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Unit - I (Basics of Probability)

$\boxed{\text{BAYE'S THEOREM}}$

§ Theorem of total probability: If $E_1, E_2, \ldots, E_n$ be n mutually disjoint events with $P(E_i) \neq 0$ $(i = 1, 2, \ldots, n)$, then for any arbitrary event A which is a subset of $\bigcup_{r=1}^{n} E_r$, we have –

$$P(A) = \underbrace{\sum_{i=1}^{n} P(E_i)\, P(A|E_i)}_{\text{total probability}} = P(E_1)P(A|E_1) + P(E_2)P(A|E_2) + \cdots + P(E_n)P(A|E_n)$$

Example 4·30. *In 1989 there were three candidates for the position of principal – Mr. Chatterji, Mr. Ayangar and Dr. Singh – whose chances of getting the appointment are in the proportion 4:2:3 respectively. The probability that Mr. Chatterji if selected would introduce co-education in the college is 0·3. The probabilities of Mr. Ayangar and Dr. Singh doing the same are respectively 0·5 and 0·8. What is the probability that there was co-education in the college in 1990?*

Theorem of total probability.

Solⁿ: Let us define the events as follows:–

A : Introduction of coeducation in 1990.

E_1 : Mr. Chatterji is selected as Principal

E_2 : Mr. Ayangar " " " "

E_3 : Dr. Singh " " " "

Then, $P(E_1) = \frac{4}{4+2+3} = \frac{4}{9}$, $P(E_2) = \frac{2}{9}$, $P(E_3) = \frac{3}{9}$

$P(A|E_1) = 0.3$, $P(A|E_2) = 0.5$ & $P(A|E_3) = 0.8$

Applying the theorem of total probability, we get –

$$P(A) = \sum_{i=1}^{3} P(E_i)P(A|E_i) = P(E_1)P(A|E_1) + P(E_2)P(A|E_2) + P(E_3)P(A|E_3)$$

$$= \frac{4}{9} \times 0.3 + \frac{2}{9} \times 0.5 + \frac{3}{9} \times 0.8$$

$$= \frac{0.4}{3} + \frac{1}{9} + \frac{12}{45}$$

$$= \frac{6+5+12}{45}$$

$$= \frac{4}{9} \times 0.3 + \frac{2}{9} \times 0.5 + \frac{3}{9} \times 0.8$$

$$= \frac{0.4}{3} + \frac{1}{9} + \frac{12}{45}$$

$$= \frac{6+5+12}{45}$$

$$= \frac{23}{45}$$

$$= \underbrace{0.511}$$

§ BAYE'S THEOREM : If $E_1, E_2, \cdots, E_n$ are n mutually disjoint events with $P(E_i) \neq 0$ $(i = 1, 2, \cdots, n)$, then for any arbitrary event A, which is a subset of $\bigcup_{r=1}^{n} E_r$ such that $P(A) > 0$, we have —

$$P(E_i \mid A) = \frac{P(E_i) P(A \mid E_i)}{\sum_{i=1}^{n} P(E_i) P(A \mid E_i)} = \boxed{\frac{P(E_i) P(A \mid E_i)}{P(A)}}$$

$$P(E_1 \mid A) = \frac{P(E_1) P(A \mid E_1)}{P(A)}, \quad P(E_2 \mid A) = \frac{P(E_2) P(A \mid E_2)}{P(A)}, \quad \cdots \text{ etc.}$$

Proof :- Since $A \subset \bigcup_{i=1}^{n} E_i$, we have —

$$A = A \cap \left(\bigcup_{i=1}^{n} E_i \right) = \bigcup_{i=1}^{n} (A \cap E_i) \quad \text{[Distributive law]}$$

Since $(A \cap E_i) \subset E_i$ $(i = 1, 2, \cdots, n)$ are mutually disjoint events, we have by the addition theorem of probability (Axiom 3) that

$$P(A) = P\left[\bigcup_{i=1}^{n} (A \cap E_i) \right] \qquad P(A \mid E_i) = \frac{P(A \cap E_i)}{P(E_i)}$$

$$= \sum_{i=1}^{n} P(A \cap E_i)$$

$$\Rightarrow P(A) = \sum_{i=1}^{n} P(E_i) P(A \mid E_i) \longrightarrow (*)$$

$$\sum_{i=1}$$

Again,

$$P(A \cap \mathcal{E}_i) = P(A)\, P(\mathcal{E}_i \backslash A)$$

$$\boxed{P(\mathcal{E}_i \backslash A) = \frac{P(\mathcal{E}_i \cap A)}{P(A)}}$$

$$\therefore P(\mathcal{E}_i \backslash A) = \frac{P(A \cap \mathcal{E}_i)}{P(A)}$$

