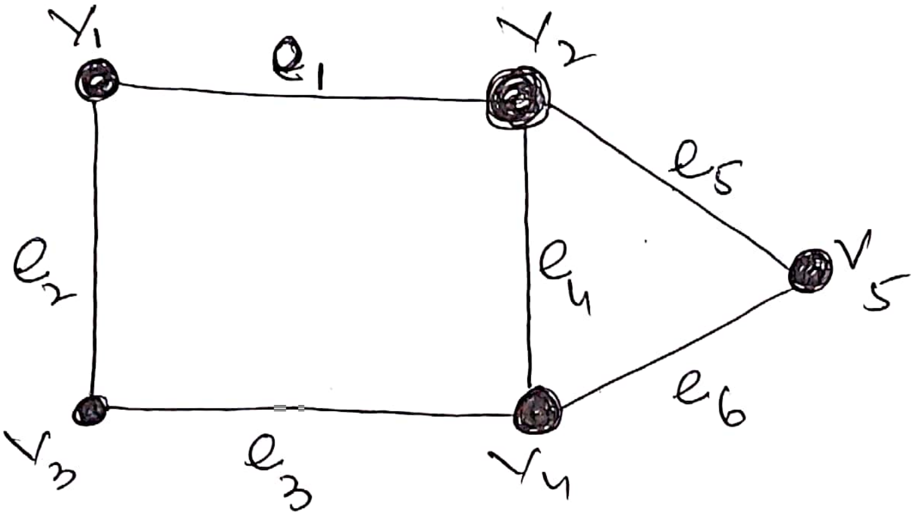
Unit IV: Graph Theory I

A graph $G=(V,E)$ Consists of a non empty set of objects $V=\{v_1, v_2, \dots\}$ called vertices and another set $E=\{e_1, e_2, \dots\}$ whose elements are called edges, such that each edge e_k is identified with an unordered pair (v_i, v_j) of vertices. The vertices v_i and v_j associated with edge e_k are called the end vertices of e_k.

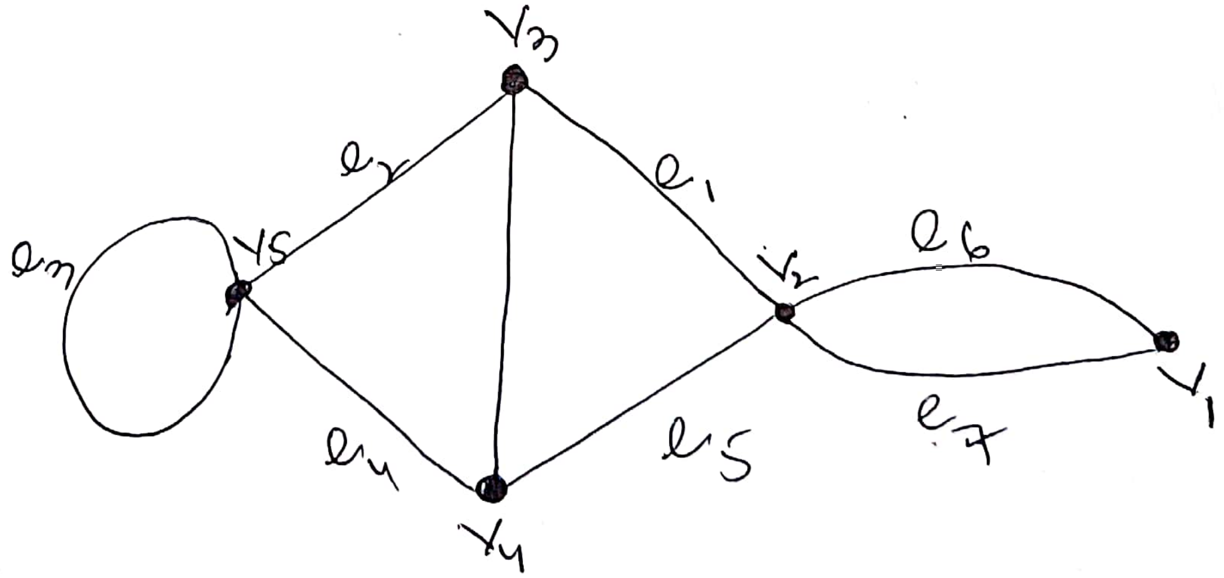
Examples:

(1)

(2)



(3)



Trivial Graph:→ A graph consisting only one vertex and no edge is called a trivial graph
e.g

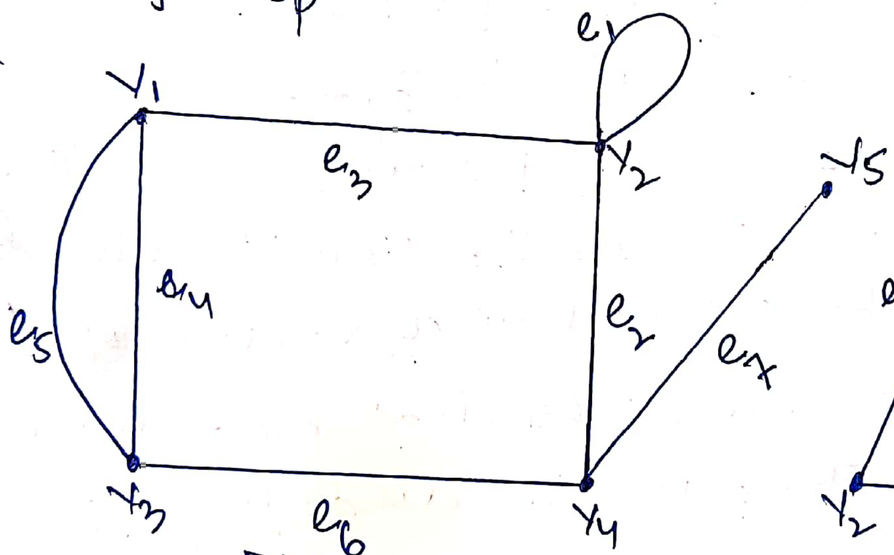
Null graph:→ A graph consisting n vertices and no edge is called a Null graph
e.g

Remark:→ Every trivial graph is a null graph.

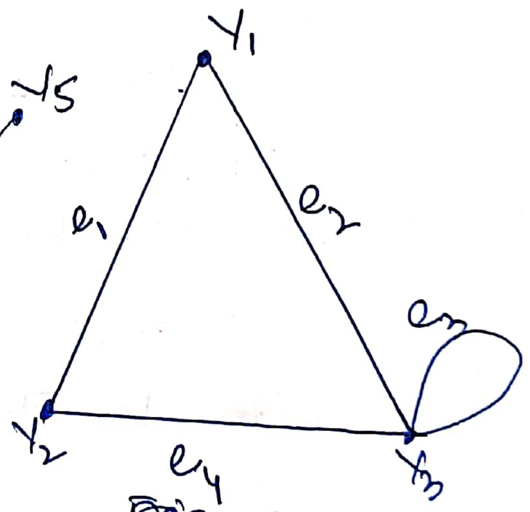
Self-loop in a graph:→ An edge having the same vertex as both its end vertices is called a self-loop

Ex

①



~~Fig~~ Graph G



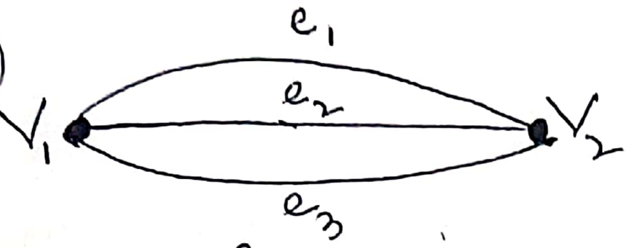
~~Fig~~ Graph H

Edges e_1 and e_3 are self-loops in Graphs G and H.

Multigraph:→ Graphs that have multiple edges connecting the same pair of vertices are called multigraphs. Multiple edges are also called parallel edges. Self-loops are not allowed in multigraphs.

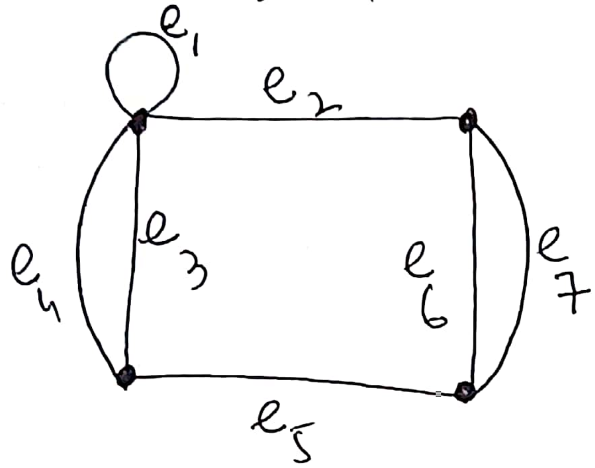
are also called parallel edges.

e.g. (1)



e_1, e_2 and e_3 are multiedges.

(2)

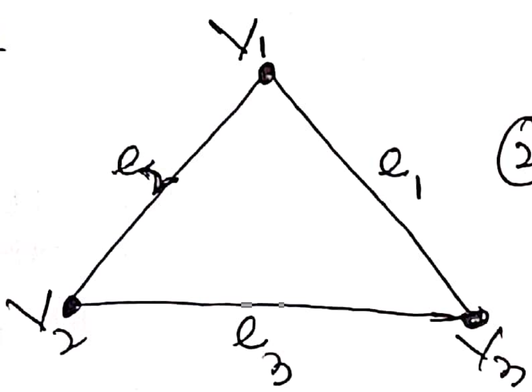


e_3 and e_4, and e_6 and e_7 are multiedges.

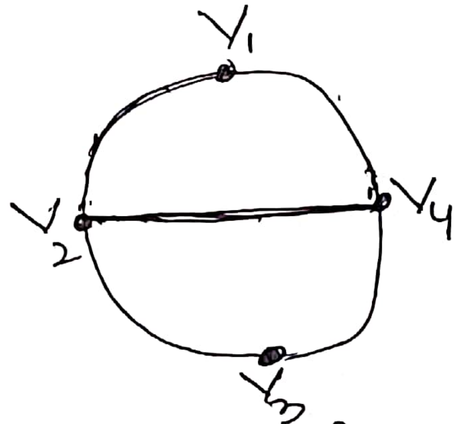
Simple Graph: → A graph that has neither self-loops nor multiple edges is called a simple graph.

Example

(1)

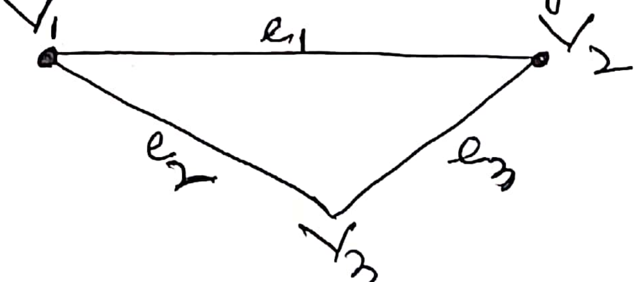


(2)



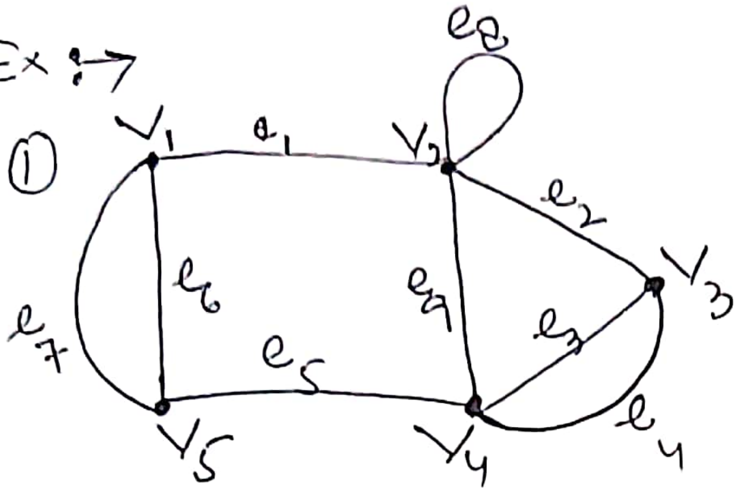
Proper Edge: → An edge which is not self-loop is called proper edge.

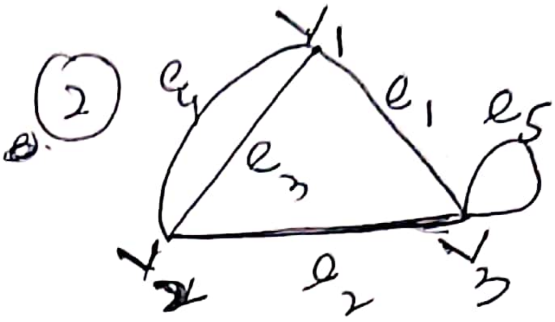
Example;



Pseudo Graph: → A graph containing both self-loops and multiple edges is called ~~Psed~~ Pseudo-graph.

Ex :→

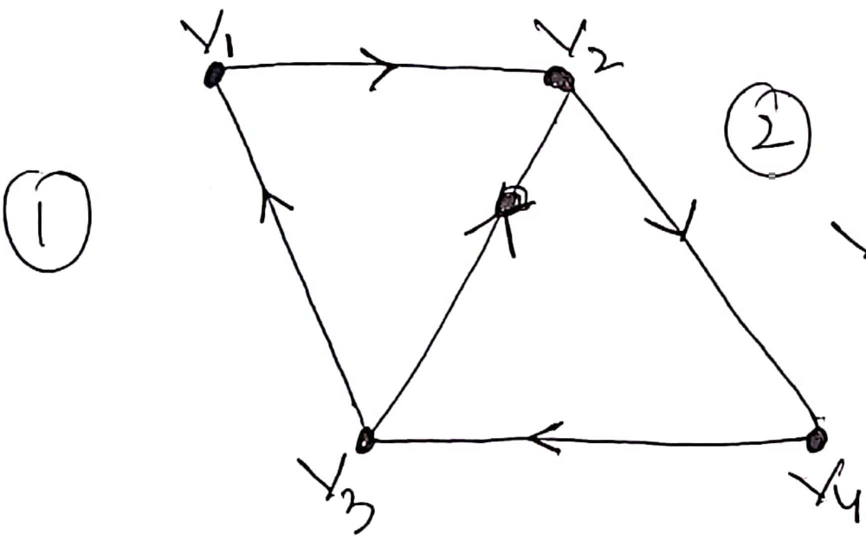


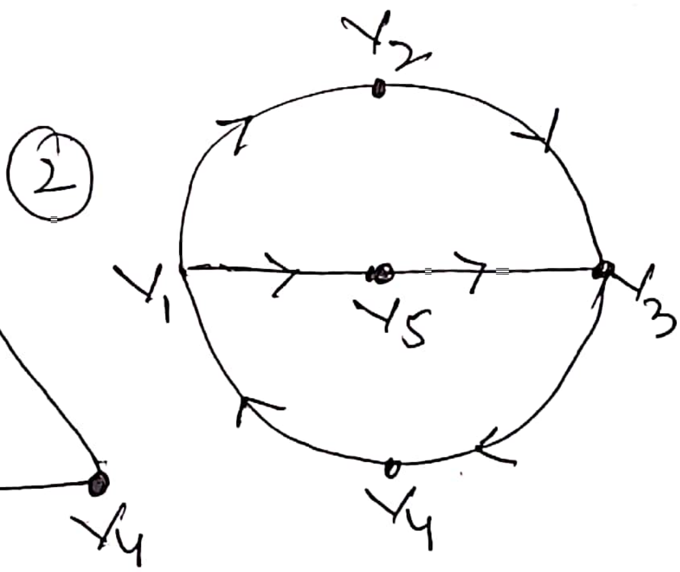


Directed Graph or Digraph Graph :→

A directed graph ~~G(V,E)~~ $G=(V, E)$ Consists of a non empty set of vertices V and a set of directed edges E. Each directed edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair (v_1, v_2) is start at v_1 and end at v_2.

Ex:





Incidence and Adjacency :-

When a vertex v_j is an end vertex of some edge e_j, v_j and e_j are said to be incident to each other. For example, in Fig 1, e_2, ~~and~~ e_6 and e_7 are incident with vertex v_4.

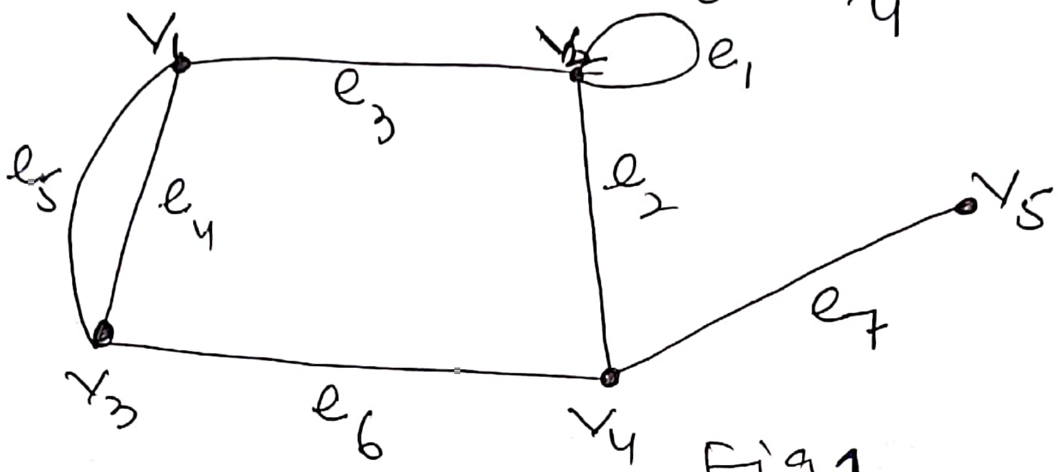


Fig 1

Two vertices u and v in an undirected graph G are said to be adjacent in G ~~if u and v are end vertices of the same edge of G~~. if there exists an edge joining these vertices. Similarly, two edges are said to be adjacent if they are incident on a common vertex.

In Fig 1, v_4 and v_5 are ~~adjacent~~ adjacent, but v_1 and v_4 are not adjacent. Also, edges e_7 and e_7 are adjacent.

<u>Degree of a vertex</u>: The degree of a vertex V in an undirected graph G is the number of edges incident on it, with self-loops counted twice. The degree of the vertex V is denoted by $deg(V)$ or $d(V)$.

For example, in Fig 1, $d(V_1) = d(V_3) = d(V_4) = 3$, $d(V_2) = 4$ and $d(V_5) = 1$.

<u>Remark</u>:→ Let us now consider a graph G with m edges and n vertices $V_1, V_2, \ldots, V_n$. Since each edge contributes two ~~degree~~ degrees, the sum of the degree of all vertices in G is twice the number of edges in G.

i.e $\sum_{i=1}^{n} deg(V_i) = 2m$

Take Fig 1 as an example

$d(V_1) + d(V_2) + d(V_3) + d(V_4) + d(V_5)$

$= 3 + 4 + 3 + 3 + 1 = 14 = 2 \times 7$

$= 2 \times$ no. of edges

Therefore, we have the following theorem:

Handshaking Theorem :→

Let $G = (V, E)$ be an undirected graph with m edges, then

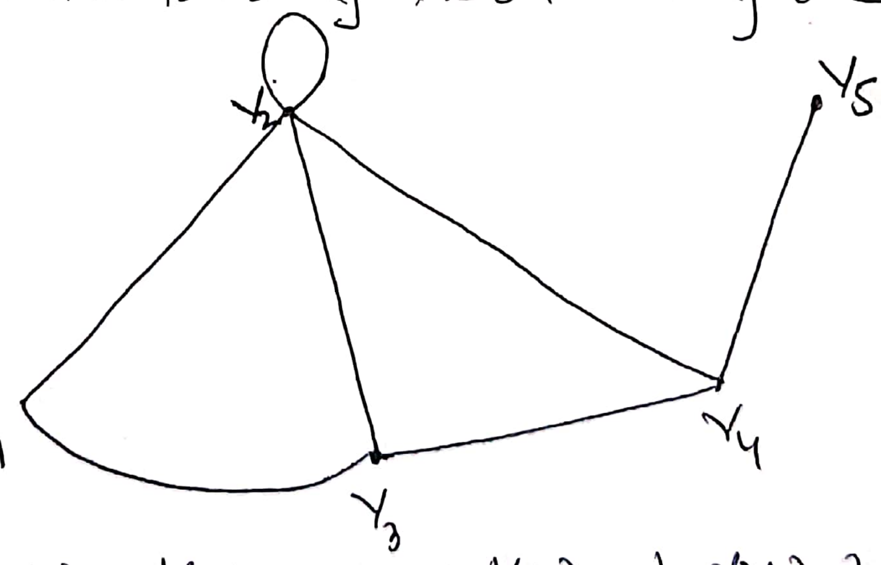
$$2m = \sum_{v \in V} \deg v.$$

Ex: How many edges are there in a graph with 10 vertices each of degree six.

Sol:→ Since the sum of the degrees of the vertices is $6 \cdot 10 = 60$, it follows that by Handshaking theorem that $2m = 60$, where m is the number of edges. Therefore, $m = 30$.

Theorem:→ An undirected graph has an even number of vertices of odd degree.

Ex



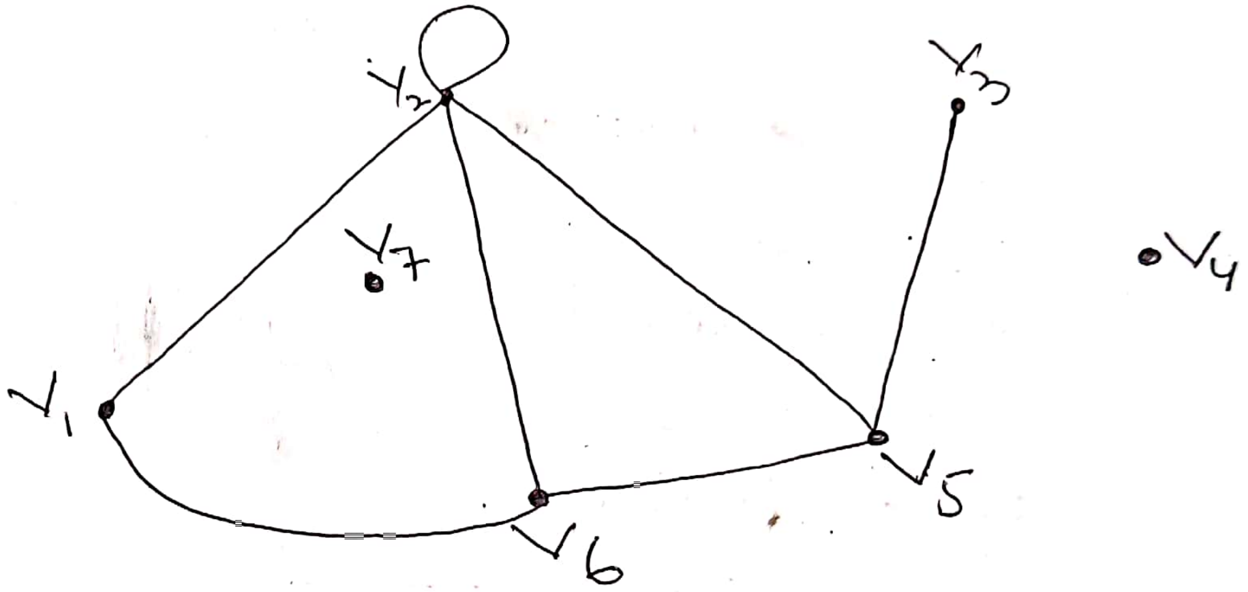
$d(v_1) = 2$, $d(v_2) = 5$, $\deg(v_3) = \deg(v_4) = 3$, $\deg(v_5) = 1$

Hence, there are four (even) vertices V_2, V_3, V_4 and V_5 of odd degree.

