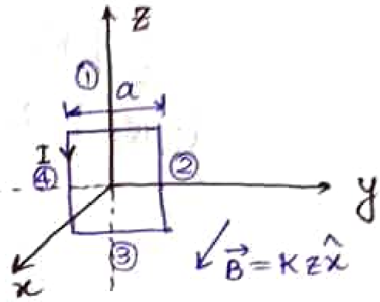
PHYSICS-II Assignment

Q① If the magnetic field $\vec{B} = kz\hat{x}$ in one region, then find the force on a square loop of length 'a' lying in the yz-plane and centered at the origin. Assume that it carries a current I flowing counter clockwise when you look down upon the x-axis.

Soln: $\vec{B} = kz\hat{x}$



Forces on square loop,

$$F_1 = BIl\hat{z} = I\,k\left(\frac{a}{2}\right)a = \frac{ka^2I}{2}\hat{z}$$

$$\vec{F}_2 = \int_{-a/2}^{a/2} (BIdl)\,\hat{y} = \int_{-a/2}^{a/2} k\,Iz\,dz\,\hat{y} = \frac{kI}{2}\left[z^2\right]_{-a/2}^{a/2}\hat{y} = \vec{0}$$

$$\vec{F}_3 = k\left(-\frac{a}{2}\right)a\,I = -\frac{ka^2I}{2}\hat{z}$$

Also, $\vec{F}_4 = 0$

$$\therefore \vec{F}_{net} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4$$

$$= ka^2I\,\hat{z}$$

Q.② The vector potential $\vec{A}(\vec{r}) = -\frac{1}{2}(\vec{r}\times\vec{B})$, then find the magnetic field $\vec{B}$ and $\text{div}\,\vec{A}$ values.

Soln: $\vec{A}(\vec{r}) = -\frac{1}{2}(\vec{r}\times\vec{B})$

As, $\vec{B} = \vec{\nabla}\times\vec{A}$

$$= -\frac{1}{2}\vec{\nabla}\times(\vec{r}\times\vec{B}) \quad , \text{ where, } \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$= -\frac{1}{2}\left[(\vec{B}.\vec{\nabla})\vec{r} - (\vec{r}.\vec{\nabla})\vec{B} + \vec{r}(\vec{\nabla}.\vec{B}) - \vec{B}(\vec{\nabla}.\vec{r})\right]$$

Now, by maxwell eqn,

$$\vec{\nabla}.\vec{B} = 0 \quad ; \quad \vec{\nabla}.\vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

$$(\vec{B}.\vec{\nabla})\vec{r} = B_x\frac{\partial\vec{r}}{\partial x} + B_y\frac{\partial\vec{r}}{\partial y} + B_z\frac{\partial\vec{r}}{\partial z}$$

$$= B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

$$= \vec{B}$$

$$\therefore \vec{B} = -\frac{1}{2}\left[\vec{B} - (\vec{r}.\vec{\nabla})\vec{B} + 0 - 3\vec{B}\right]$$

$$\Rightarrow \vec{B} = \vec{B} + \frac{1}{2}(\vec{r}.\vec{\nabla})\vec{B} \Rightarrow \frac{1}{2}(\vec{r}.\vec{\nabla})\vec{B} = 0$$

$$\Rightarrow r_x \frac{\partial B}{\partial x} + r_y \frac{\partial B}{\partial y} + r_z \frac{\partial B}{\partial z} = 0$$

$\because \vec{B}$ is a constant function.

$\therefore \vec{B} = a\hat{i} + b\hat{j} + c\hat{k}$, where a, b and c are constants.

Now,

$$\vec{\nabla}\cdot\vec{A} = -\frac{1}{2}\vec{\nabla}\cdot(\vec{r}\times\vec{B})$$

$$= -\frac{1}{2}\left[\vec{B}\cdot(\vec{\nabla}\times\vec{r}) - \vec{r}\cdot(\vec{\nabla}\times\vec{B})\right]$$

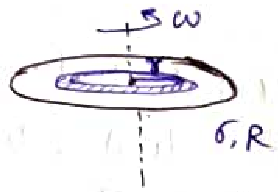
where, $\vec{\nabla}\times\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0$ and $\vec{\nabla}\times\vec{B} = 0$ as $\vec{B}$ is a constant fn.

$$\therefore \vec{\nabla}\cdot\vec{A} = 0$$

Q.3 A phonograph record of radius 'R', carrying a uniform surface charge σ is rotating with a constant angular velocity 'ω'; then find its magnetic dipole moment.

