AV224 - Control System
Assignment - 2

SAURABH KUMAR
SC22B146

(1) The open-loop transfer function of a conditionally stable LTI system is given by,

$$G(s)H(s) = \frac{K(s+1)}{s(s-1)(s^2+4s+16)}.$$

Draw the root locus plot for the closed loop system by applying all the relevant rules and find the range of gain, K, for stability.

Soln: Open-loop poles: $s = 0,\ 1,\ \frac{-4 \pm \sqrt{16-64}}{2} = -2 \pm j2\sqrt{3} = -2 \pm j3.46$

Open-loop zeros: $s = -1$

$\therefore$ no. of open-loop poles, $n = 4$

no. of open-loop zeros, $m = 1$

$\therefore$ no. of root loci $= \cancel{n+m}\ 4\ [= \max[n,m]]$

Asymptotes: $[n-m = 3$ asymptotes$]$

$$\text{Centroid, } -\sigma_A = \frac{[0+(+1)+(-2)+(-2)] - [(-1)]}{n-m}$$

$$= -\frac{2}{3} = -0.67$$

$$\text{Angle, } \phi_A = \frac{(2q+1)}{n-m} \times 180^\circ,\quad q = 0, 1, 2.$$

$$= \frac{2q+1}{3} \times 180^\circ,\quad q = 0, 1, 2.$$

$$= 60^\circ, 180^\circ, 300^\circ.$$

Intersection with imaginary axis:

Characteristic equation: $1 + G(s)H(s) = 0$

$$\Rightarrow s(s-1)(s^2+4s+16) + K(s+1) = 0$$

$$\cancel{\Rightarrow s^3 + s^2(16-4)}$$

$$\Rightarrow s^4 + s^3(4-1) + s^2(16-4) + s(-16) + Ks + K = 0$$

$$\Rightarrow s^4 + 3s^2 + 12s^2 - (16 - K)s + K = 0$$

Routh stability criteria:

$$
\begin{array}{lllll}
s^4 & : & 1 & 12 & K \\
s^3 & : & 3 & (K-16) & \\
s^2 & : & \frac{36-K+16}{3}=\left(\frac{52-K}{3}\right) & K & \\
s^1 & : & \dfrac{\left(\frac{52-K}{3}\right)(K-16)-3K}{\frac{52-K}{3}} & & \\
s^0 & : & K & &
\end{array}
$$

For stability,

$$\frac{52-K}{3} > 0 \quad \Rightarrow K < 52, \quad -(i)$$

$$\frac{\left(\frac{52-K}{3}\right)(K-16)-3K}{\frac{52-K}{3}} > 0$$

$$\Rightarrow \frac{(52-K)(K-16)-9K}{52-K} > 0$$

$$\Rightarrow \frac{52K-832-K^2+16K-9K}{52-K} > 0$$

$$K = \frac{59 \pm \sqrt{59^2-4\times 832}}{2} = 23.32,\ 35.68$$

$$\Rightarrow \frac{K^2-59K+832}{K-52} > 0 \Rightarrow \frac{(K-23.32)(K-35.68)}{K-52} > 0$$

$$\Rightarrow K \in (23.32, 35.68) \cup (52, \infty) \quad -(ii)$$

− + − +
23.32 35.68 52

and, $K > 0$ $-(iii)$

From (i), (ii) and (iii),

$$\boxed{23.32 < K < 35.68} \rightarrow \text{for stability}$$

For $K = 35.68$,

auxiliary eqn:

$$\left(\frac{52-K}{3}\right)s^2 + K = 0$$

$$\Rightarrow 5.44\,s^2 + 35.68 = 0$$

$$\Rightarrow s^2 = \frac{-35.68}{5.44} = -6.56$$

$$\Rightarrow s = \pm j\sqrt{6.56} = \pm j\,2.56$$

For $K = 23.32$,

auxiliary eq^n:

$$9.56 s^2 + 23.32 = 0$$

$$\Rightarrow s = \pm j\sqrt{\frac{23.32}{9.56}} = \pm j\sqrt{2.44} = \pm j1.56$$

$\therefore$ Point of intersection of root loci with imaginary axis : $\pm j1.56, \pm j2.56$.

