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INTEGRAL TRANSFORM

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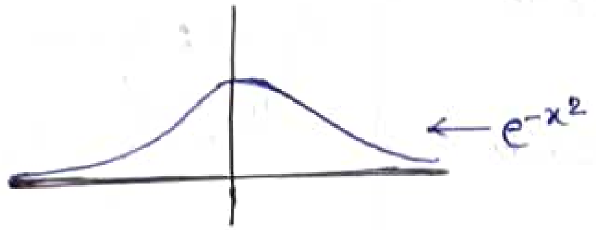
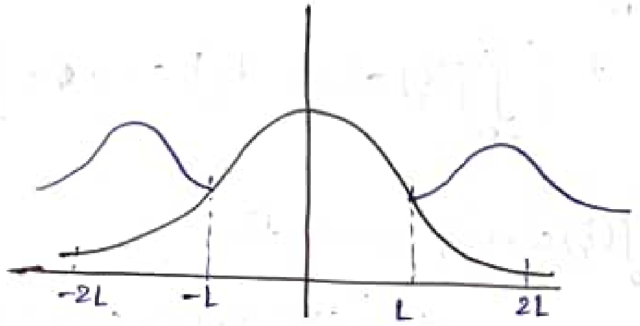


In fourier series, the domain is restricted.



$\boxed{b-a=2L}$

$$f(x) = \sum \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right) + a_0$$

$$x \in [-L, L]$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x . dx$$

Now, due to this, between $-L<x<L$ only, the fourier series will converge to $f(x)$.

Outside $|x|>L$, $f(x)$ will not converge as fourier series. Now, to get a F.S. which will converge to $f(x)$, $L \to \infty$.

As $L \to \infty \Rightarrow$ F.S. of $f \longrightarrow$ F.I. of f

$$f(x) = \text{F.I. of } f = \int_0^\infty \left(a(\omega) \cos(\omega x) + b(\omega) \sin(\omega x) \right) dx$$

$$= F_T^{-1}(F_T(f(x)))$$

$$= (F_T^{-1} \circ F_T) f(x)$$

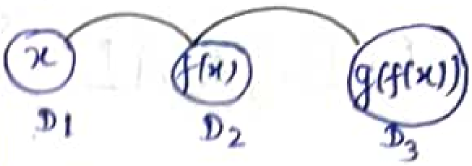
Considering F_T to be an invertible.

Also,

$$a(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x\,dx$$

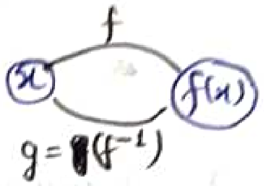
$$b(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\omega x\,dx$$

3 domains:



$(g\circ f) : D_1 \to D_3$

If $g = f'$



Now,

$$\text{F.I. of } f = \frac{1}{\pi}\int_0^{\infty}\Big[\Big(\int_{-\infty}^{\infty} f(\xi)\cos\omega\xi\,d\xi\Big)\cos\omega x + \Big(\int_{-\infty}^{\infty} f(\xi)\sin\omega\xi\,d\xi\Big)\sin\omega x\Big]d\omega$$

$$= \frac{1}{\pi}\int_0^{\infty}\Big[\int_{-\infty}^{\infty} f(\xi)\cos\omega\xi\cos\omega x\,d\xi + \int_{-\infty}^{\infty} f(\xi)\sin\omega\xi\sin\omega x\,d\xi\Big]d\omega.$$

$$= \frac{1}{\pi}\int_0^{\infty}\Big[\int_{-\infty}^{\infty} f(\xi)\cos\omega(\xi-\omega)\,d\xi\Big]d\omega$$

$$= \frac{1}{\pi}\int_0^{\infty}\Big[\int_{-\infty}^{\infty} f(\xi)\,\frac{e^{-i\omega(\xi-x)} + e^{i\omega(\xi-x)}}{2}\,d\xi\Big]d\omega$$

$$= \frac{1}{2\pi}\int_0^{\infty}\Big(\int_{-\infty}^{\infty} f(\xi)e^{-i\omega(\xi-x)}\,d\xi\Big)d\omega + \frac{1}{2\pi}\int_0^{\infty}\Big(\int_{-\infty}^{\infty} f(\xi)e^{i\omega(\xi-x)}\,d\xi\Big)d\omega.$$

Put $\omega \to -\omega$,

then $\int_0^{\infty}\int_{-\infty}^{\infty} + \int_{-\infty}^{0}\int_{-\infty}^{\infty} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}$

Now,

$$\text{F.I. of } f = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty} f(\xi)e^{-i\omega(\xi-x)}\,d\xi\Big)d\omega$$

$$\Rightarrow f(x) = \text{F.I. of } f = \frac{1}{2\pi}\int_{-\infty}^{\infty}\underbrace{\left[\underbrace{\int_{-\infty}^{\infty} f(\xi)\, e^{-i\omega\xi}\, d\xi}_{F_T(f)=\hat{f}(\omega)}\right] e^{i\omega x}\, d\omega}_{F_T^{-1}(F_T(f))}$$

$$= F_T^{-1}\left(F_T(f(x))\right).$$

<u>Define Fourier Transform :</u>

$$\boxed{\begin{aligned} F_T(f) &= \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx = \hat{f}(\omega) \\ F_T[\hat{f}(\omega)] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega x}\, d\omega = f(x) \end{aligned}}$$

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Defn: $f: \mathbb{R} \to \mathbb{R}$

$$\boxed{\begin{aligned} F_T(f) &= \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx \\ &= \hat{f}(\omega) \end{aligned}}$$

Eg. $f(x) = \begin{cases} 1, & -1 \le x \le 1 \\ 0, & \text{elsewhere} \end{cases}$

Find $\hat{f}(\omega)$.

Soln: $\hat{f}(\omega) = \int_{-1}^{1} e^{-i\omega x} dx$

$$= \left. \frac{e^{-i\omega x}}{-i\omega} \right|_{-1}^{1}$$

$$= \frac{e^{-i\omega} - e^{i\omega}}{-i\omega}$$

$$= \frac{e^{i\omega} - e^{-i\omega}}{i\omega}$$

$$= 2\,\frac{\sin\omega}{\omega}$$

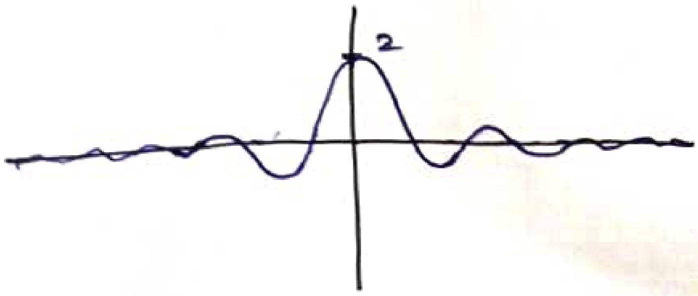
↳ Is $\hat{f}(0)$ defined? → Yes!

$$\boxed{\hat{f}(0) = 2 = \lim_{\omega \to 0} \hat{f}(\omega)}$$

⇝ directly put the value in integration.

→ $\omega \to \pm\infty$

$\hat{f}(\omega) \to 0$



Q For any $\hat{f}(\omega)$, can we say $|\hat{f}(\omega)| \to 0$ as $\omega \to \pm\infty$?

↳ Yes!

$f: [a,b] \to \mathbb{C}$

$f(t) = u(t) + iv(t)$

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$$

$f: \mathbb{C} \to \mathbb{C}$

$z \to f(z) = u(z) + iv(z)$

$\int_C f(z)dz$

$\int_a^b f(z)dz$ → ✗

~~f(x)~~

$f(x) = \frac{\sin x}{x}, x \neq 0$

$f(0)$ is not defined.

$z_1 < z_2$ ✗

$|z_1| < |z_2|$ ✓

Q) Whether $\hat{f}(\omega)$ is continuous function ?

↳ Yes !

Eg. $f(x) = 1 \ \forall x \in (-\infty, \infty)$.

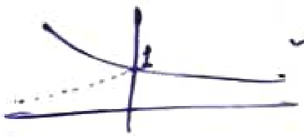
↳ Fourier transform of this is not possible.

$$L_1(\mathbb{R}) = \left\{ g : \mathbb{R} \to \mathbb{R} \text{ such that } \int_{-\infty}^{\infty} |g(x)|\,dx < \infty \right\}$$

→ $F_T : L_1(\mathbb{R}) \to C_0(\mathbb{R})$, $f \longrightarrow F_T(f)$

$$C_0(\mathbb{R}) = \left\{ g : \mathbb{R} \to \mathbb{R} \text{ s.t. } g \text{ is continuous and } g(x) \to 0 \text{ as } x \to \pm\infty \right\}$$

Eg. $f(x) = e^{-ax}, \ a > 0$



⇝ Not possible

Modify ↓ (trivial option)

$$f(x) = \begin{cases} e^{-ax}, & x \geq 0 \\ 0, & x < 0 \end{cases} \quad (a > 0)$$

$= H(x)\,e^{-ax}$, $H(x)$: Unit step function

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

$$F_T(f) = \int_{-\infty}^{\infty} H(x)\, e^{-ax} e^{-i\omega x}\, dx$$

$$= \int_{0}^{\infty} e^{-(a+i\omega)x}\, dx$$

$$= -\left. \frac{e^{-(a+i\omega)x}}{(a+i\omega)} \right|_{0}^{\infty}$$

$$= \frac{1}{a+i\omega}$$

Q) For any $\hat{f}(\omega) \in C_0(\mathbb{R})$, does $F_T^{-1}(\hat{f}(\omega))$ exist?

↳ No!

For $F_T^{-1}(\hat{f}(\omega))$ to exist the mapping

$$F_T : L_1(\mathbb{R}) \to C_0(\mathbb{R})$$

must be one-one and onto.

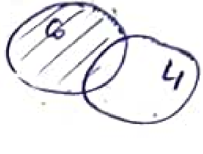
Q) Is F_T a linear map? → Yes!

f, g : functions ; α, β : scalars

$F_T(\alpha f + \beta g)$

$= \alpha F_T(f) + \beta F_T(g)$.

