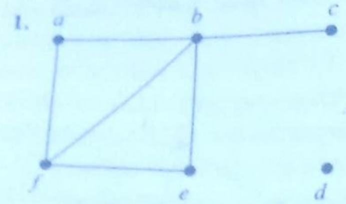
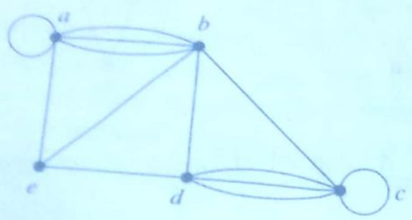
In Exercises 1–3 find the number of vertices, the number of edges, and the degree of each vertex in the given undirected graph. Identify all isolated and pendant vertices.

1.



2.



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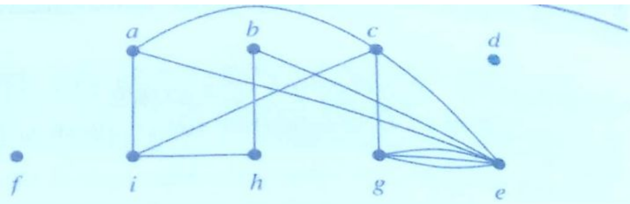


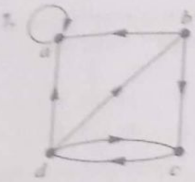
4. Find the sum of the degrees of the vertices of each graph in Exercises 1–3 and verify that it equals twice the number of edges in the graph.

5. Can a simple graph exist with 15 vertices each of degree five?

6. Show that the sum, over the set of people at a party, of the number of people a person has shaken hands with, is even. Assume that no one shakes his or her own hand.

In Exercises 7–9 determine the number of vertices and edges and find the in-degree and out-degree of each vertex for the given directed multigraph.

7.



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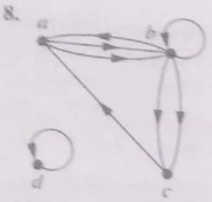

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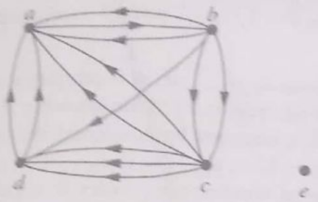
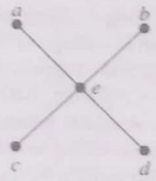


10. For each of the graphs in Exercises 7–9 determine the sum of the in-degrees of the vertices and the sum of the out-degrees of the vertices directly. Show that they are both equal to the number of edges in the graph.
11. Construct the underlying undirected graph for the graph with directed edges in Figure 2.
12. What does the degree of a vertex represent in the acquaintanceship graph, where vertices represent all the people in the world? What do isolated and pendant vertices in this graph represent? In one study it was estimated that the average degree of a vertex in this graph is 1000. What does this mean in terms of the model?
13. What does the degree of a vertex represent in a collaboration graph? What do isolated and pendant vertices represent?
14. What does the degree of a vertex in the Hollywood graph represent? What do the isolated and pendant vertices represent?
15. What do the in-degree and the out-degree of a vertex in a telephone call graph, as described in Example 7 of Section 8.1, represent? What does the degree of a vertex in the undirected version of this graph represent?
16. What do the in-degree and the out-degree of a vertex in the Web graph, as described in Example 8 of Section 8.1, represent?
17. What do the in-degree and the out-degree of a vertex in a directed graph modeling a round-robin tournament represent?
18. Show that in a simple graph with at least two vertices there must be two vertices that have the same degree.
19. Use Exercise 18 to show that in a group, there must be two people who know the same number of other people in the group.
20. Draw these graphs.
    a) K_7 b) $K_{1,8}$ c) $K_{4,4}$
    d) C_7 e) W_7 f) Q_4

In Exercises 21–25 determine whether the graph is bipartite. You may find it useful to apply Theorem 4 and answer the question by determining whether it is possible to assign either red or blue to each vertex so that no two adjacent vertices are assigned the same color.

21.



