Topics

1) Introduction to Proof, Direct proof, proof by Contraposition.
2) Vacuous and trivial proof, proof strategy, proof by contradiction
3) Proof of equivalence and counterexample, mistakes in Proof.

A proof is a valid argument that establishes the truth of a mathematical statement. A proof can use the hypothesis of the theorem, if any, axioms assumed to be true and previously proven theorems.

Some Terminology:

Formally, a theorem is a statement that can be shown to be true. In mathematical writing, the term theorem is usually reserved for a statement that is considered at least somewhat important.

Less important theorems are sometimes called propositions.

A proof is a valid argument that establishes the truth of a theorem.

Axioms are statements we assume to be true.

A less important theorem that is helpful in the proof of other results is called a lemma.

A corollary is a theorem that can be established directly from a theorem that has been proved.

A conjecture is a statement that is being proposed to be a true statement, usually on the basis of some partial evidence, a heuristic argument, or the intuition of an expert. When a proof of a conjecture is found, the conjecture becomes a theorem. Many times conjectures are shown to be false, so they are not theorems

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Direct Proofs:- A direct proof of a conditional statement $p \to q$ is constructed when the first step is the assumption that p is true. Subsequent steps are constructed using rules of inference, with the final step showing that q must be true.

Ex: (I) Give direct proof of the theorem "If n is an odd integer, then n^2 is odd"

Sol: To begin a direct proof of this theorem, we assume that hypothesis of this conditional statement is true. namely assume that n is odd.

By definition of an odd integer, it follows that $n = 2k+1$, where k is an integer

$$\Rightarrow n^2 = (2k+1)^2 = 4k^2+4k+1 = 2(2k^2+2k)+1$$

By definition we can conclude that n^2 is an odd integer (it is one more than twice an integer)

Consequently we have proved that if n is an odd integer then n^2 is an odd integer.

(II) Give a direct proof that if m and n are both perfect square, then nm is also a perfect square

Sol: To produce a direct proof of this theorem we consider the hypothesis is true, namely we assume that m and n are both perfect square.

By definition of perfect squares it follows that there are integers s and t such that $m = s^2$ and $n = t^2$

$$\Rightarrow mn = s^2t^2 \Rightarrow mn = (st)^2 \text{ (using commutativity and associativity)}$$

By the definition of a perfect square it follows that mn is a perfect square because it is the square of st, which is an integer.

~~we have proved that if m and n are integers then mn~~

Proof by Contraposition: Direct proofs lead from the hypothesis of a theorem to the conclusion. They begin with the premises, continue with a sequence of deductions, and end with the conclusion. However we will see that attempts at direct proofs often reach dead ends. So we need other methods of proving theorems. Proof of the theorem that do not start with the hypothesis and end with conclusion are called indirect proofs.

An extremely useful type of indirect proof is known as proof by contraposition

Proof by contraposition make use of the fact that $p \to q \equiv \neg q \to \neg p$.

In a proof by contraposition, we take $\neg q$ as hypothesis and using axioms, definitions, and previously proven theorem together with the rules of inference, we show that $\neg p$ must follow.

Ex(1) Prove that if n is an integer ~~then~~ and $3n+2$ is odd, then n is odd.

Sol: (First by direct proof)

We assume that $3n+2$ is odd

$\Rightarrow 3n+2 = 2k+1,\ k \in \mathbb{Z}$, can we use this to show n is odd

$\Rightarrow 3n+1 = 2k$

but there does not seem a direct way to conclude that n is odd.

Now use contraposition:

Assume that If $3n+2$ is odd, then n is even.

~~By def $3n+2 = 2k$ for some $k \in \mathbb{Z}$~~

Then by def of an even integer $n = 2k$ for some integer k

So, $3n+2 = 3(2k)+2 = 6k+2 = 2(3k+1)$

This tells that $3n+2$ is even (because it is a multiple of 2)

This is negation of the hypothesis of the theorem.

i.e $\neg q \to \neg p$. Hence the prove of $p \to q$ is over.

