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Unit-2 (Random variables)

Continuous random variables

$F(x)$

Q.
$$f(x) = \begin{cases} 0, & x<0 \\ x/4, & 0 \le x \le 2 \\ (4-x)\frac{1}{4}, & 2 \le x \le 4 \\ 0, & x>4 \end{cases}$$

Find the c.d.f of X.

Sol^n:

$$\boxed{\begin{aligned} F(x) &= P(X \le x) \\ F(x) &= \int_{-\infty}^{x} f(x)\,dx \end{aligned}}$$

$x \in (-\infty, 0)$
$x \in [0,2]$
$x \in [2,4]$
$x \in (4, \infty)$

$x \in (-\infty,0)$ or $-\infty < x < 0$, $F(x) = \int_{-\infty}^{x} f(x)\,dx = 0$

$x \in [0,2]$ or $0 \le x \le 2$, $F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{x} f(x)dx$

$$\boxed{f(x) = \begin{cases} 0, & x<0 \\ x/4, & 0 \le x \le 2 \\ (4-x)\frac{1}{4}, & 2 \le x \le 4 \\ 0, & x>4 \end{cases}}$$

$= 0 + \int_{0}^{x} x/4\,dx$

$= \frac{x^2}{8}\Big]_0^x = \frac{x^2}{8}$

$x \in [2,4]$ or $2 \le x \le 4$, $F(x) = \int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{2} f(x)\,dx + \int_{2}^{x} f(x)\,dx$

$= 0 + \int_0^2 \frac{x}{4}dx + \int_2^x \left(1 - \frac{x}{4}\right)dx$

$= \frac{x^2}{8}\Big]_0^2 + \left[x - \frac{x^2}{8}\right]_2^x$

$= \left(\frac{1}{2} - 0\right) + \left(\left(x - \frac{x^2}{8}\right) - \left(2 - \frac{1}{2}\right)\right)$

$= \frac{1}{2} + x - \frac{x^2}{8} - 2 + \frac{1}{2}$

$= x - x^2/8 - 1$

$x \in (4,\infty)$ or $4 < x < \infty$, $F(x) = \int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{0} 0\,dx + \int_0^2 \frac{x}{4}dx + \int_2^4 \left(1 - \frac{x}{4}\right)dx + \int_4^{\infty} 0\,dx$

$= 0 + \frac{x^2}{8}\Big]_0^2 + \left[x - \frac{x^2}{8}\right]_2^4 + 0$

$= \frac{1}{2} + \{(4-2) - (2 - \frac{1}{2})\}$

$= \frac{1}{2} + \left\{ (4-2) - \left(2 - \frac{1}{2}\right) \right\}$

$= \frac{1}{2} + 2 - \frac{3}{2} = \frac{5}{2} - \frac{3}{2} = 1$ //

$$\therefore \boxed{F(x) = \begin{cases} 0, & -\infty < x < 0 \\ x^2/8, & 0 \le x \le 2 \\ x - \frac{x^2}{8} - 1, & 2 \le x \le 4 \\ 1, & 4 < x < \infty \end{cases}}$$

Q. A continuous random variables has p.d.f :-

$$f(x) = ax^3, \quad 0 \le x \le 1$$
$$= 0, \quad \text{otherwise}$$

Find a.

(a) 1
(b) 2
(c) 3
(d) 4.

Solⁿ: $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\Rightarrow \int_{-\infty}^{0} 0\,dx + \int_0^1 ax^3\,dx + \int_1^{\infty} 0\,dx = 1$

$\Rightarrow a\int_0^1 x^3\,dx = 1$

$\Rightarrow a\left[\frac{x^4}{4}\right]_0^1 = 1$

$\Rightarrow a\left(\frac{1}{4} - 0\right) = 1 \Rightarrow \boxed{a = 4.}$

Q// Find $P(X < \frac{1}{4})$ & $P(X > \frac{1}{2})$.

(a) 15/16, 1/16
(b) 15/16, 1/256
(c) 1/256, 15/16
(d) 1/16, 15/16

$$\boxed{\begin{aligned} f(x) = & \ ax^3, \ 0 \le x \le 1 \\ & \ 0, \ \text{otherwise} \end{aligned}}$$

$= 4x^3, \quad 0 \le x \le 1$

$\quad 0, \quad \text{otherwise}$

$P(X < \frac{1}{4})$

$= \int_{-\infty}^{1/4} f(x)\,dx$

$= \int_0^{1/4} 4x^3\,dx = 4\left[\frac{x^4}{4}\right]_0^{1/4}$

$= \left(\left(\frac{1}{4}\right)^4 - 0^4\right) = \frac{1}{256}$

& $P(X > \frac{1}{2}) = \int_{1/2}^{\infty} f(x)\,dx = \int_{1/2}^{1} 4x^3\,dx + 0 = x^4\Big]_{1/2}^{1} = 1^4 - \left(\frac{1}{2}\right)^4$

$= 1 - \frac{1}{16} = \frac{15}{16}$ //

§ Mathematical Expectation OR Expected value of a random variable :-

The expected value of a discrete r.v is a weighted average

§ Mathematical Expectation OR Expected value of a random variable :-

The expected value of a discrete r.v is a weighted average of all possible values of the r.v.

