MEC-107: Basic Engineering Mechanics

Unit- 3

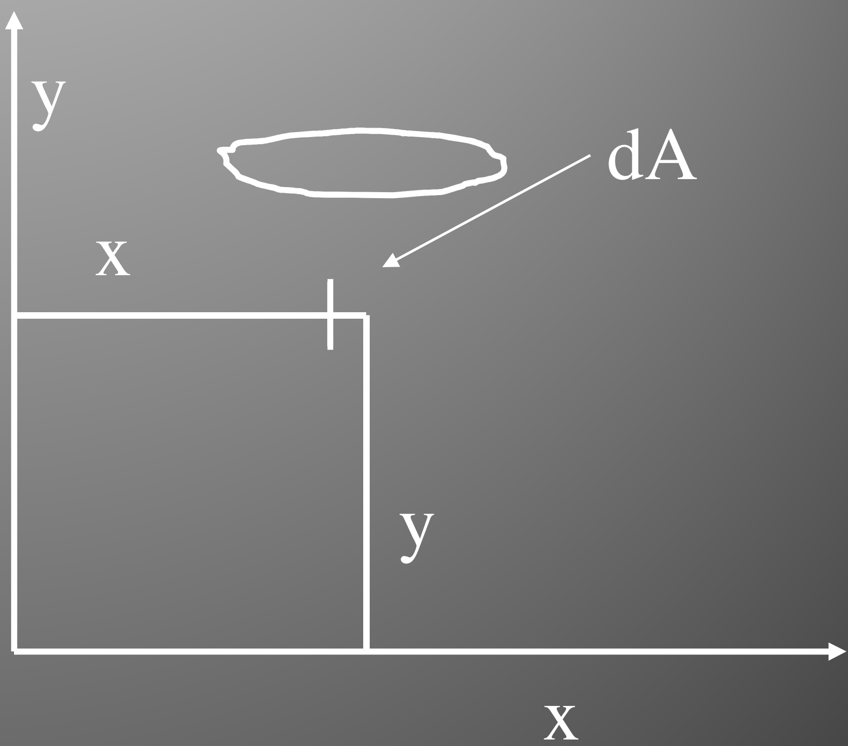
MOMENT OF INERTIA

- It is defined as the algebraic sum of product of area and the square of distance from the particular axis.
- It is denoted by I_{XX} or I_{YY}
- It is given by $I=Ah^2$
- Unit- mm^4.
- It is second moment of area which is measure of résistance to bending & forms basic to strength of material.
- Mass MI is the résistance to rotation & forms basic to dynamics to rigid bodies.

MOMENT OF INERTIA

The product of the elemental area and square of the perpendicular distance between the centroid of area and the axis of reference is the "**Moment of Inertia**" about the reference axis.

$$I_{xx} = \int dA.\ y^2$$
$$I_{yy} = \int dA.\ x^2$$

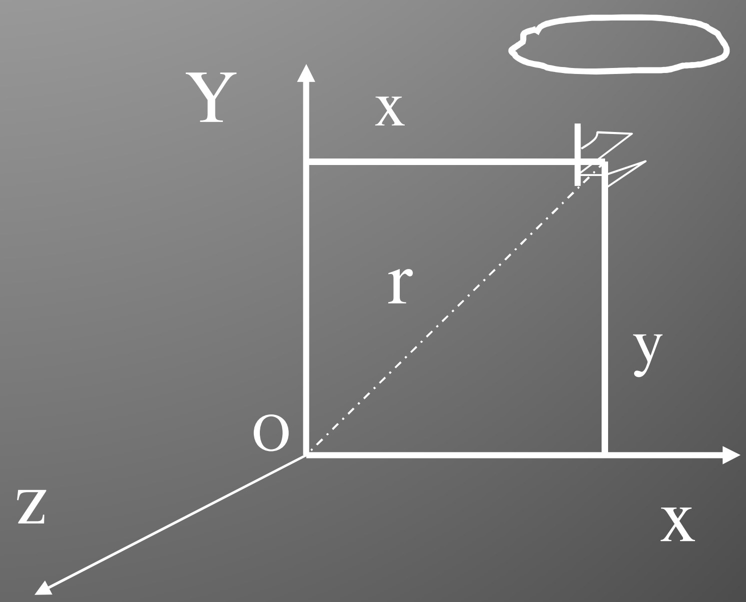


It is also called **second moment of area** because first moment of elemental area is dA.y and dA.x; and if it is again multiplied by the distance, we get second moment of elemental area as (dA.y)y and (dA.x)x.

Polar Moment of Inertia (Perpendicular Axes theorem)

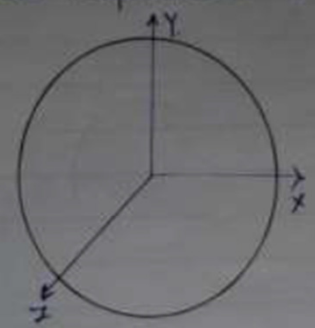
- The moment of inertia of an area about an axis **perpendicular to the plane of the area** is called '**Polar Moment of Inertia**',
- It is denoted by symbol **I_{zz} or J or I_p**,
- The moment of inertia of an area in xy plane w.r.t z axis is

$$I_{zz} = I_p = J = \int r^2 dA$$
$$= \int (x^2 + y^2)\, dA$$
$$= \int x^2 dA + \int y^2 dA$$
$$= I_{xx} + I_{yy}$$



Polar Moment of Inertia (Perpendicular Axes theorem)

