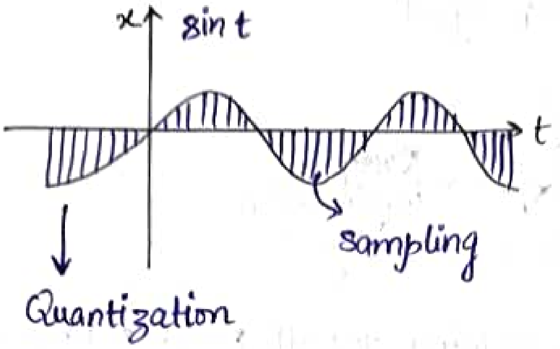
सङ्केतसंविधाः

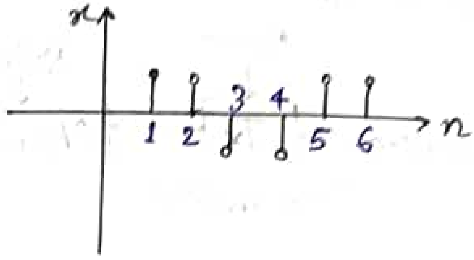
SIGNALS AND SYSTEMS

SIGNALS

Continuous Signals: $x(t)$



Discrete Signal: $x[n]$

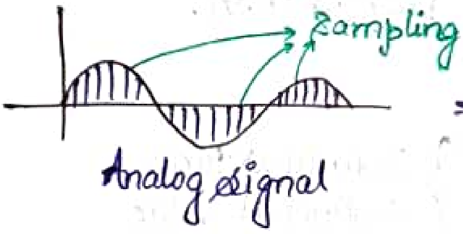


→ Noise is continuous time signal.

Eg.

→ All continuous signals are continuous-time signals but not vice-versa.

→ We need digital signal because it is more useful than analog signal like easy to store, no noise, etc.
For that we need to convert analog signals to digital.



⇒ Quantization ⇒ Digital signal

Signal: Any time varying physical phenomenon that is intended to convey information is signal.
Eg., human voice, sign language, traffic light.

Noise: Noise is like a signal time varying physical phenomenon but usually does not carry any useful information.
↳ Undesirable.

Excitation ← System → Response

i/p signal — could be multiple i/p signal

o/p signal — could be multiple o/p signal

<u>Communication</u>:

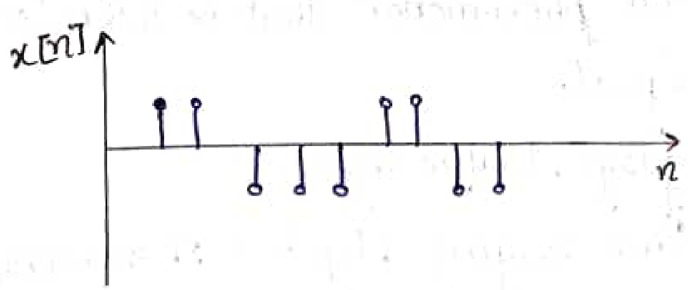
Information → Tx (Transmitter) → Channel → Rx (Receiver) → Noisy information signal

(Noise enters at Tx, Channel and Rx)

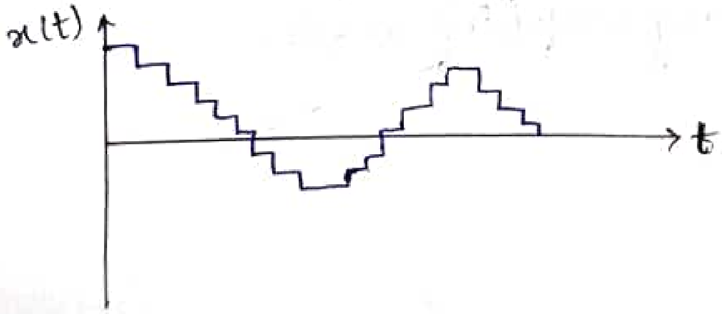
1. Continuous Time Signal (CT) ⇝ varying for all possible time and has value for all possible time.
2. Discrete Time Signal (DT)
3. Continuous Value Signal (CV)
4. Discrete Value Signal (DV)
5. Random signal (R)
6. Non-Random signal (NR)

<u>Analog Signal</u>: Variation of signal with time is analogous (proportional) to some physical phenomenon.

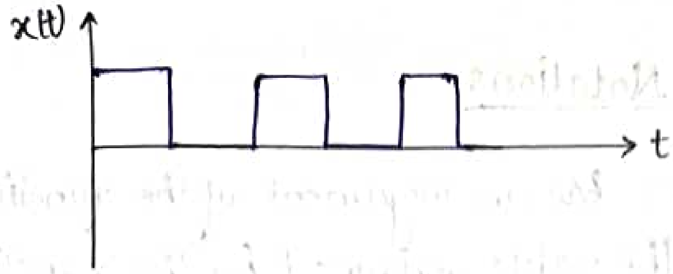
} Continuous time, Continuous value



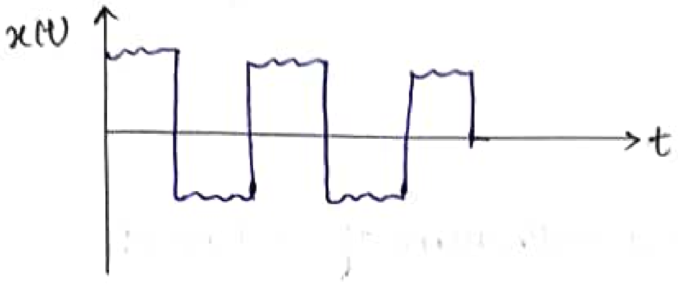
} Discrete time, Continuous value



} continuous time, Discrete value



} continuous time
Discrete value



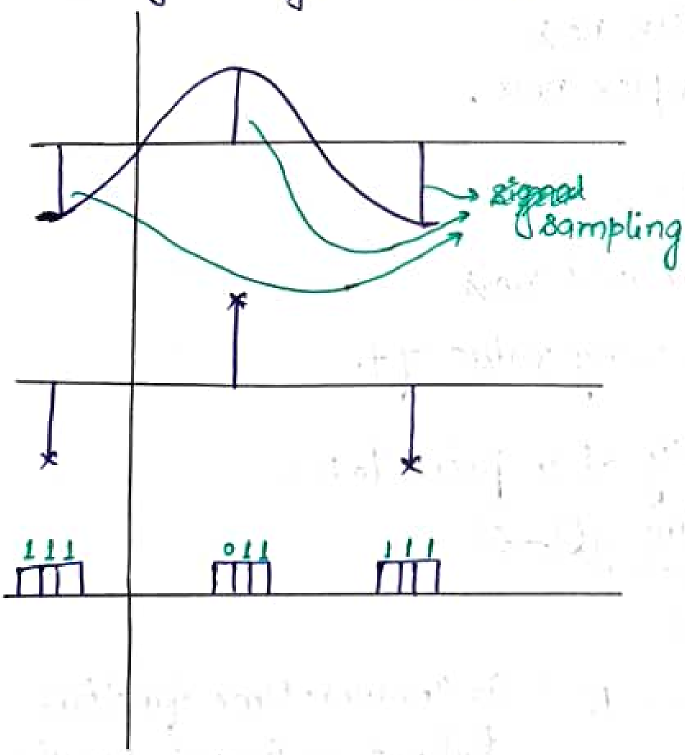
} Continuous time
Discrete value

Sampling: Acquiring values from a continuous time signal at discrete points in time.

Random signal: Signal that cannot be predicted or described by a mathematical function.

Deterministic signal: Not random signal.

ADC: Analog-to-digital converter.



CDMA, GSM technology used in telecommunication.

Functions and Functional Notations:

A function is a correspondance b/w an argument of the function which lies in its domain and the value returned by the function which lies in its range.

$g(x)$
↳ independent variable

$x \in \mathbb{R}$: any real value in a continuum of real no.s;
the value returned also lies in a continuum of real no.s.

① Domain: Continuum of real no.s.
Range: Continuum of real no.s

② Domain: Integers
Range: Continuum of complex no.s

③ Domain: Cont. of real no.s
Range: cont. of complex no.s

④ Domain: Cont. of complex no.s
Range: cont. ~~of~~ of complex no.s.

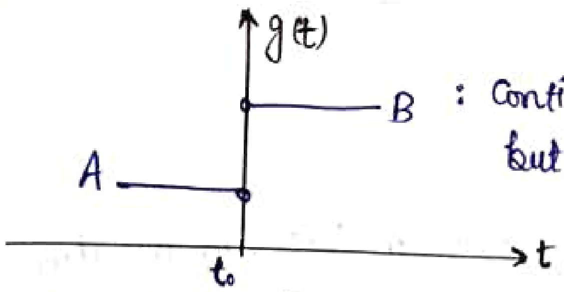
Continuous Time Function:

$g(t)$: domain → all real no.s

$g(t)$ is defined for every value of t.

To check discontinuity at a point t_0:

$$\lim_{\epsilon \to 0} g(t_0+\epsilon) \neq \lim_{\epsilon \to 0} g(t_0-\epsilon)$$



: Continuous-time function but not continuous function

→ All continuous functions are continuous time functions but not all continuous time functions are continuous functions.

Complex Exponential/Sinusoid:

$$g(t) = A\cos(2\pi f_0 t + \theta)$$
$$= A\cos(2\pi \frac{1}{T_0} t + \theta)$$
$$= A\cos(\omega_0 t + \theta)$$

$$g(t) = Ae^{(\sigma_0 + j\omega_0)t}$$
$$= Ae^{\sigma_0 t}\left[\cos(\omega_0 t) + j\sin(\omega_0 t)\right]$$

f_0: fundamental frequency

θ: phase difference

σ_0: attenuation constant

ω_0: angular frequency

A: ~~am~~ magnitude

\# spectrum analyser

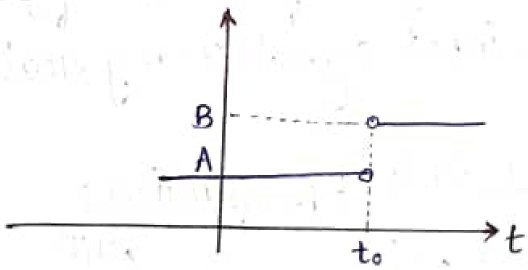
Singularity functions:

The singularity functions are related through integral and derivative.

Unit step function:

$$u(t) = \begin{cases} A, & t < t_0 \\ B, & t > t_0 \end{cases}$$

$A \neq B$



$$g(t) = \begin{cases} A, & t < t_0 \\ \frac{A+B}{2}, & t = t_0 \\ B, & t > t_0 \end{cases}$$

$$h(t) = \begin{cases} A, & t \leq t_0 \\ B, & t > t_0 \end{cases}$$

$g(t)$ and $h(t)$ are different functions but their definite integrals are the same.

$$\int_{\alpha}^{\beta} g(t)\,dt = \int_{\alpha}^{\beta} h(t)\,dt \text{ , for any } \alpha, \beta \text{ including } \alpha < t_0 < \beta.$$

$$\int_{\alpha}^{\beta} g(t)\,dt = \int_{\alpha}^{t_0-\epsilon} g(t)\,dt + \int_{t_0-\epsilon}^{t_0+\epsilon} g(t)\,dt + \int_{t_0+\epsilon}^{\beta} g(t)\,dt$$

As $\epsilon \to 0$,

$\int_{t_0-\epsilon}^{t_0+\epsilon} g(t)\,dt'$ approaches zero $\left[\text{same as } \epsilon \to 0 \quad \int_{t_0-\epsilon}^{t_0+\epsilon} h(t)\,dt \to 0\right]$

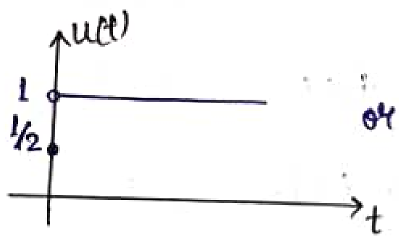
→ The area under a single point is zero regardless of point's value as long as it is finite.

<u>Generalization</u>:

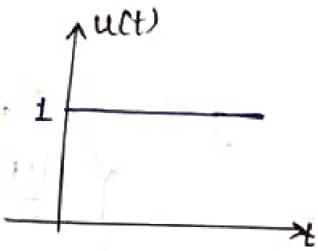
Any two functions that have finite value everywhere and differ in values only at finite no. of points are equivalent in their effects as input signal on any real physical system.

<u>Continuous time unit step function</u>:

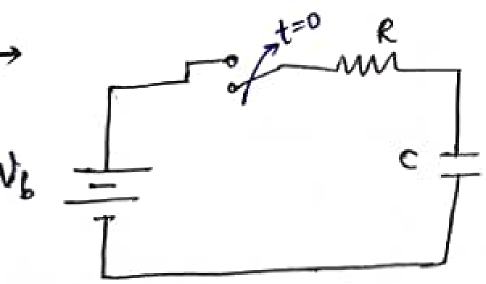
$$u(t) = \begin{cases} 1, & t > 0 \\ 1/2, & t = 0 \\ 0, & t < 0 \end{cases}$$



or



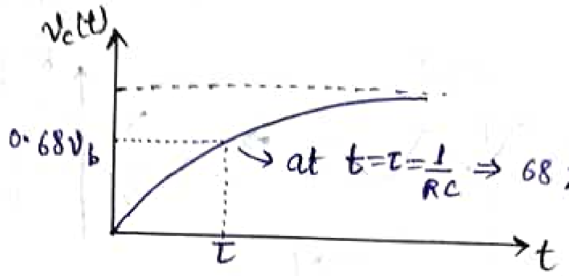
→



Unit step as switch:

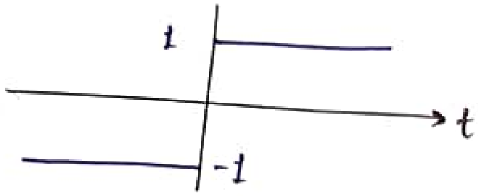
$$v_c = v_b u(t)(1-e^{-t/RC}) \quad | \quad v_{RC} = v_b u(t)$$

$$i_R = \frac{v_b}{R} e^{-t/RC} u(t)$$



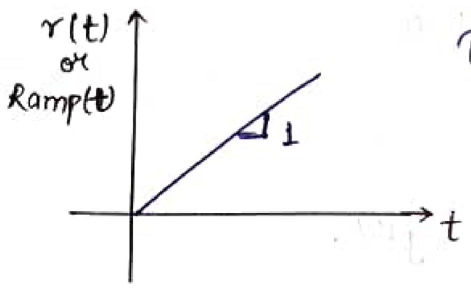
at $t=\tau=\frac{1}{RC} \Rightarrow 68\%$ of v_b

Signum Function:



$$\text{Sgn}(t) = \begin{cases} 1, & t>0 \\ 0, & t=0 \\ -1, & t<0 \end{cases} = 2u(t)-1$$

Unit Ramp Function:



The i/p is changing linearly after some time.

$$\text{Ramp}(t) = \begin{cases} t, & t>0 \\ 0, & t\leq 0 \end{cases}$$

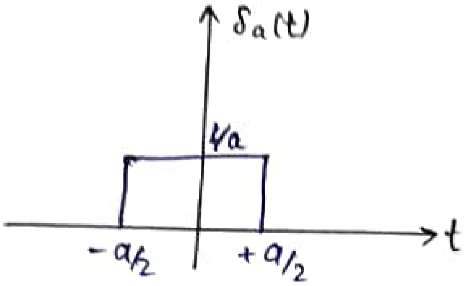
→ Ramp function is the integral of unit step function.

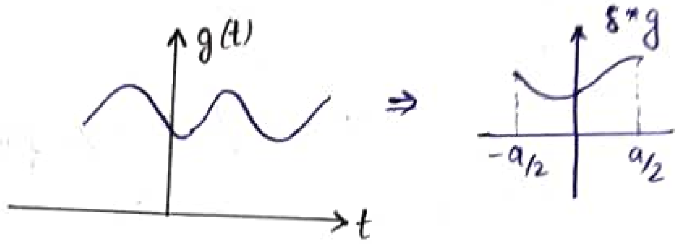
$$\text{Ramp}(t) = \int_{-\infty}^{t} u(\tau)\,d\tau = t\,u(t)$$
$$= t$$

Unit Impulse Function

$$\delta_a(t) = \begin{cases} 1/a, & |t| < a/2 \\ 0, & |t| > a/2 \end{cases}$$

$\boxed{11\text{-}01\text{-}2024}$





$$A = \int_{-\infty}^{\infty} \delta_a(t) g(t)\, dt$$

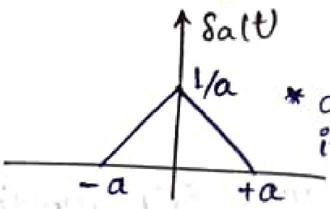
$$= \frac{1}{a} \int_{-a/2}^{a/2} g(t)\, dt$$

$a \to 0$

as 'a' approaches 0, the value of g(t) approaches the same value at both limits and everywhere b/w them because g(t) is continuous and finite at $t=0$

~~and~~ $g(t) \to g(0)$.

$$A = \frac{g(0)}{a} \int_{-a/2}^{a/2} dt = \frac{g(0)}{a} \times a = g(0).$$



* as long as it is unit area. } shape of function is not important.

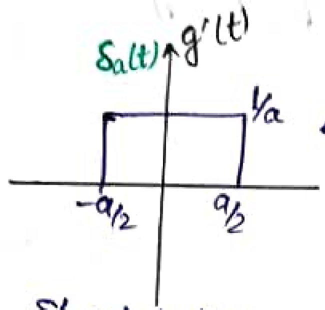
$\Rightarrow$ Same result as for above fn.

$$\rightarrow \int_{-\infty}^{\infty} \delta_a(t) g(t)\, dt = g(0).$$

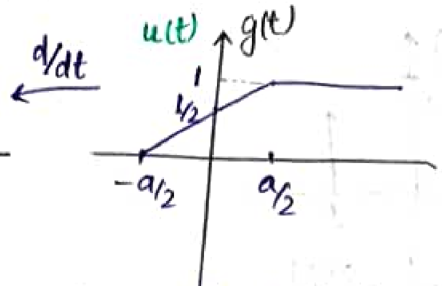
In the limit as 'a' approaches zero, the function $\delta_a(t)$ extracts the value of any continuous finite function g(t) at time $t=0$ when the product of $\delta_a(t)$ and g(t) is integrated b/w the two limits which included $t=0$.

$$\boxed{g(0) = \int_{-\infty}^{\infty} \delta(t)\, g(t)\, dt}$$

$$= \lim_{a \to 0} \int_{-\infty}^{\infty} \delta_a(t)\, g(t)\, dt$$



Short pulse



generalised derivative
$a \to 0$

→ Unit impulse function is the generalised derivative of unit step function.

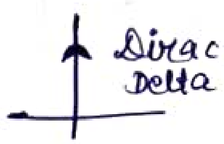
$$\frac{d}{dt}(g(t)) = \frac{d}{dt}(g(t))_{t \neq t_0} + \lim_{\substack{\epsilon \to 0 \\ \text{at } t = t_0}} [g(t+\epsilon) - g(t-\epsilon)]\, \delta(t-t_0)$$

→ $u(t) = \int_{-\infty}^{\infty} \delta(t)\, dt$

→ $\delta(t) = 0, \quad t \neq 0$

$$\int_{t_1}^{t_2} \delta(t)\, dt = \begin{cases} 1, & t_1 < 0 < t_2 \\ 0, & \text{otherwise.} \end{cases}$$

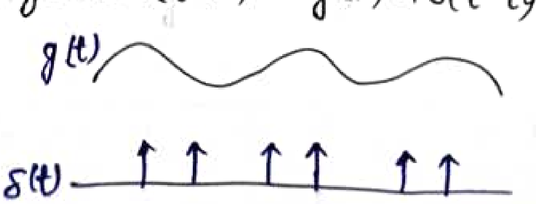
Area under the $\delta(t)$ f^n $\equiv$ strength of $\delta(t)$ f^n



→ Product of an impulse f^n with another f^n is

$A\,\delta(t-t_0)\, g(t)$ ~~= A δ(t-t₀)~~

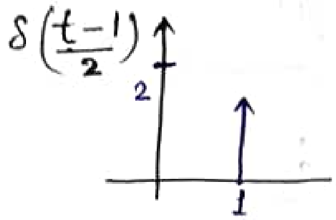
→ $g(t)\, A\, \delta(t-t_0) = g(t_0)\, A\, \delta(t-t_0)$: Equivalence property of impulse



→ $\int_{-\infty}^{\infty} g(t)\,\delta(t-t_0)\,dt = g(t_0)$: *Sampling property*

$$= g(t_0)\int_{-\infty}^{\infty}\delta(t-t_0)\,dt$$

→ $\delta(a(t-t_0)) = \frac{1}{|a|}\,\delta(t-t_0)$: *Scaling property*



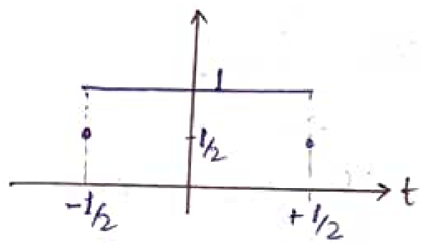
Proof: $\delta(\alpha t) = \frac{1}{|\alpha|}\,\delta(t)$

$\alpha t = \tau \Rightarrow t = \tau/\alpha \Rightarrow dt = d\tau/\alpha$

$$\frac{1}{\alpha}\int_{-\infty}^{\infty}\delta(\tau)\,d\tau = \frac{1}{\alpha}$$

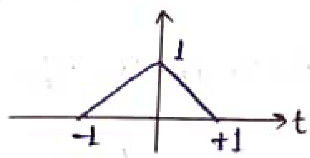
Unit Rectangular Function:

$$\text{rect}(t) = \begin{cases} 1, & |t| < 1/2 \\ 1/2, & |t| = 1/2 \\ 0, & |t| > 1/2 \end{cases}$$



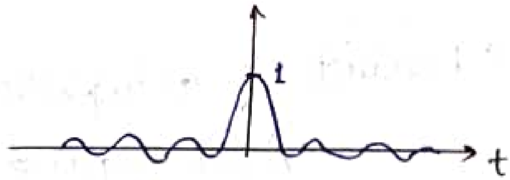
Triangular Function:

$$\text{tri}(t) = \begin{cases} 1-|t|, & |t| < 1 \\ 0, & |t| \geq 1 \end{cases}$$



Sinc Function:

$$\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$



↳ It is the fourier transform of rectangular function.

15-01-2024

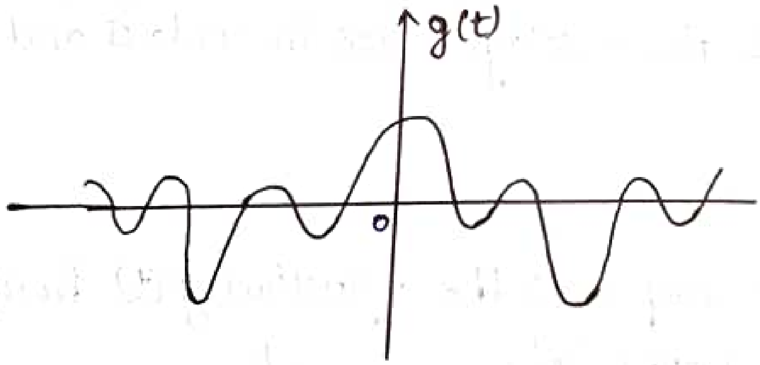
The Dirichlet functions

$$\text{dricl}(t_N) = \frac{\sin(\pi N t)}{N\sin(\pi t)}$$

↳ For N odd, it is similar to sinc function.
↳ Infinite sum of uniformly spaced sinc function.
↳ The numerator, $\sin(\pi N t)$, is zero when t is integral multiple of $1/N$; the denominator will be zero for integral t.

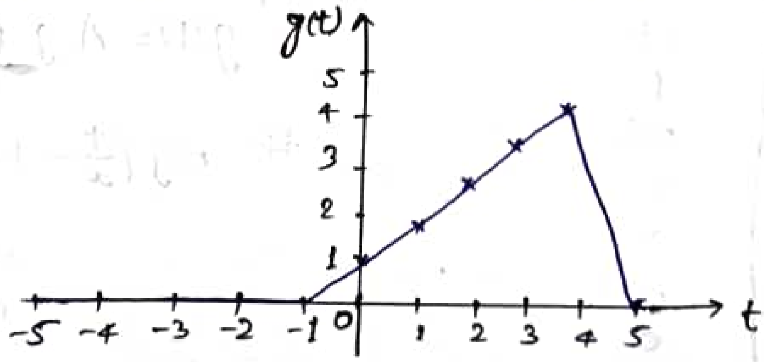
$$\lim_{\substack{t\to m \\ m:\text{integer}}} \frac{\sin(N\pi t)}{N\sin(\pi t)} = \frac{N\pi\cos(N\pi t)}{N\pi\cos(\pi t)} = \pm 1.$$

when N is an integer, the extrema of the dirichlet function alternates b/w ±1.
when N is odd, the extrema is +1.



Scaling and Shifting of the Function

t= -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5
g(t) = 0, 0, 0, 0, 0, 1, 2, 3, 4, 5, 0



$g(t) = 0, \ |t| > 5$

Time-shifting / Time-translating:

Find $g(t-1)$.

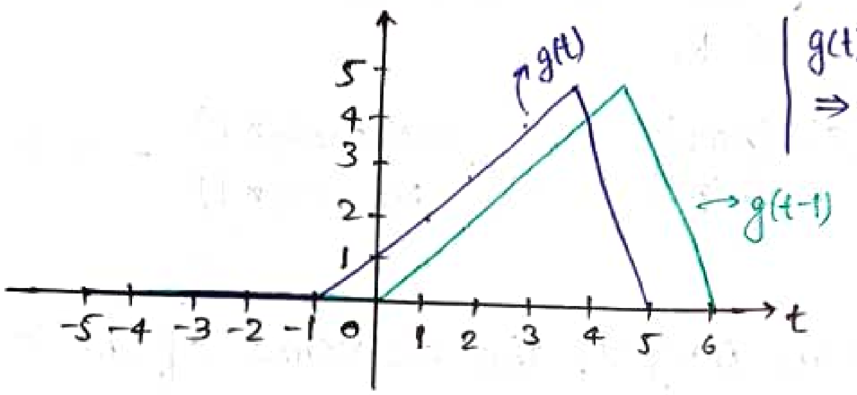
We need to replace t by t-1 and understand the effect of shifting the function by one unit to the right.

$t = -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5$

$g(t) = 0, 0, 0, 0, 0, 1, 2, 3, 4, 5, 0$

$t-1 = -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6$

$g(t-1) = 0, 0, 0, 0, 0, 0, 1, 2, 3, 4, 5, 0, 0$



$g(t) - 1 = t, \ t \geq -1, t \leq 4$

$\Rightarrow g(t) = t+1$

Eg. $rect(t) = u(t+1/2) - u(t-1/2)$

$tri(t) = ramp(t+1) - 2\,ramp(t) + ramp(t-1)$

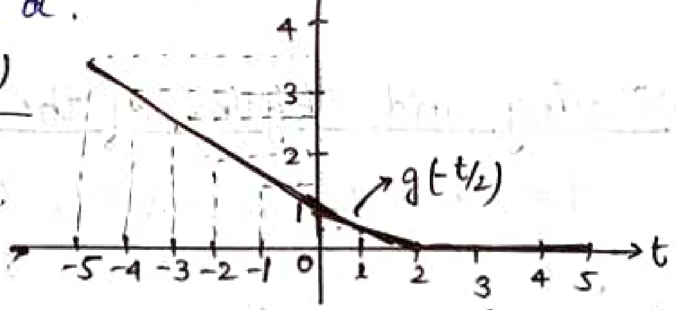
Use: To generate time-delays : use in radars and sonars.

Time-Scaling:

$t = t/a$

$g(t/a) \rightarrow$ time expands the function $g(t)$ horizontally by the factor 'a'.

t	$-t/2$	$g(-t/2)$
−5	2.5	3.5
−4	2	3
−3	1.5	2.5
−2	1	2
−1	0.5	1.5
0	0	1
1	−0.5	0.5
2	−1	0
3	−1.5	0
4	−2	0
5	−2.5	0



$g(t) = A\, g\left(\frac{t-t_0}{a}\right)$

$A g\left(\frac{t}{a} - t_0\right) \neq A g\left(\frac{t-t_0}{a}\right)$

shifting involved

16-01-2024

→ Derivative of a function at any time t is its slope at that time.

→ Integral of a function at any time t is the accumulated area under the function upto that time.

→ The zero crossing of all the derivatives show the maxima and minima of the corresponding function.

Derivative
↳ all pass filter
Integral
↳ high pass filter
↳ not all f^ns have integral.

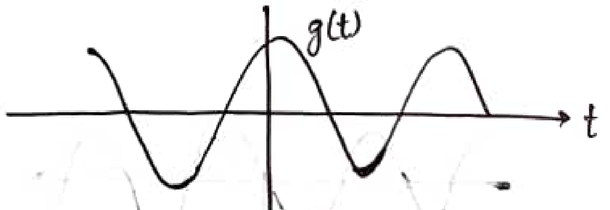
Even and Odd Functions

→ Some of the functions have property that when they undergo certain changes of independent or dependent variable, the function does not change.
↳ Invariant under the change.

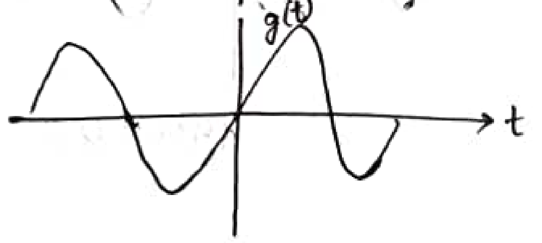
→ Even functions of 't' is one of the function that is invariant under the scaling $t \to -t$.

→ odd function is a function of 't' that is invariant under amplitude and time scaling.

$$g(t) = g(-t) \quad \dots \text{Even } f^n$$
$$g(t) = -g(-t) \quad \dots \text{odd } f^n$$



→ Even f^n



→ odd f^n

→ Any function can be written as the sum of even and odd f^ns.

$$g(t) = g_e(t) + g_o(t)$$

Every function is composed of an even part plus an odd part.

$$g_e(t) = \frac{g(t) + g(-t)}{2}$$

$$g_o(t) = \frac{g(t) - g(-t)}{2}$$

Eg. $g(t) = 4\cos(3\pi t)$

$g_e(t) = 4\cos(3\pi t)$

$g_o(t) = 0$.

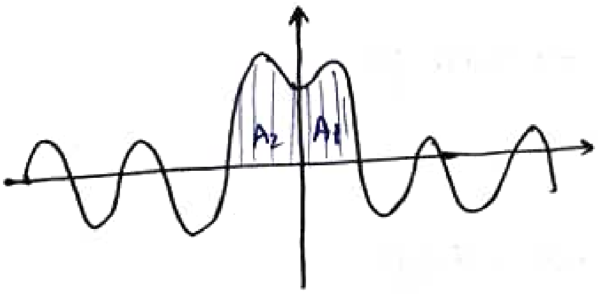
→ The sum of 2 even f^ns → Even

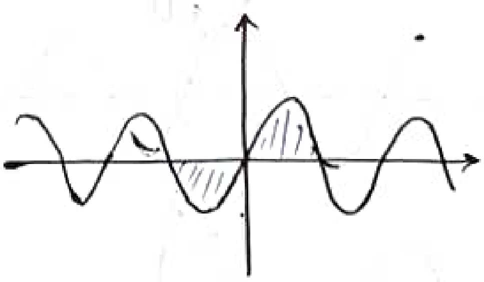
The sum of 2 odd f^ns → odd

The product of 2 odd f^ns → Even

	Sum	Difference	Product	Quotient
Both even	Even	even	Even	Even
Both odd	odd	odd	Even	Even
One even One odd	Neither	Neither	odd	odd

	Derivative	Integral
Even	odd	Odd + K
odd	Even	Even + K





Area = 0

$2(A_i)$

$$\int_{-a}^{a} g(t)\,dt = \int_{0}^{a} g(t)\,dt$$

Periodic function : The periodic function is one that has been repeating an exact pattern for an infinite time.

