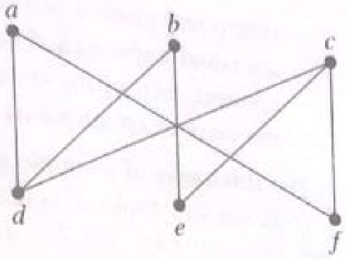

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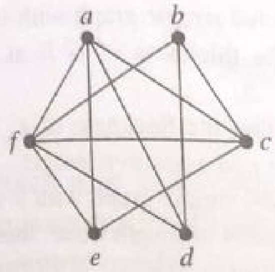

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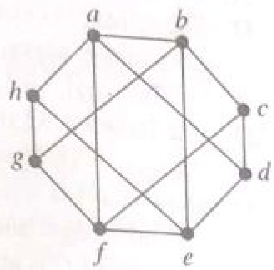

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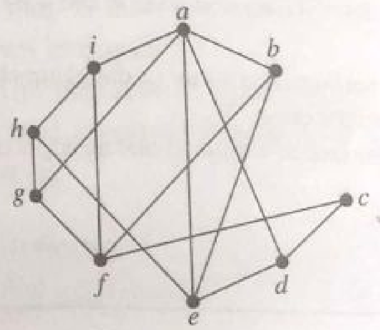


10. Complete the argument in Example 3.
11. Show that K5 is nonplanar using an argument similar to that given in Example 3.
12. Suppose that a connected planar graph has eight vertices, each of degree three. Into how many regions is the plane divided by a planar representation of this graph?
13. Suppose that a connected planar graph has six vertices, each of degree four. Into how many regions is the plane divided by a planar representation of this graph?
14. Suppose that a connected planar graph has 30 edges. If a planar representation of this graph divides the plane into 20 regions, how many vertices does this graph have?
15. Prove Corollary 3.
16. Suppose that a connected bipartite planar simple graph has e edges and v vertices. Show that $e \leq 2v - 4$ if $v \geq 3$.

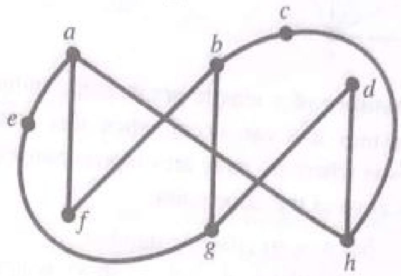
*17. Suppose that a connected planar simple graph with e edges and v vertices contains no simple circuits of length 4 or less. Show that $e \leq (5/3)\,v - (10/3)$ if $v \geq 4$.

18. Suppose that a planar graph has k connected components, e edges, and v vertices. Also suppose that the plane is divided into r regions by a planar representation of the graph. Find a formula for r in terms of e, v, and k.
19. Which of these nonplanar graphs have the property that the removal of any vertex and all edges incident with that vertex produces a planar graph?

    a) K_5 b) K_6 c) $K_{3,3}$ d) $K_{3,4}$

In Exercises 20–22 determine whether the given graph is homeomorphic to $K_{3,3}$.

20.



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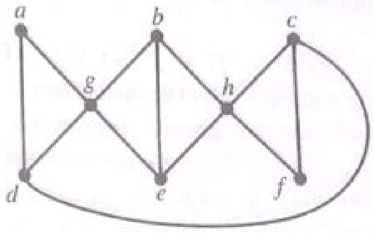

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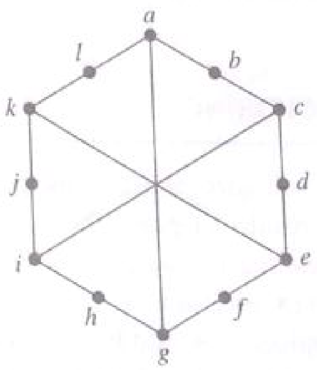
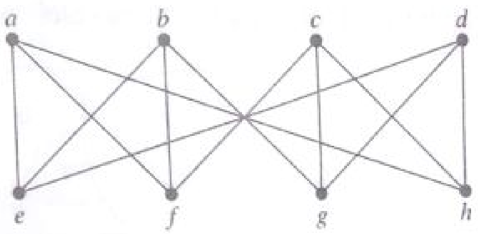
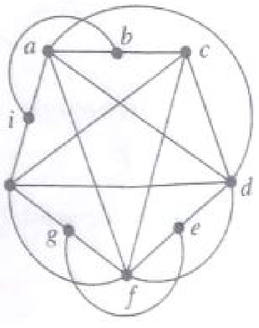


In Exercises 23–25 use Kuratowski's Theorem to determine whether the given graph is planar.