Isolated vertex and Pendant vertex:→

A vertex having no incident edge is called an Isolated vertex. In other words, isolated vertices are vertices with zero degree.

Vertices V_4 and V_7 in Figure below are Isolated vertices

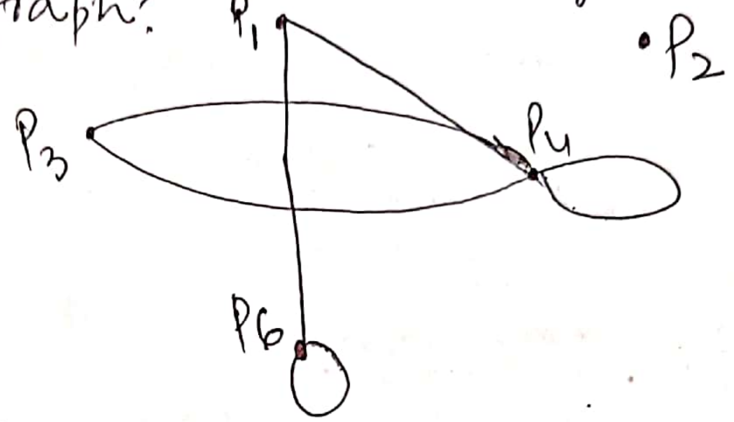


A vertex of degree one is called a pendant vertex or an end vertex. Vertex V_3 in above Figure is a pendant vertex. Hence, a pendant vertex is adjacent to exactly one other vertex.

Finite and Infinite Graphs :

A graph G with a finite number of vertices as well as ~~...~~ edges is called a finite graph, otherwise it is an infinite graph.

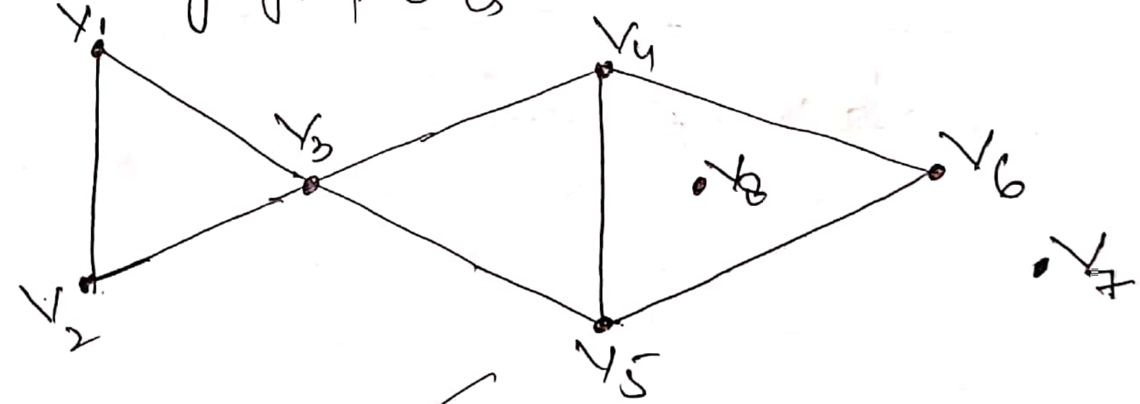
Ex: → ① Which of the following is true for a given graph?



(A) $deg(P_6) = 4$ (B) $deg(P_2) = 1$

✓(C) $deg(P_4) = 5$ (D) $deg(P_3) = 3$

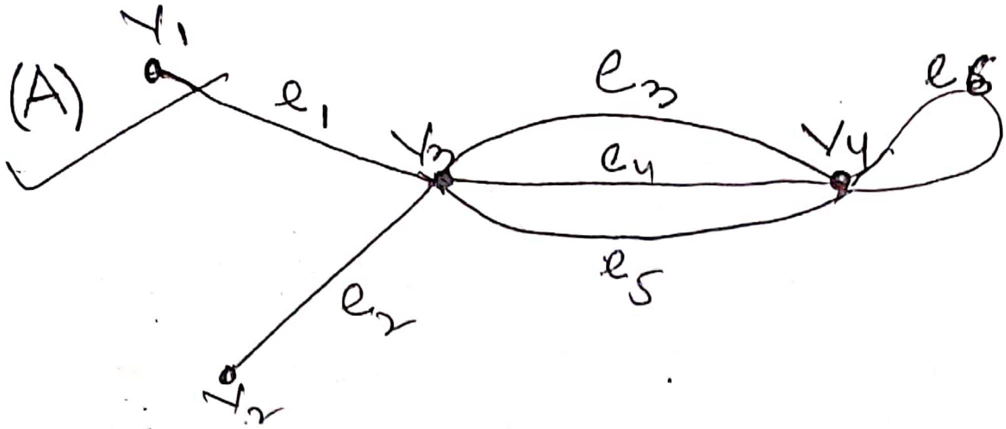
② Number of Isolated vertex in the following graph is



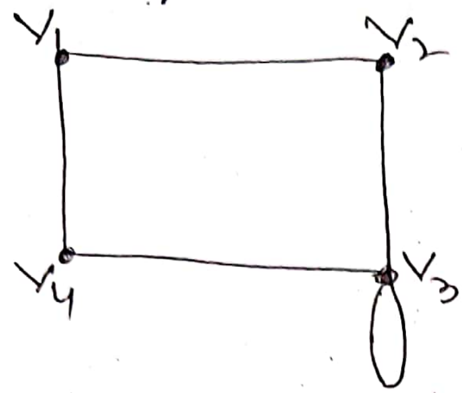
(A) 0 (B) 2 ✓ (C) 1 (D) 3

③ Which of the following is pseudo graph?

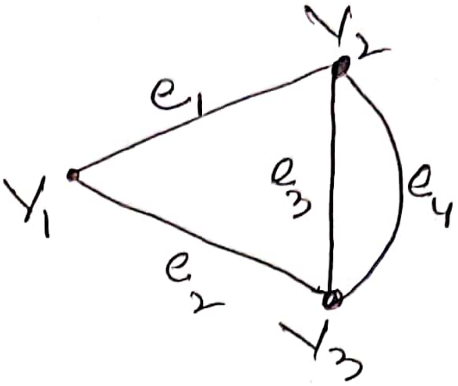
(A)



(B)

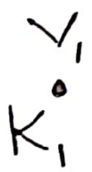


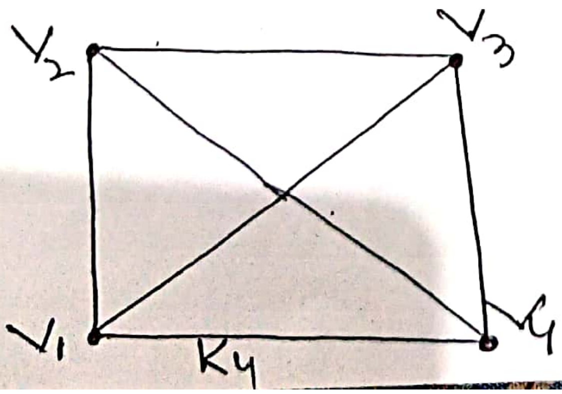
(C)

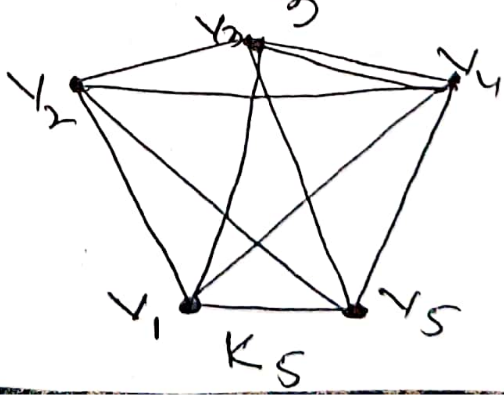


(D) None of the these.

<u>Complete Graph</u>:→ A simple graph is said to be complete if each vertex is connected with every other vertex. A complete graph for n vertices is denoted by ~~Kn~~. The graph K_n, for $n=1,2,3,4,5$ are given as follows:

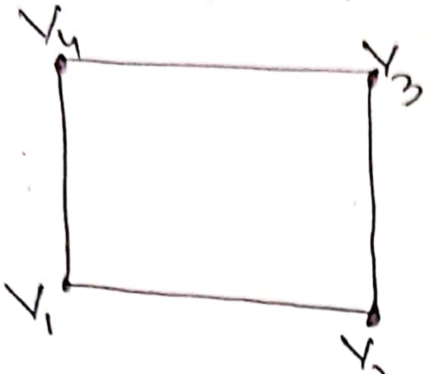




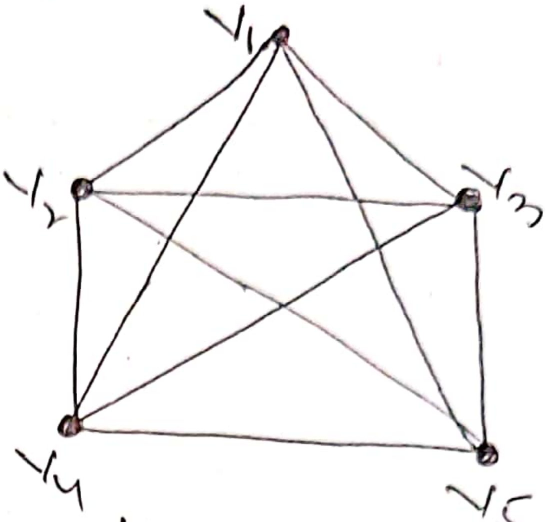


Regular Graph:→ A graph G is said to be regular if every vertex has the same degree. If degree of each vertex of a graph G is K, then it is called K-regular graph.

Ex: ①



2-Regular Graph



4-Regular Graph.

Remark:→ Every complete graph is a regular graph but not conversely.

Ex. (1)

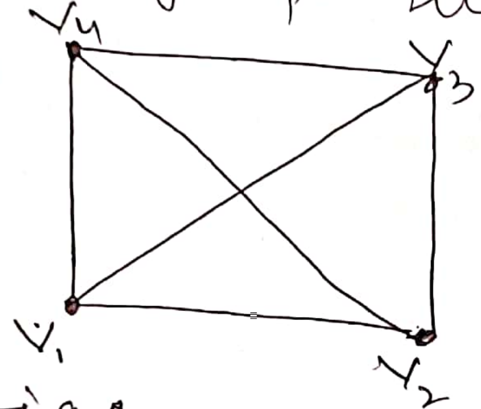


Fig A

(2)

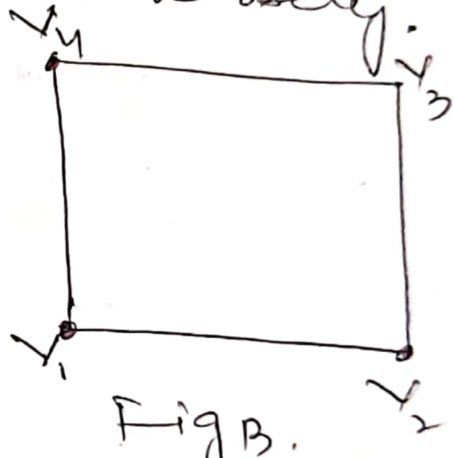


Fig B.

Note that graph displayed in Fig A is both complete and regular but the graph displayed in Fig B. is regular but not complete.

Cycle or cycle graph :→

A cycle $C_n, n \geq 3$, Consists of n vertices $V_1, V_2, \ldots, V_n$ and edges $\{V_1, V_2\}, \{V_2, V_3\}, \ldots, \{V_{n-1}, V_n\}$ and $\{V_n, V_1\}$. The cycles C_3, C_4, C_5 and C_6 are displaced in Fig B.

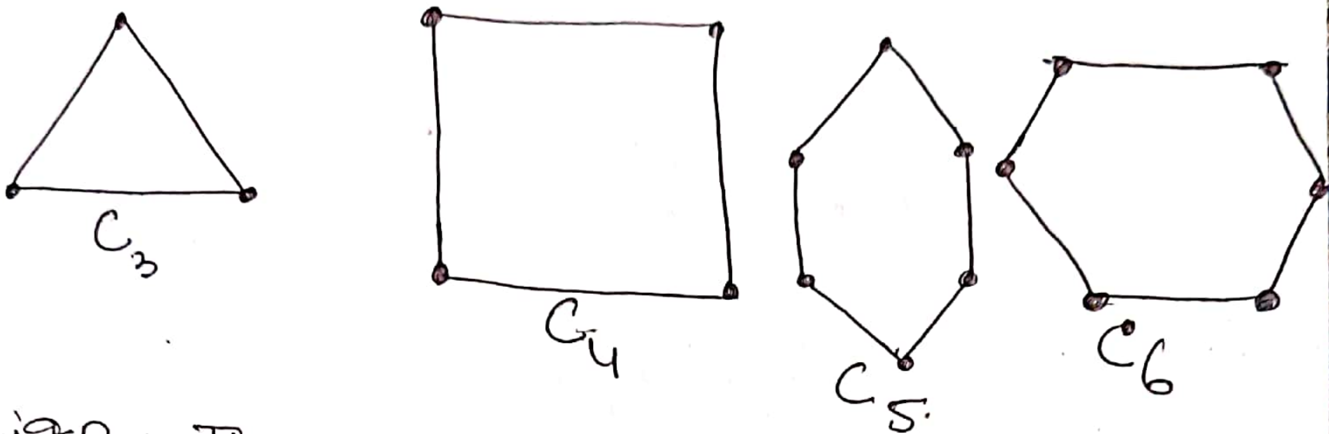


Fig B: The cycles C_3, C_4, C_5 and C_6.

Wheel :→ We obtain a wheel W_n when we add an additional vertex to a cycle C_n, for $n \geq 3$, and Connect this new vertex to each of the n vertices in C_n, by new edges. The wheels W_3, W_4, W_5 and W_6 are displaced in Figure C.

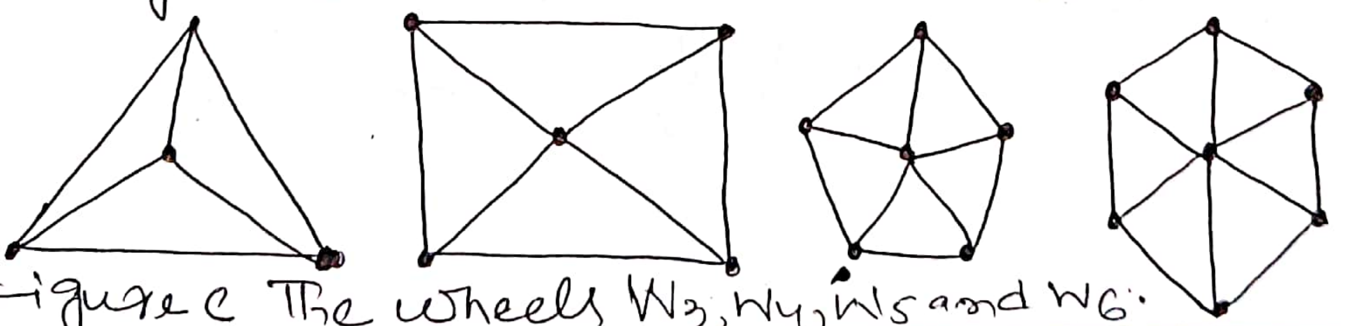
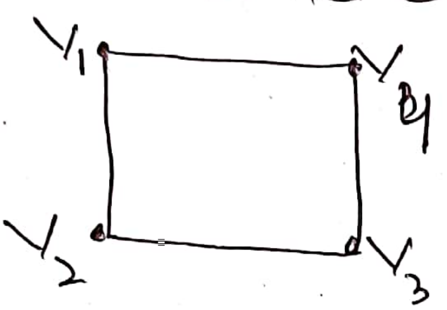


Figure C The wheels W_3, W_4, W_5 and W_6.

Bipartite Graph :→

A simple graph G is called bipartite if its vertex set V can be decomposed into two disjoint subsets V_1 and V_2 such that every edge in the graph G joins a vertex in V_1 with a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2)

Ex! Consider the cycle of length 4 i.e C_4.

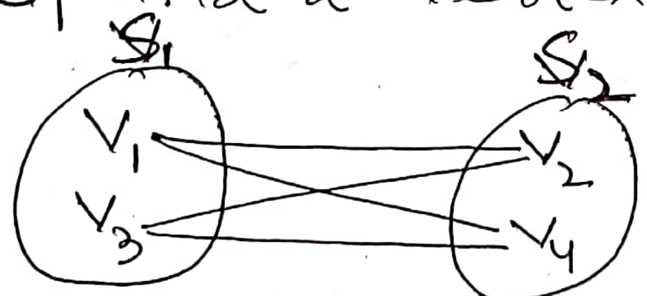


Vertex set $V = \{V_1, V_2, V_3, V_4\}$

The vertex set V can be decomposed into two disjoint subsets

$S_1 = \{V_1, V_3\}$ and $S_2 = \{V_2, V_4\}$

and every edge of C_4 connects a vertex in S_1 and a vertex in S_2.



Hence, C_4 ~~Bipartite~~ is bipartite.

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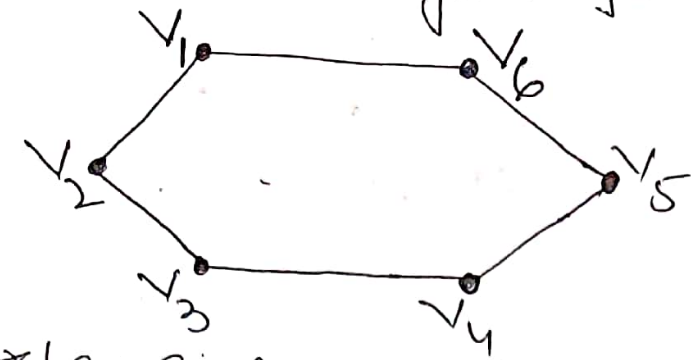
Ex: Consider the cycle of length 6 i.e C_6

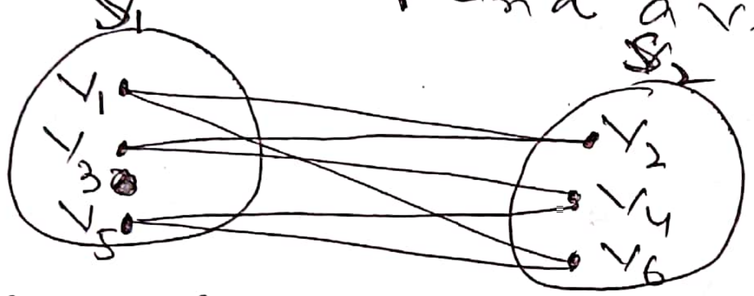


Vertex set $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$

The vertex set V can be decomposed into disjoint subsets

$S_1 = \{v_1, v_3, v_5\}$ and $S_2 = \{v_2, v_4, v_6\}$

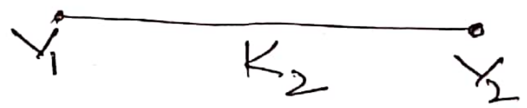
and every edge of C_6 connects a vertex in S_1 and a vertex in S_2.



Hence, C_6 is bipartite.

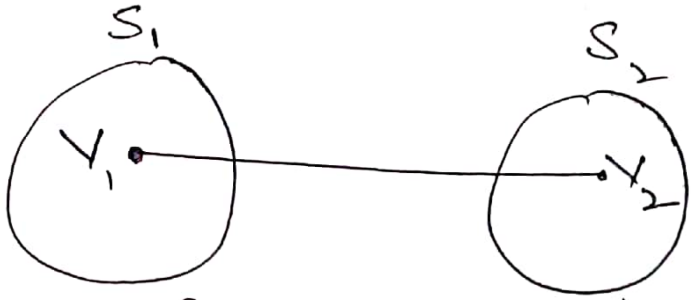
Ex:→ Prove the K_2 is bipartite.

Sol:→



The vertex set $V = \{v_1, v_2\}$ can be decomposed into disjoint subsets $S_1 = \{v_1\}$ and $S_2 = \{v_2\}$.

and every edge of K_2 connects a vertex in S_1 and a vertex in S_2.



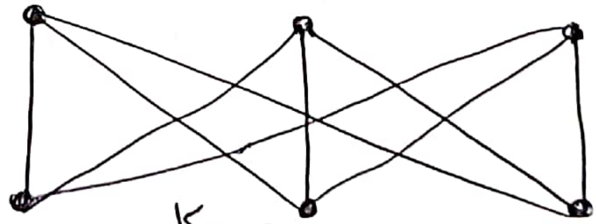
Ex:→ Show that K_3 is not Bipartite.

Sol:→ If we divide the vertex set of K_3 into two disjoint sets, one of the two sets must contain two vertices. If the graph were bipartite, these two vertices could not be connected by an edge, but in K_3 each vertex is connected to every other vertex by an edge. Hence, K_3 is not bipartite.

<u>Complete Bipartite graph:→</u>

A bipartite graph is said to be complete if every vertex of the first set is connected to every vertex of the second set. If the vertex set is decomposed into two subsets of m and n elements respectively, then its complete bipartite graph is denoted by $K_{m,n}$. The complete bipartite graphs $K_{2,3}$ and $K_{3,3}$ are displayed as follows:

$K_{2,3}$



$K_{3,3}$

Ex: C_4 is a complete bipartite graph

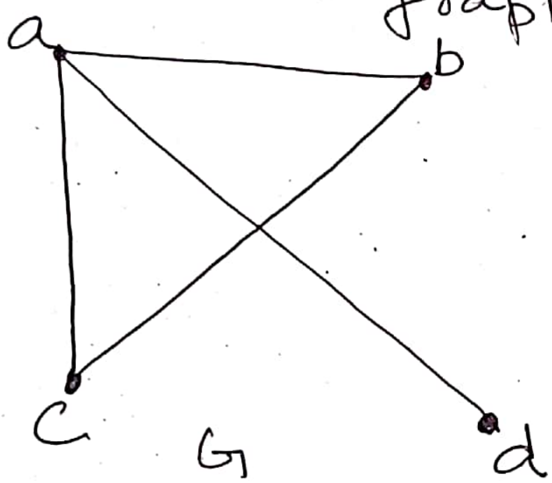
<u>Adjacency Matrix</u> :→ Suppose that $G=(V,E)$ is a simple graph with n vertices i.e $V=\{v_1, v_2, \dots, v_n\}$

Let a_{ij} denote the number of edges $\{v_i, v_j\}$, then $A=[a_{ij}]_{n\times n}$ is said to be adjacency matrix of G if

$$a_{ij} = \begin{cases} 1 & \text{, if } \{v_i, v_j\} \text{ is an edge of } G \\ 0 & \text{, otherwise} \end{cases}$$

~~The adjacency of the node~~

Ex:→ Use an adjacency matrix to represent the graph showing below



G

Sol:→ The matrix representing this graph is

$$A(G) = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Ex:→ Draw the graph with the adjacency matrix

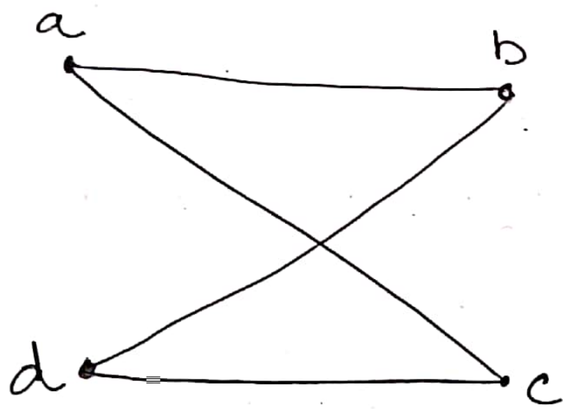
$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

with respect to the ordering of vertices a, b, c, d.