Nothing is truly periodic ; they have a starting and an ending point.

$$g(t) = g(t + nT), \text{ for any integral value of } n.$$

T is ~~the~~ a period of the function.

Sum of two periodic signals/functions :

Let $x_1(t)$ be a periodic signal with T_{01} as period.

$$x_2(t) \longrightarrow T_{02}$$

$$x(t) = x_1(t) + x_2(t)$$

If a time T_0 can be found is an integral multiple of T_{01} and T_{02}, then T_0 is a period of both $x_1(t)$ and $x_2(t)$.

$$x_1(t) = x_1(t + T_0)$$
$$x_2(t) = x_2(t + T_0)$$

If $\frac{T_{01}}{T_{02}}$ = rational no., then LCM of T_{01} & $T_{02} = T_0$.

If $\frac{T_{01}}{T_{02}}$ = irrational no., then $x(t)$ is aperiodic sum.

Eg (a) $g(t) = 7\sin(400\pi t)$

↳ Periodic with period $T_0 = \frac{1}{200}$.

(b) $g(t) = 3 + t^2$

↳ Not periodic

(c) $x(t) = e^{-j60\pi t}$

↳ Periodic

(d) $g(t) = 10\sin(12\pi t) + 4\cos(18\pi t)$

$\frac{1}{6}$ $\frac{1}{9}$

$\text{LCM} = \frac{1}{3}$.

Signal Energy and Power

Signal energy for signal $x(t)$,

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Eg. $x(t) = 3\,\text{tri}\,(t/4)$.

Soln: $E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$

$\text{tri}(t) = \begin{cases} 1-|t|, & |t|<1 \\ 0, & |t|>1 \end{cases}$

$\text{tri}(t/4) = \begin{cases} 1-|t/4|, & |\frac{t}{4}|<1 \text{ or } |t|<4 \\ 0, & |\frac{t}{4}|>1 \text{ or } |t|>4 \end{cases}$

$$= 9\int_{-\infty}^{\infty} |\text{tri}(t/4)|^2 dt$$

$$= 9\int_{-4}^{4} (1-|t/4|)^2 dt$$

$$= 18\int_{0}^{4} (1-t/4)^2 dt$$

$$= 18\int_{0}^{4} \left(1+\frac{t^2}{16}-\frac{t}{2}\right) dt$$

$$= 18\left[t+\frac{t^3}{48}-\frac{t^2}{4}\right]_0^4$$

$$= \overset{3}{\cancel{18}}\left[4+\frac{\overset{8}{\cancel{64}}}{\underset{8}{\cancel{48}}}-\frac{16}{4}\right]$$

$$= 24.$$

17-01-2024

→ The signal energy for some signals can be infinite over the time interval.
For such type of signals, it is more convenient to deal with average signal power.

$$P_x = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt$$

For periodic signals,

$$P_x = \int_{t_0}^{t_0+T} |x(t)|^2 dt$$

$$= \frac{1}{T}\int_{T} |x(t)|^2 dt$$

Eg. $x(t) = A\cos(2\pi f_0 t + \theta)$.

$$P_x = \frac{1}{T}\int_{T} [A\cos(2\pi f_0 t + \theta)]^2 dt$$

$$= \frac{A^2}{T}\int_{-T/2}^{T/2} \cos^2\left(\frac{2\pi}{T_0}t + \theta\right) dt = \frac{A^2}{T}\int_{-T/2}^{T/2} \frac{1+\cos\left(\frac{4\pi}{T_0}t + 2\theta\right)}{2} dt$$

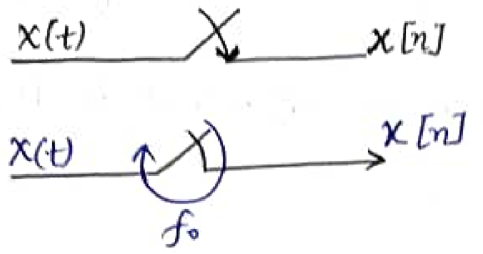
$$= \frac{A^2}{2T}\left[t + \sin\left(\frac{4\pi}{T_0}t + 2\theta\right)\right]_{-T/2}^{T/2} = \frac{A^2}{2T}\left[T + \sin(2\theta) - \sin 2\theta\right]$$

$$= \frac{A^2}{2}.$$

→ Signals that have finite signal energy are referred as energy signals.

→ Signals that have infinite signal energy but finite signal power are referred as power signals.

Discrete Time Signal



$x(t)$ ⇝ continuous time signal
$x[n]$ ⇝ Discrete time signal

↳ The signal values occuring as time passes in a discrete time (DT).
↳ Signals are countable and in a continuous time signal, they are un-countable.

$$g[n] = g(nT_s)$$

n: integer value

$$T_s = \frac{1}{f_s} = \frac{2\pi}{\omega_s} \quad \text{(time b/w samples)}$$

→ To decide the no. of samples, it depends on factors like the frequency of the signal.

→ $g[n] = A e^{\beta n}$

$$= A z^n, \quad z = e^{\beta}$$

$$g[n] = A \cos\left(\frac{2\pi n}{N_0} + \theta\right)$$

$$= A \cos(2\pi F_0 n + \theta)$$

$$= A \cos(\Omega_0 n + \theta)$$

z and β : complex constants
θ : Real phase shift in radians
N_0 : Real no.
F_0, Ω_0 : Related to N_0

$n \neq$ time

$$\frac{1}{N_0} = F_0 = \frac{\Omega_0}{2\pi}$$

Discrete Time Functions

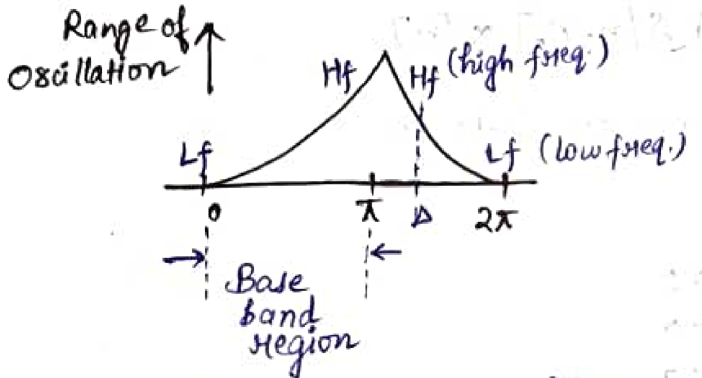
Discrete periodic: $e^{j\Omega_0 n}$, n : discrete no. (integer) $\neq$ time
↳ index of sampling

- $e^{j(\Omega_0+2\pi)n} = e^{j\Omega_0 n}$
- $e^{j2\Omega n} = 1$

Possible values of Ω_0 : 0 to 2π

$$0 \leq \Omega_0 \leq 2\pi$$

$$-\pi \leq \Omega_0 \leq +\pi$$



$$e^{j(\pi+\Delta)n} = e^{j\pi n} e^{j\Delta n}$$
$$= \pm e^{j\Delta n}$$

Periodicity: $x[n] = x[n+N]$, where N : integer

$$e^{j\Omega_0 n} = e^{j(n+N)\Omega_0}$$

$$\Rightarrow e^{j\Omega_0 n} = e^{j\Omega_0 n} e^{jN\Omega_0}$$

$$\Rightarrow N\Omega_0 = 2\pi$$

$$\Rightarrow \Omega_0 = \frac{2\pi}{N} m$$

$$\Rightarrow \Omega_0 = \frac{m}{N} 2\pi, \quad \frac{m}{N} : \text{rational}.$$

Eg. $\cos\left(\frac{n}{6}\right)$ ~~cos(2πn/12)~~ ⇝ Non-periodic

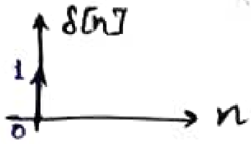
$\cos\left(\frac{2\pi n}{12}\right)$ ⇝ Period = 12

$\cos\left(\frac{5\pi n}{31}\right)$ ⇝ Period = 62 $\neq 62/5$

Singularity Function

(Kronecker Delta function)

$$\delta[n] = \begin{cases} 1, & n=0 \\ 0, & n\neq 0 \end{cases}$$



$$A\,x[n_0] = \sum_{n=-\infty}^{\infty} A\,\delta[n-n_0]\,x[n]$$

At $n=n_0$ → Kronecker delta fn

Signum function:

$$sgn\,[n] = \begin{cases} 1, & n>0 \\ 0, & n=0 \\ -1, & n<0 \end{cases}$$

Unit Ramp Function:

$$Ramp[n] = \begin{cases} n, & n\geq 0 \\ 0, & n<0 \end{cases}$$

$$Ramp[n] = n\,u[n]$$

Rectangular Function:

$$rect_{N_\omega}[n] = \begin{cases} 1, & |n|\leq N_\omega \\ 0, & |n|> N_\omega \end{cases}$$

→ Scaling/Shifting properties are application in discrete time functions.

→ Scaling involves decimation effect.

23-01-2023

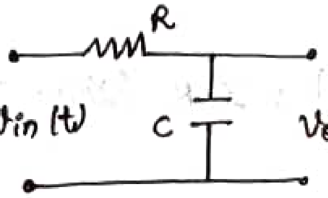
SYSTEMS

$x(t) \rightarrow y(t)$

Zero state response or **Zero input response:**

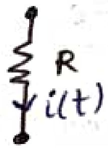
- Zero state response: Response (output) of the system when initial conditions are zero.
- Zero input response: Output of the system when there is no excitation (input).

Eg.

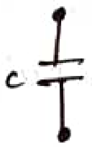


→ Low Pass Filter

$$RC\frac{dV_{out}(t)}{dt} + V_{out}(t) = V_{in}(t)$$



$v(t) = Ri(t)$



$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)\, d\tau \qquad i(t) = \frac{v(t)}{R}$$

$$i(t) = C\frac{dv}{dt}$$

Solution: $V_{out,h}(t) = K_h e^{-t/RC}$

↑ unknown

The particular solution depends on the form of $V_{in}(t)$.

Let $V_{in}(t)$ is a constant.

$$V_{out,p}(t) = K_p = A$$

$$V_{out}(t) = V_{out,h}(t) + V_{out,p}(t)$$

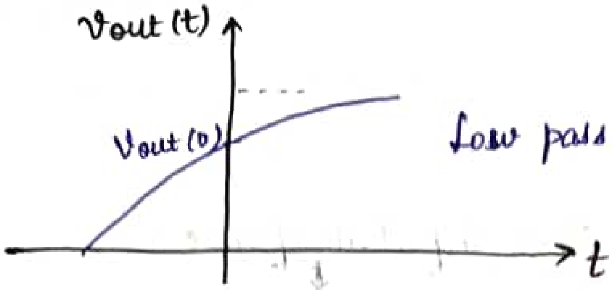
$$= K_h e^{-t/RC} + A$$

At $t=0$,

$$V_{out}(0) = K_h + A$$

$$\Rightarrow \boxed{K_h = V_{out}(0) - A}$$

$$\therefore \; V_{out}(t) = V_{out}(0)\, e^{-t/RC} + A\,(1 - e^{-t/RC})$$



low pass filter

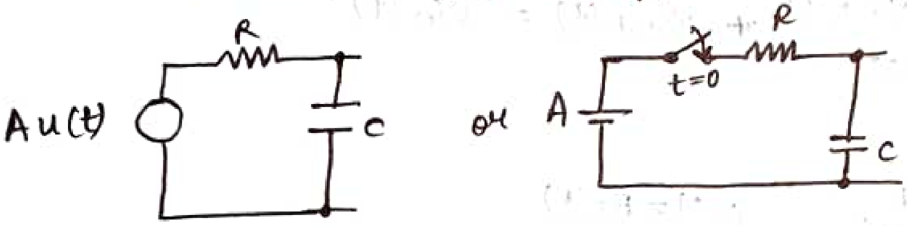
If $A = 0$,

$$V_{out}(t) = V_{out}(0)\, e^{-t/RC} \; : \; \text{ZIR (Zero Input Response)}$$
$$t > 0$$

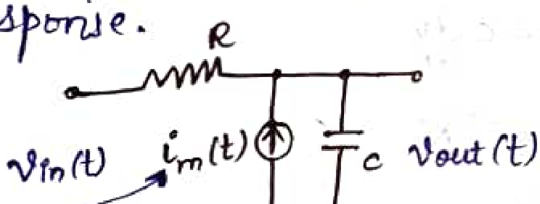
Let $V_{out}(0) = 0$,

$$V_{out}(t) = A\,(1 - e^{-t/RC}) \; : \; \text{ZSR (Zero State Response)}$$
$$t > 0$$

$$\therefore \; V_{out}(t) = ZIR + ZSR$$



If we assume that the circuit has been connected with the excitation b/w the i/p terminals for an infinite time since $t = -\infty$, the initial capacitor voltage at time $t = 0$ would be zero state response.



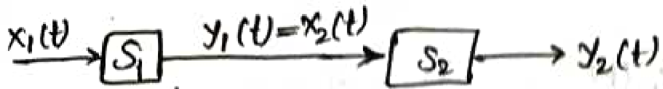
given for a very small time

$$\Delta V_{out} = \frac{1}{C} \int_{0^-}^{t} i_m(t)\, dt$$

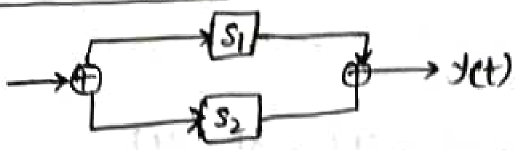
$$= \frac{1}{C} \int_{0^-}^{0^+} Q\, \delta(t)\, dt$$

$$\Delta V_{out} = Q/C$$

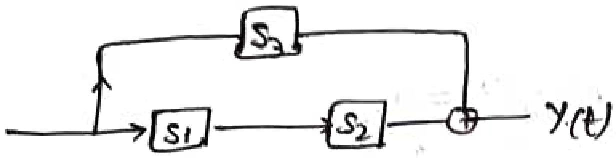
Series connection (cascading):



Parallel connection:

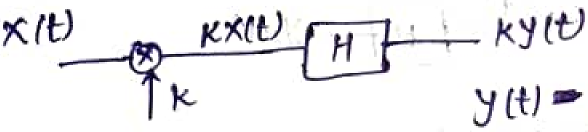


Combination of series & Parallel:



Homogeneity:

In a homogeneous system, multiplying the i/p signal by any constant (complex values) multiplies the zero-state response by the same constant:



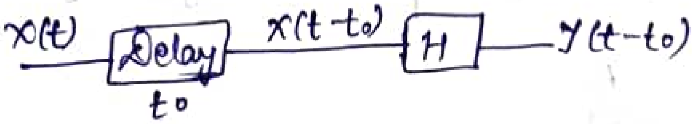
$y(t) = 1 = x(t)$ ⇝ not homogeneous

Replace $x_{new}(t) = x(t) + 1$

$y(t) = x(t)$ ⇝ homogeneous

Time Invariance System:

Delaying the excitation delays the response by the same amount without changing the functional form of the response.



24-01-2024

Additivity

$$v_{in}(t) = v_{1in}(t) + v_{2in}(t)$$

If $v_{2in}(t) = 0$

RC circuit: $$RC\frac{dv_{out}(t)}{dt} + v_{out}(t) = v_{1in}(t)$$

ZSR: $v_{out}(0) = 0$

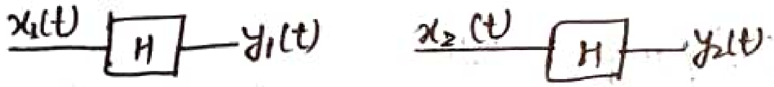
If $v_{2in}(t)$ acts alone, $v_{1in}(t) = 0$,

$$RC\frac{dv_{out}(t)}{dt} + v_{out}(t) = v_{2in}(t)$$

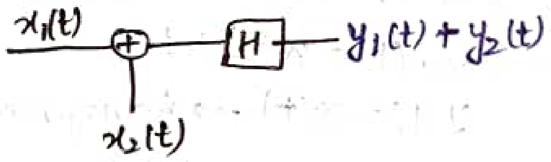
Overall response:

$$RC\left(\frac{dv_{out_1}(t)}{dt} + \frac{dv_{out_2}(t)}{dt}\right) + v_{out_1}(t) + v_{out_2}(t) = v_{1in}(t) + v_{2in}(t).$$

Eg.

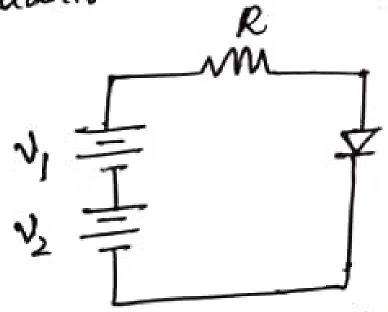


$\Rightarrow$



If a system when excited by arbitrary $x_1(t)$ produces ZSR $y_1(t)$ and when excited by $x_2(t)$ produces ZSR $y_2(t)$, the excitation $x_1(t) + x_2(t)$ produces $y_1(t) + y_2(t)$, the system is additive.

Eg.



doesn't permit -ve V

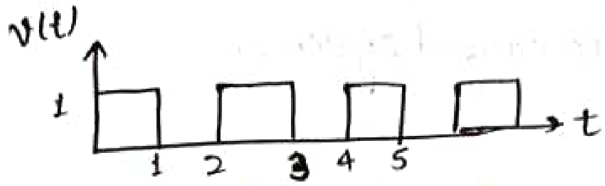
$v_1 > 0$
$v_2 = -v_1$

$\Rightarrow$ Not-additive

Linearity and Superposition

$$\alpha x_1(t) + \beta x_2(t) \xrightarrow{ZSR} \alpha y_1(t) + \beta y_2(t)$$

→ Linear + TI System ⟶ LTI (Linear Time Invariant) System



If this input is fed to the RC circuit, what is the response?

RC circuit: $\frac{dv_o}{dt} + \frac{v_o}{RC} = \frac{v_{in}}{RC}$

$$\Rightarrow v_o e^{\frac{t}{RC}} = \int e^{\frac{t}{RC}} \frac{v_{in}}{RC} dt$$

$$= \frac{1}{RC}\left[\int_0^1 e^{\frac{t}{RC}} dt + \int_2^3 e^{\frac{t}{RC}} dt + \int_4^5 e^{\frac{t}{RC}} dt + \dots \right]$$

$$= \frac{1}{(RC)^2}\left[\left(e^{t/RC}\right)\Big|_0^1 + e^{t/RC}\Big|_2^3 + e^{t/RC}\Big|_4^5 + \dots \right]$$

$$= \frac{1}{(RC)^2}\left[\left(e^{1/RC} - e^0\right) + \left(e^{3/RC} - e^{2/RC}\right) + \left(e^{5/RC} - e^{4/RC}\right) + \dots \right]$$

Stability: BIBO → Bounded Input lead Bounded Output.

$$K_1 e^{s_1 t} + K_2 e^{s_2 t} + \dots + K_N e^{s_n t}$$

K_i : constants

$s = \sigma + j\omega$

$e^{st} = e^{\sigma t} e^{j\omega t}$

$\sigma > 0$: signal amplifies
$\sigma < 0$: signal decays
$\sigma = 0$: periodic signal

Causality

We observe that each system responds during or after the time it is excited: causal System.

Any system for which ZSR occurs during or after the time in which it is excited.

~~Mem~~

Memory

Systems with memory : dynamical system
Systems without memory : static system.
↳ Eg. voltage divider, simple R-circuit.

→ All static systems are causal systems.

Invertibility

A system is said to be invertible if unique excitation produces unique ZSR.

$$x \longrightarrow \boxed{H} \xrightarrow{\;y\;} \boxed{H^{-1}} \longrightarrow x$$

Eg $y(t) = \sin(x(t))$ ⇝ not invertible b/c of many-to-one mapping.

$$a_K y^{(K)}(t) + a_{K-1} y^{(K-1)}(t) + \dots + a_1 y'(t) + a_0 y(t) = x(t)$$

↳ Invertible

↳ $y(t)$: known

Eg. $y(n) = \sum_{K=-\infty}^{\infty} x(K)$ → Non-causal

$$\left.\begin{aligned} y(n) &= \sum_{K=-\infty}^{n} x(K) \\ y(n) &= x(n-1) \\ y(t) &= \frac{1}{C}\int_{-\infty}^{t} x(\tau)\,d\tau \end{aligned}\right\} \text{causal}$$

$y(n) = x(n+1)$ → Non-causal

25-01-2024

Eg: $y(t) = \int_{-\infty}^{3t} x(\tau)\,d\tau$ → linear

For i/p $x_1(t)+x_2(t)$,

$$y(t) = \int_{-\infty}^{3t} \left(x_1(\tau)+x_2(\tau)\right)d\tau = \int_{-\infty}^{3t} x_1(\tau)\,d\tau + \int_{-\infty}^{3t} x_2(\tau)\,d\tau = y_1(t)+y_2(t)$$

For i/p $Kx(t)$,

$$y(t) = \int_{-\infty}^{3t} K\,x(\tau)\,d\tau = K\int_{-\infty}^{3t} x(\tau)\,d\tau = K\,y(t)$$

↳ Memory

↳ Non-Causal

↳ Time Invariance (Delayed i/p delays the o/p).

$$y(t+t_0) = \int_{-\infty}^{3t} x(\tau+t_0)\,d\tau$$

$$= \int_{-\infty}^{3t+t_0} x(v)\,dv$$

$$= \int_{-\infty}^{3t} x(\tau)\,d\tau + \int_{3t}^{t_0} x(\tau)\,d\tau$$

↳ Unstable

↳ Invertible.

LTI System

$$\frac{d^N y}{dt^N} + a_1 \frac{d^{N-1} y}{dt^{N-1}} + \dots + a_{N-1} \frac{dy}{dt} + a_N y(t)$$

$$= b_{N-M} \frac{d^M x}{dt^M} + b_{N-M+1} \frac{d^{M-1} x}{dt^{M-1}} + \dots + b_{N-1} \frac{dx}{dt} + b_N x(t),$$

where a_i and b_i are constants.

or $$(D^N + a_1 D^{N-1} + \dots + a_{N-1} D + a_N) y(t) = (b_{N-M} D^M + b_{N-M+1} D^{M-1} + \dots + b_{N-1} D + b_N) x(t)$$

$(M > N)$

$Q(D)\, y(t) = P(D)\, x(t)$

Differentiator
↳ High pass filter (can pass noise)
↳ Unstable

For practical, $M \leq N$.

→ Total response, $TR = ZIR + ZSR$

$x(t) = 0 \Rightarrow Q(D) y_0(t) = 0$

$\Rightarrow y_0(t) = c e^{\lambda t}$ ⇝ Eigen function

$D y_0(t) = \frac{d}{dt} y_0(t) = c \lambda e^{\lambda t}$

$D^2 y_0(t) = \frac{d^2}{dt^2} y_0(t) = c \lambda^2 e^{\lambda t}$

$\vdots$

$D^N y_0(t) = \frac{d^{(N)}}{dt^{(N)}} y_0(t) = c \lambda^N e^{\lambda t}$

$$c(\underbrace{\lambda^N + a \lambda^{N-1} + \dots + a_{N-1} \lambda + a_N}_{\text{polynomial} = 0}) e^{\lambda t} = 0$$

$Q(\lambda) = 0$

$(\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_N) = 0$

↳ Eigenvalues of the system
↳ Characteristic modes of the system.

Solution : $c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \dots$

$e^{\lambda_i t} \rightarrow ZIR$

Repeated roots:

$$(D-\lambda)^2 y_0(t)=0$$
$$\Rightarrow y_0(t) = (c_1 + c_2 t) e^{\lambda t}$$

For r repeated λ_i,

$$(D-\lambda)^r y_0(t)=0$$
$$\Rightarrow y_0(t)=(c_1+ c_2 t + \dots + c_r t^{r-1}) e^{\lambda t}$$
$$Q(\lambda)= (\lambda-\lambda_1)^r (\lambda-\lambda_{r+1})(\lambda-\lambda_N).$$

Book
→ BP Lathi

(31-01-2023)

→ The total response, $y(t) = y_0(t)$ (ZIR) + Zero state Response.

→ At $t=0^-$, the total response, $y(t)$ consists of ZI component, $y_0(t)$.

- when I/p signal is not given.

$$y(0^-) = y_0(0^-)$$
$$\dot{y}(0^-) = \dot{y}_0(0^-)$$

- $y_0(t)$ only depends on the initial conditions.
- when we give i/p $x(t)$ at $t=0$

$y_0(t)$ at $t=0^+$

$$y_0(0^+) = y_0(0^-)$$
$$\dot{y}_0(0^+) = \dot{y}_0(0^-)$$

- $y_0(t)$ will be identical at $t=0$, $t=0^-$, $t=0^+$.

Total response:

$$y(0^-) \neq y(0^+)$$
$$\dot{y}(0^-) \neq \dot{y}(0^+)$$

Unit Impulse Response

$$Q(D)\, y(t) = P(D)\, x(t)$$
$$M \le N$$

$$(D^N + a_1 D^{N-1} + \dots + a_{N-1} D + a_N)\, y(t) = (b_0 D^M + b_1 D^{M-1} + \dots + b_{M-1} D + b_N)\, x(t)$$

$\delta(t)$: Impulse at $t=0$.

All initial conditions are zero at $t=0^-$.

Impulse Response of a system: $h(t)$

↳ ZSR

→ The IR, $h(t)$, must consist of the system characteristic modes for $t \geq 0^+$.

$h(t)$ = characteristic modes at $t \geq 0^+$.

At $t=0$, the i/p is an impulse.

$h(t) = a_0 \delta(t)$ + characteristic mode terms at $t \geq 0$.

$y(t) = h(t)$

$x(t) = \delta(t)$

$h(t) = b_0 \delta(t)$ ~~oth~~ + other char. mode terms at $t \geq 0$.

for $M < N$, $b_0 = 0$

Eg. $(D^2 + 5D + 6)\, y(t) = (D+1)\, x(t)$ | $D = \frac{d}{dt}$, $D^2 = \frac{d^2}{dt^2}$

$\Rightarrow b_0 = 0$ [$\because$ No D^2 term in RHS]

$h(t)$ consists only of other char. modes.

$\lambda^2 + 5\lambda + 6 = 0$

$\Rightarrow \lambda = -2, -3$

$\therefore h(t) = (c_1 e^{-2t} + c_2 e^{-3t})\, u(t)$

$\therefore \ddot{h}(t) + 5\dot{h}(t) + 6h(t) = \dot{\delta}(t) + \delta(t)$

$h(0^-) = \dot{h}(0^-) = 0$.

At $t=0$, impulse response is given.

↳ Create new initial condition.

$h(0^+) = K_1$

$\dot{h}(0^+) = K_2$

$\dot{h}_0(0) = K_1 \delta(t)$

$\ddot{h}(0) = K_1 \dot{\delta}(t) + K_2 \delta(t)$.

$\ddot{h}(0) = K_1\dot{\delta}(t) + K_2\,\delta(t)$

As $K_1 = 1$ and $5K_1 + K_2 = 1$

$\Rightarrow K_2 = -4$

$h(0^+) = 1 = K_1$

$\dot{h}(0^+) = K_2 = -4$

As $h(t) = (c_1 e^{-2t} + c_2 e^{-3t})u(t)$,

at $t = 0^+$, $c_1 + c_2 = 1$ — (I)

$\dot{h}(t)$ at $t = 0^+$,

$-2c_1 - 3c_2 = -4$ — (II)

$\Rightarrow c_1 = -1$

$c_2 = 2$

$\therefore h(t) = (-e^{-2t} + 2e^{-3t})u(t)$.

Impulse Matching Method:

$h(t) = b_0\,\delta(t) + [P(D)\,y_n(t)]\,u(t)$

$y_n(t)$: linear combination of characteristic modes.

$y_n(0) = 0 = \dot{y}_n(0) = \ddot{y}_n(0) = \ldots = y_n^{N-2} = 0$

$y_n^{N-1}(0) = 1$,

where y_n^K: K^{th} derivative of $y_n(t)$ at $t = 0$.

N: order of the system

$N = 1$: $y_n(0) = 1$

$N = 2$: $\dot{y}_n(0) = 1$.

Eg. $(D^2 + 3D + 2)\,y(t) = D\,x(t)$

$\hookrightarrow N = 2$

$\lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2)$

$\Rightarrow \lambda = -1, -2$

$y_n(t) = c_1 e^{-t} + c_2 e^{-2t}$

$\dot{y}_n(t) = -c_1 e^{-t} - 2c_2 e^{-2(t)}$

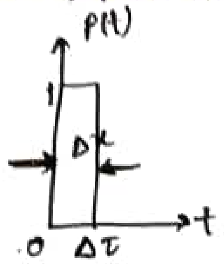
$\dot{y}_n(0) = 1$

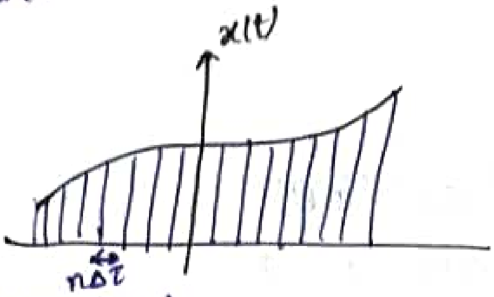
$y_n(0) = 0$.

$\left.\begin{array}{l} 0 = c_1 + c_2 \\ 1 = -c_1 + 2c_2 \end{array}\right\} \Rightarrow \begin{array}{l} c_1 = 1 \\ c_2 = -1 \end{array}$ $\quad \therefore \boxed{y_n(t) = e^{-t} - e^{-2t}}$.

ZSR of an LTI System

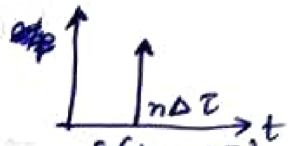
The ZSR of an LTI system to an i/p x(t) when the system is in zero state condition:





$$x(n\Delta\tau)\, p(t-n\Delta\tau)$$

$$x(t) = \lim_{\Delta\tau \to 0} \sum \frac{x(n\Delta\tau)}{\Delta\tau} p(t-n\Delta\tau)\Delta\tau$$



$\delta(t-n\Delta\tau)$

$$\underline{\text{I/P}} \qquad \underline{\text{O/P}}$$
$$\delta(t) \longrightarrow h(t)$$
$$\delta(t-n\Delta\tau) \longrightarrow h(t-n\Delta\tau)$$

$$\lim_{\Delta\tau\to 0}\sum_{\tau} (x(n\Delta\tau)\,\Delta\tau)\,\delta(t-n\Delta\tau) \Rightarrow \lim_{\Delta\tau\to 0}\sum_{\tau} x(n\Delta\tau)\,\Delta\tau\, h(t-n\Delta\tau)$$

$$\Rightarrow x(t) = y(t)$$

$$y(t) = \underbrace{\lim_{\Delta\tau\to 0}\sum_{\tau} x(n\Delta\tau)\, h(t-n\Delta\tau)\,\Delta\tau}_{x(t)}$$

$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \quad \text{... Convolution}$$

| x: function
| h: filter

Eg. $h(t) = e^{-2t} u(t)$
$x(t) = e^{-t} u(t)$.

$$y(t) = \int_{-\infty}^{\infty} e^{-\tau} u(\tau)\, e^{-2(t-\tau)}\, u(\tau)\, d\tau$$

$$= e^{-2t}\int_{0}^{\infty} e^{-3\tau}\, d\tau$$

$$= e^{-2t}\left[\frac{e^{-3\tau}}{-3}\right]_0^{\infty}$$

$$= e^{-2t}\left(\frac{0-1}{-3}\right)$$

$$= \frac{e^{-2t}}{3}.$$

01-02-2024

→ For linear time-varying system,

$$y(t) = \int x(\tau)\, h(t, \tau)\, d\tau$$

$h(t, \tau)$ ↳ The response at instant 't' to a unit impulse i/p located at τ.