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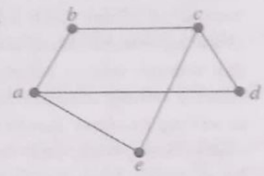

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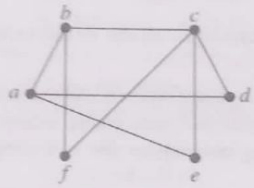

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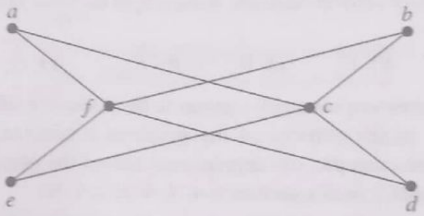
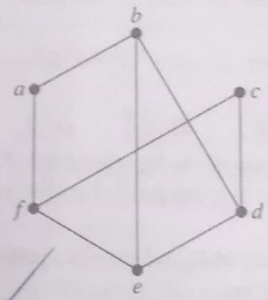


25.



26. For which values of n are these graphs bipartite?
    a) K_n 1, 2 b) C_n even c) W_n not d) Q_n all
27. Suppose that a new company has five employees: Zamora, Agraharam, Smith, Chou, and Macintyre. Each employee will assume one of six responsiblities: planning, publicity, sales, marketing, development, and industry relations. Each employee is capable of doing one or more of these jobs: Zamora could do planning, sales, marketing, or industry relations; Agraharam could do planning or development; Smith could do publicity, sales, or industry relations; Chou could do planning, sales, or industry relations; and Macintyre could do planning, publicity, sales, or industry relations.

a) Model the capabilities of these employees using a bipartite graph.

b) Find an assignment of responsibilites such that each employee is assigned a responsibility.

28. Suppose that there are five young women and six young men on an island. Each woman is willing to marry some of the men on the island and each man is willing to marry any woman who is willing to marry him. Suppose that Anna is willing to marry Jason, Larry, and Matt; Barbara is willing to marry Kevin and Larry; Carol is willing to marry Jason, Nick, and Oscar; Diane is willing to marry Jason, Larry, Nick, and Oscar; and Elizabeth is willing to marry Jason and Matt.

a) Model the possible marriages on the island using a bipartite graph.

b) Find a matching of the young women and the young men on the island such that each young woman is matched with a young man whom she is willing to marry.

29. How many vertices and how many edges do these graphs have?

a) K_n b) C_n c) W_n d) $K_{m,n}$ e) Q_n

The **degree sequence** of a graph is the sequence of the degrees of the vertices of the graph in nonincreasing order. For example, the degree sequence of the graph G in Example 1 in this section is 4, 4, 4, 3, 2, 1, 0.

30. Find the degree sequences for each of the graphs in Exercises 21–25.

31. Find the degree sequence of each of the following graphs.

a) K_4 b) C_4 c) W_4 d) $K_{2,3}$ e) Q_3

32. What is the degree sequence of the bipartite graph $K_{m,n}$ where m and n are positive integers? Explain your answer.

33. What is the degree sequence of K_n, where n is a positive integer? Explain your answer.

34. How many edges does a graph have if its degree sequence is 4, 3, 3, 2, 2? Draw such a graph.

35. How many edges does a graph have if its degree sequence is 5, 2, 2, 2, 2, 1? Draw such a graph.

A sequence $d_1, d_2, \ldots, d_n$ is called *graphic* if it is the degree sequence of a simple graph.

36. Determine whether each of these sequences is graphic. For those that are, draw a graph having the given degree sequence.

a) 5, 4, 3, 2, 1, 0 b) 6, 5, 4, 3, 2, 1 c) 2, 2, 2, 2, 2, 2
d) 3, 3, 3, 2, 2, 2 e) 3, 3, 2, 2, 2, 2 f) 1, 1, 1, 1, 1, 1
g) 5, 3, 3, 3, 3, 3 h) 5, 5, 4, 3, 2, 1

37. Determine whether each of these sequences is graphic. For those that are, draw a graph having the given degree sequence.

a) 3, 3, 3, 3, 2 b) 5, 4, 3, 2, 1 c) 4, 4, 3, 2, 1
d) 4, 4, 3, 3, 3 e) 3, 2, 2, 1, 0 f) 1, 1, 1, 1, 1

*38. Suppose that $d_1, d_2, \ldots, d_n$ is a graphic sequence. Show that there is a simple graph with vertices $1, 2, \ldots, n$ such that $\deg(i) = d_i$ for $i = 1, 2, \ldots, n$ and 1 is adjacent to $2, \ldots, d_1 + 1$.

