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SEQUENCES AND SERIES OF FUNCTIONS

Sequence and Series of Functions

$f_n(x) = x^n, \quad x \in [0,1]$

$f_1(x) = x,$

$f_2(x) = x^2$

$f_3(x) = x^3$

$\vdots$

when $x = 1/3$,

$f_1(1/3) = 1/3$

$f_2(1/3) = (1/3)^2$

$\vdots$

$f_n(1/3) = \left(\frac{1}{3}\right)^n \to 0$ as $n \to \infty$

when $x = 1/4$,

$f_1(1/4) = 1/4$

$f_2(1/4) = (1/4)^2$

$\vdots$

$x = x_0 \in [0,1)$

$f_n(x_0) \to 0$

$f_1(1) \to 1$

For a fixed x, it becomes a sequence of numbers.

Pointwise Convergence

$$f_n(x) \xrightarrow[\text{(pointwise convergence)}]{p} f(x) = \begin{cases} 0, & \text{if } x \neq 1 \\ 1, & \text{if } x = 1 \end{cases}$$

Defn (Pointwise convergence):

Let ~~function~~ $f_n : E \to \mathbb{R}$ be a sequence of functions. Then, f_n is said to converge pointwise to f [if $\forall$ fixed $x \in E$, $\underbrace{f_n(x)}_{a_n} \to \underbrace{f(x)}_{l}$] if for a given $\epsilon > 0$, $\exists\, n(\epsilon, x)$ such that

$$|f_n(x) - f(x)| < \epsilon \quad \forall \quad n \geq n(\epsilon, x).$$

Defn (Uniform Convergence):

A sequence $f_n : E \to \mathbb{R}$ of functions is said to converge uniformly to f if for a given $\epsilon > 0$, $\exists\, \underbrace{n(\epsilon)}_{\text{indep. of } x \in E}$ such that

$$|f_n(x) - f(x)| < \epsilon \quad \forall \quad n \geq n(\epsilon).$$

Eg. Let $E = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{1\,\text{Billion}}\}$

$f_n : E \to \mathbb{R}$

$f_n(x) = x^n$

$|f_n(\ ) - 0| < \epsilon \quad \forall\ n \geq (\quad)$

When $x = 1/2$, $|f_n(\frac{1}{2}) - 0| < \epsilon \quad \forall \quad n \geq n(\epsilon, \frac{1}{2})$

$\vdots$

$|f_n(\frac{1}{1B}) - 0| < \epsilon \ \forall \ n \geq n(\epsilon, \frac{1}{1B})$

$\therefore$ Stage: $\max(\frac{1}{2}, \frac{1}{1B}) = \frac{1}{2}$

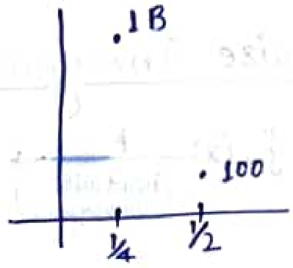
stage $= (\epsilon, \frac{1}{2})$

Remark: If the domain E is finite, then ~~pointwise con~~

pointwise convergence $\Leftrightarrow$ Uniform convergence.

Eg. $f : E = \{\frac{1}{2}, \frac{1}{4}\} \to \mathbb{R}$ if

$f(\frac{1}{2}) = 100$, $f(\frac{1}{4}) = 1B$



Given $\epsilon > 0$, $\exists\ \delta > 0$ such that

$|f(x) - f(x_0)| < \epsilon \ \forall \ |x - x_0| \cap E < \delta$

$\Rightarrow x \in (x_0 - \delta, x_0 + \delta) \cap \{\frac{1}{4}, \frac{1}{2}\}$

$\rightarrow f_n(x) \xrightarrow[\text{(Uniform convergence)}]{u} f(x)$

$\Rightarrow f$ is also continuous.