23.



24.



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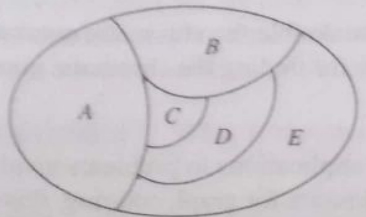
Exercises

In Exercises 1–4 construct the dual graph for the map shown. Then find the number of colors needed to color the map so that no two adjacent regions have the same color.

1.



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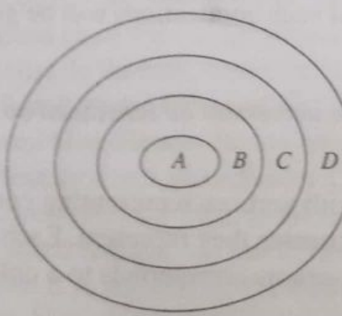

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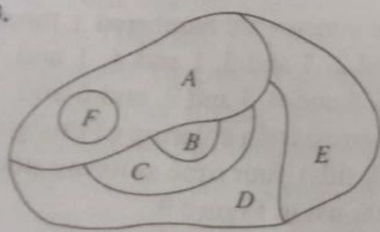

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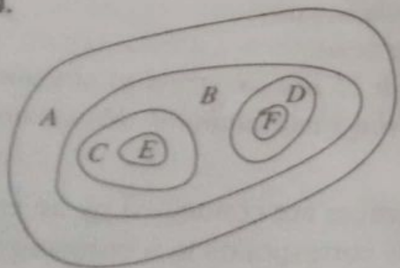
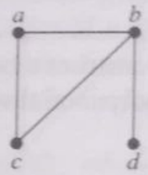


In Exercises 5–11 find the chromatic number of the given graph.

5.



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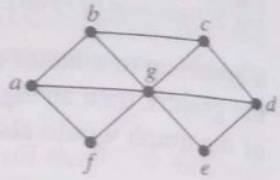

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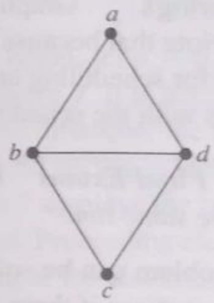

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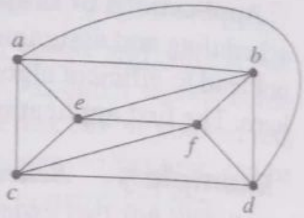

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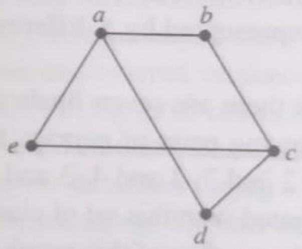

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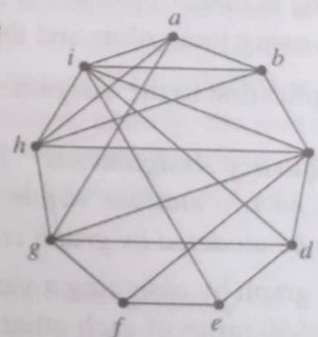
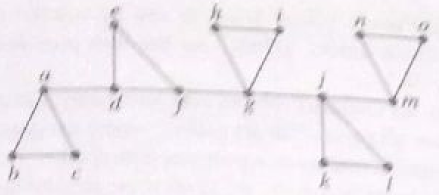


11.



12. For the graphs in Exercises 5–11, decide whether it is possible to decrease the chromatic number by removing a single vertex and all edges incident with it.
13. Which graphs have a chromatic number of 1?
14. What is the least number of colors needed to color a map of the United States? Do not consider adjacent states that meet only at a corner. Suppose that Michigan is one region. Consider the vertices representing Alaska and Hawaii as isolated vertices.
15. What is the chromatic number of W_n?
16. Show that a simple graph that has a circuit with an odd number of vertices in it cannot be colored using two colors.
17. Schedule the final exams for Math 115, Math 116, Math 185, Math 195, CS 101, CS 102, CS 273, and CS 473, using the fewest number of different time slots, if there are no students taking both Math 115 and CS 473, both Math 116 and CS 473, both Math 195 and CS 101, both Math 195 and CS 102, both Math 115 and Math 116, both Math 115 and Math 185, and both Math 185 and Math 195, but there are students in every other combination of courses.
18. How many different channels are needed for six stations located at the distances shown in the table, if two stations cannot use the same channel when they are within 150 miles of each other?