Polar or Perpendicular Axis theorem



$$I_{zz} = I_{xx} + I_{yy}$$

$$= \left(\frac{\pi}{64} \times D^4\right) + \left(\frac{\pi}{64} \times D^4\right)$$

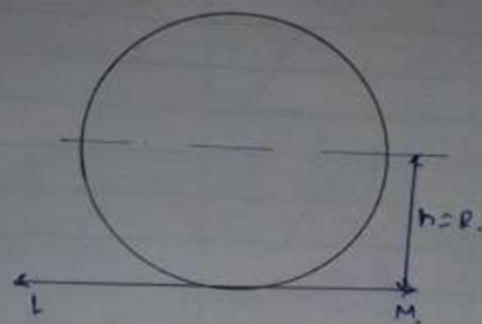
$$= 2 \times \frac{\pi}{64} \times D^4$$

$$\boxed{I_{zz} = \frac{\pi}{32} \times D^4}$$

- The moment of inertia of polar axis is equal to algebraic sum of MI of two axis about which axis is perpendicular.

Parallel Axes Theorem

Parallel Axis theorem.



$$I_{LM} = I_{xx} + Ah^2$$

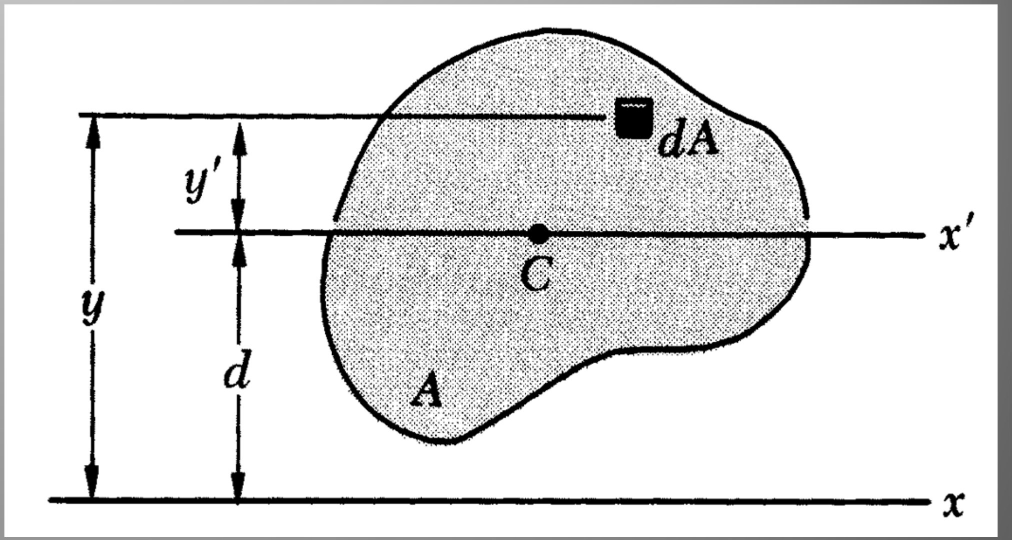
$$= I_{xx} + A \times (R)^2$$

$$= I_{xx} + A \times \left(\frac{D}{2}\right)^2$$

$$\boxed{I_{LM} = \left(\frac{\pi}{64} \times D^4\right) + \left(\frac{\pi}{4} \times D^2\right) \times \left(\frac{D}{2}\right)^2}$$

- M.I of plane section about any axis is equal to MI of plane section about the parallel axis parallel axis passing through its CG plus the product of area and square of distance between two axes.

Parallel Axes Theorem

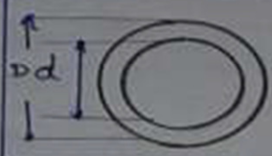
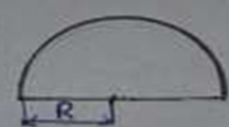
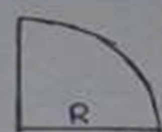
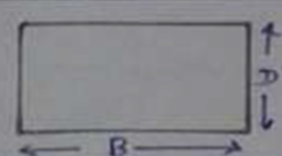
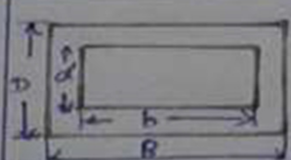


$$I_x = \int_A y^2 dA \qquad I_x = \int_A y^2 dA = \int_A (y' + d)^2 dA$$

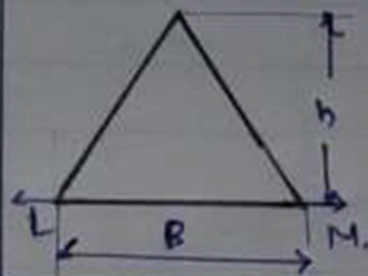
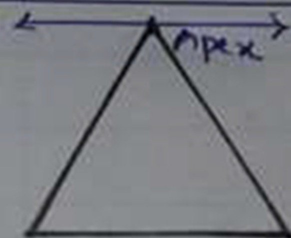
$$I_x = \int_A (y')^2 dA + 2d \int_A y' dA + \int_A d^2 dA$$