→ $$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$: Convolution Integral

Properties of Convolution:

① Commutative Property:

$$x_1(t) * x_2(t) = x_2(t) * x_1(t)$$

$$x_2(t) * x_1(t) = \int_{-\infty}^{\infty} x_2(\tau)\, x_1(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

② Distributive Property:

$$x_1(t) * [x_2(t) + x_3(t)] = x_1(t) * x_2(t) + x_1(t) * x_3(t)$$

③ Associative Property:

$$x_1(t) * [x_2(t) * x_3(t)] = [x_1(t) * x_2(t)] * x_3(t)$$

④ Shift Property:

$$x_1(t) * x_2(t) = c(t)$$

$$\Rightarrow x_1(t) * x_2(t-T_1) = c(t-T_1) = x_1(t-T_1) * x_2(t)$$

and, $$x_1(t-T_1) * x_2(t-T_2) = c(t-T_1-T_2)$$

Convolution with Impulse:

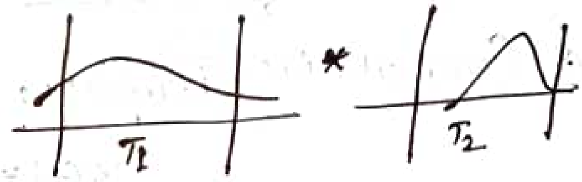
↳ Results into the same function $x(t)$.

$$x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

$$= x(t)$$

⑤ Width Property: If duration width of $x_1(t)$ and $x_2(t)$ are finite given by T_1 and T_2.

$$x_1(t) * x_2(t) = T_1 + T_2$$



06-02-2024

ZSR and Causality

LTI:
$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$
$$= x(t) * h(t)$$

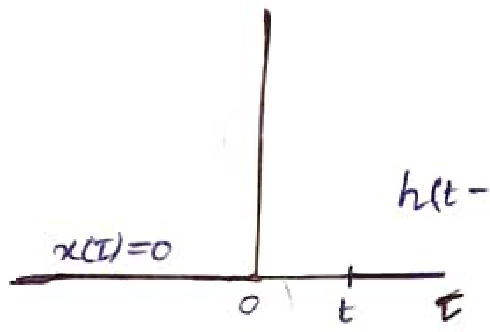
Chaotic system:
↳ Any perturbation in i/p can bring a large change in o/p

→ Most systems are causal so their response cannot begin before the input is given.

→ Any system for which ZSR occurs, only during or after the time in which it is excited are called causal system.

→ Response to a unit impulse $\delta(t)$ which is located at $t=0$ cannot begin before $t=0$.

$x(\tau) = 0, \quad \tau < 0$ (input to be causal)



For $h(t)$ to be causal,

$h(t-\tau) = 0$ for $t - \tau < 0$
$\Rightarrow \tau > t.$

→ $x(t) * h(t-\tau) = 0$, for $\tau < 0$ and $\tau > t$.

→ $y(t) = x(t) * h(t)$
$$= \int_{0^-}^{t} x(\tau)\, h(t-\tau)\, d\tau, \quad t \geq 0$$
$$= 0, \quad t < 0$$

→ $h(t)$ is real system.

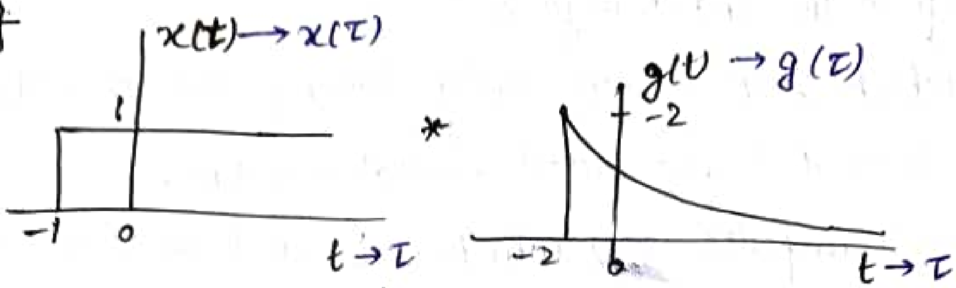
$x(t) = x_r(t) + jx_i(t)$

$y(t) = y_r(t) + jy_i(t)$

$x_r(t) \longrightarrow y_r(t)$

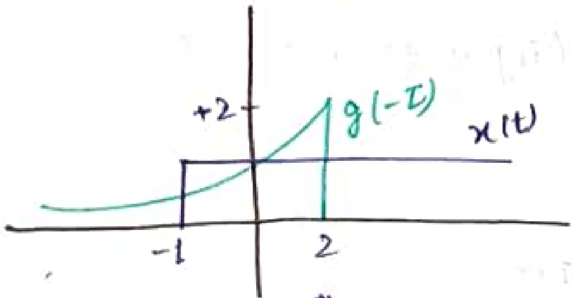
$x_i(t) \longrightarrow y_i(t)$

Eg.

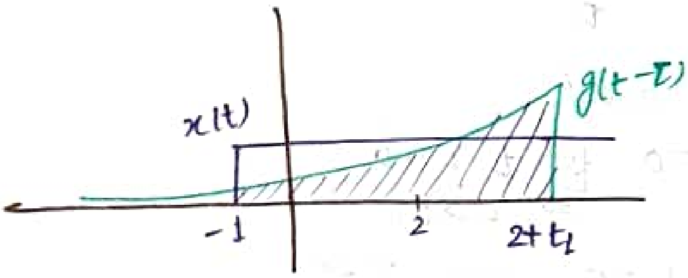


$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

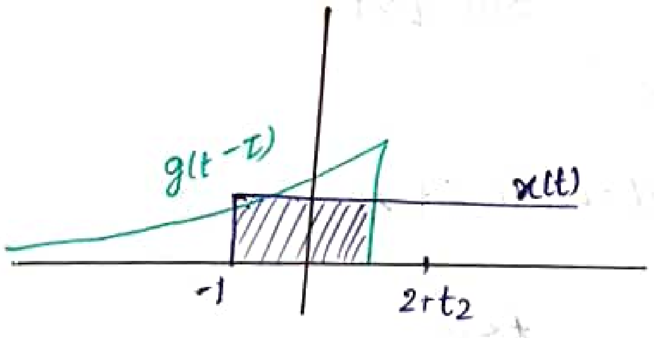
⇓



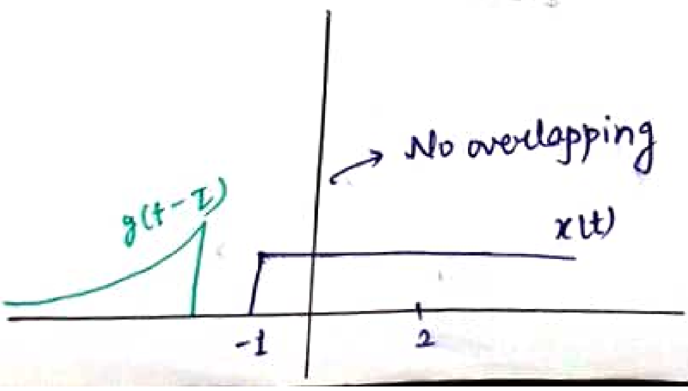
⇓



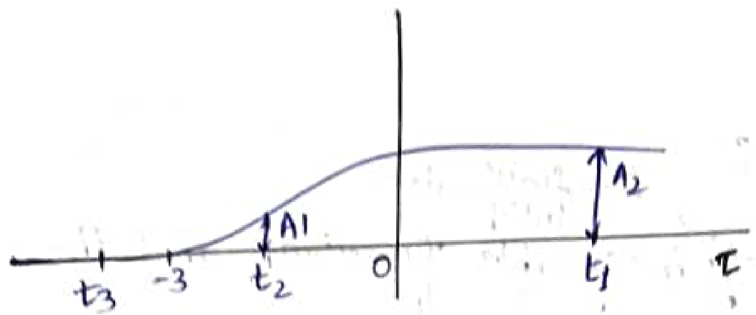
$g(t-\tau)$, $t = t_1 > 0$



$g(t-\tau)$, $t = t_2 < 0$



$g(t-\tau)$, $t = t_3 < -3$



→ Tells the influence of one signal over the response of the other.

Derivative Property:

$y(t) = x(t) * h(t)$

$$\frac{dy(t)}{dt} = \left(\frac{d}{dt}x(t)\right) * h(t) = x(t) * \frac{dh(t)}{dt}$$

07-02-2024

Area Property:

Area of $y(t)$ = area of $x(t)$ × area of $h(t)$

Scaling Property:

$$y(at) = |a|\, x(at) * h(at)$$

→ $$y(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)\,d\tau$$

Let $x(t)$ be bounded.

$$|x(t-\tau)| < B \quad \forall \tau$$
(finite value)

$$|y(t)| = \left|\int_{-\infty}^{\infty} x(t-\tau)h(\tau)\,d\tau\right|$$

The magnitude of an integral of a function is less than or equal to the integral of the magnitude of the function, i.e.,

$$\left|\int_{\alpha}^{\beta} g(x)\,dx\right| \leq \int_{\alpha}^{\beta} |g(x)|\,dx$$

and the magnitude of a product of two functions is equal to the product of their magnitudes, i.e.,

$$|g(x)h(x)| = |g(x)||h(x)|$$

$$\Rightarrow |y(t)| \le \int_{-\infty}^{\infty} B\,|h(\tau)|\,d\tau$$

$$\Rightarrow |y(t)| \le B \int_{-\infty}^{\infty} |h(\tau)|\,d\tau$$

has finite integration

$h(\tau)$ is absolutely integrable function, and the convolution is commutative.

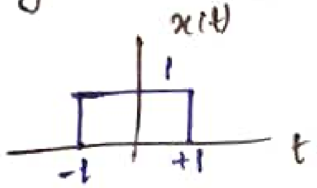
We can say that if $h(t)$ is bounded. The condition for convergence is that $x(t)$ be absolutely integrable.

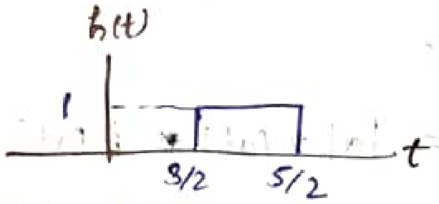
For convolution, integral to be converging, the signal being convolved must both be bounded and atleast one of them must be absolutely integrable.

Eg. $x(t) = \text{rect}(t/2)$

$h(t) = \text{rect}(t-2)$

$y(t) = x(t) * h(t)$





$$y'(t) = x'(t) * h(t)$$

$$y''(t) = x'(t) * h'(t)$$

$$\delta(t+1) - \delta(t-1) = x'(t)$$

$$h'(t) = \delta(t-3/2) - \delta(t-3/2)$$

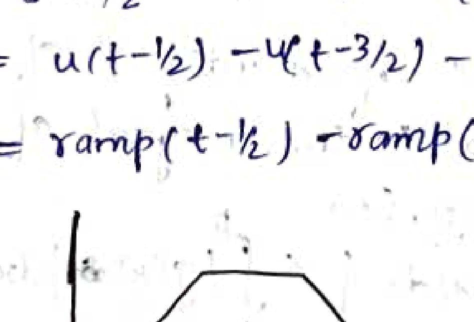
$\underline{\delta(t+1) * \delta(t-3/2)} = \delta(t+1-3/2)$

$$y''(t) = [\delta(t+1) - \delta(t-1)] * [\delta(t-3/2) - \delta(t-5/2)]$$

$$= \delta(t-1/2) - \delta(t-3/2) - \delta(t-5/2) + \delta(t-7/2)$$

$$\Rightarrow y'(t) = u(t-1/2) - u(t-3/2) - u(t-5/2) + u(t-7/2)$$

$$\Rightarrow y(t) = \text{ramp}(t-1/2) - \text{ramp}(t-3/2) - \text{ramp}(t-5/2) + \text{ramp}(t-7/2)$$



$\rightarrow$ $x(t) \longrightarrow \boxed{h(t)} \xrightarrow{x(t) * h_1(t)} \boxed{h(t)} \longrightarrow y(t) = [x(t) * h_1(t)] * h_2(t)$

$$\Downarrow$$

$x(t) \longrightarrow \boxed{h_1(t) * h_2(t)} \longrightarrow y(t)$

$\rightarrow$ $x(t) \longrightarrow \boxed{h_1(t)}$, $\boxed{h_2(t)}$ (in parallel) $\longrightarrow \oplus \longrightarrow y(t) = x(t) * h_1(t) + x(t) * h_2(t)$

$$\Downarrow$$

$x(t) \longrightarrow \boxed{h_1(t) + h_2(t)} \longrightarrow y(t).$

Convolution with e^{st}: (s : complex exponential)

$$y(t) = x(t) * h(t)$$

$$y(t) = e^{st} H(s) \qquad \Big| \quad H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau$$

$$= e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$$

08-02-2024

e^{st}, s: complex variable.

$e^{st} \longrightarrow \boxed{LTI} \longrightarrow y(t)$

ZSR → also an exponential with a multiplicative constant.

- NO OTHER function can make such claim.
- An input for which the system response is also of the same form: characteristic function / eigen function of the function.

$$y(t) = h(t) * e^{st}$$

$$= \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)} d\tau$$

$$= e^{st} \int_{\infty}^{\infty} h(\tau)\, e^{-s\tau} d\tau$$

$$\Rightarrow y(t) = H(s)\, e^{st} \quad \ldots (A)$$

where $H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$

(A) is valid only for valid of s for which H(s) exists.

H(s) is a constant, given s.

H(s): Transfer function of the system.

$$H(s) = \frac{\text{O/p Signal}}{\text{I/p signal}}$$

Input $= e^{st}$

TF is only meaningful/exist for LTI system, NOT for time varying system/non-linear system.

$$Q(D)\, y(t) = P(D)\, x(t)$$

$$H(s) = \frac{P(s)}{Q(s)} \quad \text{when } x(t) = e^{st}$$

$$H(s)\, (Q(D)\, e^{st})\, y(t) = H(s) e^{st}$$

$$= P(D) e^{st}$$

$D^r e^{st} = s^r e^{st}$

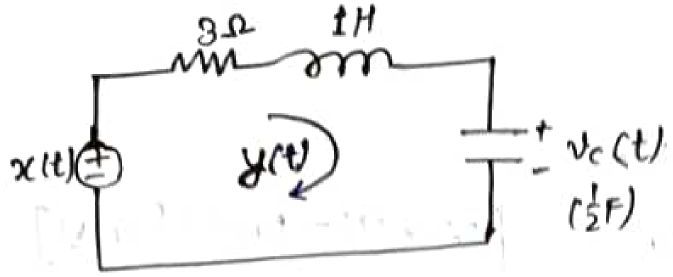
$$P(D) e^{st} = P(s)\, e^{st}$$

$$Q(D) e^{st} = Q(s)\, e^{st}$$

$$H(s) = \frac{P(s)}{Q(s)}$$

$\rightarrow$ $\boxed{y(t) = H(s) e^{st}}$: Fundamental property of LTI system.

Eg.



$x(t) = 10e^{-3t}u(t)$.

Find loop current $y(t)$ for $t \geq 0$.

The initial inductor current is zero, $y(0^-) = 0$.

The initial capacitor voltage is 5V, $V_c(0^-) = 5$.

Solⁿ: $v_L(t) + v_R(t) + v_c(t) = x(t)$

$$\Rightarrow L\frac{dy(t)}{dt} + 3y(t) + 2\int_{-\infty}^{t} y(\tau)d\tau = x(t)$$

$$\Rightarrow L\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt}.$$

$$\Rightarrow (D^2 + 3D + 2)\,y(t) = D\,x(t)$$

ZIR: we find two initial conditions: $y_0(0)$ & $\dot{y}_0(0)$.

$y_0(0^-) = 0$

$V_c(0^-) = 5V$

$x(t) = 0$

At $t=0$, $x(t) = 0$, $(\because$ Input $= 0)$

$V_c(0) = 5V$,

$y_0(0) = 0$.

$\dot{y}_0(0) + 3y_0(0) + V_c(0) = 0$

$$\Rightarrow \boxed{\dot{y}_0(0) = -5V}$$

$(D^2 + 3D + 2)y(t) = D\,x(t)$

$y(0) = 0$

$\dot{y}_0(0) = -5$.

$\lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2)$

$y_0(t) = c_1 e^{-t} + c_2 e^{-2t}$

$\dot{y}_0(t) = -c_1 e^{-t} - 2c_2 e^{-2t}$

$c_1 = -5$, $c_2 = 5$.

$\therefore y_0(t) = -5e^{-t} + 5e^{-2t}$.

The initial condition at

$t=0^-$, i/p, $x(t)=0$

$t=0^+$, $x(t)=10$ $\quad [\because x(t)=10e^{-3t}u(t)]$

$\dot{y}(0^-)+3y(0^-)+v_c(0^-)=0$

$\dot{y}(0^+)+3y(0^+)+v_c(0^+)=10$

$y(0^-)=0$

$\dot{y}(0^-)=-5$

$y(0^+)=0$

$\dot{y}(0^+)=5$

ZSR

$(D^2+3D+2)\,y(t)=Dx(t)$

$h(t)=(2e^{-2t}-e^{-t})u(t)$

$x(t)=10e^{-3t}u(t)$

$y(t)=x(t)*h(t)$

Ans:

$$TR=\underbrace{(-5e^{-t}+5e^{-2t})}_{ZIR}+\underbrace{(5e^{-t}+20e^{-2t}-15e^{-3t})}_{ZSR}, \quad t\geq 0.$$

$$\boxed{e^{\lambda_1 t}u(t)*e^{\lambda_2 t}u(t)=\frac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2}}, \quad \lambda_1\neq\lambda_2$$

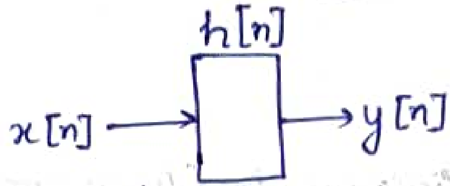
12-02-2024

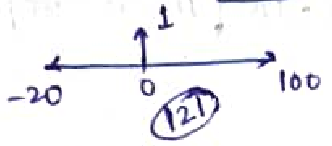
MATLAB

Impulse Response of an LTI System:

$$y[n] = x[n] + x[n-1] - 0.7\,y[n-1]$$

$$-20 \le n \le 100$$





$$y[-19] = x[-19] + x[-20] - 0.7\,y[-20]$$

$$\vdots$$

$$y[1] = x[1] + x[0] - 0.7\,y[0]$$

$$y[2] = x[2] + x[1] - 0.7\,y[1]$$

$$\vdots$$

$$y[100] = x[100] + x[99] - 0.7\,y[99].$$

13-02-2024

Discrete Time Signals

↳ $x[n]$: discrete

$\downarrow$ Sampling

$x(nT)$: Continuous

Decimation / Interpolation

↳ Deleting some of the signals/information (Down sampling in continuous time signals)

→ $x[n]$ compressed by a factor M,

$$x_d(n) = x(Mn)$$

M takes integer values only,

$n = 0, 1, 2, \ldots$

$x(Mn) = x(0), x(M), x(2M), x(3M), \ldots$

Selecting every M^{th} sample of $x(n)$.

Interpolation: Expanding/adding extra values to the signals.

↳ The interpolated signal is generated in two steps.

First, we expand $x(n)$ by an integer factor L.

$$x_e(n) = \begin{cases} x(n/L), & n = 0, \pm L, \pm 2L \\ 0, & \text{otherwise} \end{cases}$$

$L = 2$

$$\begin{cases} x_e(n) = x(n/L), & \text{for even } n. \\ x_e = 0 & \text{for odd values of } n. \end{cases}$$

→ $\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0. \end{cases}$

→ $u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0. \end{cases}$

Discrete Time Exponential Function

$$\gamma^n$$

Let $e^{\lambda t} = \gamma^t$

$e^{\lambda} = \gamma$

$\lambda = \ln \gamma$

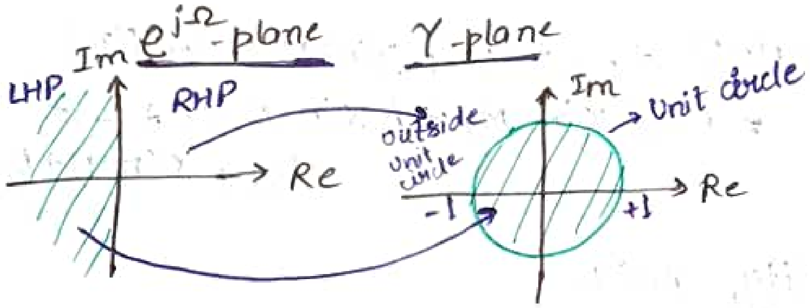
$4^t = e^{1.386t}$

$e^{1.386} = 4 \Rightarrow \ln 4 = 1.366$

$\rightarrow e^{j\Omega n} = \gamma^n$

$\Rightarrow e^{\lambda n} = \gamma^n, \quad \lambda = j\Omega$

$\Rightarrow \gamma = e^{j\Omega}$



$\rightarrow \lambda = a + jb, \; a < 0.$

The signal decays exponentially.

$\gamma = e^{\lambda} = e^{a+jb} = e^{a} \cdot e^{jb}$

$\Rightarrow |\gamma| = |e^{a}||e^{jb}| = |e^{a}| \qquad [\because |e^{jb}| = 1]$

$\rightarrow |\gamma| < 1, \; a < 0 \Rightarrow \gamma$ lies inside unit circle.

Discrete Time Sinusoids

$$C \cos(\Omega n + \theta) = C \cos(2\pi F n + \theta)$$

C: amplitude

θ: phase in radians

Ωn: angle in radians

$2\pi F n = \Omega n$

$\Rightarrow \boxed{F = \frac{\Omega}{2\pi}}$ $\frac{\text{radian}}{2\pi}$ per sample [cycles per sample]

No is the period sample cycle of the sinosoid.

$$\boxed{F = \frac{1}{N_0}} = \frac{2\pi}{\Omega}.$$

Eg. A person makes a deposit (i/p) at the end of month in a bank at an interval T (every month) and the bank pays interest. Account balance = o/p.

$x(n)$: deposit made at n^{th} instant.

$y(n)$: balance at n^{th} time.

γ : ~~instant~~ interest rate per rupees per time T.

$$y(n) = y(n-1) + \gamma(n-1) + x(n).$$

$$\Rightarrow y(n) = (1+\gamma)\,\gamma(n-1) + x(n)$$

$$\Rightarrow \gamma(n) - a\gamma(n-1) = x(n)$$

$$\# \boxed{\begin{aligned} y(n+N) + a_1 y(n+N-1) + \dots + a_{N-1}\, y(n+1) + a_N\, y(n) \\ = b_{N-M}\, x(n+M) + b\, x(n+M-1) + \dots + \\ b_{N-1}\, x(n+1) + b_N\, x(n) \end{aligned}}$$

$a_0 = 0$.

For causality, $M \le N$.

14-02-2024

DTS General Equation:

$$y(n+N) + a_1 y(n+N-1) + \dots + a_{N-1} y(n+1) + a_N y(n)$$
$$= b_{N-M} x(n+M) + b_{N-M+1} x(n+M-1) + \dots + b_{N-1} x(n+1) + b_N x(n)$$

Order: $\max(M,N)$

$M \le N$: Causality condition

$a_0 = 1$

$M = N$

$n \to n-N$

Recursive Iterative Solution:

$$y(n) = -a_1 y(n-1) - a_2 y(n-2) - \dots + a_N y(n-N) + b_0 x(n)$$
$$+ b_1 x(n-1) + \dots + b_N x(n-N)$$

Eg. $y(n) - 0.5 y(n-1) = x(n)$

$y(-1) = 16$, $x(n) = n^2$

$\therefore y(n) = 0.5 y(n-1) + x(n)$

$\Rightarrow$ $n=0$: $y(0) = 0.5 y(-1) + x(0)$
$= 8$

$n=1$: $y(1) = 0.5 y(0) + x(1)$
$= 4+1$
$= 5$

→ E denotes advancing sequence.

$E x(n) = x(n+1)$

$E^2 x(n) = x(n+2)$

$\vdots$

$E^N x(n) = x(n+N)$

$y(n+1) - a y(n) = x(n+1)$

$\Rightarrow E y(n) - a y(n) = E x(n)$

$\Rightarrow (E-a) y(n) = E x(n)$

$\rightarrow (E^N + a_1 E^{N-1} + \dots + a_{N-1} E + a_n)\, y(n) = (b_0 E^N + b_1 E^{N-1} + \dots + b_{N-1} E + b_N)\, x(n)$

$\Rightarrow Q(E)\, y(n) = P(E)\, x(n)$

ZIR : $x(n) = 0$

$(E^N + a_1 E^{N-1} + \dots + a_{N-1} E + a_N)\, y_0(n) = 0$

i.e., $Q(E)\, y_0(n) = 0$

$y_0(n+N) + a_1 y_0(n+N-1) + \dots + a_{N-1} y_0(n+1) + a_N y_0(n) = 0$

ZIR : linear combination of $y_0(n)$ and advanced $y_0(n)$ is zero for all values of n.

↳ Only exponential function has this property.

$E^K(\gamma^n) = \gamma^{n+K}$

$= \gamma^n \gamma^K \rightsquigarrow \gamma$ advanced by K.

$Q(E)\, y_0(n) = 0$

$\Rightarrow y_0(n) = c\gamma^n$

$E^K y_0(n) = c\gamma^{n+K}$

$\rightarrow \gamma^N + a_1 \gamma^{N-1} + \dots + a_{N-1}\gamma + a_N = 0$

or $Q(\gamma) = 0$

$\Rightarrow (\gamma - \gamma_1)(\gamma - \gamma_2) \dots (\gamma - \gamma_N) = 0$

$\Rightarrow y_0(n) = c_1 \gamma_1^n + c_2 \gamma_2^n + \dots + c_n \gamma_N^n$

$\gamma_1, \gamma_2, \dots, \gamma_N$ are the roots of characteristic polynomial $Q(\gamma)$.

For p repeated roots :

$Q(\gamma) = (\gamma - \gamma_1)^p (\gamma - \gamma_{p+1}) \dots (\gamma - \gamma_N)$

$y_0(n) = (c_1 + c_2 n + \dots + c_p n^{p-1})\gamma_1^n + c_{p+1}\gamma_{p+1}^n + \dots + c_N \gamma_N^n$

20-02-2024

Unit Impulse Response

$$(E^N + a_1 E^{N-1} + \dots + a_{N-1} E + a_N) y(n) = (b_0 E^N + b_1 E^{N-1} + \dots + b_{N-1} E + b_N) x(n)$$

$$\Rightarrow Q(E) y(n) = P(E) x(n).$$

Input : $x(n) = \delta(n)$

$Q(E) y(n) = P(E) \delta(n)$, such that initial condition

$$h(-1) = h(-2) = \dots = h(-N) = 0.$$

$$\delta(n) = 0, \quad n > 0.$$

$$h(n) = A_0 \delta(n) + y_c(n) \cdot u(n),$$

$y_c(n)$: characteristic modes for $n>0$.

Then form $\gamma_j^n u(n-1)$,

$$u(n-1) = u(n) - \delta(n)$$

$$\therefore Q(E) [A_0 \delta(n) + y_c(n) u(n)] = P(E) \delta(n)$$

$\because Q(E) y_c(n) u(n) = 0$, as it is char. modes of the system.

$$\Rightarrow Q(E) A_0 \delta(n) = P(E) \delta(n)$$

$$A_0 (\delta(n+N) + a_1 \delta(n+N-1) + \dots + a_N \delta(n)) = b_0 \delta(n+N) + \dots + b_N \delta(n).$$

$n=0 \Rightarrow A_0 a_N = b_N$ | $\delta(0) = 1$

$$\Rightarrow \boxed{A_0 = \frac{b_N}{a_N}}$$

$$\therefore \boxed{h(n) = \frac{b_N}{a_N} \delta(n) + y_c(n) u(n)}$$

Zero State Response

$$x(n) = x(0)\,\delta(n) + x(1)\,\delta(n-1) + \dots + x(-1)\,\delta(n+1) + x(-2)\,\delta(n+2) + \dots$$

or, $$x(n) = \sum_{m=-\infty}^{\infty} x(m)\,\delta(n-m)\ ;$$

For linear system, knowing the system response to impulse $\delta(n)$, the system response to any arbitrary input could be obtained.

$$\underset{(i/p)}{\delta(n)} \longrightarrow \underset{(o/p)}{h(n)}$$

$$\underset{(i/p)}{x(n)} \longrightarrow \underset{(o/p)}{y(n)}$$

$$\Rightarrow \delta(n-m) \rightarrow h(n-m)$$

$$\Rightarrow \underbrace{\sum_{m=-\infty}^{+\infty} x(m)\,\delta(n-m)}_{x(n)} \rightarrow \underbrace{\sum_{m=-\infty}^{\infty} x(m)\,h(n-m)}_{y(n)}$$

$$\therefore\ y(n) = \sum_{m=-\infty}^{\infty} x(m)\,h(n-m)$$ ... Convolution in discrete-time system.

Properties of DT Convolution

① Commutative : $h_1(K) * h_2(K) = h_2(K) * h_1(K)$

② Distributive : $x(K) * [h_1(K) + h_2(K)] = x(K) * h_1(K) + x(K) * h_2(K)$

Proof: $y(K) = h_1(K) * h_2(K)$

$$= \sum_{m=-\infty}^{\infty} h_1(m)\,h_2(K-m)$$

let $n = K-m \Rightarrow m = K-n$

$m = -\infty \Rightarrow n = +\infty$

$m = +\infty \Rightarrow n = -\infty$

$$y(K) = \sum_{n=+\infty}^{-\infty} h_1(K-n)\,h_2(n) = \sum_{n=-\infty}^{\infty} h_1(K-n)\,h_2(n)$$

$$y(K) = \sum_{n=-\infty}^{+\infty} h_2(n)\,h_1(K-n) = h_2(K) * h_1(K)$$

③ Associative: $x(K) * h_1(K) * h_2(K) = x(K) * [h_1(K) * h_2(K)]$

④ Time-shifting:

If $h_1(K) * h_2(K) = c(K)$,

then $h_1(n-m) * h_2(n-m) = c(K-m-n)$.

⑤ Convolution with an impulse:

Places the signal at the impulse location.

$$h(K) * \delta(K-n) = h(K-n).$$

⑥ Width Property:

$h_1(K) \rightarrow m$ length

$h_2(K) \rightarrow n$ length

$c(K) = h_1(K) * h_2(K) \rightarrow m+n-1$.