Eg. $x \in [0,1]$

$f_n(x) = x^n \xrightarrow{x} f(x) = \begin{cases} 0, & \text{if } x \neq 1 \\ 1, & \text{if } x = 1 \end{cases}$

$E = \{\frac{1}{2^i}, i = 0, 1, 2, \dots, 1B\}$

In E, $f_n \xrightarrow{u} f = \begin{cases} 0, & \text{if } x \neq 1 \\ 1, & \text{if } x = 1 \end{cases}$

Eg. $f_n(x) = \frac{1}{1+nx}$, $x \in (0,1)$

$f_n(x) \xrightarrow{P} f(x) = 0 \quad \forall x \in (0,1)$

Suppose $f_n(x) \xrightarrow{u} f(x) = 0$

$\Rightarrow$ For given $\epsilon > 0$ $\forall N \geq N(\epsilon)$ $\forall x \in (0,1)$.

In particular, take $\epsilon = \frac{1}{2}$.

$\Rightarrow \exists N(\frac{1}{2})$ such that

$|f_n(x)| < \frac{1}{2} \quad \forall n \geq N(\frac{1}{2}) \ \forall x (0,1)$

i.e, $\left|\frac{1}{1+nx}\right| < \frac{1}{2}$

Take $n = N(\frac{1}{2})$, $x = \frac{1}{2N(\frac{1}{2})} \in (0,1)$

$$\frac{1}{1 + N(\frac{1}{2}) \times \frac{1}{2N(\frac{1}{2})}} < \frac{1}{2}$$

$\frac{2}{3} < \frac{1}{2} \Rightarrow$ Contradiction

Eg. $f_n(x) = \frac{1}{1+nx}$, $x \in [10^{-100T}, 1)$

$f_n(x) \xrightarrow{u} f(x) = 0 \quad \forall x \in (0,1)$.

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→ $f_n : E \to \mathbb{R}$

$f_n \xrightarrow{p} f$ i.e., $f_n(x) \longrightarrow f(x)$ $\forall x \in E$

Define, $\delta_1 = \max\limits_{x\in E} |f_1(x) - f(x)|$

$\delta_2 = \max\limits_{x\in E} |f_2(x) - f(x)|$

$\vdots$

$\delta_n = \max\limits_{x\in E} |f_n(x) - f(x)|$

<u>Theorem:</u> $f_n \xrightarrow{u} f$ iff $\delta_n \to 0$

<u>Proof:</u> Suppose $f_n \xrightarrow{u} f$ in E

<u>To prove:</u> Given $\epsilon > 0$, $\exists\, N(\epsilon)$ such that

$|\delta_n - 0| < \epsilon \quad \forall n \geq N(\epsilon)$.

$f_n \xrightarrow{u} f$ in E

$\Rightarrow$ Given $\epsilon > 0$, $\exists\, N(\epsilon)$ such that

$|f_n(x) - f(x)| < \epsilon \quad \forall n \geq N(\epsilon)$
$\forall x \in E$

$\Leftrightarrow \max\limits_{x\in E} |f_n(x) - f(x)| < \epsilon$

i.e., $\delta_n < \epsilon \quad \forall n \geq N(\epsilon)$.

Conversely, suppose $\delta_n \to 0$ as $n \to \infty$

$\Rightarrow$ Given $\epsilon > 0$ $\exists\, N(\epsilon)$ such that

$|\delta_n - 0| < \epsilon \quad \forall n \geq N(\epsilon)$

i.e., $\delta_n < \epsilon \quad \forall n \geq N(\epsilon)$

$\Rightarrow \max\limits_{x\in E} |f_n(x) - f(x)| < \epsilon \ \forall n \geq N(\epsilon)$

$\Rightarrow |f_n(x) - f(x)| < \epsilon \ \forall n \geq N(\epsilon)$
$\forall x \in E$