$$I_x = \bar{I}_x + Ad^2$$

M.I. of Simple Shapes

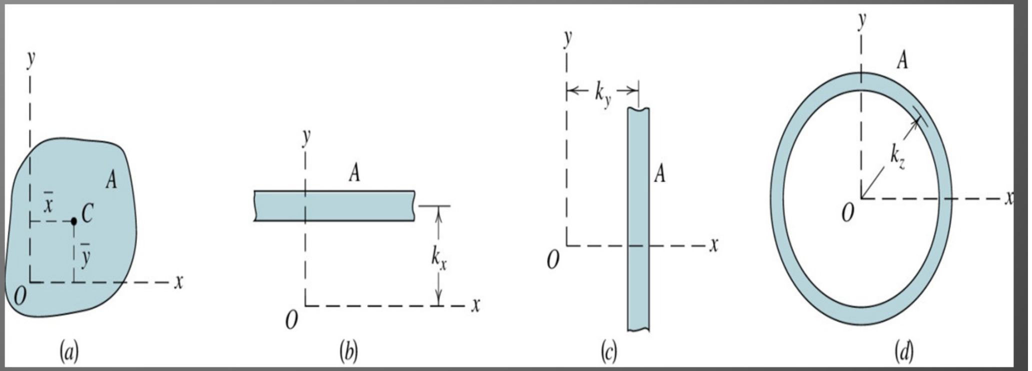
Sr no	Name of Section	Sketch	M I about X axis	M I about Y axis	Area of Section
1	Solid Circular (Circle)		$\frac{\pi}{64} \times (D)^4$	$\frac{\pi}{64} \times (D)^4$	$\frac{\pi}{4} \times (D)^2$
2	Hollow Circular		$\frac{\pi}{64} \times (D^4 - d^4)$	$\frac{\pi}{64} \times (D^4 - d^4)$	$\frac{\pi}{4} (D^2 - d^2)$
3	Semi Circle		$0.11 R^4$	$\frac{\pi}{8} \times (R)^4$ or $\frac{\pi}{128} \times (D)^4$	$\frac{\pi}{2} \times (R)^2$ or $\frac{\pi}{8} \times (D)^2$
4	Quarter Circle		$0.055 R^4$	$0.055 R^4$	$\frac{\pi R^2}{4}$ or $\frac{\pi}{16} \times D^2$
5	Rectangle		$\frac{BD^3}{12}$	$\frac{DB^3}{12}$	$B \times D$
6	Hollow Rectangular (Rectangle.		$\frac{BD^3 - bd^3}{12}$	$\frac{DB^3 - db^3}{12}$	$(B \times D) - (b \times d)$

M.I. of Simple Shapes

Sr	Name of section	Sketch	M.I about	Area.
7	Triangle		Base $I_{LM} = \frac{Bh^3}{12}$	$\frac{1}{2}$ base × height
8	Triangle		Apex $= \frac{Bh^3}{4}$	$\frac{1}{2}$ base × height
9	Triangle		C.G. $= \frac{Bh^3}{36}$	$\frac{1}{2}$ × base × height.

Radius of Gyration

- It is the **perpendicular distance** at which the **whole area** may be assumed to be **concentrated**, yielding the **same second moment of the area** about the axis under consideration,
- It is denoted by **K**,
- It is given by **$I = AK^2$**
- Therefore, **$K = (I/A)^{1/2}$**
- **Units – mm, m**



