Sequence:→ A Sequence is a function from the set of Integers (usually either the set $\{0,1,2,\dots\}$ or the set $\{1,2,3,\dots\}$) to a set R of real numbers. We use the notation a_n to denote the image of the integer n. We call a_n a nth term of the sequence.

Ex:→ Consider the sequence $a_n = \frac{n-1}{n}, n \in N$. Find the terms of the sequence a_n.

Sol:→ We have

$$a_n = \frac{n-1}{n}, n \in N.$$

Putting $n = 1, 2, 3, \dots,$ in $a_n = \frac{n-1}{n}$, we get

$$a_1 = \frac{1-1}{1} = \frac{0}{1} = 0, \quad a_2 = \frac{2-1}{2} = \frac{1}{2}, \quad a_3 = \frac{3-1}{3} = \frac{2}{3}, \dots$$

$$\therefore a_1 = 0, \; a_2 = \frac{1}{2}, \; a_3 = \frac{2}{3}, \dots$$

Arithmetic Progression:→ An arithmetic progression is a sequence of the form

$a, a+d, a+2d, \dots, a+(n-1)d, \dots$, where a is the first term and d is a common difference.

e.g.: ① The sequence $1, 2, 3, \dots$ is a Arithmetic Progression.

② The sequence $2, 4, 6, 8, \dots$ is an Arithmetic progression.

Geometric progression:→

A geometric progression is a sequence of the form $a, ar, ar^2, \cdots$, where a is a first term and r is a common difference.

e.g. (1) The sequence $1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \cdots$ is a geometric progression.

Recurrence Relation:→

A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \cdots, a_{n-1}$ for all non-negative integers n.

Ex: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1, 2, 3, \cdots$ and suppose that $a_0 = 2$. Find a_1, a_2 and a_3?

Sol:→ We have

$$a_n = a_{n-1} + 3 \text{ for } n = 1, 2, 3, \cdots, \; a_0 = 2$$

$$\therefore a_1 = a_{1-1} + 3 = a_0 + 3 = 2 + 3$$

$$\Rightarrow \boxed{a_1 = 5}$$

Also, $a_2 = a_{2-1} + 3 = a_1 + 3 = 5 + 3 = 8$

$$\Rightarrow \boxed{a_2 = 8}$$

Finally, $a_3 = a_{3-1} + 3 = a_2 + 3 = 8 + 3 = 11$

$$\Rightarrow \boxed{a_3 = 11}$$

Ex:→ Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n=2,3,\ldots$ and suppose that $a_0 = 3$ and $a_1 = 5$. Find a_2 and a_3.

Sol:→ We have

$a_n = a_{n-1} - a_{n-2}$ for $n=2,3,\ldots$. Also $a_0 = 3$ and $a_1 = 5$.

Putting $n=2$, we get

$$a_2 = a_{2-1} - a_{2-2} = a_1 - a_0 = 5-3 = 2$$

$$\Rightarrow \boxed{a_2 = 2}$$

Putting $n=3$, we get

$$a_3 = a_{3-1} - a_{3-2} = a_2 - a_1 = 2-5 = -3$$

<u>Solution of a recurrence relation</u>:→

A sequence is called a solution of a recurrence relation if its terms satisfy the ~~equation~~ recurrence relation.

Ex:→ Determine whether the sequence $\{a_n\}$, where $a_n = 3n$ for every non-negative integer n, is a solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2,3,4,\ldots$ Answer the same question where $a_n = 2^n$ and where $a_n = 5$.

Sol:→ Suppose $a_n = 3n$ for every non-negative integer n. Then, for $n \geq 2$, we have

$$2a_{n-1} - a_{n-2} = 2[3(n-1)] - 3(n-2)$$
$$= 6n - 6 - 3n + 6 = 3n = a_n$$

Therefore, $\{a_n\}$ where $a_n = 3n$, is a solution of the recurrence relation.

Suppose that $a_n = 2^n$ for every non-negative integer n. Note $a_0 = 1$, $a_1 = 2$ and $a_2 = 4$.

For $n = 2$, we have

$$2a_{n-1} - a_{n-2} = 2a_{2-1} - a_0 = 2a_1 - a_0$$
$$= 2 \times 2 - 1 = 3 \neq 4 = a_2$$

Hence, $a_n = 2^n$ is not a solution of the recurrence relation

Suppose that $a_n = 5$ for every non negative n. Then for $n \geq 2$, we see that

$$2a_{n-1} - a_{n-2} = 2 \times 5 - 5 = 5 = a_n$$

Therefore $a_n = 5$ is a solution of the recurrence relation.

Modelling with Recurrence Relations:→
(Rabbit and the Fibonacci Numbers):

A young pair of rabbits (one of each sex) is placed on an island. A pair of rabbits does not breed until they are 2 months old. After they are 2 months old, each pair of rabbits produces another pair each month. Find a recurrence relation for the number of pairs of rabbits on the island after n months, assuming that no rabbits ever die.

Sol. Let M_i denote the male rabbit of age i and F_j denote the female rabbit of age j, so that $M_i F_i$ denote a rabbit pair of age i.

Thus, according to this situation, the number of pairs produced on the island in the k^{th} month can be modelled as follows:

Month 1	$M_1 F_1$	1 pair
Month 2	$M_2 F_2$	1 pair
Month 3	$M_3 F_3$ $M_1 F_1$ (New born)	2 pairs
Month 4	$M_4 F_4$ $M_2 F_2$ $M_1 F_1$ (New born from $M_3 F_3$)	3 pairs
Month 5	$M_5 F_5$ $M_1 F_1$ $M_3 F_3$ $M_1 F_1$ $M_2 F_2$	5 pairs
Month 6	$M_6 F_6$ $M_1 F_1$ $M_2 F_2$ $M_4 F_4$ $M_1 F_1$ $M_2 F_2$ $M_3 F_3$ $M_1 F_1$	8 pairs.

Denote by f_n the number of pairs of rabbits after n months. We will show that f_n, $n = 1, 2, 3, \ldots$, are the terms of the Fibonacci Sequence.

The rabbit population can be modeled using a recurrence relation. At the end of the first month, the number of pairs of rabbits on the island is $f_1 = 1$. Because this pair does not breed during the second month, $f_2 = 1$ also. To find the number of pairs after n months, add the number on the island the previous month, f_{n-1}, and the number of new born pairs, which equals f_{n-2}, because each new pair comes from a pair atleast 2 months old.

Consequently, the sequence $\{f_n\}$ satisfies the recurrence relation

$$f_n = f_{n-1} + f_{n-2}$$

for $n \geq 2$ together with the intial conditions $f_1 = 1$ and $f_2 = 1$. Because this recurrence relation and the intial conditions uniquely determines this sequence, the number of pairs of rabbits on the island after n months is given by the nth Fibonacci number.

Fibonacci Number:→ The Fibonacci numbers, commonly denoted by F_n, form a sequence, the Fibonacci sequence, in which each is the sum of the two preceding ones. The sequence commonly starts from 0 and 1 or from 1 and 1. In general, we have $F_n = F_{n-1} + F_{n-2}$, with $F_0 = 1, F_1 =$ and $n \geq 3$. e.g 1, 1, 2, 3, 5, 8, 13 and 21.