\# $C_{0(\mathbb{R})} \subset L_1(\mathbb{R})$ ⇝ then, no issue



Properties:

① Linearity:

$F_T(\alpha f + \beta g) = \alpha F_T(f) + \beta F_T(g)$, f, g : functions
α, β : scalars

② $F_T(f(\alpha x)) = \frac{1}{\alpha}\underbrace{\hat{f}\left(\frac{\omega}{\alpha}\right)}$, $\alpha > 0$.

$\left(F_T(f)\big|_{\omega \leftarrow \frac{\omega}{\alpha}}\right)$: F.T. of f where ω is replaced by ω/α.

$$= \int_{-\infty}^{\infty} f(\alpha x)\, e^{-i\omega x}\, dx \quad \begin{bmatrix} \alpha x = t \\ \Rightarrow \alpha dx = dt \end{bmatrix}$$

$$= \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t/\alpha}\, \frac{dt}{\alpha} = \frac{1}{\alpha}\int_{-\infty}^{\infty} f(t) e^{-i\omega t/\alpha}\, dt = \frac{1}{\alpha}\hat{f}\left(\frac{\omega}{\alpha}\right)$$

→ $F_T(f(\alpha x)) = -\frac{1}{\alpha}\hat{f}\left(\frac{\omega}{\alpha}\right)$, $\alpha < 0$.

- $\alpha \in \mathbb{R} - \{0\}$,

$$\boxed{F_T(f(\alpha x)) = \frac{1}{|\alpha|}\hat{f}\left(\frac{\omega}{\alpha}\right)}$$

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③ $$\boxed{F_T(f(x-\alpha)) = \overset{\text{Phase shift}}{e^{-i\omega\alpha}}\hat{f}(\omega)}$$

→ Delay

$$F_T(f(x-\alpha)) = \int_{-\infty}^{\infty} f(x-\alpha)\, e^{-i\omega x}\, dx$$

$$\left[\begin{array}{l}\text{Take } x-\alpha = t \\ \Rightarrow x = \alpha + t\end{array}\right]\left[\frac{dx}{dt} = 1\right]$$

$$= \int_{-\infty}^{\infty} f(t)\, e^{-i\omega(\alpha+t)}\, dt$$

$$= e^{-i\omega\alpha}\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$= \underbrace{e^{-i\omega\alpha}\hat{f}(\omega)}_{\text{exponential modulation}}$$

Books:

- JAIN & IYENGER
- GREENBERG (Engineering Mathematics)
- MATHEWS & HALL (Complex)
- Bachmann and Narichi (Fourier & wavelet Analysis)

④ $$\boxed{F_T(e^{i\alpha x} f(x)) = \hat{f}(\omega-\alpha)} = F_T(f)\Big|_{\omega \leftarrow \omega-\alpha}$$

$$\underbrace{F_T(e^{i\alpha x} f(x))}_{\substack{\text{exponential}\\ \text{modulation}}} = \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, e^{-i\omega x}\, dx$$

$$= \int_{-\infty}^{\infty} f(x)\, e^{-i\underbrace{(\omega-\alpha)}_{\bar{\omega}} x}\, dx$$

$$= \hat{f}(\omega-\alpha)$$

Eg. $f(x) = e^{-|x|}$ ⇝ absolutely integrable

$$F_T(e^{-|x|}) = \frac{2}{1+\omega^2}$$

Eg. $f(x) = e^{-3|x|}$

$$F_T(e^{-3|x|}) = \frac{1}{3} F_T(e^{-|x|})\Big|_{\omega \leftarrow \frac{\omega}{3}}$$

$$= \frac{1}{3}\cdot\frac{2}{1+\frac{\omega^2}{9}}$$

$$= \frac{6}{9+\omega^2}$$

Eg $f(x) = e^{-3|x-2|}$

$$F_T\left(e^{-3|x-2|}\right) = e^{-iw2}\; F_T\left(e^{-3|x|}\right)$$

$$= e^{i2w}\; \frac{6}{9+w^2}$$

L_1 : absolutely integrable

⑤ $f, f' \in L_1(\mathbb{R})$

f is piecewise smooth on every closed interval in $(-\infty, \infty)$.

and, $f(x) \to 0$ as $x \to \pm\infty$.

$$\boxed{F_T(f') = (iw)\,\hat{f}(w)}$$

Proof: $F_T(f') = \int_{-\infty}^{\infty} \underbrace{f'(x)}_{II}\, \underbrace{e^{-iwx}}_{I}\, dx$

$$= \underset{\overset{//}{0}}{f(x)\, e^{-iwx}}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (-iw)\, e^{-iwx} f(x)\, dx$$

$$= (iw)\; F_T(f)$$

$$= (iw)\,\hat{f}(w)$$

⑥ f, f' are P-S on every closed interval in $(-\infty, \infty)$.

$f, f', f'' \in L_1(\mathbb{R})$

and, $\begin{cases} f(x) \to 0 \text{ as } x \to \pm\infty \\ f'(x) \to 0 \text{ as } x \to \pm\infty \end{cases}$

$$\boxed{F_T(f'') = (iw)^2\,\hat{f}(w)}$$

Proof: $F_T(f'') = \int_{-\infty}^{\infty} \underbrace{f''(x)}_{II}\, \underbrace{e^{-iwx}}_{I}\, dx$

$$= \cancelto{0}{f'(x)\, e^{-iwx}}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (-iw)\, e^{-iwx} f'(x)\, dx$$

$$= (iw)\; F_T(f')$$

$$= (iw)^2\,\hat{f}(w)$$

$F_T(\cos\alpha x\; f(x))$

$F_T(\sin\alpha x\; f(x))$

Eg. $F_T(e^{-ax^2})$, $a>0$

$$f(x) = e^{-ax^2}$$

$$f'(x) = -2ax\, e^{-ax^2} = -2ax f(x)$$

$$\Rightarrow \boxed{f'(x) + 2axf(x) = 0}$$

$$\Rightarrow F_T(f') + 2aF_T(xf) = 0$$

⑦ $F_T(xf) = i\dfrac{d\hat{f}(\omega)}{d\omega}$, $f, xf \in L_1(\mathbb{R})$

$$\Rightarrow i\omega\hat{f}(x) + 2a \cdot i \cdot \frac{d\hat{f}(\omega)}{d\omega} = 0$$

$$\Rightarrow \boxed{\frac{d\hat{f}(\omega)}{d\omega} + \frac{\omega}{2a}\hat{f}(\omega) = 0}$$

$$\hat{f}(\omega) = c\, e^{-\omega^2/4a}$$

$$c = \hat{f}(0) = \int_{-\infty}^{\infty} e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}$$

$$\therefore \hat{f}(\omega) = \sqrt{\frac{\pi}{a}}\, e^{-\omega^2/4a}$$

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→ $F_T(0)=0$

$F_T(1)$ → doesnot exist ×

$a>0$, $F_T(e^{-ax})$ ×

$F_T(x)$ ×

$F_T(\sin x)$ ×

Fourier Cosine Transform

Defn: Suppose $f \in L_1(0,\infty)$

$$F_T^c(f) = \int_0^\infty f(x)\cos wx\,dx = \hat{f}_c(w)$$: Fourier Cosine Transform

↳ even function

$$F_T^s(f) = \int_0^\infty f(x)\sin wx\,dx = \hat{f}_s(w)$$: Fourier Sine Transform

↳ odd function

Observation:

① Suppose $f \in L_1(\mathbb{R}) \Rightarrow F_T(f)$ exists.

- $f \in L_1(0,\infty)$

$\Rightarrow F_T^c(f)$ and $F_T^s(f)$ both exist.

$$F_T(f) = \int_{-\infty}^{\infty} f(x)\,e^{-iwx}\,dx$$

$$= \int_{-\infty}^{\infty} f(x)\cos wx\,dx - i\int_{-\infty}^{\infty} f(x)\sin wx\,dx$$

Suppose f is an even function.

$$F_T(f) = \int_{-\infty}^{\infty} f(x)\cos wx\,dx$$

$$= 2\int_0^\infty f(x)\cos wx\,dx$$

$$\therefore \boxed{F_T(f) = 2F_T^c(f)}$$

f is an odd function.

$$F_T(f) = -i\int_{-\infty}^{\infty} f(x)\sin wx\,dx$$

$$= -2i\int_0^\infty f(x)\sin wx\,dx$$

$$\therefore \boxed{F_T(f) = -2i\,F_T^s(f)}$$

② $\hat{f}_c(-w) = \hat{f}_c(w)$ [Even function]

$\hat{f}_s(-w) = -\hat{f}_s(w)$ [Odd function]

→ fourier transform of an even function is an ~~odd~~ even function.

Eg. $f(x) = e^{-|x|}$ (even f^n)

$\Rightarrow F_T(f) = \frac{2}{1+w^2}$ (even f^n in ω)

Eg. $a>0$, $e^{-ax} = f(x)$

$F_T(f)$ does not exist.

$$\int_0^\infty e^{-ax}dx = \lim_{t\to\infty}\int_0^t e^{-ax}dx$$

$$= \lim_{t\to\infty} \frac{e^{-ax}}{-a}\Big|_0^t = \frac{1}{a}.$$

Area under curve is finite in $(0,\infty)$, we can talk about fourier cosine, sine transform.

$$F_T^c(f) = \int_0^\infty e^{-ax}\cos wx\,dx = \frac{a}{a^2+w^2}$$

$$F_T^s(f) = \int_0^\infty e^{-ax}\sin wx\,dx = \frac{w}{a^2+w^2}$$

Eg. (Rectangular Pulse)

$$f(x) = \begin{cases} 1, & -1 \le x \le 1 \\ 0, & \text{elsewhere.} \end{cases}$$

$$F_T^c(f) = \int_0^\infty \cos wx\,dx$$

$$= \frac{\sin wx}{w}\Big|_0^1 = \frac{\sin w}{w} - 0$$

$$= \frac{\sin w}{w},\ w \neq 0$$

$\hat{f}_c(w) = F_T^c(f)$

$\hat{f}_c(0) = 1$

Since $f \in L_1(\mathbb{R})$,

$$\boxed{F_T(f) = 2F_T^c(f)}.$$

Eg. Find f such that $\int_0^\infty f(x)\cos\omega x\,dx = e^{-\omega}$ $(0<\omega<\infty)$ (Inverse problem)

Find f such that $\frac{df}{dx} = e^{-x}$.

