Indian Institute of Space Science and Technology

Thiruvananthapuram-695547

Summer Supplementary Examination, June-July 2014

B.Tech 2nd Semester

MA121 - Vector Calculus and Differential Equations

Date : 23^{rd}**June, 2014** **Time: 9.30 am to 12.30 pm** **Max. Marks: 100**

SECTION A (Answer all 10 questions - 10x5= 50 marks.)

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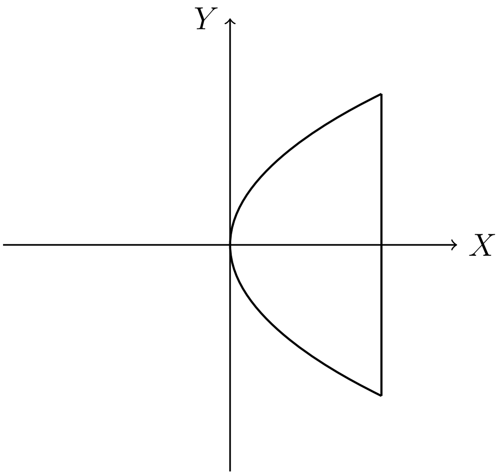
1. Determine whether the sequence $f_n(x) = \dfrac{n^2 \ln x \sin nx}{x^n}$, $x \in [2, \infty)$ converges uniformly on the given interval? (5)

2. Evaluate the following limit with appropriate justification: (5)

$$\lim_{n\to\infty} \int_1^2 \left(\frac{x^2+1}{8}\right)^n \sin nx \; dx.$$

3. Define arc length function of a smooth curve. Find the arc length function of the curve C : $3x = \left(\dfrac{y}{2\sqrt{z}} + 2\right)^3 = (\ln z)^3$ with initial point $(0, -4, 1)$. [Hint: take $z = e^t$] [2+3]

4. Let $\vec{F} = (xye^z + ze^x, \frac{1}{2}x^2e^z, \frac{1}{2}x^2ye^z + e^x + 1)$ be a vector field and C be a curve joining straight line segments $(0,1,0) \longrightarrow (0,0,1) \longrightarrow (1,0,0)$ and oriented accordingly. Evaluate $\int_C \vec{F} \cdot d\vec{r}$. Is the integral independent of the path? Justify your answer. [3+2]

5. 

Using Green's theorem find the area of the region G bounded by the parabola $x = y^2$ and the straight line $x = 2$ on xy-plane. [5]

6. Let a surface S be given by $\vec{r}(u,v) = (\cos u, \sin u, 2v)$ where $u \in [0, 2\pi]$ and $v \in [-1, 1]$. Represent the surface S pictorially. Using surface integral, find the surface area of S. [1+4]

7. By solving the differential equation

$$(x^2 - 2x)\,\frac{dy}{dx} = 2(x-1)y,$$

find out initial points (x_0, y_0) such that the given differential equation with the initial condition $y(x_0) = y_0$, has
(i) no solution, (ii) unique solution, (iii) infinitely many solutions.

8. (a) If the Wronskian of two functions f and g is $W(f,g)(x) = 3e^{4x}$, where $f(x) = e^{2x}$, then find $g(x)$.
(b) Determine whether the pair of functions $\{f(x) = x^5,\ g(x) = x^2|x|^3\}$ can be solutions of the differential equation $y'' + p(x)y' + q(x)y = 0$ with p and q continuous on $[-a, a]$, $a > 0$.

9. Find the singular point of the following differential equation and classify it. Also determine the interval of convergence of the corresponding series solution of

$$(x-\pi)y'' + \frac{1+\pi-x}{2\pi - 2\pi x + 2\pi^2}y' + \frac{1}{\frac{3}{2}(x-\pi)^2 + \frac{2}{3}}y = 0,$$

about that point.