*39. Show that a sequence $d_1, d_2, \ldots, d_n$ of nonnegative integers in nonincreasing order is a graphic sequence if and only if the sequence obtained by reordering the terms of the sequence $d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n$ so that the terms are in nonincreasing order is a graphic sequence.

*40. Use Exercise 39 to construct a recursive algorithm for determining whether a nonincreasing sequence of positive integers is graphic.

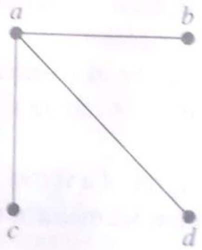
41. Show that every nonincreasing sequence of nonnegative integers with an even sum of its terms is the degree sequence of a pseudograph, that is, an undirected graph where loops are allowed. [*Hint:* Construct such a graph by first adding as many loops as possible at each vertex. Then add additional edges connecting vertices of odd degree. Explain why this construction works.]

42. How many subgraphs with at least one vertex does K_2 have?

43. How many subgraphs with at least one vertex does K_3 have?

44. How many subgraphs with at least one vertex does W_3 have?

45. Draw all subgraphs of this graph.



46. Let G be a graph with v vertices and e edges. Let M be the maximum degree of the vertices of G, and let m be the minimum degree of the vertices of G. Show that

a) $2e/v \geq m$. b) $2e/v \leq M$.

A simple graph is called **regular** if every vertex of this graph has the same degree. A regular graph is called n-**regular** if every vertex in this graph has degree n.

47. For which values of n are these graphs regular?

a) K_n b) C_n c) W_n d) Q_n

48. For which values of m and n is $K_{m,n}$ regular?

49. How many vertices does a regular graph of degree four with 10 edges have?

7. Represent the graph in Exercise 3 with an adjacency matrix.

8. Represent the graph in Exercise 4 with an adjacency matrix.

9. Represent each of these graphs with an adjacency matrix.

a) K_4 b) $K_{1,4}$ c) $K_{2,3}$
d) C_4 e) W_4 f) Q_3

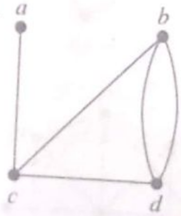
In Exercises 10–12 draw a graph with the given adjacency matrix.

10. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

11. $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

In Exercises 13–15 represent the given graph using an adjacency matrix.

13.


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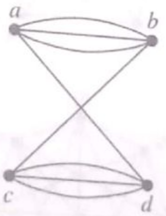

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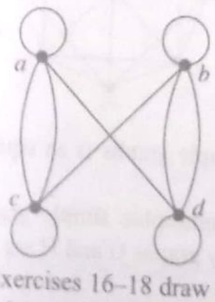


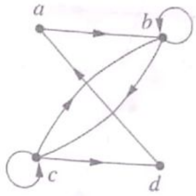
In Exercises 16–18 draw an undirected graph represented by the given adjacency matrix.

16. $\begin{bmatrix} 1 & 3 & 2 \\ 3 & 0 & 4 \\ 2 & 4 & 0 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

18. $\begin{bmatrix} 0 & 1 & 3 & 0 & 4 \\ 1 & 2 & 1 & 3 & 0 \\ 3 & 1 & 1 & 0 & 1 \\ 0 & 3 & 0 & 0 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{bmatrix}$

In Exercises 19–21 find the adjacency matrix of the given directed multigraph.

19.