$f \in L_1(\mathbb{R})$

$$f(x) = \int_0^\infty (a(\omega)\cos\omega x + b(\omega)\sin\omega x)\,d\omega$$

[→ The point where function is discontinuous → average value.]

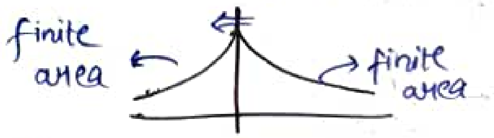
$$= (F_T^{-1}\circ F_T)(f)$$

Eg. Suppose $f\in L_1(\mathbb{R}^+) \Rightarrow F_T^c(f)$ exists.

Let f_{ev} be an even extension of f.

$\Rightarrow f_{ev}\in L_1(\mathbb{R})$?

↳ yes!



⇒ ~~[illegible]~~ $F_T(f_{ev})$ exists.

→ $f_{ev}(x) = (F_T^{-1}\circ F_T)(f_{ev})$

$$= F_T^{-1}(\hat f_{ev}(\omega)) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat f_{ev}(\omega)e^{i\omega x}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}(2\hat f_c(\omega))e^{i\omega x}\,d\omega$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\hat f_c(\omega)\,e^{i\omega x}\,d\omega$$

On $0<x<\infty$,

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\hat f_c(\omega)\,e^{i\omega x}\,d\omega$$

$$= \frac{1}{\pi}\left[\int_{-\infty}^{\infty}\hat f_c\cos\omega x\,dx + i\int_{-\infty}^{\infty}\hat f_c(\omega)\sin\omega x\,d\omega\right]$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\hat f_c(\omega)\cos\omega x\,d\omega.$$

$$\therefore \boxed{f(x) = \frac{2}{\pi}\int_0^{\infty}\hat f_c(\omega)\cos\omega x\,d\omega}$$

Eg. $f(x) = \frac{2}{\pi}\int_0^\infty e^{-\omega}\cos\omega x\,d\omega = \frac{2}{\pi}\times\frac{1}{1+x^2}$

Find the derivative of sine.

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Eg. ① Find f such that

$$\frac{df}{dx} = e^{-x}$$

② Find f such that

$$\int_0^\infty f(x) \cos wx \, dx = e^{-w} \Rightarrow \hat{f}_c(w) = e^{-w}.$$

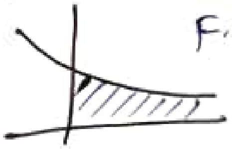
Defn: $f \in L_1[0,\infty)$: $\int_0^\infty |f(x)| dx < \infty$

$$F_T^{\,C}(f) = \int_0^\infty f(x) \cos wx \, dx = \hat{f}_c(w)$$

$$F_T^{\,S}(f) = \int_0^\infty f(x) \sin wx \, dx = \hat{f}_s(w)$$

$$\boxed{F_T(f) = \int_{-\infty}^\infty f(x) e^{-iwx} dx, \quad f \in L_1(\mathbb{R})}$$

Eg. $f(x) = e^{-x} \notin L_1(\mathbb{R})$



F.T. of e^{-x} does not exist.

$e^{-x} \in L_1[0,\infty)$

$\left.\begin{array}{l} F_T^{\,C}(e^{-x}) \\ F_T^{\,S}(e^{-x}) \end{array}\right\}$ both exist.

→ $$f(x) = \frac{2}{\pi} \int_0^\infty \hat{f}_c(w) \cos wx \, dw$$

$$= F_T^{\,C^{-1}}(\hat{f}_c(w))$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \hat{f}_s(w) \sin wx \, dx$$

$$= F_T^{\,S^{-1}}(\hat{f}_s(w))$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty \hat{f}(w) e^{iwx} dx$$

$$= F_T^{-1}(\hat{f}(w)).$$

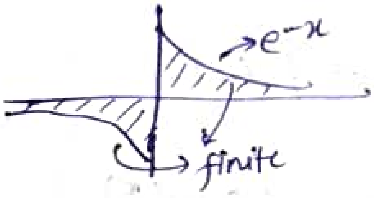
$$\rightarrow \boxed{\begin{aligned} g(x) &= F_T^{-1}(F_T(g)) \\ &= F_T^{-1}(\hat{g}(w)), \\ &-\infty < x < \infty \end{aligned}}, \ g \in L_1(\mathbb{R})$$

Suppose $f \in L_1[0, \infty)$,

$F_T^{s}(f)$ exists.

Let f_{odd} be an odd extension of f.

$\Rightarrow f_{odd} \in L_1(\mathbb{R}) \Rightarrow F_T(f_{odd})$ exists.



$$\begin{aligned} f_{odd} &= F_T^{-1}\left(F_T(f_{odd})\right) \\ &= F_T^{-1}\left(\hat{f}_{odd}(w)\right) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}_{odd}(w)\, e^{iwx}\, dw \end{aligned}$$

Q1 $\frac{d^2u}{dx^2} - u = e^{-2x}$, $0 < x < \infty$. : Semi-infinite domain

$$\begin{cases} u(0) = 0 \\ u(x) \to 0 \text{ as } x \to \infty. \\ \frac{du(x)}{dx} \to 0 \text{ as } x \to \infty. \end{cases}$$

Sol^n: Step-1: Apply F_T^s / F_T^c based on given condition, and convert DE → AE ($\hat{u}_c(\omega)$) or $\hat{u}_s(\omega)$).

Step-2: Find $u(x)$ using inverse transform.

Result-I: Assume that:

① f is differentiable on $[0, \infty)$.

② $f, f' \in L_1[0, \infty)$.

③ $f(x) \to 0$ as $x \to \infty$.

Aim: To find $F_T^c(f'(x))$, and $F_T^s(f'(x))$.

$$F_T^c(f') = \int_0^\infty \underbrace{f'(x)}_{II} \underbrace{\cos \omega x}_{I} \, dx.$$

$$= \cos \omega x \, f(x) \Big|_0^\infty - \int_0^\infty (-\omega \sin \omega x) f(x) dx$$

$$\therefore F_T^c(f') = \omega \hat{f}_s(\omega) - f(0). \qquad [\because f(x)|_{x\to\infty} = 0]$$

$$F_T^s(f') = \int_0^\infty \underbrace{f'(x)}_{II} \underbrace{\sin \omega x}_{I} \, dx$$

$$= \sin \omega x \, f(x) \Big|_0^\infty - \int_0^\infty (\omega \cos \omega x) f(x) dx$$

$$= \overset{\to 0}{\cancel{\sin \omega f(x)}} \Big|_{x \to \infty} - \omega \int_0^\infty \cos \omega x \, f(x) dx$$

$$\therefore F_T^s(f') = -\omega \hat{f}_c(\omega).$$

Result-II: Assume that:

① f is twice differentiable.

② $f, f', f'' \in L_1[0, \infty)$.

③ $\begin{cases} f(x) \to 0 \text{ as } x \to \infty \\ f'(x) \to 0 \text{ as } x \to \infty. \end{cases}$

$$F_T^c(f'') = -\omega^2 \hat{f}_c(\omega) - \frac{df(0)}{dx} \quad ; \quad F_T^s(f'') = -\omega^2 \hat{f}_s(\omega) + \omega f(0).$$

10-03-2024

$$F_T^c(f'') = \int_0^\infty \underbrace{f''(x)}_{\text{II}} \underbrace{\cos wx}_{\text{I}}\, dx$$

$$F_T^s(f'') = \int_0^\infty \underbrace{f''(x)}_{\text{II}} \underbrace{\sin wx}_{\text{I}}\, dx$$

$$-\int_0^\infty \frac{d}{dx}(\cos wx) f'(x)\, dx$$

$$= +\omega \underbrace{\int_0^\infty \sin \omega x\, f'(x)\, dx}$$

$$= \omega F_S^T(f'(x)) = -\omega^2 \hat{f}_c(\omega)$$

$$\boxed{\begin{array}{l} F_S^T(f'(x)) = -\omega \hat{f}_c(\omega) \\ f(x) \to 0 \text{ as } x \to \infty \end{array}}$$

As $x \to \infty$,

$$\underbrace{f'(x)}_{0} \cos wx \Big|_{x\to\infty} = 0$$

$$f'(x) \cos wx \Big|_{x=0} = 0$$

$$f'(x) \sin wx \Big|_{x=\infty} = 0$$

$$f'(x) \sin \omega x \Big|_{x=0} = 0$$

$$F_T^c(f'') = \cos \omega x\, f'(x) \Big|_0^\infty - \int_0^\infty \frac{d}{dx}(\cos \omega x) f'(x)\, dx$$

$$\boxed{F_T^c(f'') = -f'(0) - \omega^2 \hat{f}_c(\omega)}$$

Similarly,

$$\boxed{F_T^s(f'') = -\omega^2 \hat{f}_s(\omega) + \omega f(0)}$$

Boundary Value Problem

Q) $u'' - u = e^{-2x}$, $\quad 0 < x < \infty$

$u(0) = 0$

$$\begin{cases} u(x) \to 0 \text{ as } x \to \infty \\ u'(x) \to 0 \text{ as } x \to \infty \end{cases}$$

Soln: $F_s^T(u'' - u) = F_s^T(e^{-2x})$

$$\Rightarrow F_s(u'') - F_s(u) = \frac{\omega}{4+\omega^2}$$

$$\Rightarrow -\omega^2 \hat{u}_s(\omega) + \omega \cancelto{0}{u(0)} - \hat{u}_s(\omega) = \frac{\omega}{4+\omega^2}$$

$$\Rightarrow \boxed{\hat{u}_s(\omega) = \frac{-\omega}{(4+\omega^2)(1+\omega^2)}}$$

$F_T(e^{-ax})$, $a > 0$
does not exist

$$F_T^s(e^{-ax}) = \int_0^\infty e^{-ax} \sin \omega x\, dx = \frac{\omega}{a^2+\omega^2}$$

Task: Find $u(x)$.