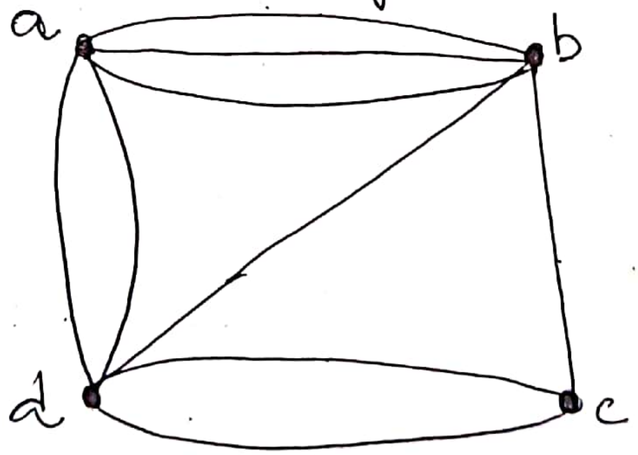
Sol:→ The graph with this ~~adjacency~~ adjacency matrix ~~is shown below~~

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

is shown below



Q:→ Find the adjacency matrix of the pseudograph



G

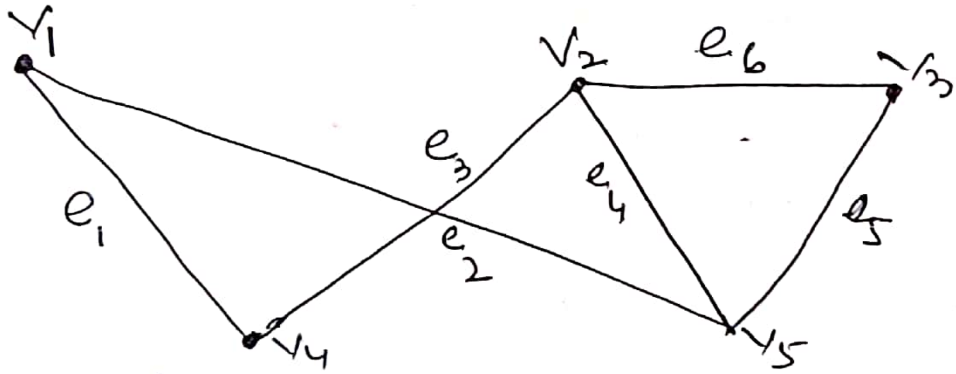
Sol:→ The adjacency matrix using the ordering the ordering of vertices a, b, c, d ie

$$A(G) = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

Incidence Matrices:→ Let $G=(V,E)$ be a simple graph with n vertices $v_1, v_2, \dots, v_n$ and m edges $e_1, e_2, \dots, e_m$, then $M=[m_{ij}]_{n\times m}$ is said to be incidence matrices of G if

$$m_{ij} = \begin{cases} 1, & \text{if the edge } e_j \text{ is incident on a vertex } v_j \\ 0, & \text{otherwise.} \end{cases}$$

Ex:→ Represent the graph shown in fig. A with an incidence matrix.



Sol:→ The incidence matrix is

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}_{5\times 6}.$$

Ex:→ Represent the pseudograph shown in fig. B using an incidence matrix

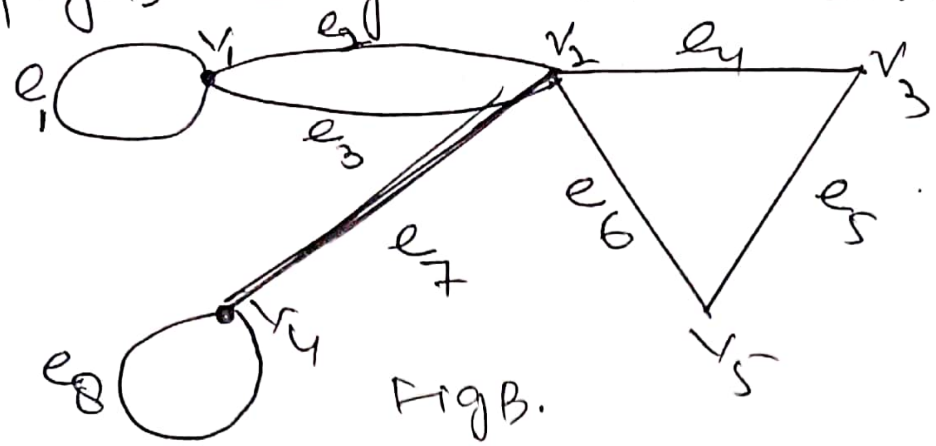


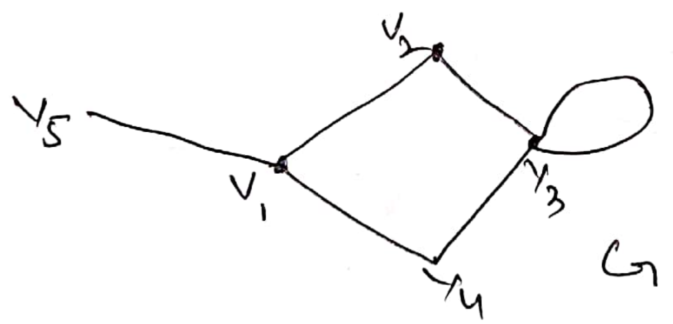
Fig B.

Sol:→ The incidence matrix for this graph is

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array}\begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Degree sequence in a graph:→

The degree sequence of an undirected graph is the non-increasing sequence of its vertex degrees.



G

Degree sequence of graph G is (4, 3, 2, 2, 1)

Isomorphism Of Graphs :→

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be Isomorphic if there is one-to-one and onto function from V_1 and V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1. Such a function f is called an Isomorphism

Remark :→ Two graphs G_1 and G_2 are Isomorphic if

1. G_1 and G_2 has the same number of vertices.
2. G_1 and G_2 has the same number of edges.
3. Degree sequence of G_1 = Degree sequence of G_2.

Ex :→ Show that the graphs $G = (V, E)$ and $H = (W, F)$, displayed in Figure A are Isomorphic.

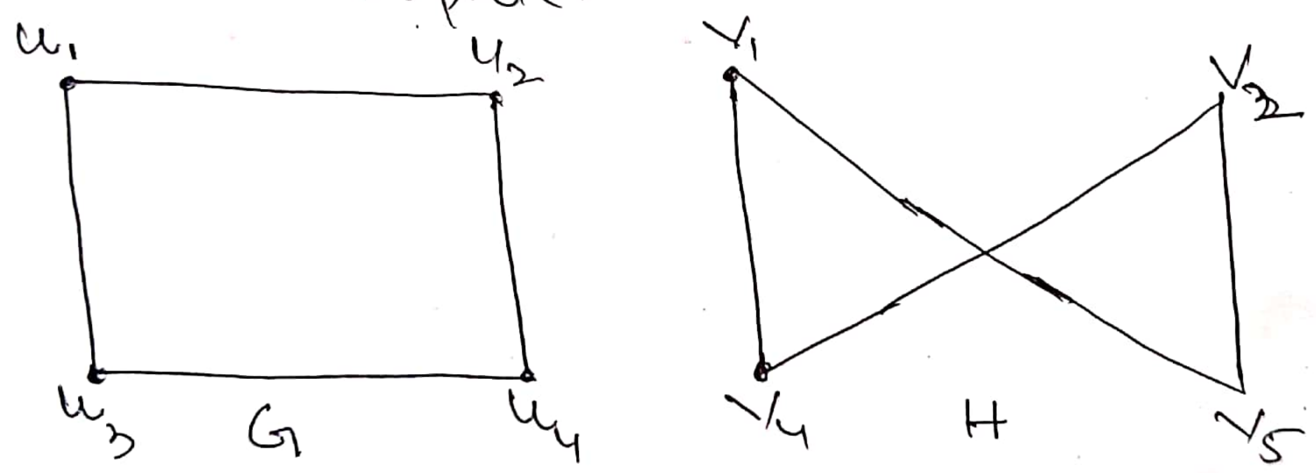
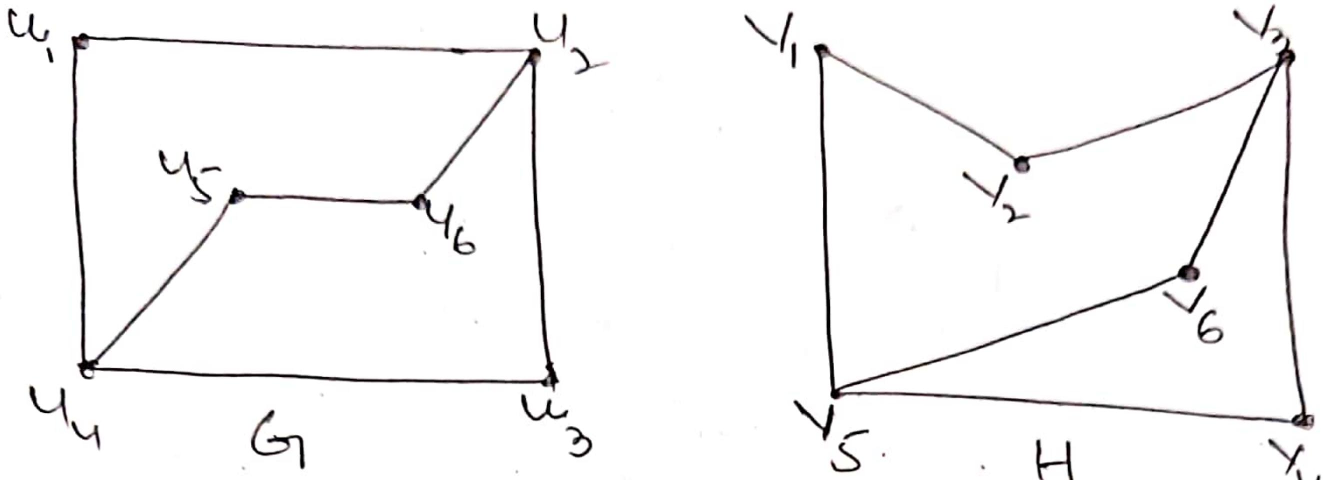


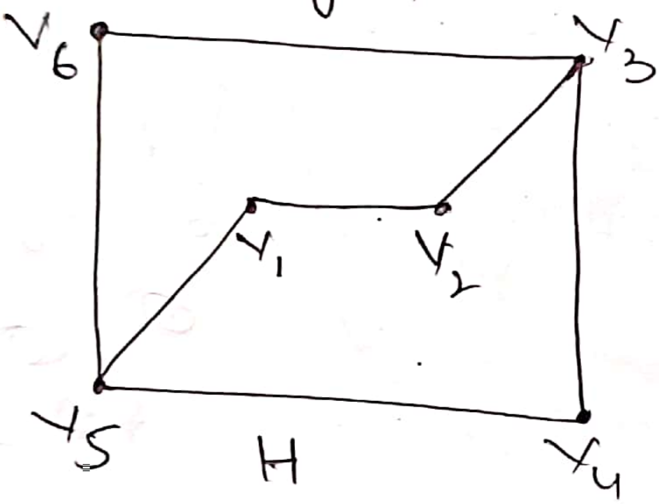
Figure A

Clearly, $A(G) = A(H)$. Hence, the graphs G and H are Isomorphic.

Ex:→ Determine whether the graphs G and H displayed in Figure e are Isomorphic



Sol:→ Note that graph H can be drawn in the following form



Define a function $f: V \to W$ by
$f(u_1) = v_6$, $f(u_2) = v_3$, $f(u_3) = v_4$
$f(u_4) = v_5$, $f(u_5) = v_1$ and $f(u_6) = v_2$
where $V = \{u_1, u_2, u_3, u_4, u_5, u_6\}$
and $W = \{v_1, v_2, v_3, v_4, v_5, v_6\}$.

Clearly, the function f is one-to-one and onto. To see that this correspondence ~~[illegible]~~ preserves adjacency, we show that $A(G) = A(H)$

$$
\begin{array}{c|c}
 & \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\
\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} &
A(G) = \begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0
\end{bmatrix}
\end{array}
$$

$$
\begin{array}{c|c}
 & \begin{array}{cccccc} v_6 & v_3 & v_4 & v_5 & v_1 & v_2 \end{array} \\
\begin{array}{c} v_6 \\ v_3 \\ v_4 \\ v_5 \\ v_1 \\ v_2 \end{array} &
A(H) = \begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0
\end{bmatrix}
\end{array}
$$

Clearly, $A(G) = A(H)$. Hence, the graphs G and H are Isomorphic.

Ex:→ Show that the graphs displayed in Figure D are not Isomorphic.

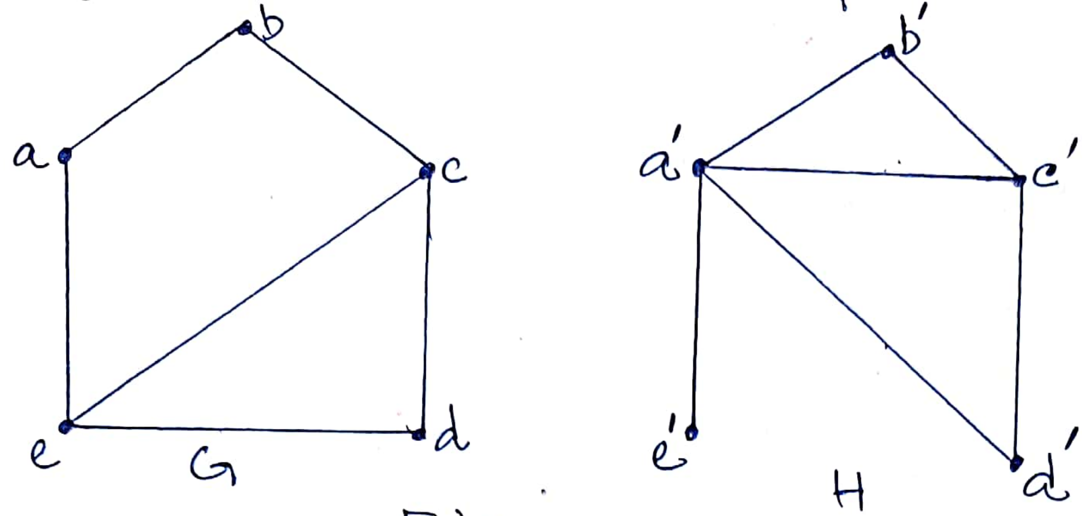


Figure D

Sol:→ Both G and H have five vertices and six edges. Degree sequence of G is [3,3,2,2,2] and Degree sequence of H is [4,3,2,2,1]. Thus, G and H have different degree sequences i.e H has a vertex a' of degree 4, whereas G has no vertex of degree 4. Thus, we conclude that G and H are not Isomorphic.

Ex:→ Determine whether the graphs shown in Figure B are Isomorphic

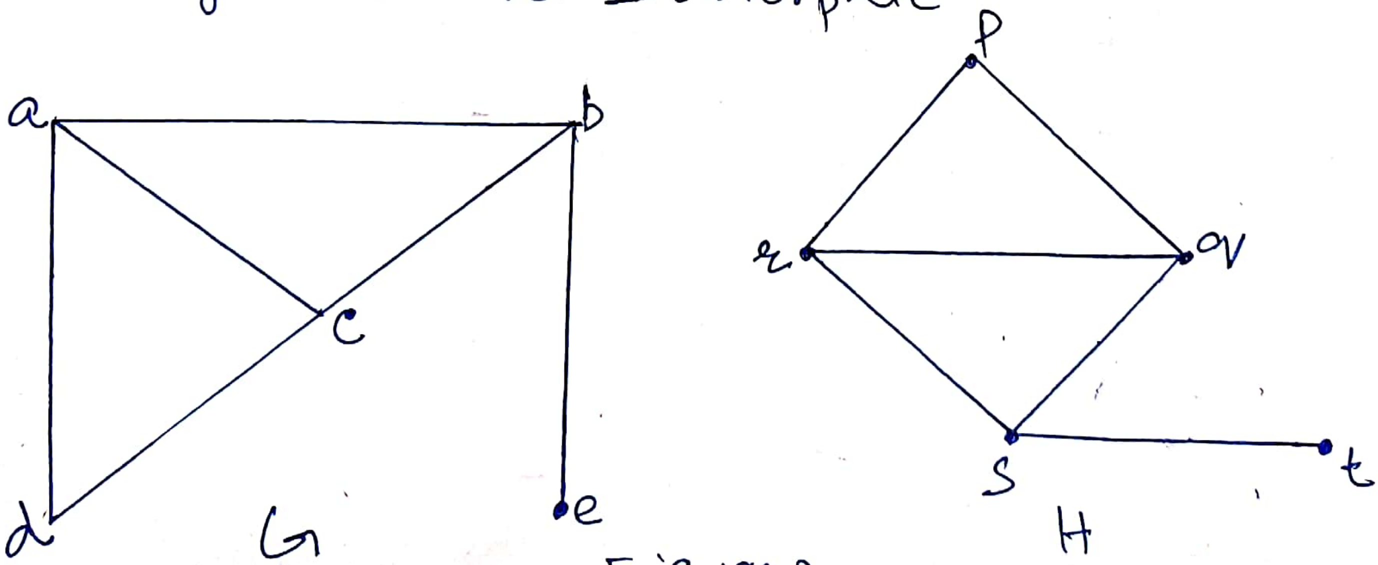
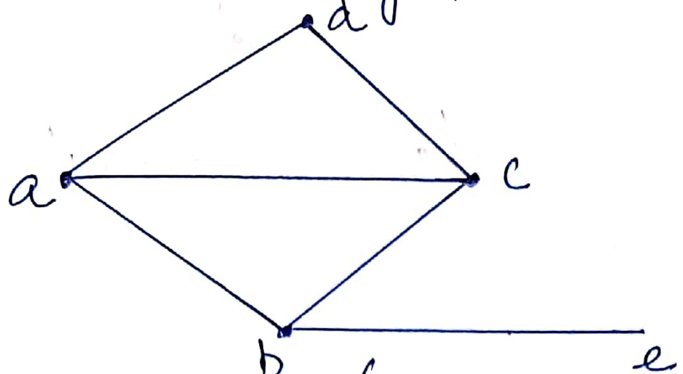


Figure B

Sol:→ Note that the graph G can be drawn in the ~~form~~ following form



G

Define a function $f: V \to W$ by
$f(d) = p$, $f(a) = r$, $f(c) = q$, $f(b) = s$ and $f(e) = t$
where V and W are vertex sets of G and H, respectively. Clearly, the function f is one-to-one and onto. To see that this correspondence preserves adjacency, we show that $A(G) = A(H)$.

$$A(G) = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{c} \\ r \\ s \\ q \\ p \\ t \end{array} \begin{array}{c} \begin{array}{ccccc} r & s & q & p & t \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Clearly, $A(G) = A(H)$. Hence the graphs G and H are Isomorphic.

Ex:→ Determine whether the graphs in Figure 1 are Isomorphic

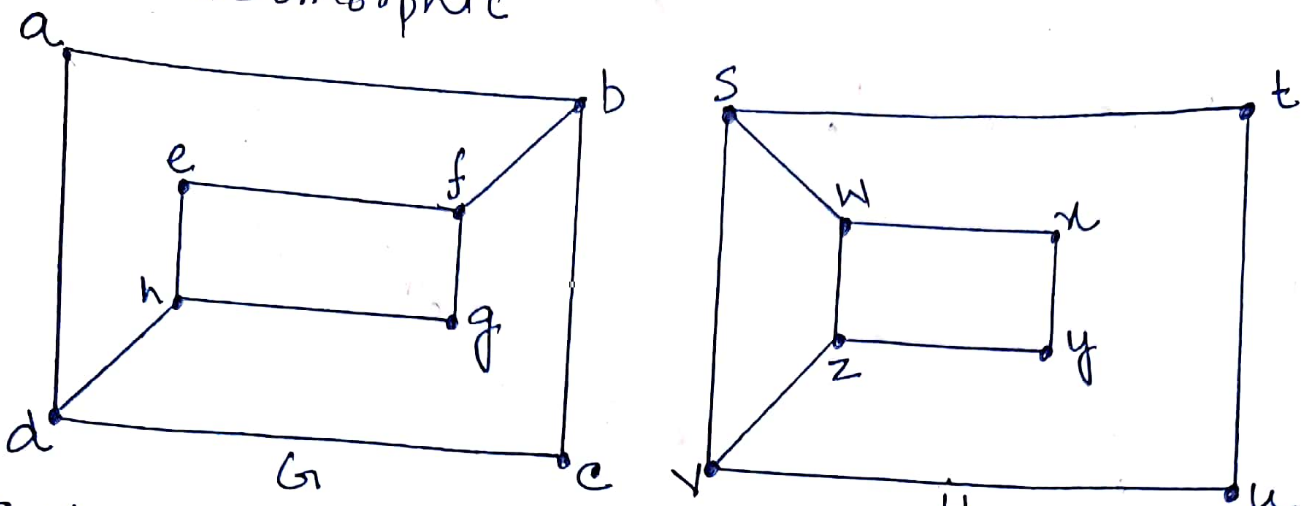


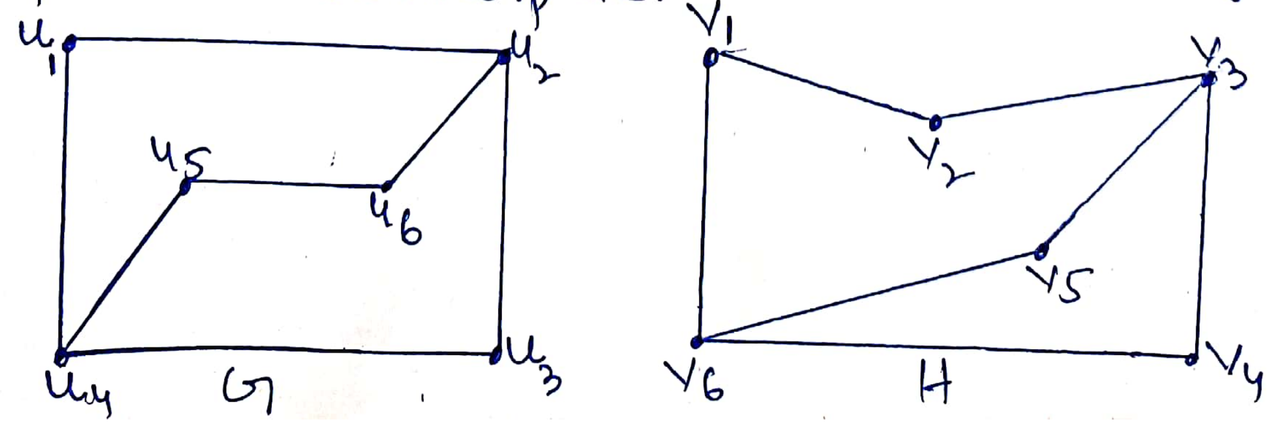
Figure 1

Sol:→ The graph's G and H both have eight and 10 edges. Degree sequence of G is [3,3,3,3,2,2,2,2] and degree sequence of H is [3,3,3,3, 2,2,2,2]. Hence, G and H satisfies the necessary conditions for the two graphs to be Isomorphic.