$$\Rightarrow P(\mathcal{E}_i \backslash A) \boxed{= \frac{P(\mathcal{E}_i)\, P(A \backslash \mathcal{E}_i)}{\sum_{i=1}^{n} P(\mathcal{E}_i) P(A \backslash \mathcal{E}_i)}} \quad [\text{from } \circledast]$$

$$= \frac{P(\mathcal{E}_i) P(A \backslash \mathcal{E}_i)}{P(A)}$$

Proved

Example. 4·31. *The contents of urns I, II and III are as follows:*

I — *1 white, 2 black and 3 red balls,*

II — *2 white, 1 black and 1 red balls, and*

III — *4 white, 5 black and 3 red balls.*

One urn is chosen at random and two balls drawn. They happen to be white and red. What is the probability that they come from urns I, II or III ?

Solⁿ: Let us define the events as follows:

$\mathcal{E}_1$: event that urn I is chosen

$\mathcal{E}_2$: " " " II " "

$\mathcal{E}_3$: " " " III " "

A : event that the two selected balls are white & red.

Then, $P(\mathcal{E}_1) = P(\mathcal{E}_2) = P(\mathcal{E}_3) = \frac{1}{3}$

$$P(A \backslash \mathcal{E}_1) = \frac{{}^1C_1 \times {}^3C_1}{{}^6C_2} = \frac{1 \times 3}{15} = \frac{1}{5}$$

$$P(A \backslash \mathcal{E}_2) = \frac{{}^2C_1 \times {}^1C_1}{{}^4C_2} = \frac{2 \times 1}{6} = \frac{1}{3}$$

$$P(A \backslash \mathcal{E}_3) = \frac{{}^4C_1 \times {}^3C_1}{{}^{12}C_2} = \frac{4 \times 3}{6 \times 11} = \frac{2}{11}$$

$P(\mathcal{E}_1 \backslash A) = ?$ $\quad P(\mathcal{E}_2 \backslash A) = ?$ $\quad P(\mathcal{E}_3 \backslash A) = ?$

we have —

$$P(A) = \sum_{i=1}^{3} P(E_i)P(A|E_i) = P(E_1)P(A|E_1) + P(E_2)P(A|E_2) + P(E_3)P(A|E_3)$$

$$= \frac{1}{3}\times\frac{1}{5} + \frac{1}{3}\times\frac{1}{3} + \frac{1}{3}\times\frac{2}{11}$$

$$= \frac{1}{3}\left(\frac{1}{5}+\frac{1}{3}+\frac{2}{11}\right)$$

$$= \frac{1}{3}\cdot\frac{33+55+30}{165} = \frac{1}{3}\cdot\frac{118}{165} = \frac{1}{3}\cdot\frac{118}{165}$$

$\therefore$ From Baye's theorem,

$$P(E_1|A) = \frac{P(E_1)P(A|E_1)}{P(A)} = \frac{\frac{1}{3}\times\frac{1}{5}}{\frac{1}{3}\cdot\frac{118}{165}} = \frac{1}{5}\times\frac{165}{118} = \frac{33}{118}$$

$$P(E_2|A) = \frac{P(E_2)P(A|E_2)}{P(A)} = \frac{\frac{1}{3}\times\frac{1}{3}}{\frac{1}{3}\cdot\frac{118}{165}} = \frac{1}{3}\times\frac{165}{118} = \frac{55}{118}$$

$$\frac{33}{118}+\frac{55}{118}+\frac{30}{118} = \frac{118}{118} = 1$$

$$P(E_3|A) = \frac{P(E_3)P(A|E_3)}{P(A)} = \frac{\frac{1}{3}\times\frac{2}{11}}{\frac{1}{3}\cdot\frac{118}{165}} = \frac{\cancel{2}}{\cancel{11}}\times\frac{\cancel{165}^{15}}{\cancel{118}_{59}} = \frac{15}{59}.$$

Example 4·33. *In a bolt factory machines A, B and C manufacture respectively 25%, 35% and 40% of the total. Of their output 5, 4, 2 per cent are defective bolts. A bolt is drawn at random from the product and is found to be defective. What are the probabilities that it was manufactured by machines A, B and C?*