~~Angle of~~

Breakaway points:

$$K = -\frac{s(s-1)(s^2+4s+16)}{(s+1)}$$

$$= -\frac{s^4+3s^3+12s^2-16s}{s+1}$$

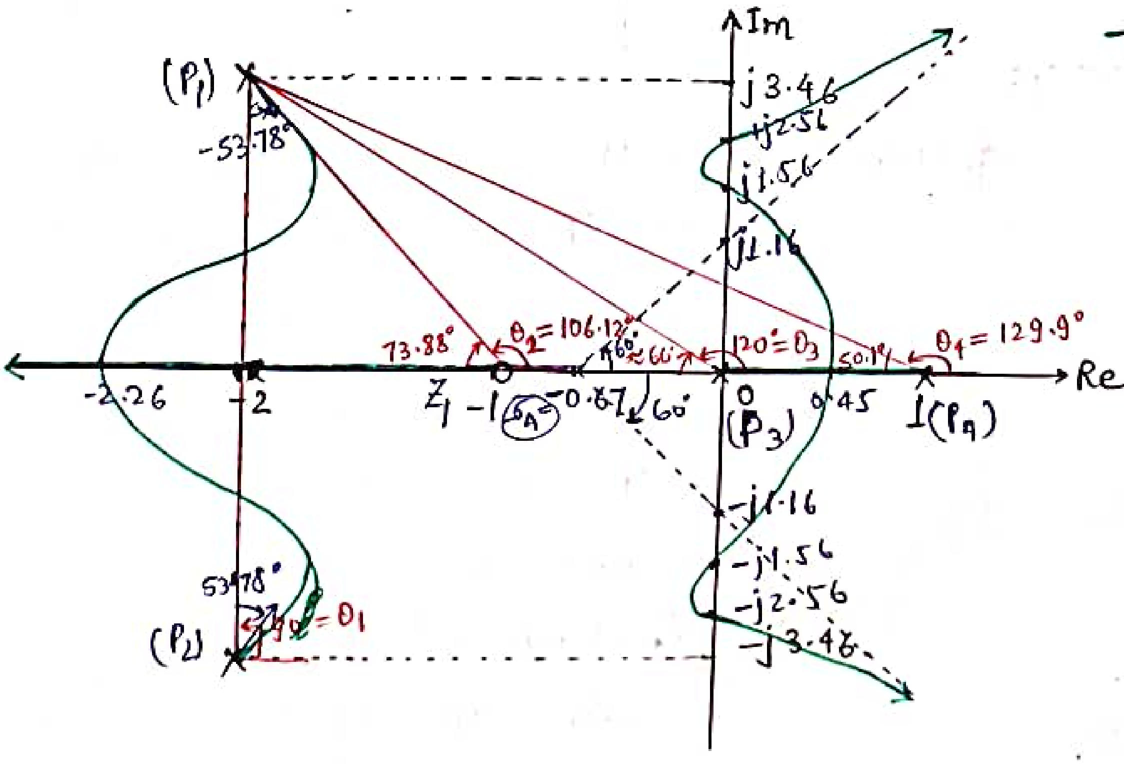
$$\frac{dK}{ds} = 0 \Rightarrow \frac{(s+1)(4s^3+9s^2+24s-16) - (s^4+3s^3+12s^2-16s)}{(s+1)^2} = 0$$

$$\Rightarrow (4s^4+9s^3+24s^2-16s+4s^3+9s^2+24s-16)-(s^4+3s^3+12s^2-16s) = 0$$

$$\Rightarrow 3s^4+10s^3+21s^2+24s-16 = 0$$

$$\Rightarrow s \approx 0.4482 \approx 0.45,$$

$$s \approx -2.26$$



— Root locus plot for closed-loop system

Angle of departure:

$$\phi_{P_1} = 180^\circ - (\Sigma P_0 - \Sigma Z_0) \qquad [\text{for } P_1 = -2 + j3.46]$$

$$= 180^\circ - (\theta_1 + \theta_3 + \theta_4 - \theta_2)$$

$$= 180^\circ - (90^\circ + 120^\circ + 129.9^\circ - 106.12^\circ)$$

$$= 180^\circ - 233.78^\circ$$

$$= -53.78^\circ$$

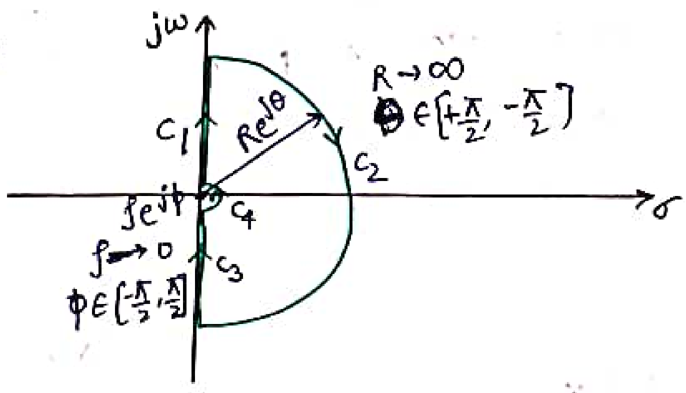
For $P_2 = -2 - j3.46$, $\quad \phi_{P_2} = +53.78^\circ$

Range of K for stability : $\boxed{K \in (23.32, 35.68)}$.