Corollary: Suppose $\underbrace{|f_n(x) - f(x)| \leq a_n}_{\delta_n \leq a_n}$ $\forall\, x \in E$

and $a_n \to 0$ as $n \to \infty$

$\Rightarrow f_n \xrightarrow{u} f$ in E

Cor: Suppose for each n, $\exists\, x_n \in E$ such that

$$|f_n(x_n) - f(x_n)| \geq b_n \;\&\; b_n \not\to 0$$

$$\underbrace{\max_{x \in E} |f_n(x) - f(x)|}_{\delta_n \geq b_n} \geq |f_n(x_n) - f(x_n)|$$

Eg. $f_n(x) = \dfrac{1}{1+nx}$, $x \in (0,1)$

$\xrightarrow{p} f(x) = 0$ $\left[\begin{array}{l}\#\text{ Fix } x \in (0,1). \\ f_n(x) \to 0 \text{ as } n \to \infty\end{array}\right]$

$$|f_n(x) - f(x)| = \frac{1}{1+nx}$$

Fix n and take $x_n = 1/n$.

$$\left|f_n(1/n) - f(1/n)\right| = \frac{1}{2} \quad \forall n \Rightarrow \max|f_n(1/n) - f(1/n)| \geq 1/2$$

Eg $f_n(x) = \dfrac{1}{1+nx}$, $x \in \underbrace{[10^{-18}, 1]}_{E}$

$\xrightarrow{p} f = 0$

$$\delta_n = \max_{x \in I} |f_n(x) - f(x)|$$

$$= \max_{x \in [10^{-18}, 1]} \frac{1}{1+nx} = \frac{1}{1+n \times 10^{-18}}$$

Eg. $f_n(x) = \dfrac{x}{1+nx}$, $x \in (0,1)$

$\xrightarrow{p} f(x) = 0$

$1 + nx > nx$

$$\Rightarrow \frac{x}{1+nx} \leq \frac{x}{nx} \leq \frac{1}{n} \;\&\; \frac{1}{n} \to 0$$

Eg. $f_n(x) = \underbrace{\frac{x^n}{1+x^{2n}}}_{\le x^n \to 0}$, $0 \le x < 1$

For fixed n,

$$\max_{x \in [0,1)} f_n(x) = \frac{1}{2}$$

$$\delta_n = \frac{1}{2} \not\to 0$$

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Eg. Let $p > 1$, $f_n(x) = \frac{nx}{1+n^p x^2}$, $x \in (-\infty, \infty)$ such that

$f_n \xrightarrow{u} f = 0$ iff $p > 2$.

Case-I: $p > 2$

When $x = 0$, $f_n(x) = 0 \longrightarrow f(x) = 0$

Assume $x \neq 0$

$$f_n(x) = \frac{nx}{n^p(n^{-p} + x^2)} = \frac{n^{1-p}\, x}{n^{-p} + x^2}$$

$$\le \frac{n^{1-p}\, x}{x^2} = \frac{n^{1-p}}{x} \to 0$$

$$\delta_n = \max_{x \in (-\infty,\infty)} |f_n(x) - f(x)|$$

$$= \max_{x \in (-\infty,\infty)} \frac{n|x|}{1+n^p x^2}$$

$$\le \max_{x \in (-\infty,\infty)} \frac{n\cancel{|x|}}{2n^{p/2}\cancel{|x|}}$$

$$= \max_{x \in (-\infty,\infty)} \frac{1}{2n^{(p/2-1)}} \to 0 \text{ as } n \to \infty.$$

$a^2 + b^2 \ge 2ab$

$1 + n^p x^2 = 1 + (n^{p/2} x)^2$

$\ge 2 \cdot n^{p/2} |x|$

case: $1 < p \le 2$

Take $x_n = \frac{1}{n}$

$$|f_n(x_n) - \underset{0}{\underset{\shortparallel}{f(x_n)}}| = \frac{1}{1+n^{p-2}} \ge \frac{1}{2}$$

→ Suppose $f_n \xrightarrow{u} f$ in E, $g_n \xrightarrow{u} g$ in E.

Then $f_n + g_n \xrightarrow{u} f+g$ in E.

Proof: Let $\epsilon > 0$ be given.