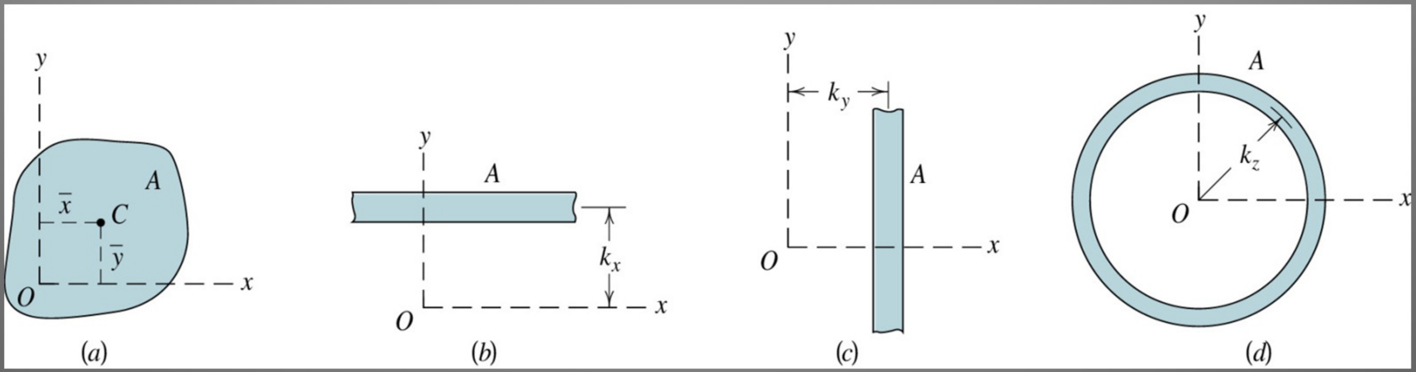
Radius of Gyration

- Similarly,

$$k_y = \sqrt{I_y / A}$$

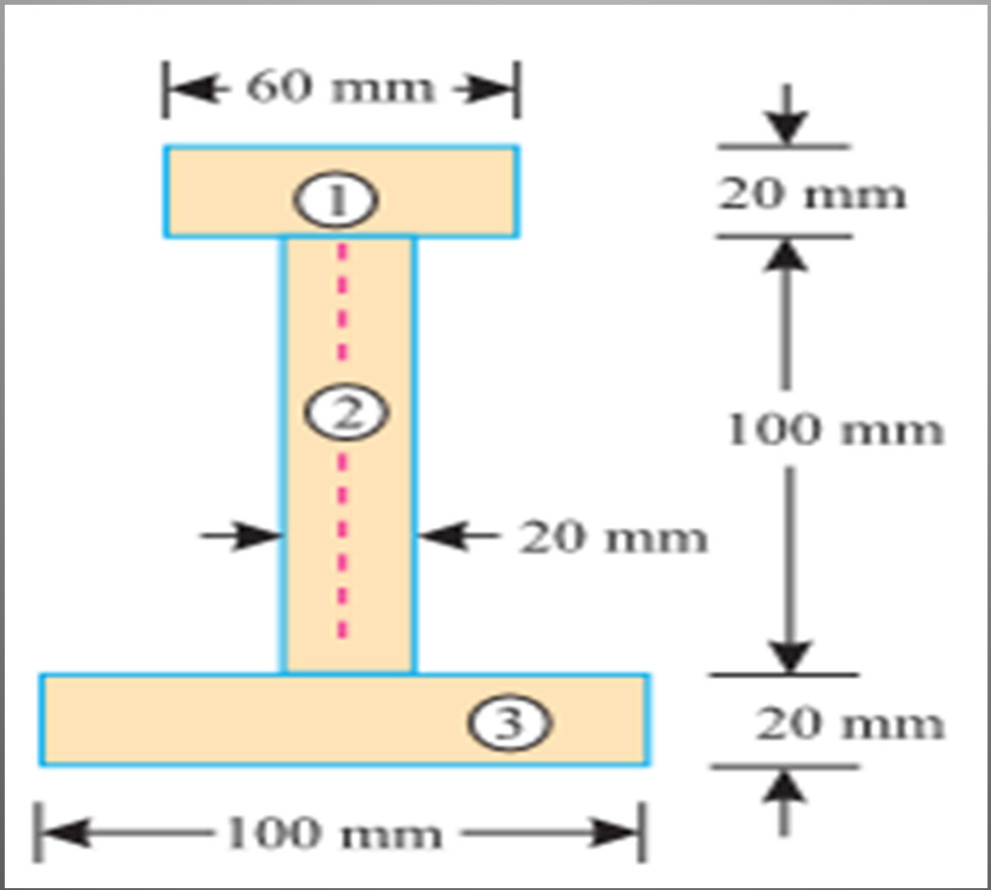
$$k_z = \sqrt{I_z / A}$$

$$k_z^2 = k_x^2 + k_y^2$$



Numerical- 1

Example 7.11. *An I-section is made up of three rectangles as shown in Fig. 7.15. Find the moment of inertia of the section about the horizontal axis passing through the centre of gravity of the section.*



Numerical- 1

(i) *Rectangle* 1

$$a_1 = 60 \times 20 = 1200 \text{ mm}$$

and $$y_1 = 20 + 100 + \frac{20}{2} = 130 \text{ mm}$$

(ii) *Rectangle* 2

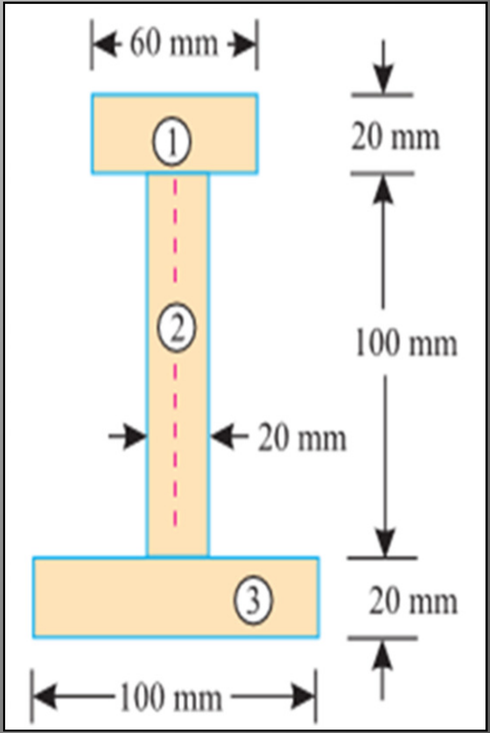
$$a_2 = 100 \times 20 = 2000 \text{ mm}^2$$

and $$y_2 = 20 + \frac{100}{2} = 70 \text{ mm}$$

(iii) *Rectangle* 3

$$a_3 = 100 \times 20 = 2000 \text{ mm}^2$$

and $$y_3 = \frac{20}{2} = 10 \text{ mm}$$



We know that the distance between centre of gravity of the section and bottom face,

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2 + a_3 y_3}{a_1 + a_2 + a_3} = \frac{(1200 \times 130) + (2000 \times 70) + (2000 \times 10)}{1200 + 2000 + 2000} \text{ mm}$$

$$= 60.8 \text{ mm}$$

Numerical- 1

$$I_{G1} = \frac{60 \times (20)^3}{12} = 40 \times 10^3 \text{ mm}^4$$

and distance between centre of gravity of rectangle (1) and *X-X* axis,

$$h_1 = 130 - 60.8 = 69.2 \text{ mm}$$

∴ Moment of inertia of rectangle (1) about *X-X* axis,

$$= I_{G1} + a_1 h_1^2 = (40 \times 10^3) + [1200 \times (69.2)^2] = 5786 \times 10^3 \text{ mm}^4$$

Similarly, moment of inertia of rectangle (2) about an axis through its centre of gravity and parallel to *X-X* axis,

$$I_{G2} = \frac{20 \times (100)^3}{12} = 1666.7 \times 10^3 \text{ mm}^4$$

and distance between centre of gravity of rectangle (2) and *X-X* axis,

$$h_2 = 70 - 60.8 = 9.2 \text{ mm}$$

∴ Moment of inertia of rectangle (2) about *X-X* axis,

$$= I_{G2} + a_2 h_2^2 = (1666.7 \times 10^3) + [2000 \times (9.2)^2] = 1836 \times 10^3 \text{ mm}^4$$