__Ex:__ (The Tower of Hanoi)

Tower of Hanoi puzzle consists of three pegs mounted on a board together with disks of different sizes. Intially these disks are placed on the first peg in order of size, with the largest on the bottom (as shown in Figure 1). The rules of the puzzle allow disks to be moved one at a time from one peg to another as long as a disk is never placed on top of a smaller disk. The goal of the puzzle is to have all the disks on the second peg in order of size with the largest on the bottom.

Sol:→ Let H_n denote the number of moves needed to solve the puzzle with n disks.

We start with n disks on peg 1. We can transfer the top $n-1$ disks ~~following the rules of the puzzle~~ ~~to peg 3 by applying the rules of the puzzle~~ to peg 3 using H_{n-1} moves by applying the rules of the puzzle. We keep the largest disk fixed during these moves. Then, we use one move to transfer the largest disk to the second peg. Finally, we can ~~transfer~~ the $n-1$ disks on peg 3 to peg 2 using H_{n-1} additional moves, placing them on top of the largest disk, which always stays fixed on the bottom of peg 2. This shows that

$$H_n = 2H_{n-1} + 1..$$

with the intial condition $H_1 = 1$, because one disk can be transfered from peg 1 to peg 2, in one move, according to the rules of the puzzle.

We can use an iterative approach to solve the recurrence relation. Note that

$$H_n = 2H_{n-1} + 1$$

$$= 2(2H_{n-2}+1) + 1 = 2^2 H_{n-2} + 2 + 1$$

$$= 2^2(2H_{n-3}+1) + 2 + 1 = 2^3 H_{n-3} + 2^2 + 2 + 1$$

$$\vdots$$

$$= 2^{n-1} H_{n-(n-1)} + 2^{n-2} + 2^{n-3} + \cdots + 2^2 + 2 + 1$$

$$= 2^{n-1} H_1 + 2^{n-2} + 2^{n-3} + \cdots + 2^2 + 2 + 1$$

$$= 2^{n-1} \times 1 + 2^{n-2} + 2^{n-3} + \cdots + 2^2 + 2 + 1 \quad (\because H_1 = 1)$$

$$= 2^{n-1}\left[\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}\right] \quad \left[\begin{array}{l}\because \text{Given series is a G.P} \\ \text{with first term } 2^{n-1} \text{ and} \\ \text{Common ratio } \frac{1}{2} < 1\end{array}\right]$$

$$= \frac{2^{n-1}\left[1-\frac{1}{2^n}\right]}{\frac{1}{2}} = 2^n\left[1-\frac{1}{2^n}\right] = 2^n - 1$$

$$\therefore H_n = 2^n - 1.$$

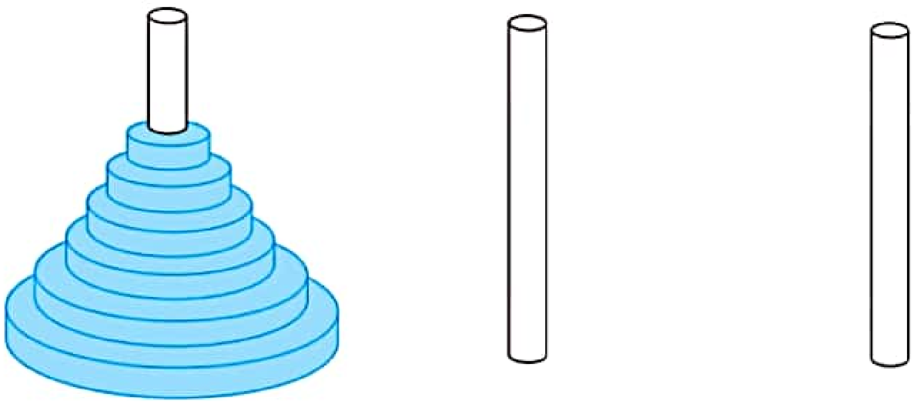

FIGURE 2 **The Initial Position in the Tower of Hanoi.**

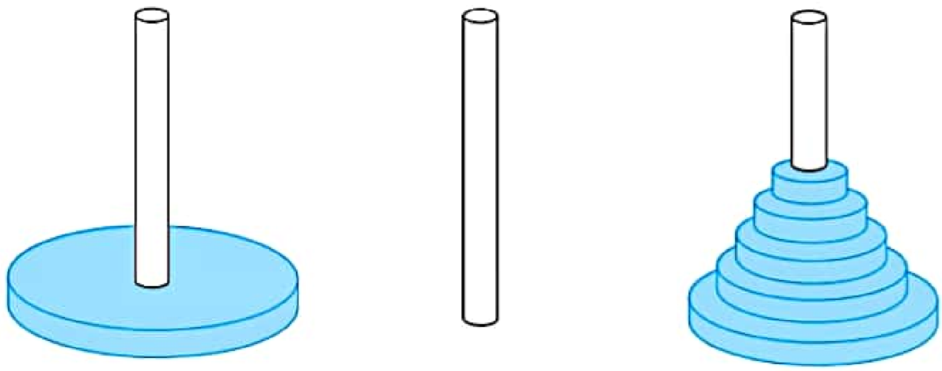

FIGURE 3 **An Intermediate Position in the Tower of Hanoi.**

Definition:→ A linear homogeneous recurrence relation of degree K with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_K a_{n-K} \quad \text{—— (1)}$$

where $c_1, c_2, \ldots, c_K$ are real numbers and $c_K \neq 0$.

The recurrence relation (1) is linear because the right-hand side is a sum of previous terms of the sequence each multiplied by a function of n. The recurrence relation (1) is also homogeneous because no terms occur that are not multiples of the a_j's. The coefficients of the terms of the sequence are all constants, rather than functions that depend on n. The degree is K ~~because~~ because a_n is expressed in terms of the previous K terms of the sequence.

Examples:→

(1) $a_n = (1.5)\, a_{n-1}$ is a linear homogeneous recurrence relation.

(2) $f_n = f_{n-1} + f_{n-2}$ is a linear homogeneous recurrence relation.

(3) $a_n = 5a_{n-1} + n^2$ is not homogeneous recurrence relation

(4) $a_n = a_{n-1} + a_{n-2}^2$ is a homogeneous but not linear recurrence relation.

⑤ $a_n = a_{n-1}^3 + a_{n-2} + a_{n-3}a_{n-4}$ is homogenous but not linear recurrence relation.

⑥ $H_n = 2H_{n-1} + 1$ is not homogeneous

Solving Linear Homogeneous Recurrence Relations with Constant Coefficients:→

The basic approach for solving linear homogeneous recurrence relation is to look for solutions of the form

$a_n = r^n$, where r is a constant.