Soln:

$$m = \int_0^R A\cdot dI \quad , \text{where } A = \pi r^2$$



$$dq = \sigma(2\pi r dr)$$

Let time period, $T = \frac{2\pi}{\omega}$

$$\therefore dI = \frac{dq}{T} = \frac{\sigma 2\pi r dr \omega}{2\pi} = \sigma\omega r dr$$

$$\therefore m = \int_0^R (\pi r^2)\sigma\omega r dr$$

$$= \pi\sigma\omega\int_0^R r^3 dr = \frac{\pi\sigma\omega R^4}{4}$$

Q.4 An infinitely long cylinder of radius R carries a frozen-in magnetization parallel to the axis, which is, $\vec{M} = ks\hat{z}$, where k is constant and s is radial distance from the axis. Then, find magnetic field inside and outside the cylinder.

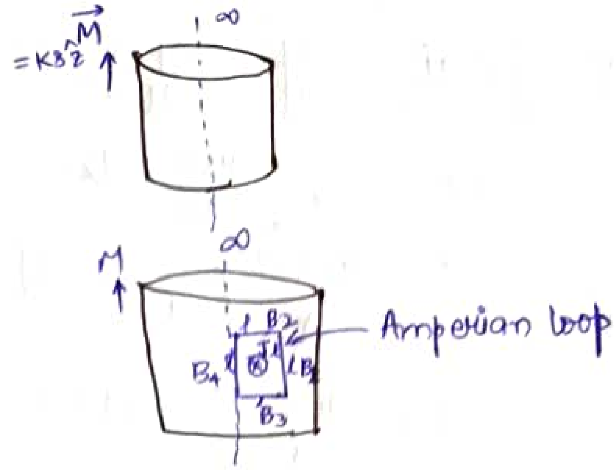
Sol^n: Inside the cylinder

$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{encl.}$

$= \mu_0 \int \vec{J_b}\cdot d\vec{a}$

where, $\vec{J_b} = \vec{\nabla}\times\vec{M}$

$= -\frac{\partial}{\partial s}(Ks)\,\hat{\phi} = -K\hat{\phi}$



$\therefore \oint \vec{B}\cdot d\vec{l} = B_1 l + B_2 l - B_3 l - B_4 l$

$= -\mu_0 \int K\cdot da = -\mu_0 K l^2$

$B_4 = 0$ as B_4 lies on the symmetry axis of the cylinder

$B_2 = B_3$ as cylinder is infinitely long and symmetric.

$\therefore B_1 l = -\mu_0 K l^2$

$\Rightarrow B_1 = -\mu_0 K l$

or $B_{in} = (\mu_0 K s)\,\hat{\phi}$, where s: distance from the axis.

Outside the cylinder

$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{encl.} = \mu_0 \int \vec{J_b}\cdot d\vec{a} = \mu_0 \int \vec{K_s}\, d\vec{l}$

where, $\vec{K_s} = \vec{M}\times\vec{n} = \vec{M}\times\vec{s}$

$= Ks(\hat{z}\times\hat{n})$

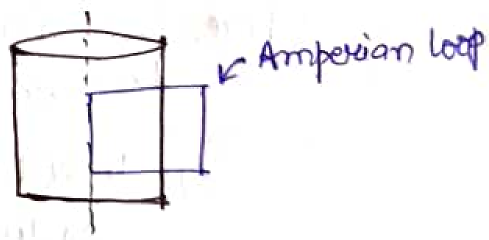
$= Ks(\hat{z}\times\hat{s})$

$= Ks\,\hat{\phi}$

$= KR\,\hat{\phi}$ [as K_s exists at surface, $s=R$]

$\therefore \oint \vec{B}\cdot d\vec{l} = -\mu_0 \int_0^R K\cdot da + \mu_0 \int_0^R KR\, dl$