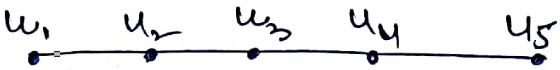
However, G and H are not Isomorphic, because in G there is a vertex a of degree 2 which is adjacent to vertices b and d each of degree 3. But there is no vertices in H of degree 2 adjacent to two vertices of degree 3.

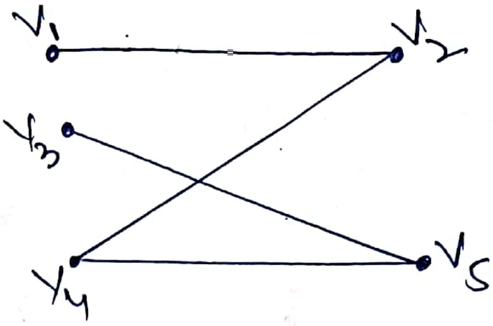
Ex:→ Determine whether the following graphs are Isomorphic.

(1)

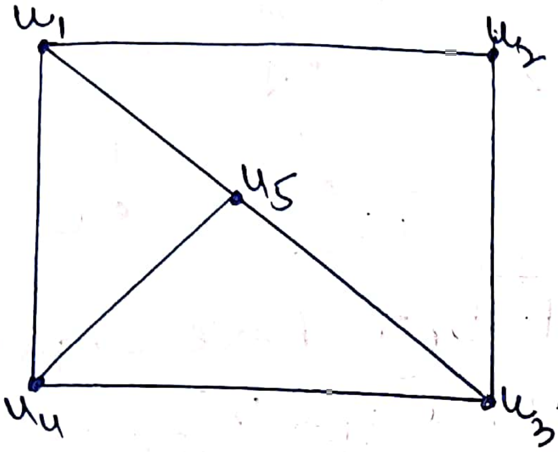


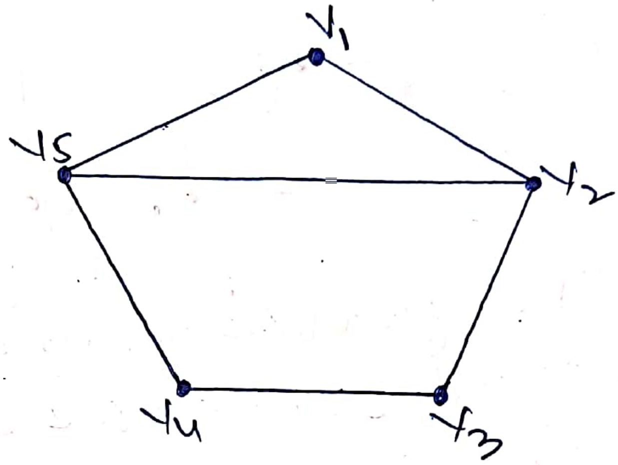
(2)


u1
u2
u3
u4
u5

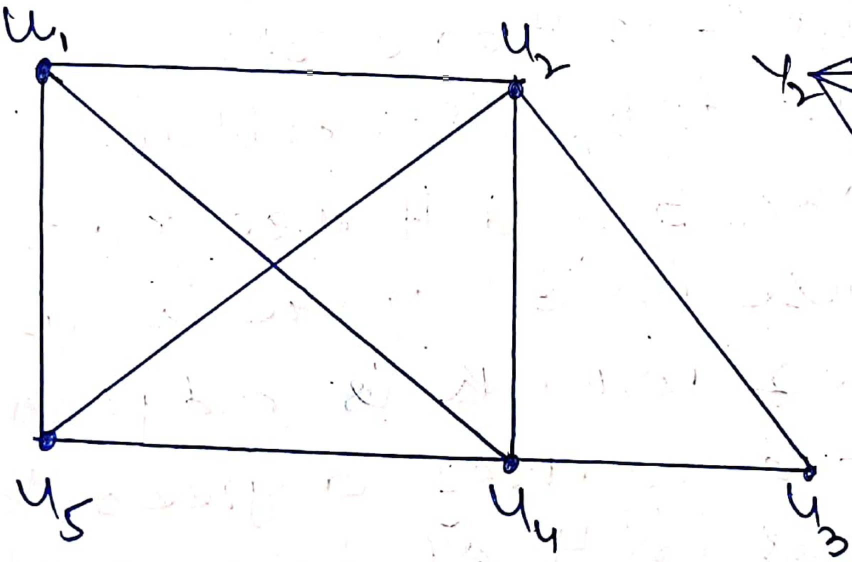

v1
v2
v3
v4
v5

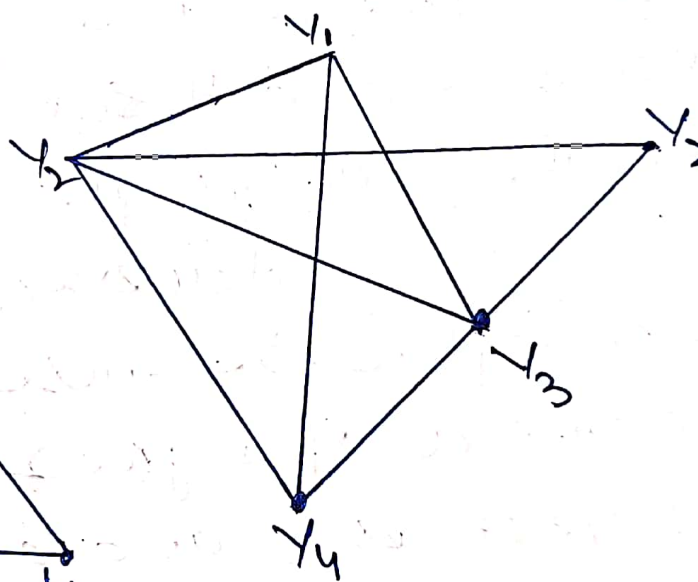
(3)


u1
u2
u5
u4
u3


v1
v5
v2
v4
v3

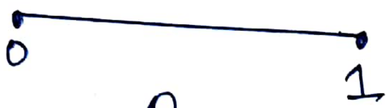
(4)


u1
u2
u5
u4
u3

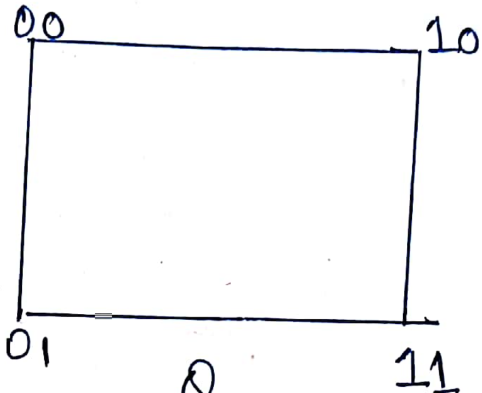

v1
v2
v3
v4

Cube:→ A graph Q_n whose vertex set is the set of binary numbers of length n and two vertices are adjacent if the number of 1's between the two binary numbers differ by 1.

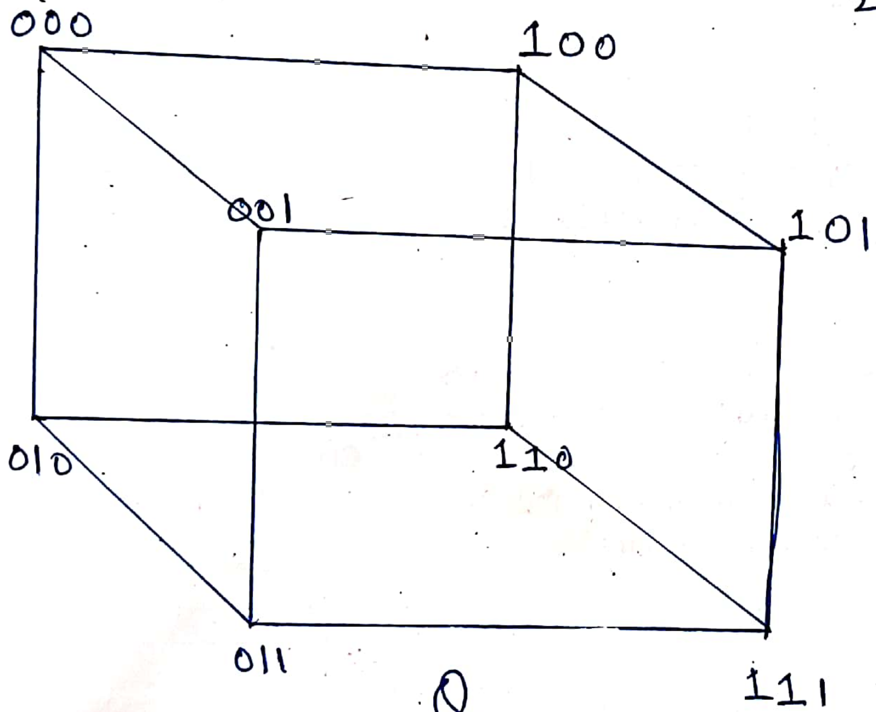
Ex:



Q_1



Q_2



Q_3

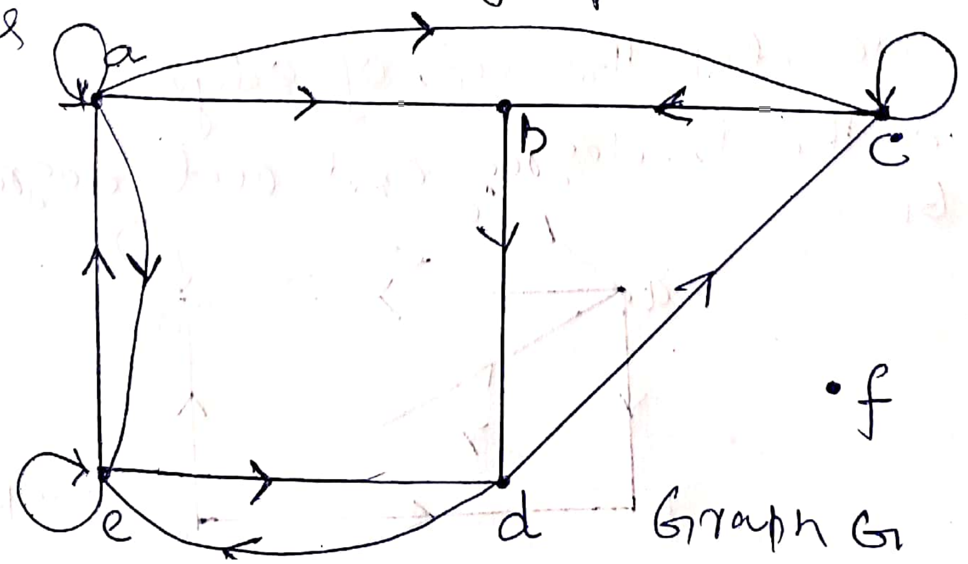
Remark:→

1. order of Q_n is 2^n
2. Q_n is n-regular
3. size of Q_n is $n2^{n-1}$.

Definition:→ When (u, v) is an edge of a directed graph G, then u is said to be adjacent to v and v is said to be adjacent from u. The vertex u is called the initial vertex of (u, v) and v is called the terminal vertex or end vertex of (u, v). The initial vertex and terminal vertex of a self-loop are the same.

Definition:→ In a connected graph G, the in-degree of a vertex v, denoted by $deg^-(v)$, is the number of edges with v as their terminal vertex. The out-degree of v, denoted by $deg^+(v)$, is the ~~...~~ number of edges with v as their initial vertex. Note that a self-loop contributes 1 to both the in-degree and the out-degree of this vertex.

Ex!→ Find the in-degree and out-degree of each vertex in the graph G with directed edges



Graph G

Sol:→ The in-degree of G are

$deg^-(a)=2$, $deg^-(b)=2$, $deg^-(c)=3$, $deg^-(d)=2$, $deg^- e=3$ and $deg^-(f)=0$.

The out-degree of G are

$deg^+(a)=4$, $deg^+(b)=1$, $deg^+(c)=2$, $deg^+(d)=2$, $deg^+(e)=3$ and $deg^+(f)=0$

Note that sum of in-degree's of G are

$deg^-(a)+deg^-(b)+deg^-(c)+deg^-(d)+deg^-(e)$
$+deg^-(f) = 2+2+3+2+3+0 = 12$

and sum of out-degree's of G are

$deg^+(a)+deg^+(b)+deg^+(c)+deg^+(d)+deg^+(e)$
$+deg^+(f) = 4+1+2+2+3 = 12$

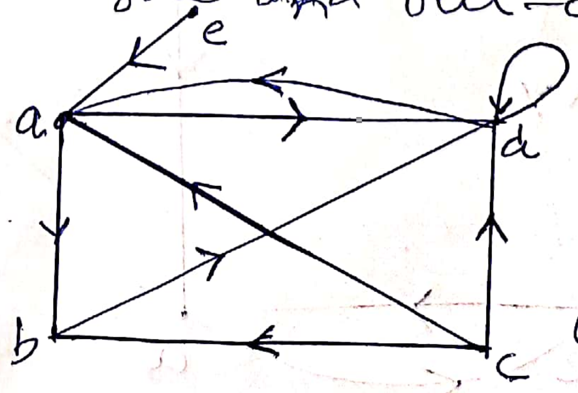
Hence, we have the following theorem:

<u>Theorem</u>:→ Let $G=(V,E)$ be a directed graph, then

$$\sum_{v\in V} deg^-(v) = \sum_{v\in V} deg^+(v) = |E|$$

where $|E|$ denotes the no. of edges.

Ex:→ Find the in-degree and out-degree of the graph G



Graph G

Walk :→ A walk is defined as a continuous sequence of vertices and edges, begining and ending with vertices. Vertices and edges can be repeated in a walk.

Trail :→ A trail is a walk in which no edge is repeated.

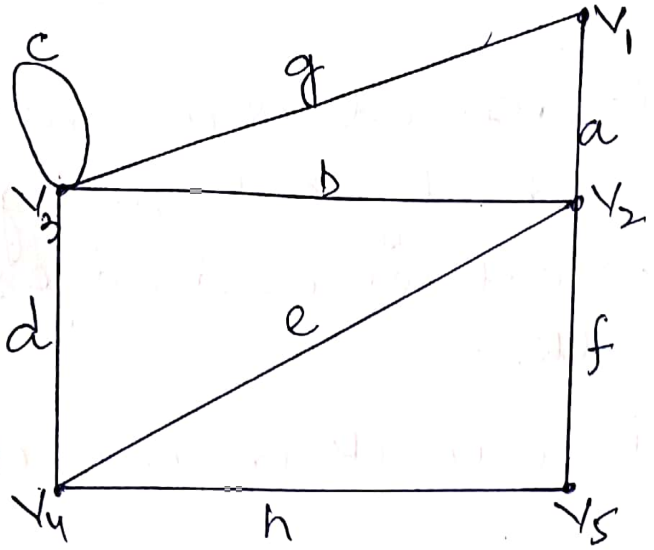
Path :→ A path is a trail in which no vertex is repeated i.e a path is a walk in which no vertex and edge is repeated. A path of order n is denoted by P_n.

Open walk :→ A walk is said to be an open walk if its intial and terminal vertex are distinct.

closed walk :→ A walk is said to be an closed walk if its intial and terminal vertex are same.

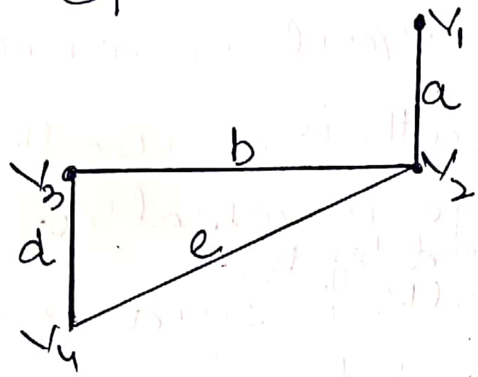
Circuit :→ A closed walk in which no edge is repeated is ~~called~~ called a circuit. Vertices can be repeated in a circuit.

cycle :→ A circuit in which no vertex is repeated except the intial vertex (i.e the intial and terminal vertex are same) is called a ~~circuit~~ cycle. A cycle of order n is denoted by C_n.



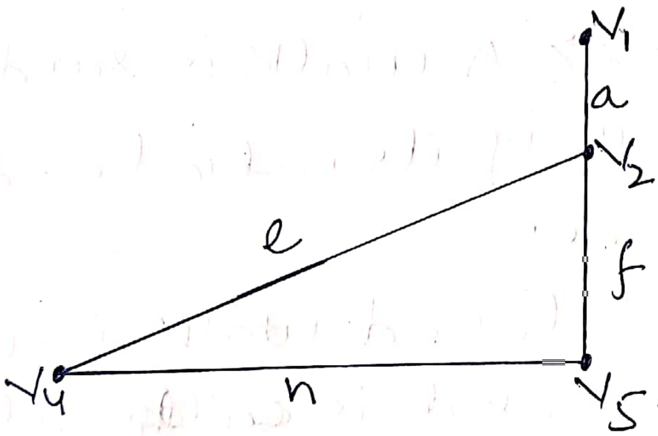
G

Ex:→



$v_1 a v_2 b v_3 d v_4 e v_2 a v_1$ is a walk in G.

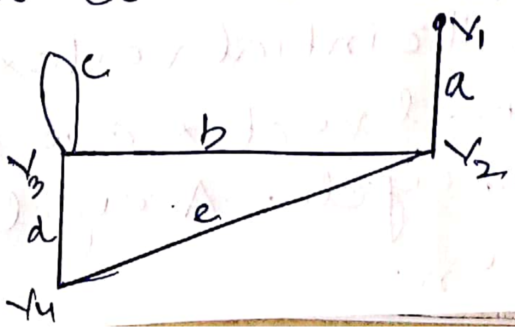
Ex:→



$v_2 e v_4 h v_5 f v_2 a v_1$ is a wak in G.

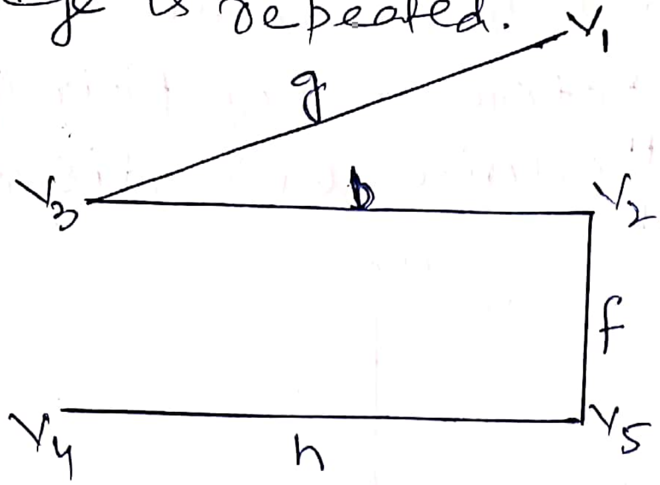
Ex:→ $v_4 h v_5 f v_2 a v_1 g v_3 c v_3 g$ is not a walk but it did not terminated at a vertex.

Ex:→



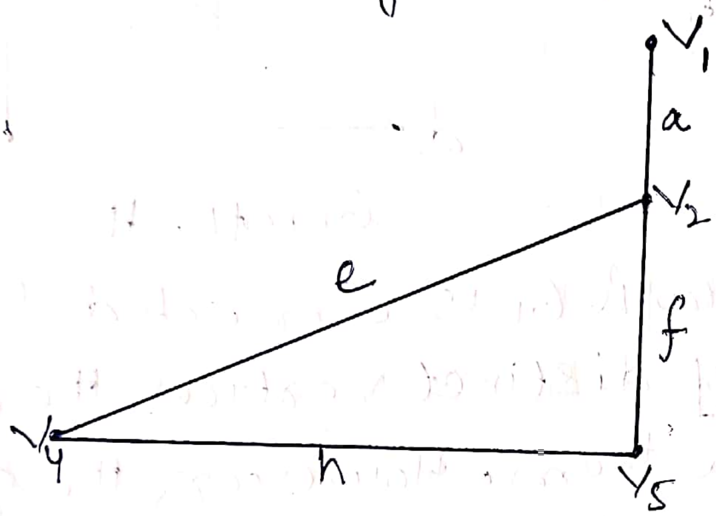
$v_1 a v_2 b v_3 c v_3 d v_4 e v_2$ is a trail because no edge is repeated.

Ex:→



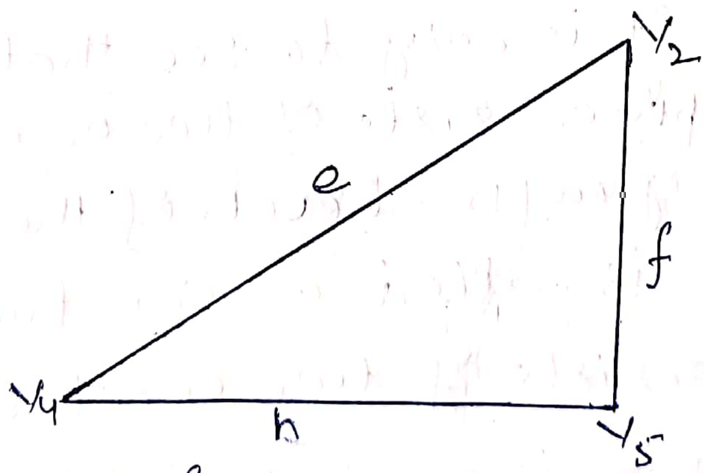
$v_1 g v_3 b v_2 f v_5 h v_4$ is a path because no vertex and edge is repeated.

Ex:→



$v_1 a v_2 e v_4 h v_5 f v_2 a v_1$ is a circuit in G

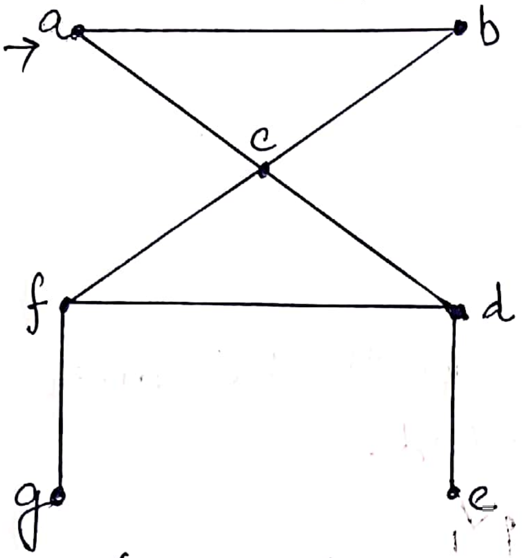
Ex:→



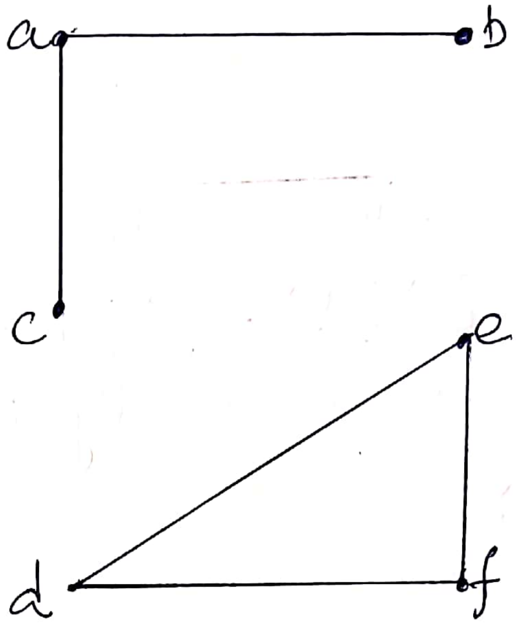
$v_2 e v_4 h v_5 f v_2$ is a cycle in G.