Since $f_n \xrightarrow{u} f$ in E, $\exists$ $N_1(\epsilon)$ such that

$$|f_n(x) - f(x)| < \epsilon \ \forall \ n \geq N_1(\epsilon) \ \forall \ x \in E$$

$||^{ly}$, since $g_n \xrightarrow{u} g$ in E, $\exists$ $N_2(\epsilon)$ s.t.

$$|g_n(x) - g(x)| < \epsilon \ \forall \ n \geq N_2(\epsilon) \ \forall \ x \in E.$$

Take $N(\epsilon) = \text{Max}\{N_1(\epsilon), N_2(\epsilon)\}$

Then,

$$|f_n(x) + g_n(x) - f(x) - g(x)|$$
$$= |(f_n(x) - f(x)) + (g_n(x) - g(x))|$$
$$\leq |f_n(x) - f(x)| + |g_n(x) - g(x)|$$

→ $f_n \xrightarrow{u} f$ in E, $g_n \xrightarrow{u} g$ in E

$$\not\Rightarrow f_n \cdot g_n \xrightarrow{u} f \cdot g$$

Eg. Define $f: E \to \mathbb{R}$ be unbounded.

$$\left(\begin{array}{l} f:(0,1) \to \mathbb{R} \\ f(x) = \frac{1}{x} \end{array} \right)$$

Define $f_n(x) = f(x) + 1/n$

Then, $f_n(x) \xrightarrow{P} f(x)$ & $f_n \xrightarrow{u} f$

Define $g_n(x) = f(x) + \frac{1}{n}$

Then, $g_n \xrightarrow{P} f(x)$ & $g_n \xrightarrow{u} f$

Let $h_n(x) = f_n(x) g_n(x) = (f + \frac{1}{n})(f + \frac{1}{n})$

$$h(x) = f(x) g(x) = f(x)^2 \quad (\because g(x) = f(x))$$

$$|h_n(x) - h(x)| = |\cancel{f^2} + 2\frac{1}{n} f + \frac{1}{n^2} - \cancel{f^2}|$$
$$= \left| \frac{2f(x)}{n} + \frac{1}{n^2} \right|$$
$$\geq \frac{2|f(x)|}{n} - \frac{1}{n^2}$$

Since f is unbounded, $\exists\ x_n \in E$ such that

$$f_n(x_n) \geq n.$$

$$|h_n(x_n) - h(x_n)| \geq \frac{2|f(x_n)|}{n} - \frac{1}{n^2}$$

$$\geq 2 \cdot \frac{n}{n} - \frac{1}{n^2}$$

$$\geq 2 - \frac{1}{n^2}$$

Theorem: If $f_n \xrightarrow{u} f$ in E and each f_n is continuous at $x_0 \in E$ then f is also continuous at x_0.

Proof: Let $\epsilon > 0$ be given.

Since $f_n \xrightarrow{u} f$, $\exists\ N(\epsilon)$ s.t.

$$|f_n(x) - f(x)| < \frac{\epsilon}{3} \quad \forall\ n \geq N(\epsilon), \ \forall\ x \in E$$

In particular,

$$|f_{N(\epsilon)}(x) - f(x)| < \epsilon/3 \quad \forall\ x \in E$$

In particular,

$$|f_{N(\epsilon)}(x_0) - f(x_0)| < \epsilon/3$$

Continuity of $f_{N(\epsilon)}$ at x_0

$\Rightarrow \exists\ \delta > 0$ s.t.

$$|f_{N(\epsilon)}(x) - f_{N(\epsilon)}(x_0)| < \epsilon/3 \quad \forall\ |x - x_n| < \delta$$

Now,

$$|f(x) - f(x_0)| \leq |f(x) - f_{N(\epsilon)}(x)| \leq \epsilon/3$$
$$+ |f_{N(\epsilon)}(x) - f_{N(\epsilon)}(x_0)| \leq \epsilon/3$$
$$+ |f_{N(\epsilon)}(x_0) - f(x_0)| \leq \epsilon/3$$