Now moment of inertia of rectangle (3) about an axis through its centre of gravity and parallel to *X-X* axis,

$$I_{G3} = \frac{100 \times (20)^3}{12} = 66.7 \times 10^3 \text{ mm}^4$$

and distance between centre of gravity of rectangle (3) and *X-X* axis,

$$h_3 = 60.8 - 10 = 50.8 \text{ mm}$$

∴ Moment of inertia of rectangle (3) about *X-X* axis,

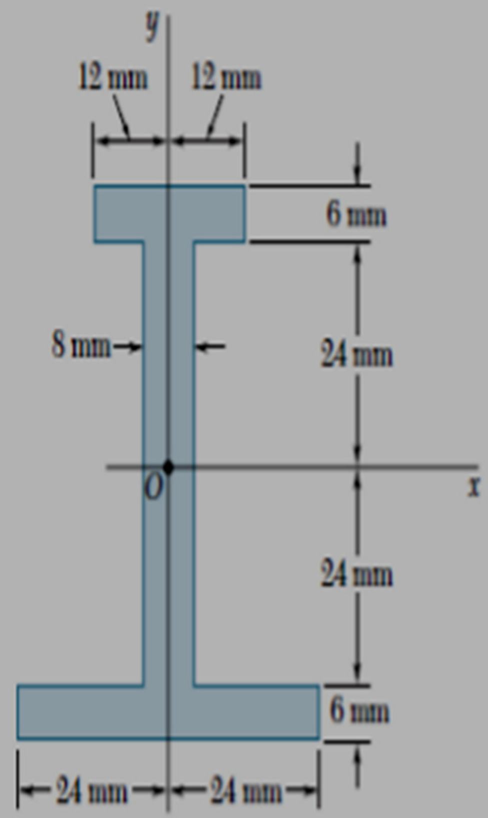
$$= I_{G3} + a_3 h_3^2 = (66.7 \times 10^3) + [2000 \times (50.8)^2] = 5228 \times 10^3 \text{ mm}^4$$

Now moment of inertia of the whole section about *X-X* axis,

$$I_{XX} = (5786 \times 10^3) + (1836 \times 10^3) + (5228 \times 10^3) = 12\,850 \times 10^3 \text{ mm}^4 \qquad \textbf{Ans.}$$

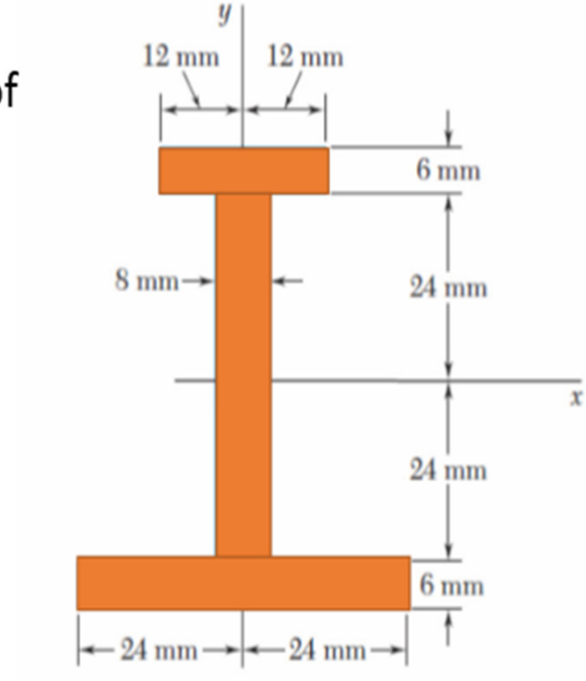
Numerical- 2

Determine the moment of inertia and the radius of gyration of the shaded area with respect to the x axis as shown in figure **(I_{xx}= 390000 mm^4 and k_x=21.9 mm)**



Numerical- 2

Determine the moment of inertia and the radius of gyration of the area shown in the fig.



$$\overline{I1_x} = \frac{1}{12}bd^3 = \frac{1}{12} \times 24 \times 6^3 = 432\ mm^4$$

$$\overline{I2_x} = \frac{1}{12}bd^3 = \frac{1}{12} \times 8 \times 48^3 = 73728\ mm^4$$

$$\overline{I3_x} = \frac{1}{12}bd^3 = \frac{1}{12} \times 48 \times 6^3 = 864\ mm^4$$

$$I1_x = \overline{I1_x} + Ah^2 = 432 + 24 \times 6 \times (24+3)^2 = 105408\ mm^4$$

$$I3_x = \overline{I3_x} + Ah^2 = 864 + 48 \times 6 \times (24+3)^2 = 210816\ mm^4$$

$$I_x = 105408 + 73728 + 210816 = 390 \times 10^3 mm^4$$

$$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{390 \times 10^3}{816}} = \mathbf{21.9\ mm}$$

Numerical- 3

Example 7.10. *Find the moment of inertia of a T-section with flange as 150 mm × 50 mm and web as 150 mm × 50 mm about X-X and Y-Y axes through the centre of gravity of the section.*

(*i*) *Rectangle* (1)