Connected Graphs:→ An undirected graph G is said to be connected if there is atleast one path between every pair of distinct vertices in G. Otherwise, G is disconnected.

Ex:→



Graph G



Graph H

The graph G is connected because for every pair of distinct vertices there is a path between them. However, the graph H is not connected. For instance, there is no path in H between the vertices c and d.

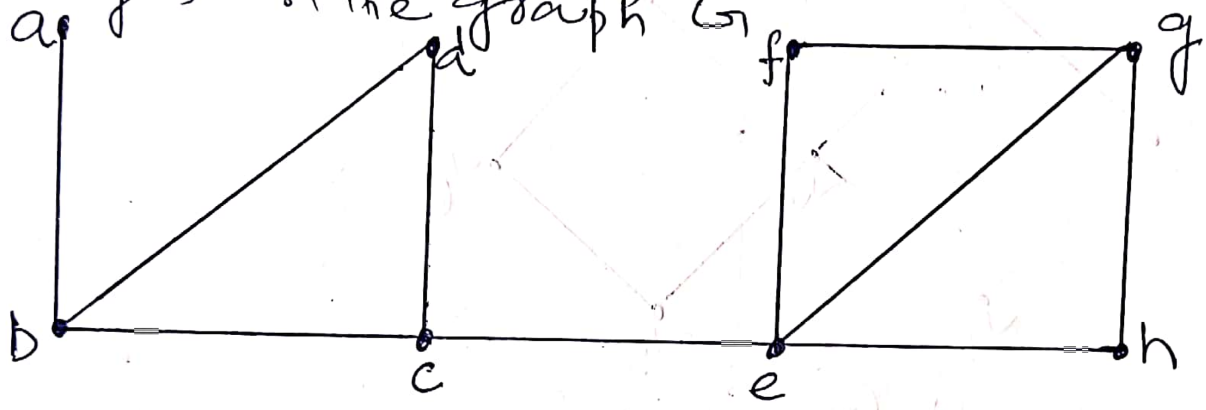
It is easy to see that a disconnected graph consists of two or more connected graphs. Each of these connected subgraphs is called a component. The graph H consists of two components.

Theorem:→ There is a simple path between every pair of distinct vertices of a connected undirected graph.

Cut Vertex :→ Let G be a connected graph. A vertex $v \in G$ is said to be a cut vertex of G if its removal from the graph G leaves the graph G disconnected.

Cut Edge :→ Let G be a connected graph. An edge $E \in G$ is said to be cut edge of G if its removal from the graph G leaves the graph G disconnected.

Ex:→ Find the cut vertices and cut edges in the graph G

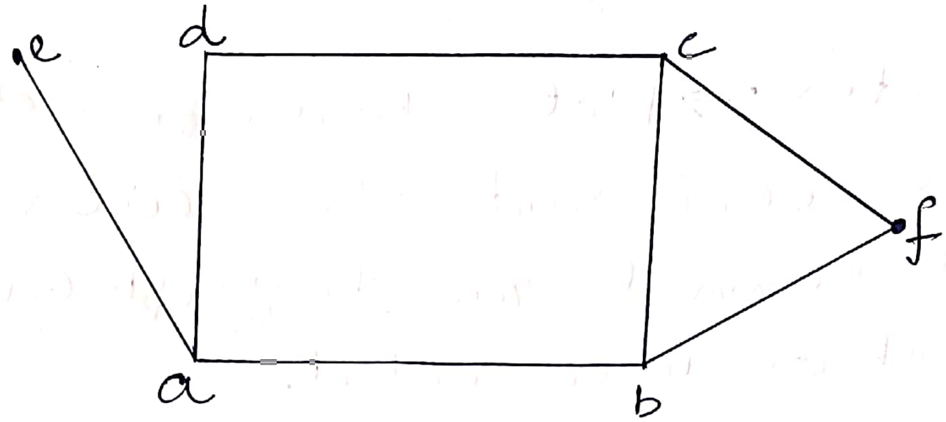


The Graph G

Sol:→ The cut vertices of G are b, c and e. The removal of one of these vertices (and its adjacent edges) disconnects the graph G. The cut edges are {a, b} and {c, e}. Removing either one of these edges disconnects the graph G.

Q:→ Find the ~~xxx~~ cut vertex and cut edge of the ~~xxxxx xxxx~~ Graph G

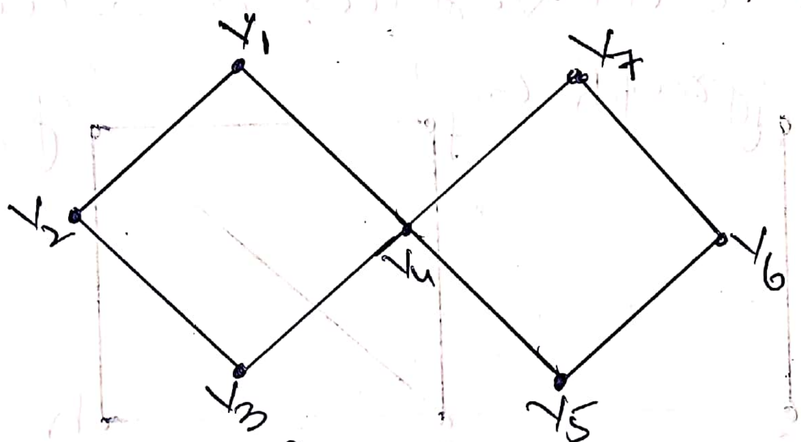
Sol:→



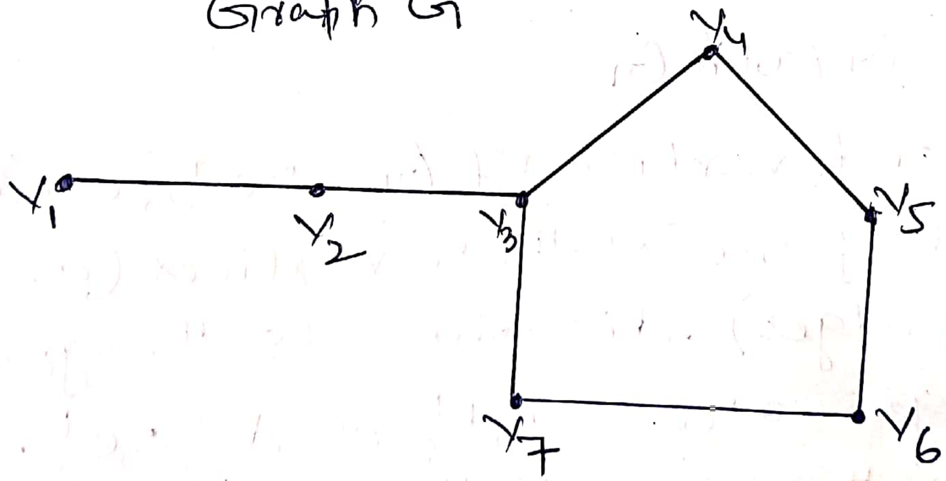
Graph G

Sol:→ Clearly, the graph G has cut vertex a and cut edge {a, e}.

Ex:→ Find the cut vertex and cut edge (if they exist) of the following graphs:



Graph G



Graph H.

Remark:→ Not all graphs have cut vertices and cut edges. For instance, the complete graph K_n, $n \geq 3$, has no cut vertices and cut edges.

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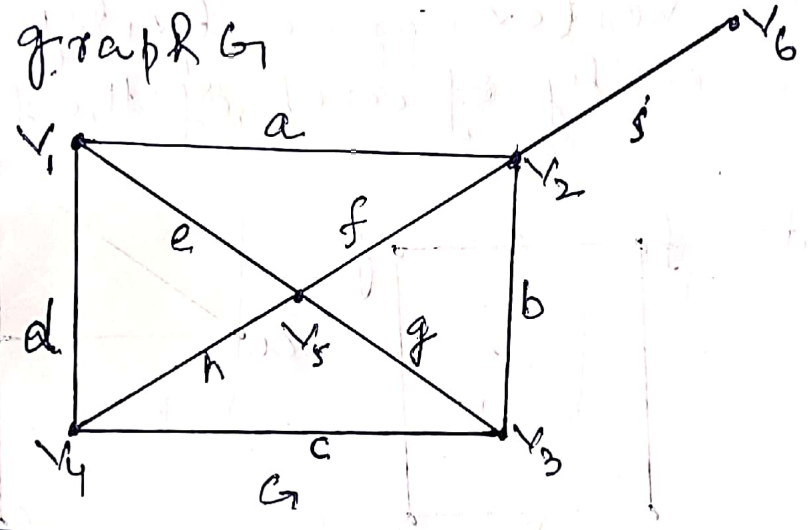
Vertex cut or cut set: → In a connected graph G, a cut set is a set of vertices whose removal from G leaves the graph G disconnected.

Edge cut or cut set: → In a connected graph G, a cut set is a set of edges whose removal from G leaves the graph G disconnected.

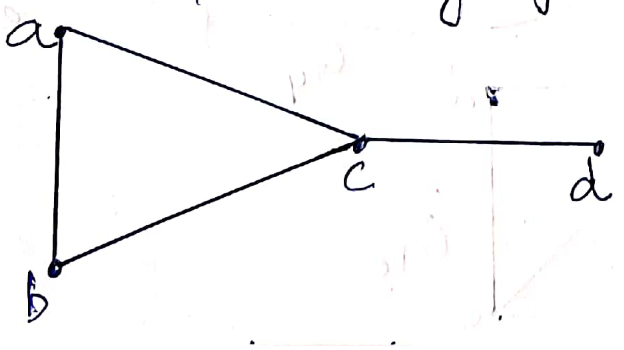
Q:→ Find the vertex cut and edge cut of the graph G

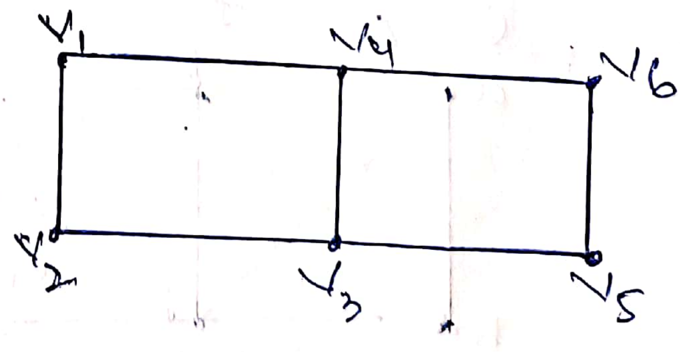


G

Sol:→ The vertex ~~of~~ cut of G are $\{v_1, v_3, v_5\}$, $\{v_2, v_4, v_5\}$, $\{v_2\}$ and $\{v_4\}$. The edge cut of G are $\{a, e, h, c\}$, $\{e, h, f, g\}$ and $\{v_2, v_6\}$.

Q:→ Find the vertex cut and edge cut of the following graphs

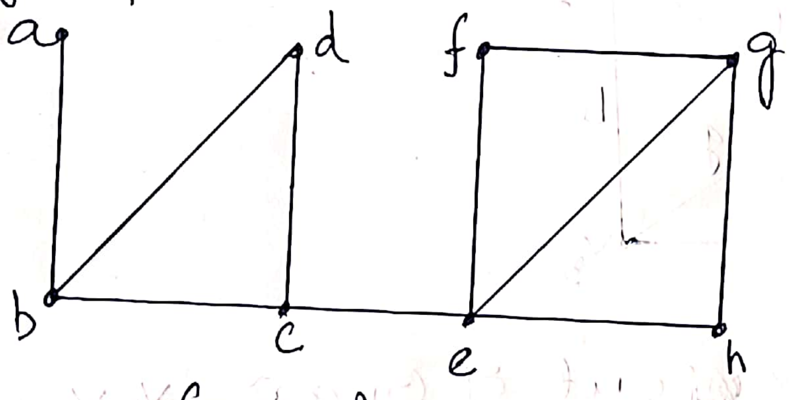




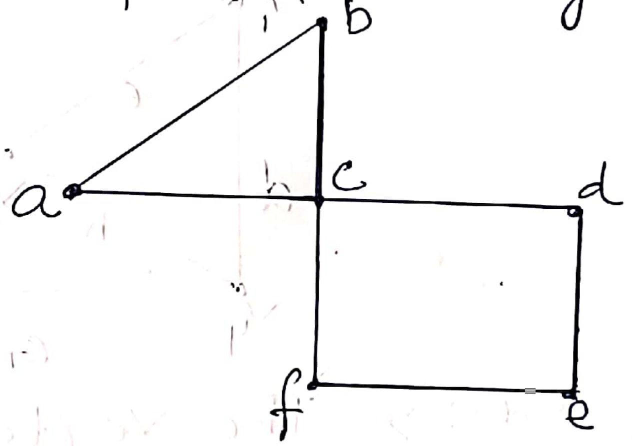
Vertex Connectivity:→ The vertex connectivity of a connected graph G is defined as the minimum number of vertices in a cut set. It is denoted by K(G)

Edge Connectivity:→ The edge connectivity of a connected graph G is the minimum number of edge in a cutset (Edge cut). It is denoted by λ(G).

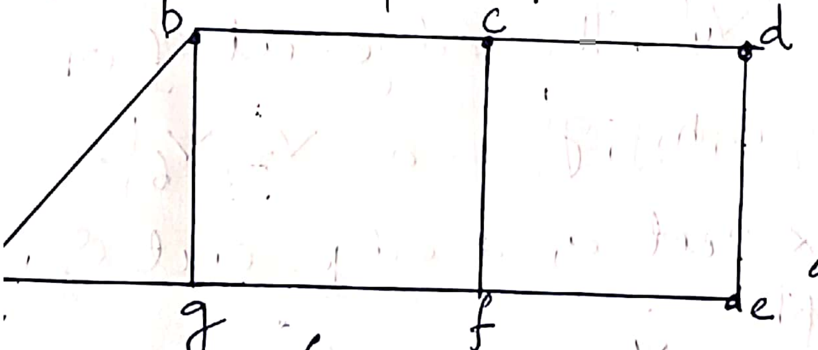
Ex:→ Find the vertex connectivity and Edge connectivity of each of the following graphs:



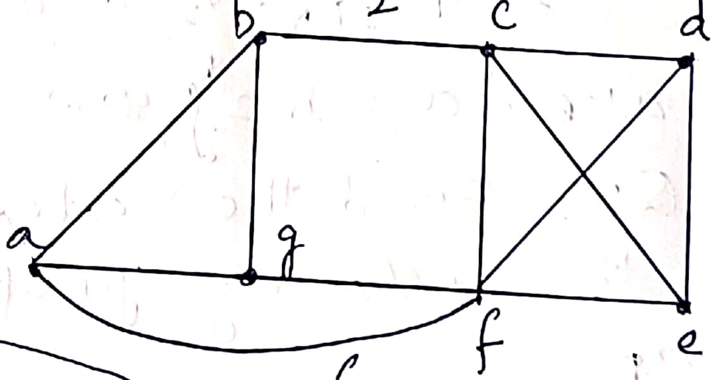
Graph G_1



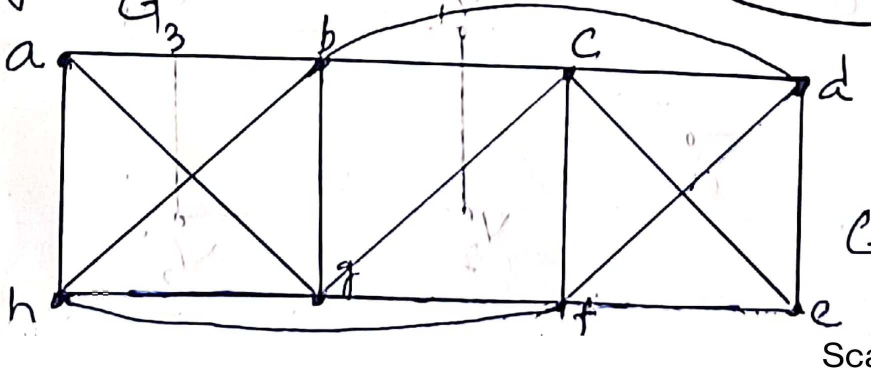
G_2



G_3



G_4



G_5

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Sol:→ Each of the five graphs is connected and has more than one vertex, so each of these graphs has positive vertex connectivity. Because G_1 is a connected with a cut sets ~~vertices~~ $\{b\}$, $\{c\}$ and $\{e\}$, hence $K(G_1)=1$. Also, $\{c\}$ is a cut set of G_2, so $K(G_2)=1$

The graph G_3 has no cut vertices, but has cut sets $\{b,g\}$, $\{c,f\}$ and $\{d,e\}$, so $K(G_3)=2$. Similarly, G_4 has cut sets of size two such as $\{c,f\}$ but no cut vertices, therefore $K(G_4)=2$. The graph G_5 has no cut set of size two, but $\{b,c,f\}$ is a cut set of G_5. Hence, $K(G_5)=3$.

G_1 has cut edges $\{a,b\}$ and $\{c,e\}$, therefore $\lambda(G_1)=1$. The graph G_2 has no cut edges, but the removal of the two edges $\{a,b\}$ and $\{a,c\}$ disconnects it. Hence, $\lambda(G_2)=2$. Similarly, $\lambda(G_3)=2$ because G_3 has no cut edges, but the removal of the two edges $\{b,c\}$ and $\{f,g\}$ disconnects it.

Also, the graph G_4 has no cut edges, but the removal of the three edges

$\{b,c\}$, $\{a,f\}$ and $\{f,g\}$ disconnects it. Hence, $\lambda(G_4)=3$. The removal of any two edges in G_5 does not disconnect it; Therefore, $\lambda(G_5)=3$ because removal of three ~~edges~~ edges $\{a,b\}$, $\{a,g\}$ and $\{a,h\}$ disconnects it.

Whitney's Theorem :→ For any graph G,
$K(G) \le \lambda(G) \le \delta(G)$, where $\delta(G)$ denotes the smallest vertex degree of G.

Note :→ ① $K(G)=0$ iff G has no edge.

② $K(P_n) = 1$, $n \geq 2$

③ $K(C_n)=2$, $n \geq 3$

④ $K(W_n)=3$, $n \geq 3$

⑤ $K(K_n) = n-1$

⑥ $K(Q_n) = n$.

Note :→ ① $\lambda(G)=0$ iff G has no edges

② $\lambda(P_n) = 1$, $n \geq 2$

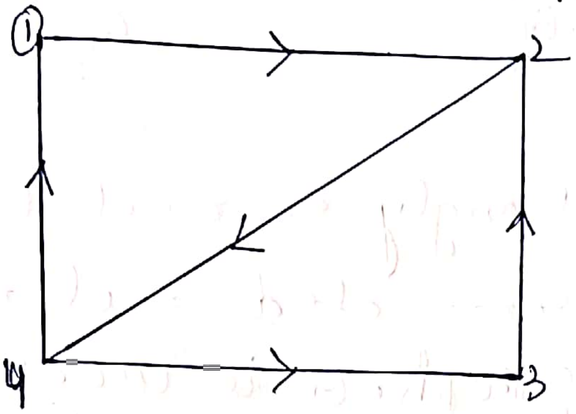
③ $\lambda(C_n) = 2$, $n \geq 3$

④ $\lambda(W_n) = 3$, $n \geq 3$

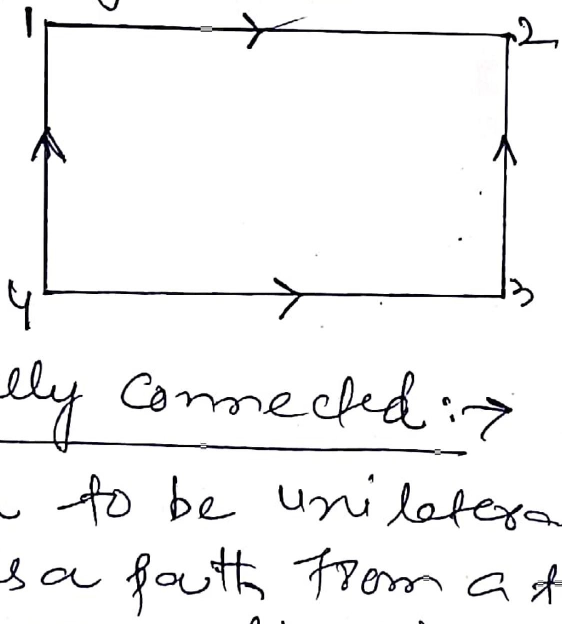
⑤ $\lambda(K_n) = n-1$

⑥ $\lambda(Q_n) = n$.

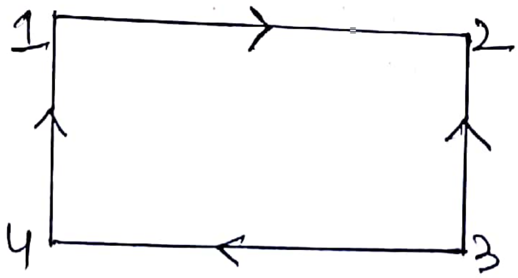
Strongly Connected:→ A directed graph G is said to be strongly connected if there is a path from a to b and from b to a, whenever a and b are vertices in a graph G.



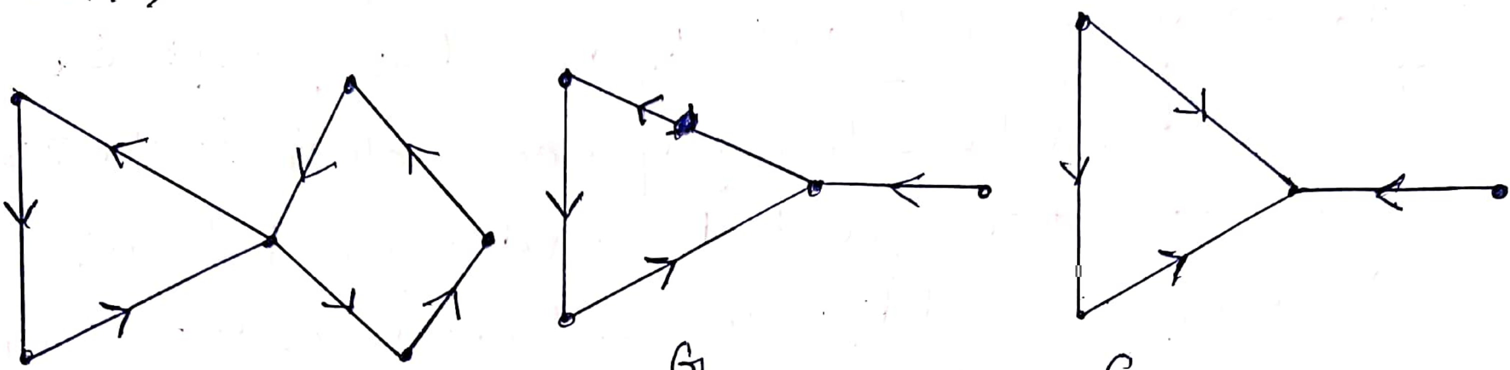
Weakly Connected Graph:→ A directed graph G is said to be weakly connected if there is a path between every two vertices in the underlying undirected graph.