Note that $a_n = r^n$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$ —— ①

if and only if

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \dots + c_{k-1} r^{n-k+1} + c_k r^{n-k}$$

$$\Rightarrow \frac{r^n}{r^{n-k}} = \frac{c_1 r^{n-1} + c_2 r^{n-2} + \dots + c_{k-1} r^{n-k+1} + c_k r^{n-k}}{r^{n-k}}$$

$$\Rightarrow r^k = c_1 r^{k-1} + c_2 r^{k-2} + \dots + c_{k-1} r + c_k$$

or $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \dots - c_{k-1} r - c_k = 0$ —— ②

Hence, the sequence $\{a_n\}$, where $a_n = r^n$ is a solution if and only if r is a solution of equation ②. Equation ② is called the characteristic equation of the recurrence relation ①. The solutions of ② are called the characteristic roots of ①.

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Theorem 1:→ Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then, the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n \text{ for } n = 0, 1, 2, \ldots,$$

where α_1 and α_2 are constants.

Ex:→ Find the solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$.

Sol:→ The characteristic equation of the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ is

$$r^2 - 1 \cdot r^{2-1} - 2 = 0 \quad \text{i.e } r^2 - r - 2 = 0$$

$$\Rightarrow r^2 - 2r + r - 2 = 0 \Rightarrow r(r-2) + 1(r-2) = 0$$

$$\text{or } (r-2)(r+1) = 0 \Rightarrow r = 2, -1.$$

Hence, r has two distinct values. Hence, the sequence $\{a_n\}$ is a solution to the recurrence relation if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

$$\text{i.e } a_n = \alpha_1 2^n + \alpha_2 (-1)^n \quad \text{—— (1)}$$

From the intial conditions, we get $a_0 = 2$. But from $a_0 = \alpha_1 2^0 + \alpha_2 (-1)^0$

$$\Rightarrow \alpha_1 + \alpha_2 = 2$$

Also $a_1 = 7$, there $a_1 = \alpha_1 2^1 + \alpha_2 (-1)^1 = 7$

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$\Rightarrow 2\alpha_1 - \alpha_2 = 7$

Hence, we have

$$\alpha_1 + \alpha_2 = 2$$
$$2\alpha_1 - \alpha_2 = 7$$

$3\alpha_1 = 9 \Rightarrow \boxed{\alpha_1 = 3}$.

Also $\alpha_2 = 2 - \alpha_1 = 2 - 3 \Rightarrow \boxed{\alpha_2 = -1}$

Subs. the values of α_1 & α_2 in (1), we get

$$a_n = 3.2^n - (-1)^n.$$

Ex:→ Solve the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ with the intial conditions

$$a_1 = 1.5 \text{ and } a_2 = 3.$$

Sol:→ Consider the recurrance relation

$$a_n = 2a_{n-1} - a_{n-2} \quad \text{——} (1)$$

Its characteristic equation is

$$q^2 - 2q - (-1) = 0 \text{ or } q^2 - 2q + 1 = 0$$
$$\Rightarrow (q-1)^2 = 0 \quad \text{or } q = 1, 1$$

Hence, the solution of the recurrance relation (1) is

$$a_n = \alpha_1 (1)^n + n\alpha_2 (1)^n \text{ or } a_n = \alpha_1 + n\alpha_2 \text{ —} (2)$$
(using Theorem 2)

Since $a_1 = 1.5$, $\therefore$ $a_1 = 1.5 = \alpha_1 + 1.\alpha_2$

$\Rightarrow \alpha_1 + \alpha_2 = 1.5$

Also, $a_2 = 3$, $\therefore$ $a_2 = 3 = \alpha_1 + 2\alpha_2 \Rightarrow \alpha_1 + 2\alpha_2 = 3.$

Hence, we have

$$\alpha_1+\alpha_2=1.5$$
$$\underline{\overset{-}{\alpha_1}+\overset{-}{2\alpha_2}=\overset{-}{3}}$$

$-\alpha_2=-1.5 \Rightarrow \boxed{\alpha_2=1.5}$

Also, $\alpha_1=1.5-\alpha_2=1.5-1.5$
$\Rightarrow \boxed{\alpha_1=0}$.

Substituting the values of α_1 & α_2 in ②, we get

$$\boxed{a_n=1.5n}$$

Theorem 2:→ Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2-c_1r-c_2=0$ has only one root r_0. A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n=c_1a_{n-1}+c_2a_{n-2}$ if and only if $a_n=\alpha_1 r_0^n+\alpha_2 n r_0^n$ for $n=0,1,2,\ldots$, where α_1 and α_2 are constants.

Ex:→ Find an explicit formula for the Fibonacci numbers.

Sol:→ Fibonacci numbers satisfies the recurrence relation $f_n=f_{n-1}+f_{n-2}$ with the intial conditions $f_0=0$ and $f_1=1$. Its characteristic equation is

$$r^2-r-1=0$$

$$r=\frac{1\pm\sqrt{(-1)^2-4(1)(-1)}}{2\times1}=\frac{1\pm\sqrt{5}}{2}$$

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$\therefore r_1 = \frac{1+\sqrt{5}}{2}$ and $r_2 = \frac{1-\sqrt{5}}{2}$

Hence, the roots are distinct. Hence, the sequence $\{f_n\}$ is a solution to the recurrence relation if and only if

$$\cancel{a_n = \alpha_1 2^n + \alpha_2 (-1)^n} \quad f_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

$$\Rightarrow f_n = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right)^n \text{ --- } (1) \quad \text{(By theorem 1)}$$

Put $n = 0$, we get

$$f_0 = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right)^0 + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right)^0 = \alpha_1 \times 1 + \alpha_2 \times 1$$

$\Rightarrow f_0 = \alpha_1 + \alpha_2$. But $f_0 = 0$

$\therefore \alpha_1 + \alpha_2 = 0 \quad \cancel{\Rightarrow \boxed{\alpha_1 = -\alpha_2}}$

$$\Rightarrow \boxed{\alpha_1 = -\alpha_2}$$

Put $n=1$, we get

$$f_1 = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right)^1 + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right)^1$$

$$f_1 = -\alpha_2\left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right) \quad (\because \alpha_1 = -\alpha_2)$$

$$f_1 = \alpha_2\left[\frac{-1-\sqrt{5}+1-\sqrt{5}}{2}\right] = \alpha_2 \times \frac{-2\sqrt{5}}{2}$$

$$\Rightarrow f_1 = -\sqrt{5}\,\alpha_2 \quad \text{or} \quad \alpha_2 = -\frac{f_1}{\sqrt{5}}$$

But $f_1 = 1$, hence $\boxed{\alpha_2 = \frac{-1}{\sqrt{5}}}$

But $\alpha_1 = -\alpha_2 = -\left(\frac{-1}{\sqrt{5}}\right)$ or $\alpha_1 = \frac{1}{\sqrt{5}}$

Substituting the values of α_1 and α_2 in (1), we get

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Q:→ Find the solution of the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with intial conditions $a_0 = 1$ and $a_1 = 6$.

Sol:→ The given recurrence relation is

$$a_n = 6a_{n-1} - 9a_{n-2}$$

Its characteristic eqn is

$$r^2 - 6r - (-9) = 0$$

or $r^2-6r+9=0$

or $(r-3)^2=0 \Rightarrow r=3,3.$

Hence, the characteristic roots are of multiplicity two (i.e repeated roots).