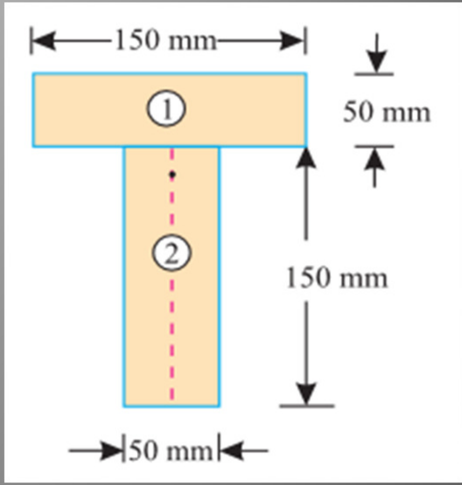
$$a_1 = 150 \times 50 = 7500 \text{ mm}^2$$

and $$y_1 = 150 + \frac{50}{2} = 175 \text{ mm}$$

(*ii*) *Rectangle* (2)

$$a_2 = 150 \times 50 = 7500 \text{ mm}^2$$

and $$y_2 = \frac{150}{2} = 75 \text{ mm}$$



We know that distance between centre of gravity of the section and bottom of the web,

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2} = \frac{(7500 \times 175) + (7500 \times 75)}{7500 + 7500} = 125 \text{ mm}$$

Moment of inertia about X-X axis

We also know that M.I. of rectangle (1) about an axis through its centre of gravity and parallel to *X-X* axis.

$$I_{G1} = \frac{150\,(50)^3}{12} = 1.5625 \times 10^6 \text{ mm}^4$$

and distance between centre of gravity of rectangle (1) and *X-X* axis,

$$h_1 = 175 - 125 = 50 \text{ mm}$$

Numerical- 3

$\therefore$ Moment of inertia of rectangle (1) about X-X axis

$$I_{G1} + a_1 h_1^2 = (1.5625 \times 10^6) + [7500 \times (50)^2] = 20.3125 \times 10^6 \text{ mm}^4$$

Similarly, moment of inertia of rectangle (2) about an axis through its centre of gravity and parallel to X-X axis,

$$I_{G2} = \frac{50\,(150)^3}{12} = 14.0625 \times 10^6 \text{ mm}^4$$

and distance between centre of gravity of rectangle (2) and X-X axis,

$$h_2 = 125 - 75 = 50 \text{ mm}$$

$\therefore$ Moment of inertia of rectangle (2) about X-X axis

$$= I_{G2} + a_2 h_2^2 = (14.0625 \times 10^6) + [7500 \times (50)^2] = 32.8125 \times 10^6 \text{ mm}^4$$

Now moment of inertia of the whole section about X-X axis,

$$I_{XX} = (20.3125 \times 10^6) + (32.8125 \times 10^6) = 53.125 \times 10^6 \text{ mm}^4 \quad \textbf{Ans.}$$

Moment of inertia about Y-Y axis

We know that M.I. of rectangle (1) about Y-Y axis

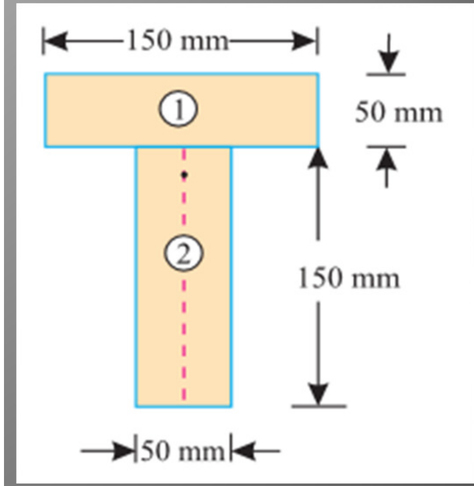
$$= \frac{50\,(150)^3}{12} = 14.0625 \times 10^6 \text{ mm}^4$$

and moment of inertia of rectangle (2) about Y-Y axis,

$$= \frac{150\,(50)^3}{12} = 1.5625 \times 10^6 \text{ mm}^4$$

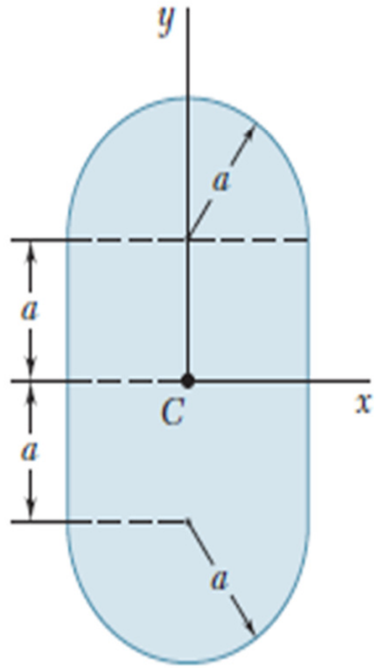
Now moment of inertia of the whole section about Y-Y axis,

$$I_{YY} = (14.0625 \times 10^6) + (1.5625 \times 10^6) = 15.625 \times 10^6 \text{ mm}^4 \quad \textbf{Ans.}$$



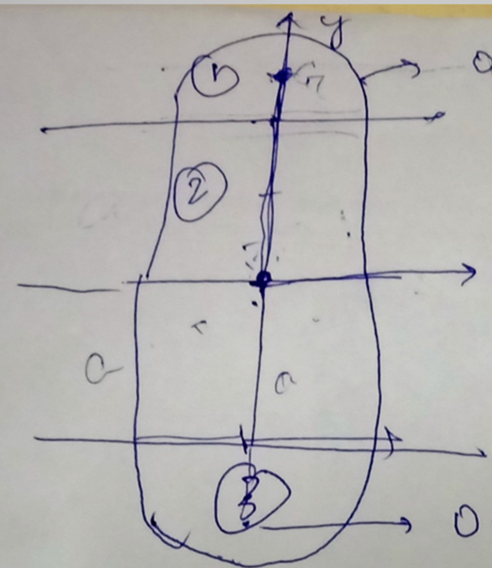
Numerical- 4

Determine the moments of inertia of the shaded area shown with respect to the *x* and *y* axes when a=20 mm as shown in figure **(I_{xx}=1268000 mm^4 and I_{yy}= 339000 mm^4)**