$h_1(K) \longrightarrow \underline{\text{Length}}\ m \rightarrow [0, m-1]$

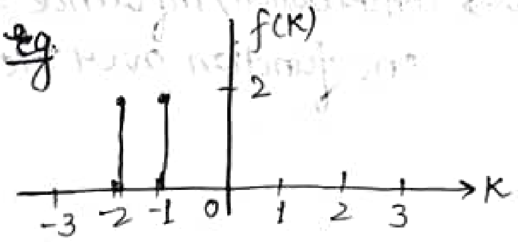
$h_2(K) \longrightarrow n \longrightarrow [0, n-1]$

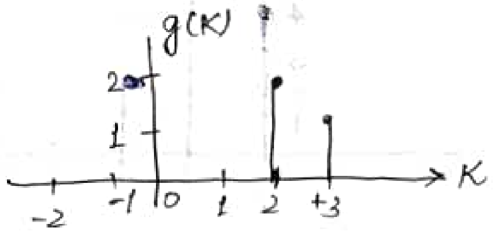
Start time $\rightarrow 0+0 \rightarrow 0$; $c(K)=0,\ K<0$

Stop time $\rightarrow (m-1)+(n-1) = m+n-2$; $c(K)=0,\ K>m+n-2$, $K<0$

$= m+n-1$ (o/p signal length)

eg





Start time $= -2+2 = 0$

Stop time $= -1+3 = 2$

$K = 0, 1, 2$.

$h(K) = 0$ for $K<0$ & $K>2$

$$h(K) = \sum_{m=-\infty}^{\infty} f(m)\, g(K-m)$$

$$= \sum_{m=-2}^{-1} f(m)\, g(K-m)$$

$$K=0 \Rightarrow h(0) = \sum_{m=-2}^{-1} f(m)\, g(0-m)$$

$$= \sum_{m=-2}^{-1} f(m)\, g(-m)$$

$$= f(-2)\, g(+2) + f(-1)\, g(+1)$$

$$= 2\times 2 + 2\times 0$$

$$= 4$$

$$K=1 \Rightarrow h(1) = \sum_{m=-2}^{-1} f(m)\, g(1-m)$$

$$= f(-2)\, g(3) + f(-1)\, g(2)$$

$$= 2\times 1 + 2\times 2$$

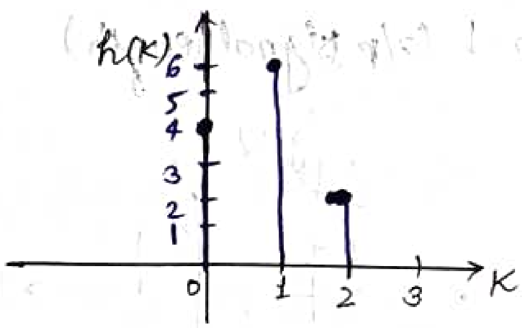
$$= 6$$

$$K=2 \Rightarrow h(2) = \sum_{-2}^{-1} f(m)\, g(2-m)$$

$$= f(-2)\, g(4) + f(-1)\, g(3)$$

$$= 2\times 0 + (2)\times 1$$

$$= 2$$



⇝ Contribution/Influence of one function over the other.

<u>Sliding Tape Method:</u>

① Plot $x(m)$ and $h(K-m)$.

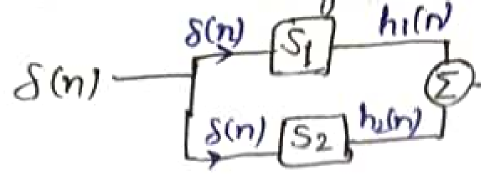
② Shift $h(K-m)$ to far left (large K).

③ Multiply and sum signals.

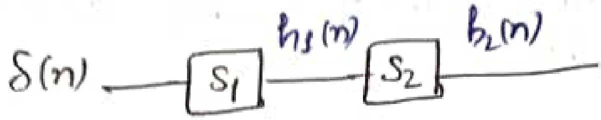
④ Shift K until new intermediate signal results.

⑤ Repeat 3 and 4 until done.

Inter-Connected System



$h(n) = h_1(n) + h_2(n)$



$h_1(n) * h_2(n)$

$\rightarrow \quad h(n) \xrightarrow{\text{inverse}} h_i(n)$

$$h(n) * h_i(n) = \delta(n)$$

Everlasting Exponential z^n

$$y(t) = H(s)\, e^{st}$$

$$y(n) = h(n) * z^n$$

$$= \sum_{m=-\infty}^{\infty} h(m)\, z^{n-m}$$

$$= z^n \sum_{m=-\infty}^{\infty} h(m)\, z^{-m}$$

$$y(n) = H(z)\, z^n$$

$$H(z) = \sum_{m=-\infty}^{\infty} h(m)\, z^{-m}, \text{ for constant given } z.$$

$$H(z) = \left.\frac{\text{O/P signal}}{\text{I/P signal}}\right|_{\text{I/P } z^n}$$

$$= \frac{P(z)}{Q(z)}$$

$$\Rightarrow H(z)\left[Q(E)\, z^n\right] = P(E)\, z^n$$

$$= E^k z^n$$

$$E^k z^n = z^{n+k} = z^n z^k$$

$$P(E)\, z^n = P(z)\, z^n$$

$$Q(E)\, z^n = Q(z)\, z^n$$

$$\Rightarrow H(z) = \frac{P(z)}{Q(z)}.$$

22-02-2024

THE CONTINUOUS TIME FOURIER SERIES

$x(t) = x(t+T_0) \quad \forall t$

↳ Periodic function

Fundamental period : Smallest value of T_0 which satisfies the condition.

$$\int_a^{a+T_0} x(t)\,dt = \int_b^{b+T_0} x(t)\,dt$$

Eg. $\cos(2\pi f_0 t)$ or $\sin(2\pi f_0 t)$

f_0 : ~~fundamental~~ frequency of sinusoid

$$T_0 = \frac{1}{f_0}$$

$$\cos(\omega_0 t) = \cos(2\pi f_0 t)$$

$$\omega_0 = 2\pi f_0 \text{ rad/s.}$$

nf_0 : n^{th} harmonic of sinusoid with frequency 'f_0'.

Eg. $x(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right]$

ω_0 : fundamental frequency.

Show $x(t+T_0) = x(t)$ for above.

$$n\omega_0 T_0 = 2\pi n$$

$$\Rightarrow T_0 = \frac{2\pi}{\omega_0}$$

$$x(t+T_0) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega_0 t + 2\pi n) + b_n \sin(n\omega_0 t + 2\pi n)\right]$$

$$= a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right]$$

$$= x(t)$$

→ In one fundamental period T_0, the n^{th} harmonic executes n complete cycle.

Orthogonal Function

$$\int_{T_0} f_1(x) f_2(x)\,dx = 0$$

Eg. Show if $\int_{-\pi}^{\pi} \cos(mx)\sin(nx)\,dx$, $m \neq n$, is orthogonal.

26-02-2024

Trigonometric Fourier Series

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right] \quad \ldots (A)$$

ω_0 : fundamental frequency

$x(t+T) = x(t)$; $n\omega_0$: n^{th} harmonic.

$$I = \int_{T_0} \cos n\omega_0 t \ \sin m\omega_0 t \ dt$$

$$= \frac{1}{2}\left[\int_{T_0} \underbrace{\cos(m+n)\omega_0 t}_{=0} \ dt + \int_{T_0} \underbrace{\cos(m-n)\omega_0 t}_{=0} \ dt\right] \ \forall m \neq n$$

$$I = \begin{cases} 0, & m \neq n \\ \frac{T_0}{2}, & m = \pm n \neq 0 \end{cases}$$

$\rightarrow \int_{T_0} \cos n\omega_0 t \cdot \cos m\omega_0 t \cdot dt = \begin{cases} 0, & m \neq n \\ T_0/2, & m = \pm n \neq 0 \end{cases}$

$\rightarrow \int_{T_0} \sin n\omega_0 t \cdot \sin m\omega_0 t \cdot dt = \begin{cases} 0, & m \neq n \\ T_0/2, & m = n \neq 0 \end{cases}$

$\rightarrow \int_{T_0} \sin n\omega_0 t \cdot \cos m\omega_0 t \cdot dt = \{ 0 \ \forall m, n.$

$$\therefore \int_{T_0} x(t)\, dt = a_0 \int_{T_0} dt + \sum_{n=1}^{\infty} a_n \int_{T_0} \cancel{\cos n\omega_0 t\, dt}^{\,0} + b_n \int_{T_0} \cancel{\sin n\omega_0 t\, dt}^{\,0}$$

$$= a_0 T_0$$

$$\Rightarrow \boxed{a_0 = \frac{1}{T_0}\int_{T_0} x(t)\, dt}$$

[Integrate F.S. both sides in T_0 to get a_0]

$$\int_{T_0} x(t)\, dt = a_0 \int dt$$

Multiply by $\cos m\omega_0 t$ and integrate ~~with~~ (A).

$$\int_{T_0} x(t) \cos m\omega_0 t\, dt = a_0 \int_{T_0} \cancel{\cos m\omega_0 t\, dt}^{\,0} + \sum_{n=1}^{\infty} a_n \int_{T_0} \cos n\omega_0 t \cdot \cos m\omega_0 t\, dt$$

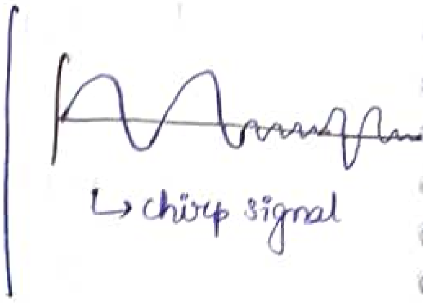
$$+ \sum_{n=1}^{\infty} b_n \int_{T_0} \cancel{\sin n\omega_0 t \cdot \sin m\omega_0 t\, dt}^{\,0}$$

$$= \sum_{n=1}^{\infty} a_n \int_{T_0} \cos n\omega_0 t \cdot \cos m\omega_0 t\, dt$$

$$\hookrightarrow \begin{cases} = 0, & \text{for } m \neq n \\ = T_0/2, & \text{for } m = n \neq 0 \end{cases}$$

$$\therefore \int_{T_0} x(t) \cos m\omega_0 t\, dt = a_m \frac{T_0}{2}$$

$$\therefore \boxed{\begin{aligned} a_n &= \frac{2}{T_0}\int_{T_0} x(t)\cos n\omega_0 t\, dt \\ b_n &= \frac{2}{T_0}\int_{T_0} x(t)\sin n\omega_0 t\, dt \\ a_0 &= \frac{1}{T_0}\int_{T_0} x(t)\, dt \end{aligned}}$$



↳ chirp signal

Dirichlet Conditions:

① Signal should have finite no. of maxima and minima over the range of time.

② Signal should have finite no. of discontinuities over the range of time.

③ Signal should be absolutely integrable over the range of time period.

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$$

↳ Trigonometric-type fourier series.

Exponential fourier series

28-02-2024

LG model
Cocktail problems
Blind source separation problem.

→ Finding a_0, a_n, b_n : Analysis

Finding summation of series : Synthesis

→ Sinusoidal functions are orthogonal in nature.

$$\boxed{\omega_0 = \frac{2\pi}{T_0}}$$

Compact Trigonometric Fourier Series

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$

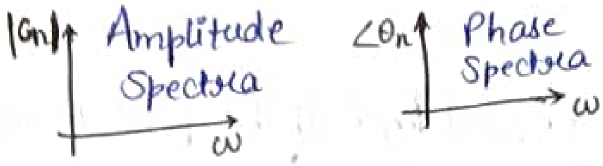
$$C_0 = a_0$$

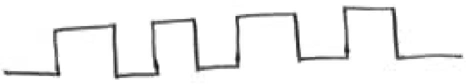
$$C_n = \sqrt{a_n^2 + b_n^2}$$

$$\theta_n = \tan^{-1}\left(\frac{b_n}{a_n}\right)$$

Exponential Form of Fourier Series

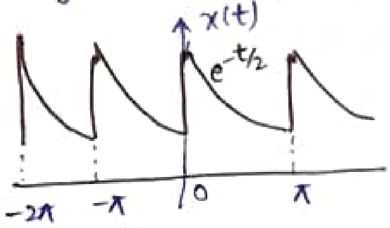
$$f(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}, \quad D_n = \frac{1}{T_0}\int_{T_0} f(t) e^{-jn\omega_0 t}\, dt$$





→ has frequency components due to sudden changes at some points

→ A signal has dual identity. The time domain description, $x(t)$, and frequency domain identity (Fourier spectra); they are complement to each other.



$T_0 = \pi$ (s)

$f_0 = \frac{1}{\pi}$ (Hz)

$\omega_0 = 2\pi \times \frac{1}{\pi} = 2$ (rad/s)

$$a_0 = \frac{1}{\pi}\int_{T_0} x(t)\,dt = \frac{1}{\pi}\int_0^{\pi} e^{-t/2}\,dt = 0.504$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} e^{-t/2}\cos 2nt\,dt = 0.504\left(\frac{2}{1+16n^2}\right)$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} e^{-t/2}\sin 2nt\,dt = 0.504\left(\frac{8n}{1+16n^2}\right)$$

$$\therefore x(t) = 0.054 + \sum_{n=1}^{\infty}\left(0.504\,\frac{2}{1+16n^2}\cos 2nt + 0.504\,\frac{8n}{1+16n^2}\sin 2nt\right)$$

The effect of Symmetry

→ The F.S. of an even periodic function, $x(t)$, consists of cosine terms only.

→ For any odd-periodic function, $x(t)$, F.S. consists of sine terms only.

→ The information of one period of $x(t)$ is implicit in only half period.

$\cos n\omega_0 t \longrightarrow$ even

$\sin n\omega_0 t \longrightarrow$ odd

Hidden Symmetry:

$$\left.\begin{aligned} a_0 &= \frac{2}{T_0}\int_0^{T_0} x(t)\,dt \\ a_n &= \frac{4}{T_0}\int_0^{T_0/2} x(t)\cos n\omega_0 t\,dt \\ b_n &= 0 \end{aligned}\right\} \text{(if } x(t) \text{ is even)}$$

If $x(t)$ is odd,

$$a_0 = 0$$
$$a_n = 0, \ \forall n$$
$$b_n = \frac{4}{T_0} \int_0^{T_0/2} x(t) \sin n\omega_0 t \, dt.$$

Half wave Symmetry:

If a periodic signal, $x(t)$, shifted by half the period, remains unchanged, except for a sign.

$$x(t - T_0/2) = -x(t)$$

Half wave symmetry : All the even-numbered harmonics vanish.

Every periodic signal, $x(t)$, can be expressed as the sum of sinusoids of a fundamental freq ω_0 and its harmonics.

$$x_1(t) = 2 + 7\cos\left(\frac{t}{2} + \theta_1\right) + 3\cos\left(\frac{2}{3}t + \theta_2\right) + 5\cos\left(\frac{7}{6}t + \theta_3\right)$$

$$\Rightarrow \omega_0 = \frac{1}{6}$$

→ The fundamental frequency is the greatest common factor (GCF) of all the frequencies in series.

$$x_2(t) = 2\cos(2t + \theta_1) + 5\sin(\pi t + \theta_2)$$

↳ Not periodic

ω_2 : irrational

$\therefore$ No GCF

$$x_3(t) = 3\sin(3\sqrt{2}t + \theta) + 7\cos(6\sqrt{2}t + \phi)$$

$$e^{st} \rightarrow \boxed{LTI} \longrightarrow \int_{-\infty}^{\infty} e^{s(t-\tau)} h(\tau) d\tau$$

$$= e^{st} \int_{-\infty}^{\infty} e^{-s\tau} h(\tau) d\tau$$

$$= H(s) e^{st}$$

$$x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + \dots$$

$$y(t) = a_1 H(s_1) e^{s_1 t} + a_2 H(s_2) e^{s_2 t} + \dots$$

Put $s = j\omega$.

$$a e^{j\omega_0 t} = a e^{st}$$

Synthesis: $x(t) = \sum_{K=-\infty}^{\infty} a_K e^{jK\omega_0 t}$, a_K : complex

$$\int_{T_0} e^{-j\omega_0 t} x(t)\, dt = \int_{T_0} \sum_{K=-\infty}^{\infty} a_K e^{-jn\omega_0 t + jK\omega_0 t} dt$$

$$= \sum_{K=-\infty}^{\infty} \int_{T_0} a_K e^{j(K-n)\omega_0 t} dt \quad , K \neq n$$

$$k = n : \quad = \sum_{K=-\infty}^{\infty} a_K T_0$$

Analysis:

$$a_n = \frac{1}{T_0} \int_{T_0} e^{-jn\omega_0 t} x(t)\, dt$$

$$a_0 = \frac{1}{T_0} \int_{T_0} x(t)\, dt$$

Eg $x_a(t) = \cos\omega_0 t$

$$x(t) = \sum_{K=-\infty}^{\infty} a_K e^{j\omega_0 K t}$$

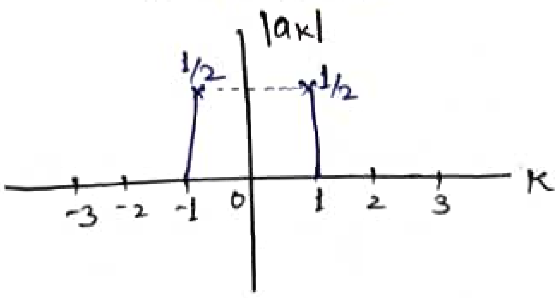
$$a_K = \frac{1}{T} \int_T x(t) e^{-jK\omega_0 t} dt$$

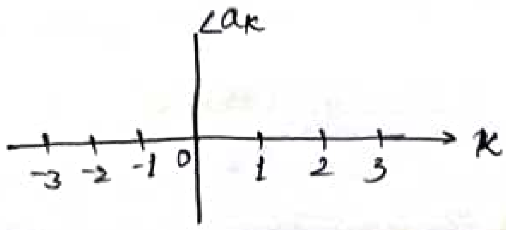
$$\therefore x_a = \frac{1}{2} e^{j\omega_0 t} + \frac{1}{2} e^{-j\omega_0 t}$$

$$\Rightarrow a_1 = \frac{1}{2}$$

$$a_{-1} = \frac{1}{2}$$

$$a_K = 0 , \quad k \neq \pm 1$$





Eg. Find fourier series, plot magnitude and phase spectra.

$x_b(t) = \sin(100t)$, $x_c(t) = 1 + \sin 100t + 2\cos 100t + \cos\left(200t + \frac{\pi}{4}\right)$

$$x_b(t) = \frac{e^{j100t} - e^{-j100t}}{2j}$$

$$a_1 = \frac{1}{2j}, \quad a_{-1} = \frac{-1}{2j}$$

$$\Rightarrow |a_1| = 1/2, \quad |a_{-1}| = 1/2,$$

$$\angle a_1 = -\pi/2, \quad \angle a_{-1} = \frac{\pi}{2}.$$

05-03-2024

Properties of Fourier Series

1. Linearity
2. Property conjugation
3. Time Reversal
4. Time Scaling
5. Time shifting
6. Frequency shifting
7. Convolution in time
8. Multiplication in time
9. Differentiation in time
10. Integration in time
11. Parsavells Theorem

→ Orthogonal series of function can be used as basis signal for generalised Fourier series.

1. Trigonometric (Sinusoids)
2. Exponential
3. Walsh Function
4. Bessel function
5. Legendre polynomials
6. Lagurre function
7. Jacobi ~~function~~ polynomials
8. Hermite polynomials
9. Chebyshev polynomials

① **Linearity:** Complex exponential fourier coefficients

$$x_1(t) \rightarrow C_{1n}$$
$$x_2(t) \rightarrow C_{2n}$$

(Both the signals having a same fundamental period T_0.

$$LCM(T_0, T_0) = T_0$$

$$\alpha x_1(t) + \beta x_2(t) \rightarrow x(t) \rightarrow \boxed{C_n = \alpha C_{1n} + \beta C_{2n}}$$

$$C_n = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-jn\omega_0 t}\, dt = \frac{1}{T_0}\int_0^{T_0} (\alpha x_1(t) + \beta(x_2)(t))\, e^{-jn\omega_0 t} dt$$

$$= \frac{1}{T_0}\int_0^{T_0} \alpha x_1(t)\, e^{-jn\omega_0 t} dt + \frac{1}{T_0}\int_0^{T_0} \beta x_2(t)\, e^{-jn\omega_0 t} dt$$

$$= \alpha C_{1n} + \beta C_{2n}.$$

② **Property of conjugation:**

$$x(t) \rightarrow C_n$$

$$C_n = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-jn\omega_0 t} dt$$

$$C_n^* = \frac{1}{T_0}\int_0^{T_0} x^*(t)\, (e^{-jn\omega_0 t})^* dt$$

$$= \frac{1}{T_0}\int_0^{T_0} x^*(t)\, e^{jn\omega_0 t} dt$$

$$n \rightarrow -n$$

$$C_{-n}^* = \frac{1}{T_0}\int_0^{T_0} x^*(t)\, e^{-jn\omega_0 t} dt$$

$$\boxed{x^*(t) \longrightarrow C_{-n}^*}$$

③ **Time Reversal:**

$$x(t) \longrightarrow C_n$$

$$C_n = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-jn\omega_0 t} dt$$

$x(t) \rightarrow x(-t)$ [Time reversal of $x(t)$]

$$C_n' = \frac{1}{T_0}\int_0^{T_0} x(-t)\, e^{-jn\omega_0 t} dt$$

Let $-t = \tau$ | $t = 0 \Rightarrow \tau = 0$

$\Rightarrow -dt = d\tau \Rightarrow dt = -d\tau$ | $t = T_0 \Rightarrow \tau = -T_0$

$$\Rightarrow C_n' = \frac{1}{T_0}\int_0^{-T_0} x(\tau)\, e^{-jn\omega_0(-\tau)} (-d\tau)$$

$$\Rightarrow C_n' = \frac{1}{T_0}\int_{-T_0}^{0} x(\tau)\, e^{-j(-n)\omega_0 \tau} d\tau$$

$$\Rightarrow C_n' = C_{-n}$$

$$\boxed{x(-t) \longrightarrow C_{-n}}$$

④ <u>Time Scaling</u> :

$x(t) \longrightarrow C_n$

Time period : T_0

$$\omega_0 = \frac{2\pi}{\omega_0}$$

$$C_n = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-jn\omega_0 t} dt$$

$x(t) \rightarrow x(at)$, $|a| > 1$

$T_0' = \frac{T_0}{a}$: Time period decreases

$$\omega_0' = \frac{2\pi}{T_0'} = \frac{2\pi}{T_0/a} = a \cdot \frac{2\pi}{T_0} = a\omega_0$$

$$C_n' = \frac{1}{T_0'}\int_0^{T_0'} x(at)\, e^{-jn\omega_0' t} dt$$

$$= \frac{1}{(T_0/a)}\int_0^{T_0/a} x(at)\, e^{-jn(\omega_0 a)t} dt$$

Let $\tau = at$
$\Rightarrow a\,dt = d\tau$
$\Rightarrow dt = \frac{d\tau}{a}$

$t = 0 \Rightarrow \tau = 0$
$t = T_0/a \Rightarrow \tau = T_0$

$$\Rightarrow C_n' = \frac{\alpha}{T}\int_0^{T_0} x(\tau)\, e^{-jn\omega_0 \alpha \cdot \frac{\tau}{\alpha}} \frac{d\tau}{\alpha}$$

$$\Rightarrow C_n' = \frac{1}{T_0}\int_0^{T_0} x(\tau)\, e^{-jn\omega_0 \tau} d\tau$$

$$\therefore \boxed{C_n = C_n'}$$

⑤ Time Shifting

$$x(t) \rightarrow C_n$$
$$(T_0) \; : \; \omega_0 = \frac{2\pi}{T_0}$$
$$C_n = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-jn\omega_0 t}\, dt$$
$$x(t) \longrightarrow x(t-t_0)$$
$$\boxed{C_n' = e^{-jn\omega_0 t_0}\, C_n}$$

⑥ Frequency Shifting [Harmonic number shifting property] 07-03-2024

$$x(t) \longrightarrow C_n \quad (T_0)$$
$$C_n = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-jn\omega_0 t}\, dt$$

Shifting in the frequency,

$$C_{n-m} = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-j(n-m)\,\omega_0 t}\, dt$$
$$= \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-jn\omega_0 t}\, e^{jm\omega_0 t}\, dt$$
$$= \frac{1}{T_0}\int_0^{T_0} x_1(t)\, e^{-jn\omega_0 t}\, dt$$
$$\boxed{x_1(t) = x(t)\, e^{jm\omega_0 t}}$$

Frequency shift causes exponential term getting multiplied with $x(t)$.

⑦ Convolution in Time Period 'T_0'

$$x_1(t) \longrightarrow C_{1n}$$
$$x_2(t) \longrightarrow C_{2n}$$

$x_1(t) * x_2(t) \rightarrow x(t) \rightarrow C_n'$

$$x_1(t) * x_2(t) \rightleftharpoons T_0 \cdot (C_{1n} \cdot C_{2n})$$
$$C_n' = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-jn\omega_0 t}\, dt$$
$$= \frac{1}{T_0}\int_0^{T_0} (x_1(t) * x_2(t))\, e^{jn\omega_0 t}\, dt$$
$$= \frac{1}{T_0}\int_0^{T_0}\left(\int_0^{T_0} x_1(\tau)\, x_2(t-\tau)\, d\tau\right) e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T_0} \int_0^{T_0} \int_0^{T_0} x_1(\tau)\, x_2(t-\tau)\, d\tau\, e^{-jn\omega_0 t} \frac{e^{+jn\omega_0 \tau}}{e^{+jn\omega_0 \tau}}\, dt$$

$$= \left(\frac{1}{T_0} \int_0^{T_0} \cancel{x_1(\tau)} \int_0^{T_0} x_1(\tau)\, x_2(t-\tau)\, e^{-jn\omega_0 \tau} e^{-jn\omega_0 (t-\tau)}\, dt\, d\tau \right)$$

$$= \left(\frac{1}{T_0} \int_0^{T_0} x_1(\tau) e^{-jn\omega_0 \tau} d\tau \right) \left(\int_0^{T_0} x_2(t-\tau)\, e^{-jn\omega_0 (t-\tau)} dt \right)$$

$$\left[\begin{array}{l|l} t-\tau = \alpha & t=0 \Rightarrow \alpha = -\tau \\ \Rightarrow dt = d\alpha & t = T_0 \Rightarrow \alpha = T_0 - \tau \end{array} \right]$$

$$\frac{T_0}{T_0} \int_{-T}^{T_0 - T} x_2(\alpha)\, e^{-jn\omega_0 \alpha} d\alpha$$

$$= T_0 \,(C_{1n} \cdot C_{2n})$$

⑧ Multiplication in Time

$$x_1(t) \cdot x_2(t) \longrightarrow C_{1n} * C_{2n}$$

$$x_1(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$x_2(t) = \sum_{n=-\infty}^{\infty} B_n e^{jn\omega_0 t}$$

$$x_1(t)\, x_2(t) = \sum_{K=-\infty}^{\infty} C_K e^{jK\omega_0 t} \sum_{m=-\infty}^{\infty} B_m e^{jm\omega_0 t}$$

$$= \sum_{m=-\infty}^{\infty} \sum_{K=-\infty}^{\infty} C_K B_m e^{j(m+K)\omega_0 t}$$

$$\left[\begin{array}{l|l} m+K = n & m = -\infty \Rightarrow n = -\infty \\ \Rightarrow m = n-K & m = +\infty \Rightarrow n = \infty \end{array} \right]$$

$$= \sum_{n=-\infty}^{\infty} \underbrace{\sum_{K=-\infty}^{\infty} C_K B_{n-K}}_{\text{convolution of } C_n \,\&\, B_n} e^{jn\omega_0 t}$$

↳ convolution of C_n & B_n.

⑨ Differentiation in Time

$$x(t) \longrightarrow C_n \qquad \left[T_0 = \frac{2\pi}{\omega_0}\right]$$

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$\frac{dx(t)}{dt} = jn\omega_0 C_n$$

$$\frac{d^2x(t)}{dt} = (jn\omega_0)^2 C_n$$

$$\vdots$$

$$\boxed{\frac{d^k x(t)}{dt} = (jn\omega_0)^k C_n}$$

⑩ Integration in Time

$$x(t) \longrightarrow C_n$$

$$\boxed{\int_{-\infty}^{t} x(\tau)\, d\tau = \frac{C_n}{jn\omega_0}}$$

Parsavells Theorem

$$\frac{1}{T_0}(x(t))^2 = \sum_{n=-\infty}^{\infty} (C_n)^2$$

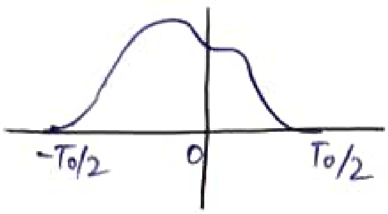
$$x(t) \rightarrow C_n,$$

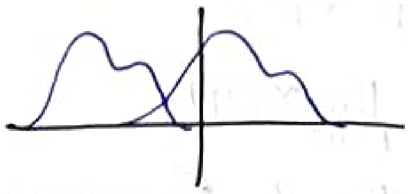
$$\frac{1}{T_0}\int_0^{T_0} |x(t)|^2 dt \longrightarrow x(t)\, x^*(t)$$

11-03-2024

→ A periodic signal can be expressed as a continuous sum (integral) of everlasting exponential.

If $x(t)$ is aperiodic, reconstruct a new periodic signal $x_{T_0}(t)$ formed by repeating the signal $x(t)$ at interval of T_0 seconds.





$x_{T_0}(t) \rightarrow$ F.S.

$T_0 \rightarrow \infty$

then, $x(t)$ repeats after an infinite interval.

$$\lim_{T \to \infty} x_{T_0}(t) = x(t)$$

$$x_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

$$D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_{T_0}(t)\, e^{-jn\omega_0 t} dt$$

$$\omega_0 = \frac{2\pi}{T_0}$$

For $x(t)$, $\quad D_n = \frac{1}{T_0} \int_{-\infty}^{\infty} x(t)\, e^{-jn\omega_0 t} dt$

Continuous fn: $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$, $D_n = \frac{1}{T_0} X(n\omega_0)$

$T_0 \rightarrow \infty$

$\Rightarrow D_n \rightarrow 0$

$\omega_0 \rightarrow 0$

$$x_{T_0}(t) = \sum_{n=-\infty}^{\infty} \frac{X(n\omega_0)}{T} e^{jn\omega_0 t}$$

$T_0 \rightarrow \infty$, $\omega_0 \rightarrow 0$

$$\Delta\omega = \frac{2\pi}{T_0}$$

$$x_{T_0}(t) = \sum_{n=-\infty}^{\infty} X(n\Delta\omega)\frac{\Delta\omega}{2\pi} e^{jn\Delta\omega t}$$

$0, \pm\Delta\omega, \pm 2\Delta\omega, \pm 3\Delta\omega, \pm \cdots$

$$X(n\Delta\omega)\frac{\Delta\omega}{2\pi}$$

$T_0 \to \infty$

$\Delta\omega \to 0$

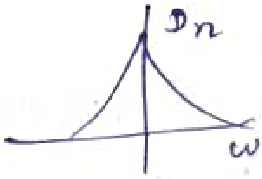
$$x_{T_0}(t) = x(t)$$

$$x(t) = \lim_{T_0\to\infty} x_{T_0}(t)$$

$$= \lim_{\Delta\omega\to 0} \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} X(n\Delta\omega)\, e^{jn\Delta\omega}\,\Delta\omega$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

... Fourier integral



$$\boxed{\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \\ x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \\ &\text{FS } \Delta\omega \to 0 \end{aligned}}$$

(Analysis)

(Synthesis)

$e^{-at}u(t)$

$$X(\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)\, e^{-j\omega t} dt$$

$$= \int_{0}^{\infty} e^{-at}\, e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt = \frac{-1}{a+j\omega}\, e^{-(a+j\omega)t}\Big|_0^{\infty}$$

$$X(\omega) = \frac{1}{a+j\omega}, \quad a>0$$

$$X(\omega) = \frac{1}{(a^2+\omega^2)^{1/2}}\, e^{-j\tan^{-1}(\omega/a)}$$

Not all signals are F. T.

