17 - 02 - 2022

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Unit-3

* Binomial distribution : $X \sim B(n,p)$

$$P(X=x) = {}^nC_x\, p^x q^{n-x}$$

* Moment about origin & moments about mean : -

$$E(x) = np \qquad V(x) = npq$$

$$= \mu_1' \qquad\qquad = \mu_2 = \mu_2' - \mu_1'^2.$$

§ Remarks

(1) $E(x) = np$, $V(x) = npq$ $X \sim B(n,p)$ $q < 1$

$\Rightarrow npq < np$

$$\boxed{V(X) < E(X)}$$

$\Rightarrow$ Variance of a Binomial variate is less than the expectation (mean).

Pb :- The mean of a binomial distribution is 3 & variance is 4. Is it a valid information ?

Sol^n: $E(X) > V(X)$ $X \sim B(n,p)$

Not a valid Statement.

$$E(X) = 3 = np$$

$$V(X) = 4 = npq$$

Now, $q = \frac{npq}{np} = \frac{V(X)}{E(X)} = \frac{4}{3} = 1.33\ldots > 1$

X.

Q The mean & variance of X where $X \sim B(n,p)$ are 4 & $\frac{4}{3}$ resp. Find $P(X \geq 1)$, 2nd moment & 3rd moments about origin.

Sol^n: $E(X) = 4$ $V(X) = \frac{4}{3}$

$\Rightarrow np = 4$ $\Rightarrow npq = \frac{4}{3}$

$\Rightarrow q = \frac{npq}{\ldots} = \frac{4/3}{\ldots}$ ...

$$\Rightarrow np = 4 \qquad \Rightarrow npq = \frac{4}{3}$$

$$\Rightarrow q = \frac{npq}{np} = \frac{4/3}{4} = 1/3 \qquad \Rightarrow p = 1 - \frac{1}{3} = \frac{2}{3}$$

$$np = 4$$

$$\Rightarrow n \cdot \frac{2}{3} = 4 \qquad n = 6, \; p = \frac{2}{3}$$

$$\Rightarrow n = 4 \times \frac{3}{2} = 6$$

Now,

$$P(X \geq 1) = 1 - P(X < 1)$$

$$= 1 - P(X = 0)$$

$$= 1 - {}^6C_0 \, p^0 q^6$$

$$= 1 - 1.1 \left(\frac{1}{3}\right)^6 = 1 - \frac{1}{3^6} = 0.99863.$$

$\mu_2' = ?$ (a) $\frac{17}{3}$ (b) $\frac{29}{3}$ (c) $\frac{37}{3}$ (d) $\frac{52}{3}$.

$$\mu_2' = n(n-1)p^2 + np = 6 \cdot 5 \cdot \frac{2^2}{3^2} + 6 \cdot \frac{2}{3} = \frac{40}{3} + 4 = \frac{40+12}{3} = \frac{52}{3} \;//$$

$\mu_3' = ?$

$$= n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np$$

$$= 6 \cdot 5 \cdot 4 \cdot \frac{8}{27} + 18 \cdot 5 \cdot \frac{4}{9} + 4$$

$$= \frac{320}{9} + 40 + 4$$

$$= \frac{320}{9} + 44 = \frac{320 + 396}{9} = \frac{716}{9} \;//$$

§ Moment Generating Function of Binomial distribution:

Let $X \sim B(n,p)$. Then –

$$M_X(t) = E(e^{tX}) = \sum_{x=0}^{n} e^{tx} p(x)$$

$$= \sum_{x=0}^{n} e^{tx} \, {}^nC_x \, p^x q^{n-x}$$

$$= \sum^{n} {}^nC \, (pe^t)^x \, q^{n-x}$$

$$x=0$$

$$= \sum_{x=0}^{n} {}^nC_x (pe^t)^x q^{n-x}$$

$$= {}^nC_0 (pe^t)^0 q^n + {}^nC_1 (pe^t)^1 q^{n-1} + {}^nC_2 (pe^t)^2 q^{n-2} + \cdots + {}^nC_{n-1} (pe^t)^{n-1} q + {}^nC_n (pe^t)^n q^0$$

$$\Rightarrow \boxed{M_X(t) = (q+pe^t)^n}$$

$X \sim B(n,p)$

Eg If $X \sim B(n,p)$ with $n = 6$ & $p = \frac{1}{5}$. Then -

$$M_X(t) = \left(\frac{4}{5} + \frac{1}{5}e^t\right)^6 = \frac{1}{5^6}(4+e^t)^6 \ //$$

Q Let $X \sim B(n,p)$ where $p = \frac{1}{4}$ & $n = 3$. What is the m.g.f and the fourth moment about origin.