Series of Function

$$\sum_{n=1}^{\infty} f_n(x)$$

Let $S_1(x) = f_1(x)$

$S_2(x) = f_1(x) + f_2(x)$

$\vdots$

$S_n(x) = f_1(x) + \dots + f_n(x)$

If $S_n(x) \xrightarrow{p} S(x)$

Cauchy (कॉशी) theorem:

(a_n) converges if for $\epsilon>0$, $\exists\, N(\epsilon)$ such that

$$|a_n - a_m| < \epsilon \quad \forall\, n,m \geq N(\epsilon).$$

Cauchy Criteria:

A sequence of function f_n converges uniformly if given $\epsilon>0$, $\exists\, N(\epsilon)$ such that

$$|f_n(x) - f_m(x)| < \epsilon \quad \forall\, m,n \geq N(\epsilon)$$
$$\forall\, x \in E$$

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$\rightarrow \sum_{n=1}^{\infty} u_n(x)$

$S_1(x) = u_1(x)$

$S_2(x) = u_1(x) + u_2(x)$

$\vdots$

$S_n(x) = \sum_{i=1}^{n} u_i(x)$

$S_n(x) \longrightarrow u \Leftrightarrow \sum u_n \rightarrow u$

Theorem (Weierstrass):

Let $\sum u_n$ be a series of function.
Suppose $\exists\, M_n$ such that

(i) $|u_n(x)| \leq M_n \quad \forall\, x \in E$

(ii) $\sum M_n$ converges.

Then, $\sum u_n$ converges uniformly.

__Proof:__

It is enough to show that

$$|S_n(x) - S_m(x)| < \epsilon \quad \forall\, n \ge N(\epsilon)$$

where $S_n(x) = \sum_{i=1}^{n} u_i(x)$

Let $a_n = \sum_{i=1}^{n} M_i = M_1 + M_2 + \dots + M_n$

Now, assume $n > m$.

$$\begin{aligned} |S_n(x) - S_m(x)| &= |u_{m+1}(x) + \dots + u_n(x)| \\ &\le |u_{m+1}(x)| + \dots + |u_n(x)| \\ &\le M_{m+1} + \dots + M_n \\ &= |a_n - a_m| < \epsilon \quad \forall\, n, m \ge N(\epsilon). \end{aligned}$$ (by Cauchy's theorem)

__Recall, Ratio test:__

Consider $\sum a_n$, if $\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$

then, $\sum a_n$ converges.

Eg. $\sum_{n=1}^{\infty} x^n$ converges for $0 \le x_0 \le 1$

Eg $\sum_{n=1}^{\infty} a_n x^n$

Fix x_0, $\sum_{n=1}^{\infty} a_n x_0^n$

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{a_{n+1}\, x_0^{n+1}}{a_n\, x_0^n} \right|$$

$$\Rightarrow \frac{1}{R} = \lim \left|\frac{a_{n+1}}{a_n}\right| |x_0|$$

$$\Rightarrow \lim \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{R}|x_0| < 1, \text{ for convergence}$$

$$\Rightarrow |x_0| < R$$

Eg for what values of x, the convergence of $\sum a_n x^n$ is uniform?

Ans: For any $r < R$, then (Here $\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$)

$\sum a_n x^n$ converges uniformly for $|x| < r$

Take $u_n(x) = a_n x^n$

$$\sum u_n(x)$$

$$|u_n(x)| = |a_n x^n| \le |a_n x|^n$$

Eg. Show that $\sum \frac{(-1)^n}{\sqrt{n}} \sin\left(\frac{x}{n}\right)$, $x \in \underset{\text{bounded set}}{E}$, converges uniformly.

Soln: Take $u_n(x) = \frac{(-1)^n}{\sqrt{n}} \sin\left(\frac{x}{n}\right)$

$$|u_n(x)| \le \frac{1}{\sqrt{n}} \cdot \left|\frac{x}{n}\right|$$

$$\le \frac{1}{n^{3/2}} \cdot K$$

$$\le \frac{K}{n^{3/2}}$$

Theorem: $\sum |u_n(x)|$ converges uniformly $\Rightarrow \sum u_n(x)$ converges uniformly.