$= -\mu_0 K R^2 + \mu_0 K R^2 = 0$



$\therefore \vec{B}_{out} = 0$

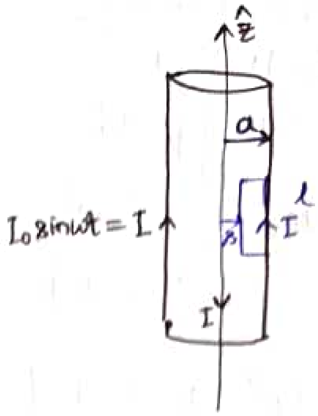
Thus, $\vec{B} = \begin{cases} -\mu_0 K s\,\hat{\phi} & , s < a \\ 0 & , s \geq a \end{cases}$

Q.5 An alternating current $I = I_0 \sin\omega t$ flows down a long wire and returns along a co-axial conducting tube of radius 'a'. Find the dir^n of induced electric field whether it longitudinal or radial or circumferential.

Soln: $\int \vec{E}\cdot\vec{dl} = -\frac{d\phi}{dt} = -\frac{d}{dt}\int \vec{B}\cdot\vec{da}$

$$El = -\frac{\partial}{\partial t}\int_{s}^{R}\frac{\mu_0 I}{2\pi s}\, l\, ds$$

$$El = -\frac{\mu_0 I_0 l}{2\pi}\left(\frac{\partial}{\partial t}\sin\omega t\right)\ln\left(\frac{R}{s}\right)$$



$$\therefore E = -\frac{\mu_0 I_0}{2\pi}\,\omega\cos(\omega t).\ln\left(\frac{R}{s}\right)$$

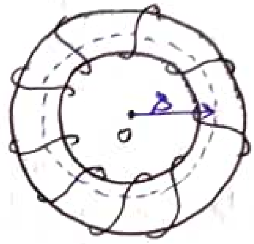
$\vec{E}$ will flow along the dirn of current (ie, longitudinally).

Q.6 Calculate the energy stored in a toroidal coil derived in our class.

Soln: $\oint \vec{B}\cdot\vec{dl} = \mu_0 NI$

$$\Rightarrow B(2\pi s) = \mu_0 NI$$

$$\therefore \vec{B}_{inside} = \frac{\mu_0 NI}{2\pi s}\hat{\phi}$$



and, $B_{outside} = 0$ (as $I_{encl.} = 0$)

$$\therefore \text{Energy stored} = \frac{1}{2\mu_0}\int_V |B|^2 d\tau$$

$$W = \frac{1}{2\mu_0}\int\left(\frac{\mu_0 NI}{2\pi s}\right)^2 s\, d\phi\, dz\, ds$$

$$= \frac{1}{2\mu_0}\int\frac{\mu_0^2 N^2 I^2}{4\pi^2 s^2}\, s.ds\,(2\pi)\,[2R\sin\theta] \qquad \left[\because \int_{-R\sin\theta}^{R\sin\theta} dz = 2R\sin\theta\right]$$

where, $s = r - R\cos\theta$

$ds = R\sin\theta\, d\theta$

$$\therefore W = \frac{\mu_0^2 N^2 I^2}{8\pi^2\mu_0}(2\pi)(2R^2)\int_0^{\pi}\frac{\sin^2\theta\, d\theta}{r - R\cos\theta}$$

$$= \frac{\mu_0 N I^2 r}{\pi}\ln\left(\frac{r+R}{r-R}\right).$$

Q.7 In free space, the electric field $\vec{E} = A\cos(\omega t - 50z)\,\hat{x}$ V/m. Then, calculate (i) displacement current $\vec{J}_d$, (ii) the magnetic field $\vec{H}$ & ω.

Soln: (i) $\vec{J}_d = \epsilon_0\frac{\partial \vec{E}}{\partial t} = -\omega\epsilon_0 A\sin(\omega t - 50z)\,\hat{x}$

(ii) $\vec{H} = \frac{\vec{B}}{\mu_0}$

and, $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$$\Rightarrow \frac{\partial}{\partial z}(E_x)\hat{y} = -\frac{\partial \vec{B}}{\partial t}$$

$$\Rightarrow -\int 50\, A \sin(\omega t - 50z)\, dt\, \hat{y} = \vec{B}$$

$$\Rightarrow \vec{B} = \frac{50A}{\omega}\cos(\omega t - 50z)\,\hat{y}$$

$$\therefore \vec{H} = \frac{50A}{\mu_0 \omega}\cos(\omega t - 50z)\,\hat{y}$$

Now,

$$\vec{\nabla} \times \vec{H} = \epsilon_0 \frac{\partial \vec{E}}{\partial t} \Rightarrow -\frac{\partial H_y}{\partial z} = -\epsilon_0 A\omega \sin(\omega t - 50z)$$