The mathematical expression for computing the expected value of a discrete r.v X with probability mass function f(x) is :-

$$\boxed{\begin{aligned} E(X) &= \sum_{x} x f(x) \quad (\text{for discrete r.v}) \\ &= \sum_{x} x p(x) \end{aligned}}$$

pmf (p(x))

- The mathematical exp. for computing the expected value of a continuous r.v X with p.d.f f(x) is :

$$\boxed{E(X) = \int_{-\infty}^{\infty} x f(x)\, dx \quad (\text{for cont. r.v})}$$

Q Let X be discrete r.v with probability distribution :-

$X=x$	-3	6	9
$P(X=x)$	$1/6$	$1/2$	$1/3$

$$\boxed{E(X) = \sum_{x} x p(x)}$$

Calculate the expectation of X.

Sol^n:

$$E(X) = \sum_{x} x p(x)$$

$$= (-3)\times\frac{1}{6} + 6\times\frac{1}{2} + 9\times\frac{1}{3}$$

$$= -\frac{1}{2} + 3 + 3$$

$$= 6 - \frac{1}{2} = \frac{11}{2}$$

Q An unbiased die is thrown & the no. is noted. Calculated the expectation.

Sol^n: X = no. appeared on the die

$X=x$	1	2	3	4	5	6
$P(X=x)$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

(a) $7/2$ (b) $5/2$ (c) $3/2$ (d) $9/2$

$$E(X) = \sum_{x} x p(x) = 1\times\frac{1}{6} + 2\times\frac{1}{6} + 3\times\frac{1}{6} + 4\times\frac{1}{6} + 5\times\frac{1}{6} + 6\times\frac{1}{6}$$

$$E(X) = \sum_{x} x p(x) = 1\times\frac{1}{6} + 2\times\frac{1}{6} + 3\times\frac{1}{6} + 4\times\frac{1}{6} + 5\times\frac{1}{6} + 6\times\frac{1}{6}$$

$$= \frac{1}{6}(1+2+3+4+5+6)$$

$$= \frac{21}{6}$$

$$= 7/2$$

§ Properties of Expectation:

(1) Addition theorem of Expectation: If X and Y are random variables, then $E(X+Y) = E(X) + E(Y)$, provided all the expectation exists.

$$E(X_1 + X_2 + \cdots + X_n) = E(X_1) + E(X_2) + \cdots + E(X_n).$$

(2) Multiplication theorem of expectation: If X and Y are independent r.vs, then $E(XY) = E(X)\,E(Y)$.

If $X_1, X_2, \ldots, X_n$ are independent r.vs, then -

$$E\left(\prod_{i=1}^{n} X_i\right) = \prod_{i=1}^{n} E(X_i)$$

(3) If X is a r.v and a is constant, then -

(a) $E[a\psi(X)] = a\,E[\psi(X)]$

(b) $E[\psi(X) + a] = E[\psi(X)] + a$, where $\psi(X)$ is a fun' of X ; is a r.v & all the expectations exists.

(4) If X is a r.v and a, b are constants, then -

$E(aX+b) = a\,E(X) + b$, provided all the expectation exists.

$$\boxed{E(b) = b}$$

(5) Expectation of a linear combination of R.Vs:

If X_i's are r.vs & a_i's are constants, then -

$$E(a_1X_1 + a_2X_2 + \cdots + a_nX_n) = a_1E(X_1) + a_2E(X_2) + \cdots + a_nE(X_n)$$

or, $E\left(\sum_{i=1}^{n} a_iX_i\right) = \sum_{i=1}^{n} a_i E(X_i)$, provided all the expectation exists.

(6) If $X \geqslant 0$, then $E(X) \geqslant 0$.

(7) If X and Y are two r.vs such that $Y \leq X$, then -

$$E(Y) \leq E(X),$$ provided all the exp. exists.

(8) $|E(X)| \leq E|X|$, provided the exp. exists.

---x---