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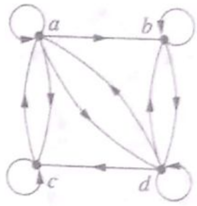

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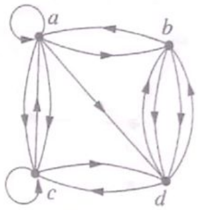


In Exercises 22–24 draw the graph represented by the given adjacency matrix.

22. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

23. $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix}$

24. $\begin{bmatrix} 0 & 2 & 3 & 0 \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$

25. Is every zero–one square matrix that is symmetric and has zeros on the diagonal the adjacency matrix of a simple graph?

26. Use an incidence matrix to represent the graphs in Exercises 1 and 2.

27. Use an incidence matrix to represent the graphs in Exercises 13–15.

*28. What is the sum of the entries in a row of the adjacency matrix for an undirected graph? For a directed graph?

*29. What is the sum of the entries in a column of the adjacency matrix for an undirected graph? For a directed graph?

30. What is the sum of the entries in a row of the incidence matrix for an undirected graph?

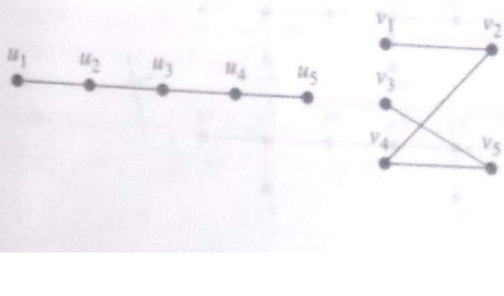
31. What is the sum of the entries in a column of the incidence matrix for an undirected graph?

*32. Find an adjacency matrix for each of these graphs.

a) K_n b) C_n c) W_n d) $K_{m,n}$ e) Q_n

*33. Find incidence matrices for the graphs in parts (a)–(d) of Exercise 32.

In Exercises 34–44 determine whether the given pair of graphs is isomorphic. Exhibit an isomorphism or provide a rigorous argument that none exists.

34.


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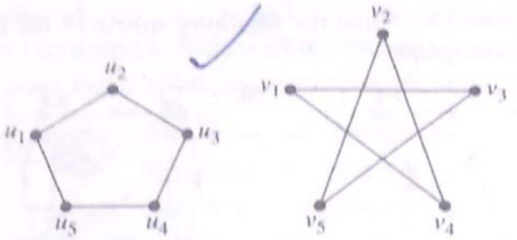

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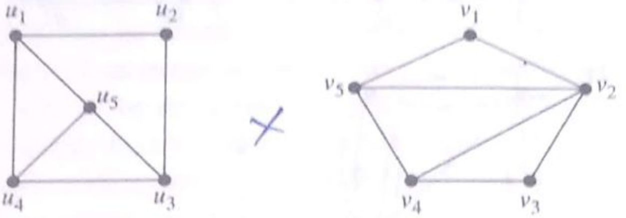

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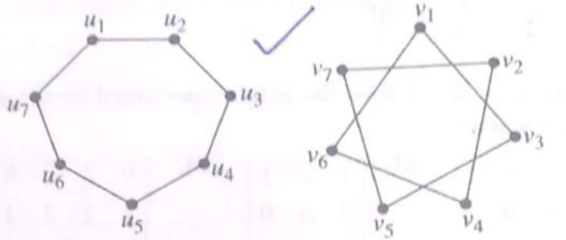

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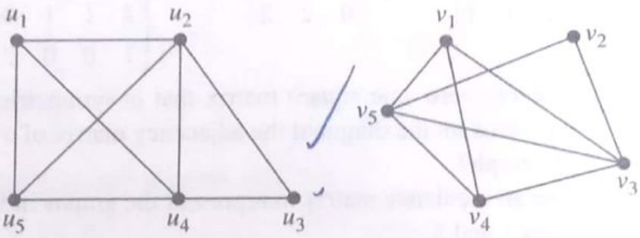