$$\hat{u}_s(\omega) = \frac{1}{3}\left[\frac{\omega}{4+\omega^2} - \frac{\omega}{1+\omega^2}\right]$$

$$u(x) = F_s^{-1}(\hat{u}_s(\omega))$$

$$\Rightarrow \boxed{u(x) = \frac{2}{\pi}\int_0^\infty \hat{u}_s(\omega)\sin\omega x\, d\omega}, \quad 0<x<\infty. \quad \cancel{-\infty<x<\infty}$$

$$= \frac{1}{3}\left[\frac{2}{\pi}\int_0^\infty \frac{\omega}{4+\omega^2}\sin\omega x\, d\omega - \frac{2}{\pi}\int_0^\infty \frac{\omega}{1+\omega^2}\sin\omega x\, d\omega\right.$$

$$\therefore\ u(x) = \frac{1}{3}\left(e^{-2x} - e^{-x}\right), \quad 0<x<\infty$$

$$F_T^S(f') = -\omega \hat{f}_s(\omega) \times$$
$$= -\omega \hat{f}_c(\omega)$$

Eg. $u'' - u' = e^{-2x}, \ 0<x<\infty$

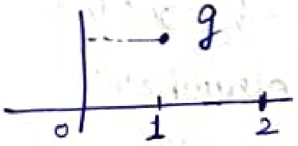
$u(0)=0$

$$\begin{cases} u(x) \to 0 \text{ as } x\to\infty \\ u'(x) \to 0 \text{ as } x \to \infty. \end{cases}$$

Defn:
$$(f * g)(t) = \int_{x\to-\infty}^{\infty} \underbrace{f(t-x)}_{\substack{\uparrow \\ \text{weight}}}\ \underbrace{g(x)}_{\substack{\uparrow \\ \text{input}}}\, dx$$

[Convolution]

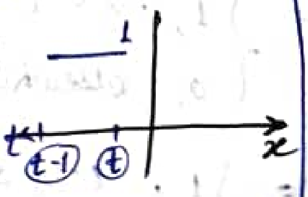
Eg. $g(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{elsewhere} \end{cases}$



$f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{elsewhere}. \end{cases}$

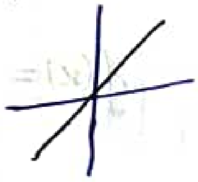
$f(t-x) = \begin{cases} 1, & 0 \le t-x \le 1 \\ 0, & \text{elsewhere}. \end{cases}$

$= \begin{cases} 1, & t-1 \le x \le t \\ 0, & \text{elsewhere} \end{cases}$

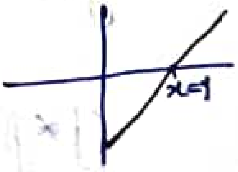


$t=1$

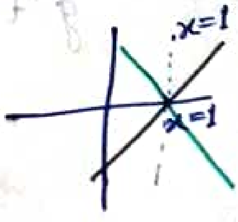
$f(x)=x$



$f(x) = (x-1)$



$f(x) = (1-x)$



① $\underline{t \le 0}$: $-1 \le x \le 0$ but $g(x)=0$ when $-1 \le x \le 0$.

$$\therefore (f*g)(t) = 0$$

② $\underline{0 < t \le 1}$: $t=1 \Rightarrow 0 \le x \le 1 = t \,;\, g(x) = 1$

$t = \frac{1}{2} \Rightarrow -\frac{1}{2} \le x \le \frac{1}{2} = t \,;\, g(x)=1,\ 0 \le x \le \frac{1}{2}$

$$(f*g)(t) = \int_0^t f(t-x)\, g(x)\, dx = t$$

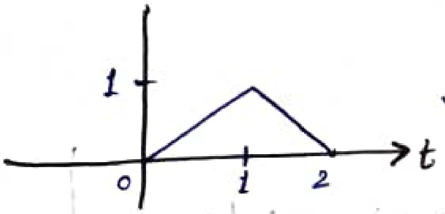
③ $\underline{1 < t \le 2}$: $t=2 \Rightarrow \overset{t-1=}{1} \le x \le 2 \,;\, g(x)=0$

$t=1.5 \Rightarrow 0.5 \le x \le 1.5 \,;\, g(x)=1,\ \overset{t-1=}{0.5} \le x \le 1.5$

$$(f*g)(t) = \int_{t-1}^{1} f(t-x)\, g(x)\, dx = [x]_{t-1}^{1}$$
$$= 2-t$$

④ $\underline{t > 2}$:

$$(f*g)(t) = 0$$



⇝ Smoothening

Eg. $f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{elsewhere} \end{cases}$: input

$g(x) = \begin{cases} 1, & -\frac{1}{2} \le x \le \frac{1}{2} \\ 0, & \text{elsewhere} \end{cases}$

$$(f*g)(t) = \int_{-\infty}^{\infty} f(t-x)\, g(x)\, dx = \int_{-\infty}^{\infty} f(x)\, g(t-x)\, dx$$

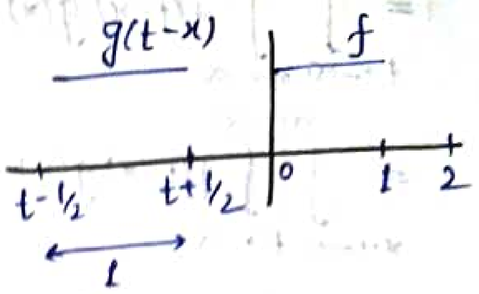
$$g(t-x) = \begin{cases} 1, & -\frac{1}{2} \le t-x \le \frac{1}{2} \\ 0, & \text{elsewhere} \end{cases}$$

$$= \begin{cases} 1, & t-\frac{1}{2} \le x \le t+\frac{1}{2} \\ 0, & \text{elsewhere} \end{cases}$$

: weight function

① $t \le -\frac{1}{2}$:

$(f*g)(t)=0$



② $-\frac{1}{2} < t \le \frac{1}{2}$:

$t=\frac{1}{2} \Rightarrow 0 \le x \le 1$

$t=\frac{1}{4} \Rightarrow \left(\frac{1}{4}-\frac{1}{2}\right) \le x \le \left(\frac{1}{4}+\frac{1}{2}\right)$ | lower limit = 0

$$(f*g)(t) = \int_{0}^{t+1/2} 1.\,dx = [x]_0^{t+1/2} = t+1/2$$

③ $\frac{1}{2} < t \le \frac{3}{2}$:

$t=3/2 \Rightarrow 1 \le x \le 2$

$t=1 \Rightarrow 1/2 \le x \le 3/2$.

$$(f*g)(t) = \int_{t-1/2}^{1} 1.\,dx$$

$$= [x]_{t-1/2}^{1} = 3/2 - t$$

④ $t > 3/2$:

$(f*g)(t) = 0$

Convolution Theorem

$$(f*g)(t) = \int_{-\infty}^{\infty} f(t-x).g(x)\,dx$$ | Given $F_T(f) = \hat{f}(\omega)$, $F_T(g) = \hat{g}(\omega)$ } exists and given as.

$f, g \in L_1(\mathbb{R})$ [Absolutely integrable]

$$\boxed{F_T(f*g) = \hat{f}(\omega) \times \hat{g}(\omega)}$$

$$F_T(f*g) = \int_{t\to-\infty}^{\infty} (f*g)(t)\, e^{-i\omega t}\, dt$$ ← not 'ω'

$$= \int_{t\to-\infty}^{\infty} \left[\int_{x\to\infty}^{\infty} f(t-x) g(x)\,dx \right] e^{-i\omega t}\, dt$$

$$= \int_{t\to-\infty}^{\infty}\int_{x\to-\infty}^{\infty} f(t-x)\, g(x)\, e^{-iwt}\, dx\, dt$$

$$= \int_{x\to-\infty}^{\infty}\left[\int_{t\to-\infty}^{\infty} f(t-x)\, g(x)\, e^{-iwt}\, dt\right] dx$$

$$= \int_{x\to-\infty}^{\infty} g(x)\left[\int_{t\to-\infty}^{\infty} f(\underbrace{t-x}_{z})\, e^{-iwt}\, dt\right] dx$$

[Take $t-x=z \Rightarrow dt=dz$
$\Rightarrow t=x+z$]

$$= \int_{x\to-\infty}^{\infty} g(x)\left[\int_{z\to-\infty}^{\infty} f(z)\, e^{-iw(x+z)}\, dz\right] dx$$

$$= \int_{x\to-\infty}^{\infty} g(x)\, e^{-iwx}\left[\underbrace{\int_{-\infty}^{\infty} f(z)\, e^{-iwz}\, dz}_{\hat{f}(w)}\right] dx$$

$$= \hat{f}(w)\,\hat{g}(w).$$

Fourier Convolution Theorem:

$\boxed{20\text{-}03\text{-}2024}$

Given $\left.\begin{array}{l} F_T(f) = \hat{f}(w) \\ F_T(g) = \hat{g}(w) \end{array}\right\}$ exists and given as.

$$F_T(f*g) = \hat{f}(w)\cdot\hat{g}(w)$$

$$\Downarrow$$

$$F_T^{-1}\left(\hat{f}(w)\cdot\hat{g}(w)\right) = (\hat{f}*\hat{g})(t).$$

Eg. $F_T^{-1}\left(\dfrac{1}{2+3iw-w^2}\right).$

$(2+3iw-w^2) = (1+iw)(2+iw)$

$$\Rightarrow \frac{1}{2+3iw-w^2} = \frac{1}{1+iw} - \frac{1}{2+iw}$$

$$\Rightarrow F_T^{-1}\left(\frac{1}{2+3iw-w^2}\right) = F_T^{-1}\left(\frac{1}{1+iw}\right) - F_T^{-1}\left(\frac{1}{2+iw}\right)$$

$F_T(e^{-x})$ does not exist.

$F_T(e^{-|x|})$ exists.