	1	*2*	*3*	*4*	*5*	*6*
1	—	85	175	200	50	100
2	85	—	125	175	100	160
3	175	125	—	100	200	250
4	200	175	100	—	210	220
5	50	100	200	210	—	100
6	100	160	250	220	100	—

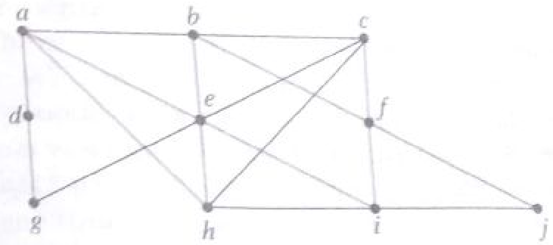
19. The mathematics department has six committees each meeting once a month. How many different meeting times must be used to ensure that no member is scheduled to attend two meetings at the same time if the committees are C_1 = {Arlinghaus, Brand, Zaslavsky}, C_2 = {Brand, Lee, Rosen}, C_3 = {Arlinghaus, Rosen, Zaslavsky}, C_4 = {Lee, Rosen, Zaslavsky}, C_5 = {Arlinghaus, Brand}, and C_6 = {Brand, Rosen, Zaslavsky}?
20. A zoo wants to set up natural habitats in which to exhibit its animals. Unfortunately, some animals will eat some of the others when given the opportunity. How can a graph model and a coloring be used to determine the number of different habitats needed and the placement of the animals in these habitats?

An **edge coloring** of a graph is an assignment of colors to edges so that edges incident with a common vertex are assigned different colors. The **edge chromatic number** of a graph is the smallest number of colors that can be used in an edge coloring of the graph.

21. Find the edge chromatic number of each of the graphs in Exercises 5–11.
*22. Find the edge chromatic numbers of
a) K_n **b)** $K_{m,n}$ **c)** C_n **d)** W_n
23. Seven variables occur in a loop of a computer program. The variables and the steps during which they must be stored are t: steps 1 through 6; u: step 2; : steps 2 through 4; w: steps 1, 3, and 5; x: steps 1 and 6; y: steps 3 through 6; and z: steps 4 and 5. How many different index registers are needed to store these variables during execution?
24. What can be said about the chromatic number of a graph that has K_n as a subgraph?

This algorithm can be used to color a simple graph: First, list the vertices ${}_1, {}_2, {}_3, \ldots, {}_n$ in order of decreasing degree so that $\deg_{(1)} \geq \deg_{(2)} \geq \cdots \geq \deg_{(n)}$. Assign color 1 to ${}_1$ and to the next vertex in the list not adjacent to 1 (if one exists), and successively to each vertex in the list not adjacent to a vertex already assigned color 1. Then assign color 2 to the first vertex in the list not already colored. Successively assign color 2 to vertices in the list that have not already been colored and are not adjacent to vertices assigned color 2. If uncolored vertices remain, assign color 3 to the first vertex in the list not yet colored, and use color 3 to successively color those vertices not already colored and not adjacent to vertices assigned color 3. Continue this process until all vertices are colored.

25. Construct a coloring of the graph shown using this algorithm.



*26. Use pseudocode to describe this coloring algorithm.
*27. Show that the coloring produced in this algorithm may use more colors than are necessary to color a graph.

A connected graph G is called **chromatically k-critical** if the chromatic number of G is k, but for every edge e of

*41.

32. Find these values:

a) $\chi_2(K_3)$ **b)** $\chi_2(K_4)$ **c)** $\chi_2(W_4)$
d) $\chi_2(C_5)$ **e)** $\chi_2(K_{3,4})$ **f)** $\chi_3(K_5)$
***g)** $\chi_3(C_5)$ **h)** $\chi_3(K_{4,5})$

