10 - 02 - 2022

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Unit - 2

* Expectation & variance :

(HA) **Q** Let X = sum of the nos appeared on the two die when thrown. Find variance of X.

Solⁿ:

X	2	3	4	5	6	7	8	9	10	11	12
P(X)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$E(x)$, $E(x^2)$

$$E(X) = \frac{1}{36}(2\times1 + 3\times2 + 4\times3 + 5\times4 + 6\times5 + 7\times6 + 8\times5 + 9\times4 + 10\times3 + 11\times2 + 12)$$

$$= \frac{1}{36}(2+6+12+20+30+42+40+36+30+22+12)$$

$$= \frac{252}{36} = \frac{126}{18} = \frac{63}{9} = \frac{21}{3} = 7.$$

$$E(X^2) = \frac{1}{36}(2^2\times1 + 3^2\times2 + 4^2\times3 + 5^2\times4 + 6^2\times5 + 7^2\times6 + 8^2\times5 + 9^2\times4 + 10^2\times3 + 11^2\times2 + 12^2\times1)$$

$$= 54.83$$

$$\therefore V(X) = E(X^2) - \{E(X)\}^2 = 54.83 - 49 = 5.83 \text{ Ans}$$

Q If the probability density function of a continuous random variable is given by –

$$f(x) = e^{-x}, \quad 0 \le x < \infty \quad , x \in [0, \infty).$$

Calculate the expectation & variance.

(a) $E(X) = 0$, $V(X) = 1$

(b) $E(X) = 1$, $V(X) = 0$

(c) $E(X) = 1$, $V(X) = 1$ ✓

(d) $E(X) = 0$, $V(X) = 0$

Solⁿ:

$$\boxed{E(X) = \int_{-\infty}^{\infty} x f(x)\,dx \qquad V(X) = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \left(\int_{-\infty}^{\infty} x f(x)\,dx\right)^2}$$

$$E(X) = \int^{\infty} x f(x)\,dx = \int^{\infty} x e^{-x}\,dx$$

(ILATE)

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\infty} x e^{-x}\,dx$$

ILATE

$$= x\int_0^{\infty} e^{-x}\,dx - \int_0^{\infty}\left\{\frac{d}{dx}x\int e^{-x}\,dx\right\}dx$$

$$= x\left[-e^{-x}\right]_0^{\infty} - \int^{\infty}(-e^{-x})\,dx$$

$$= \left[-xe^{-x} - e^{-x}\right]_0^{\infty}$$

$$= -\left[(0+0) - (0+1)\right] = 1$$

$$V(X) = E(X^2) - \{E(X)\}^2$$

$$= \int_{-\infty}^{\infty} x^2 e^{-x}\,dx - 1^2$$

$$= \int_0^{\infty} x^2 e^{-x}\,dx - 1$$

$$= \left[x^2\int e^{-x}\,dx - \int_0^{\infty}\left\{\frac{d}{dx}x^2\int e^{-x}\,dx\right\}dx\right] - 1$$

$$= \left[-x^2 e^{-x} + 2\left(\int_0^{\infty} x e^{-x}\,dx\right)\right]_0^{\infty} - 1$$

$$= \left[-(x^2 e^{-x})_0^{\infty} + 2\cdot 1\right] - 1$$

$$= \left[-(0-0) + 2\right) - 1$$

$$= 2-1$$

$$= 1$$

$E(X)=1$, $V(X)=1$

Q A coin is tossed repeatedly until a head appears. What is the expectation of the number of tosses required?

Let X = no. of tosses reqd to get the 1st head.

Events	H	TH	TTH	TTTH	
X	1	2	3	4	. . .
$P(X=x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	. . .