$F_T(e^{-x}H(x))$ exists.

$$F_T(e^{-|x|}) = \frac{2}{1+w^2}$$

$$F_T(e^{-x}H(x)) = \frac{1}{1+iw}$$

$$F_T(e^{-2x}H(x)) = \frac{1}{2+iw}$$

$= e^{-x} H(x) - e^{-2x} H(x)$

$= (e^{-x} - e^{-2x}) H(x).$

$F_T^{-1}\left(\frac{1}{2+3i\omega-\omega^2}\right)$

$= F_T^{-1}\left(\frac{1}{(1+i\omega)} \cdot \frac{1}{(2+i\omega)}\right) = (f*g)(t).$

$\hat{f}(\omega) = \frac{1}{1+i\omega}, \hat{g}(\omega) = \frac{1}{2+i\omega}.$

$f(x) = e^{-x} H(x).$

$g(x) = e^{-2x} H(x).$

$(f*g)(t) = \int_{-\infty}^{\infty} f(t-x)\, g(x)\, dx$

$= \int_{-\infty}^{\infty} e^{-(t-x)} H(t-x) \cdot e^{-2x} \cdot H(x)\, dx$

$= e^{-t} \int_0^t e^{-x}\, dx$

$= e^{-t} \left(-e^{-x}\right)\Big|_0^t$

$= -e^{-t}\left[e^{-t} - 1\right]$

$= \left[e^{-t} - e^{-2t}\right] H(x).$

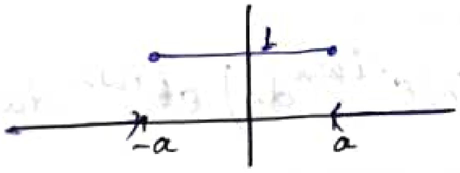
differential operator

→ $\boxed{L\, u(t) = f(t)}$

$\downarrow$ F.T

$\boxed{\text{Algebraic equation in } \hat{u}(\omega)}$

Eg. $f(x) = \begin{cases} 1, & -a \le x \le a \\ 0, & \text{elsewhere.} \end{cases}$



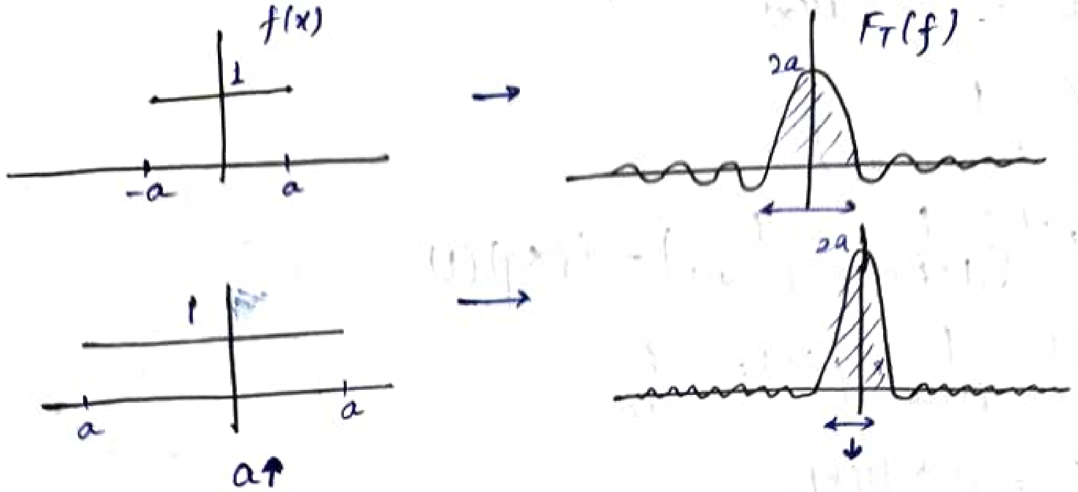
<u>$\alpha = 1$</u>

$F_T(f) = \begin{cases} 2\frac{\sin \cancel{a\omega}\,\omega}{\cancel{a}\omega}, & \omega \ne 0 \\ 2, & \omega = 0 \end{cases}$

<u>$a \ne 1$</u>

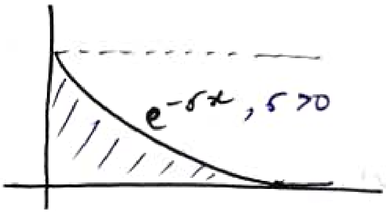
$F_T(f) = \begin{cases} 2a\left(\frac{\sin a\omega}{a\omega}\right), & \omega \ne 0 \\ 2a, & \omega = 0 \end{cases}$

→ $F_T(1)$ does not exist, according to definition:



$$\left.\begin{array}{l} \rightarrow F_T(1) \\ F_T(e^{-x}), F_T(e^{x}) \\ F_T(\sin x) \\ F_T(\cos x) \\ F_T(x) \\ F_T(x^n) \end{array}\right\} \text{does not exist.}$$

#



Unilateral Laplace transform : $\int_0^\infty$

→ Suppose $g \in L_1(\mathbb{R})$, $\int_{-\infty}^{\infty} |g(x)|\, dx < \infty$.

$$g(x) = (F_T^{-1} \circ F_T)\, g(x)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F_T(g)\, e^{i\omega x}\, d\omega.$$

$$\therefore \boxed{g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(x) e^{-i\omega x} dx \right) e^{i\omega x}\, d\omega.}$$

Set/choose $f : [0, \infty) \to \mathbb{R}$.

$$g(x) = \begin{cases} e^{-\sigma x} f(x), & x \geq 0 \\ 0, & x < 0 \end{cases}$$ such that $g \in L_1(\mathbb{R})$.

$$= e^{-\sigma x} f(x) H(x).$$

$$e^{-\sigma x} f(x) H(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-\sigma x} f(x) e^{-i\omega x} H(x)\, dx \right) e^{i\omega x}\, d\omega$$

$$\Rightarrow f(x) H(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{0}^{\infty} e^{-\sigma x} f(x) e^{-i\omega x}\, dx \right) e^{i\omega x} e^{\sigma x}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{0}^{\infty} f(x) e^{-(\sigma + i\omega)x}\, dx \right) e^{(\sigma + i\omega)x}\, d\omega.$$

$$\sigma + i\omega = s$$
$$\Rightarrow \frac{ds}{d\omega} = i$$

$$\therefore f(x) H(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \left[\int_{0}^{\infty} f(x) e^{-sx}\, dx \right] ds$$

$$= \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \hat{f}(s) e^{sx}\, ds$$

$$= (L_T^{-1} \circ L_T)(x)$$

Defⁿ: $f : [0, \infty) \to \mathbb{R}$

$$L_T(f) = \int_0^\infty f(x) e^{-sx}\, dx = \hat{f}(s).$$

LAPLACE TRANSFORM

(21-03-2024)

Defn: Let $f : [0,\infty) \to \mathbb{R}$

$$\boxed{L_T(f) = \int_0^\infty f(x) e^{-sx}\, dx = \hat{f}(s)}$$, provided it exists.

$s = \sigma + i\omega$

Eg. $f(x) = K, \ x \in [0,\infty)$

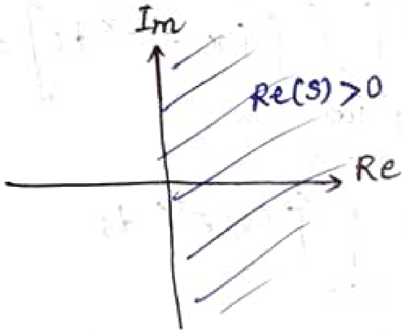
$$L_T(K) = K \int_0^\infty e^{-sx} dx = \frac{K}{-s} e^{-sx}\Big|_0^\infty$$

$$= \frac{K}{s} - \frac{K}{s} e^{-sx}\Big|_{x\to\infty}$$

$$e^{-sx} = e^{-(\sigma+i\omega)x}$$
$$= e^{-\sigma x} e^{-i\omega x} \to 0 \ , \text{ as } x \to 0$$

(if $\sigma > 0$)

OR (if $Re(s) > 0$)



$$\therefore \boxed{L_T(K) = \frac{K}{s} \ , \text{ if } Re(s) > 0}$$

For simplicity, we choose $s \in \mathbb{R}$.

$$\boxed{L_T(K) = \frac{K}{s} \ , \text{ if } s > 0}$$

Note: ① $f : (-\infty,\infty) \to \mathbb{R}$.

$f(x) = K$.

$L_T(f) = \frac{K}{s}$.

$(-\infty, 0)$ is not playing any role.

② $g(x) \in L_1(\mathbb{R})$

$g(x) = (F_T^{-1} \circ F_T)\, g(x)$, $-\infty < x < \infty$.

$f:[0,\infty)\to\mathbb{R} \,/\, f:(-\infty,\infty)\to\mathbb{R}$.

$f(x)H(x) = (L_T^{-1}\circ L_T)(f), \ \forall -\infty, x<\infty$

$f(x) = (L_T^{-1}\circ L_T)(f), \ \forall \ x\geq 0$.

Eg. $F_T^{-1}\left(\frac{1}{2+3i\omega-\omega^2}\right) = \left(e^{-x}-e^{-2x}\right)(H(x))$

Eg. $F_T^{-1}\left(\frac{4}{(1+\omega^2)^2}\right) = F_T^{-1}\left(\frac{2}{1+\omega^2}\cdot\frac{2}{1+\omega^2}\right) \cancel{H(x)}$ | $F_T(e^{-|x|})=\frac{2}{1+\omega^2}$

Eg. $L_T(e^x) = \int_0^\infty e^x e^{-sx}\,dx$

$$= \int_0^\infty e^{-(s-1)x}\,dx$$

$$= -\frac{e^{-(s-1)x}}{(s-1)}\Big|_0^\infty$$

$$= \frac{1}{s-1} - \cancel{\frac{e^{-(s-1)x}}{s-1}} \to 0 \text{ for } s>1 \Big|_{x\to\infty}$$

$$= \frac{1}{s-1}, \text{ if } s>1$$

$\therefore \ L_T(e^{ax}) = \frac{1}{s-a}, \ s>a$

Eg. $L_T(x) = \int_0^\infty x\,e^{-sx}\,dx$

$$= \frac{e^{-sx}}{-s}\cdot x\Big|_0^\infty - \int_0^\infty \frac{e^{-sx}}{-s}\,dx$$

$$= -\frac{x}{s}e^{-sx}\Big|_0^\infty - \frac{e^{-sx}}{s^2}\Big|_0^\infty$$

$$= \cancel{-\frac{x}{s}e^{-sx}} \to 0, s>0 \Big|_{x\to\infty} + \cancel{\frac{x}{s}e^{-sx}} \to 0 \Big|_{x\to 0} - \cancel{\frac{e^{-sx}}{s^2}} \to 0, s>0 \Big|_{x\to\infty} + \frac{e^{-sx}}{s^2}\Big|$$

$$= \frac{1}{s^2}, \ s>0$$

$$L_T(\sin x) = \int_0^\infty \left(\frac{e^{ix}-e^{-ix}}{2i}\right) e^{-sx}\, dx$$

$$= \frac{1}{2i}\int_0^\infty \left(e^{-(s-i)x} - e^{-(s+i)x}\right) dx$$

$$= \frac{1}{2i}\,\frac{e^{-(s-i)x}}{-(s-i)}\Bigg|_0^\infty - \frac{1}{2i}\,\frac{e^{-(s+i)x}}{-(s+i)}\Bigg|_0^\infty$$