***33.** Let G and H be the graphs displayed in Figure 3. Find

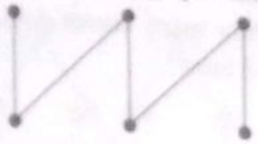
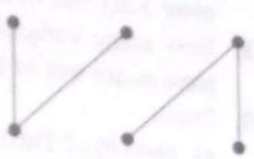
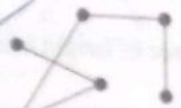
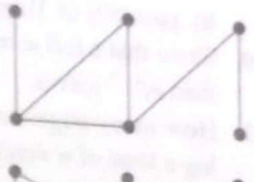
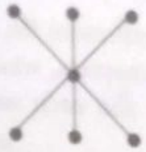
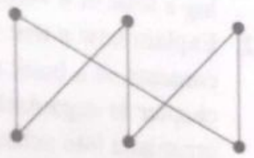
a) $\chi_2(G)$. **b)** $\chi_2(H)$. **c)** $\chi_3(G)$. **d)** $\chi_3(H)$.

34. What is $\chi_k(G)$ if G is a bipartite graph and k is a positive integer?

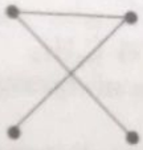
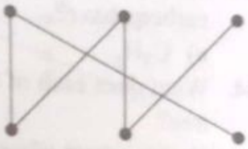
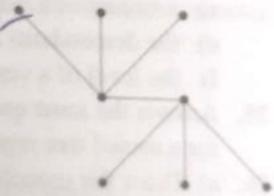
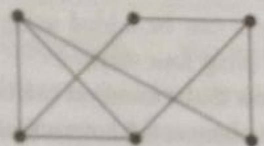
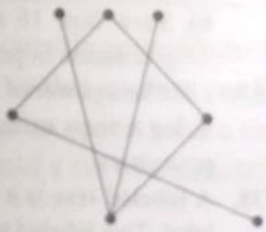
*42.

Exercises 9·1

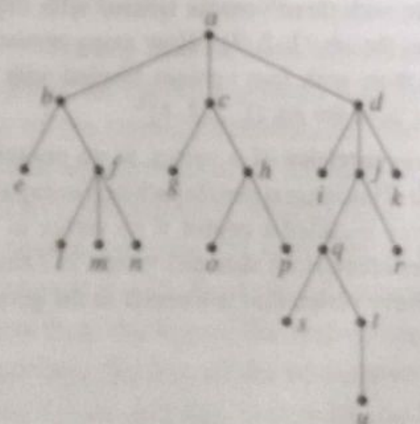
1. Which of these graphs are trees?

a) 
b) 
c) 
d) 
e) 
f) 

2. Which of these graphs are trees?

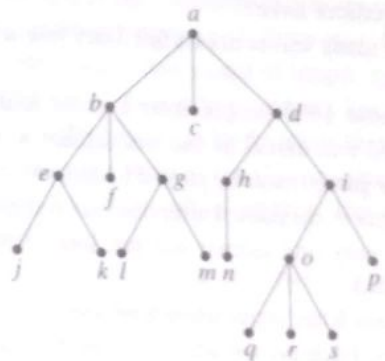
a) 
b) 
c) 
d) 
e) 
f) 

3. Answer these questions about the rooted tree illustrated.



a) Which vertex is the root? a
b) Which vertices are internal?
c) Which vertices are leaves?
d) Which vertices are children of j? q, r
e) Which vertex is the parent of h? c
f) Which vertices are siblings of o? p
g) Which vertices are ancestors of m? f, b, a
h) Which vertices are descendants of b? l, n, m, f, e

4. Answer the same questions as listed in Exercise 3 for the rooted tree illustrated.



NO

5. Is the rooted tree in Exercise 3 a full m-ary tree for some positive integer m?
6. Is the rooted tree in Exercise 4 a full m-ary tree for some positive integer m?
7. What is the level of each vertex of the rooted tree in Exercise 3?
8. What is the level of each vertex of the rooted tree in Exercise 4?
9. Draw the subtree of the tree in Exercise 3 that is rooted at
   a) a. b) c. c) e.
10. Draw the subtree of the tree in Exercise 4 that is rooted at
   a) a. b) c. c) e.
11. a) How many nonisomorphic unrooted trees are there with three vertices?
   b) How many nonisomorphic rooted trees are there with three vertices (using isomorphism for directed graphs)?

*12. a) How many nonisomorphic unrooted trees are there with four vertices?
   b) How many nonisomorphic rooted trees are there with four vertices (using isomorphism for directed graphs)?

*13. a) How many nonisomorphic unrooted trees are there with five vertices?
   b) How many nonisomorphic rooted trees are there with five vertices (using isomorphism for directed graphs)?