Therefore, the solution to the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ is

$$a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$$

or $a_n = \alpha_1 3^n + \alpha_2 n . 3^n$ ——— ①

Putting $n=0$, we get

$$a_0 = \alpha_1 . 3^0 + \alpha_2 \times 0 \times 3^0 \Rightarrow a_0 = \alpha_1$$

But $a_0 = 1$, hence $\boxed{\alpha_1 = 1}$

Putting $n=1$, we get

$$a_1 = \alpha_1 3^1 + \alpha_2 \times 1 \times 3^1 = 3(\alpha_1 + \alpha_2)$$

$$= 3(1+\alpha_2) \quad (\because \alpha_1 = 1).$$

But $a_1 = 6$, therefore $6 = 3(1+\alpha_2)$

$\Rightarrow 2 = 1+\alpha_2$ or $\boxed{\alpha_2 = 1}$.

Subs. the values of α_1 & α_2 in ①, we get

$$a_n = 1.3^n + 1 \times n \times 3^n$$

$$a_n = (1+n)3^n.$$

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Theorem 3:→ Let $c_1, c_2, \cdots, c_k$ be real numbers. Suppose that the characteristic equation
$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$
has k distinct roots $r_1, r_2, \cdots, r_k$. Then, the sequence $\{a_n\}$ is a solution of the recurrence relation
$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$
if and only if
$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$$
for $n = 0, 1, 2, \cdots$, where $\alpha_1, \alpha_2, \cdots, \alpha_k$ are constants.

Ex:→ Find the solution to the recurrence relation
$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$
with the intial conditions $a_0 = 2$, $a_1 = 5$ and $a_2 = 15$.

Sol:→ The characteristic equation of the recurrence relation $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ is
$$r^3 - 6r^2 - (-11)r - 6 = 0$$
i.e $r^3 - 6r^2 + 11r - 6 = 0$

Clearly, $r = 1$ is a root.

$\therefore$

1)	1	-6	11	-6
		1	-5	6
	1	-5	6	0

$\therefore r^2 - 5r + 6 = 0$ or $r^2 - 3r - 2r + 6 = 0$

$\Rightarrow r(r-3) - 2(r-3) = 0$ or $(r-3)(r-2) = 0$

Hence, $r = 1, 2, 3$.

Therefore, the solutions of the recurrence relation are of the form

$$a_n = \alpha_1 (1)^n + \alpha_2 (2)^n + \alpha_3 (3)^n \quad \text{—— (1)}$$

Put $n = 0$, we get

$a_0 = \alpha_1 (1)^0 + \alpha_2 2^0 + \alpha_3 \cdot 3^0$. But $a_0 = 2$.

$\Rightarrow \alpha_1 + \alpha_2 + \alpha_3 = 2$

Put $n = 1$, we get $a_1 = \alpha_1 (1)^1 + \alpha_2 2^1 + \alpha_3 3^1$

But $a_1 = 5$, we get

$$\alpha_1 + 2\alpha_2 + 3\alpha_3 = 5$$

Puting $n = 2$ in (1), we get $a_2 = \alpha_1 (1)^2 + \alpha_2 2^2 + \alpha_3 \cdot 3^2$

But $a_2 = 15$, therefore

$$\alpha_1 + 4\alpha_2 + 9\alpha_3 = 15$$

Hence, we have three equations as follows

$$\alpha_1 + \alpha_2 + \alpha_3 = 2 \quad \text{—— (2)}$$

$$\alpha_1 + 2\alpha_2 + 3\alpha_3 = 5 \quad \text{—— (3)}$$

$$\alpha_1 + 4\alpha_2 + 9\alpha_3 = 15 \quad \text{—— (4)}$$

Substracting (2) & (3), we get

$$\begin{array}{r} \alpha_1 + \alpha_2 + \alpha_3 = 2 \\ \alpha_1 + 2\alpha_2 + 3\alpha_3 = 5 \\ - \quad - \quad - \quad - \\ \hline -\alpha_2 - 2\alpha_3 = -3 \end{array}$$

or $\boxed{\alpha_2 + 2\alpha_3 = 3}$ —— (5)

Substracting ③ & ④, we get

$$\begin{array}{r} \alpha_1 + 2\alpha_2 + 3\alpha_3 = 5 \\ -\alpha_1 \pm 4\alpha_2 \pm 9\alpha_3 = \pm 15 \\ \hline -2\alpha_2 - 6\alpha_3 = -10 \end{array}$$

or $\boxed{\alpha_2 + 3\alpha_3 = 5}$ — ⑥

From ⑤ & ⑥, we get

$$\begin{array}{r} \alpha_2 + 2\alpha_3 = 3 \\ \alpha_2 + 3\alpha_3 = 5 \\ - \quad - \quad - \\ \hline -\alpha_3 = -2 \end{array} \Rightarrow \boxed{\alpha_3 = 2}$$

From ⑥, $\alpha_2 = 5 - 3\alpha_2 = 5 - 3 \times 2$

$\Rightarrow \boxed{\alpha_2 = -1}$

From ②, we get $\alpha_1 = 2 - \alpha_2 - \alpha_3$

$= 2 - (-1) - 2$

$\Rightarrow \boxed{\alpha_1 = 1}$.

Substituting the values of α_1, α_2 and α_3 in ①, we get the unique solution to this recurrence relation as

$$a_n = 1 - 2^n + 2 \cdot 3^n.$$

Theorem 4:→ Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$, so that $m_i \geq 1$ for $i = 1, 2, \ldots, t$ and $m_1 + m_2 + \cdots + m_t = k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$a_n = (\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n$$
$$+ (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n$$
$$+ \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n$$

for $n = 0, 1, 2, \ldots$, where α_{ij} are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i - 1$.

Ex:→ Suppose that the roots of the characteristic equation of a linear homogeneous recurrence relation are 2, 2, 2, 5, 5 and 9. What is the form of the general solution.

Sol:→ The general solution is given by

$$(\alpha_1 + \alpha_2 n + \alpha_3 n^2) 2^n + (\alpha_4 + \alpha_5 n) 5^n + \alpha_6 9^n$$

Ex:→ Find the solution to the recurrence relation (using theorem 4)

$$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$$

with initial conditions $a_0 = 1$, $a_1 = -2$ and $a_2 = -1$.

Sol:→ The characteristic equation of the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ is

$$r^3 - (-3)r^2 - (-3)r - (1) = 0$$

or $r^3 + 3r^2 + 3r + 1 = 0$ or $(r+1)^3 = 0$

$\therefore r = -1, -1, -1$

Hence, the solution of this recurrence relation are of the form

$$a_n = \alpha_1 (-1)^n + \alpha_2 n \cdot (-1)^n + \alpha_3 n^2 (-1)^n \text{ —— } *$$

$$\Rightarrow a_n = (\alpha_1 + n\alpha_2 + n^2\alpha_3)(-1)^n \text{ —— } (1)$$

Put $n=0$, $a_0 = (\alpha_1 + 0 + 0)(-1)^0$

But $a_0 = 1$ $\Rightarrow \boxed{\alpha_1 = 1}$ —— (2)

Put $n=1$, $\therefore a_1 = (\alpha_1 + \alpha_2 + \alpha_3)(-1)^1$

But $a_1 = -2$, $\therefore -2 = -(\alpha_1 + \alpha_2 + \alpha_3)$

$\Rightarrow \boxed{1 + \alpha_2 + \alpha_3 = 2}$ $(\because \alpha_1 = 1)$.