If $x(t)$ has a finite $\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$ ..

12-03-2024

$x(t)$: Discrete Spectrum.

Sum of discrete exponentials with finite amplitude,

$$x(t) = \sum_n D_n e^{jn w_0 t}$$

For an aperiodic, the spectrum becomes continuous, i.e., the spectrum exists for every value of w.

But the amplitude of each component in the spectrum is zero.

$$x(t) = \lim_{\Delta w \to 0} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} x(n\Delta w)\, e^{jn\Delta w t} \Delta w \qquad | \, n\Delta w \to w$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} x(w)\, e^{jwt} dw$$

↳ Spectrum lies for all w.

$x(w)$: Fourier Spectrum

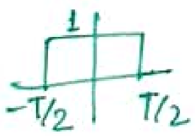
or

Fourier Transform

or

Spectral density

rect (t/T) :



$$\text{rect}(t/T) = \begin{cases} 1, & \text{for } |t| \le T/2 \\ 0, & \text{elsewhere.} \end{cases}$$

$$x(w) = \int_{-\infty}^{\infty} x(t)\, e^{-jwt} dt$$

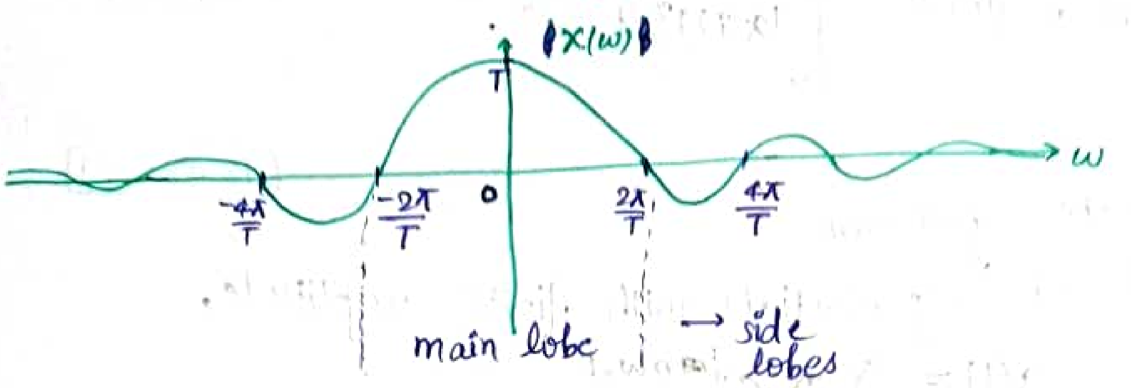
$$= \int_{-T/2}^{T/2} 1 \cdot e^{-jwt} dt$$

$$= \left. \frac{e^{-jwt}}{-jw} \right|_{-T/2}^{T/2}$$

$$= \frac{1}{jw}\left(e^{jwT/2} - e^{-jwT/2}\right) = \frac{1}{jw}\left[j\sin(wT/2) + j\sin(wT/2)\right]$$

$$\therefore x(w) = T \,\text{sinc}(wT/2) \qquad \left[\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}\right]$$

$X(\omega) \Leftrightarrow T\, \text{sinc}\left(\frac{\omega T}{2}\right)$

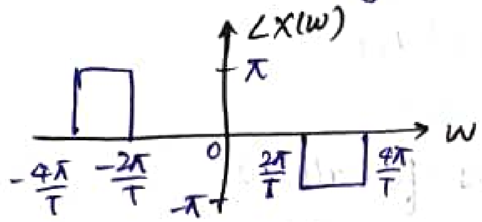


$\omega = 0$: $|X(\omega)| = T$

$\frac{\omega T}{2} = \pm n\pi \Rightarrow |X(\omega)| = 0$

$\angle X(\omega) = 0$ for positive $|X(\omega)|$

$\angle X(\omega) = \pm\pi$ for negative $|X(\omega)|$.



$\omega = \pm \frac{2n\pi}{T}$, $n = 1, 2, 3$

$\Rightarrow \sin(\omega T/2) = 0$

$$\angle X(\omega) = \begin{cases} 0, & \text{sinc}(\omega T/2) > 0 \\ \pm\pi, & \text{sinc}(\omega T/2) < 0 \end{cases}$$

→ Not all signals are fourier transform.
If $x(t)$ is having a finite energy,

$$\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$$

The fourier-transform $X(\omega)$ is finite and converges to $x(t)$ in the mean.

$$\hat{x}(t) = \lim_{\omega \to \infty} \frac{1}{2\pi} \int_{-\omega}^{\omega} X(\omega) e^{j\omega t} d\omega$$ [Approximation]

$$\int_{-\infty}^{\infty} |x(t) - \hat{x}(t)|^2 dt = 0. \quad \text{[For ideal approximation]}$$

If $x(t)$ satisfies Dirichlet condition, its F.T. is guaranteed to converge pointwise at all points, where $x(t)$ is continuous. At the point of discontinuity, $x(t)$ converges to the value midway b/w two values of $x(t)$ on either side of discontinuity.

Condition for F.T.:

① $x(t)$ should be absolutely integrable

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty.$$

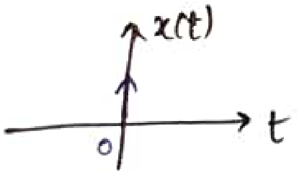
② $x(t)$ must have only finite number of discontinuity within any finite interval.

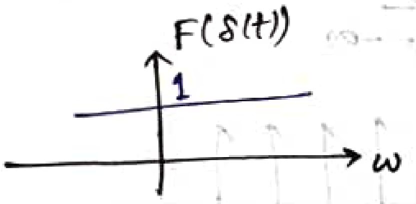
③ $x(t)$ must contain finite no. of minima and maxima within any finite interval.

Sufficient but not necessary conditions

F.T. of Impulse function:

$$F(\delta(t)) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$



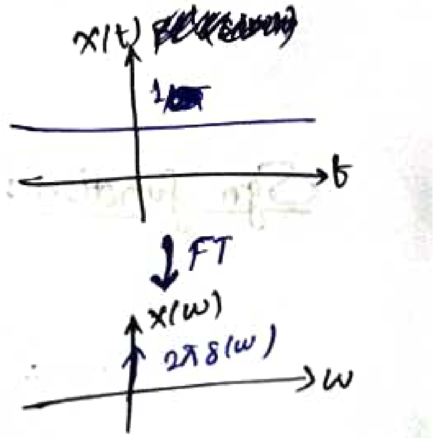


I.F.T. ($\delta(\omega)$):

$$F^{-1}(\delta(\omega)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega)\, e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi}$$

$$1 \Leftrightarrow 2\pi\,\delta(\omega)$$



Delay in impulse:

$$IFT\left(\delta(\omega-\omega_0)\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega-\omega_0)\, e^{j\omega t}\, dt$$

$$= \frac{1}{2\pi} e^{j\omega_0 t}$$

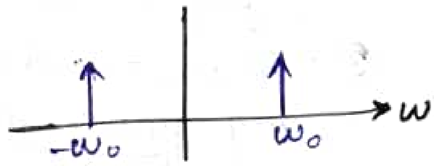
$$e^{j\omega_0 t} \Leftrightarrow 2\pi\,\delta(\omega-\omega_0)$$

$$\delta(\omega+\omega_0) : \quad e^{-j\omega_0 t} \Leftrightarrow 2\pi\,\delta(\omega+\omega_0).$$

F.T. $\cos(\omega_0 t)$

$$= \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) e^{j\omega t}\, dt$$

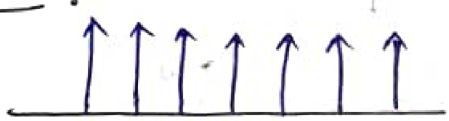
$$= \pi\left[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right]$$



$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}, \quad \omega_0 = \frac{2\pi}{T_0}$$

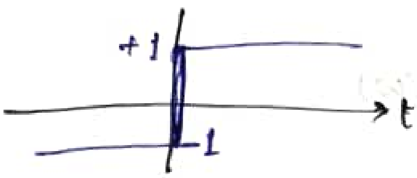
$$X(\omega) = 2\pi \sum_{n=-\infty}^{\infty} D_n\,\delta(\omega - n\omega_0)$$

Comb function:



$$X(\omega) = \frac{2\pi}{T_0}\sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0), \quad \omega_0 = \frac{2\pi}{T_0}$$

Sgn function:



19-03-2024

Wavelet Transform:

$\Delta t . \Delta w = \frac{h}{\pi} = 4\pi$

- Resolution in time-domain is inversely proportional to that in frequency-domain. [Gaussian f^n]

Uncertainity Principle:

$$\Delta T . \Delta W = 4\pi$$

A function becomes more localized in time (ΔT decreases), it becomes less localized in frequency (ΔW increases) and vice-versa.

Properties of Fourier Transform (FT):

① Linearity:

$$x_1(t) \longleftrightarrow X_1(w)$$
$$x_2(t) \longleftrightarrow X_2(w)$$
$$a_1 x_1(t) + a_2 x_2(t) \Longleftrightarrow a_1 X_1(w) + a_2 X_2(w)$$

② Time-shifting property:

If $x(t) \longrightarrow X(w)$,

$$x(t-t_0) \longleftrightarrow e^{-jwt_0} X(w)$$

When a signal is shifted in time, a phase shift introduced into its F.T., the magnitude remains unchanged

③ Frequency-shifting:

$$x(t) \longleftrightarrow X(w)$$
$$x(t) e^{jw_0 t} \longleftrightarrow X(w-w_0)$$

Multiplication of signal by a complex exponent to a frequency shift in frequency-domain.

④ Time-scaling:

$$x(t) \longleftrightarrow X(w)$$
$$x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{w}{a}\right)$$
$$x(t/a) \longleftrightarrow |a| X(wa)$$

Time-scaling by 'a' results in frequency-scaling by $\frac{1}{|a|}$ and amplitude-scaling by $\frac{1}{|a|}$.

⑤ Time Reversal:

$$x(t) \longleftrightarrow X(\omega)$$
$$x(-t) \longleftrightarrow X(-\omega)$$

⑥ Differentiation in time-domain:

$$x(t) \longleftrightarrow X(\omega)$$
$$\frac{dx(t)}{dt} \longleftrightarrow j\omega X(\omega)$$
$$\frac{d^n x(t)}{dt^n} \longleftrightarrow (j\omega)^n X(\omega)$$

⑦ Differentiation in frequency-domain:

$$x(t) \longleftrightarrow X(\omega)$$
$$t x(t) \longleftrightarrow j \frac{dX(\omega)}{d\omega}$$

⑧ Integration property:

$$x(t) \longleftrightarrow X(\omega)$$
$$\int_{-\infty}^{t} x(\tau)\, d\tau \longleftrightarrow \frac{1}{j\omega} X(\omega), \quad X(0) = 0.$$

If $X(0) \neq 0$,

$$\int_{-\infty}^{t} x(\tau) d\tau \longleftrightarrow \frac{X(\omega)}{j\omega} + \pi X(0)\, \delta(\omega).$$

Proof: $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$

$$\Rightarrow \frac{dx(t)}{dt} \overset{F.T.}{\longleftrightarrow} \frac{1}{2\pi} \int_{-\infty}^{\infty} j\omega X(\omega)\, e^{j\omega t} d\omega$$

$$\longleftrightarrow j\omega X(\omega)$$

$$\frac{d^2 x(t)}{dt^2} \longleftrightarrow (j\omega)^2 X(\omega).$$

$$x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau)\, u(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{t} x(\tau)\, u(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{t} x(\tau) d\tau + \int_{t}^{\infty} x(\tau).0\, d\tau$$

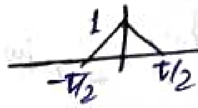
$$= \int_{-\infty}^{t} x(\tau) d\tau$$

$$F.T. (u(t)) = \frac{1}{j\omega} + \pi\,\delta(\omega)$$

$$F.T. (x(t) * u(t)) = F.T. (x(t)) . F.T. (u(t)).$$

$$= X(\omega)\left(\frac{1}{j\omega} + \pi\,\delta(\omega)\right)$$

Eg. Find F.T. of $e^{-a|t|}$, $e^{-a|t-t_0|}$, a triangular pulse



$x(t) = \Delta(t/\tau)$

Soln: (i)

$$F_T(e^{-a|t|}) = \int_{-\infty}^{\infty} e^{-a|t|}\, e^{-j\omega t} dt$$

$$= \int_{-\infty}^{0} e^{(a-j\omega)t} dt + \int_{0}^{\infty} e^{-t(a+j\omega)} dt$$

$$= \frac{e^{(a-j\omega)t}}{a-j\omega}\Big|_{-\infty}^{0} + \frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\Big|_{0}^{\infty}$$

$$= \left[\frac{1-0}{a-j\omega}\right] + \left[\frac{0-1}{-(a+j\omega)}\right]$$

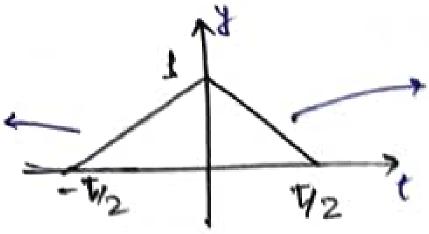
$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega}$$

$$= \frac{2a}{a^2+\omega^2}$$

(ii) $F_T(e^{-a|t-t_0|}) = e^{-j\omega t_0} . \dfrac{2a}{a^2+\omega^2}$ [By using time-shifting property]

(iii)

$y = 1+\frac{2t}{\tau}$



$\frac{x}{\tau/2} + \frac{y}{1} = 1$

$\Rightarrow x + \frac{\tau}{2}y = \frac{\tau}{2}$

$\Rightarrow y = 1 - \frac{2t}{\tau}$

$$\int_{-\tau/2}^{0}\left(1+\frac{2t}{\tau}\right)e^{-j\omega t}dt + \int_{0}^{\tau/2}\left(1-\frac{2t}{\tau}\right)e^{-j\omega t}dt$$

$$= \left(\frac{e^{-j\omega t}}{-j\omega}\Big|_{-\tau/2}^{\tau/2} + \frac{2}{\tau}\left[\frac{e^{-j\omega t}}{-j\omega}t - \int\frac{e^{-j\omega t}}{-j\omega}dt\right]_{-\tau/2}^{0}\right)$$

$$- \frac{2}{\tau}\left[\frac{e^{-j\omega t}}{-j\omega}t - \int\frac{e^{-j\omega t}}{-j\omega}dt\right]_{0}^{\tau/2}$$

$$= \frac{e^{-j\omega\tau/2} - e^{j\omega\tau/2}}{-j\omega} + \frac{2}{\tau}\left[\frac{e^{-j\omega t}}{-j\omega}t\right]_{-\tau/2}^{0} - \frac{2}{\tau}\left[\frac{e^{-j\omega t}}{-j\omega}t\right]_{0}^{\tau/2}$$

$$= \cancel{\frac{e^{j\omega\tau/2} - e^{-j\omega\tau/2}}{j\omega}} + \frac{2}{\tau j\omega}\frac{e^{-j\omega t}}{(-j\omega)}\Big|_{-\tau/2}^{0} - \frac{2}{\tau j\omega}\frac{e^{-j\omega t}}{(-j\omega)}\Big|_{0}^{\tau/2}$$

$$= \frac{e^{j\omega\tau/2} - e^{-j\omega\tau/2}}{j\omega} + \left(\frac{-e^{j\omega\tau/2}}{j\omega}\right) + \left(\frac{e^{-j\omega\tau/2}}{j\omega}\right)$$

$$+ \left(\frac{2(1)}{\tau\omega^2} - \frac{2e^{j\omega\tau/2}}{\tau\omega^2}\right) - \left(\frac{2(1)}{\tau\omega^2} - \frac{2e^{-j\omega\tau/2}}{\tau\omega^2}\right)$$

$$= \cancel{\frac{2e^{j\omega\tau/2}}{j\omega} + \frac{4e^{-j\omega\tau/2}}{\tau\omega^2}}$$

$$= \frac{2}{\tau\omega^2}\left(e^{-j\omega\tau/2} - e^{j\omega\tau/2}\right)$$

20-08-2024

Convolution Property:

$$x_1(t) \longleftrightarrow X_1(\omega)$$
$$x_2(t) \longleftrightarrow X_2(\omega)$$
$$x_1(t) * x_2(t) \longleftrightarrow X_1(\omega) \cdot X_2(\omega)$$
$$x_1(t) \cdot x_2(t) \longleftrightarrow \frac{1}{2\pi}(X_1(\omega) * X_2(\omega)).$$

Multiplication / Modulation Property:

$$x_1(t) \longleftrightarrow X_1(\omega)$$
$$x_2(t) \longleftrightarrow X_2(\omega)$$
$$x_1(t) \cdot x_2(t) \longleftrightarrow \frac{1}{2\pi} \cdot (X_1(\omega) * X_2(\omega)).$$

Parseval's Relation

Rayleigh Energy Theorem

$$x_1(t) \longleftrightarrow X_1(\omega)$$
$$x_2(t) \longleftrightarrow X_2(\omega)$$

$$\underbrace{\int_{-\infty}^{\infty} x_1(t) \cdot \underset{\substack{\uparrow \\ \text{conjugate}}}{x_2^*(t)}\, dt}_{\text{Inner product}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\omega) \cdot X_2^*(\omega)\, d\omega .$$

$$x_1(t) = x_2(t) = x(t).$$

$$\boxed{E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega}$$

$\therefore$ Energy in one domain is ~~the~~ equal to that in other.

Area under $x(t)$ from $X(\omega)$:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$\Rightarrow X(\omega)\Big|_{\omega=0} = \int_{-\infty}^{\infty} x(t) e^{-0} \cdot dt$$

$$\Rightarrow X(0) = \int_{-\infty}^{\infty} x(t)\, dt$$

Area under $X(\omega)$ from $x(t)$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$\Rightarrow x(t)\Big|_{t=0} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, d\omega$$

$$\Rightarrow 2\pi\, x(0) = \int_{-\infty}^{\infty} X(\omega)\, d\omega .$$

Duality (Similarity Theorem)

$$x(t) \longleftrightarrow X(\omega)$$

then, $X(t) \longleftrightarrow 2\pi\, x(-\omega)$

Modulation Property

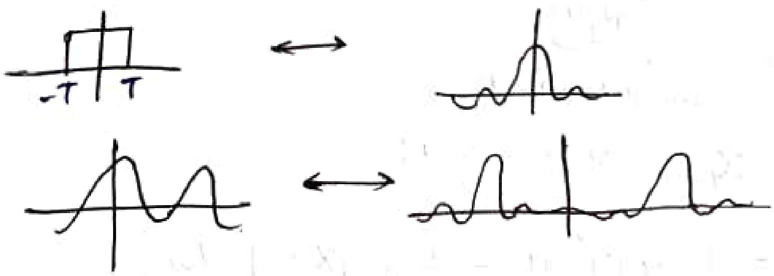
$$x(t) \longleftrightarrow X(\omega)$$

$$x(t)\cos\omega_c t \longleftrightarrow \frac{1}{2}\left[X(\omega-\omega_c) + X(\omega+\omega_c)\right]$$

$$F.T.\left(x(t)\cos\omega_0 t\right) = F.T.\left[x(t)\left(\frac{e^{j\omega_c t} + e^{-j\omega_c t}}{2}\right)\right]$$

$$= \frac{1}{2}\left[F.T\left(x(t)\, e^{j\omega_c t}\right) + F.T.\left(x(t)\, e^{-j\omega_c t}\right)\right]$$

$$= \frac{1}{2}\left(X(\omega-\omega_c) + X(\omega+\omega_c)\right)$$



21-03-2024

Duality Property:

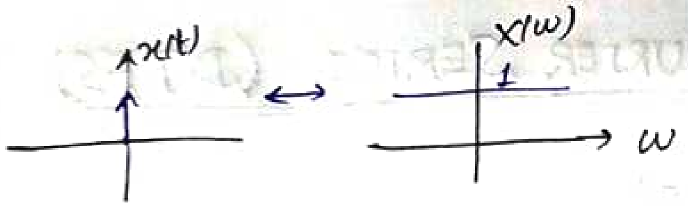
$$x(t) = \delta(t)$$

$$\Rightarrow X(\omega) = 1$$

If $y(t) = 1$, $Y(\omega) =$

$$x(t) \longleftrightarrow X(\omega)$$

$$X(\omega = t) \longleftrightarrow 2\pi\, x(t = -\omega)$$



Eg. $F.T.\left(6\,\text{sinc}(4t)\right) = \frac{3}{2}\pi\,\text{rect}\left(\frac{\omega}{8}\right)$

DISCRETE TIME FOURIER SERIES (DTFS)

↳ $\cos \Omega n$ or $\exp(j\Omega n)$

$\frac{\Omega}{2\pi}$ is rational number.

N_0 : Time period

$$\cos \Omega(n+N_0) = \cos \Omega n$$

$$\Omega N_0 = 2\pi m \quad , \quad m, N_0 : \text{integer}$$

$$\Rightarrow \frac{\Omega}{2\pi} = \frac{m}{N_0} \rightarrow \text{Rational no.}$$

→ $\cos \Omega n$ $(e^{j\Omega n})$ is periodic.

Period : $N_0 = \frac{2\pi}{\Omega} m$

Eg. $\Omega = \frac{4\pi}{17}$, $m = ?$

$$N_0 = \frac{2\pi}{\Omega} m = \cancel{2\pi} \cdot \frac{17}{\cancel{4\pi}_{2}} \times m = \frac{17m}{2}$$

Min^m value of m for N_0 to be integer : $m = 2$
(f^n to be periodic)

$\Rightarrow N_0 = 17$.

→ $x[n] = x[n+N_0]$

The fundamental period is N_0.

Angular frequency (fundamental) : $\Omega_0 = \frac{2\pi}{N_0}$ rad/sample

~~Trig ex~~

Exponential Fourier Series :

$$e^{j0n}, e^{\pm j\Omega_0 n}, e^{\pm j2\Omega_0 n}, \ldots \quad [e^{\pm jn\Omega_0 n}]$$

$$e^{j(\Omega \pm 2\pi m)n} = e^{j\Omega n} \underbrace{e^{\pm j2\pi mn}}_{1} = e^{j\Omega n}$$

r^{th} harmonic is identical to $(r+N_0)^{th}$ harmonic.

$$g_{r+N_0} = e^{j(r+N_0)\Omega_0 n}$$
$$= e^{jr\Omega_0 n}$$

$\Rightarrow g_r = g_{r+N_0} = g_{r+2N_0} = \dots = g_{r+mN_0}$

$\rightarrow$ All DT signals are inherently band limited.
↳ defined in $(-\pi$ to $\pi)$
$(0$ to $2\pi)$.

Fourier Series:

$$x[n] = \sum_{r=0}^{N_0-1} D_r e^{jr\Omega_0 n}$$

$$D_r = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n] e^{-jr\Omega_0 n}, \quad \Omega_0 = \frac{2\pi}{N_0}.$$

26-03-2024

DTFT:

$$x(n) = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(\Omega)\, e^{jn\Omega} d\Omega$$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-jn\Omega n}$$

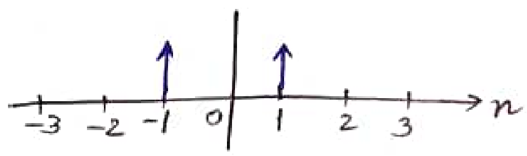
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-jn\omega}$$

$$x(n) = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(e^{j\omega})\, e^{jn\omega} d\omega.$$

$\Delta\Omega = \frac{2\pi}{N_0}$

$\Rightarrow \Delta\Omega \times N_0 = 2\pi$

Eg. $x(n) = \delta(n-1) + \delta(n+1)$



$$X(\omega) = 1.\, e^{-j(-1)\omega} + 1.\, e^{-j(1)\omega}$$
$$= e^{j\omega} + e^{-j\omega}$$
$$= 2\cos(\omega).$$

To get back $x(n)$,

$$x(n) = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} 2\cos\omega \,.\, e^{jn\omega} d\omega \qquad [\because e^{jn\omega} = \cos n\omega + j\sin n\omega]$$

$$= \frac{1}{2\pi} \int \Big[\frac{1}{2}\cos(n+1)\omega + \frac{1}{2}\cos(n-1)\omega$$
$$+ \frac{1}{2}\sin(n+1)\omega - \frac{1}{2}\sin(n-1)\omega\Big] d\omega$$

Eg: Find F.T. of :

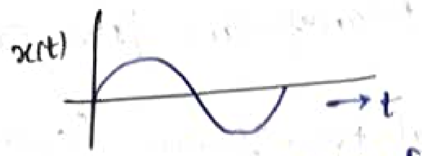
① $x(n) = 2^n \cdot u(-n)$

② $x(n) = \left(\frac{1}{3}\right)^n \cdot u(n-3)$

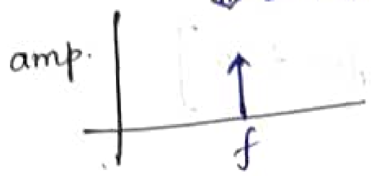
③ $x(n) = \delta(n)$.

④ $x(n) = (n-5)\, u(n-5) - u(n-6)$.

LAPLACE TRANSFORM



⇓ Transform



[Computation reduced]

→ Laplace transform is applicable only for continuous time-signal.

[Discrete time-signal → Z-transform]

↳ Integral transform:

$$g(\alpha) = \int x(t)\, f(\alpha, t)\, dt$$

↳ Derived from fourier-transform.

<u>Fourier-transform (FT):</u>

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$\boxed{X(s) = \int_{-\infty}^{\infty} x(t) e^{-st}\, dt}$$: Laplace Transform.

[$s = \sigma + j\omega$, σ = Real quantity.]

$$= \int_{-\infty}^{\infty} [x(t) e^{-\sigma t}] e^{-j\omega t}\, dt$$ → F.T. of $x(t)e^{-\sigma t}$.

Eg. $x(t) = e^{2t} u(t)$

↳ Not absolutely integrable.

∴ F.T. does not exit.

<u>L.T:</u>

$$X = \int_{0}^{\infty} (e^{2t} e^{-\sigma t}) e^{-j\omega t}\, dt.$$

↳ For some values of σ, L.T. converges.

<u>Region of Convergence (ROC):</u> In s-plane, the value of σ for which the L.T. converges.

Bilateral / Unilateral Transform:

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt \quad \text{... Bilateral transform [Non-Causal]}$$

$$X(s) = \int_{0}^{\infty} x(t)\, e^{-st}\, dt \quad \text{... Unilateral transform. [Causal]}$$

[Eg., $x(t) = e^{-at}u(t) \rightarrow$ causal : $x(t) = 0$ for $t<0$]

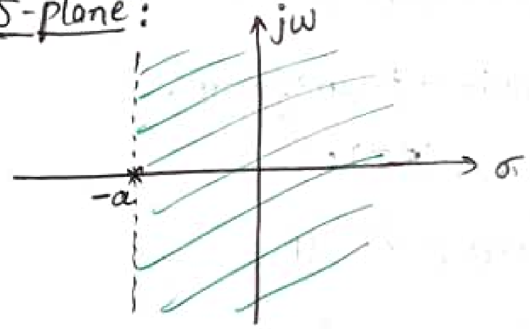
Eg. $x(t) = e^{-at}.u(t)$

LT : $X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$

$$= \int_{0}^{\infty} e^{-at} e^{-st}\, dt$$

$$\boxed{X(s) = \frac{1}{s+a}}$$

S-plane :



RoC : $Re\{s\} > -a$

or $\sigma > -a$

Eg. $x(t) = -e^{-at}u(-t)$, $a>0$.

$\cancel{x(t)=}$ $X(s) = -\int_{-\infty}^{0} e^{-at} e^{-st}\, dt$

$= \cancel{\int_{\infty}^{0} e^{ay} e^{sy} dy}$

[$\cancel{-t = y \Rightarrow -dt = dy}$
$\cancel{t = 0 \Rightarrow y = 0}$
$\cancel{t = -\infty \Rightarrow y = \infty}$]

$$-\int_{-\infty}^{0} e^{-(a+s)t}\, dt = -\left[\frac{e^{-(a+s)t}}{-(a+s)}\right]_{-\infty}^{0}$$

$$= \frac{1}{a+s}\left[1 - \cancel{e^{-(a+s)(-\infty)}}\right]$$

$a+s<0$

$$= \frac{1}{s+a}, \quad \boxed{s < -a}$$

Eg. Find the L.T. of $x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$.

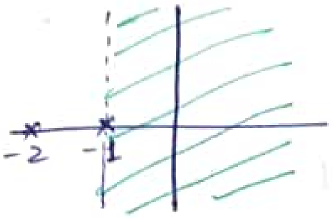
Soln:

$$3e^{-2t}u(t) \longleftrightarrow \frac{3}{s+2} \quad ; \text{R.O.C.} : Re\{s\} > -2 \equiv R_1$$

$$2e^{-t}u(t) \longleftrightarrow \frac{2}{s+1} \quad ; \text{R.O.C} : Re\{s\} > -1 \equiv R_2$$

ROC will always be parallel strip(s) to $j\omega$ axis.

Overall R.O.C : $R \equiv R_1 \cap R_2$
$\equiv R\{s\} > -1$.



$$X(s) = \frac{3}{s+2} - \frac{2}{s+1}$$

$$= \frac{s-1}{s^2+3s+2} \rightarrow \text{R.O.C} : Re\{s\} > -1.$$

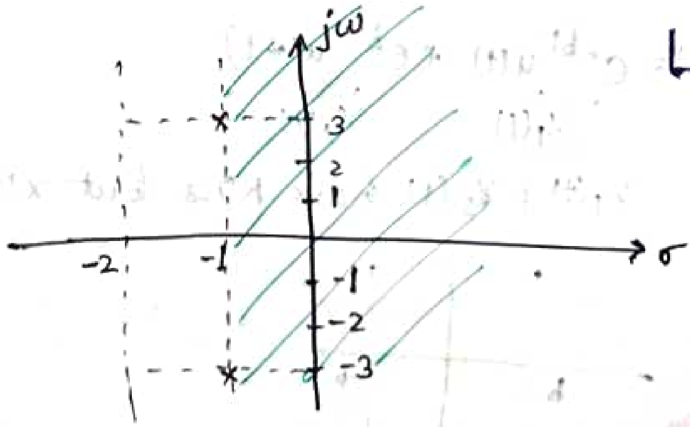
Eg. $x(t) = e^{-2t}u(t) + e^{-t}(\cos 3t)\,u(t)$

$$e^{-2t}u(t) \longleftrightarrow \frac{1}{s+2}$$

$$\left[e^{-t}\cos(3t)\,u(t) \longrightarrow \frac{(s+1)}{(s+1)^2+9} \right]$$

$$\mathcal{L}\{e^{-t}\underbrace{\cos(3t)}_{\left(\frac{e^{j3t}+e^{-j3t}}{2}\right)}u(t)\} = \mathcal{L}\left\{\frac{1}{2}\left[e^{-(1-3j)t} + e^{-(1+3j)t}\right]\right\}$$

$$= \frac{1}{2}\left[\frac{1}{s+(1-3j)} + \frac{1}{s+(1+3j)}\right]$$

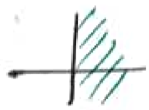


↳ Look only the real part.