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*14. Show that a simple graph is a tree if and only if it is connected, but the deletion of any of its edges produces a graph that is not connected.

15. Let G be a simple graph with n vertices. Show that G is a tree if and only if G is connected and has $n - 1$ edges. [*Hint:* For the "if" part, use Exercise 14 and Theorem 2.]

16. Which complete bipartite graphs $K_{m,n}$, where m and n are positive integers, are trees?

17. How many edges does a tree with 10,000 vertices have?

18. How many vertices does a full 5-ary tree with 100 internal vertices have?

19. How many edges does a full binary tree with 1000 internal vertices have?

20. How many leaves does a full 3-ary tree with 100 vertices have?

21. Suppose 1000 people enter a chess tournament. Use a rooted tree model of the tournament to determine how many games must be played to determine a champion, if a player is eliminated after one loss and games are played until only one entrant has not lost. (Assume there are no ties.)

22. A chain letter starts when a person sends a letter to five others. Each person who receives the letter either sends it to five other people who have never received it or does not send it to anyone. Suppose that 10,000 people send out the letter before the chain ends and that no one receives more than one letter. How many people receive the letter, and how many do not send it out?

23. A chain letter starts with a person sending a letter out to 10 others. Each person is asked to send the letter out to 10 others, and each letter contains a list of the previous six people in the chain. Unless there are fewer than six names in the list, each person sends one dollar to the first person in this list, removes the name of this person from the list, moves up each of the other five names one position, and inserts his or her name at the end of this list. If no person breaks the chain and no one receives more than one letter, how much money will a person in the chain ultimately receive?

*24. Either draw a full m-ary tree with 76 leaves and height 3, where m is a positive integer, or show that no such tree exists.

*25. Either draw a full m-ary tree with 84 leaves and height 3, where m is a positive integer, or show that no such tree exists.

*26. A full m-ary tree T has 81 leaves and height 4.
 a) Give the upper and lower bounds for m.
 b) What is m if T is also balanced?

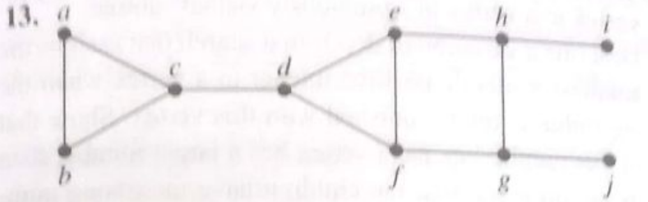
a) K_3 b) K_4 c) $K_{2,2}$ d) C_5

*12. How many nonisomorphic spanning trees does each of these simple graphs have?

a) K_3 b) K_4 c) K_5

In Exercises 13–15 use depth-first search to produce a spanning tree for the given simple graph. Choose a as the root of this spanning tree and assume that the vertices are ordered alphabetically.

13.



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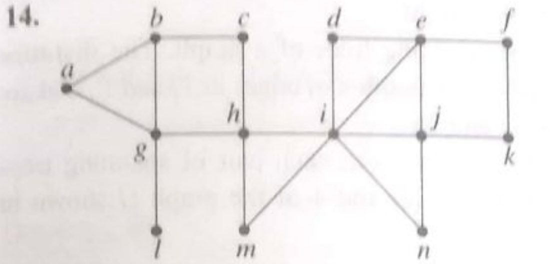

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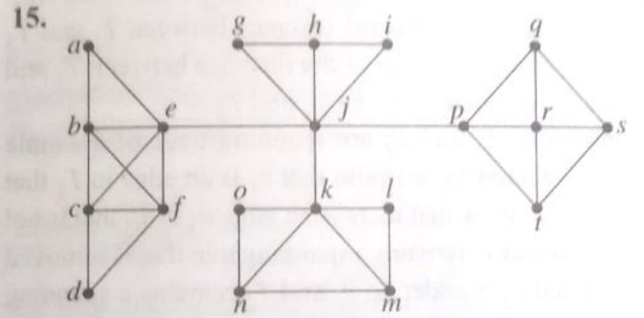


16. Use breadth-first search to produce a spanning tree for each of the simple graphs in Exercises 13–15. Choose a as the root of each spanning tree.