$\Rightarrow \alpha_2 + \alpha_3 = 1$ —— (3)

Put $n=2$, $\therefore a_2 = (\alpha_1 + 2\alpha_2 + 4\alpha_3)(-1)^2$

But $a_2 = -1$, $\therefore -1 = (1 + 2\alpha_2 + 4\alpha_3)$

$\Rightarrow 2\alpha_2 + 4\alpha_3 = -2$ —— (4).

Multiplying (3) by 2 and adding (4), we get

$$\begin{array}{r} 2\alpha_2 + 2\alpha_3 = 2 \\ 2\alpha_2 + 4\alpha_3 = -2 \\ - \quad - \quad + \\ \hline -2\alpha_3 = 4 \end{array} \Rightarrow \boxed{\alpha_3 = -2}$$

From (3), we get $\alpha_2 = 1-\alpha_3 = 1-(-2)$

$$\boxed{\alpha_2 = 3}$$

Substituting values of α_1, α_2 and α_3 in (*), we get a unique solution to the given recurrence relation as

$$a_n = (1+3n-2n^2)(-1)^n.$$

Ex:→ Consider the recurrence relation

$$a_{n+2} = 2a_{n+1} + 4a_n - 8a_{n-1}$$

Find explicit formula.

Ex:→ Solve the recurrence relation

$$a_n = 4a_{n-2}$$

where (1) $a_0 = 4$ and $a_1 = 6$

(2) $a_0 = 6$ and $a_2 = 20$

(3) $a_1 = 6$ and $a_2 = 20$

Linear Non-Homogeneous Recurrence Relation:→

A linear non-homogeneous recurrence relation with constant coefficients is a recurrence of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + g(n)$$

where $c_1, c_2, \ldots, c_k$ are real numbers and $g(n)$ is a function of n only such that $g(n) \neq 0$.

The recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ is the associated homogeneous recurrence relation. It plays an important role in the solution of the non-homogeneous recurrence relation.

Ex:→ $a_n = 3a_{n-1} + 2n$ is a linear ~~non homogeneou~~ non-homogeneous recurrence relation with constant coefficients. Its associated homogeneous recurrence relation is $a_n = a_{n-1}$;

Ex:→ The recurrent relation

$a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$ is a linear non-homogeneous recurrence relation with constant coefficients. Its associated homogeneous recurrence relation is

$$a_n = a_{n-1} + a_{n-2}$$

Theorem:→ ~~The shift operator E is defined~~

If $\{a_n^p\}$ is a particular solution of the non-homogeneous linear recurrence relation with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$$

then every solution is of the form $\{a_n^p + a_n^h\}$, where $\{a_n^h\}$ is a solution of the associated homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-1} + \cdots + c_k a_{n-k}$$

Shift operator:→ The shift operator E is defined as the operator that increases the argument of a function by one tabular interval. Thus,

$$E a_n = a_{n+1}$$

$$\Rightarrow E^2 a_n = E(E a_n) = E(a_{n+1}) = a_{n+2}$$

$$\therefore E^2 a_n = a_{n+2}$$

Also, $E^3 a_n = E(E^2 a_n) = E(a_{n+2}) = a_{n+3}$

$$\therefore E^3 a_n = a_{n+3}$$

In general, we have $E^k a_n = a_{n+k}$

Difference operator:→ The difference operator Δ is defined by $\Delta a_n = a_{n+1} - a_n$

Relation between shift operator and Difference operator :

we have $\Delta a_n = a_{n+1} - a_n$

$$\Rightarrow \Delta a_n = E a_n - a_n$$

$$\Rightarrow \Delta a_n = (E-1) a_n$$

$$\Rightarrow \Delta = E-1$$

Ex: The recurrence relation $a_n = a_{n-1} + a_{n-2}$ can also be written as

$$a_{n+2} = a_{n+1} + a_n$$

Here, we used the shift operator E two times.

Ex: The recurrence relation

$a_n = 5a_{n-1} - 6a_{n-2} + 7^n$ can also be written as

$$a_{n+2} = 5a_{n+1} - 6a_n + 7^{n+2}$$

Here also, we used the shift operator two times.

Use of shift operator :→

The recurrence relation

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + g(n)$ can be written as, using shift operator as follows:

$$a_{n+k} = c_1 a_{n+k-1} + c_2 a_{n+k-2} + \dots + c_k a_{n+k-k} + g(n+k)$$

$$\Rightarrow a_{n+k} - c_1 a_{n+k-1} - c_2 a_{n+k-2} - \dots - c_{k-1} a_{n+1} - c_k a_n = g(n+k)$$

$$\Rightarrow E^K a_n - c_1 E^{K-1} a_n - c_2 E^{K-2} a_n - \cdots - c_{K-1} E^1 a_n - c_K a_n = f(n)$$

where $f(n) = g(n+K)$ $\quad (\because E^K a_n = a_{n+K})$

$$\Rightarrow \left(E^K - c_1 E^{K-1} - c_2 E^{K-2} - \cdots c_{K-1} E - c_K\right) a_n = f(n)$$

or $\quad F(E) a_n = f(n)$

where $F(E) = E^K - c_1 E^{K-1} - c_2 E^{K-2} - \cdots - c_{K-1} E - c_K$

Here, $F(m)$ is the characteristic equation for the associated recurrence relation and particular solution is given by

$$P.S = \frac{1}{F(E)} f(n)$$

<u>Case I</u>:→ If $f(n) = a^n$, then

$$P.S = \frac{1}{F(E)} a^n = \frac{1}{F(a)} a^n, \quad F(a) \neq 0.$$

Ex:→ Solve $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$

Sol:→ Using the shift operator, the given recurrence relation is given by

$$a_{n+2} = 5a_{n+2-1} - 6a_{n+2-2} + 7^{n+2}$$

or $a_{n+2} = 5a_{n+1} - 6a_n + 7^{n+2}$

$$\Rightarrow E^2 a_n = 5Ea_n - 6a_n + 7^{n+2} \quad (\because E^K a_n = a_{n+K})$$

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$\Rightarrow (E^2 - 5E + 6)a_n = 7^{n+2}$

Here, the characteristic eqn is given by

$m^2 - 5m + 6 = 0$

$\Rightarrow (m-3)(m-2) = 0$

$\Rightarrow m = 2, 3$

Therefore, the general solution of the associated homogeneous recurrence relation is given by

$$a_n^h = \alpha_1 3^n + \alpha_2 2^n$$

where α_1 and α_2 are constants.