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2) Prove that if $n = ab$, where a and b are positive integers; then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Sol: Here we notice that there is no direct way of showing that $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$ from the eqn $\underline{n = ab}$

Proof by contraposition: We assume that

$(a \leq \sqrt{n}) \vee (b \leq \sqrt{n})$ is false.

$\Rightarrow \neg[(a \leq \sqrt{n}) \vee (b \leq \sqrt{n})]$ is true

$\Rightarrow \neg(a \leq \sqrt{n}) \wedge \neg(b \leq \sqrt{n})$ is true

$\Rightarrow a > \sqrt{n}$ and $b > \sqrt{n}$.

$\Rightarrow ab > \sqrt{n} \cdot \sqrt{n} = n$.

This shows that $ab \neq n$ which contradicts the statement $n = ab$.

So negation of conclusion implies ~~that negation~~ the hypothesis is false;

Therefore the original conditional statement is true.

Our proof by contraposition succeeded; we have proved that if $n = ab$, where a and b are positive integers then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

§ Vacuous and Trivial proof:

We can quickly prove a conditional statement $p \to q$ is true when we know that p is false, because $p \to q$ must be true when p is false. Consequently, if we can show that p is false, then we have a proof, called vacuous proof, of the conditional statement $p \to q$.

Ex: Show that the proposition $P(0)$ is true, when $P(n)$ is "If $n > 1$ then $n^2 > n$" and the domain consists of all integers.

Sol: Note that $P(0)$ is "If $0 > 1$ then $0^2 > 0$". We can show $P(0)$ using a vacuous proof. Indeed the hypothesis $0 > 1$ is false, so $P(0)$ true

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We can also quickly prove a conditional statement $p \to q$ if we know that the conclusion q is true. By showing that q is true, it follows that $p \to q$ must also be true. A proof of $p \to q$ that uses the fact that q is true is called a trivial proof

Ex:- Let $P(n)$ be "If a and b are positive integers with $a \geq b$, then $a^n \geq b^n$", where the domain consists of all non-negative integers. Show that $P(0)$ is true

Sol: The proposition $P(0)$ is ~~true~~. "If $a \geq b$, then $a^0 \geq b^0$". Because $a^0 = b^0 = 1$, the conclusion of the conditional statement, "If $a \geq b$ then $a^0 \geq b^0$ is true". Hence this conditional statement, which is $P(0)$, is true.

This is an example of trivial proof.

Proof strategy: When we want to prove a statement of the form $\forall x (P(x) \to Q(x))$, first we should look for a direct method. If a direct proof does not seem to go anywhere, try the method of proof by contraposition.

§ Proof by Contradiction:

Suppose we want to prove that p is true. So if we can find a contradiction q such that $\neg p \to q$ is true. Because q is false, but $\neg p \to q$ is true, we conclude that $\neg p$ is false, which means that p is true.

Since the statement $r \wedge \neg r$ is a contradiction whenever r is a proposition, we can prove that p is true if we can show that $\neg p \to (r \wedge \neg r)$ is true for some proposition r. Proofs of this type are called proof by contraposition.

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Q) Show that atleast four of any 22 days must fall on the same day of the week. (24)

Sol: Let p be the proposition "Atleast four of 22 chosen days fall on the same day of the week". Suppose that $\neg p$ is true. This means that at most three of the 22 days fall on the same day of the week. Because there are seven days of the week, this implies that atmost 21 days could have been chosen, as for each of of the days of the week, at most three of the chosen days can fall on that day. This contradicts the premise that we have 22 days under consideration. That is if r is the statement that 22 days are chosen, then we have shown that $\neg p \to (r \wedge \neg r)$. Consequently, we know that p is true. We have proved that 22 chosen days fall on the same day of the week.

Q) Prove that $\sqrt{2}$ is irrational by giving a proof by ~~contradict~~ contradiction.

Sol: Let p be the proposition "$\sqrt{2}$ is irrational".

Let $\neg p$ is true i.e. $\sqrt{2}$ is rational

i.e. $\sqrt{2} = \frac{a}{b}$, where $b \neq 0$ and a and b have no common factors.