Semicircle Moment of inertia	
Ix (about c.g)	Iy(about c.g)
$0.11R^4$	$\pi R^4/8$

Numerical- 4



$$0.11a^4 + \frac{\pi a^2}{2}\left(\frac{4a}{3\pi} + a\right)^2$$

$$\frac{2a(2a)^3}{12}$$

$$0.11a^4 + \frac{\pi a^2}{2}\left(\frac{4a}{3\pi} + a\right)^2$$

$$I_{yy} = \frac{\pi a^4}{8} + \frac{\pi a^4}{8} + \frac{(2a)(2a)^3}{12}$$

$$= a^4\left(\frac{\pi}{8} + \frac{\pi}{8} + \frac{2 \times 8}{12}\right)$$

$$= 20^4(0.785 + 1.333) = 338$$

Numerical- 5

Example 7.15. *A rectangular hole is made in a triangular section as shown in Fig. 7.19.*

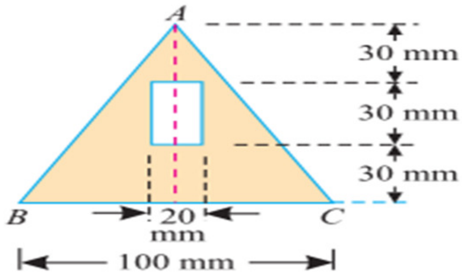


Fig. 7.19.

Determine the moment of inertia of the section about X-X axis passing through its centre of gravity and the base BC.

Solution. As the section is symmetrical about *Y-Y* axis, therefore centre of gravity of the section will lie on this axis. Let $\bar{y}$ be the distance between the centre of gravity of the section and the base *BC*.

(*i*) *Triangular section*

$$a_1 = \frac{100 \times 90}{2} = 4500 \text{ mm}^2$$

and $$y_1 = \frac{90}{3} = 30 \text{ mm}$$

(*ii*) *Rectangular hole*

$$a_2 = 30 \times 20 = 600 \text{ mm}^2$$

and $$y_2 = 30 + \frac{30}{2} = 45 \text{ mm}$$

We know that distance between the centre of gravity of the section and base *BC* of the triangle,

$$\bar{y} = \frac{a_1 y_1 - a_2 y_2}{a_1 - a_2} = \frac{(4500 \times 30) - (600 \times 45)}{4500 - 600} = 27.7 \text{ mm}$$

Numerical- 5

Moment of inertia of the section about X-X axis.

We also know that moment of inertia of the triangular section through its centre of gravity and parallel to X-X axis,

$$I_{G1} = \frac{bd^3}{36} = \frac{100 \times (90)^3}{36} = 2025 \times 10^3 \text{ mm}^4$$

and distance between the centre of gravity of the section and X-X axis,

$$h_1 = 30 - 27.7 = 2.3 \text{ mm}$$

$\therefore$ Moment of inertia of the triangular section about X-X axis

$$= I_{G1} + a_2 h_1^2 = 2025 \times 10^3 + [4500 \times (2.3)^2] = 2048.8 \times 10^3 \text{ mm}^4$$

Similarly moment of inertia of the rectangular hole through its centre of gravity and parallel to the X-X axis

$$I_{G2} = \frac{bd^3}{12} = \frac{20 \times (30)^3}{12} = 45 \times 10^3 \text{ mm}^4$$

and distance between the centre of gravity of the section and X-X axis

$$h_2 = 45 - 27.7 = 17.3 \text{ mm}$$

$\therefore$ Moment of inertia of rectangular section about X-X axis

$$= I_{G2} + a_2 h_2^2 = (45 \times 10^3) + [600 \times (17.3)^2] = 224.6 \times 10^3 \text{ mm}^4$$

Now moment of inertia of the whole section about X-X axis.

$$I_{xx} = (2048.8 \times 10^3) - (224.6 \times 10^3) = 1824.2 \times 10^3 \text{ mm}^4 \quad \textbf{Ans.}$$

Moment of inertia of the section about the base BC

We know that moment of inertia of the triangular section about the base BC

$$I_{G1} = \frac{bd^3}{12} = \frac{100 \times (90)^3}{12} = 6075 \times 10^3 \text{ mm}^4$$

Similarly moment of inertia of the rectangular hole through its centre of gravity and parallel to X-X axis,

$$I_{G2} = \frac{bd^3}{12} = \frac{20 \times (30)^3}{12} = 45 \times 10^3 \text{ mm}^4$$

and distance between the centre of gravity of the section about the base BC,

$$h_2 = 30 + \frac{30}{2} = 45 \text{ mm}$$

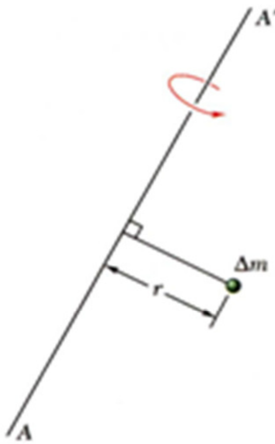
$\therefore$ Moment of inertia of rectangular section about the base BC,

$$= I_{G2} + a_2 h_2^2 = (45 \times 10^3) + [600 \times (45)^2] = 1260 \times 10^3 \text{ mm}^4$$

Now moment of inertia of the whole section about the base BC.