$$P.S = \frac{1}{E^2 - 5E + 6} 7^{n+2} = 49 \frac{1}{E^2 - 5E + 6} 7^n$$

$$= 49 \frac{1}{(7)^2 - 5\times 7 + 6} 7^n = \frac{49}{20} 7^n = a_n^p$$

Thus, the required solution is

$$a_n = a_n^p + a_n^h = \frac{49}{20} 7^n + \alpha_1 3^n + \alpha_2 2^n.$$

<u>Case II:→</u> <u>Failure case:</u>

If $F(a) = 0$, then

$$P.S = \frac{1}{(E-a)^r} a^n = {}^nC_r \, a^{n-r}$$

Ex:→ Solve $a_n = 12a_{n-1} - 48a_{n-2} + 64a_{n-3} + 5.4^{n-3}$

<u>Sol:→</u> Using the shift operator, the given recurrence relation is given by

$$a_{n+3} = 12a_{n+3-1} - 48a_{n+3-2} + 64a_{n+3-3} + 5\,4^{n+3-3}$$

or $a_{n+3} = 12a_{n+2} - 48a_{n+1} + 64a_n + 5\,4^n$

$\Rightarrow a_{n+3} - 12a_{n+2} + 48a_{n+1} - 64a_n = 5 \cdot 4^n$

using $E^k a_n = a_{n+k}$, we get

$E^3 a_n - 12E^2 a_n + 48E^1 a_n - 64a_n = 5 \cdot 4^n$

$\Rightarrow (E^3 - 12E^2 + 48E - 64)a_n = 5 \cdot 4^n$

The characteristic equation is

$m^3 - 12m^2 + 48m - 64 = 0$

$\Rightarrow (m-4)^3 = 0$ or $m = 4, 4, 4$

Hence, the general solution of the associated homogeneous recurrence relation is given by

$$a_n^h = \alpha_1 4^n + \alpha_2 n 4^n + \alpha_3 n^2 4^n$$

where $\alpha_1, \alpha_2, \alpha_3$ are constants.

$$P.S = \frac{1}{(E-4)^3} 5 \cdot 4^n = 5 \frac{1}{(E-4)^3} 4^n \quad \text{(Case of failure)}$$

$$= 5 \cdot {}^nC_3\, 4^{n-3} = 5 \cdot \frac{n!}{3!(n-3)!} 4^{n-3} = a_n^p$$

$$= \frac{5}{3!} \frac{n(n-1)(n-2)(n-3)!}{(n-3)!} 4^{n-3} = \frac{5}{6} n(n-1)(n-2) 4^{n-3}$$

Hence, the required solution is

$$a_n = a_n^p + a_n^h = (\alpha_1 + \alpha_2 n + \alpha_3 n^2)4^n + \frac{5}{6} n(n-1)(n-2) 4^{n-3}$$

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Factorial Polynomial:→ It is defined as follows:

$$x^{(m)} = x(x-1)(x-2)(x-3)\cdots(x-(m-1))$$

So, $x^{(0)} = 1$ and $x^{(1)} = x$

Now, $x^{(2)} = x(x-1) = x^2 - x$

$\Rightarrow x^2 = x^{(2)} + x$ or $\boxed{x^2 = x^{(2)} + x^{(1)}}$

Also, $x^{(3)} = x(x-1)(x-2)$

$= x^3 - 3x^2 + 2x$

$\Rightarrow x^3 = x^{(3)} + 3x^2 - 2x$

$\Rightarrow x^3 = x^{(3)} + 3(x^{(2)} + x^{(1)}) - 2x^{(1)}$

$\Rightarrow \boxed{x^3 = x^{(3)} + 3x^{(2)} + x^{(1)}}$

Important Formulae

① $\Delta x^{(m)} = m x^{(m-1)}$

② $\frac{1}{\Delta} x^{(m)} = \frac{x^{(m+1)}}{(m+1)}$

Case 3:→ If $f(n) = n^d$ or any polynomial in n, replace E by $1+\Delta$ and expand $\frac{1}{F(1+\Delta)}$ in binomial series in ascending powers of Δ up to Δ^d.

Ex:→ Solve the recurrence relation $a_n = 3a_{n-1} + 2n$

Sol:→ Using the shift operator, the given recurrence relation $a_n = 3a_{n-1} + 2n$ is given by

$\therefore a_{n+1} = 3a_{n+1-1} + 2(n+1)$

or $a_{n+1} - 3a_n = 2(n+1)$

$E a_n - 3a_n = 2(n+1)$

or $(E-3)a_n = 2(n+1)$

The characteristic equation is

$m - 3 = 0 \Rightarrow m = 3.$

Therefore, the general solution of the associated homogeneous recurrence relation is given by

$a_n^h = \alpha 3^n$, where α is a constant

$P.S = a_n^p = \frac{1}{E-3} 2(n+1) = 2 \frac{1}{E-3} n+1$

$= 2 \frac{1}{1+\Delta-3}(n^{(1)}+1) = 2 \frac{1}{\Delta-2}(n^{(1)}+1)$

$= 2 \frac{1}{-2(1+\frac{\Delta}{-2})}(n^{(1)}+1) = -(1-\frac{\Delta}{2})^{-1}(n^{(1)}+1).$

$= -(1+\frac{\Delta}{2}+\frac{\Delta^2}{4}+\cdots)(n^{(1)}+1).$

$= -[n^{(1)}+1+\frac{1}{2}\Delta(n^{(1)}+1)+\frac{1}{4}\Delta^2(n^{(1)}+1)+\cdots]$

$= -[n^{(1)}+1+\frac{1}{2}\times 1 n^{(1-1)}+0+\frac{1}{4}\times 0+0+0+\cdots+0]$

$\cancel{= [n^{(1)}+\frac{1}{2}]} = -[n^{(1)}+1+\frac{1}{2}] = -[n+\frac{3}{2}]$

Hence, the required solution is

$a_n = a_n^{(h)} + a_n^{(p)} = \alpha 3^n - (n+\frac{3}{2}).$

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Case 4 :→ If $f(n) = a^n V(n)$, where $V(n)$ is a polynomial function in n, then

$$P.S = \frac{1}{F(E)}\{a^n V(n)\} = a^n \frac{1}{F(aE)} V(n).$$

Ex: Solve the recurrence relation

$$a_n = 20a_{n-1} - 100a_{n-2} + n \cdot 5^n$$

Sol:→ Using the shift operator, the given recurrence relation is given by

$$a_{n+2} = 20a_{n+2-1} - 100a_{n+2-2} + (n+2)5^{n+2}$$

or $a_{n+2} - 20a_{n+1} + 100a_n = (n+2)5^{n+2}$

or $(E^2 - 20E + 100)a_n = (n+2)5^{n+2}$

The characteristic equation is given by

$$m^2 - 20m + 100 = 0$$

or $(m-10)^2 = 0 \Rightarrow r = 10, 10$

Hence, the general solution of the associated recurrence relation is given by

$$a_n^h = \alpha_1 10^n + \alpha_2 n 10^n, \text{ where } \alpha_1 \text{ and } \alpha_2 \text{ are constants.}$$

