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Unit-3

Poisson distribution (Discrete distribution)

$$P(X=x) = p(x,\lambda) = \begin{cases} \dfrac{e^{-\lambda}\lambda^x}{x!}, & x = 0,1,2,3,\ldots\infty \\ 0, & \text{otherwise} \end{cases}$$

$X \sim P(\lambda)$

* Expectation = Variance = λ.

* Moments about origin:

$\mu_1' = \lambda$, $\mu_2' = \lambda^2+\lambda$, $\mu_3' = \lambda^3 + 3\lambda^2+\lambda$

$= E(X)$ $\mu_4' = \lambda^4 + 7\lambda^3 + 6\lambda^2 + \lambda$. etc...

$\text{var}(X) = \mu_2' - \mu_1'^2 = \lambda$.

Expec = Var = λ

§ Central moments of Poisson distribution:

$\mu_1 = 0$

$\mu_2 = \mu_2' - \mu_1'^2 = \lambda$

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = (\lambda^3+3\lambda^2+\lambda) - 3(\lambda^2+\lambda)\lambda + 2\lambda^3$

$= \cancel{\lambda^3} + \cancel{3\lambda^2} + \lambda - \cancel{3\lambda^3} - \cancel{3\lambda^2} + \cancel{2\lambda^3}$

$\mu_3 = \lambda$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

$\mu_4 = 3\lambda^2+\lambda$

$$\boxed{\begin{aligned} \mu_1 &= 0 \\ \mu_2 &= \lambda \\ \mu_3 &= \lambda \\ \mu_4 &= 3\lambda^2+\lambda \end{aligned}}$$

Central moments.

Pb Let $X \sim P(\lambda)$ with $\lambda^2 - \lambda - 2 = 0$. Find mean and variance of X.

Pb Let $X \sim P(\lambda)$ with $\lambda^2-\lambda-2=0$. Find mean and variance of X.

(a) 1, 2
(b) 1, 1
(c) 2, 1
(d) 2, 2 ✓

$\lambda^2-\lambda-2=0$
$\Rightarrow \lambda^2-2\lambda+\lambda-2=0$
$\Rightarrow \lambda(\lambda-2)+1(\lambda-2)=0$
$\Rightarrow (\lambda-2)(\lambda+1)=0$
$\Rightarrow \lambda=2, -1$

$\lambda>0$

$\lambda=2$

$\lambda=2=E(X)=\text{mean}$
$=2$

μ_4 ??
$=3\cdot4+2$
$=14$

$\mu_3' = \lambda^3+3\lambda^2+\lambda$
$=8+12+2$
$=22$

Q Let $X \sim P(\lambda)$ with $4\lambda^2+3\lambda-1=0$. Calculate mean (exp), variance & 3rd central moment.

(a) $4, 4, \frac{1}{4}$
(b) $4, \frac{1}{4}, \frac{1}{4}$
(c) 4, 4, 4
(d) $\frac{1}{4}, \frac{1}{4}, \frac{1}{4}$ ✓

$\lambda=\frac{1}{4}>0$
-1

$\lambda=\frac{1}{4}$

$\mu_3=\lambda=\frac{1}{4}$

* Moment generating function of Poisson distribution:

$$M_X(t)=E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} p(x,\lambda)$$
$$= \sum_{x=0}^{\infty} e^{tx}\cdot\frac{e^{-\lambda}\lambda^x}{x!}$$
$$= \sum_{x=0}^{\infty} \frac{e^{-\lambda}(\lambda e^t)^x}{x!}$$
$$= e^{-\lambda}\sum \frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda} \left\{ 1 + \frac{\lambda e^t}{1!} + \frac{(\lambda e^t)^2}{2!} + \frac{(\lambda e^t)^3}{3!} + \cdots \right\}$$

$$= e^{-\lambda} e^{\lambda e^t}$$

$$= e^{\lambda e^t - \lambda}$$

$$\Rightarrow M_X(t) = e^{\lambda(e^t - 1)}$$

$$M_X(t) = e^{\lambda(e^t - 1)}$$

Pb Let $X \sim P(\lambda)$ with $\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$. Calculate m.g.f of X.

× (a) $e^{-2(e^t - 1)}$

× (b) $e^{2(e^t - 1)}$

× (c) $e^{1 - e^t}$

✓ (d) $e^{(e^t - 1)}$

$\Rightarrow (\lambda - 1)^3 = 0$

$\Rightarrow \lambda = 1 > 0$

$$M_X(t) = e^{\lambda(e^t - 1)} = e^{e^t - 1}$$

§ Mode of Poisson distribution:

we have –

$$\frac{p(x)}{p(x-1)} = \frac{\dfrac{e^{-\lambda}\lambda^x}{x!}}{\dfrac{e^{-\lambda}\lambda^{x-1}}{(x-1)!}} = \frac{e^{-\lambda}\lambda^x}{x!} \times \frac{(x-1)!}{e^{-\lambda}\lambda^{(x-1)}}$$

$$= \frac{\lambda}{x}$$

$$\Rightarrow \frac{p(x)}{p(x-1)} = \frac{\lambda}{x}.$$

Case I :- If λ is not an integer, then $\lambda = \underset{\text{int}}{m} + \underset{\text{frac.}}{f}$

In this case, the distribution is unimodal & the mode $= m =$ integral part of λ.

Case 2 :- If λ is an integer, then $\lambda = m$, int.

In this case, the distribution is bimodal & the

mode = m & $m-1$

$= \lambda$ & $\lambda - 1$

Q. Let $X \sim P(\lambda)$ with $5\lambda^2 + 6\lambda - 1 = 0$. Find the mode of the distribution.

(a) $1/5$

(b) 0 ✓

(c) 5

(d) 0 & 1.

$$5\lambda^2 + 5\lambda + \lambda - 1 = 0$$

$$\Rightarrow \lambda = \frac{-6 \pm \sqrt{36+20}}{2.5}$$

$$= \frac{-6 \pm \sqrt{56}}{10}$$

$$= \frac{-6 \pm 2\sqrt{14}}{10}$$

$$= \frac{-3 \pm \sqrt{14}}{5}$$

$$= -\frac{3}{5} + \frac{\sqrt{14}}{5}, \quad -\frac{3}{5} - \frac{\sqrt{14}}{5} \times$$

$$= \frac{-3 + 3.74}{5}$$

$$= \frac{.74}{5} = 0.148$$

$\lambda = 0.148$

$= 0 + .148$

Pb Let $X \sim P(\lambda)$ with $\lambda^3 - 8 = 0$. Mode = ?

$$\Rightarrow \lambda^3 - 2^3 = 0$$

$$\Rightarrow (\lambda - 2)(\lambda^2 + 2\lambda + 4) = 0$$

$\lambda = 2$

$\lambda, \lambda - 1$

2, 1