Proof: Let $S_n(x) = u_1(x) + \dots + u_n(x)$

$$K_n(x) = |u_1(x)| + \dots + |u_n(x)|$$

Now, assume $n > m$,

then $|S_n(x) - S_m(x)| = |u_{m+1}(x) + \dots + u_n(x)|$

$$\le |u_{m+1}(x)| + \dots + |u_n(x)|$$

$$= K_n(x) - K_m(x)$$

Since $K_n(x)$ converges uniformly,

given $\epsilon > 0$, $\exists N(\epsilon)$ such that

$$K_n(x) - K_m(x) < \epsilon \quad \forall n, m \ge N(\epsilon)$$

$$\Rightarrow |S_n(x) - S_m(x)| < \epsilon \quad \forall n, m \ge N(\epsilon).$$

Theorem: Let $g : [0,1] \to \mathbb{R}$ be a continuous function defined by

$$f_n(x) = x^n g(x)$$

Then, f_n converges uniformly iff $g(1) = 0$.

Proof:

Suppose $f_n \xrightarrow{u} f(x) = \begin{cases} 0, & \text{if } x \in [0,1) \\ g(1), & \text{if } x = 1. \end{cases}$

For function to be continuous, $g(1) = 0$.

Eg. If $f_n(x) \xrightarrow{p} f(x)$ in $[a,b]$. can we conclude

$$\int_a^b f_n(x)dx \longrightarrow \int_a^b f(x)dx \ ?$$

Ans: No.

↳ Eg. $f_n(x) = n(n+1)x^{n-1}(1-x)$, $0 \le x \le 1$

Fix $x_0 \in (0,1)$.

$$\underbrace{n(n+1)x_0^{n-1}(1-x_0)}_{a_n}$$

$\sum a_n$

such $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)(n+2)x_0^n(1-x_0)}{n(n+1)x_0^{n-1}(1-x_0)}$

$$= \lim_{n\to\infty} \frac{n-2}{n} x_0$$

$$= x_0 < 1$$

$$\int_0^1 f_n(x)dx = \int_0^1 n(n+1)(x^{n-1} - x^n)dx$$

$$= n(n+1)\left(\frac{x^n}{n} - \frac{x^{n+1}}{n+1}\right)_0^1$$

$$= 1$$

and, $\int_0^1 f(x)dx = 0$.

Theorem:

If $f_n \xrightarrow{u} f$ in $[a,b]$,

then $\underbrace{\int_a^b f_n(x)dx}_{a_n} \longrightarrow \underbrace{\int_a^b f(x)dx}_{l}$.

Proof:- $\left|\int_a^b f_n(x)dx - \int_a^b f(x)dx\right|$

$$\le \int_a^b |f_n(x) - f(x)|dx$$

$$\le \int_a^b \frac{\epsilon}{b-a}dx \quad \forall\, n \ge N(\epsilon)$$

$$\forall\, x \in E$$

$|f_n(x) - f(x)| < \epsilon$

" $< \frac{\epsilon}{b-a}$

Cor. : If $\sum u_n(x) \xrightarrow{u} u$ and $u_n(x)$ is integrable.

Then $\sum u_n(x)$ is also integrable.

$$\sum_{n=1}^{\infty} \int_a^b u_n(x)\,dx = \int_a^b \sum_{n=1}^{\infty} u_n(x)\,dx$$

Proof:

Let $S_n(x) = \sum_{i=1}^{n} u_i(x)$

$$\xrightarrow{u} u = \sum_{n=1}^{\infty} u_n(x)$$

$$\int_a^b S_n(x)\,dx \longrightarrow \int_a^b u(x)\,dx$$

$$\Rightarrow \lim_{n\to\infty} \int_a^b S_n(x)\,dx = \int_a^b u(x)\,dx$$

$$\Rightarrow \lim \left[\int_a^b \sum_{i=1}^{n} u_i(x)\,dx \right] = \int_a^b \sum_{n=1}^{\infty} u_n(x)\,dx$$

$$\Rightarrow \lim_{n\to\infty} \left(\sum_{i=1}^{n} \int_a^b u_i(x)\,dx \right) = \int_a^b \sum_{n=1}^{\infty} u_n(x)\,dx .$$

→ In general,

$$\sum_{n=1}^{\infty} \int_a^b u_n(x)\,dx \neq \int_a^b \sum_{n=1}^{\infty} u_n(x)\,dx$$

Eg. Define

$$f_n(x) = n x e^{-nx^2}, \quad 0 \le x \le 1.$$

$$u_n(x) = f_n(x) - f_{n-1}(x)$$

claim: $\displaystyle \sum_{n=1}^{\infty} \int_0^1 u_n(x)\,dx \neq \int_0^1 \sum_{n=1}^{\infty} u_n(x)\,dx.$

$$\begin{aligned} S_n(x) &= \sum_{i=1}^{n} u_i(x) \\ &= u_1 + u_2 + \dots + u_n \\ &= (f_1 - f_0) + (f_2 - f_1) + \dots + (f_n - f_{n-1}) \\ &= f_n - f_0 \\ &= f_n \quad (\because f_0(x) = 0) \end{aligned}$$

$$\lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} f_n(x) = 0$$

$$\text{i.e., } \sum_{n=1}^{\infty} u_n(x) = 0$$

$$\therefore \int_0^1 \left(\sum_{n=1}^{\infty} u_n(x)\right) dx = 0$$

$$\sum_{n=1}^{\infty} \int_0^1 u_n(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \int_0^1 u_i(x)\,dx$$

$$= \lim_{n\to\infty} \int_0^1 \sum_{i=1}^{n} u_i(x)\,dx$$

$$= \lim_{n\to\infty} \int_0^1 S_n(x)\,dx$$

$$= \lim_{n\to\infty} \int_0^1 f_n(x)\,dx$$

$$= \frac{1}{2}$$

Theorem:

If ① $f_n' \xrightarrow{u} f'$ in $[a,b]$
and ⓘⓘ $f_n(x_0) \xrightarrow{p} f(x_0)$ for some $x_0 \in [a,b]$
then $f_n \xrightarrow{u} f$ in $[a,b]$.

↳ converse need not be true.

Eg. $f_n(x) = \dfrac{\sin(nx)}{\sqrt{n}}$, $-\infty < x < \infty$

$\xrightarrow{p} f(x) = 0 \ \forall x$.

$$\delta_n = \max_{x \in \mathbb{R}} |f_n(x)|$$

$$= \max_{x \in \mathbb{R}} \left| \frac{\sin(nx)}{\sqrt{n}} \right|$$

$$\leq \frac{1}{\sqrt{n}} \to 0$$

$\Rightarrow f_n \xrightarrow{u} f = 0 \quad \Rightarrow f'(x) = 0$

$f_n'(x) = \sqrt{n}\cos(nx) \not\to 0 \ \forall x$.

Eg. $f_n(x) = \dfrac{x^n}{n}$, $0 \le x \le 1$

Show that:
(i) $f_n \xrightarrow{u} f \to$ (diff) f^n
(ii) $f_n'(x) \xrightarrow{p} g(x)$
$g(x) \neq f'(x)$
$\forall x \in [0,1]$.