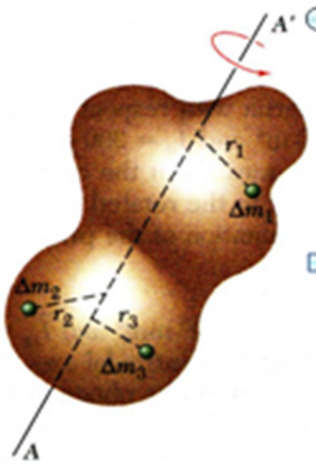
$$I_{BC} = (6075 \times 10^3) - (1260 \times 10^3) = 4815 \times 10^3 \text{ mm}^4 \quad \textbf{Ans.}$$

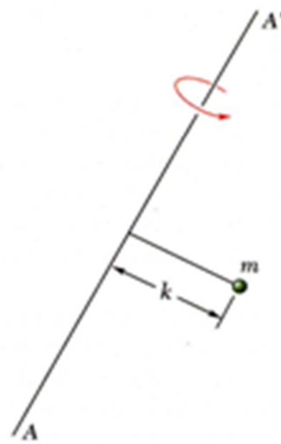
Moment of Inertia of a Mass



- Angular acceleration about the axis *AA'* of the small mass Δm due to the application of a couple is proportional to $r^2 \Delta m$.

  $r^2 \Delta m =$ *moment of inertia* of the mass Δm with respect to the axis *AA'*





- For a body of mass m the resistance to rotation about the axis *AA' is*

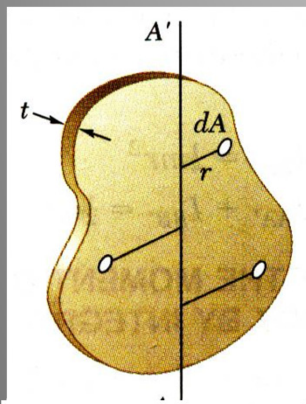
$$I = r_1^2 \Delta m + r_2^2 \Delta m + r_3^2 \Delta m + \cdots$$

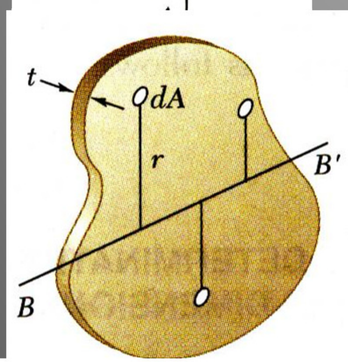
$$= \int r^2 dm = mass\ moment\ of\ inertia$$

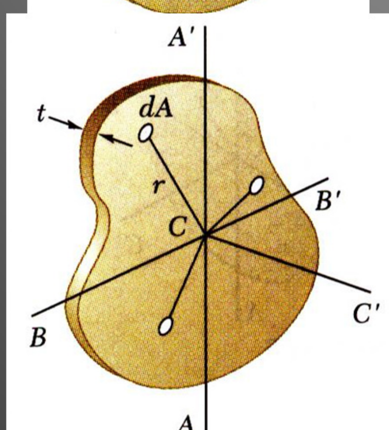
- The radius of gyration for a concentrated mass with equivalent mass moment of inertia is

$$I = k^2 m \qquad k = \sqrt{\frac{I}{m}}$$

Moments of Inertia of Thin Plates







- For a thin plate of uniform thickness *t* and homogeneous material of density *ρ*, the mass moment of inertia with respect to axis *AA'* contained in the plate is

$$\boxed{\begin{aligned} I_{AA'} &= \int r^2 dm = \rho t \int r^2 dA \\ &= \rho t \, I_{AA',area} \end{aligned}}$$

- Similarly, for perpendicular axis *BB'* which is also contained in the plate,

$$\boxed{I_{BB'} = \rho t \, I_{BB',area}}$$

- For the axis *CC'* which is perpendicular to the plate,

$$\boxed{\begin{aligned} I_{CC'} &= \rho t \, J_{C,area} = \rho t \left(I_{AA',area} + I_{BB',area} \right) \\ &= I_{AA'} + I_{BB'} \end{aligned}}$$

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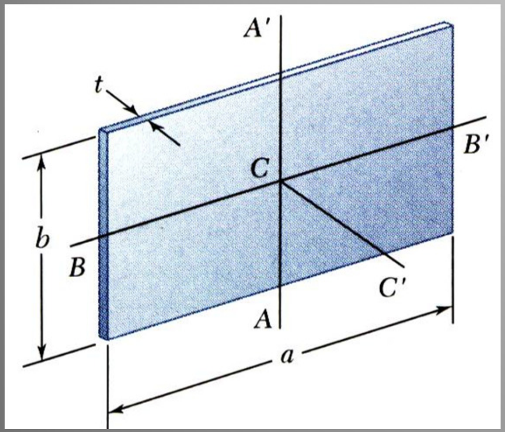

- For the principal centroidal axes on a rectangular plate,

$$I_{AA'} = \rho t \, I_{AA',area} = \rho t \left(\tfrac{1}{12} a^3 b\right) = \tfrac{1}{12} m a^2$$

$$I_{BB'} = \rho t \, I_{BB',area} = \rho t \left(\tfrac{1}{12} a b^3\right) = \tfrac{1}{12} m b^2$$

$$I_{CC'} = I_{AA',mass} + I_{BB',mass} = \tfrac{1}{12} m \left(a^2 + b^2\right)$$

THANKS