Unilaterally Connected:→ A directed graph G is said to be unilaterally connected if there is a path from a to b or b to a, whenever a and b are vertices in a graph G.



Ex:→



G_1 G_2 G_3

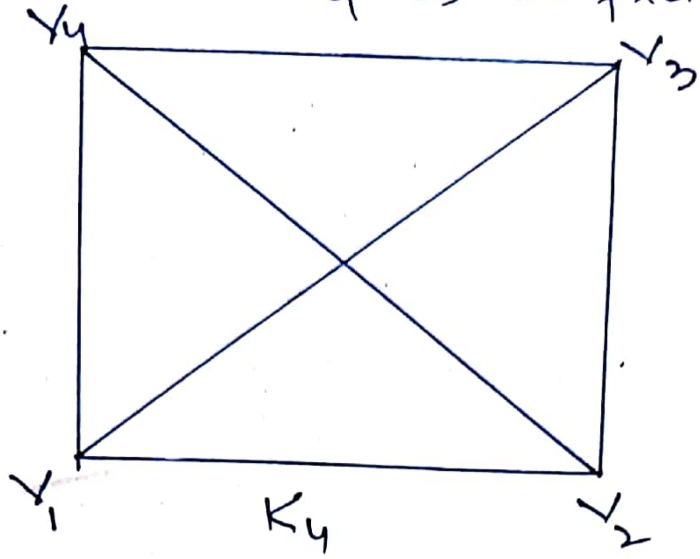
Clearly, G_1 is strongly connected graph. G_2 is unilaterally connected but not strongly connected graph. G_3 is weakly connected but neither unilaterally connected nor strongly.

Remark:→ Strongly connected ⇒ Unilaterally connected ⇒ weakly connected.

Planar Graph:→ A graph G is said to be planar if it can be drawn on a plane without a crossover of edges.

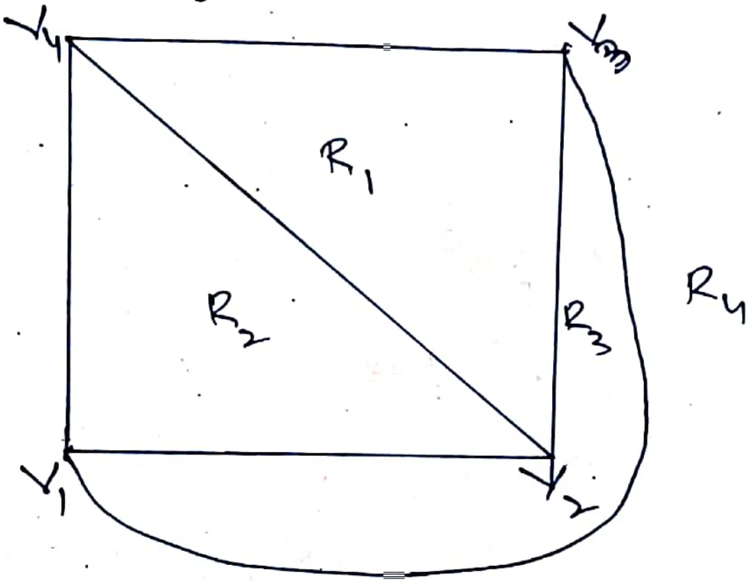
Ex:→ Show that K_4 is a planar graph.

Sol:→



K_4

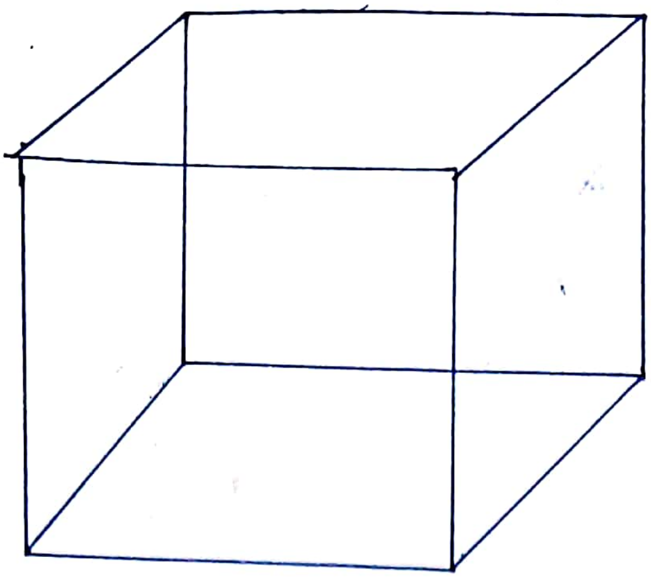
The above graph K_4 can be redrawn as



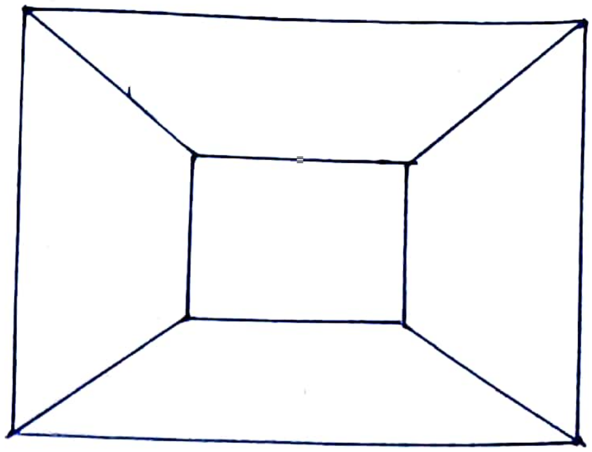
Hence, there is no crossover of the edges in K_4. Thus, K_4 is a planar graph.

Ex:→ Show that Q_3 is a planar graph.

Sol:→ Q_3 is planar, because it can be drawn without any edges crossing



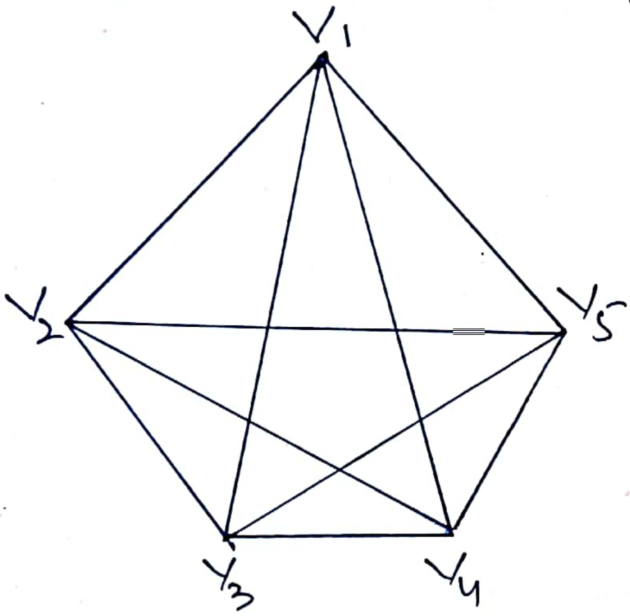
The graph Q_3



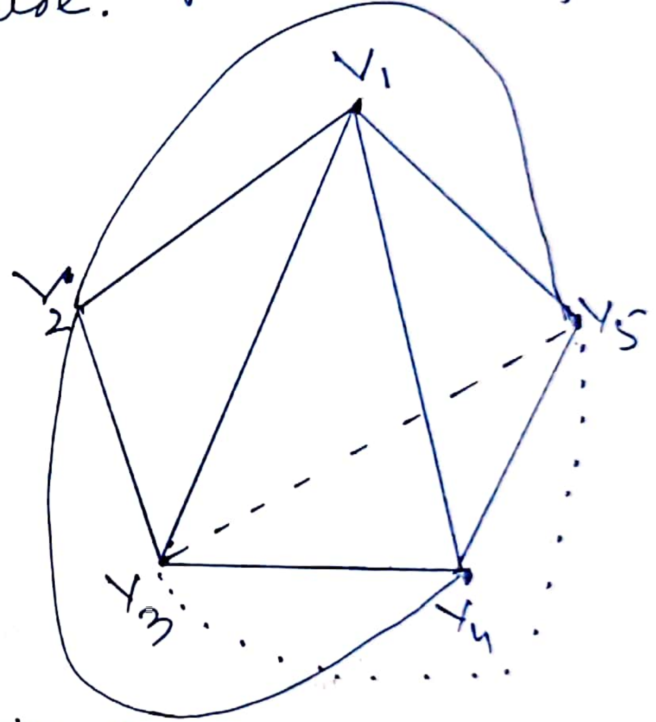
A planar representation of Q_3.

Ex: Show that K_5 is a non-planar graph.

Sol:→ K_5 is a non-planar graph, because it can be drawn with any crossover of the edge, as shown in figure.



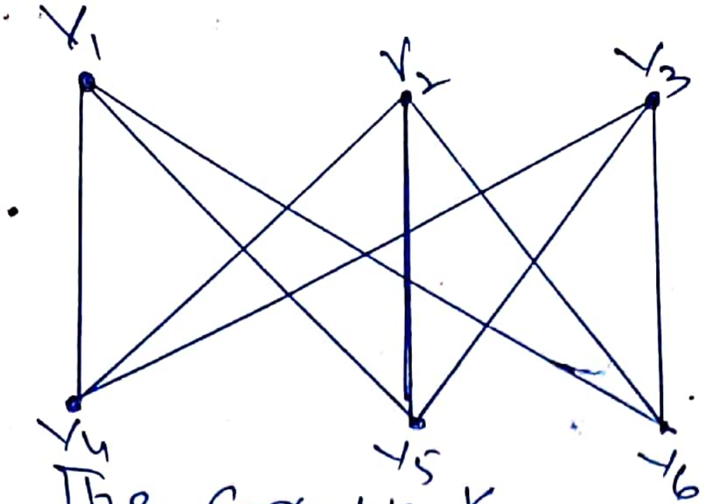
The Graph K_5



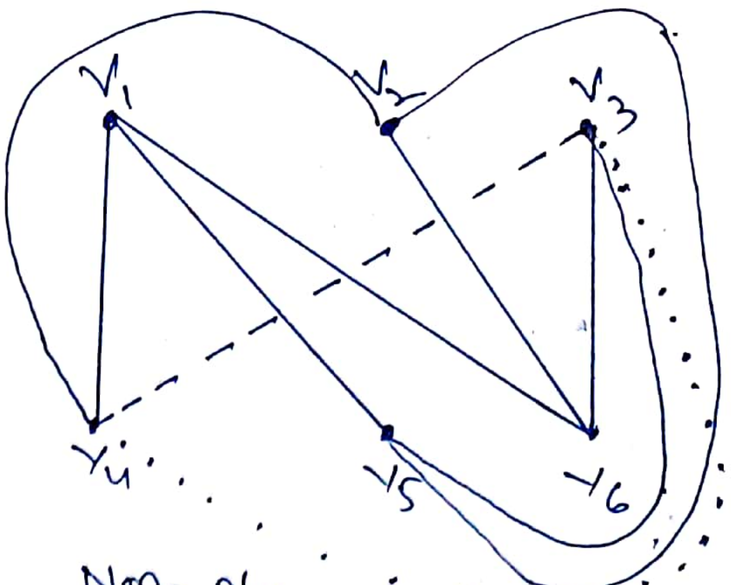
Non-planar representation of K_5

Ex:→ Show that K_{33} is a non-planar graph.

Sol:→ K_{33} is a non-planar graph, because it can be drawn with an crossover of the edge, as shown in the below figure



The Graph $K_{3,3}$



Non-planar representation of $K_{3,3}$.

Remark:→ K_5 is a non-planar graph of smallest order and $K_{3,3}$ is a non-planar of smallest size.

Crossing number of a graph:→

The crossing number of a graph G is the minimum number of edges that must be removed from the graph so that it becomes planar.

The crossing number of planar graphs is obviously zero. For K_5 and $K_{3,3}$ the crossing number is one.

Remark:→ The complete graph K_n is planar for $K=1,2,3,4$ and non-planar for $n \geq 5$.

Theorem:→ Every planar graph satisfies the inequality $m \le 3n-6$ for $n \ge 3$, where m and n denotes the size and order of the graph, respectively.

Ex:→ Show that K_4 satisfies the inequality $m \le 3n-6$.

Sol:→ K_4 has 4 vertices and $\frac{n(n-1)}{2} = \frac{4(4-1)}{2} = 6$ edges.

$\therefore$ $n = 4$ and $m = 6$.

Now $m \le 3n-6 \Rightarrow 6 \le 3\times4-6$

$\Rightarrow 6 \le 12-6$ or $6 \le 6$, which is true.

Hence, K_4 satisfies the inequality $m \le 3n-6$.

Ex:→ Prove that K_5 is non-planar.

Sol:→ K_5 has 5 vertices and $\frac{n(n-1)}{2} = \frac{5(5-1)}{2} = 10$ edges. Hence, $n=5$ and $m=10$.

We have, $m \le 3n-6$.

$\Rightarrow 10 \le 3\times5-6$ or $10 \le 15-6$

$\Rightarrow 10 \le 9$, which is absurd.

Hence, K_5 is non-planar.

Eulers Formula:→ If G is a planar graph with n vertices, m edges, r regions and K components, then

$$n - m + r = K + 1$$

Remark:→ Euler's formula for a connected graph can be restated as follows:

If G is a connected planar graph having n vertices, m edges and r regions then

$$n - m + r = 2$$

Ex:→ Find the number of edges in a connected planar G with 10 vertices and 7 regions.

Sol:→ By Euler's formula, we have

$$n - m + r = 2$$

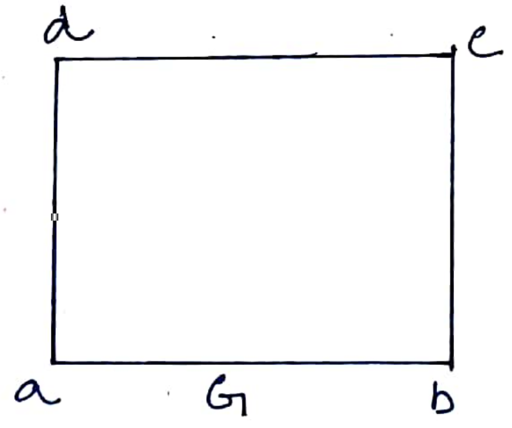
or $m = n + r - 2$

$\Rightarrow m = 10 + 7 - 2$ or $m = 15$

Hence, there are 15 edges.

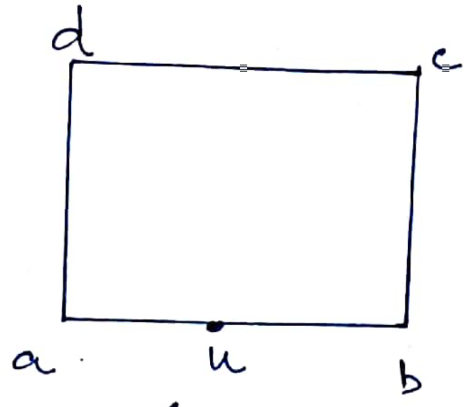
Subdivision of Edge:→ Subdivision of edge between two vertices a and b is obtained by taking one more vertex u and, edges au and ub.

Ex:→ Find the subdivisions of the graph G.

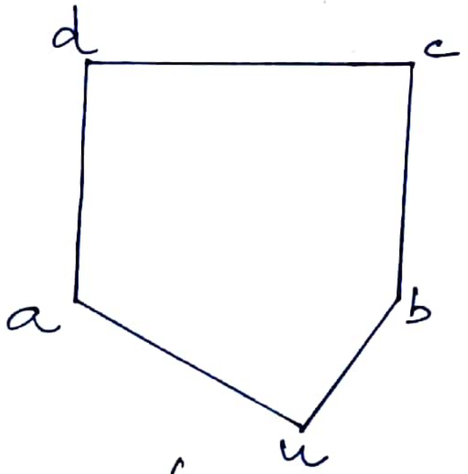


G

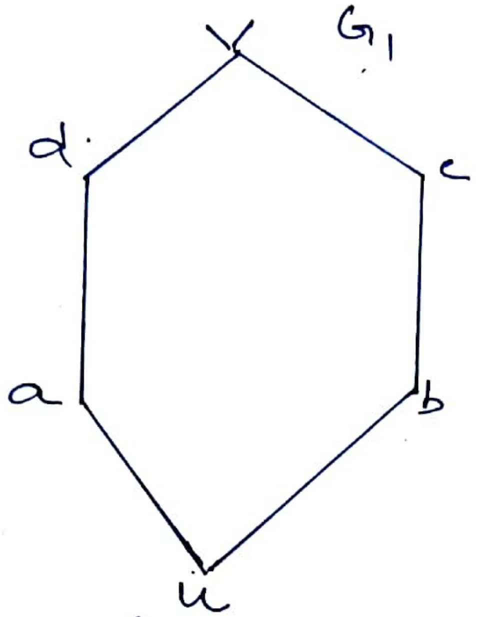
Sol:→



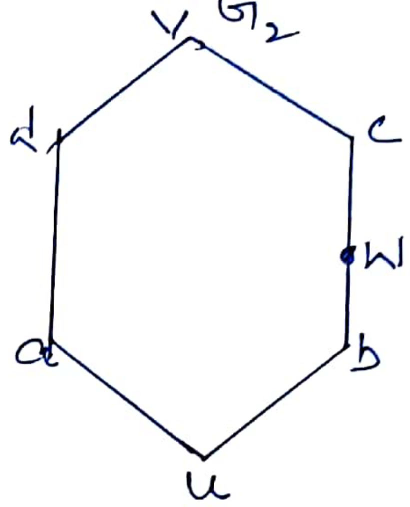
G_1



G_2



G_3

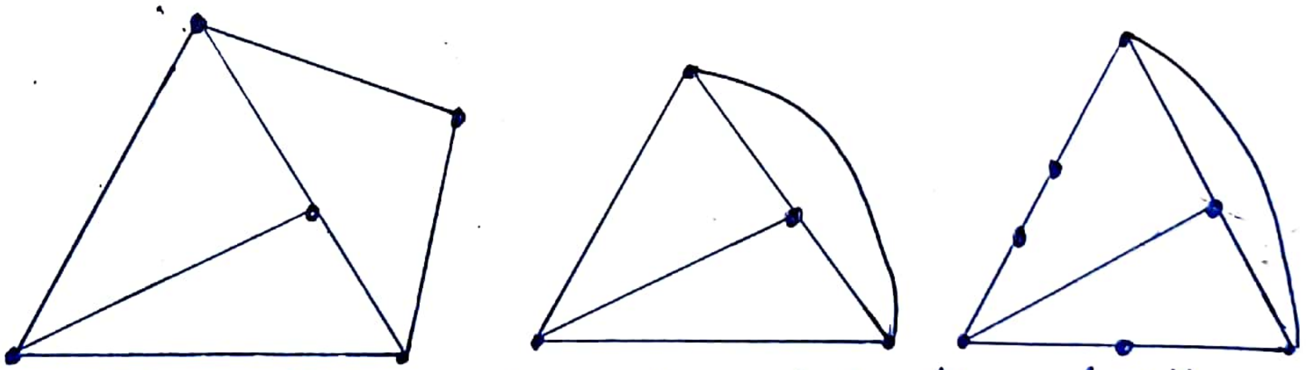


G_4

G_1, G_2, G_3 and G_4 are subdivisions of a graph G.

<u>Homeomorphic Graphs</u> :→ Two graphs are said to be homeomorphic if one graph can be obtained by other by subdivisions.

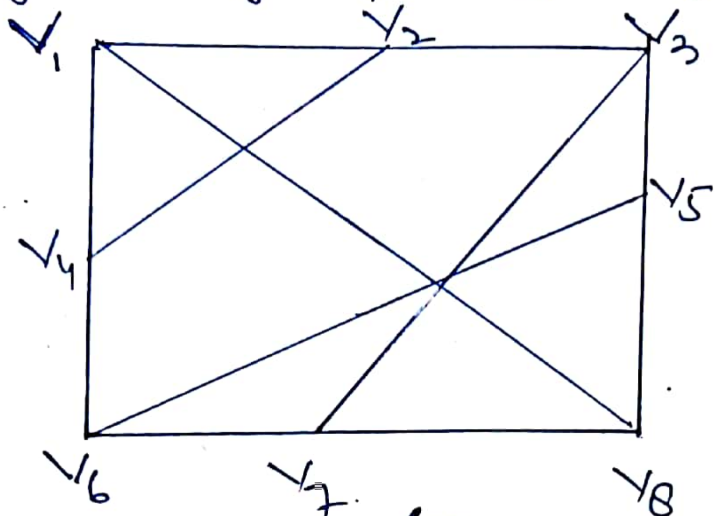
Ex:



Three graphs homeomorphic to each other.

Kuratowski's Theorem :→ A graph is planar if and only if it has no subgraph homeomorphic to K_5 or $K_{3,3}$.

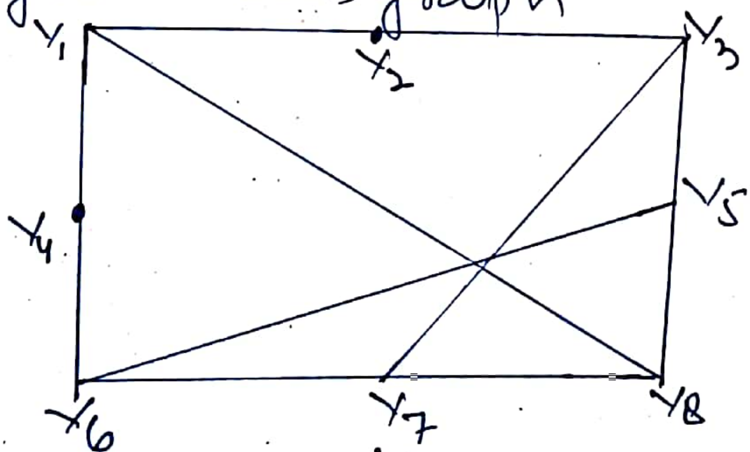
Ex: Use Kuratowski's theorem to prove the given graph is non-planar.



G

Sol:→ Note that the given graph G is a ~~regular~~ 3-regular graph.

Remove the edge $\{v_4, v_2\}$, we get a subgraph



H

The above subgraph H is Homeomorphic to the graph.