Soln: (i) $\delta_n = \max_{x \in [0,1]} \left| \dfrac{x^n}{n} \right| \le \dfrac{1}{n} \to 0$

$\therefore f_n \xrightarrow{u} f = 0 \ \forall x \in [0,1]$
$\Leftrightarrow f'(x) = 0$

(ii) $f_n'(x) = x^{n-1} \xrightarrow{p} g(x) = \begin{cases} 0, & \forall x \in [0,1) \\ 1, & \text{if } x = 1. \end{cases}$

Tutorial-1

① $f_n(x) = \frac{x}{x+n}$, $x \in (0,\infty)$

$\epsilon = 0.1$

$|f_n(x)| < \epsilon$

$\Rightarrow \frac{x}{x+n} < \epsilon$

$\Rightarrow \frac{x+n}{x} > \frac{1}{\epsilon}$

$\Rightarrow 1 + \frac{n}{x} > \frac{1}{0.1} = 10$

$\Rightarrow \frac{n}{x} \geq 9$

$\Rightarrow n \geq 9x$

(i) $x = 10 \Rightarrow n = N(\underset{x}{10}, \underset{\epsilon}{0.1}) = 9 \times 10$
$= 90$

(ii) $x = 100 \Rightarrow n = N(100, 0.1) = 9 \times 100$
$= 900$

④ (i) $f_n(x) = n^2 x e^{-nx}$, $0 \leq x \leq 1$

$\xrightarrow{p} f(x) = 0 \ \forall x$

$\delta_n = \underset{x\in[0,1]}{Max} |n^2 x e^{-nx}| \leq a_n \to 0$

~~δ_n~~ $f(x_n) \geq b_n$, $b_n \not\to 0$

Take $x_n = \frac{1}{n}$

$f_n\left(\frac{1}{n}\right) = \frac{n}{e}$

$\therefore \delta_n = \underset{x\in[0,1]}{Max} |f_n(x)| \geq f_n\left(\frac{1}{n}\right)$
$= \frac{n}{e} \to \infty$

$\Rightarrow f_n \overset{u}{\not\to} f$

(ii) $f_n(x) = x^n(1-x)^n$, $0 \leq x \leq 1$

$\xrightarrow{p} 0$

$\delta_n = \underset{x\in[0,1]}{Max} |x^n(1-x)^n| = \left(\frac{1}{4}\right)^n \to 0$

(8) $\sum u_n(x)$, $0<x<1$

$$u_n(x) = \frac{x}{((n-1)x+1)(nx+1)}$$

$$= \frac{1}{nx+1-x} - \frac{1}{nx+1}$$

$$S_n(x) = u_1(x) + u_2(x) + \dots + u_n(x)$$

$$= \left(1 - \cancel{\frac{1}{x+1}}\right) + \left(\cancel{\frac{1}{x+1}} - \cancel{\frac{1}{2x+1}}\right) + \dots + \left(\cancel{\frac{1}{nx+1-x}} - \frac{1}{nx+1}\right)$$

$$= 1 - \frac{1}{nx+1}$$

$$= \frac{nx}{nx+1}$$

$$S_n(x) \xrightarrow{p} f(x) = 1$$

$$\delta_n = \underset{x\in(0,1)}{\text{Max}} |S_n(x) - 1|$$

$$= \underset{x\in(0,1)}{\text{Max}} \left|\frac{nx}{nx+1} - 1\right|$$

$$= \underset{x\in(0,1)}{\text{Max}} \left|\frac{1}{1+nx}\right|$$

$$\therefore u_n(x) \overset{u}{\not\to}$$

(12) $\sum \frac{(-1)^{n-1}}{n+x^2}$, $x \in \mathbb{R}$

Fix $x = x_0$, $\sum \frac{(-1)^{n-1}}{n+x_0^2} \xrightarrow{P}$ converges

$$\sum \left|\frac{(-1)^{n-1}}{n+x^2}\right| = \sum \frac{1}{n+x^2}$$

Take $x=0$, $\sum \frac{1}{n} \overset{P}{\not\to}$ converges.

(18) (i) $f_n(x) = x^n$, $x \in [0,1]$

$$\xrightarrow{p} f(x) = \begin{cases} 0, & \text{if } x \in [0,1) \\ 1, & \text{if } x = 1 \end{cases}$$

(ii) $f_n(x) = e^{-nx^2}$

$$\xrightarrow{p} f(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{otherwise} \end{cases}$$