② Draw the Nyquist plot for the above system and find the range of gain, K, for stability.

Soln:
$$G(s)\,H(s) = \frac{K(s+1)}{s(s-1)(s^2+4s+16)}$$

Nyquist contour:



For C_1: $s = j\omega$

$$G(j\omega)H(j\omega) = \frac{K(j\omega+1)}{j\omega(j\omega-1)(\cancel{1}-\omega^2+4j\omega+16)} \times \frac{(j\omega+1)(16-\omega^2-4j\omega)}{(j\omega+1)(16-\omega^2-4j\omega)}$$

$$= -\frac{K(1-\omega^2+2j\omega)(16-\omega^2-4j\omega)}{j\omega(1+\omega^2)\left((16-\omega^2)^2+16\omega^2\right)}$$

$$= \frac{jK(1-\omega^2+2j\omega)(16-\omega^2-4j\omega)}{\omega(1+\omega^2)\left((16-\omega^2)^2+16\omega^2\right)}$$

$$= \frac{K\left[(1-\omega^2)(4\cancel{\omega}) + 2\cancel{\omega}(\omega^2-16)\right]}{\cancel{\omega}(1+\omega^2)\left((16-\omega^2)^2+16\omega^2\right)} - j\frac{K\left[(1-\omega^2)(16-\omega^2)+8\omega^2\right]}{\omega(1+\omega^2)\left((16-\omega^2)^2+16\omega^2\right)}$$

As $\omega \to 0$, $G(j\omega)H(j\omega) = \frac{-28K}{16^2} - j\infty$

$$= \infty \angle 90^\circ$$

As $\omega \to \infty$, $GH(j\omega) = \left| \frac{K\left(\frac{j}{\omega^3} + \frac{1}{\omega^4}\right)}{j(j-1)\left(-1 + \frac{4j}{\omega^3} + \frac{16}{\omega^4}\right)} \right| \angle \left[\tan^{-1}(\infty) - 90^\circ - (180^\circ - \tan^{-1}(-\infty)) - (180^\circ - \tan^{-1}(0)) \right]$

$$= 0 \angle (\cancel{90^\circ} - \cancel{90^\circ} - 90^\circ - 180^\circ)$$

$$= 0 \angle -270^\circ$$

For C_2: $s = Re^{j\theta}$, $R \to \infty$

$\theta \in \left[\frac{\pi}{2}, -\frac{\pi}{2}\right]$

$$\underset{R\to\infty}{GH(Re^{j\theta})} = \frac{K(Re^{j\theta}+1)}{Re^{j\theta}(Re^{j\theta}-1)(R^2e^{2j\theta} + 4Re^{j\theta} + 16)}$$

$$= \frac{K\left(1 + \cancel{\frac{1}{Re^{j\theta}}}\right)}{(Re^{j\theta})^3\left[1 - \cancel{\frac{1}{Re^{j\theta}}}\right]\left[1 + \cancel{\frac{4}{Re^{j\theta}}} + \cancel{\frac{16}{(Re^{j\theta})^2}}\right]}$$

$$= \frac{K}{R^3} e^{-3j\theta}$$

where $R \to \infty \Rightarrow$ Mag. $\to 0$

$\theta \in [\pi/2, -\pi/2] \Rightarrow$ Phase $\in \left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$

$\in [-270^\circ, 270^\circ]$

For C_4:

$s = \rho e^{j\phi}$, $\rho \to 0$

$\phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\underset{\rho\to 0}{GH(\rho e^{j\phi})} = \frac{K(\rho e^{j\phi}+1)}{\rho(e^{j\phi})(\rho e^{j\phi}-1)(\rho^2 e^{2j\phi} + 4\rho e^{j\phi} + 16)}$$

$$= \frac{K}{\rho e^{j\phi}(-1)(16)}$$

$$= -\frac{K}{16\rho} e^{-j\phi}$$, where $\rho \to 0 \Rightarrow$ Mag. $\to \infty$