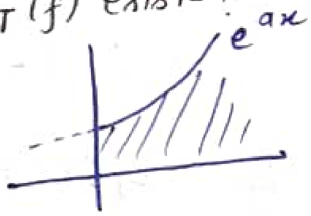
$$L_T(\sin x) = \frac{1}{s^2+1}, \quad s>0$$

$$L_T(\cos x) = \frac{s}{s^2+1}, \quad s>0$$

$$L_T(\sin ax) = \frac{a}{s^2+a^2}, \quad s>0$$

$$L_T(\cos ax) = \frac{s}{s^2+a^2}, \quad s>0.$$

Q] Suppose $f : (-\infty, \infty) \to \mathbb{R}$ such that $F_T(f)$ does not exist. we are looking for laplace transform of f. What is the property of f such that $L_T(f)$ exists?



Theorem

(I) Suppose f is P-C on $[0, \infty)$.

$$f(x) = \begin{cases} 1/x, & (0,\infty) \ni x \\ 0, & x=0 \end{cases}$$

↳ Not P-C.

At x_0, f is discontinuous, then $\lim_{x \to x_0} f(x)$ must exists for f to be P-C. [~~atleast one-sided limit should exist~~]

⇗ there must be a finite jum in the function.

Here, $\lim_{x \to 0} f(x)$ does not exist in $\mathbb{R}$.

f should be bounded on a finite interval. | $[0, L], L>$

(II) f must be exponential order with abcissa of convergence, σ_c.

$$\Downarrow$$

$L_T(f)$ exists, if $s > \sigma_c$. [Sufficient but not necessary condition for existence of Laplace transform.]

$\boxed{22\text{-}02\text{-}2024}$

→ Suppose f is P-C on $[0, \infty)$, and f is of exponential order with σ_c (abcissa of convergence, then $L_T(f)$ exists for $s > \sigma_c$.

Defn: f is of exponential order with σ_c as $x \to \infty$ if $\exists$ a constant $M > 0$ and a real constant σ such that

$$\boxed{|f(x)| \le M e^{\sigma x} \quad \forall\ x > x_0}$$

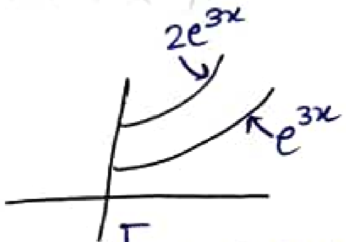
(some real no. x_0)

$$\Rightarrow f(x) \le |f(x)| \le M e^{\sigma x}$$

$$\Rightarrow e^{-\sigma x} f(x) \le M \quad \forall\ x > x_0$$

Note: σ_c is the lowest value of σ.

Eg. ① $f(x) = e^{3x}$



Choose $M = 2$, $\sigma = 3 \Rightarrow \sigma_c = 3$.

$f(x) = |f(x)| \le 2e^{3x} \quad \forall\ x > 0$.

$\therefore L_T(f)$ exists for $s > \sigma_c = 3$.

[$L_T(e^{3x}) = \frac{1}{s-3}$, $s > 3$

② $f(x) = x$.

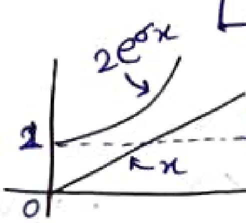
$x \le 2e^{\sigma x}$, $x > 0$

$\forall\ \sigma > 0$

$\Rightarrow \sigma_c > 0$.

[→ P-C f^ns on a finite interval are bounded.]

$\therefore L_T(f)$ exists for $s > \sigma_c > 0$.



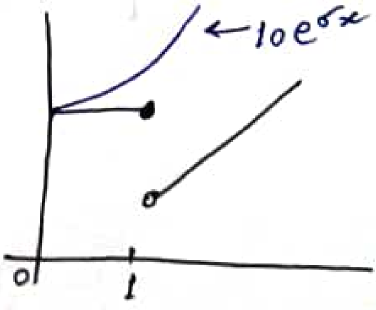
$\sigma_c \neq 0$

[$L_T(x) = \frac{1}{s^2}$, $s > 0$]

③ $f(x) = \begin{cases} 10, & 0 \le x \le 1 \\ x, & x > 1. \end{cases}$

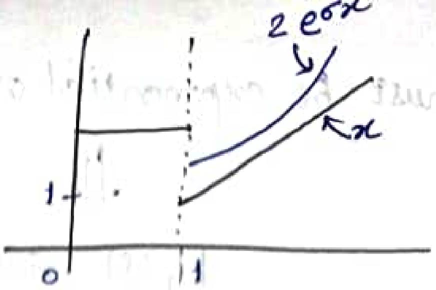
$f(x) \le 10 e^{\sigma x} \quad \forall\ x > 0$

$\hookrightarrow \boxed{x_0 = 0}$



OR $f(x) \leq 2e^{\sigma x} \; \forall \; x > 1$
$\hookrightarrow \boxed{x_0 = 1}$

$\forall \; \sigma > 0$
$\Rightarrow \sigma_c > 0$

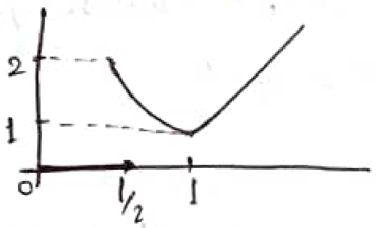


④ $$f(x) = \begin{cases} 0, & x = 0 \\ \frac{1}{x}, & 0 < x \leq 1 \\ x, & x > 1 \end{cases}$$

$\hookrightarrow$ f is not P-C.
$\Rightarrow L_T(f)$ is not guaranteed.
$\Rightarrow L_T(f)$ may or may not exist.
$\Rightarrow f$ is not exponential order.

⑤ $$f(x) = \begin{cases} 0, & 0 \leq x \leq 1/2 \\ 1/x, & 1/2 < x \leq 1 \\ x, & x > 1 \end{cases}$$

$\rightarrow$ f is P-C. (finite jump at $x = 1/2$)
$\Rightarrow$ f is exponential order.



⑥ $f(x) = x^2, \quad x \in [0, \infty)$

Counter-Example:

$$f(x) = \begin{cases} \frac{1}{\sqrt{x}}, & 0 < x < \infty \\ 0, & x = 0 \end{cases}$$

$\rightarrow$ Not P-C, but LT exists.

$$\int_0^\infty f(x)\, e^{sx} dx$$

$$= \underbrace{\int_0^{x_0} f(x)\, e^{sx} dx}_{\substack{\swarrow A \\ (\text{Cond. I})}} + \underbrace{\int_{x_0}^\infty f(x)\, e^{sx} dx}_{\substack{B \searrow \\ (\text{Cond. II})}}$$

28-03-2024

Properties of Laplace Transform

Suppose $L_T(f) = \hat{f}(s)$, $s>0$, exists.

① First Shift Theorem:

$$L_T\left(e^{ax}f(x)\right) = L_T(f)\big|_{s\leftarrow s-a}$$
$$= \hat{f}(s-a).$$

Eg. $L_T(e^{2x}\underbrace{x}_{f}) = L_T(x)\big|_{s\leftarrow s-2}$
$$= \frac{1}{(s-2)^2}$$

② Derivative of Laplace Transform:

$$L_T\left(x^n f(x)\right) = (-1)^n \frac{d^n \hat{f}(s)}{ds^n}.$$

Eg $L_T(\underbrace{e^{2x}}_{f}.x) = (-1)^n \frac{d}{ds} L_T(e^{2x})$ | $L_T(e^{2x}) = \frac{1}{s-2}$.

$$= (-1)\cdot(-1).\ \frac{1}{(s-2)^2} = \frac{1}{(s-2)^2}$$

③ Laplace transform of derivative:

(i) f must be continuous on $[0,\infty)$ and f is of exponential order.

(ii) f' is p-c on $[0,\infty)$. [f' is of ~~exponential~~ order.] (No need)

⇒ Sufficient conditions

$$L_T\left(f'(x)\right) = \int_0^\infty \underbrace{f'(x)}_{II}\,\underbrace{e^{-sx}}_{I}\,dx = -f(0) + s\hat{f}(s)$$

LT of double derivative:

(i) f, f' must be continuous on $[0,\infty)$ and f, f' are of exponential order.

(ii) f'' is p-c on $[0,\infty)$.

$$L_T\left(f''(x)\right) = \int_0^\infty \underbrace{f''(x)}_{I}\,\underbrace{e^{-sx}}_{II}\,dx = -f'(0) - sf(0) + s^2\hat{f}(s).$$

Eg. $y'' + y = 1$

$y(0) = 0$

$y'(0) = 0$

} Initial Value Problem (IVP)

Apply Transform

$$\downarrow FT/LT$$

$$DE(x) : \frac{d}{ds}\hat{f}(s) + 2\hat{f}(s) = \frac{1}{s^2+1}$$

OR

$$\frac{d}{dw}\hat{f}(w) + 2\hat{f}(w) = \frac{1}{w^2+1}$$

$$\downarrow$$

$$\hat{f}(s) \,/\, \hat{f}(w)$$

$$\downarrow LT^{-1}/FT^{-1}$$

$$f(x)$$

Soln: $L_T(y'') + L_T(y) = L_T(1)$

$$\Rightarrow s^2\hat{y}(s) + \hat{y}(s) = \frac{1}{s}$$

$$\Rightarrow \hat{y}(s) = \frac{1}{s(s^2+1)}$$

→ $F(s) = \frac{P(s)}{Q(s)}$, $\deg(P(s)) < \deg(Q(s))$.