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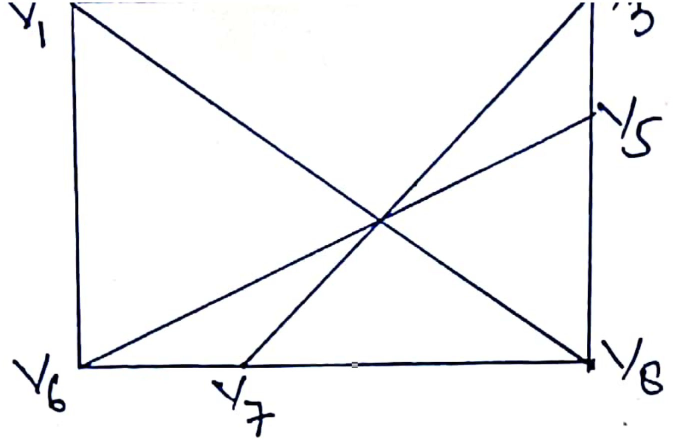

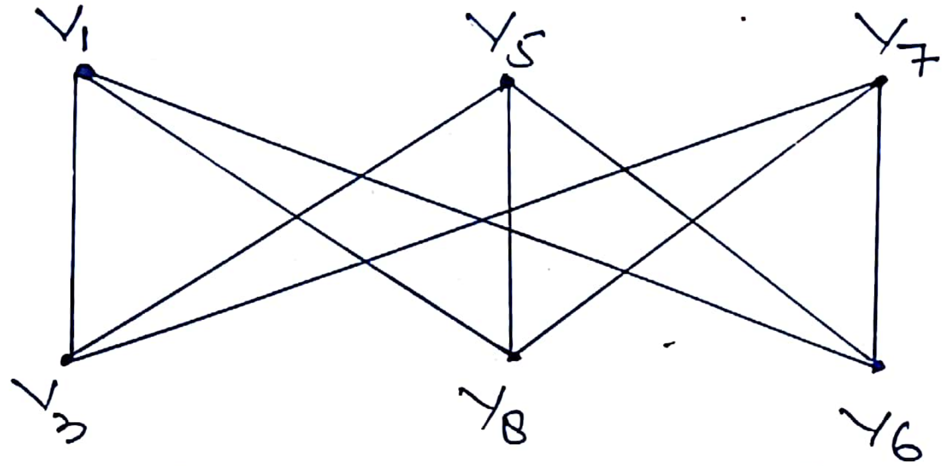

which is a complete bipartite graph $K_{3,3}$. Hence, the given graph G contains a subgraph homeomorphic to $K_{3,3}$. Therefore, by Kuratowski's theorem, the given graph is non-planar.

Dual of a Planar Graph:→

If G is a planar graph with n vertices having the regions $r_1, r_2, \ldots, r_t$. The dual of G is a graph G^* whose vertex set is $V^* = \{r_1^*, r_2^*, \ldots, r_t^*\}$ and two vertices r_i^* and r_j^* are adjacent if and only if regions r_i and r_j have an edge in common. And for every edge common between regions r_i and r_j, there is an edge between vertices r_i^* and r_j^*.

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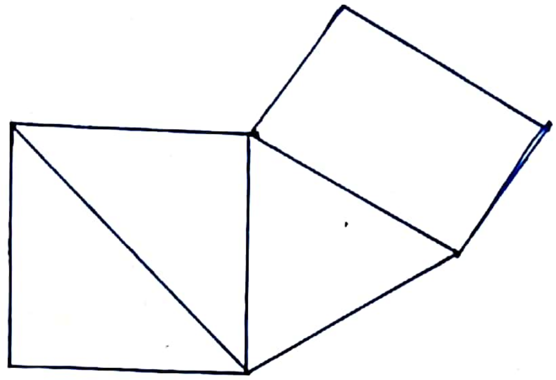
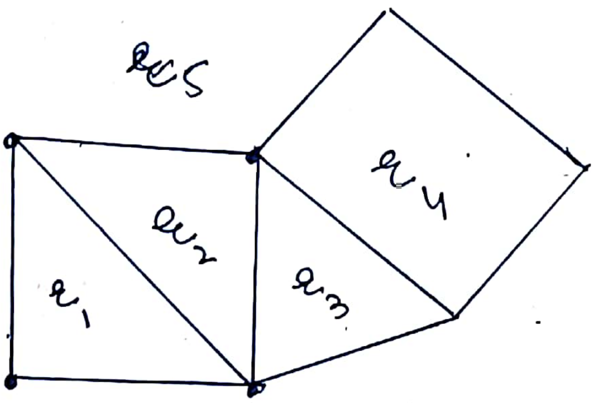
Ex:→ Find the dual of the graph G

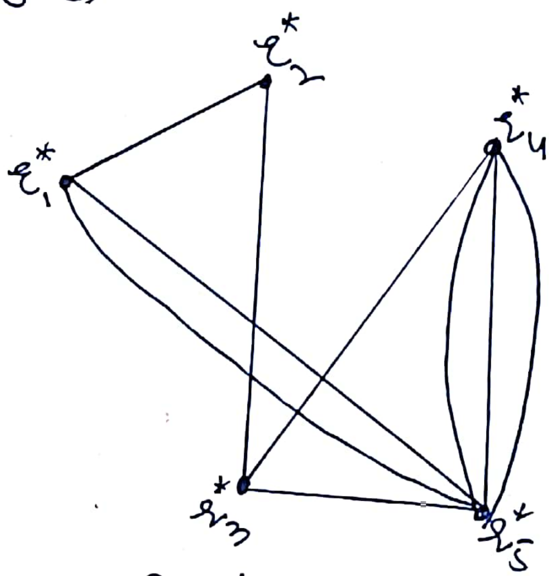


G

Sol:→ The given graph is

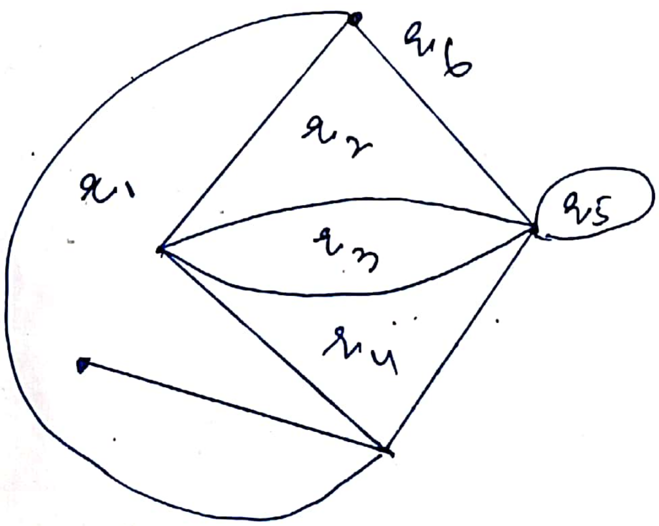


Graph G

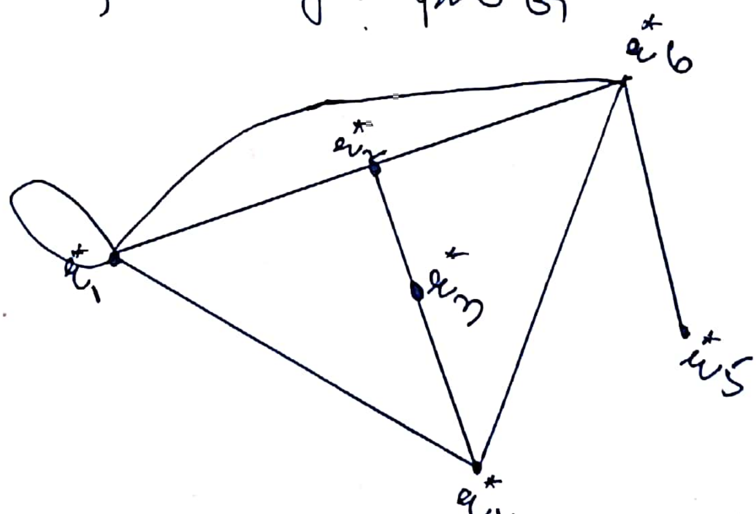


Dual of Graph G.

Ex:→ Find the dual of the graph G



Graph G



Dual of G

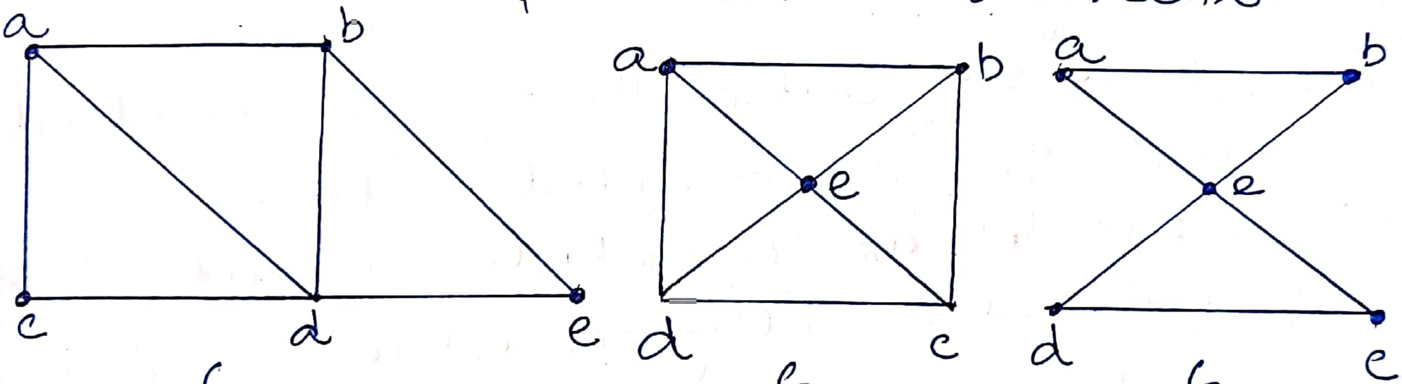
Remark:→ ① An edge forming a self-loop in G yields a pendant edge in dual of G.

② A pendant edge in G yields a self-loop in dual of G.

Euler path:→ A trail that contains each edge of a graph G but intial and ~~final~~ terminal vertices are distinct is called a Euler path or an open Euler path.

Euler circuit:→ A closed Euler path is called a Euler circuit. Thus, an Euler circuit begins and ends at the same vertex and contains each edge of a graph G exactly once.

Ex:→ Which of the undirected graphs in Figure A have an Euler path and Euler circuit.



G_1 G_2 G_3

Sol:→ The graph G_1 has an Euler path, namely, a c d e b d a b. The graph G_1 ~~G~~ does not contain an Euler circuit. The graph G_2 neither contains an Euler path nor an Euler circuit. The graph G_3 contains an ~~Euler~~ Euler circuit, namely, a e c d e b a but does not an Euler path.

Euler Graph:→ A graph that has an Euler circuit is called an Euler graph or Eulerian graph.

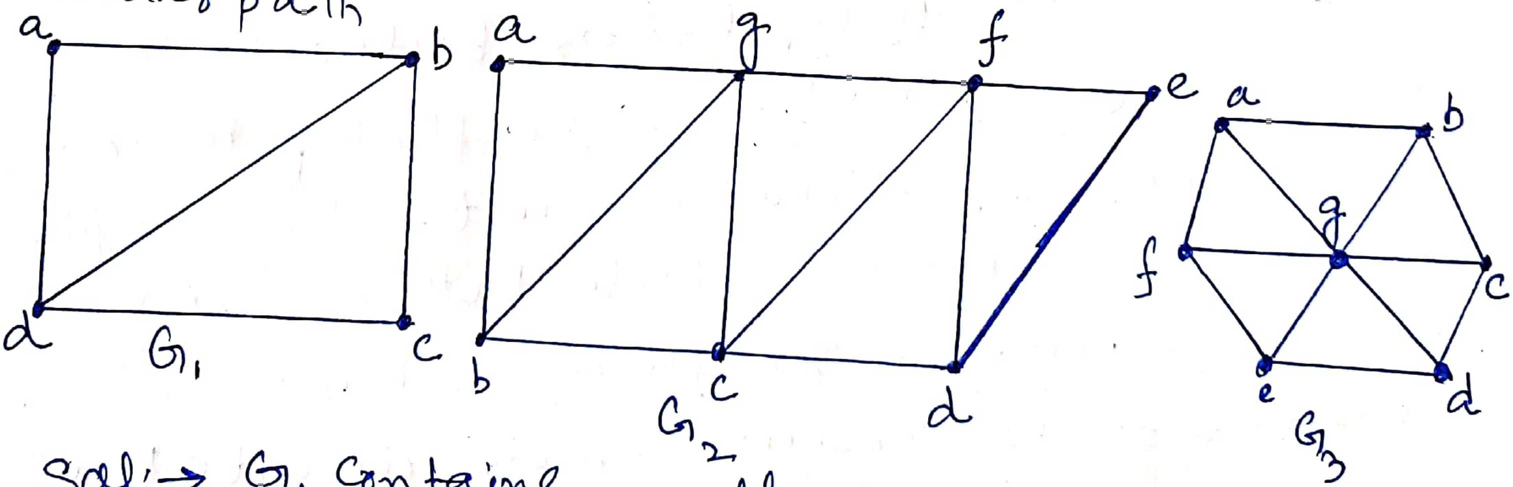
Unicursal Graph:→ A graph that contains an Euler path is called a unicursal graph.

Euler's Theorem:→ A graph is an ~~Euler graph~~ Eulerian graph if and only if each vertex of G has even degree.

Theorem:→ A graph G is unicursal graph if and only if G has exactly two vertices of odd degree.

Remark:→ From above theorem, we ~~conclude~~ conclude that a connected graph has an Euler path but ~~an~~ not an Euler circuit if and only if it has exactly two vertices of odd degree.

Ex: which graph's shown in figure A have an Euler path



Sol:→ G_1 contains exactly two vertices of odd degree, namely, b and d. Hence, it has an Euler path that must ~~be~~ have b and d as its endpoints.

One such Euler path is d, a, b, c, d, b. Similarly, G_2 has exactly two vertices of odd degree, namely b and d. So it has an Euler path that must have b and d as endpoints. One such Euler path is b, a, g, f, e, d, c, g, b, c, f, d. G_3 has no Euler path because it has six vertices of odd degree.

Hence, the graph G_1 and G_2 are called unicursal Graphs.

Ex:→ Which graphs shown in Figure B have an Euler circuit

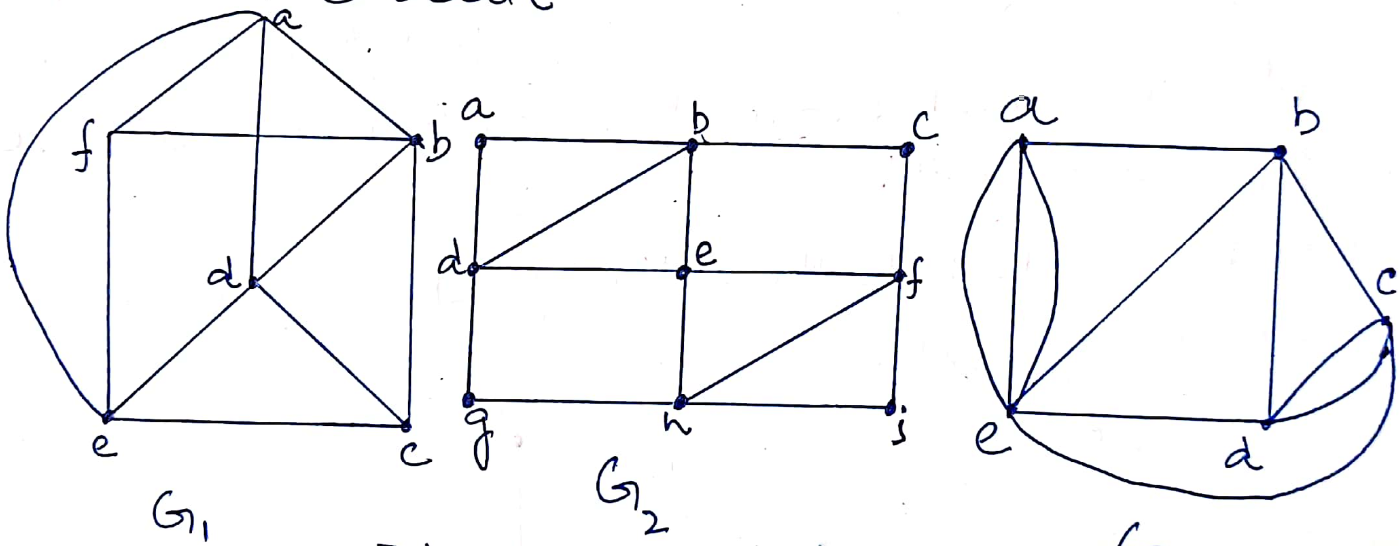
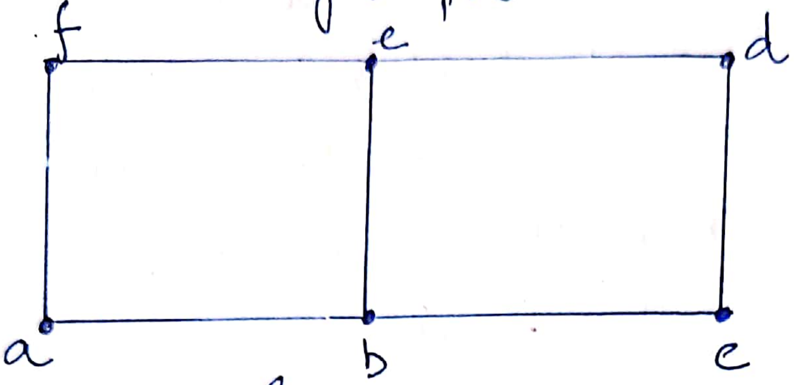


Figure B

Sol:→ G_1 has no Euler circuit because it contains vertex c of odd degree. All the vertices in G_2 are of even degree. Hence, G_2 has an Euler circuit a, b, d, e, b, c, f, e, h, f, i, h, g, d, a. Similarly, all the vertices in G_3 are of even degree. Hence, G_3 has an Euler circuit a, b, e, d, b, c, d, c, e, a, e, a, e, a.

Hence, G_2 and G_3 are ~~called~~ Euler graphs.

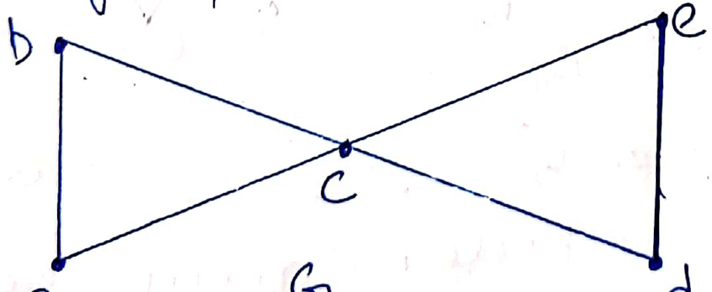
Ex:→ check whether the graph G_1 is a unicursal graph



G_1

The graph G_1 is a unicursal graph because it contains two vertices b and e of odd degree and all other vertices have degree 2 (even).
Also, e, f, a, b, e, d, c, b is a unicursal line or Euler path.

Ex:→ Check whether the graph G_2 is a Euler Graph.

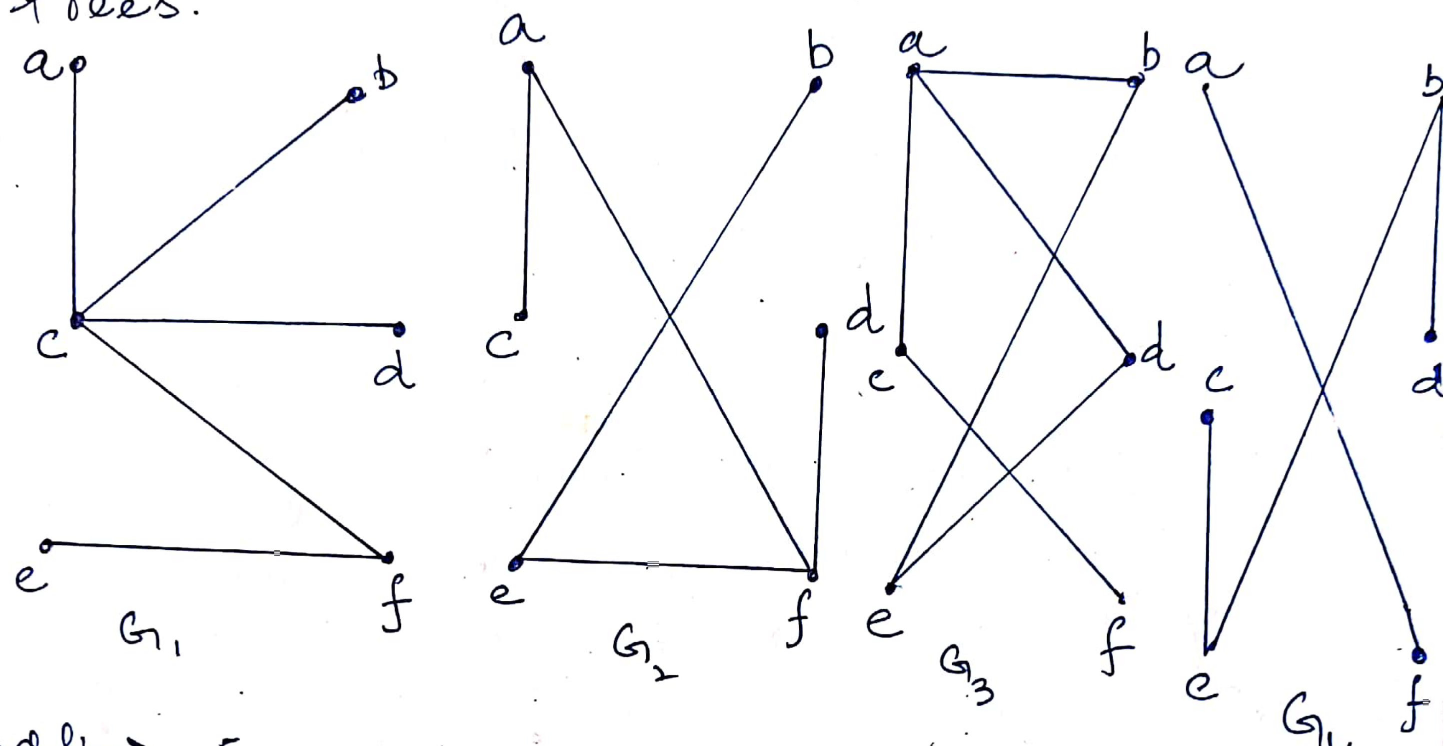


G_2

The graph G_2 is an Euler graph, because each vertex in G_2 has even degree. Also, a, c, d, e, c, b, a is a Euler line or Euler circuit.

Tree:→ A tree is a simple connected undirected graph without ~~any~~ cycle.

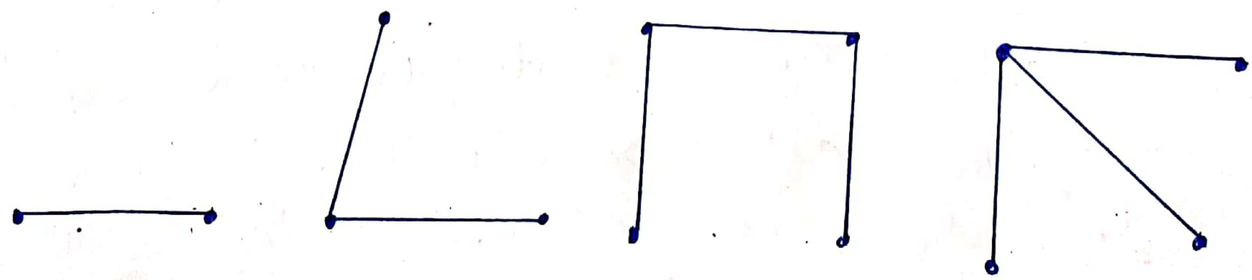
Ex:→ Which of the graphs in Figure 1 are ~~trees~~ trees.



Sol:→ G_1 and G_2 are trees, because both are connected graph's without cycles: G_3 is not a tree because e, b, a, d, e is a cycle in this graph. Finally, G_4 is not a tree because it is not connected.