$$\frac{(50)^2 A}{\mu_0 \omega} = \epsilon_0 A\omega \Rightarrow \omega^2 = \frac{50^2}{\mu_0 \epsilon_0}$$

$$\Rightarrow \omega = \frac{50}{\sqrt{\mu_0 \epsilon_0}}$$

Q.⑧ In a medium characterised by $\sigma = 0$, $\mu = \mu_0$, $\epsilon = 3\epsilon_0$, the electric field is given by $\vec{E} = E_0 \cos(\omega t - kz)\,\hat{x}$ V/m, then calculate $\vec{H}, \vec{B}, \vec{D}$ using Maxwell's equations.

Soln: $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$

$$\Rightarrow \frac{\partial \vec{H}}{\partial t} = -\frac{kE_0}{\mu_0}\sin(\omega t - kz)\,\hat{y}$$

$$\Rightarrow \vec{H} = \frac{kE_0}{\mu_0 \omega}\cos(\omega t - kz)\,\hat{y}$$

and, $\vec{B} = \frac{kE_0}{\omega}\cos(\omega t - kz)\,\hat{y}$

and, $\vec{D} = \epsilon \vec{E} + \vec{P}$, where $\vec{P} = 0$

$$= 3\epsilon_0 E_0 \cos(\omega t - kz)\,\hat{x}$$

Q.⑨ If $\vec{A} = (2\hat{x} + 3\hat{y} - 4\hat{z})$, then find magnetic field $\vec{B}$, div. $\vec{A}$ and div. $\vec{B}$.

Soln: $\vec{B} = \vec{\nabla} \times \vec{A} = \hat{i}(0) - \hat{j}(0) + \hat{k}(0) = \vec{0}$ (as $\vec{A}$ = constant)

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial(2)}{\partial x} + \frac{\partial(3)}{\partial y} + \frac{\partial(-4)}{\partial z} = 0$$

$$\vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot (\vec{0}) = 0$$

Q. ⑩ The electric field and magnetic field in free space are given by, $\vec{E} = \frac{1}{\rho}\cos(10^8 t + kz)\hat{\phi}$, $\vec{H} = \frac{H_0}{\rho}\cos(10^8 t + kz)\hat{\rho}$, then show that they satisfy Maxwell's eq^n and also find from that the values of H_0 and propagation constant K.

Soln: $\vec{\nabla}\cdot\vec{E} = \frac{1}{\rho}\frac{\partial E_\phi}{\partial \phi} = 0$ (as $\vec{E}$ doesn't depend on ϕ)

$$\vec{\nabla}\times\vec{E} = \frac{-\partial E_\phi}{\partial z} = \frac{k}{\rho}\sin(10^8 t + kz) = -\mu_0\frac{\partial H}{\partial t}$$

$$= +\frac{\mu_0 H_0 \times 10^8}{\rho}\sin(10^8 t + kz)\hat{\rho}$$

where, $H_0 = \frac{K}{10^8\mu_0}$, $\frac{\omega}{k} = c$ and $\omega = 10^8 \Rightarrow \boxed{K = \frac{10^8}{c}}$

$$\Rightarrow \boxed{H_0 = \frac{1}{\mu_0 c}}$$

Also,

$\vec{\nabla}\cdot\vec{H} = 0$

and, $\vec{\nabla}\times\vec{H} = \epsilon_0\frac{\partial\vec{E}}{\partial t}$

as $\vec{\nabla}\times\vec{H} = \frac{\partial H_\rho}{\partial z}\hat{\phi} = \frac{-kH_0}{\rho}\sin(10^8 t + kz)\hat{\phi}$

$$= \frac{-10^8}{\rho\mu_0 c^2}\sin(10^8 t + kz)$$

$$= -\frac{\epsilon_0 10^8}{\rho}\sin(10^8 t + kz)$$

and, $\epsilon_0\frac{\partial\vec{E}}{\partial t} = -\frac{\epsilon_0 10^8}{\rho}\sin(10^8 t + kz)\hat{\phi}$

$$\therefore\ \vec{\nabla}\times\vec{H} = \epsilon_0\frac{\partial\vec{E}}{\partial t}$$