28-03-2024

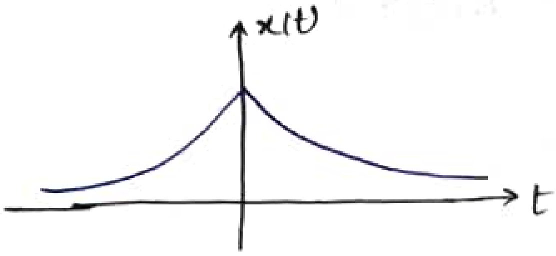
Unit Impulse: $LT[\delta(t)] \longleftrightarrow 1$ → ROC is entire s-plane

Unit Step: $LT[u(t)] \longleftrightarrow \frac{1}{s}$ → ROC : $s>0$



→ Two signals can have same LT but different ROC.

Eg. $x(t) = e^{-b|t|}$, $b>0$



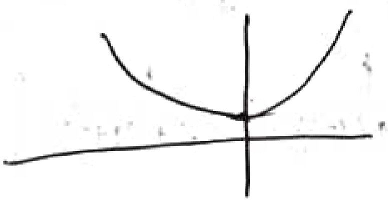
$$x(t) = \underbrace{e^{-bt}u(t)}_{x_1(t)} + \underbrace{e^{bt}u(-t)}_{x_2(t)}$$

$$e^{-bt}u(t) \xleftrightarrow{LT} \frac{1}{s+b}, \text{ ROC: } Re\{s\} > -b \equiv R_1$$

$$e^{bt}u(-t) \xleftrightarrow{LT} \frac{1}{s-b}, \text{ ROC: } Re\{s\} < b \equiv R_2$$

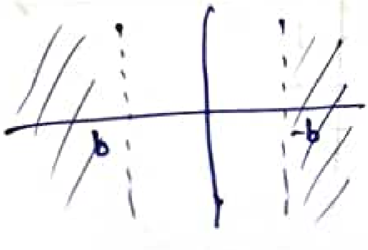
ROC of $x(t)$: $R \equiv R_1 \cap R_2$
↳ strip

b<0:



$$x(t) = \underbrace{e^{-bt}u(t)}_{x_1(t)} + \underbrace{e^{bt}u(-t)}_{x_2(t)}$$

$x_1(t)$ & $x_2(t)$ have ROCs but $x(t)$ doesn't.



Q) Draw the pole-zero plot and determine the ROC for signal:

$$x(t) = \delta(t) - \frac{4}{3} e^{-t} u(t) + \frac{1}{3} e^{2t} u(t).$$

Soln:

$$\delta(t) \xleftrightarrow{LT} 1$$

$$-\frac{4}{3} e^{-t} u(t) \longleftrightarrow \frac{-4/3}{s+1}, \quad \text{ROC: } Re\{s\} > -1.$$

$$\frac{1}{3} e^{2t} u(t) \longleftrightarrow \frac{1/3}{s-2}; \quad \text{ROC: } Re\{s\} > 2.$$

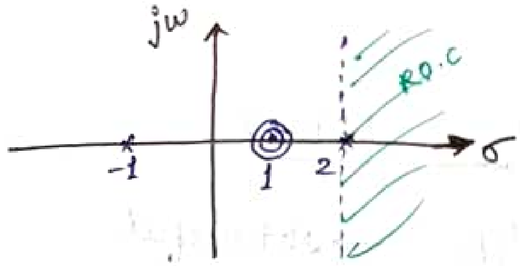
$$X(s) = 1 - \frac{4}{3}\frac{1}{s+1} + \frac{1}{3}\frac{1}{s-2}$$

$$= \frac{3(s+1)(s-2) - 4(s-2) + (s+1)}{3(s+1)(s-2)}$$

$$= \frac{3s^2 - 6s + 3}{3(s+1)(s-2)}$$

$$= \frac{s^2 - 2s + 1}{(s+1)(s-2)}$$

$$\therefore X(s) = \frac{(s-1)^2}{(s+1)(s-2)} \rightarrow$$ Zeroes: +1, +1
Poles: −1, 2



Properties of ROC :

① ROC is parallel to the $j\omega$ axis.

② The ROC does not contain any poles.

③ If the signal is finite duration, then the ROC is the entire s-plane.

④ If the signal is right-sided and $Re\{s\} = \sigma_0$, then ROC is $Re\{s\} > \sigma_0$.

⑤ If the signal is left-sided and $Re\{s\} = \sigma_0$, then ROC is $Re\{s\} < \sigma_0$.

⑥ If the function is right-sides, then ROC is right of the right-most pole.

⑦ If the function is left-sided, then ROC is left of the left-most pole.

⑧ If $x(t)$ is two-sided, i.e., $x(t)$ is infinite duration signal, then ROC is $\sigma_1 < Re\{s\} < \sigma_2$.

Inverse Laplace Transform (ILT)

$$x(t) = \frac{1}{2\pi} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$$ → Not useful

$$X(s) = \sum_{i=1}^{m} \frac{A_i}{s + a_i}$$

If x is right-sided,

$$x(t) = \sum A_i e^{-a_i t} u(t)$$

If x is left-sides,

$$x(t) = -\sum A_i e^{-a_i t} u(-t).$$

Eg. Find $x(t)$ for $X(s) = \frac{1}{(s+1)(s+2)}$, (i) $\underbrace{Re\{s\} > -1}_{ROC}$.

Soln: $X(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)} \Rightarrow A = 1, B = -1$

$$= \frac{1}{s+1} - \frac{1}{s+2}$$

ILT $\Rightarrow x(t) = (e^{-t} - e^{-2t})u(t)$.

(ii) If $Re\{s\} < -2$.

$$X(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

$$\Rightarrow x(t) = -(e^{-t} - e^{-2t})\,u(-t).$$

03-04-2024

Properties of LT:

① Linearity:

$$x_1(t) \longleftrightarrow X_1(s) \xrightarrow{ROC} R_1$$
$$x_2(t) \longleftrightarrow X_2(s) \longrightarrow R_2$$

$$a_1x_1(t) + a_2x_2(t) \longleftrightarrow a_1X_1(s) + a_2X_2(s)$$ [Scaling and additivity]

$$\boxed{R : R_1 \cap R_2}$$

Eg $X_1(s) = \frac{1}{s+1}$, $Re\{s\} > -1$

$X_2(s) = \frac{1}{(s+1)(s+2)}$, $Re\{s\} > -1$

$X(s) = \frac{1}{s+1} - \frac{1}{(s+1)(s+2)} = \frac{1}{(s+2)}$, $Re\{s\} > -2$.

② Time shifting:

$$x(t) \longleftrightarrow X(s)$$
$$x(t-t_0) \longleftrightarrow e^{-st_0}X(s).$$

Proof: $X_1(s) = \int_{-\infty}^{\infty} x(t-t_0)\, e^{-st}\, dt$

let $t - t_0 = \tau \Rightarrow dt = d\tau$

$\Rightarrow t = \tau + t_0$

$$\therefore X_1(s) = \int_{-\infty}^{\infty} x(\tau) e^{-s(t_0+\tau)} d\tau$$

$$= e^{-st_0} \int_{-\infty}^{\infty} x(\tau) e^{-s\tau} d\tau$$

$$= e^{-st_0} X(s)$$

$$x(t+t_0) \longleftrightarrow e^{+st_0} X(s)$$

③ Frequency shifting:

$$x(t) \longleftrightarrow X(s)$$

$$e^{s_0 t} x(t) \longleftrightarrow X(s-s_0)$$

Proof: $X_1(s) = \int_{-\infty}^{\infty} x(t) e^{s_0 t} e^{-st} dt$

$$= \int_{-\infty}^{\infty} x(t) e^{-(s-s_0)t} dt$$

$$= X(s-s_0)$$

$$e^{-s_0 t} x(t) \longleftrightarrow X(s+s_0)$$

→ $\cos bt \xleftrightarrow{LT} \frac{s}{s^2+b^2}$

$$e^{j\omega_0 t} x(t) \xleftrightarrow{LT} X(s-j\omega_0)$$

$$e^{-at} \cos bt \xleftrightarrow{LT} \frac{s+a}{(s+a)^2+b^2}$$

④ Scaling:

$$x(t) \longleftrightarrow X(s)$$

$$x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{s}{a}\right)$$

Proof: $X_1(s) = \int_{-\infty}^{\infty} x(at) e^{-st} dt$

$$\left[at = u \Rightarrow dt = \frac{du}{a} \quad \Rightarrow t = \frac{u}{a} \right]$$

$$= \int_{-\infty}^{\infty} x(u) e^{-\left(\frac{s}{a}\right)u} \frac{du}{|a|}$$

$$= \frac{1}{|a|} X\left(\frac{s}{a}\right)$$

⑤ Convolution :

$$x_1(t) \longleftrightarrow X_1(s)$$
$$x_2(t) \longleftrightarrow X_2(s)$$
$$x_1(t) * x_2(t) = X_1(s).X_2(s)$$

⑥ Time differentiation :

$$x(t) \longleftrightarrow X(s)$$
$$\frac{dx(t)}{dt} \longleftrightarrow sX(s) - x(0)$$

Proof: $X_1(s) = \int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st} dt$

$$\left[\begin{array}{l} u = e^{-st}, \; dv = dx(t) \\ \Rightarrow du = -se^{-st}, \; v = x(t) \end{array} \right]$$

$$= \int u \, dv$$
$$= uv - \int v \, du$$
$$= \left[e^{-st} x(t)\right]_0^{\infty} - \int_{0^-}^{\infty} x(t) \cdot (-se^{-st} dt)$$
$$= -x(0^-) + sX(s)$$
$$= sX(s) - x(0^-)$$

$$\frac{d^2x(t)}{dt^2} \longleftrightarrow s^2X(s) - sx(0^-) - \dot{x}(0^-)$$

$$\frac{d^nx(t)}{dt^n} \longleftrightarrow s^nX(s) - s^{n-1}x(0^-) - s^{n-2}x(0^-) - \dots - x^{(n-1)}(0^-)$$

diff.

⑦ Differentiation in s-domain :

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$
$$\frac{dX(s)}{ds} = \int_{-\infty}^{\infty} -t\,x(t) e^{-st} dt$$
$$= LT\{-t\,x(t)\}$$
$$\therefore -t\,x(t) \xleftrightarrow{LT} \frac{dX(s)}{ds}$$

Eg. $x(t) = te^{-at}u(t)$.

$$e^{-at} \xleftrightarrow{LT} \frac{1}{s+a}$$

$$te^{-at} \longleftrightarrow -\frac{d}{ds}\left[\frac{1}{s+a}\right]$$

$$= +(s+a)^{-2}$$

$$= \frac{1}{(s+a)^2}, \quad \text{ROC: } Re\{s\} > -a.$$

Eg. $x(t) = \frac{t^2}{2} e^{-at}u(t) \longleftrightarrow \frac{1}{(s+a)^3}$

$$\frac{t^{n-1}}{(n-1)!} e^{-at}u(t) \longleftrightarrow \frac{1}{(s+a)^n}$$

$\boxed{08\text{-}04\text{-}2024}$

Time Integration :

$$x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau)\, u(t-\tau)\, d\tau \quad 1, -\infty \text{ to } t$$

$$[u(t-\tau) = 1, \quad \tau \le t]$$

$$\Rightarrow x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

Applying LT both sides,

$$\frac{X(s)}{s} = LT\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right]$$

$$\int_{-\infty}^{t} x(\tau)\, d\tau \longleftrightarrow \frac{X(s)}{s}.$$

Initial Value Theorem :

Conditions : (i) It is applicable only when $x(t) = 0,\ t < 0$.

(ii) $x(t)$ must not contain any impulse or higher order singularities (discontinuity) at $t = 0$.

$$x(t) \xleftrightarrow{LT} X(s)$$

$$\boxed{x(0^+) = \lim_{s \to \infty} sX(s)}$$

Proof:

$$LT\left\{\frac{dx(t)}{dt}\right\} = sX(s) - x(0^-), \quad \text{[Differentiation property]}$$

$$\int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st} dt = \int_{0^-}^{0^+} \frac{dx(t)}{dt} \underbrace{e^{-st}}_{1 \text{ in } 0^- \to 0^+} dt + \int_{0^+}^{\infty} \frac{dx(t)}{dt} e^{-st} dt$$

$$\Rightarrow sX(s) - \cancel{x(0^-)} = x(0^+) - \cancel{x(0^-)} + \int_{0^+}^{\infty} \frac{dx(t)}{dt} e^{-st} dt$$

$$\Rightarrow \lim_{s\to\infty} sX(s) = x(0^+). \qquad \left[\because \lim_{s\to\infty} \int_{0^+}^{\infty} \frac{dx(t)}{dt} e^{-st} dt = 0\right]$$

Final Value Theorem

Conditions: (i) It is applicable only when $x(t)=0$, $t<0$.

(ii) $sX(s)$ must have poles in the left half of s-plane.

$$x(t) \xleftrightarrow{LT} X(s)$$

$$\boxed{\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)}$$

Proof: $$\frac{dx(t)}{dt} \xleftrightarrow{LT} sX(s) - x(0^-)$$

$$\Rightarrow \int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st} dt = sX(s) - x(0^-)$$

$$\Rightarrow \lim_{s\to 0} sX(s) - \lim_{s\to 0} x(0^-) = \int_{0^-}^{\infty} \frac{dx(t)}{dt} \lim_{s\to 0} \cancel{e^{-st}}^{\,1} dt = x(\infty) - \lim_{s\to 0} x(0^-)$$

$$\Rightarrow \lim_{s\to 0} sX(s) = \lim_{t\to\infty} x(t)$$

Eg. $x(t) = e^{-2t} u(t) + e^{-t} \cos t . u(t)$

$x(0) = 1+1 = 2$

$$\Rightarrow X(s) = \frac{2s^2+2s+12}{(s^2+2s+10)(s+2)}$$

$$\therefore \lim_{s\to\infty} sX(s) = 2$$

Eg. Find the initial and final value of:

$$X(s) = \frac{s+5}{s^2+3s+2}$$

Soln: $\lim_{t\to 0^+} x(t) = \lim_{s\to\infty} sX(s)$

$$= \lim_{s\to\infty} \frac{s(s+5)}{s^2+3s+2} = \lim_{s\to\infty} \frac{1+5/s}{1+3/s+2/s^2} = 1.$$

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} s.X(s)$$

$$= \lim_{s\to 0} \frac{s(s+5)}{s^2+3s+2} = 0.$$

→ For a causal LTI system, the impulse response is zero for $t<0$ and thus right-sides. The ROC associated with the system function for a causal system is a right-half plane.

— Rational function: right of right-most plane.

→ LTI system is stable iff ROC of $H(s)$ includes $j\omega$ axis.

09-04-2024

→ Consider a system described as

$$H(s) = \frac{s-1}{(s+1)(s-2)}$$

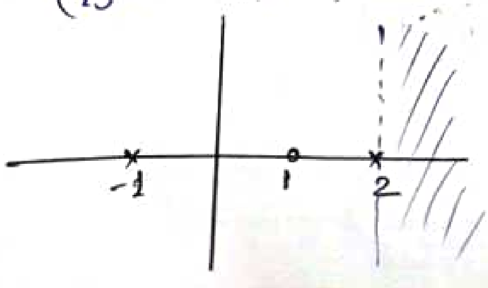
$$= \frac{A}{s+1} + \frac{B}{s-2}$$

$$\Rightarrow H(s) = \frac{2/3}{s+1} + \frac{1/3}{s-2}$$

$$\Rightarrow h(t) = \tfrac{2}{3}e^{-t} + \tfrac{1}{3}e^{2t}$$

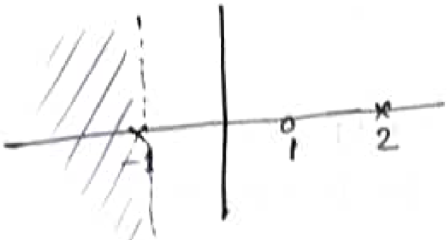
Case-1: Causal

$h(t) = (\tfrac{2}{3}e^{-t} + \tfrac{1}{3}e^{2t})u(t)$ ← Causal, unstable



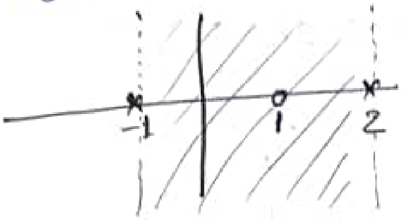
Case-2: Non-causal [Anti-causal] ⇝ purely non-causal

$h(t) = -\left(\frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}\right)u(-t)$ ← Non-causal, unstable



case-3:

$h(t) = \frac{2}{3}\, e^{-t}u(t) - \frac{1}{3}\, e^{2t}\, u(-t)$ ← Non-causal, stable.



Q) If i/p and o/p of a system are given by

$x(t) = e^{-3t}\, u(t)$, $y(t) = (e^{-t} - e^{-2t})\, u(t)$.

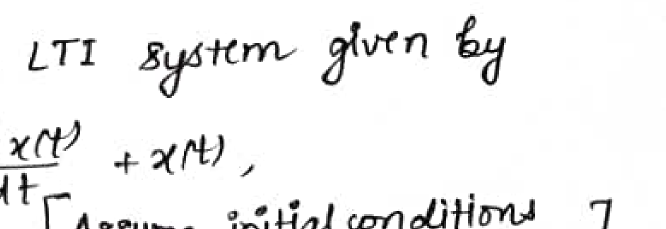
Is the system causal and stable.

Soln: $x(t) = e^{-3t}u(t) \longleftrightarrow X(s) = \frac{1}{s+3}$

$y(t) = e^{-t}u(t) - e^{-2t}u(t) \longleftrightarrow Y(s) = \frac{1}{s+1} - \frac{1}{s+2}$.

System T.F., $H(s) = \frac{Y(s)}{X(s)} = \frac{s+3}{s+1} - \frac{s+3}{s+2} = \frac{s+3}{(s+1)(s+2)} = \frac{s+3}{s^2+3s+2}$

∴ stable and causal.



$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y(t) = \frac{dx}{dt} + 3x(t)$.

Q) Find the response, $y(t)$, of an LTI system given by

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y(t) = \frac{dx(t)}{dt} + x(t),$$

if $x(t) = 3e^{-5t}\, u(t)$. [Assume initial conditions are zero.]

Soln: $H(s) = \frac{Y(s)}{X(s)} = \frac{s+1}{s^2+5s+6} = \frac{s+1}{(s+2)(s+3)}$

$x(t) = 3e^{-5t}u(t) \longleftrightarrow X(s) = \frac{3}{s+5}$.

$$\therefore Y(s) = H(s) \cdot X(s) = \frac{3(s+1)}{(s+2)(s+3)(s+5)}$$

$$= \frac{A}{s+2} + \frac{B}{s+3} + \frac{C}{s+5}$$

$$= \frac{A(s^2+8s+15) + B(s^2+7s+10) + C(s^2+5s+6)}{(s+2)(s+3)(s+5)}$$

$$\Rightarrow A+B+C=0$$

$$8A+7B+5C=3 \Rightarrow 3A+2B=3$$

$$15A+10B+6C=3 \Rightarrow 9A+4B=3$$

$$3A=-3$$

$$A=-1$$

$$\Rightarrow B=3$$

$$\Rightarrow C=-2$$

$$\therefore Y(s) = \frac{-1}{s+2} + \frac{3}{s+3} + \frac{-2}{s+5}$$

$$\text{ILT} \Rightarrow y(t) = -e^{-2t} + 3e^{-3t} - 2e^{-5t}.$$

Z TRANSFORM

↳ Discrete-time form of ~~Fourier~~ Laplace transform.
↳ Used to find algebraic form of signal.
↳ Generalization of FT.

signal : $x(n)$

Z-transform :

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}, \quad z = re^{j\omega}$$

$$= \sum_{n=-\infty}^{\infty} x(n) (re^{j\omega})^{-n}$$

$$= \sum_{n=-\infty}^{\infty} (x(n) r^{-n}) e^{-j\omega n}$$

$$= \text{F.T. of } [x(n) r^{-n}]$$

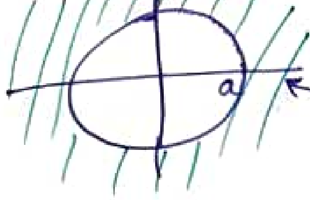
If $r=1$, ZT converges to F.T.

Eg. $x(n) = a^n u(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{z}{z-a} = \frac{1}{1-az^{-1}}.$$

↳ converges for $|az^{-1}| < 1$



ROC: $|z| > |a|$.

Eg. $x(n) = -b^n u(-n-1)$.

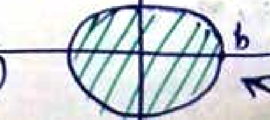
$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} b^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} b^{-n} z^n = -\left[\sum_{n=0}^{\infty} (b^{-1}z)^n - 1\right]$$

→ converges for $|b^{-1}z| < 1$

$$= \frac{z}{z-b} = \frac{1}{(1-bz^{-1})}$$



ROC: $|z| < |b|$.

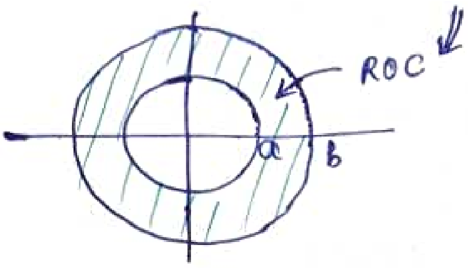
Eg. $x(n) = a^n u(n) - b^n u(-n-1)$, where $b > a$.

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} = \sum_{n=0}^{\infty} (aZ^{-1})^n - \left[\sum_{n=0}^{\infty} (b^{-1}Z)^n - 1\right]$$

$$= \frac{Z}{Z-a} - \frac{Z}{Z-b}$$

$$= \frac{1}{1-aZ^{-1}} - \frac{1}{1-bZ^{-1}}$$

↳ converges for $|Z| < |b|$ and $|Z| > |a|$.



11-04-2024

Finite Duration Signal

Eg. $x(n) = \{ \underset{\uparrow}{2}, -1, 3 \}$ ← sequence

$n=0$, $n=1$, $n=2$

$$\Rightarrow x(n) = \begin{cases} 2, & n=0 \\ -1, & n=1 \\ 3, & n=2 \end{cases}$$

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

$$= \sum_{n=0}^{2} x(n) Z^{-n}$$

$$= x(0)Z^{0} + x(1)Z^{-1} + x(2)Z^{-2}$$

$$= 2 - Z^{-1} + 3Z^{-2}$$

↳ ROC: Entire Z-plane except at $Z=0$.

Eg. $x(n) = \{ 3, -2, \underset{\uparrow}{1} \}$

$$x(n) = \begin{cases} 3, & n=-2 \\ -2, & n=-1 \\ 1, & n=0. \end{cases}$$

$$X(Z) = \sum_{n=-2}^{0} x(n) Z^{-n}$$

$$= x(-2) Z^2 + x(-1) Z^1 + x(0) Z^0$$

$$= 3Z^2 - 2Z + 1$$

↳ ROC: Entire Z-plane except $Z = \infty$.

Eg: $x(n) = \{2, -1, \underset{\uparrow}{1}, 3, 4\}$

$$\Rightarrow X(Z) = 2Z^2 - Z + 1 + 3Z^{-1} + 4Z^{-2}$$

↳ ROC: Entire Z-plane except $Z = 0$ and $Z = \infty$.

Properties of ROC

① ROC consists of a ring in Z-plane, centered about the origin.

② ROC does not contain poles.

③ If $x(n)$ is finite duration, ROC is entire Z-plane except at $Z = 0$ and/or $Z = \infty$.

④ If $x(n)$ is right-sided sequence and has the transform $X(Z)$ with $|Z| = r_0$, then ROC: $|Z| > r_0$.

⑤ If $x(n)$ is left-sided sequence and has the transform $X(Z)$ with $|Z| = r_0$, then ROC: $|Z| < r_0$.

⑥ If $x(n)$ is two-sided sequence and has the transform $X(Z)$, then the ROC is a concentric ring.

⑦ If $X(Z)$ is a rational function and $x(n)$ is right-sided with pole of highest magnitude R_0, then ROC: $|Z| > R_0$.

⑧ If $X(Z)$ is a rational function and $x(n)$ is left-sided with pole of lowest magnitude r_0, then ROC: $|Z| < r_0$.

Eg ① $\delta(n) \longleftrightarrow 1$

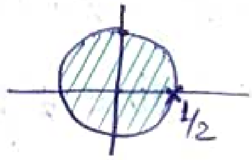
ROC: entire Z-plane

② $x(n) = (1/2)^n u(-n)$.

$$X(Z) = \sum_{n=-\infty}^{0} (1/2)^n Z^{-n}$$

$$= \frac{1}{1-2Z}$$

↳ ROC: $|Z| < 1/2$



Eg. $$X(Z) = \frac{3 - \frac{5}{6}Z^{-1}}{(1-\frac{1}{4}Z^{-1})(1-\frac{1}{3}Z^{-1})}$$

$$= \frac{A}{1-\frac{1}{4}Z^{-1}} + \frac{B}{1-\frac{1}{3}Z^{-1}}$$

$$A = \frac{3-\frac{5}{6}\times 4}{(1-\frac{1}{3}\times 4)} = \frac{(18-20)\frac{1}{2}}{(3-4)} = 1$$

$$B = \frac{3-\frac{5}{6}\times 3}{1-\frac{1}{4}\times 3} = \frac{18-\cancel{20}^{15}}{4-3} \times \frac{\cancel{4}^{2}}{\cancel{6}_{3}} = \cancel{\frac{4}{3}}\ 2$$

$$X(Z) = \frac{1}{1-\frac{1}{4}Z^{-1}} + \frac{2}{1-\frac{1}{3}Z^{-1}}$$

$\Rightarrow x(n) = (1/4)^n u(n) + 2(1/3)^n u(n)$.

⟶ ROC: $|Z| > 1/3$.

Properties of Z-Transform

① **Linearity:**

$$x_1(n) \longleftrightarrow X_1(Z) \rightarrow R_1$$
$$x_2(n) \longleftrightarrow X_2(Z) \rightarrow R_2$$
$$a x_1(n) + b x_2(n) \longleftrightarrow aX_1(Z) + bX_2(Z) \rightarrow R: R_1 \cap R_2.$$

② **Time Shifting:**

$$x(n) \longleftrightarrow X(Z)$$
$$x(n-n_0) \longleftrightarrow Z^{-n_0} X(Z)$$

Proof:

$$ZT\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

$$ZT\{x(n-n_0)\} = \sum_{n=-\infty}^{\infty} x(n-n_0) Z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(p) Z^{-p} Z^{-n_0} \quad \Big|\, \begin{array}{l} n-n_0=p \\ \Rightarrow n=p+n_0 \end{array}$$

$$= Z^{-n_0} \sum_{n=-\infty}^{\infty} x(p) Z^{-p}$$

$$= Z^{-n_0} X(Z)$$

③ **Scaling in Z-domain:**

$$x(n) \longleftrightarrow X(Z) \quad \rightarrow R$$
$$Z_0^n x(n) \longleftrightarrow X(Z/Z_0) \rightarrow R' = |Z_0| R$$

Proof: Take $Z = e^{jw}$, $Z_0 = e^{jw_0}$

$$ZT\{Z_0^n x(n)\} = \sum_{n=-\infty}^{\infty} e^{jw_0 n} x(n) Z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) (e^{-jw_0} Z)^{-n}$$

$$= X(Z/Z_0).$$

15-04-2024

④ Time Reversal

$$x(n) \longleftrightarrow X(z) \rightarrow ROC: R$$

$$x(-n) \longleftrightarrow X(1/z) \rightarrow ROC: \frac{R}{2}$$

Proof:

$$Z\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n) z^{-n}$$

[Let $-n = l$.]

$$= \sum_{l=-\infty}^{\infty} x(l)(z^{-1})^{-l}$$

$$= X(z^{-1})$$

Eg. Find the Z-transform of:

$$x(n) = (1/3)^n \sin\left(\frac{\pi}{4}n\right) u(n)$$

Soln: $\sin\left(\frac{\pi}{4}n\right) = \frac{1}{2j}\left[e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n}\right]$

& $Z\{x(n)\} = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n \frac{1}{2j}\left[e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n}\right] z^{-n}$

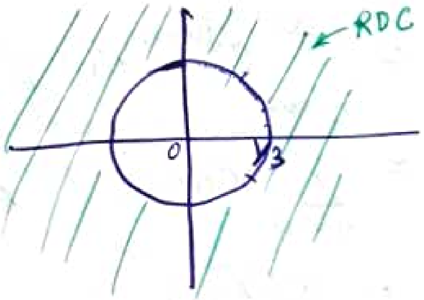
$$= \frac{1}{2j}\left[\sum_{n=0}^{\infty}\left(\frac{1}{3}e^{j\frac{\pi}{4}}z^{-1}\right)^n - \sum_{n=0}^{\infty}\left(\frac{1}{3}e^{-j\frac{\pi}{4}}z^{-1}\right)^n\right]$$

$$= \frac{1}{2j}\left[\frac{1}{1-\frac{1}{3}e^{j\frac{\pi}{4}}z^{-1}} - \frac{1}{1-\frac{1}{3}e^{-j\frac{\pi}{4}}z^{-1}}\right]$$

↳ converges for $\left|\frac{1}{3}e^{j\frac{\pi}{4}}z^{-1}\right| < 1$

ROC : $|z| > 1/3$.

$$X(z) = \frac{\frac{1}{3\sqrt{2}}z}{\left(z - \frac{1}{3}e^{j\pi/4}\right)\left(z - \frac{1}{3}e^{-j\pi/4}\right)}$$



⑤ Differentiation

$$x(n) \longleftrightarrow X(z)$$
$$n\,x(n) \longleftrightarrow -z\frac{dX(z)}{dz}$$

Proof:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} x(n)\cdot(-n)\, z^{-n-1}$$

$$= -z^{-1}\sum_{n=-\infty}^{\infty} n\,x(n) z^{-n}$$

$$\Rightarrow -z\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} n x(n) z^{-n} = ZT\{n\,x(n)\}$$

Eg. Find the Z-transform of:

$$x(n) = n a^n u(n).$$

Soln:

$$x_1(n) = a^n u(n)$$

$$X_1(z) = \frac{1}{1-az^{-1}}, \quad ROC: |z|>a$$

$$\frac{dX_1(z)}{dz} = \cancel{-(1-az^{-1})} \ \frac{az^{-1}(-z^{-1})}{(1-az^{-1})^2}$$

$$\Rightarrow -z\frac{dX_1(z)}{dz} = \frac{az^{-1}}{(1-az^{-1})^2}.$$

Convolution

$$x_1(n) \longleftrightarrow X_1(z) \rightarrow ROC: R_1$$
$$x_2(n) \longleftrightarrow X_2(z) \rightarrow ROC: R_2$$

$$x_1(n) * x_2(n) \longleftrightarrow X_1(z)\,.\,X_2(z) \rightarrow ROC: R_1 \cap R_2.$$

Eg. $h(n) = \delta(n) - \delta(n-1)$

$x(n) \longleftrightarrow X(z)$

$y(n) = ?$

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) * h(n)$$

$$Y(z) = X(z)H(z)$$

$$H(z) = 1 - z^{-1}$$

$$Y(z) = X(z)(1-z^{-1})$$

$$\therefore\ y(n) = x(n) - x(n-1)$$

Eg. $x_1(n) = \{\underset{\uparrow}{1}, 2, -4\}$

$x_2(n) = \{\underset{\uparrow}{-1}, 3, 5\}$

Find $y(n) = x_1(n) * x_2(n)$

Soln: Take ZT,

$$X_1(Z) = 1 + 2Z^{-1} - 4Z^{-2}$$

$$X_2(Z) = -1 + 3Z^{-1} + 5Z^{-2}$$

$$Y(Z) = X_1(Z) \cdot X_2(Z)$$

$$= -1 + 3Z^{-1} + 5Z^{-2} - 2Z^{-1} + 6Z^{-2} + 10Z^{-3} + 4Z^{-2} - 12Z^{-3} - 20Z^{-4}$$

$$= -1 + Z^{-1} + 15Z^{-2} - 2Z^{-3} - 20Z^{-4}$$

$$\therefore y(n) = \{-1, 1, 15, -2, -20\}.$$

Eg. $X(Z) = \log(1 - aZ^{-1})$, $|Z| > a$.