17. Use depth-first search to find a spanning tree of each of these graphs.

a) W_6 (see Example 7 of Section 8.2), starting at the vertex of degree 6

b) K_5

c) $K_{3,4}$, starting at a vertex of degree 3

d) Q_3

18. Use breadth-first search to find a spanning tree of each of the graph in Exercise 17.

19. Describe the trees produced by breadth-first search and depth-first search of the wheel graph W_n, starting at the vertex of degree n, where n is an integer with $n \geq 3$. (See Example 7 of Section 8.2.) Justify your answers.

20. Describe the trees produced by breadth-first search and depth-first search of the complete graph K_n, where n is a positive integer. Justify your answers.

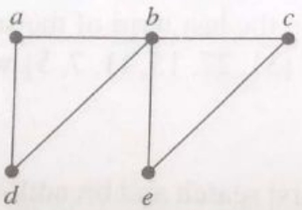
21. Describe the trees produced by breadth-first search and depth-first search of the complete bipartite graph $K_{m,n}$,

Exercises

1. How many edges must be removed from a connected graph with n vertices and m edges to produce a spanning tree?

In Exercises 2–6 find a spanning tree for the graph shown by removing edges in simple circuits.

2.



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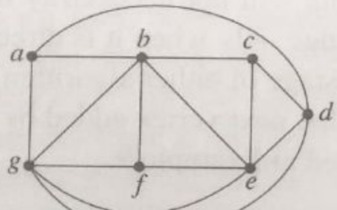

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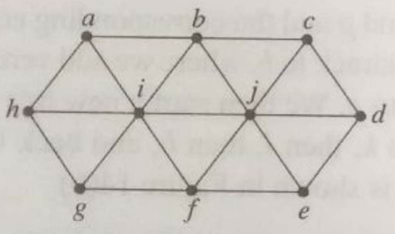

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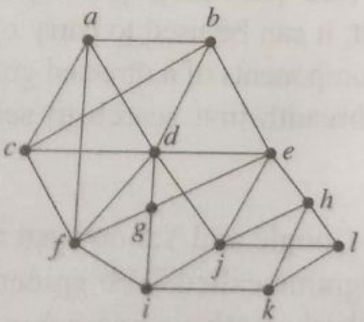
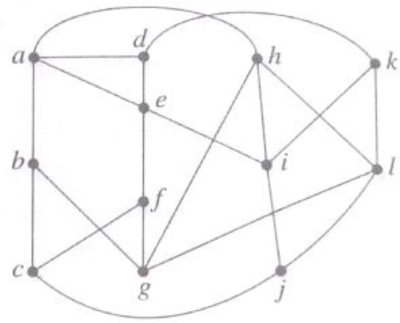


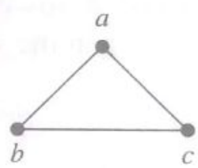
6.



7. Find a spanning tree for each of these graphs.
   a) K_5 b) $K_{4,4}$ c) $K_{1,6}$
   d) Q_3 e) C_5 f) W_5

In Exercises 8–10 draw all the spanning trees of the given simple graphs.

8.



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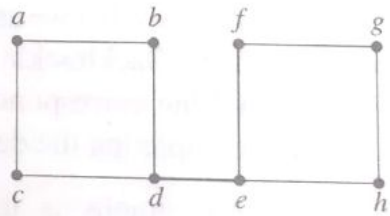

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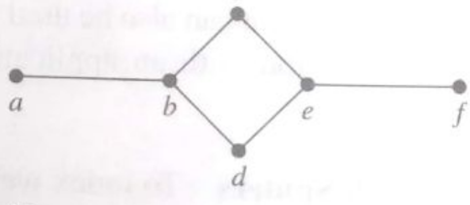
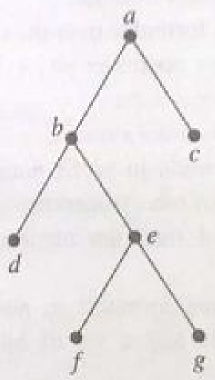


*11. How many different spanning trees does each of these simple graphs have?

In Exercises 7–10 determine the order in which a preorder traversal visits the vertices of the given ordered rooted tree.

7.

8.