$$P.S = \frac{1}{(E-10)^2}(n+2)5^{n+2} = 5^2 \frac{1}{(E-10)^2} 5^n(n+2)$$

$$= 5^2 5^n \frac{1}{(5E-10)^2}(n+2) = 25\,5^n \frac{1}{5^2(E-2)^2}(n+2)$$

$$= 5^n \frac{1}{(E-2)^2} n+2 = 5^n \frac{1}{(1+\Delta-2)^2}(n+2) = 5^n \frac{1}{(\Delta-1)^2}(n+2)$$

$$= 5^n \frac{1}{(1-\Delta)^2}(n+2) = 5^n (1-\Delta)^{-2}(n+2)$$

$$= 5^n (1+2\Delta+3\Delta^2+4\Delta^3+\dots)(n^{(1)}+2)$$

$$= 5^n \left(n^{(1)}+2 + 2\Delta(n^{(1)}+2) + 3\Delta^2(n^{(1)}+2) + \dots \right)$$

$$= 5^n \left(n^{(1)}+2+2(1\cdot n^{1-1}+0) + 0+0+\dots+0 \right)$$

$$= 5^n (n^{(1)}+2+2\times 1) = 5^n(n+4) = a_n^p$$

$$P.S = a_n^p = 5^n(n+4).$$

Hence, the required solution is

$$a_n = a_n^h + a_n^p = \alpha_1 10^n + \alpha_2 n 10^n + 5^n(n+4).$$

H.A

Ex:→ Find the general solution of the recurrence relation

① $a_n = 2a_{n-1} - a_{n-2} + 2^n$

② $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3} + 2^{n-3}$

③ $a_n = 8a_{n-1} - 16a_{n-2} + (n-2)2^n$

~~Ex→ Identify~~

Ex: Identify which of the following recurrence are linear, homogeneous, non-linear and non-homogeneous.

① $a_n = a_{n-1} + a_{n-2} + n + 3$

② $a_n = 3$ ③ $a_n = a_{n-1}^2$

④ $a_n = 3a_{n-1} + 4a_{n-2} + 6a_{n-3}$

⑤ $a_n = \frac{1}{n} a_{n-1}$ ⑥ $a_n = 2n a_{n-1} + a_{n-2}$

Ex! Solve the recurrence relation

$$a_{n+1} - 2a_n = n^2$$

Sol:→ Using the shift operator, the ~~given~~ recurrence relation $a_{n+1} - 2a_n = n^2$ is given by

$$\cancel{a_{n+1} - 2a_n = n^2}$$

$$E^1 a_n - 2a_n = n^2 \text{ or } (E-2)a_n = n^2$$

Hence, the characteristic eqn is

$$m - 2 = 0 \quad \text{or} \quad m = 2$$

Hence, the general solution of the associated homogeneous recurrence relation is given by

$$a_n^h = \alpha\, 2^n$$

Also, $P.S = a_n^p = \frac{1}{E-2} n^2$

$$= \frac{1}{1+\Delta-2} n^2 = \frac{1}{\Delta - 1} n^2 = -\frac{1}{(1-\Delta)} n^2$$

$$\cancel{= (\Delta-1)^{-1}} \quad = -(1-\Delta)^{-1} n^2$$

$$= -(1+\Delta+\Delta^2+\Delta^3+\cdots) n^2$$

$$= -(1+\Delta+\Delta^2+\Delta^3+\cdots)(n^{(2)} + n^{(1)})$$

$$(\because n^2 = n^{(2)} + n^{(1)}).$$

$$= -[n^{(2)} + n^{(1)} + \Delta(n^{(2)} + n^{(1)}) + \Delta^2(n^{(2)} + n^{(1)}) + \cdots]$$

$$= -[n^{(2)} + n^{(1)} + 2n^{(1)} + 1\, n^{(1-1)} + 2 + 0 + \cdots]$$

$$(\because \Delta n^{(m)} = m\, n^{(m-1)})$$

$$= -[n^{(2)} + 3n^{(1)} + n^{(0)} + 2] = -[n^2 - n^{(1)} + 3n^{(1)} + 1 + 2]$$

$$= -[n^2 - n + 3n + 3] = -[n^2 + 2n + 3] \quad (\because n^2 = n^{(2)} + n^{(1)})$$

Hence, the required solution is

$$a_n = \alpha 2^n - (n^2 + 2n + 3).$$

Generating Function:→

The generating function for the sequence $a_0, a_1, a_2, \ldots, a_k, \ldots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1x + a_2x^2 + \cdots + a_kx^k + \cdots = \sum_{k=0}^{\infty} a_k x^k$$

Ex:→ The generating function for the sequence $\{a_k\}$, with $a_k = 3$ is

$$\sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} 3x^k$$

Ex: The generating function for the sequence $\{a_k\}$, with $a_k = 2^k$ is

$$\sum_{k=0}^{\infty} 2^k x^k.$$

Remark:→ We can define generating functions for finite sequences of real numbers by extending a finite sequence $a_0, a_1, \ldots, a_n$ into an infinite sequence by setting $a_{n+1} = 0$, $a_{n+2} = 0$ and so on. The generated function $G(x)$ of the infinite sequence $\{a_n\}$ is a polynomial of degree n because no terms of the form a_jx^j with $j > n$ occur

i.e $G(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$.

Ex: What is the generated function for the sequence 1, 1, 1, 1, 1, 1

Sol:→ The generating function of 1, 1, 1, 1, 1, 1 is

$G(x) = 1 + x + x^2 + x^3 + x^4 + x^5$

Also, we have

$$\frac{x^6-1}{x-1} = 1+x+x^2+x^3+x^4+x^5, \text{ when } x \neq 1 \text{ and } x<1$$

Consequently, $G(x) = \frac{x^6-1}{x-1}$ is the generating function of the sequence 1, 1, 1, 1, 1, 1.

Ex:→ What is the generating function of the sequence 1, 2, 3, 4, 5, . . . ?

A. $\frac{1}{1-x}$ B. $\frac{1}{(1+x)^2}$ C. $\frac{1}{1+x}$ D. $\frac{1}{(1-x)^2}$

Sol:→ We have

A. $\frac{1}{1-x} = (1-x)^{-1} = 1+x+x^2+x^3+\ldots$

which is the generating function of the sequence

1, 1, 1, 1, 1, . . .

B. $\frac{1}{(1+x)^2} = (1+x)^{-2} = 1-2x+3x^2-4x^3+5x^4 \ldots$

which is the generating function of the sequence

1, −2, 3, −4, 5, −6, . . .

C. $\frac{1}{1+x} = (1+x)^{-1} = 1-x+x^2-x^3+\ldots$

which is the generating function of the sequence

1, −1, 1, −1, 1, −1, . . .

D. $\frac{1}{(1-x)^2} = (1-x)^{-2} = 1+2x+3x^2+4x^3+\ldots$

which is the generating function of the sequence

1, 2, 3, 4, 5, ... Hence, option D is correct.

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Q:→ What is the generating function for the sequence 1, 1, 1, 1, . . .