$\Rightarrow 2 = \frac{a^2}{b^2} \Rightarrow 2b^2 = a^2$

$\Rightarrow a^2$ is even $\Rightarrow a$ is even

Since a is even, so $a = 2c$ for some integer 'c'

Thus $2b^2 = 4c^2 \Rightarrow b^2 = 2c^2 \Rightarrow b^2$ is even $\Rightarrow b$ is even

So $\sqrt{2} = \frac{a}{b}$ leads to contradiction as both a and b being even will be divided by 2

So $\neg p$ must be false. That $\sqrt{2}$ is irrational is true.

Proof of equivalence:-

To prove a theorem that is a biconditional statement, that is, a statement of the form $p \leftrightarrow q$ we show that $p \to q$ and $q \to p$ ~~are~~ both are true. The validity of this approach is based on the tautology $(p \leftrightarrow q) \leftrightarrow (p \to q) \wedge (q \to p)$

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Ex:- prove that "If n is an integer, then n is odd if and only if n^2 is odd".

Sol: p if and only if q

$p \to q$, i.e we first show that if n is odd then n^2 is odd.

Since n is odd, $n = 2k+1$, for some integer k.

$\Rightarrow n^2 = (2k+1)^2 = 4k^2+4k+1 = 2(2k^2+2k)+1$

$\Rightarrow n^2$ is odd

Conversely, we prove $q \to p$

let n^2 is odd, we prove this through "proof by contraposition" (i.e $\neg p \to \neg q$)

So consider n is not odd, i.e n is even

so $n = 2k$, for some integer k.

$\Rightarrow n^2 = 4k^2 = 2(2k^2)$, which implies that n^2 is even.

i.e we proved that $\neg p \to \neg q$

So if n^2 is odd then n is odd.

Because we have shown that both $p \to q$ and $q \to p$ are true, we have shown that the theorem is true.

§ when we have to prove

$p_1 \leftrightarrow p_2 \leftrightarrow \cdots \leftrightarrow p_n$

we can prove it using the tautology

$p_1 \leftrightarrow p_2 \leftrightarrow \cdots \leftrightarrow p_n \equiv (p_1 \to p_2) \wedge (p_2 \to p_3) \wedge \cdots \wedge (p_n \to p_1)$

This shows that if n conditional statements $p_1 \to p_2, p_2 \to p_3, \cdots p_n \to p_1$ can be shown to be true, then the proposition $p_1, p_2, \cdots p_n$ are all equivalent

Ex! Show that these statements about n integers are equivalent!

p_1: n is even, p_2: $n-1$ is odd, p_3: n^2 is even

Sol: We need to prove $p_1 \to p_2$, $p_2 \to p_3$ and $p_3 \to p_1$

$p_1 \to p_2$: n is even $\Rightarrow n = 2k$ for some integer k

$\Rightarrow n-1 = 2k-1 = 2(k-1)+1 \Rightarrow n-1$ is odd as

$p_2 \to p_3$: $n-1$ is odd $\Rightarrow n-1 = 2k+1$ for some integer k

$\Rightarrow n = 2k+2 \Rightarrow n^2 = 4k^2+8k+4 = 2(2k^2+4k+2) \Rightarrow n^2$ is even

$p_3 \to p_1$: prove it by contraposition and hence the result

8. We know that to show that a statement of the form $\forall x\, p(x)$ is false, we need to give a <u>counterexample</u>, that is, an example x for which $p(x)$ is false.

<u>Ex:</u> Show that "Every positive integer is the sum of the squares of two integers" is false.

Sol: Take an integer '3'.

Only perfect squares not exceeding 3 are $0^2 = 0$ and $1^2 = 1$. Furthermore 3 cannot be written as sum of 0^2 or 1^2.

<u>Mistakes in Proofs:</u>

① What is wrong with this famous supposed "proof" that $1 = 2$?

<u>"Proof":</u>

1) $a = b$
2) $a^2 = ab$
3) $a^2 - b^2 = ab - b^2$
4) $(a-b)(a+b) = b(a-b)$
5) $a + b = b$
6) $2b = b$
7) $2 = 1$

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