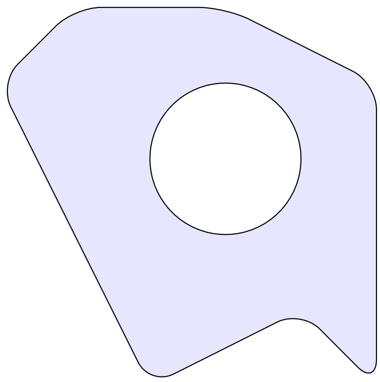
10. Find the general solution of $xy'' + (1+2\lambda)y' + xy = 0, \quad x > 0$ in terms of Bessel's functions, using the substitution $y = \dfrac{u(x)}{x^\lambda}$, where λ is a positive real number.

SECTION B (Answer any 5 questions - 5x10= 50 marks.)

11. (a) Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$ does not converge uniformly on $[0,1)$. (4)
(b) Let $\{f_n\}$ be a sequence of functions defined on $[a,b]$. Then under what conditions the following can be justified: $\displaystyle\int_a^b \sum_{n=1}^{\infty} f_n(x)dx = \sum_{n=1}^{\infty}\int_a^b f_n(x)dx$. Hence show that $\ln 2 = \displaystyle\sum_{n=1}^{\infty}\frac{1}{n\ 2^n}$.
(1+5)

12. (a) State Stoke's theorem for non-simply connected surfaces having two boundary components (i.e., only one hole). Verify the stated Stoke's theorem for the vector field $F(x,y,z) = (x^2, y^2, z)$ over the annular region $S : 1 \le x^2 + y^2 \le 9,\ z = 3$. [2+6]
(b) From definition find the directional derivative of $xy + e^z$ at the point $(1,0,1)$ along the unit vector $\vec{v} = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. [2]

13. (a) Using Green's theorem calculate the double integral $\displaystyle\iint_G (y-1)\ dx\ dy$ in terms of line integral where G is given by $G : x^2 + y^2 \le 4$. [3]

(b) 

Let G be a connected region on xy-plane with one hole as depicted in the figure with smooth boundaries C_1 and C_2 oriented positively. Let $\vec{F}$ be a smooth vector field with domain G having curl$(\vec{F}) = 0$. Show that the line integrals of $\vec{F}$ along C_1 and along C_2 are the same. [7]

14. (a) Using the Picard's method of successive approximations find the first three approximations ϕ_0, ϕ_1, ϕ_2 to the solution of the initial-value problem $\dfrac{dy}{dx} = y + y^2, \quad y(0) = 1$. Also find the exact solution of the above initial-value problem. **[(3+2) Marks]**

    (b) Find the general solution of $y'' + 6y' + 9y = \dfrac{e^{-3x}}{x}$, $x > 0$, using the method of variation of parameters. **[5 Marks]**

15. (a) Find the indicial equation, the recurrence relation and the Frobenius series solution up to <u>first two terms</u> of $(x - x^2)y'' + (1 - 5x)y - 4y = 0$, about the point $x = 0$ for $x > 0$. Also write the proper form of another independent solution. **[(5+2) Marks]**

    (b) Using the Picard's theorem, verify whether the initial-value problem

$$\frac{dy}{dx} = 1 + y^{2/3}, \; y(2014) = \frac{1}{2^{2014}},$$

has a unique solution around the point 2014.
If $y(2014) = 0$, does the Picard's theorem guarantee the existence of a unique solution? Justify your answer. **[(2+1) Marks]**

16. (a) Using the definition of Bessel function of first kind of order $p\,(p > 0)$, given by

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n + p + 1)} \left(\frac{x}{2}\right)^{2n+p}$$

establish the identity $\dfrac{d}{dx}[x^p J_p(x)] = x^p J_{p-1}(x)$. Also show that between any two positive roots of $J_{p+1}(x) = 0$, there is a root of $J_p(x) = 0$. **[(2+3) Marks]**

    (b) Find the eigenvalues and eigenfunctions of the boundary value problem

$$\frac{d}{dx}(x\frac{dy}{dx}) + \frac{\lambda}{x}y = 0, \quad 1 < x < l, \quad y(1) = 0, \, y(l) = 0, \quad \lambda \in \mathbb{R}.$$

[5 Marks]