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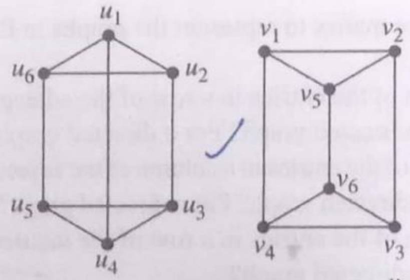

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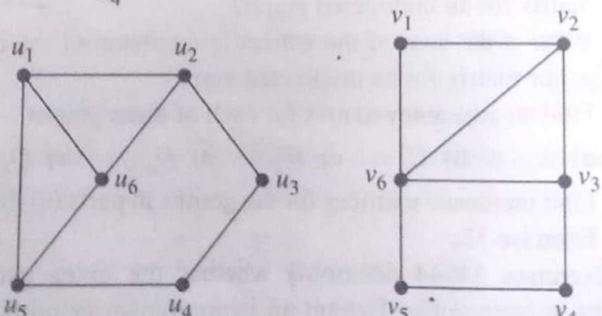

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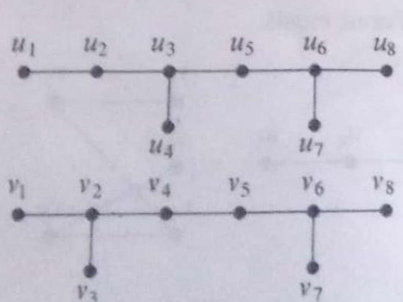

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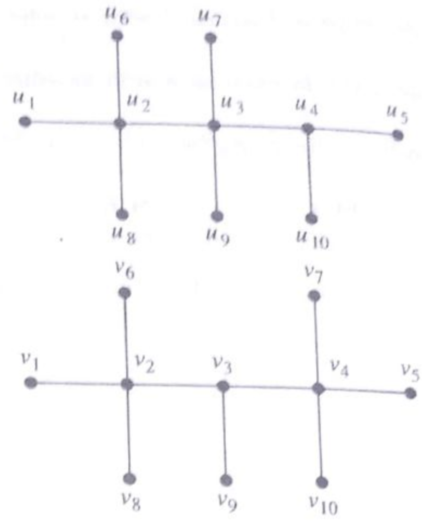

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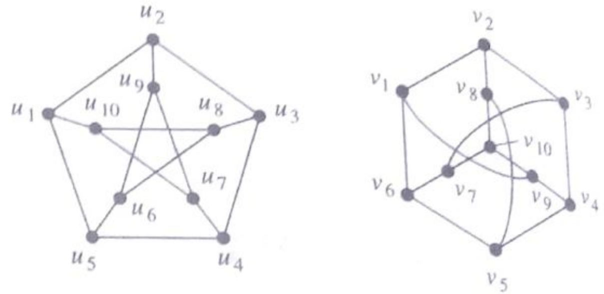

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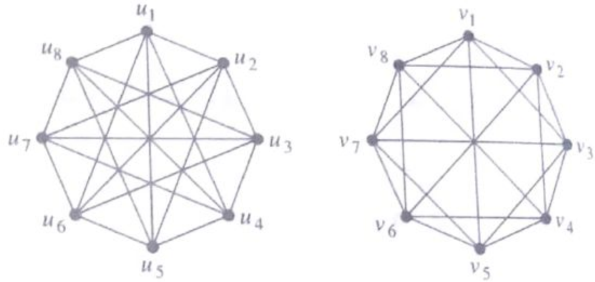

45. Show that isomorphism of simple graphs is an equivalence relation.

46. Suppose that G and H are isomorphic simple graphs. Show that their complementary graphs G and H are also isomorphic.

47. Describe the row and column of an adjacency matrix of a graph corresponding to an isolated vertex.

48. Describe the row of an incidence matrix of a graph corresponding to an isolated vertex.

49. Show that the vertices of a bipartite graph with two or more vertices can be ordered so that its adjacency matrix has the form

$$\begin{bmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{B} & \mathbf{0} \end{bmatrix},$$

where the four entries shown are rectangular blocks.

A simple graph G is called **self-complementary** if G and $\overline{G}$ are isomorphic.