Find $x(n)$.

Soln: $$\frac{dX(Z)}{dZ} = \frac{aZ^{-2}}{1 - aZ^{-1}}$$

$$\Rightarrow -Z\frac{dX(Z)}{dZ} = -\frac{aZ^{-1}}{1 - aZ^{-1}}$$

Inverse ZT,

$$n\,x(n) = -a \cdot a^{n-1}u(n-1). \qquad \left[\because ZT\{a^n u(n)\} = \frac{1}{1 - aZ^{-1}}\right]$$

$$\Rightarrow x(n) = -\frac{a^n}{n}u(n-1).$$

Eg. Find $X(Z)$ if

$$x(n) = a^{n-1}u(n-1).$$

Soln: $$Z\{a^n u(n)\} = \frac{1}{1 - aZ^{-1}}$$

By time-shifting property,

$$Z\{a^{n-1}u(n-1)\} = \frac{Z^{-1}}{1 - aZ^{-1}}$$

$$= \frac{1}{Z - a}.$$

16-04-2024

→ ZT → Unilateral transform : $0 \to \infty$ (one-sided)
→ Bilateral transform : $-\infty \to \infty$ (two-sided)

Unilateral Time-Shift:

Bilateral: $x(n-m) \longleftrightarrow z^{-m} X(z)$

Proof:

$$ZT\{x(n-m)\} = \sum_{n=0}^{\infty} x(n-m) z^{-n}$$

$$= z^{-m} \sum_{n=0}^{\infty} x(n-m) z^{-(n-m)}$$

$$\begin{bmatrix} \text{Take } n-m = p \\ n=0 \Rightarrow p=-m \\ n=\infty \Rightarrow p=\infty \end{bmatrix}$$

$$= z^{-m} \sum_{p=-m}^{\infty} x(p) z^{-p}$$

$$= z^{-m} \{ \underbrace{\sum_{p=0}^{\infty} x(p) z^{-p}}_{X(z)} + \sum_{p=-m}^{-1} x(p) z^{-p} \}$$

$$= z^{-m} X(z) + z^{-m} \sum_{p=-m}^{-1} x(p) z^{-p}$$

$$= z^{-m} X(z) + \underbrace{z^{-m} \sum_{p=1}^{m} x(p) z^{p}}_{\sum_{p=1}^{m} x(-p) z^{-(m-p)}}$$

Eg: If $m=1 \Rightarrow ZT\{x(n-m)\} = z^{-m} X(z) + x(-1)$.

Unilateral: $x(n-m) \longleftrightarrow z^{-m} X(z) + \sum_{p=1}^{m} x(-p) z^{-(m-p)}$.

Initial Value Theorem

If $x(n)$ is a causal signal, $x(n) = 0$ for $n<0$, then

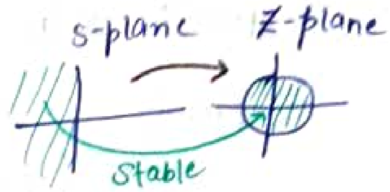
$$x(0) = \lim_{z \to \infty} X(z).$$

Proof:

$$x(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$\Rightarrow \lim_{z\to\infty} X(z) = x(0) + \underbrace{x(1)z^{-1} + x(2)z^{-2} + \dots}_{0}$$

$$\therefore x(0) = \lim_{z\to\infty} X(z).$$



Final Value Theorem [Unilateral]

If $x(n)$ is a causal signal,

$x(n) \longleftrightarrow X(z)$, then

$(1-z^{-1})X(z)$ should have a pole inside a unit circle in the z-plane.

If $X(z)$ is a rational function, all its pole should strictly lie inside the unit circle except possibly a first order pole at $z=1$.

$$\lim_{n\to\infty} x(n) = \lim_{z\to 1} (1-z^{-1})X(z).$$

Proof:

$$ZT\{x(n-1) - x(n)\} = z^{-1}X(z) + x(-1) - X(z)$$

$$= X(z)\left[z^{-1} - 1\right] + x(-1)$$

$$= \cancel{\left(\sum_{n=0}^{\infty}\left[x(n) - x(n-1)\right]\right)}$$

$$\lim_{z\to 1} \sum_{n=0}^{\infty} \left[x(n-1) - x(n)\right] z^{-n}$$

$$= \lim_{p\to\infty} \sum_{n=0}^{p} \left[x(n-1) - x(n)\right]$$

$$= \lim_{p\to\infty} \Big\{ (x(-1) - x(0)) + (x(0) - x(1)) + (x(1) - x(2)) + \dots + (x(p-1) - x(p)) \Big\}$$

$$\Rightarrow x(\infty) = \lim_{z\to 1} X(z)\left[1 - z^{-1}\right]$$

Eg. Find the inverse Z-transform of:

$$X(Z) = \frac{8Z - 19}{(Z-2)(Z-3)}.$$

Soln: Method-1:

$$X(Z) = \frac{A}{(Z-2)} + \frac{B}{(Z-3)}$$

$$= \frac{3}{Z-2} + \frac{5}{Z-3}$$

IZT $\Rightarrow$ $x(n) = \left[3(2)^{n-1} + 5(3)^{n-1}\right]u(n-1).$

$$\left[\because \frac{1}{Z-a} \leftrightarrow a^{n-1}u(n-1)\right]$$

Method-2:

$$\frac{X(Z)}{Z} = \frac{8Z-19}{Z(Z-2)(Z-3)}$$

$$= \frac{A}{Z} + \frac{B}{Z-2} + \frac{C}{Z-3}$$

$$= \frac{-19}{6Z} + \frac{3/2}{Z-2} + \frac{5/3}{Z-3}.$$

$$\Rightarrow X(Z) = -\frac{19}{6} + \frac{3/2\,Z}{Z-2} + \frac{5/3\,Z}{Z-3}.$$

$$\Rightarrow x(n) = -\frac{19}{6}\delta(n) + 3/2\,(2)^n u(n) + 5/3\,(3)^n u(n).$$

Q. Find output, $y(n)$, of a system with the input $x(n) = (1/2)^n u(n)$ and $h(n) = 6\,\delta(n) - 4(1/3)^n u(n)$.

Soln: ZT $\Rightarrow$ $X(Z) = \frac{Z}{Z - 1/2}$, $H(Z) = 6 - 4\frac{Z}{Z - 1/3}$.

$$\therefore Y = X(Z).H(Z)$$

$$= \frac{6Z}{Z - 1/2} - \frac{4Z^2}{(Z-1/2)(Z-1/3)}$$

IZT $\Rightarrow$ $y(n) = 6(1/2)^n u(n) - 12(1/2)^n u(n) + 8(1/3)^n u(n).$

$$\frac{4Z}{(Z-1/2)(Z-1/3)} = \frac{G(Z)}{Z}$$

$$\frac{G(Z)}{Z} = \frac{12}{Z-1/2} + \frac{-8}{Z-1/3}$$

$$\Rightarrow G(Z) = \frac{12Z}{Z-1/2} - \frac{8Z}{Z-1/3}$$

$$\Rightarrow g(n) = 12(1/2)^n u(n) - 8(1/3)^n u(n)$$

17-04-2024

Methods of Inverse ZT:

1. Partial fraction
2. Long division
3. Convolution
4. Residue approach.

$$\frac{Z}{Z-a} \longleftrightarrow a^n u(n)$$

$$\frac{Z}{(Z-a)^2} \longleftrightarrow n a^{n-1} u(n)$$

$$\frac{Z}{(Z-a)^3} \longleftrightarrow \frac{n(n-1)}{2!} a^{n-2} u(n)$$

$$\vdots$$

$$\frac{Z}{(Z-a)^K} \longleftrightarrow \frac{n(n-1)\cdots(n-(K-2))}{(K-1)!} \cdot a^{n-K+1} \cdot u(n).$$

Eg. Find $x(n)$ if $X(Z) = \dfrac{1+2Z^{-1}}{1-2Z^{-1}+Z^{-2}}$, if it is (a) causal, (b) non-causal.

Long-division method:

(a)

$$\begin{array}{r|l} & 1 + 4Z^{-1} + 7Z^{-2} \\ \hline 1-2Z^{-1}+Z^{-2} & 1+2Z^{-1} \\ & \underline{1 - 2Z^{-1} + Z^{-2}} \\ & 4Z^{-1} - Z^{-2} \\ & \underline{4Z^{-1} - 8Z^{-2} + 4Z^{-3}} \\ & 7Z^{-2} - 4Z^{-3} \\ & \underline{7Z^{-2} - 14Z^{-3} + 7Z^{-4}} \\ & 10Z^{-3} - 7Z^{-4} \\ & \vdots \end{array}$$

$\therefore x(n) = \{ \underset{\uparrow}{1}, 4, 7, \dots \}$ ← Preferable for finite-signal.

(b)

$$\begin{array}{l|l} & 2z+5z^2+8z^3 \\ \hline z^{-2}-2z^{-1}+1 & 2z^{-1}+1 \\ & \underline{2z^{-1} - 4 + 2z} \\ & \quad 5-2z \\ & \quad \underline{5 - 10z + 5z^2} \\ & \qquad 8z-5z^2 \\ & \qquad \underline{8z - 16z^2 + 8z^3} \\ & \qquad\quad 11z^2-8z^3 \\ & \qquad\quad \vdots \end{array}$$

$\therefore x(n) = \{\ldots, 11, 8, 5, 2, \underset{\uparrow}{0}\}$

<u>For stability</u>:

1. ROC should include unit circle.
2. If $H(z) \rightarrow$ rational function, all the poles will lie inside the unit-circle.

Eg. For the given system, plot the poles and zeroes and comment on its stability.

$$y(n) = y(n-1) - 0.5y(n-2) + x(n) + x(n-1)$$

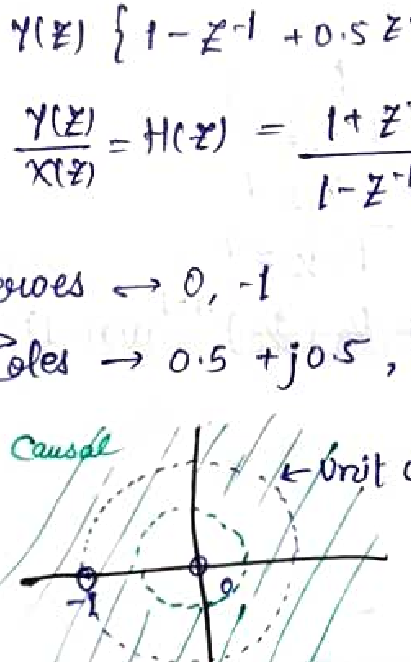
<u>Soln</u>: $Y(z) = Y(z).z^{-1} - 0.5z^{-2}Y(z) + X(z) + z^{-1}X(z)$

$\Rightarrow Y(z)\{1 - z^{-1} + 0.5z^{-2}\} = X(z)\{1+z^{-1}\}$

$\Rightarrow \dfrac{Y(z)}{X(z)} = H(z) = \dfrac{1+z^{-1}}{1-z^{-1}+0.5z^{-2}} = \dfrac{z^2+z}{0.5-z+z^2} = \dfrac{z(z+1)}{z^2-z+0.5}$

Zeroes $\rightarrow$ 0, -1

Poles $\rightarrow$ $0.5+j0.5$, $0.5-j0.5$



<u>Comment</u>: If the s/m is causal, the s/m is stable as the pole lie inside the unit circle.

LTI causal

Eg. An system is specified by the diff. eqn.,

$$y(n) - y(n-1) + \frac{1}{4}y(n-2) = x(n),$$

where $x(n) = 2(1/8)^n u(n)$,

$y(-1) = 2$ and $y(-2) = 4$. Find $y(n)$.

Soln: $ZT \Rightarrow Y(z) - [z^{-1}Y(z) + \overset{2}{y(-1)}] + \frac{1}{4}[z^{-2}Y(z) + z^{-1}\overset{2}{y(-1)} + \overset{4}{y(-2)}]$

$$= X(z) = \frac{2z}{z-1/8}$$

$$\Rightarrow Y(z) = \frac{1-0.5z^{-1}}{1-z^{-1}+\frac{1}{4}z^{-2}} + \frac{2}{1-\frac{1}{8}z^{-1}(1-z^{-1}+\frac{1}{4}z^{-2})}$$

$$= \frac{z(z-0.5)}{(z-0.5)^2} + \frac{2z^3}{(z-\frac{1}{2})^2(z-\frac{1}{8})}$$

Eg. LTI causal system:

$$y(n) - \frac{1}{2}y(n-1) = x(n) + \frac{1}{3}x(n-1). \quad \text{Find } h(n).$$

Soln: $ZT \Rightarrow Y(z) - \frac{1}{2}z^{-1}Y(z) = X(z) + \frac{1}{3}z^{-1}X(z)$

$$\Rightarrow \frac{Y(z)}{X(z)} = \frac{1+1/3\, z^{-1}}{1-1/2\, z^{-1}}$$

$$= (1+1/3\, z^{-1})\left(\frac{1}{1-1/2\, z^{-1}}\right)$$

$$= \frac{1}{1-1/2\, z^{-1}} + \frac{1/3\, z^{-1}}{1-1/2\, z^{-1}}$$

$$\therefore h(n) = (1/2)^n u(n) + 1/3\,(1/2)^{n-1} u(n-1).$$

18-04-2024

Block Representation

1. Direct form 1 → Large complexity → separate i/p and o/p blocks.
2. Direct form 2 → Optimised
3. Cascade
4. Parallel.

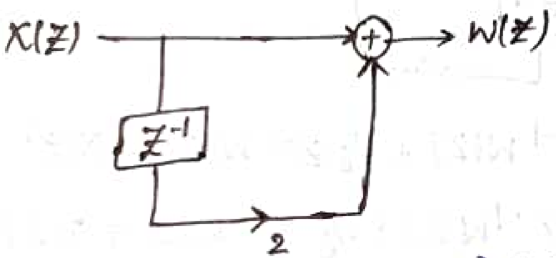
Direct form 1:

$$y(n) - 5/6\, y(n-1) + 1/6\, y(n-2) = x(n) + 2x(n-1).$$

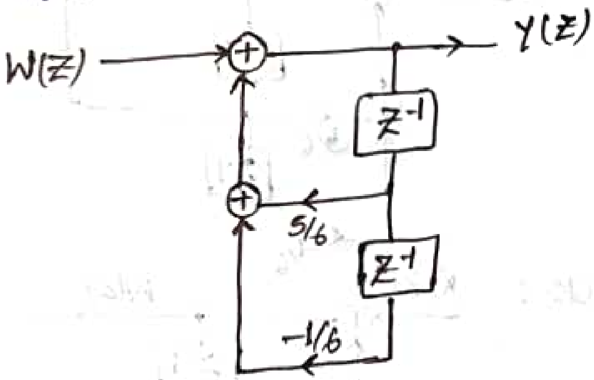
$$\Rightarrow Y(Z) - \frac{5}{6} Z^{-1} Y(Z) + \frac{1}{6} Z^{-2} Y(Z) = \underbrace{X(Z) + 2Z^{-1} X(Z)}_{W(Z)}$$

Delay block: 

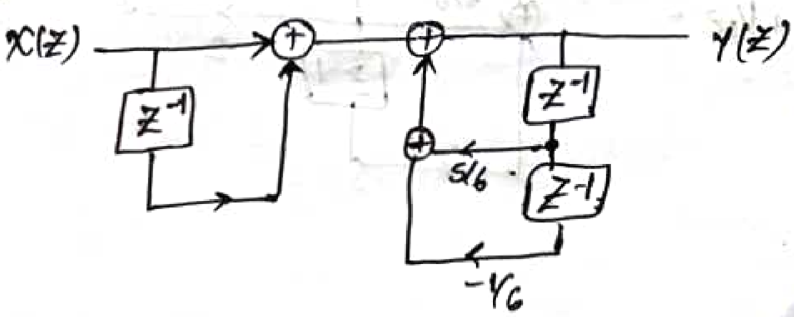
Scaling (gain) block: 



$$Y(Z) = 5/6 \cdot Z^{-1} Y(Z) - 1/6\, Z^{-2} Y(Z) + W(Z)$$



Complete Block:



→ Takes more hardware space
↓ not hardware efficient
→ Required when i/p and o/p are to be separated.

Direct form 2 :

$$H(z) = \frac{1+2z^{-1}}{1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}$$

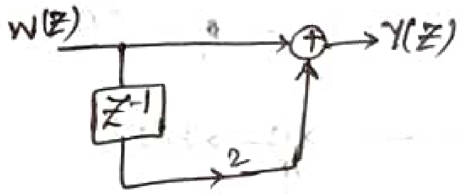
→ Reduces the no. of delay blocks and Increases hardware efficiency.

$$H(z) = \frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)}$$

$$= \frac{1+2z^{-1}}{1-5/6\, z^{-1}+1/6\, z^{-2}}$$

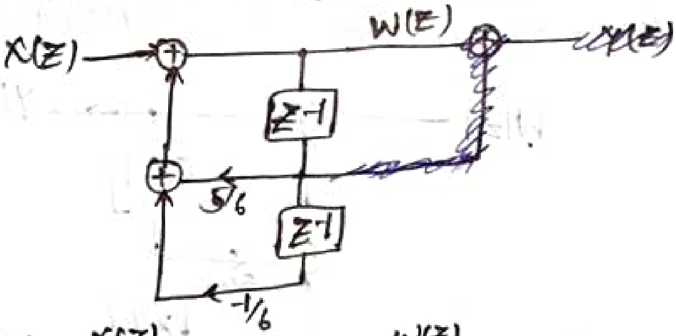
$$\frac{Y(z)}{W(z)} = 1+2z^{-1}, \quad \frac{W(z)}{X(z)} = \frac{1}{1-5/6\, z^{-1}+1/6\, z^{-2}}$$

$$\Rightarrow Y(z) = W(z) + 2z^{-1}W(z)$$



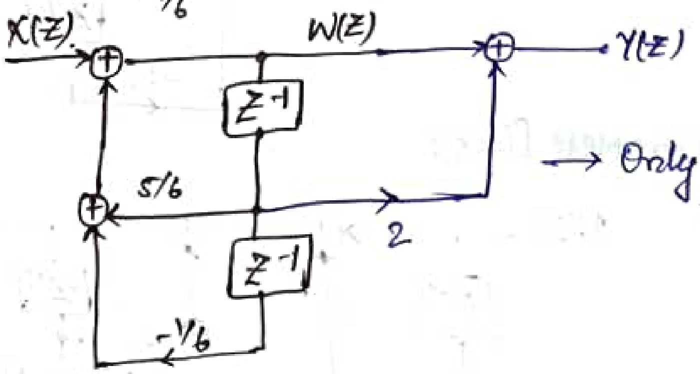
and, $W(z) - 5/6\, z^{-1}W(z) + 1/6\, z^{-2}W(z) = X(z)$

$$W(z) = 5/6\, z^{-1}W(z) - 1/6\, z^{-2}W(z) + X(z)$$



Complete Block :

$$H(z) = \frac{1+2z^{-1}}{1-5/6\, z^{-1}+1/6\, z^{-2}}$$



→ Only 2 delay blocks

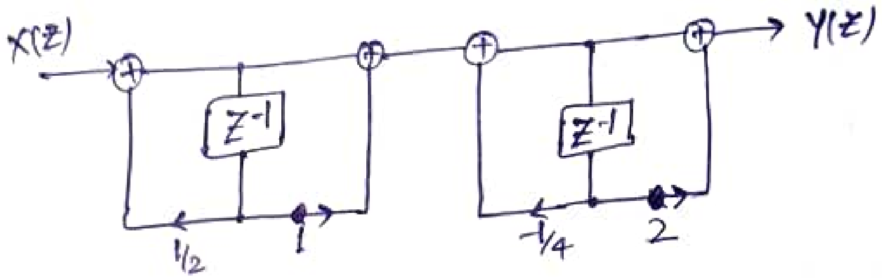
Cascade

$$H(z) = H_1(z)\, H_2(z)\, H_3(z) \quad \rightarrow \text{Decompose } H(z) \text{ into } H_1(z), H_2(z), H_3(z), \ldots$$

Eg. $$H(z) = \frac{1 + 3z^{-1} + 2z^{-2}}{1 - \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}}$$

$$= \frac{(1+z^{-1})(1+2z^{-1})}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{4}z^{-1})} \rightarrow \text{Take any combination}$$

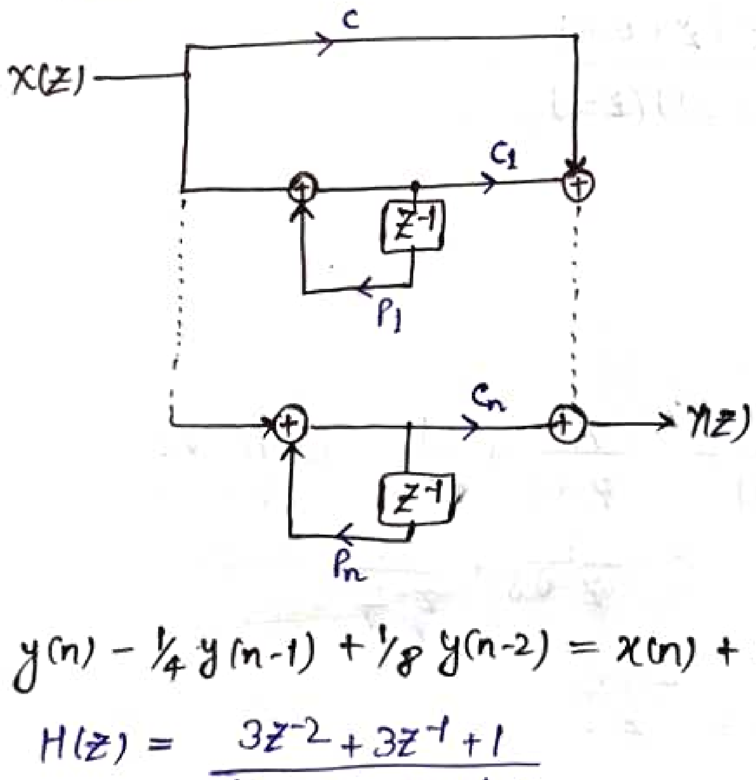
$H_1(z)$ $\quad$ $H_2(z)$



23-04-2024

Parallel Form

$$H(z) = \underbrace{c}_{\text{maybe zero}} + \frac{c_1}{1-P_1 z^{-1}} + \frac{c_2}{1-P_2 z^{-1}} + \ldots + \frac{c_n}{1-P_n z^{-1}}$$



Eg. $y(n) - \frac{1}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n) + 3x(n-1) + 2x(n-2)$.

Soln: $$H(z) = \frac{3z^{-2} + 3z^{-1} + 1}{\frac{1}{8}z^{-2} - \frac{1}{4}z^{-1} + 1}$$

$$\frac{1}{8}z^{-2} - \frac{1}{4}z^{-1} + 1 \,\Big)\, 2z^{-2} + 3z^{-1} + 1 \,\Big(\, 16$$

$$\underline{2z^{-2} - 4z^{-1} + 16}$$

$$7z^{-1} - 15$$

$$\therefore H(z) = 16 + \frac{7z^{-1} - 15}{\frac{1}{8}z^{-2} - \frac{1}{4}z^{-1} + 1}$$

$$= 16 + \frac{8(7z^{-1} - 15)}{z^{-2} - 2z^{-1} + 8}$$

↳ imaginary roots

Eg. For the given transform, determine the time-domain relation for the given conditions.

$$X(z) = \frac{-z(z + 0.4)}{(z - 0.8)(z - 2)}.$$

(a) $|z| > 2$

(b) $|z| < 0.8$

(c) $0.8 < |z| < 2$.

Soln: $$\frac{X(z)}{z} = \frac{-(z + 0.4)}{(z - 0.8)(z - 2)} = \frac{A}{z - 0.8} + \frac{B}{z - 2} \Rightarrow A = 1, \; B = -2$$

$$= \frac{1}{z - 0.8} + \frac{-2}{z - 2}$$

(i) $$X(z) = \frac{z}{z - 0.8} - \frac{2z}{z - 2}$$

$$\Rightarrow x(n) = (0.8)^n u(n) - 2(2)^n u(n)$$

(ii) $$x(n) = -(0.8)^n u(-n - 1) + 2(2)^n u(-n - 1)$$

(iii) $$x(n) = (0.8)^n u(n) + 2(2)^n u(-n - 1).$$

Eg. Find the ~~the~~ $Y(z)$ ~~Z transform~~ for the system $H(z) = \frac{z}{z-0.5}$, having an input $x(n) = (0.8)^n u(n) + (0.6)^n u(-n-1)$.

Sol^n:

$$X(z) = \frac{z}{z-0.8} - \frac{z}{z-0.6}$$

$$Y(z) = H(z)\,X(z)$$

$$= \frac{z^2}{(z-0.5)(z-0.8)} - \frac{z^2}{(z-0.5)(z-0.6)}$$

$$\frac{Y(z)}{z} = \frac{z}{(z-0.5)(z-0.8)} - \frac{z}{(z-0.5)(z-0.6)}$$

$$= \left(\frac{-0.5/0.3}{z-0.5} + \frac{0.8/0.3}{z-0.8}\right) - \left(\frac{-0.5/0.1}{z-0.5} - \frac{0.6/0.1}{z-0.6}\right)$$

$$\Rightarrow Y(z) = \left(-\frac{5/3\,z}{z-0.5} + \frac{8/3\,z}{z-0.8}\right) + \left(\frac{5z}{z-0.5} + \frac{6z}{z-0.6}\right)$$

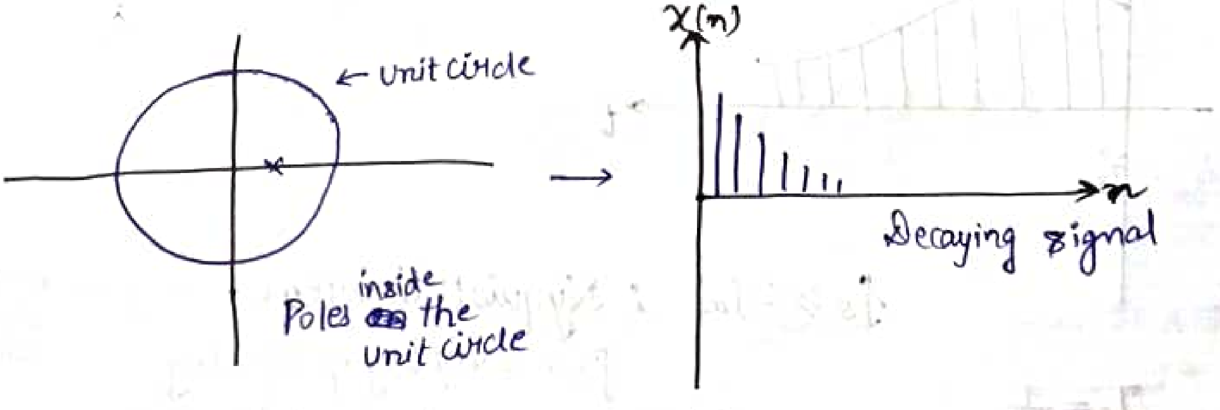
$$y(n) = \left(-\frac{5}{3}(0.5)^n + \frac{8}{3}(0.8)^n\right)u(n) + \left(5(0.5)^n + 6(0.6)^n\right)u(n)$$

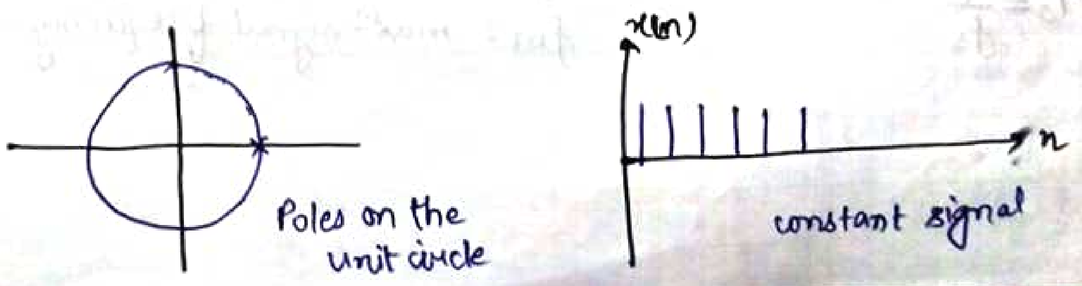
$$= \left[6(0.6)^n + 8/3\,(0.8)^n + 10/3\,(0.5)^n\right]u(n).$$

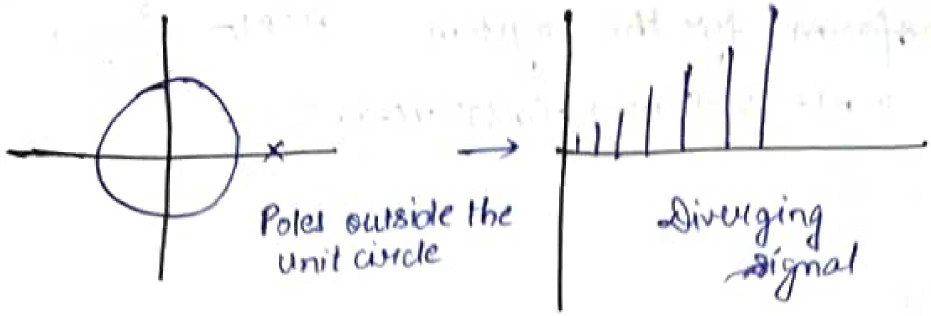
Stability

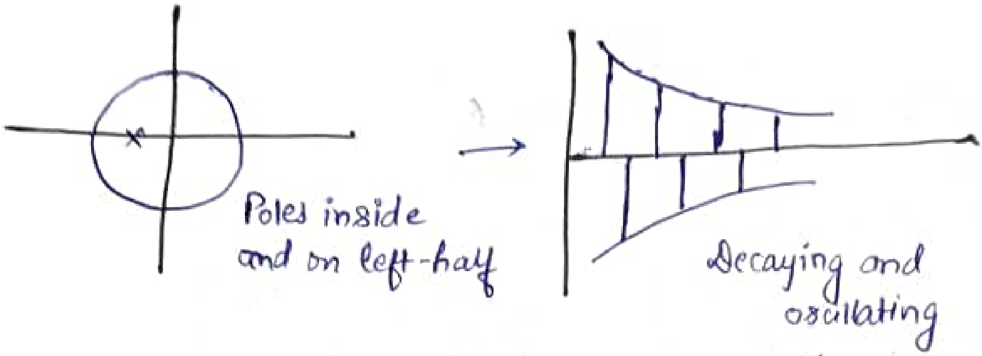
$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