① $Q(s) = as + b \longrightarrow F(s) = \frac{A}{as+b}$ → Find A

② $Q(s) = as^2 + bs + c \longrightarrow F(s) = \frac{Bs + C}{(as^2+bs+c)}$ → Find B, C.

③ $Q(s) = (as+b)^2 \longrightarrow F(s) = \frac{A_1}{(as+b)} + \frac{A_2}{(as+b)^2}$ → find A_1, A_2.

④ $Q(s) = (as^2+bs+c)^2 \longrightarrow F(s) = \frac{B_1 s + C_1}{(as^2+bs+c)} + \frac{B_2 s + C_2}{(as^2+bs+c)^2}$

→ Find B_1, B_2, C_1, C_2.

$$L_T^{-1}\left(\frac{1}{s(s^2+1)}\right) = L_T^{-1}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)$$

$$\left[\begin{aligned} \frac{1}{s(s^2+1)} &= \frac{A}{s} + \frac{Bs+C}{s^2+1} \\ &= \frac{As^2 + A + Bs^2 + cS}{s(s^2+1)} \\ &= \frac{A + Cs + (A+B)s^2}{s(s^2+1)} \end{aligned}\right]$$

$$= 1 - \cos x, \quad x \geq 0.$$

04-04-2024

Laplace Convolution

Let $f, g : [0,\infty) \longrightarrow \mathbb{R}$.

Define F, G such that

$F : \mathbb{R} \to \mathbb{R}$ such that $F(x) = \begin{cases} f(x), & x \geq 0 \\ 0, & x < 0. \end{cases}$

$G : \mathbb{R} \to \mathbb{R}$ such that $G(x) = \begin{cases} g(x), & x \geq 0 \\ 0, & x < 0. \end{cases}$

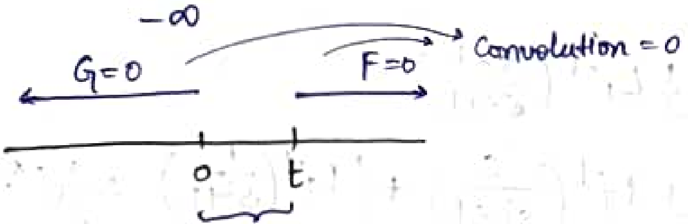
$$F(t-x) = \begin{cases} f(t-x), & t-x \geq 0 \Rightarrow x \leq t \\ 0, & t-x < 0 \Rightarrow x > t \end{cases}$$

Fourier Convolution

$g, f : \mathbb{R} \to \mathbb{R}$

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t-x).g(x)\,dx.$$

$$(F * G)(t) = \int_{-\infty}^{\infty} F(t-x)\,G(x)\,dx$$



$$\therefore (F * G)(t) = \int_0^t F(t-x)\,G(x)\,dx$$

Defn: $\boxed{(f * g)(t) = \int_0^t f(t-x)\,g(x)\,dx}$

Convolution Theorem:

$f, g \in L[0,\infty)$

$L_T(f) = \hat{f}(s)$

$L_T(g) = \hat{f}(s)$

$\Rightarrow \boxed{f * g = L_T^{-1}\left(\hat{f}(s)\,\hat{g}(s)\right)}$

Eg. $L_T^{-1}\left(\frac{1}{s(s^2+1)}\right) = L_T^{-1}\left(\underbrace{\frac{1}{s}}_{\hat{f}(s)} \cdot \underbrace{\frac{1}{s^2+1}}_{\hat{g}(s)}\right)$

$$\left[\begin{array}{l} L_T(1) = 1/s \;,\; s>0 \\ L_T(\sin x) = \frac{1}{s^2+1} \;,\; s>0 \end{array}\right]$$

$$= (f*g)(t)$$

$$= \int_0^t 1.\sin x\,dx = -\cos x\Big|_0^t$$

$$= 1-\cos t$$

Eg. $L_T^{-1}\left(\frac{1}{(s+1)(s^2+1)}\right)$

Use partial fraction:

$$\frac{1}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}$$

Find A, B, C.

$\Rightarrow A=\frac{1}{2}, C=\frac{1}{2}, B=\frac{-1}{2}.$

$$L_T^{-1}\left(\frac{1}{(s+1)(s^2+1)}\right) = \frac{1}{2}L_T^{-1}\left(\frac{1}{s+1}\right) + L_T^{-1}\left(\frac{-\frac{1}{2}s+\frac{1}{2}}{s^2+1}\right)$$

$$= \frac{1}{2}L_T^{-1}\left(\frac{1}{s+1}\right) - \frac{1}{2}L_T^{-1}\left(\frac{s-1}{s^2+1}\right)$$

$$= \frac{1}{2}L_T^{-1}\left(\frac{1}{s+1}\right) - \frac{1}{2}L_T^{-1}\left(\frac{s}{s^2+1}\right) + \frac{1}{2}L_T^{-1}\left(\frac{1}{s^2+1}\right) = \frac{1}{2}e^{-x} - \frac{1}{2}\cos x + \frac{1}{2}\sin x.$$

Apply convolution: $L_T^{-1}\left(\underbrace{\frac{1}{(s+1)}}_{\hat{f}(s)} \cdot \underbrace{\frac{1}{(s^2+1)}}_{\hat{g}(s)}\right)$

$$= (f*g)(t) = \int_0^t e^{(t-x)}\sin x\,dx$$

$$= e^{-t}\int_0^t e^{x}\sin x\,dx$$

Eg. $L_T^{-1}\left(\frac{1}{\sqrt{s}(s^2+1)}\right)$. | Use convolution theorem: Find f such that $L_T(f) = \frac{1}{\sqrt{s}}$.

Second-shift Theorem

→ How to deal with piecewise-continuous function.

$$H(x) = \begin{cases} 1, & x \geqslant 0 \\ 0, & x < 0 \end{cases}$$

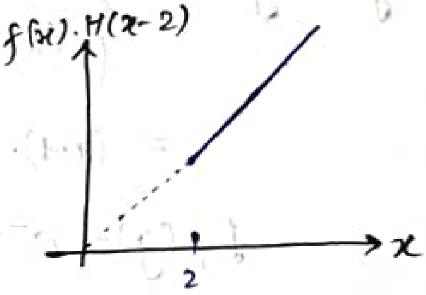
$$L_T(H(x)) = \frac{1}{s}.$$

$a \geqslant 0$,

$$H(x-a) = \begin{cases} 1, & x \geqslant 0 \\ 0, & x < 0. \end{cases}$$

$$f(x).H(x-a) = \begin{cases} f(x), & x \geqslant a \\ 0, & x < a \end{cases}$$

↳ "No shifting"



$f(x) = x$, $a = 2$

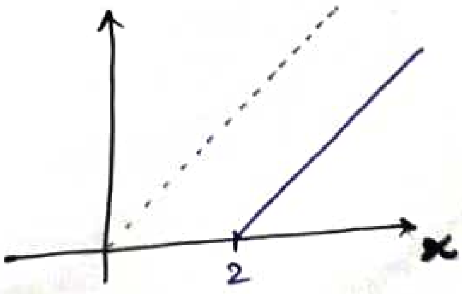
↳ "f is switched on at $x=2$"

$$f(x) = \begin{cases} 0, & x < a \\ K, & a \leq x < b \\ 0, & x \geqslant b. \end{cases}$$

$$= KH(x-a) - KH(x-b)$$

$$f(x-a)H(x-a) = \begin{cases} f(x-a), & x \geqslant a \\ 0, & x < a \end{cases}$$

↳ "f is delayed by 'a' units."



Second-shift Theorem :

Suppose $L_T(f) = \hat{f}(s)$

$$L_T(f(x-a).H(x-a)) = \int_a^\infty f(x-a)\,e^{-sx}\,dx$$

$$\left[\begin{array}{l} x-a=X \Rightarrow x=a+X \\ \quad \Rightarrow \frac{dx}{dX}=1 \end{array}\right]$$

$$= \int_0^\infty f(X)\,e^{-s(a+X)}\,dX$$

$$= e^{-as}\int_0^\infty f(X)\,e^{-sX}\,dX$$

$$= e^{-as}L_T(f)$$

$$\therefore \boxed{L_T(f(x-a).H(x-a)) = e^{-as}\underbrace{L_T(f)}}$$

*** NOT the laplace transform of $f(x-a)$

Eg. $g(x) = \begin{cases} 0, & 0 \le x < 1 \\ (x-1), & x \ge 1. \end{cases}$

$= (x-1).H(x-1)$ ⇝ "x" is delayed by 1 unit"

$[f(x)=x]$

$L_T(g) = e^{-s}L_T(x)$.

$$= \frac{e^{-s}}{s^2}.$$

07-04-2024

Discontinuous Source Function

Q1 Solve $y'' + y = f(x)$,

$y(0)=0, \; y'(0)=1$

$$f(x) = \begin{cases} 2, & x \geq 1 \\ 0, & 0 < x < 1. \end{cases}$$

Soln: $y'' + y = f(x)$

$\Rightarrow y'' + y = 2H(x-1)$

$$LT \Rightarrow s^2 Y(s) - \cancel{s y(0)}^{\,0} - \cancel{y'(0)}^{\,1} + Y(s) = 2\frac{e^{-s}}{s}$$

$$\Rightarrow s^2 Y(s) + Y(s) - 1 = 2\frac{e^{-s}}{s}$$

$$\Rightarrow Y(s)\left[s^2+1\right] = 1 + \frac{2e^{-s}}{s}$$

$$\Rightarrow Y(s) = \frac{1}{s^2+1} + \frac{2e^{-s}}{s(s^2+1)}$$

$$= \frac{1}{s^2+1} + 2e^{-s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)$$

$$y(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) + 2\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s} - e^{-s}\cdot\frac{s}{s^2+1}\right\}$$

$$= \sin x + 2H(x-1) + 2\cos(x-1).H(x-1)$$

$$= \begin{cases} \sin x, & 0 \leq x < 1 \\ 2 - \cos(x-1) + \sin x, & x \geq 1. \end{cases}$$

Q1 Find LT of:

$f(x) = x, \; 0 \leq x < 1$

and $f(x+1) = f(x)$.