$\therefore\ E(X) = \sum_{x} x\,p(x) = 1\times\frac{1}{2} + 2\times\frac{1}{4} + 3\times\frac{1}{8} + 4\times\frac{1}{16} + \cdots$

(Arithmetic- geometric series)

$\Rightarrow \frac{1}{2}E(X) = \frac{1}{4} + 2\times\frac{1}{8} + 3\times\frac{1}{16} + 4\times\frac{1}{32} + \cdots$

$$\Rightarrow E(X) - \frac{1}{2}E(X) = \left(1\times\frac{1}{2} + 2\times\frac{1}{4} + 3\times\frac{1}{8} + 4\times\frac{1}{16}\right) - \left(\frac{1}{4} + 2\times\frac{1}{8} + 3\times\frac{1}{16} + 4\times\frac{1}{32} + \cdots\right)$$

$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$$

is a G.S with $a = \frac{1}{2}$ & c.r, $r = \frac{1}{2}$

$$= \frac{a}{1-r}$$

$$\Rightarrow \frac{1}{2}E(X) = \frac{1/2}{1-1/2} = 1$$

$$\Rightarrow E(X) = 2$$

Moments & moment generating function

* The moment generating function (m.g.f) of a random variable X, about origin, having the probability function (p.m.f for discrete, p.d.f for continuous r.v) $f(x)$ is given by —

$$\boxed{M_X(t)} = E(e^{tX}) = \begin{cases} \sum_{x} e^{tx} f(x), & X - \text{discrete r.v} \\ \int e^{tx} f(x)\,dx, & X - \text{continuous r.v.} \end{cases}$$

The integration or summation being extended to the entire range of x. t being a real parameter & the R.H.S is absolutely convergent for some +ve $\mathbb{Z}$ h such that $-h < t < h$.

Thus, $M_X(t) = E(e^{tX})$

$$= E\left(1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \cdots\right)$$

$$= 1 + tE(X) + \frac{t^2}{2!}E(X^2) + \frac{t^3}{3!}E(X^3) + \cdots$$

$E(aX+b)$

$+b$

$E(ax+b) = aE(x)+b$

$$= 1 + tE(X) + \frac{t^2}{2!}E(X^2) + \frac{t^3}{3!}E(X^3) + \cdots$$

$$\Rightarrow M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \boxed{E(X^r)},$$

where, $\mu_r' = E(X^r) = \begin{cases} \sum_x x^r f(x), & \text{if } X \text{ is discrete r.v} \\ \int_{-\infty}^{\infty} x^r f(x)dx, & \text{if } X \text{ is continuous r.v} \end{cases}$

is the rth moment of X about the origin.

* The co-effecient of $\frac{t^r}{r!}$ in $M_X(t)$ is the r-th moment μ_r' about the origin.

$$M_X(t) = 1 + \frac{t}{1!}\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\mu_3' + \cdots$$

1st moment, second, third moment

/* Since $M_X(t)$ generates moments, it is called moment generating function. */

Now, $\mu_1' = E(X^1) = E(X) =$ expectation of X.

$\mu_2' = E(X^2)$

$V(X) = E(X^2) - (E(X))^2 = \mu_2' - \mu_1'^2$

$$\boxed{V(X) = \mu_2' - \mu_1'^2}$$

Q. Let the r-th moment of a random variable X be given by —

$\mu_r' = 3^r$, $r = 1, 2, 3, \ldots$ Find the m.g.f, expectation & variance of X.

Sol^n: $M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r'$

Sol:

$$M_X(t) = \sum_{r=0} \frac{t^r}{r!} \mu_r'$$

$$= 1 + \frac{t}{1!}\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\mu_3' + \cdots$$

$$= 1 + t \cdot 3^1 + \frac{t^2}{2!} 3^2 + \frac{t^3}{3!} 3^3 + \cdots$$

$$= 1 + (3t) + \frac{(3t)^2}{2!} + \frac{(3t)^3}{3!} + \cdots$$

$$= e^{3t}$$

$E(X) = \mu_1' = 3^1 = 3.$

$V(X) = \mu_2' - (\mu_1')^2$

$= 3^2 - 3^2$

$= 0$

(a) $E(X) = 3, \ V(X) = 0$

(b) $E(X) = 0, \ V(X) = 3$

(c) $E(X) = 0, \ V(X) = 0$

(d) $E(X) = 3, \ V(X) = 3$

Q Let the r.v X assumes the value 'r' with the prob. law

$P(X = r) = q^{r-1}p \ ; \ r = 1, 2, 3, \ldots$ Find the m.g.f of X, expectation of X & variance of X.

(A) m.g.f $= \frac{p}{1-e^t}$

(B) m.g.f $= \frac{pe^t}{1-qe^t}$

(C) m.g.f $= pe^t$

(D) m.g.f $= \frac{qe^t}{1-pe^t}$