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Unit-3

Q. The probability of a man hitting a target is 1/4. If he fires 7 times, what is the prob. of his hitting the target at least twice.

Solⁿ: $p = \frac{1}{4}$ $q = 1 - \frac{1}{4} = \frac{3}{4}$. $n = 7$

$P(X \geq 2) = P(X=2) + \cdots + P(X=7)$

$[P(X=2) + \cdots + P(X=7) = 1 - P(X=1)]$

$= 1 - P(X=1) - P(X=0)$

$P(X=x) = {}^nC_x p^x q^{n-x}$

$= {}^7C_x p^x q^{7-x}$

$= 1 - {}^7C_1 p^1 q^{7-1} - {}^7C_0 p^0 q^7$

$= 1 - 7 \cdot \frac{1}{4}\left(\frac{3}{4}\right)^6 - \left(\frac{3}{4}\right)^7$

$= 1 - \frac{7}{4}\left(\frac{3}{4}\right)^6 - \left(\frac{3}{4}\right)^7$

$= 1 - \frac{7 \cdot 3^6}{4^7} - \frac{2187}{16384} = 1 - \frac{5103}{16384} - 0.133 = 1 - 0.311 - 0.133 = 0.688 - 0.133 = \boxed{0.555}$ Ans

Q. If $X \sim B(n, p)$, with no. of trials 3 and $p = \frac{1}{3}$. Find $P(X \geq 2)$.

(a) 1/27

(b) 3/27

(c) 5/27

(d) 7/27

$n = 3$ $p = \frac{1}{3}$ $q = 1 - \frac{1}{3} = \frac{2}{3}$.

$P(X \geq 2) = P(X=2) + P(X=3)$

$= {}^3C_2 \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^1 + {}^3C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^0$

$= 3 \cdot \frac{1}{9} \cdot \frac{2}{3} + 1 \cdot \frac{1}{27}$

$= \frac{6+1}{27}$

$= \frac{7}{27}$

$P(X=x) = {}^nC_x p^x q^{n-x}$

§ Moments about origin of a random variable following Binomial distribution:

The first four moments about origin are given by -

$$\mu_1' = E(X) = \sum_{x=0}^{\infty} x\,p(x)$$ $\qquad \mu_r' = E(X^r)$

$$= \sum_{x=0}^{\infty} x\,P(X=x)$$

$$= \sum_{x=0}^{\infty} x\; {}^nC_x\, p^x q^{n-x}$$

$$x\,{}^nC_x = x\cdot\frac{n!}{x!(n-x)!} = \frac{n(n-1)!}{(x-1)!\{(n-1)-(x-1)\}!} = n\,{}^{n-1}C_{x-1}$$

$$(a+b)^x = {}^xC_0 a^0 b^x + {}^xC_1 a^1 b^{x-1} + \dots + {}^xC_x a^x b^0$$

$$= np \sum_{x=1}^{n} {}^{n-1}C_{x-1}\, p^{x-1} q^{(n-1)-(x-1)}$$

$$= np\{ {}^{n-1}C_0 p^0 q^{n-1} + {}^{n-1}C_1 p^1 q^{n-2} + {}^{n-2}C_2 p^2 q^{(n-3)} + \dots + {}^{n-1}C_{n-1} p^{n-1} q^0 \}$$

$$= np\,(q+p)^{n-1}$$

B.D. $\quad p+q=1$

$$= np\cdot 1^{n-1}$$

$$\Rightarrow \boxed{\mu_1' = np = E(X)}$$ Expectation or mean

Now,

$$\mu_2' = E(X^2) = \sum_{x=0}^{n} x^2 P(X=x)$$

$$= \sum_{x=0}^{n} x^2\, {}^nC_x\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \{x(x-1)+x\} \frac{n(n-1)}{x(x-1)}\, {}^{n-2}C_{x-2}\, p^x q^{n-x}$$

$${}^nC_x = \frac{n!}{x!(n-x)!} = \frac{n(n-1)}{x(x-1)}\frac{(n-2)!}{(x-2)!\{(n-2)-(x-2)\}!} = \frac{n(n-1)}{x(x-1)}\,{}^{n-2}C_{x-2}$$

$$= n(n-1)p^2 \sum_{x=2}^{n} {}^{n-2}C_{x-2}\, p^{x-2} q^{(n-2)-(x-2)} + \sum_{x=0}^{n} x\,p(x)$$

$$= n(n-1)p^2 (q+p)^{n-2} + np$$

$$= n(n-1)p^2\cdot 1 + np$$

$$\Rightarrow \boxed{\mu_2' = n(n-1)p^2 + np}$$ Second moment about origin.

μ_2 about origin.

$$\mu_3' = E(X^3) = \sum_{x=0}^{n} x^3 p(x) = \sum_{x=0}^{n} x^3 \, {}^nC_x p^x q^{n-x}$$

$$= n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np.$$

$$\mu_4' = E(X^4) = \sum_{x=0}^{n} x^4 p(x) = \sum_{x=0}^{n} x^4 \, {}^nC_x p^x q^{n-x}$$

$$= n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np$$

$$Var(X) = \mu_2' - \mu_1'^2 = n(n-1)p^2 + np - n^2p^2$$

$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$= np(1-p)$$

$$Var(X) = npq$$

§ For random variable $X \sim B(n, p)$, we have –

$$\boxed{E(X) = np \quad \& \quad Var(X) = npq.}$$

Q If $X \sim B(n, p)$ and $n = 5$, $p = \frac{1}{5}$. $E(X) = ?$ $V(X) = ?$

(a) $1, \frac{3}{5}$

(b) $\frac{4}{5}, 1$

(c) $1, \frac{4}{5}$ ✓

(d) $\frac{3}{5}, 1$

$$E(X) = np = 5 \cdot \frac{1}{5} = 1$$

$$Var(X) = npq = 5 \cdot \frac{1}{5}\left(1 - \frac{1}{5}\right) = \frac{4}{5}$$

*Moments about mean (central moments) of a Binomial distribution:

$$\mu_1 = 0$$

$$\mu_2 = V(X) = \mu_2' - \mu_1'^2 = npq$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

$$\mu_2 = V(X) = \mu_2' - \mu_1'^2 = npq$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

$$= \{n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np\} - 2\{n(n-1)p^2 + np\}np + 2(np)^3$$

$$= np\{-3np^2 + 3np + 2p^2 - 3p + 1 - 3npq\}$$

$$= np\{3np(1-p) + 2p^2 - 3p + 1 - 3npq\}$$

$$= np(\cancel{3npq} + 2p^2 - 3p + 1 - \cancel{3npq})$$

$$= np(2p^2 - 2p - p + 1)$$

$$= np(2p-1)(p-1)$$

$$= np(1-2p)(1-p)$$

$$\mu_3 = npq(1-p-p) = npq(q-p)$$

$$\boxed{\mu_3 = npq(q-p)}$$

H/W

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

$$= npq\{1 + 3(n-2)pq\}$$

Q Calculate variance & 3rd central moment of a r.v $X \sim B(n,p)$ with $n = 5$, $p = \frac{2}{3}$.

Solⁿ: $Var(X) = \mu_2 = npq = 5 \times \frac{2}{3} \times \left(1 - \frac{2}{3}\right) = 5 \times \frac{2}{3} \cdot \frac{1}{3} = \frac{10}{9}$.

$$\mu_3 = npq(q-p) = \underbrace{5 \times \frac{2}{3} \cdot \frac{1}{3}}\left(\frac{1}{3} - \frac{2}{3}\right) = \frac{10}{9}\left(-\frac{1}{3}\right) = -\frac{10}{27}$$