Theorem: An undirected graph is a tree if and only if there is a unique ~~simple~~ path between any two of its vertices

Ex:

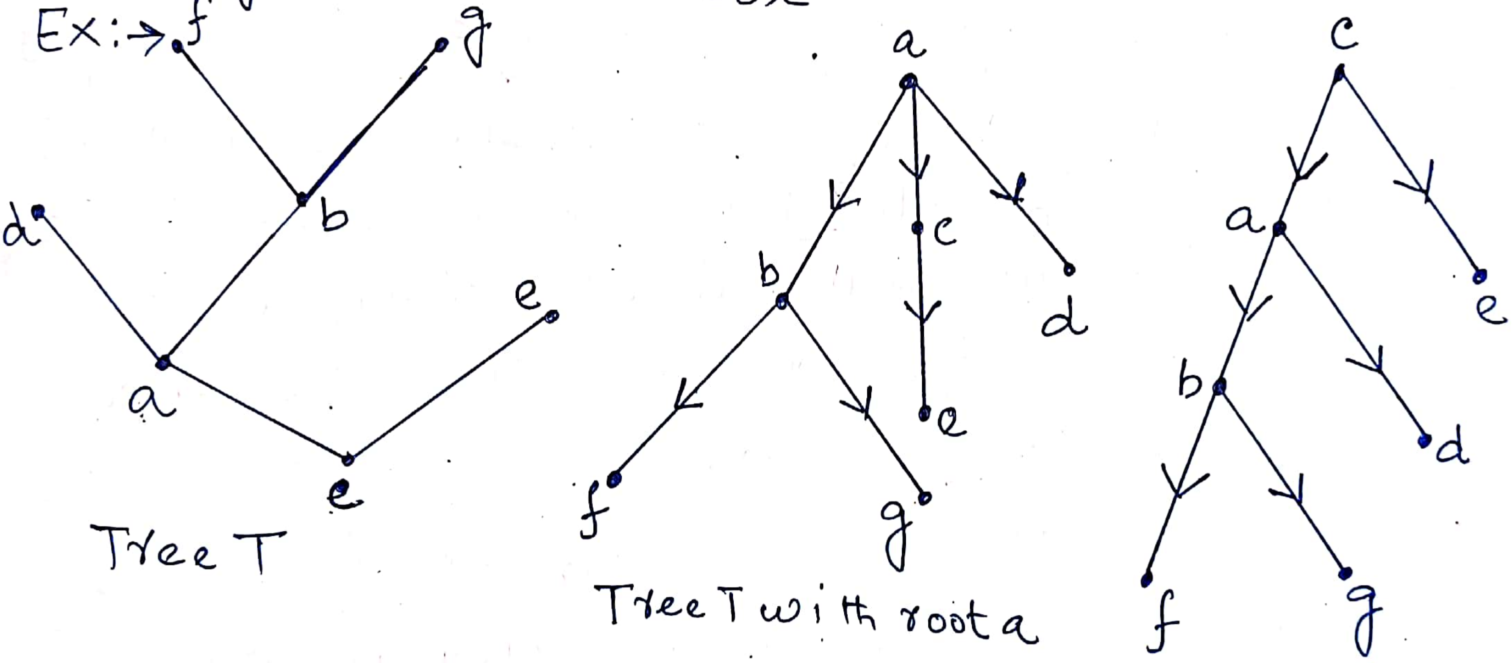


Theorem:→ If T is a tree having atleast two vertices, then T has atleast two pendent vertices.

Theorem:→ A connected graph G ~~is a tree if and~~ of order n is a tree if and only if it has size n−1 i.e A tree with n vertices has n−1 edges.

Rooted Tree:→ A rooted tree is a tree in which one vertex has been designated as the root and every edge is directed away from the root.

Ex:→



Tree T

Tree T with root a

Tree T with root c.

Definition:→ If there is an edge in a rooted ~~tree~~ tree from a vertex u to a vertex v then u is called parent of v and v is called child of u. Vertices with the same parent are called siblings.

Leaf:→ A vertex of a rooted tree is called a leaf if it has no child.

Internal vertex:→ A vertex that has a child is called a interval vertex.

Ex:→ In a root tree shown in Figure A, find the parent of c, the children of g, the siblings of h, all interval vertices and all leafs.

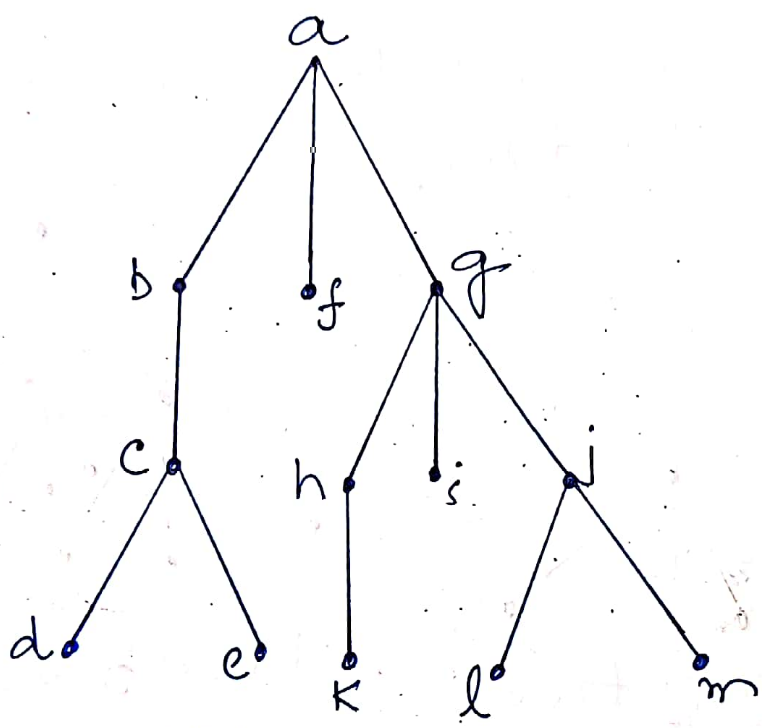
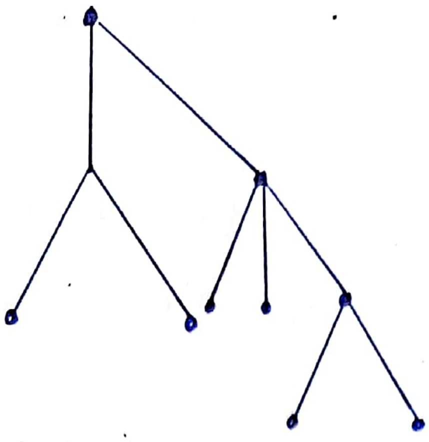


Figure A

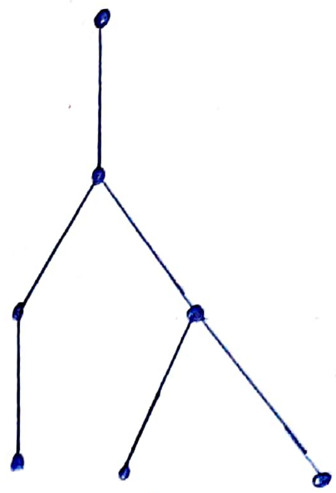
Sol:→ The parent of c is b. The children of g are h, i and j. The siblings of h are i and j. The interval vertices are a, b, c, g, h and j. The leafs are d, e, f, k, l and m.

m-ary tree:→ A rooted tree is called an m-ary tree if every internal vertex has no more than m children.

Ex:→



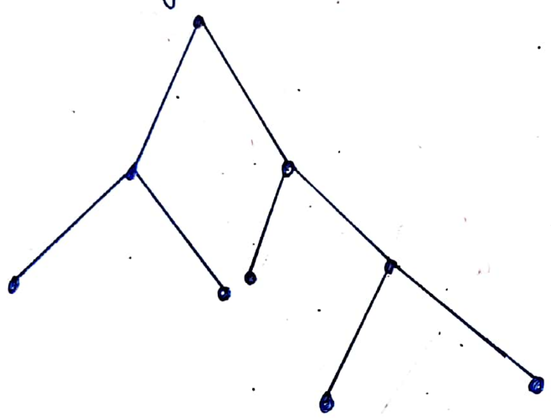
3-ary tree



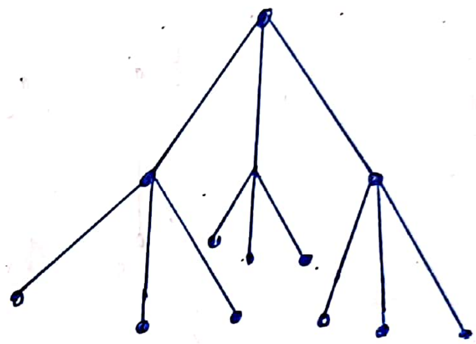
2-array tree

<u>Full m-ary tree</u>:→ A tree is called a full m-ary tree if every internal vertex has exactly m children.

Ex:



Full-2 ary tree



Full 3-ary tree.

<u>Binary tree</u>:→ A full 2-ary tree is called a binary tree.

<u>Theorem</u>:→ A binary tree has odd number of vertices.

<u>Theorem</u>:→ A binary tree having order n has $\frac{n+1}{2}$ pendent vertices.

<u>Theorem</u>:→ A binary tree having order n has $\frac{n-3}{2}$ vertices of degree 3.

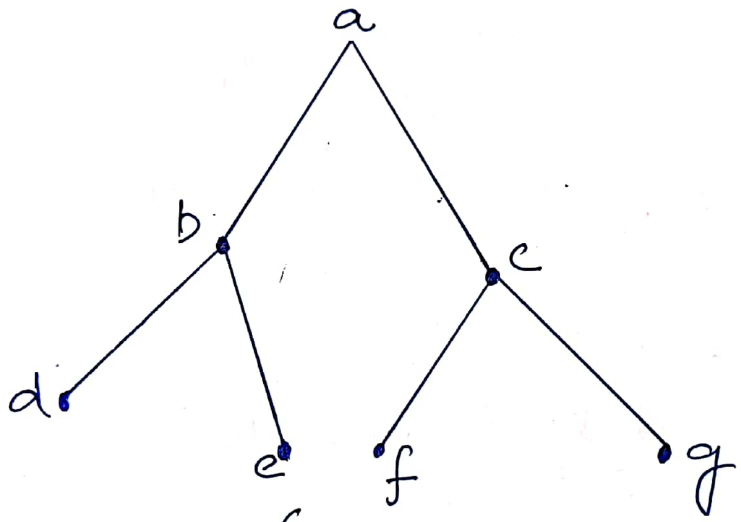
Theorem:→ A full m-ary tree with

(i) n vertices has $i = \frac{n-1}{m}$ ~~vertices~~ internal vertices and $l = \frac{(m-1)n+1}{m}$ leaves.

Imp (ii) i internal vertices has $n = mi+1$ vertices and $l = (m-1)i + 1$ leaves.

(iii) l leaves has $n = \frac{ml-1}{m-1}$ vertices and $i = \frac{l-1}{m-1}$ internal vertices.

Ex:→



G

Verify all the conditions of the above theorem in G.

Sol:→ Clearly, the graph G is a full 2-ary tree.

(i) Here, number of vertices $n = 7$

Therefore, number of internal vertices in G are $i = \frac{n-1}{m} = \frac{7-1}{2} = 3$

Clearly, a, b and c are internal vertices.

Number of leaves in G are $l = \frac{(m-1)n+1}{m}$

$= \frac{(2-1)7+1}{2} = \frac{7+1}{2} = 4$

Clearly, d, e, f and g are leaves.

(ii) G has 3 internal vertices. The no. of vertices in G are $n = mi + 1 = 2 \times 3 + 1 = 7$.

Also, $l = (m-1)i + 1 = (2-1)3 + 1 = 3 + 1 = 4$.

It is clear from the graph G that it has 7 vertices and 4 leaves.

(iii) The given graph has 4 leaves. The number of vertices in G are

$$n = \frac{ml - 1}{m - 1} = \frac{2 \times 4 - 1}{2 - 1} = 7$$

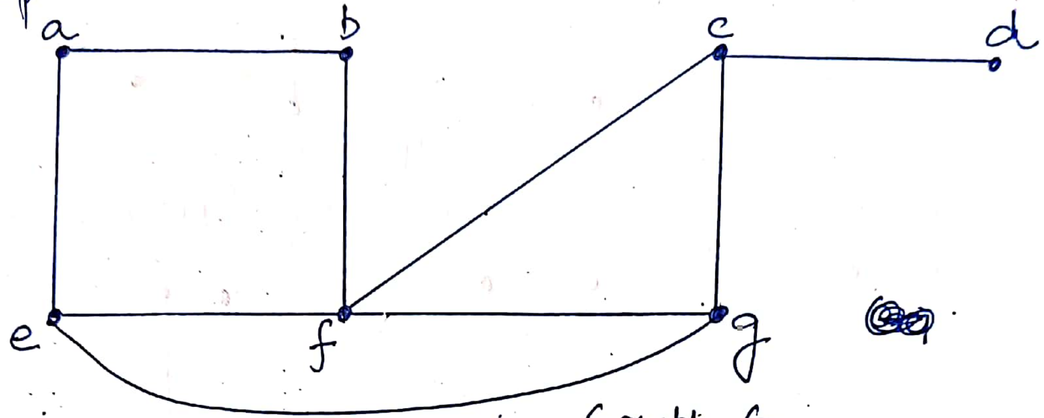
and the number of internal vertices are

$$i = \frac{l - 1}{m - 1} = \frac{4 - 1}{2 - 1} = 3.$$

Both these conditions matches with the graph.

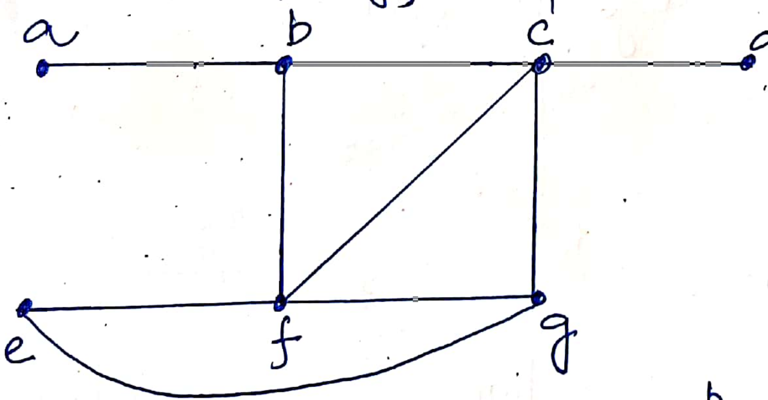
Spanning Tree :→ A tree T is said to be a spanning tree of a connected graph G if T is a subgraph of G and T contains all vertices of G.

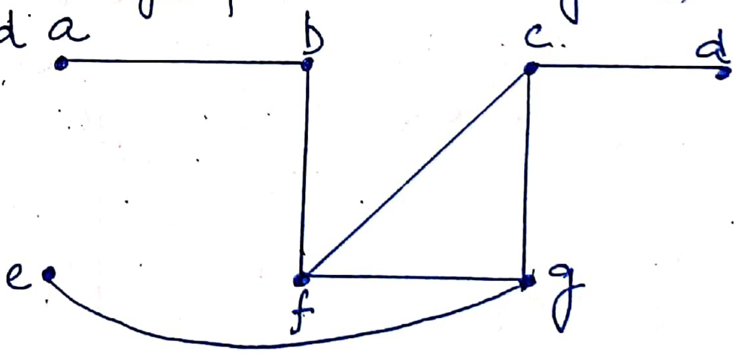
Ex:→ Find a spanning tree of the following graph

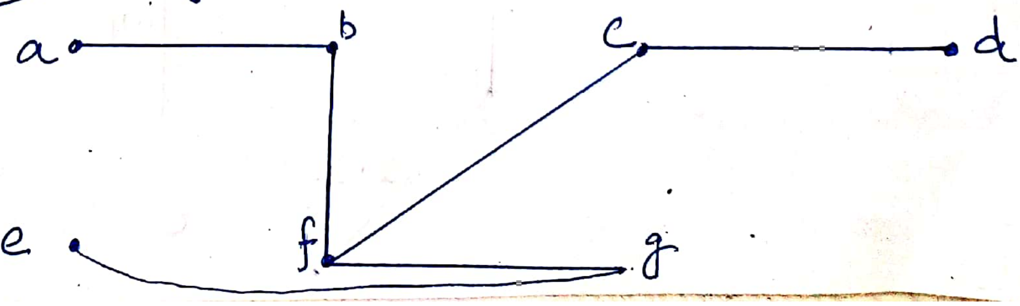


Graph G

Sol:→ The graph G is connected, but it is not a tree because it contains circles. Remove the edge {a,e}. This eliminates one cycle and the resulting subgraph is still connected and contains every every of G. Next remove the {e,f} to eliminate a second ~~simple~~ circle. Finally remove {c,g} to produce a graph with no cycles,







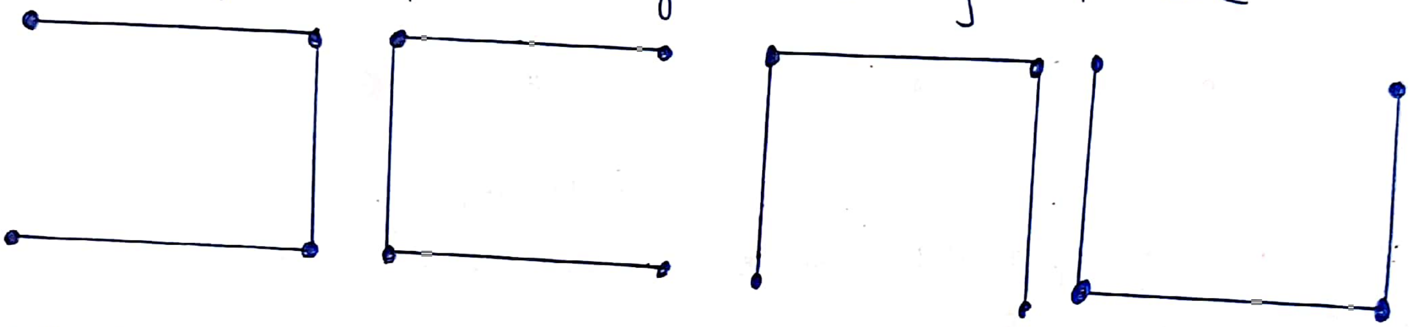
Theorem:→ A simple graph is connected if and only if it has a spanning tree.

Ex:→ Find the spanning trees of the graph C_4

Sol:→ We have

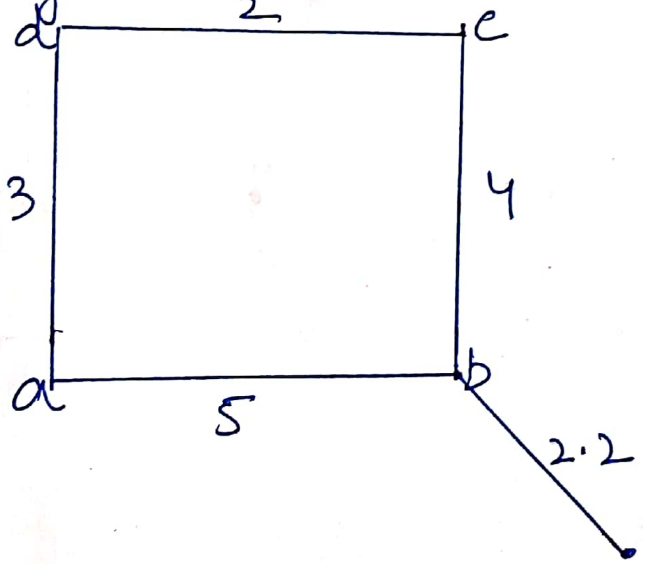
C_4

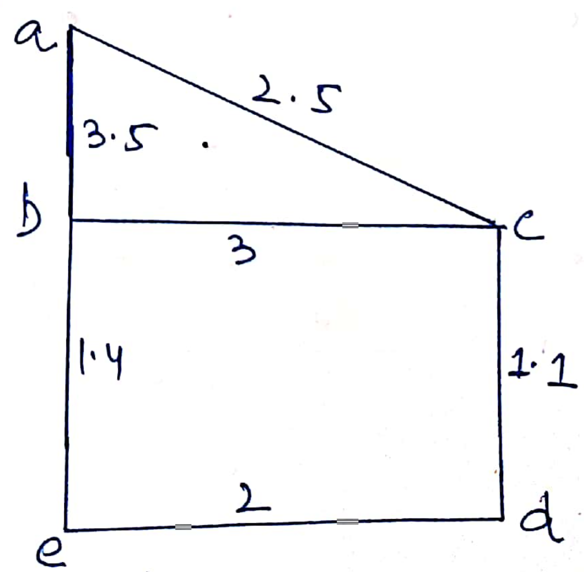
Hence, the spanning trees of C_4 are



Weighted Graph:→ A graph in which every edge is assigned a non-negative real number is called a weighted graph and the non-negative real number assigned to ~~each~~ an edge is called its weight.

Ex

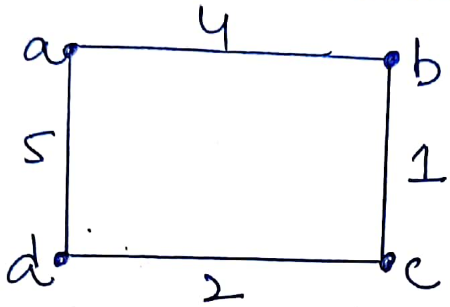




Minimum Spanning Tree :→

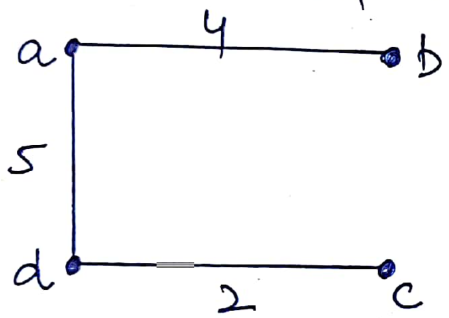
A minimum spanning tree in a connected weighted graph is a spanning tree in which the sum of the weight of the edges is as minimum as possible.

Ex:→ Find the minimum spanning tree of the graph

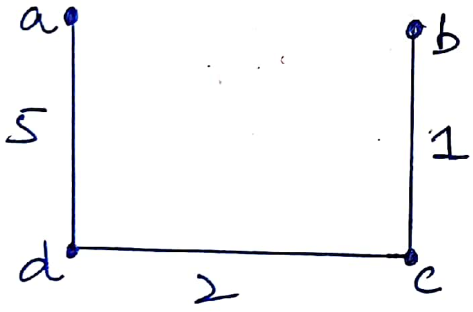


Graph G

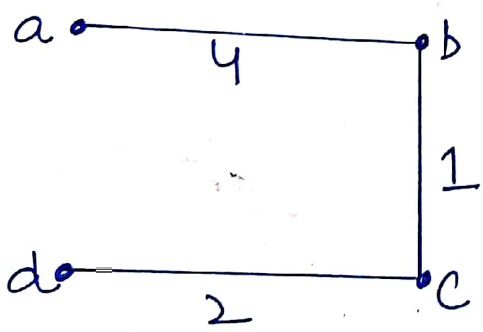
Sol:→ The spanning tree of G are



Sum of weight on Edges = 10



Sum of weight of Edges = 8



Sum of weight of Edges = 7

which is a minimum spanning tree.