12-02-2002

Saturday, February 12, 2022 12:58 PM

Unit-2

H/A

Q Let the r.v X assumes the value 'r' with the prob. law

$P(X=r) = q^{r-1}p$; $r=1,2,3,\dots$ Find the m.g.f of X, expectation of X & variance of X.

(A) m.g.f $= \frac{p}{1-e^t}$

(B) m.g.f $= \frac{pe^t}{1-qe^t}$

(C) m.g.f $= pe^t$

(D) m.g.f $= \frac{qe^t}{1-pe^t}$

Soln:

$$M_X(t) = E(e^{tX})$$
$$= \sum_{r=1}^{\infty} e^{tr} q^{r-1} p$$
$$= \frac{p}{q} \sum_{r=1}^{\infty} e^{tr} q^r$$
$$= \frac{p}{q} \sum_{r=1}^{\infty} (qe^t)^r$$
$$= \frac{p}{q} \{qe^t + (qe^t)^2 + (qe^t)^3 + \dots\}$$
$$= \frac{p}{q} \times qe^t \{1 + qe^t + (qe^t)^2 + \dots\}$$
$$= pe^t \cdot \frac{1}{1-qe^t}$$
$$= \frac{pe^t}{1-qe^t} \quad \text{Ans}$$

E(x)

V(x)

$\frac{t}{1!} = \mu_1'$

$\frac{t^2}{2!} = \mu_2'$

μ_1', μ_2' $\quad E(x) = \mu_1'$, $\quad V(x) = \mu_2' - (\mu_1')^2$

$$M_X(t) = 1 + \mu_1' \frac{t}{1!} + \mu_2' \cdot \frac{t^2}{2!} + \mu_3' \cdot \frac{t^3}{3!} + \dots$$

$p+q=1$

$p = 1-q$

$\frac{d}{dt}$ $\quad M_X'(t) = \mu_1' + t\mu_2' + \frac{t^2}{2!}\mu_3' + \dots$

$$M_X'(0) = \mu_1'$$

$$M_X(t) = \frac{pe^t}{}$$

$$M_X(t) = \frac{pe^t}{1-qe^t}$$

$$\Rightarrow M_X'(t) = \frac{(1-qe^t)pe^t - pe^t(-qe^t)}{(1-qe^t)^2} = \frac{pe^t}{(1-qe^t)^2}$$

$$\Rightarrow M_X'(0) = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p} = \mu_1' = E(X).$$

$$M_X''(t) = \frac{(1-qe^t)^2 \cdot pe^t - pe^t\, 2(1-qe^t)(-qe^t)}{(1-qe^t)^4} = \frac{(1-qe^t)\cdot pe^t + 2pqe^{2t}}{(1-qe^t)^3}$$

$$= \frac{pe^t(1+qe^t)}{(1-qe^t)^3}$$

$$M_X''(0) = \frac{p(1+q)}{p^3} = \frac{1+q}{p^2} = \mu_2'$$

$$\therefore V(X) = \mu_2' - (\mu_1')^2 = \frac{1+q}{p^2} - \frac{1}{p^2} = \frac{q}{p^2}$$

---

§ Properties of moment generating function:

(1) $M_{cX}(t) = M_X(ct)$, c – constant.

Pf: $M_{cX}(t) = E(e^{t(cX)}) = E(e^{(ct)X}) = M_X(ct).$

(2) The m.g.f of the sum of a number of independent random variables is equal to the the product of their m.g.fs.

i.e., if $X_1, X_2, \ldots, X_n$ are n random variables, then –

$$M_{X_1+X_2+\cdots+X_n}(t) = M_{X_1}(t)\, M_{X_2}(t) \cdots M_{X_n}(t).$$

Pf: $M_{X_1+X_2+\cdots+X_n}(t) = E\left(e^{t(X_1+X_2+\cdots+X_n)}\right)$

$$= E\left(e^{tX_1+tX_2+\cdots+tX_n}\right)$$

$$= E\left(e^{tX_1}\cdot e^{tX_2}\cdots e^{tX_n}\right)$$

$$= E(e^{tX_1})\, E(e^{tX_2}) \cdots E(e^{tX_n})$$

$$= M_{X_1}(t)\, M_{X_2}(t) \cdots M_{X_n}(t) \;//$$

(3) Effect of change of scale & origin on moment generating function:

Let us transform the r.v X to a new variable u by changing both origin and scale as follows:-

$$U = \frac{X-a}{h}, \quad a, h - \text{constants.}$$

Here, the m.g.f of U about the origin is given by —

$$M_u(t) = E\left(e^{tu}\right) = E\left(e^{t\left(\frac{X-a}{h}\right)}\right)$$

$$= E\left(e^{\frac{tX}{h}} \cdot e^{-\frac{ta}{h}}\right)$$

$$= e^{-\frac{ta}{h}}\, E\left(e^{\frac{tX}{h}}\right)$$

$$= e^{-\frac{ta}{h}}\, E\left(e^{\left(\frac{t}{h}\right)X}\right)$$

$$\Rightarrow \boxed{M_u(t) = e^{-ta/h}\, M_X\left(\frac{t}{h}\right)}$$

$$U = \frac{X-2}{3}.$$

$$M_u(t) = e^{-\frac{2t}{3}}\, M_X\left(\frac{t}{3}\right)$$

* Standard variate: If in particular, $a = E(X) = \mu$ (let) and $h = \sigma_X = \sigma$, then—

mean

standard deviation

$$U = \frac{X-\mu}{\sigma_X} = \frac{X-\mu}{\sigma} = Z \text{ (say)},$$ is known as standard variate.