Q] Solve using LT :

$$\begin{tikzcd} y'' + xy = f(x) \arrow[d, "LT"] \\ DE(\hat{y}(s)) \end{tikzcd}$$

Eg. $$\begin{tikzcd} \frac{d\hat{y}(s)}{ds} + \hat{y}(s) = 1 \arrow[d, "solve"] \\ \hat{y}(s) \arrow[d, "ILT"] \\ y(x) \end{tikzcd}$$

System of DEs

Eg. $\frac{dx}{dt} = -y - t,$

$\frac{dy}{dt} = x - 1,$

$x(0) = 0, \ y(0) = 0.$

$$\left[\begin{array}{c} \frac{d\vec{x}}{dt} = A\vec{x} + B \\ \Big\downarrow {\scriptstyle \text{convert to diagonal matrix (if not already)}} \\ D \end{array} \right] \longrightarrow$$ No need here!

Soln: $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \ A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \ B = \begin{bmatrix} -t \\ -1 \end{bmatrix}.$

So, $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -t \\ -1 \end{bmatrix}.$

Now, $s\hat{x}(s) = -\hat{y}(s) - \frac{1}{s^2}$ ... ⓘ

$s\hat{y}(s) = \hat{x}(s) - \frac{1}{s}$ ... ⓘⓘ

$\Rightarrow \hat{x}(s) = s\hat{y}(s) + \frac{1}{s}$

$\Rightarrow s\hat{x}(s) = s^2\hat{y}(s) + 1$

From ①, $s^2\hat{y}(s)+1=-\hat{y}(s)-\frac{1}{s^2}$

$$\Rightarrow (s^2+1)\hat{y}(s)=-1-\frac{1}{s^2}$$

$$\Rightarrow \hat{y}(s)=-\frac{(1+s^2)}{s^2(s^2+1)}=-\frac{1}{s^2}\xrightarrow{L_T^{-1}} y(t)=-t$$

$$\hat{x}(s)=0\xrightarrow{L_T^{-1}} x(t)=0$$

$$\therefore x(t)=0,\ y(t)=-t$$

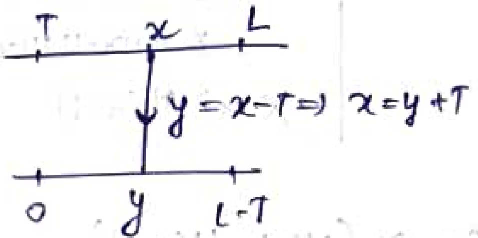
2nd order system: $\frac{d^2\vec{x}}{dt^2}+\frac{d\vec{x}}{dt}=A\vec{x}+B$.

<u>**Result**</u>: Suppose f is a periodic function with period T, and f is P-C on interval of length T.

$$\mathcal{L}(f)=\int_0^\infty e^{-sx}f(x)\,dx$$

$$=\underbrace{\int_0^T e^{-sx}f(x)\,dx}_{I}+\underbrace{\int_T^\infty e^{-sx}f(x)\,dx}_{II}$$

II : $\int_T^\infty e^{-sx}f(x)\,dx=\lim_{L\to\infty}\int_T^L e^{-sx}f(x)\,dx$



$$=\lim_{L\to\infty}\int_0^{L-T} e^{-s(y+T)}f(y+T)\,d(y+T)$$

$$=e^{-ST}\lim_{L\to\infty}\int_0^{L-T} e^{-ST}f(y)\,dy$$

$$=e^{-ST}\int_0^\infty e^{-ST}f(y)\,dy$$

$$=e^{-ST}\mathcal{L}(f)$$

$$\therefore \mathcal{L}(f)=e^{-ST}\mathcal{L}(f)+\int_0^T e^{-sx}f(x)\,dx$$

$$\Rightarrow (1-e^{-ST})\mathcal{L}(f)=\int_0^T e^{-sx}f(x)\,dx$$

$$\Rightarrow \boxed{\mathcal{L}(f)=\frac{1}{1-e^{-ST}}\int_0^T e^{-sx}f(x)\,dx}$$

$$\mathcal{L}_T(f) = \int_0^\infty e^{-sT} f(x)\,dx$$

$$= \frac{1}{1-e^{-sT}} \int_0^\infty G(x) e^{-sx}\,dx \quad , \quad G(x) = \begin{cases} f(x), & 0 \le x < T \\ 0, & x > T \end{cases}$$

$$= \frac{1}{1-e^{-sT}} L_T(G). \qquad = f(x)\left(H(x) - H(x-T)\right)$$

$\mathcal{L}_T(f) = \hat{f}(s) \longrightarrow$ Known

$\mathcal{L}_T\left(f(x-a) \cdot H(x-a)\right) = e^{-as} \hat{f}(s)$, $a \geq 0$ ... second-shift theorem.

$\hookrightarrow$ f is delayed by 'a' unit.

$\mathcal{L}_T\left(f(x) \cdot H(x-a)\right) = ?$

$\hookrightarrow$ f is switched ON at $x=a$.

$$\mathcal{L}_T\left(f(x-a+a) \cdot H(x-a)\right)$$

$$= \mathcal{L}_T\left(F(x-a) \cdot H(x-a)\right) \quad , \quad F(x) = f(x+a).$$

$$= e^{-as} \mathcal{L}_T(F(x))$$

$$= e^{-as} \mathcal{L}_T(f(x+a))$$

linear

✓ $x \, H(x-1) = (x-1+1) H(x-1)$

$= (x-1) H(x-1) + H(x-1)$

✗ $\sin x \cdot H(x-1) \overset{\sin}{=} (x-1+1) H(x-1)$

$\neq \sin(x-1) H(x-1) + H(x-1)$

✗ $x^2 H(x-1)$

Eg. LT of $f(x) = x$, $0 \le x < 1$

$\quad$ & $f(x+1) = f(x)$

$$\hookrightarrow \mathcal{L}_T(f) = \frac{1}{1-e^{-sT}} \mathcal{L}_T(G), \quad G = x\left(H(x) - H(x-1)\right).$$

$$\mathcal{L}_T(G) = \mathcal{L}_T(x H(x)) - \mathcal{L}_T(x H(x-1))$$

$$= \mathcal{L}_T(x H(x)) - \mathcal{L}_T((x-1) H(x-1)) - \mathcal{L}_T(H(x-1)).$$

→ $L_T(\frac{1}{x}) \to$ does not exist ✗

$L_T(\frac{1}{\sqrt{x}}) \to$ exists ✓

$L_T(x)$ ✓

$L_T(\sqrt{x})$ ✓

$L_T(x^n)$, $n=0,1,2,\ldots$ ✓

$L_T(x^\nu)$, $\nu=-\frac{1}{2}$ ✓
$\nu=\frac{1}{2}$ ✓
$\nu=3/2$ ✓

Piecewise continuity
OR
for every x,
$\int_0^x f(x)\,dx$ exists.

$$L_T(x^\nu) = \int_0^\infty e^{-sx} x^\nu dx$$

$[sx = t \Rightarrow x = t/s]$

$$= \int_0^\infty e^{-t} \left(\frac{t}{s}\right)^\nu d\left(\frac{t}{s}\right)$$

$$= \frac{1}{s^{(\nu+1)}} \int_0^\infty e^{-t} t^\nu dt$$

Gamma function: $\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1} dx$, $\alpha > 0$.

Properties: $\Gamma(1) = 1$
$\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$.

$\alpha = n$: $\Gamma(n+1) = n\Gamma(n)$
$= n!$

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$[\Gamma : (0,\infty) \to \mathbb{R}]$

$$= \frac{1}{s^{\nu+1}} \int_0^\infty e^{-t} t^{(\nu+1)-1} dt$$

$$= \frac{1}{s^{\nu+1}} \Gamma(\nu+1), \quad \nu+1 > 0.$$

$$\therefore L_T(x^{-1/2}) = \frac{\Gamma(-\frac{1}{2}+1)}{s^{-\frac{1}{2}+1}} = \frac{\Gamma(\frac{1}{2})}{\sqrt{s}} = \frac{\sqrt{\pi}}{\sqrt{s}}, \; s > 0.$$

$$L_T(\sqrt{x}) = \frac{1}{2} \cdot \frac{\sqrt{\pi}}{s^{3/2}}.$$

Eg. $\mathcal{L}_T^{-1}\left(\frac{1}{\sqrt{s}(s+1)}\right)$.

→ $\mathcal{L}_T\left(\frac{1}{x}\right)$

$$\boxed{\mathcal{L}_T\left(\frac{f(x)}{x}\right) = \int_s^\infty \hat{f}(\sigma)\, d\sigma}$$

Condition: ① $\lim_{x\to 0} \frac{f(x)}{x}$ must exist.

② $\mathcal{L}_T(f)$ exists $= \hat{f}(s)$.

Eg. $\mathcal{L}_T\left(\frac{1-\cos 2x}{x}\right) \cancel{= \mathcal{L}_T\left(\frac{\sin^2 x}{2x}\right)}$

$f(x) = 1-\cos 2x$

$$\mathcal{L}_T(f(x)) = \frac{1}{s} - \frac{s}{s^2+4}$$

$$\therefore \mathcal{L}_T\left(\frac{f(x)}{x}\right) = \int_s^\infty \left(\frac{1}{\sigma} - \frac{\sigma}{\sigma^2+4}\right) d\sigma$$

$$= \left[\ln\sigma - \frac{1}{2}\ln(\sigma^2+4)\right]_s^\infty$$

$$= \ln\frac{\sigma}{\sqrt{\sigma^2+4}}\Bigg|_s^\infty$$

$$= -\frac{1}{2}\ln\left(\frac{s^2}{s^2+4}\right).$$

Proof: $\int_s^\infty \hat{f}(\sigma)\, d\sigma = \int_{\sigma=s}^\infty \left(\int_{x=0}^\infty e^{-sx} f(x)\, dx\right) d\sigma$ ⇒ interchange the limit.