Sol:→ The generating function for the sequence 1, 1, 1, 1, . . . is

$$G(x) = 1 + 1\cdot x + 1\cdot x^2 + 1\cdot x^3 + \cdots$$
$$= 1 + x + x^2 + x^3 + \cdots$$
$$= \frac{1}{1-x}$$

($\because 1 + x + x^2 + \cdots$ is a G.P with $a = 1$, $r = x$)

Q:→ What is the generating function for the sequence $1, a, a^2, a^3, \ldots$

Sol:→ The given sequence $1, a, a^2, a^3, \ldots$ form a geometric progression and its generating function is

$$G(x) = 1 + ax + a^2x^2 + a^3x^3 + \cdots$$
$$= \frac{1}{1-ax}$$

($\because a = 1$, $r = ax$, Sum $= \frac{a}{1-r}$)

H.A

Ex: What are the generating functions for the sequences

① 1, 0, 1, 0, 1, 0, . . .

② 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, . . .

Solving Recurrence Relations using Generating Functions:→

Ex: Solve the recurrence relation

$a_k = 3a_{k-1}$ for $k = 1, 2, 3$ with intial condition $a_0 = 2$

Sol:→ Let $G(x)$ be the generating function for the sequence $\{a_k\}$

i.e $G(x) = \sum_{k=0}^{\infty} a_k x^k$

We have $a_k = 3a_{k-1} \Rightarrow a_k x^k = 3a_{k-1} x^k$

$$\Rightarrow \sum_{k=1}^{\infty} a_k x^k = \sum_{k=1}^{\infty} 3a_{k-1} x^k$$

$$\Rightarrow a_1 x + a_2 x^2 + a_3 x^3 + \cdots = 3(a_0 x^1 + a_1 x^2 + a_2 x^3 + \cdots)$$

$$\Rightarrow (a_0 + a_1 x + a_2 x^2 + \cdots) - a_0 = 3x(a_0 + a_1 x + a_2 x^2 + \cdots)$$

$$\Rightarrow G(x) - a_0 = 3x\,G(x)$$

$$\Rightarrow G(x) - 3x\,G(x) = a_0$$

or $G(x)(1-3x) = a_0$

$$\Rightarrow G(x) = \frac{a_0}{1-3x} \quad \text{or} \quad G(x) = \frac{2}{1-3x} \quad (\because a_0 = 2)$$

$$\Rightarrow \sum_{k=0}^{\infty} a_k x^k = 2\sum_{k=0}^{\infty} (3x)^k$$

$\left[\because \sum_{k=0}^{\infty} (3x)^k = 1 + 3x + (3x)^2 + \cdots\right.$ is a G.P

$a = 1,\ r = 3x$

$\therefore$ Sum $= \frac{1}{1-3x}$

$$\Rightarrow \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} (2\cdot 3^k) x^k. \text{ Hence } a_k = 2\cdot 3^k,\ k = 0, 1, 2, \ldots$$

Ex:→ Using the generating functions, solve the recurrence relation

$a_n = 8a_{n-1} + 10^{n-1}$ with initial condition $a_1 = 9$.

Sol:→ The given recurrence relation is

$$a_n = 8a_{n-1} + 10^{n-1}$$

Putting $n=1$, we get

$$a_1 = 8a_0 + 10^0 \quad \text{or} \quad a_1 = 8a_0 + 1$$

Since $a_1 = 9$, therefore $9 = 8a_0 + 1$

$$\Rightarrow \boxed{a_0 = 1}$$

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ be the required generating function. We have

$$a_n = 8a_{n-1} + 10^{n-1}$$

$$\Rightarrow a_n x^n = 8a_{n-1} x^n + 10^{n-1} x^n$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n x^n = 8\sum_{n=1}^{\infty} a_{n-1} x^n + \sum_{n=1}^{\infty} 10^{n-1} x^n$$

$$\Rightarrow G(x) - a_0 = 8x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} + 10^{1-1} x^1 + 10^{2-1} x^2 + 10^{3-1} x^3 + \cdots$$

$$\Rightarrow G(x) - a_0 = 8x \sum_{n=0}^{\infty} a_n x^n + x + 10x^2 + 10^2 x^3 + \cdots$$

$$\Rightarrow G(x) - a_0 = 8x\,G(x) + x(1 + 10x + 10^2 x^2 + \cdots)$$

$$\Rightarrow G(x) - 8x\,G(x) = a_0 + x \times \frac{1}{1-10x} \quad (\text{G.P with } a=1, r=10x)$$

$$\Rightarrow G(x)(1-8x) = \frac{x}{1-10x} + 1 \quad (\because a_0 = 1).$$

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$$G(x)(1-8x) = \frac{x+1-10x}{1-10x}$$

or $$G(x) = \frac{1-9x}{(1-8x)(1-10x)} \quad \text{——— (1)}$$

Now $$\frac{1-9x}{(1-8x)(1-10x)} = \frac{A}{(1-8x)} + \frac{B}{1-10x} \quad \text{——— (2)}$$

$$\Rightarrow \frac{1-9x}{(1-8x)(1-10x)} = \frac{A(1-10x)+B(1-8x)}{(1-8x)(1-10x)}$$

$$\Rightarrow 1-9x = A(1-10x)+B(1-8x)$$

Comparing the coefficients of constants terms and x, we get

$1 = A+B$ and $-9 = -10A - 8B$

$\Rightarrow \boxed{A = 1-B}$

$$-9 = -10(1-B) - 8B$$
$$= -10 + 10B - 8B$$
$$\Rightarrow -9 + 10 = 2B$$
$$\Rightarrow \boxed{B = \tfrac{1}{2}}$$

But $A = 1 - B = 1 - \frac{1}{2} \Rightarrow \boxed{A = \frac{1}{2}}$

Hence from (2), we get

$$\cancel{\frac{1-9x}{(1-8x)(1-10x)}} \quad \frac{1-9x}{(1-8x)(1-10x)} = \frac{1/2}{1-8x} + \frac{1/2}{1-10x}$$

Hence, from (1), we get

$$G(x) = \frac{1}{2}\left(\frac{1}{1-8x} + \frac{1}{1-10x}\right)$$

$$G(x) = \frac{1}{2}\left[\sum_{n=0}^{\infty}(8x)^n + \sum_{n=0}^{\infty}(10x)^n\right]$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}\left[\frac{1}{2}8^n \cdot x^n + \frac{1}{2}10^n x^n\right]$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}\left[\frac{1}{2}8^n + \frac{1}{2}10^n\right]x^n$$

Hence, $a_n = \frac{1}{2}8^n + \frac{1}{2}10^n$

or $a_n = \frac{1}{2}(8^n + 10^n)$.

Ex:→ Solve the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2} + 2, \quad n = 2, 3, \ldots$$

by the generating function method with intial conditions $a_0 = 1$ and $a_1 = 1$

Ex:→ Solve the recurrance relation

$$a_n = 7a_{n-1} + 10a_{n-2}$$

by the generating function method with intial conditions $a_0 = 3$ and $a_1 = 3$.