$\phi \in [\pi/2, \pi/2] \Rightarrow$ Phase $\in [270^\circ, 90^\circ]$

$$= \frac{K}{16\rho} e^{j(\pi - \phi)}$$

Intersection with real axis:

$$\text{Im}\{GH(j\omega)\} = 0$$

$$\Rightarrow \cancel{K}(1-\omega^2)(16-\omega^2) + 8\omega^2 = 0$$

$$\Rightarrow -17\omega^2 + \omega^4 + 16 + 8\omega^2 = 0$$

$$\Rightarrow 9\omega^2 - \omega^4 - 16 = 0$$

$$\Rightarrow \omega^4 - 9\omega^2 + 16 = 0$$

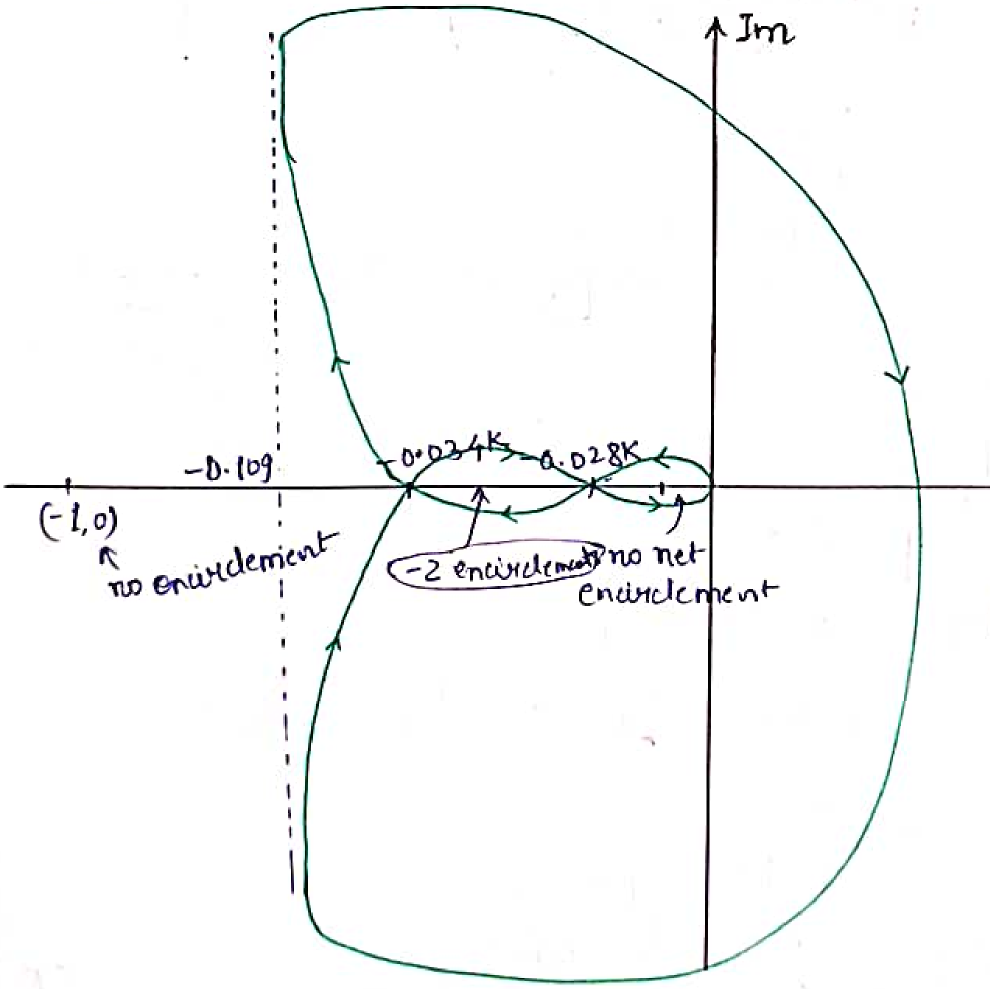
$$\Rightarrow \omega^2 = \frac{9 \pm \sqrt{81-64}}{2} = 6.56,\ 2.44$$

$$\Rightarrow \omega = \pm 2.56,\ \pm 1.56.$$

$$\therefore\ G(j\omega)H(j\omega)\Big|_{\omega^2=6.56} = \frac{-(5.56)4 + 2(6.56-16)}{(7.56)\left[(16-6.56)^2 + 16(6.56)\right]} \cdot K = -0.028K$$

$$G(j\omega)H(j\omega)\Big|_{\omega^2=2.44} = \frac{-(1.44)4 + 2(2.44-16)}{(3.44)\left[(16-2.44)^2 + 16(2.44)\right]} K = -0.034K$$

Nyquist plot:



For stability, around $(-1+j0)$, there should be no net encirclements,

so $-0.034K > -1$

$\Rightarrow K > 29.41$

or, $-0.028K < -1$

$\Rightarrow K < 35.71$

$\therefore\ \boxed{K \in (29.41,\ 35.71)}$