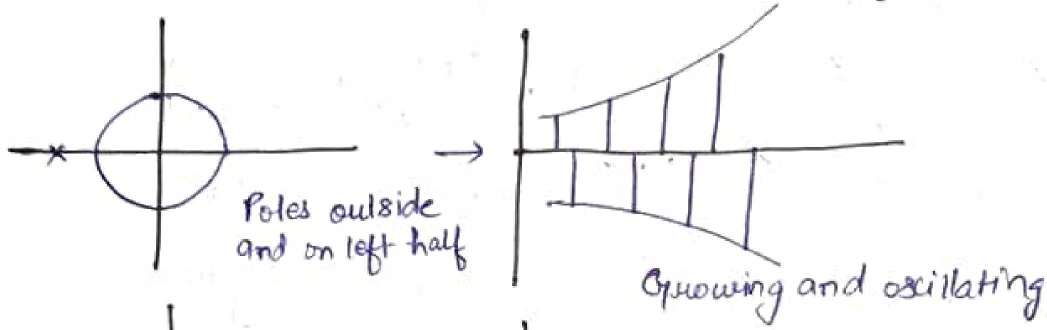
$$\Rightarrow \sum_{n=-\infty}^{\infty} |h(n)\, z^{-n}| < \infty, \quad z = re^{j\omega}$$

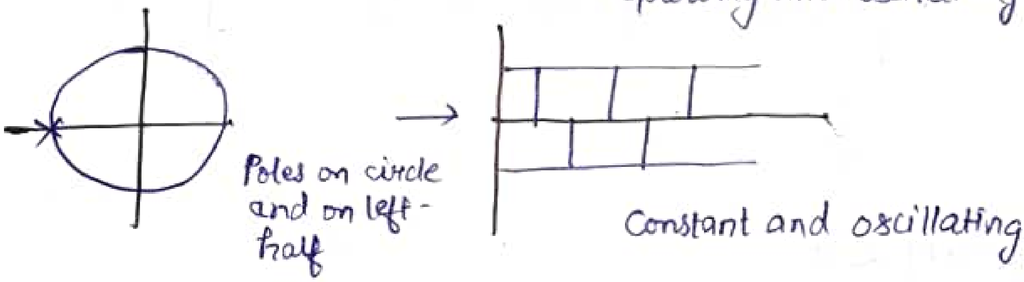




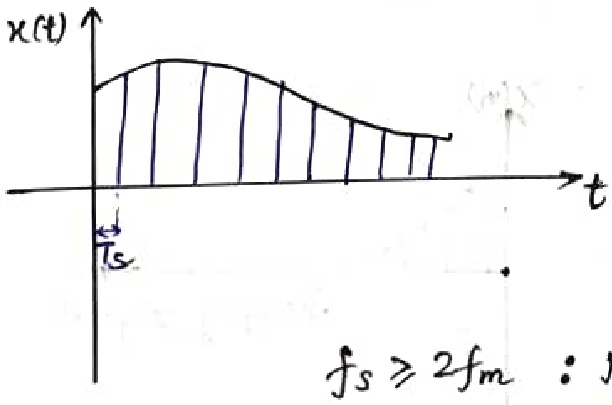








Sampling: Discretizing in time

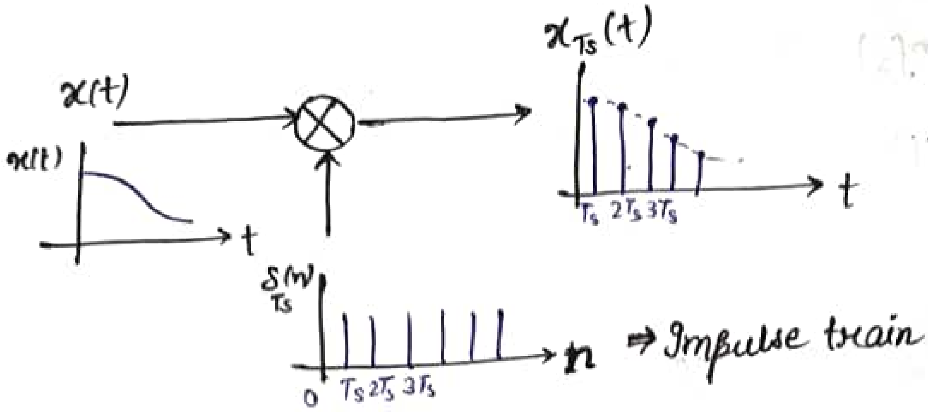


$f_s \geq 2f_m$: Nyquist theorem → gives min^m error in sampling

f_s : sampling frequency

f_m : max^m signal frequency.

$$T_s = \frac{1}{f_s}$$



$$\delta_{T_s} = \sum_{n=0}^{\infty} \delta(t - nT_s)$$

$$x(t)\,\delta(t) = x(0)$$
$$x(t)\,\delta(t-t_0) = x(t_0)$$

<u>Trigonometric F.S.:</u>

$$\delta_{T_s}(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

$$a_0 = \frac{2}{T_s}\int_{T_s} \delta_{T_s}(t)\,dt = \frac{2}{T_s}$$

$$a_n = \frac{2}{T_s}\int_{T_s} \delta_{T_s}(t)\cos \omega_0 t\, dt = 2/T_s$$

$$b_n = 0$$

$$\therefore\ \delta_{T_s}(t) = \frac{2}{T_s}\cdot\frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{T_s}\cos n\omega_s t$$

$$= \frac{1}{T_s} + \sum_{n=1}^{\infty} \frac{2}{T_s}\cos n\omega_s t$$

$$x_s(t) = x(t)\left[\frac{1}{T_s} + \sum_{n=1}^{\infty}\frac{2}{T_s}\cos n\omega_s t\right]$$

$$= \frac{x(t)}{T_s} + \frac{x(t)}{T_s}\cdot 2\cos\omega_s t + \frac{x(t)}{T_s}\cdot 2\cos 2\omega_s t + \frac{x(t)}{T_s}\cdot 2\cos 3\omega_s t + \dots$$

$\downarrow$

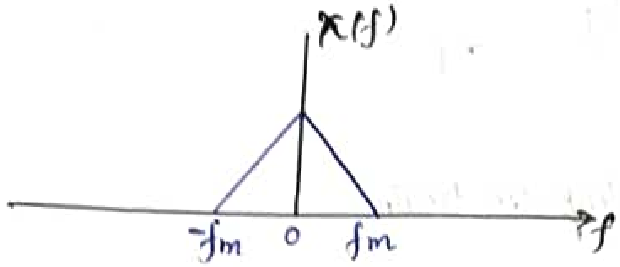
$$X_s(f) = \frac{1}{T_s}\left[X(f) + \{X(f-f_s) + X(f+f_s)\} + \{X(f-2f_s) + X(f+2f_s)\} + \dots\right]$$

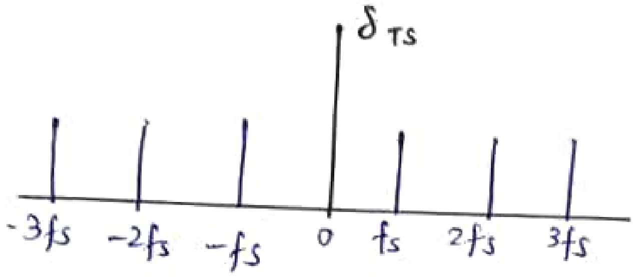
$$\left[\because\ x(t)\cos\omega_0 t \longrightarrow \frac{1}{2}\left[X(f-f_0) + X(f+f_0)\right]\right]$$

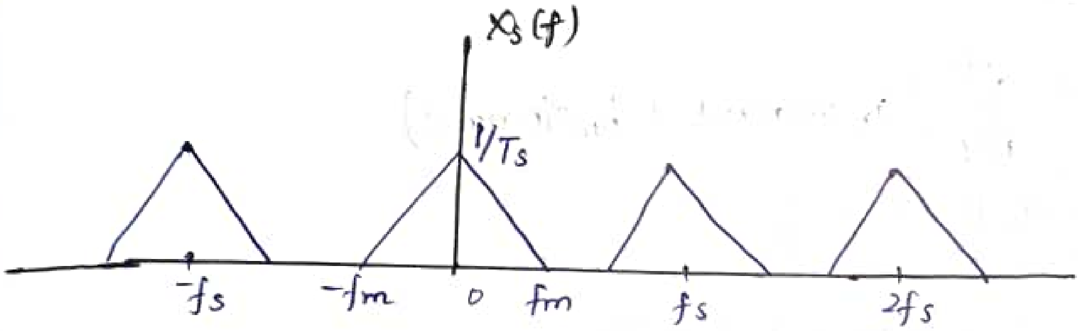
$$\Rightarrow \boxed{X_s(f) = \underbrace{\frac{1}{T_s}}_{(f_s)} \sum_{n=-\infty}^{\infty} X(f-nf_s)}$$

$\rightarrow$ $X_s(f)$ is period f^n of f, consists of superposition of shifted replicas of $X(f)$ scaled by $1/T_s$.

$$X_s(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s).$$







24-04-2024

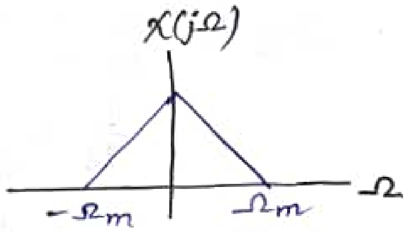
Sampling Theorem:

Any signal $x(t)$ is bandlimited,

$X(j\omega)=0$, $|\Omega| > \Omega_m$ can be completely specified by the samples, if

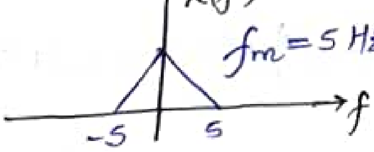
$\Omega_s \geq 2\Omega_m$.

$f_s \geq 2f_m$: Nyquist criteria.

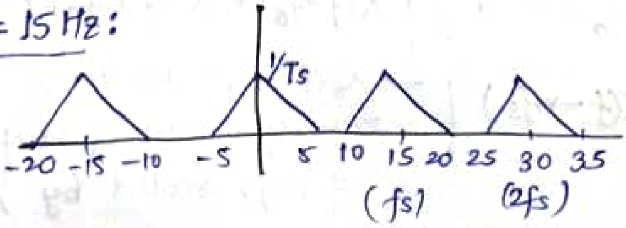


Aliasing:

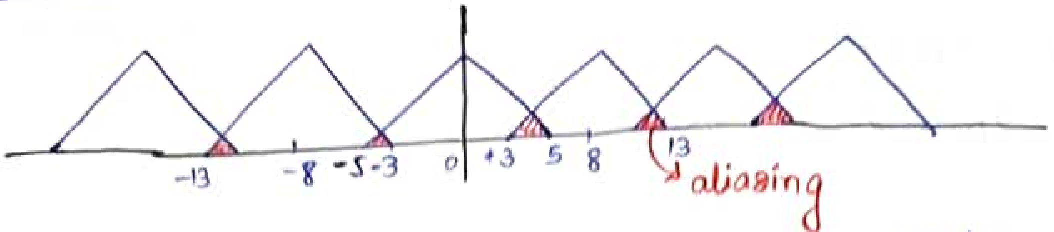
Eg.



$f_s = 15\,Hz$:

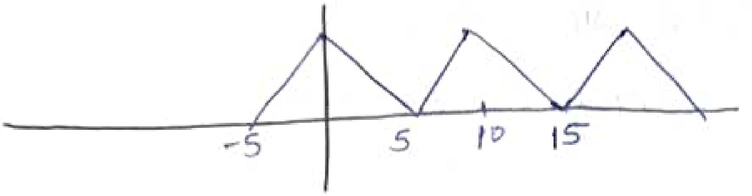


$f_s = 8 Hz$:

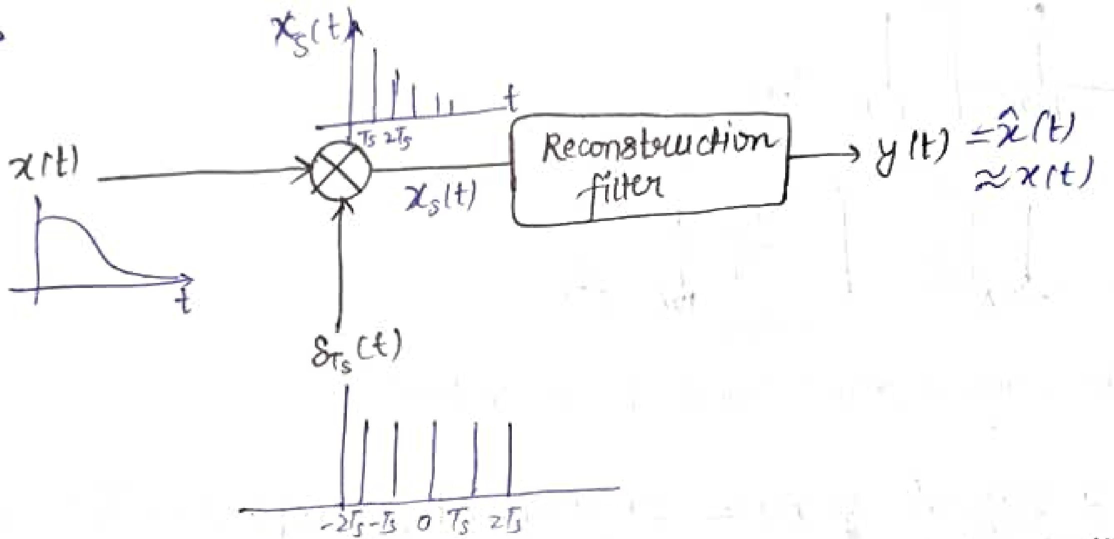


↳ The signal cannot be reconstructed.

$f_s = 10$ Hz: $[f_s = 2f_m]$ → Critically aliased signal

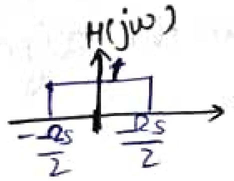


→



Filter: $H(j\Omega) = T$ for $|\Omega| \le \Omega_s/2$

$= 0$ for $|\Omega| > \Omega_s/2$.



Eg. Find the Nyquist train and f_s & f_m.

(i) $x(t) = \text{sinc}^2(200\pi t)$

(ii) $x(t) = \text{sinc}(200\pi t) + \text{sinc}^2(120\pi t)$.

Soln: (i) $f_m = 200 Hz$

$f_s = 400$ Hz

$T_s = 1/400$.

(ii) $f_{1m} = 100, f_{2m} = 120$

$\Rightarrow f_m = 120 = \max(f_{1m}, f_{2m})$

$f_s = 2 \times 120 = 240 \Rightarrow T_s = 1/240$.

(iii) $x(t) = \sin(200\pi t)\cos(400\pi)$.

Soln: $f_{1m} = 100$, $f_{2m} = 200$

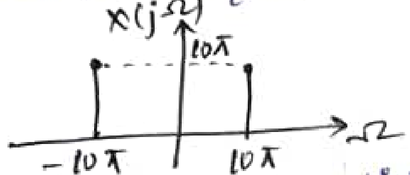
$\Rightarrow f_m = f_{1m} + f_{2m} = 300$

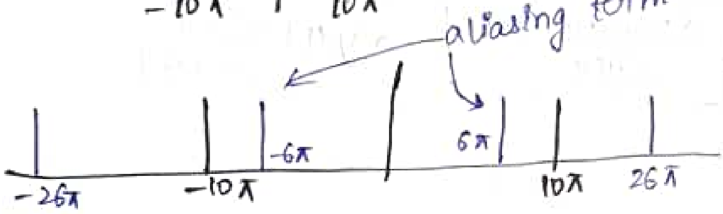
$\Rightarrow f_s = 600$

$\Rightarrow T_s = 1/600$.

Eg. Consider a signal given by $x(t) = 10\cos(10\pi t)$.
It is sampled at a rate of 8 samples/sec, for $|\Omega| \leq 30\pi$.
Can the original signal be recovered.

Soln: FT $\Rightarrow X(j\Omega) = 10\pi\left[\delta(\Omega + 10\pi] + \delta[\Omega - 10\pi]\right]$



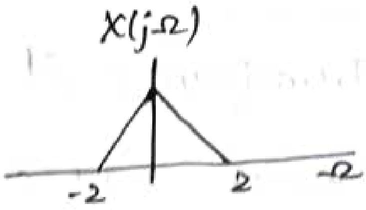


The original signal cannot be reconstructed.

Eg. Consider a signal $x(t) = 2 + \cos 100\pi t$, sampled at $T_s = 0.0125$ sec.
Can the signal be reconstructed. Show the plot.

25-04-2024

Eg. For the given signal spectrum if the sampling rates are specified as follows:



(a) $T = \pi/3$ sec.

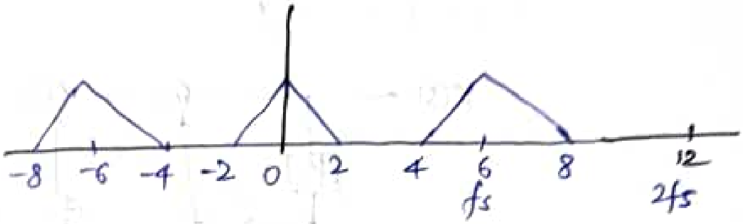
(b) $T = \pi/2$ sec.

(c) $T = 2\pi/3$ sec.

Soln: (1) $\Omega_s = 2\pi f_s$, $T = \pi/3$ sec.

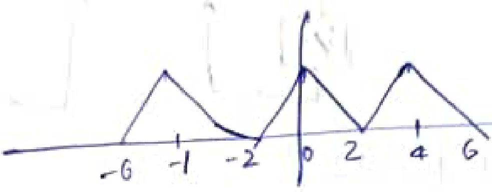
$\Omega_s = 2\pi \times 3/\pi$, $f_s = 3/\pi$.

$= 6$ rad/sec.



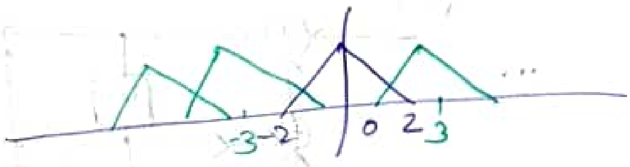
(2) $T = \pi/2$

$\Omega_s = 2\pi \times 2/\pi = 4$ rad./sec.



(3) $T = 2\pi/3$

$\Omega_s = 2\pi \times \frac{3}{2\pi} = 3$ rad/sec.

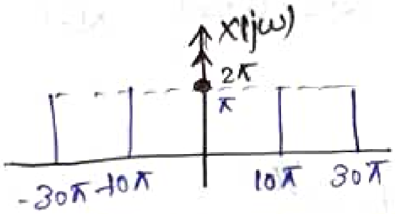


Eg. Consider the signal $x(t) = 1 + \cos(10\pi t) + \cos(30\pi t)$, $T = 0.04$ sec.

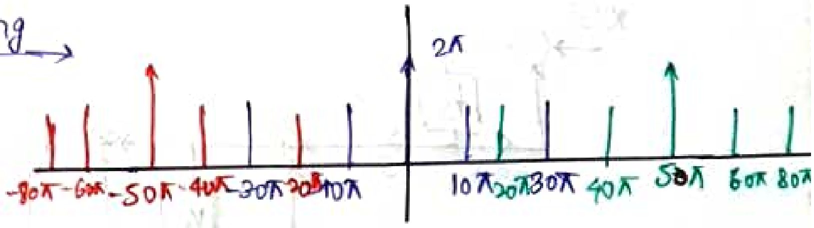
If it is filtered ~~with~~ at ideal sampling, draw the output spectrum.

Soln: FT $\Rightarrow$ $X(j\omega) = 2\pi\delta(\Omega) + \pi[\delta(\Omega + 10\pi) + \delta(\Omega - 10\pi)]$

$+ \pi[\delta(\Omega + 30\pi) + \delta(\Omega - 30\pi)]$.

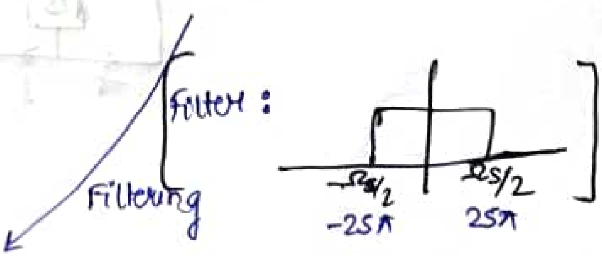


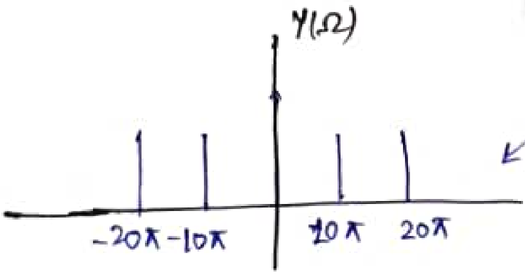
sampling



$T = 0.04$ s, $f_s = 25$ Hz

$\Omega_s = 50\pi$



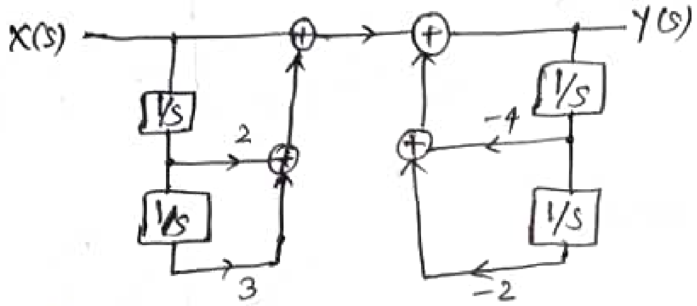


Block Representation

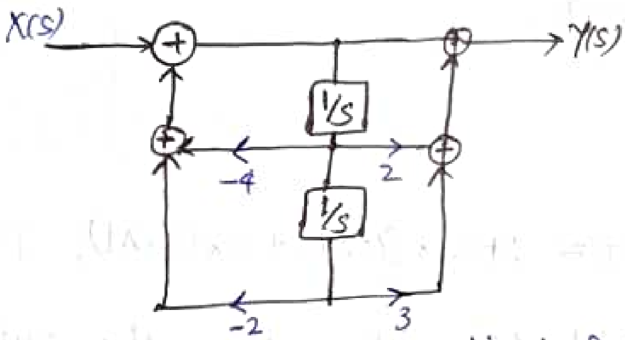
For Laplace Transform: $z^{-1} \rightarrow 1/s$

Eg. $H(s) = \dfrac{s^2+2s+3}{s^2+4s+2} = \dfrac{1+2/s+3/s^2}{1+4/s+2/s^2} = \dfrac{Y(s)}{X(s)}$

Direct-form-1:

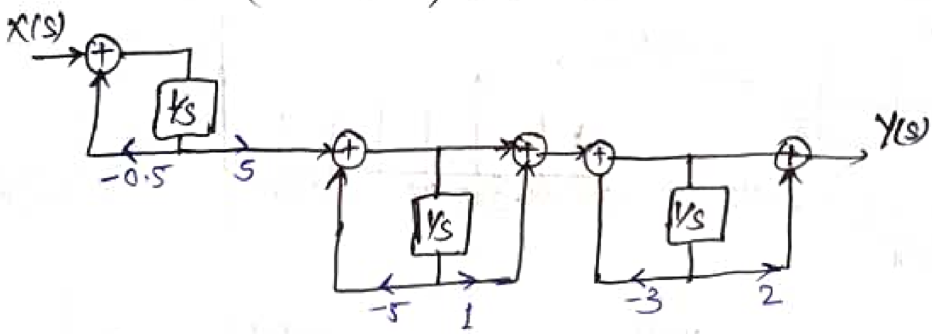


Direct form 2:



→ $H(S) = H_1(S) \cdot H_2(S) \cdots H_n(S)$ → Use direct form −2 for reduced hardware.

Eg. $H(S) = \dfrac{5(s^2+3s+2)}{(s^2+8s+15)(s+0.5)} = \dfrac{5}{(s+0.5)} \cdot \dfrac{(s+1)}{(s+5)} \cdot \dfrac{(s+2)}{(s+3)} = \dfrac{Y(S)}{X(S)}$



29-04-2024

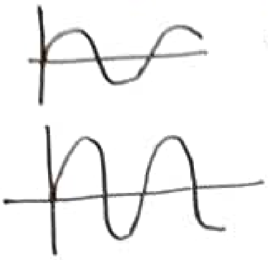
Practical Applications

Distortionless Tx :

$Tx \rightarrow x(t)$

$Rx \; y(t) = K x(t - t_d)$ → time delay

↑ scaled and delayed version of the Tx signal



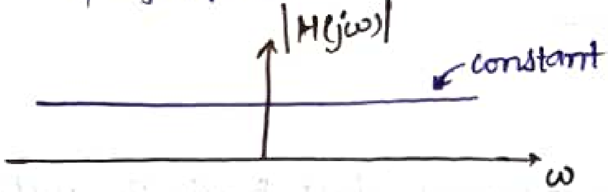
→ Amplitude distortion
→ Phase distortion

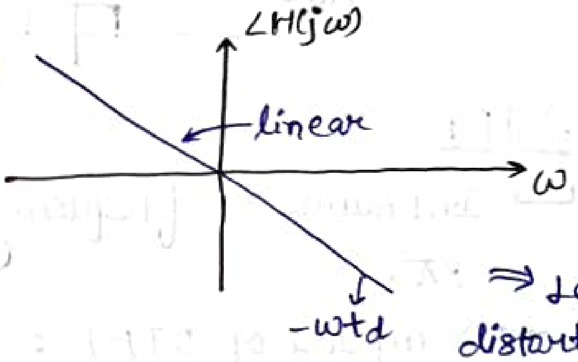
$$Y(j\omega) = K e^{j\omega t_d} X(j\omega)$$

Amplitude :

$$\frac{Y(j\omega)}{X(j\omega)} = H(j\omega) = K \cdot e^{-j\omega t_d}$$

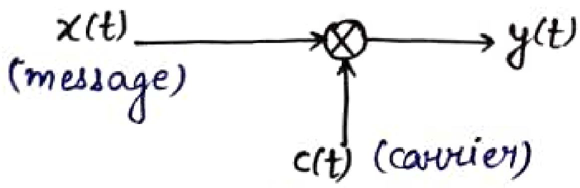
$$|H(j\omega)| = K$$





⇒ Leads to distortionless

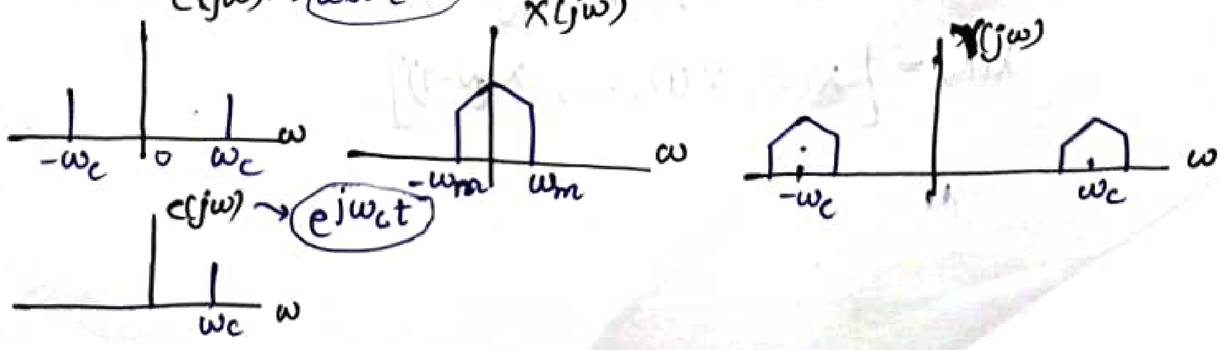
Modulation

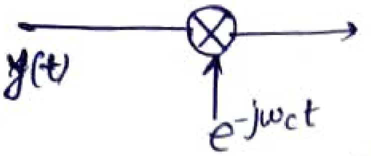


$\hookrightarrow e^{j\omega_c t} / \cos \omega_c t$

$$y(t) = \frac{1}{2\pi} \left[X(j\omega) * C(j\omega) \right]$$

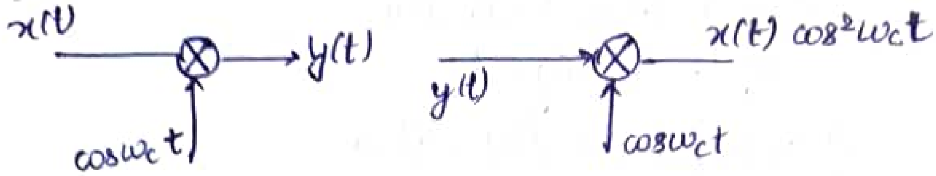
$C(j\omega) \rightarrow \cos \omega_c (t)$





$$y(t) = x(t)\, e^{+j\omega_c t}$$

Eg. $y(t) = x(t)\cos\omega_c t$ → Modulated Rx



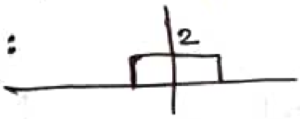
$$\cos^2\omega_c t = \frac{1}{2}\left[1 + \cos 2\omega_c t\right]$$

$$= x(t)\cdot\frac{1}{2}\left[1 + \cos 2\omega_c t\right]$$

$$= \frac{x(t)}{2} + \frac{x(t)}{2}\cos 2\omega_c t$$

(scaled) required ↓ undesirable (high frequency)

Filter :



DTFT

↳ Continuous in frequency domain but periodic with period 2π.

→ Sampling of DTFT : DFT
(Discreatising)

$$\text{DTFT}: X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

↓ Discretising

$$\text{DFT}: X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-\frac{j2\pi K n}{N}}$$

$$X(K) = X(e^{j\omega n})\Big/_{\omega_K = \frac{2\pi K}{N}}, \quad K = 0, 1, \dots, N-1.$$

$$X(K) = \left[X(0), X(1), \dots, X(N-1)\right].$$

Eg $x(n) = \begin{cases} 1/3, & n \le 2 \\ 0, & \text{otherwise.} \end{cases}$

Calculate 4-point DFT.

Soln: $x(n) = \{\underset{\uparrow\, n=0}{\tfrac{1}{3}}, \underset{1}{\tfrac{1}{3}}, \underset{2}{\tfrac{1}{3}}\}$

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi Kn}{N}}$$

$$X(0) = x(0) + x(1) + x(2) = 1$$

$$X(1) = x(0) + x(1)\, e^{-j\frac{2\pi}{4}} + x(2)\, e^{-j\frac{2\pi \times 1 \times 2}{4}} = -j/3$$

$$X(2) = 1/3$$

$$X(3) = \frac{1}{3} \angle \pi/2 .$$

→ If $x(n)$ is of length L, choose $N \geq L$.