(a) $\frac{1}{64}(3+e^t)^3$

(b) $\frac{1}{16}(3+e^t)^2$

(c) $\frac{1}{64}(3+e^t)^2$

(d) $\frac{1}{16}(3+e^t)^3$

$(q+pe^t)^3$

$$M_X(t) = (q+pe^t)^n$$
$$= \left(\frac{3}{4} + \frac{1}{4}e^t\right)^3$$
$$= \frac{1}{4^3}(3+e^t)^3$$
$$= \frac{1}{64}(3+e^t)^3$$
$$= \frac{1}{64}(27 + 27e^t + 9e^{2t} + e^{3t})$$
$$= \frac{1}{64} \times 27 + \frac{27}{64}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right) + \frac{9}{64}\left(1 + 2t + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \cdots\right) + \frac{1}{64}\left(1 + 3t + \frac{(3t)^2}{2!} + \frac{(3t)^3}{3!} + \cdots\right)$$

$\mu_4' =$ co-ef $\frac{t^4}{4!}$

$$\frac{t^4}{4!} = \frac{27}{64} + \frac{9}{64} \cdot 2^4 + \frac{1}{64} \cdot 3^4$$

$$= \frac{27 + 144 + 81}{64} = \frac{252}{64}$$

$$\mu_4' = \frac{252}{64} = \frac{126}{32} = \frac{63}{16}$$ Ans

* M.G.F about mean of Binomial distribution:

$M_{(X-0)}t = (q+pe^t)^n$

$E(X) = np$

$$M_{X-np}(t) = E\{e^{t(X-np)}\}$$

$$E(X) = np$$

$$M_{X-np}(t) = E\{e^{t(X-np)}\}$$
$$= E\{e^{tx} \cdot e^{-ntp}\}$$
$$= e^{-tnp} E\{e^{tx}\}$$
$$= e^{-tnp} M_X(t)$$
$$= e^{-tnp} (q+pe^t)^n$$
$$= \{e^{-tp}(q+pe^t)\}^n$$
$$= (qe^{-tp} + pe^{t(1-p)})^n$$
$$\Rightarrow \boxed{M_{X-np}(t) = (qe^{-tp} + pe^{tq})^n}$$

* Moment about origin:

$$\mu_1' = np = E(x)$$
$$\mu_2' = n(n-1)p^2 + np$$
$$\mu_3' = n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np$$
$$\mu_4' = n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np$$

* Central moments:

$$\mu_1 = 0$$
$$\mu_2 = V(X) = npq$$
$$\mu_3 = npq(q-p)$$
$$\mu_4 = npq\{1+3(n-2)pq\}$$

* $M_X(t) = (q+pe^t)^n$

$$M_{X-np}(t) = (qe^{-pt} + pe^{qt})^n$$

X 1 2 3

§ Mode of Binomial distribution:

$p(x)$

we have –

$p(x)$ ${}^nC_x p^x q^{n-x}$

we have –

$$\frac{p(x)}{p(x-1)} = \frac{{}^nC_x p^x q^{n-x}}{{}^nC_{x-1} p^{x-1} q^{n-x+1}}$$

$$= \frac{\frac{n!}{x!(n-x)!}}{\frac{n!}{(x-1)!(n-x+1)!}} \cdot \frac{p^x q^{n-x}}{p^{x-1} q^{n-x+1}} \checkmark$$

$$= \frac{n-x+1}{x} \cdot \frac{p}{q}$$

$$= \frac{(n-x+1)p}{xq}$$

$$= \frac{xq + (n-x+1)p - xq}{xq}$$

$$= 1 + \frac{(n-x+1)p - xq}{xq}$$

$$= 1 + \frac{(n+1)p - x(p+q)}{xq}$$

$X \sim B(n,p)$

$(n+1)p = m + f$

$$\Rightarrow \frac{p(x)}{p(x-1)} = 1 + \frac{(n+1)p - x}{xq}$$

Case 1 :- If $(n+1)p$ is not an integer, then $(n+1)p = m+f$, where m = integral part & f = fractional part.

Here, we say that the distribution is unimodal and mode $= m =$ the integral part of $(n+1)p$.

Case 2 If $(n+1)p$ is an integer, then the distribution is bimodal & the mode are m & $m-1$, where $(n+1)p = m$.

$m, m-1$

Eg: $n = 12$, $p = \frac{1}{5}$ mode = ?

$$(n+1)p = (12+1)\cdot\frac{1}{5} = \frac{13}{5} = 2 + \frac{3}{5}$$

$$(n+1)p = (12+1)\cdot\frac{1}{5} = \frac{13}{5} = 2+\frac{3}{5} //$$

$$\text{mode} = 2$$