Eg: Let X be a r.v s.t $E(X) = 3$ & $\overset{\sigma^2}{V(X)} = 16$. Find the standard variate.

$$U = \frac{X-\mu}{\sigma} = \frac{X-3}{4} = Z$$

* If the origin is changed by $E(X)$ or μ and the scale is changed by the S.D (or $\sqrt{V(X)}$), the new random variable is called a standard variate.

changed by the SD(or V(X)), the new random variable is
called a standard variate.

$$\boxed{Z = \frac{X - E(X)}{\sqrt{V(X)}}}$$

Q Two coins are tossed & X = no. of tails appeared.

X	0	1	2
$P(X)$	$1/4$	$1/2$	$1/4$

$$E(X) = \sum_x x p(x) = 0 + \frac{1}{2} + \frac{1}{2} = 1$$

$$E(X^2) = \sum_x x^2 p(x) = 0 + \frac{1}{2} + 1 = \frac{3}{2}$$

$$\Rightarrow V(X) = \frac{3}{2} - 1^2 = \frac{1}{2}$$

$$Z = \frac{X - E(X)}{\sqrt{V(X)}} = \frac{X-1}{\sqrt{\frac{1}{2}}} = \sqrt{2}(X-1)$$

$$\Rightarrow \underbrace{Z = \sqrt{2}(X-1)}$$

Q Find the m.g.f, expectation & variance of the r.v whose moments about the origin are given by -

$$\mu_r' = (r+1)!\, 2^r$$

X(a) $(1-2t)^{-2}$, 2, 4

X (b) $(1-2t)^2$, 2, 4

✓(c) $(1-2t)^{-2}$, 4, 8

X (d) $(1-2t)^2$, 4, 8

m.g.f, E(X), variance

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r'$$

$$= \sum_{r=0}^{\infty} \frac{t^r (r+1)!\, 2^r}{r!}$$

$$= \sum_{r=0}^{\infty} (r+1)(2t)^r$$

$$= 1 + 2\cdot(2t) + 3\cdot(2t)^2 + 4\cdot(2t)^3 + \cdots$$

$$= \frac{1}{(1-2t)^2}$$

$$= (1-2t)^{-2}$$

$$M_X(t) = 1 + 2\cdot(2t) + 3\cdot(2t)^2 + 4\cdot(2t)^3 + \cdots$$

$$= 1 + 4t + 12t^2 + 32t^3 + \cdots$$

$M_X(t) = 1 + 2\cdot(2t) + 3\cdot(2t)^2 + 4\cdot(2t)^3 + \cdots$

$= 1 + 4t + 12t^2 + 32t^3 + \cdots$

$= 1 + 4\cdot\frac{t}{1!} + 24\cdot\frac{t^2}{2!} + \cdots$

$\mu_1' = 4$ $\quad\quad\therefore E(X) = \mu_1' = 4$

$\mu_2' = 24$ $\quad\quad V(X) = \mu_2' - \mu_1'^2 = 24 - 16 = 8$

§ Unique theorem of M.G.F :- The m.g.f of a distribution, if exists, uniquely determines the distribution.

i.e., corresponding to a given probability distribution, there is only one m.g.f (provided it exists) and corresponding to a given m.g.f, $\exists$ only one probability distribution.

i.e, if $M_X(t) = M_Y(t) \Rightarrow X$ and Y are identically distribution.

✓

$E(X)$ $V(X)$

$\to Z$

X — r.v

$$Z = \frac{X - E(X)}{\sqrt{V(X)}}$$

$E(Z) = 0.$

* Expection and variance of a standard variate :

(a) $$E(Z) = E\left(\frac{X - E(X)}{\sqrt{V(X)}}\right) = E\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma}E(X-\mu)$$

$$= \frac{1}{\sigma}\{E(X) - \mu\}$$

$$= \frac{1}{\sigma}(\mu - \mu)$$

$$= 0$$

(b) $$V(Z) = V\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma^2}V(X-\mu) = \frac{1}{\sigma^2}V(X) = \frac{1}{\sigma^2}\sigma^2 = 1.$$

$\therefore$ Expection & variance of a standard variate for any r.v X are always 0 and 1 respectively.

$$\boxed{E(Z) = 0,\ V(Z) = 1}$$

Pb Let 3 coins are tossed & no. of heads are tossed by r.v X. Find the expression for standard variate.

Sol^n:

X	0	1	2	3
$P(X)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$E(X) = \frac{3}{8} + \frac{3}{4} + \frac{3}{8} = \frac{3}{2}$$

$$E(X^2) = \frac{3}{8} + \frac{3}{2} + \frac{9}{8} = \frac{3}{2} + \frac{3}{2} = \frac{6}{2} = 3$$

$$\therefore V(X) = E(X^2) - (E(X))^2 = 3 - \frac{9}{4} = \frac{3}{4}$$

$$Z = \frac{X - E(X)}{\sqrt{V(X)}} = \frac{X - 3/2}{\frac{\sqrt{3}}{2}} = \frac{2(2X-3)}{2 \times \sqrt{3}}$$

$$\Rightarrow Z = \frac{2X-3}{\sqrt{3}} \quad //$$

Pb :- If the moments of variable X are defined by $\mu_r' = E(X^r) = 0.6$; $r = 1, 2, 3, \ldots$ what are the value of $P(X=0)$, $P(X=1)$ & $P(X \geq 2)$?