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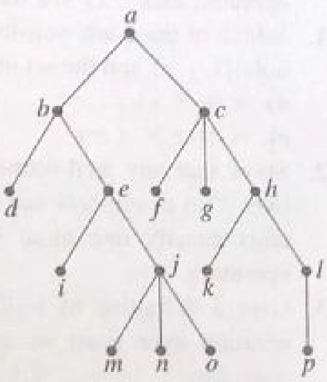

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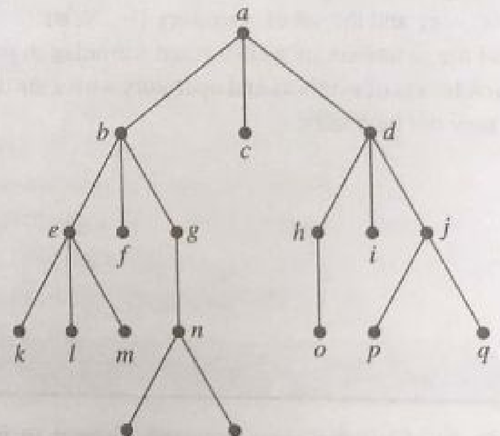

11. In which order are the vertices of the ordered rooted tree in Exercise 7 visited using an inorder traversal?

12. In which order are the vertices of the ordered rooted tree in Exercise 8 visited using an inorder traversal?

13. In which order are the vertices of the ordered rooted tree in Exercise 9 visited using an inorder traversal?

14. In which order are the vertices of the ordered rooted tree in Exercise 7 visited using a postorder traversal?

15. In which order are the vertices of the ordered rooted tree in Exercise 8 visited using a postorder traversal?

16. In which order are the vertices of the ordered rooted tree in Exercise 9 visited using a postorder traversal?

17. **a)** Represent the expression $((x + 2) \uparrow 3) * (y - (3 + x)) - 5$ using a binary tree.

Write this expression in

b) prefix notation.

c) postfix notation.

d) infix notation.

18. **a)** Represent the expressions $(x + xy) + (x/y)$ and $x + ((xy + x)/y)$ using binary trees.

Write these expressions in

b) prefix notation. **c)** postfix notation.

d) infix notation.

19. **a)** Represent the compound propositions $\neg(p \land q) \leftrightarrow (\neg p \lor \neg q)$ and $(\neg p \land (q \leftrightarrow \neg p)) \lor \neg q$ using ordered rooted trees.

Write these expressions in

b) prefix notation.

c) postfix notation.

d) infix notation.

20. **a)** Represent $(A \cap B) - (A \cup (B - A))$ using an ordered rooted tree.

Write this expression in

b) prefix notation. **c)** postfix notation.

d) infix notation.

***21.** In how many ways can the string $\neg p \land q \leftrightarrow \neg p \lor \neg q$ be fully parenthesized to yield an infix expression?

***22.** In how many ways can the string $A \cap B - A \cap B - A$ be fully parenthesized to yield an infix expression?

23. Draw the ordered rooted tree corresponding to each of these arithmetic expressions written in prefix notation. Then write each expression using infix notation.

a) $+ * + - 5\ 3\ 2\ 1\ 4$ **b)** $\uparrow + 2\ 3 - 5\ 1$

c) $* / 9\ 3 + * 2\ 4 - 7\ 6$

24. What is the value of each of these prefix expressions?

a) $- * 2 / 8\ 4\ 3$ **b)** $\uparrow - * 3\ 3 * 4\ 2\ 5$

c) $+ - \uparrow 3\ 2 \uparrow 2\ 3 / 6 - 4\ 2$

d) $* + 3 + 3 \uparrow 3 + 3\ 3\ 3$

25. What is the value of each of these postfix expressions?

a) $5\ 2\ 1 - - 3\ 1\ 4 + + *$

b) $9\ 3 / 5 + 7\ 2 - *$

c) $3\ 2 * 2 \uparrow 5\ 3 - 8\ 4 / * -$

26. Construct the ordered rooted tree whose preorder traversal is $a, b, f, c, g, h, i, d, e, j, k, l$, where a has four children, c has three children, j has two children, b and e have one child each, and all other vertices are leaves.

***27.** Show that an ordered rooted tree is uniquely determined when a list of vertices generated by a preorder traversal of the tree and the number of children of each vertex are specified.

***28.** Show that an ordered rooted tree is uniquely determined when a list of vertices generated by a postorder traversal of the tree and the number of children of each vertex are specified.

29. Show that preorder traversals of the two ordered rooted trees displayed below produce the same list of vertices. Note that this does not contradict the statement in Exercise 26, because the numbers of children of internal vertices in the two ordered rooted trees differ.