AVD611- Modern Signal Processing
Assignment-2:

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① $L = 20$

Single stage:

$$10\text{ KHz} \longrightarrow \boxed{\uparrow 20} \xrightarrow{f_x = 200\text{ KHz}} \underset{\text{LPF}}{\boxed{\text{Filter}}} \longrightarrow 200\text{ KHz}$$

$\delta_1 = 10^{-2}$
$\delta_2 = 10^{-3}$

Passband: $0 \leq F \leq 90 = f_{pc}$

Transition band: $90 \leq F \leq 100 = f_{sc}$

$$\Delta f = \frac{f_{sc} - f_{pc}}{f_x} = \frac{100-90}{200k} = \frac{10}{200k} = \frac{1}{20k} = 5\times10^{-5}$$

$$\therefore N = \frac{-10\log(\delta_1\delta_2) - 13}{14.6\Delta f} + 1$$

$$= \frac{50-13}{14.6(5\times10^{-5})} + 1$$

$$= \frac{37}{7.3}\times10^{4} + 1$$

$$\approx 50,686.$$

Two stage: $L = 20 = 2\times10$

1st stage:

$$10\text{ KHz} \longrightarrow \boxed{\uparrow 2} \xrightarrow{20\text{ KHz}} \boxed{\text{LPF}} \longrightarrow \boxed{\uparrow 10} \xrightarrow{200\text{ KHz}} \boxed{\text{LPF}} \longrightarrow$$

Passband: $0 \leq F \leq 90$

Transition: $90 \leq F \leq 10k - 100$

$$\Delta f = \frac{(10k-100) - 90}{20k} = 0.4905$$

$$\therefore N_1 = \frac{-10\log\left(\frac{1}{2}\times10^{-5}\right) - 13}{14.6\times0.4905} + 1$$

$$= \frac{40}{7.1613} + 1$$

$$\approx 7$$

2nd stage: Passband: $0 \leq F \leq 90$

Transition: $90 \leq F \leq 20k - 100$

$$\Delta f = \frac{(20k-100) - 90}{200k} = 0.09905$$

$$\therefore N_2 = \frac{-10\log(0.5\times10^{-5}) - 13}{14.6\times0.09905} + 1$$

$$= \frac{40}{1.44613} + 1 \approx 28.$$

② $x(n) = a^n u(n), \quad |a|<1.$

ⓐ $\Rightarrow X(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n}$

$$= \sum_{0}^{\infty} (az^{-1})^n$$

$$= \frac{1}{1-az^{-1}}$$

$$\Rightarrow X(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}$$
$$= X(\omega)$$

ⓑ $D=2.$

$$X_d(z) = \frac{1}{2}\sum_{k=0}^{1} X(z^{1/2}\cdot W_M^{-k}), \quad W_M^k = e^{-j\frac{2\pi k}{M}} = e^{-j\pi k}$$

$$= \frac{1}{2}\left[X(z^{1/2}\cdot W_2^0) + X(z^{1/2}\cdot W_2^{-1})\right]$$

$$= \frac{1}{2}\left[X(z^{1/2}) + X(-z^{1/2})\right] \qquad [\because W_M^0 = 1, W_M^1 = -1]$$

$$\Rightarrow X_d(\omega) = \frac{1}{2}\left[X(e^{j\omega/2}) + X(-e^{j\omega/2})\right]$$

$$= \frac{1}{2}\left[\frac{1}{1-ae^{-j\omega/2}} + \frac{1}{1+ae^{-j\omega/2}}\right]$$

$$= \frac{1}{2}\left[\frac{2}{1-a^2e^{j\omega}}\right]$$

$$= \frac{1}{1-a^2e^{j\omega}}.$$

ⓒ $x(2n) = \{a^0, a^2, a^4, \dots\}$

$$\Rightarrow X(z) = a^0 + a^2z^{-1} + a^4z^{-2} + \dots$$

$$= \frac{1}{1-a^2z^{-1}}$$

$$X(e^{j\omega}) = \frac{1}{1-a^2e^{-j\omega}} = X_d(\omega).$$
$$= X(\omega)$$

③ $$x(n) \longrightarrow \boxed{\uparrow 7} \xrightarrow{a(n)} \boxed{28\downarrow} \xrightarrow{b(n)} \boxed{\uparrow 4} \longrightarrow y(n)$$

$$a(n) = \begin{cases} x\left(\frac{n}{7}\right), & n = 0, \pm 7, \pm 14 \\ 0, & \text{otherwise} \end{cases}$$

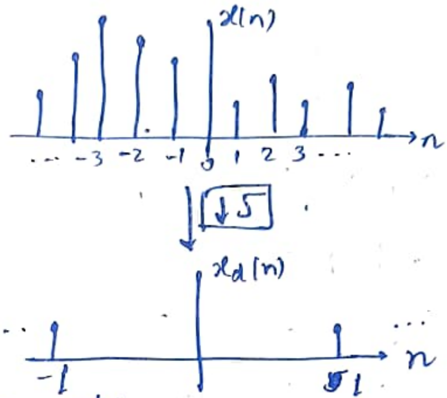
$b(n) = a(28n) = x(4n)$, $28n = 0, \pm 7, \pm 14, \ldots$ which is always true.

$$y(n) = \begin{cases} b(n/4), & n = 0, \pm 4, \pm 8, \ldots \\ 0 \end{cases}$$

$$= \begin{cases} x(n), & n = 0, \pm 4, \pm 8 \\ 0, & \text{otherwise.} \end{cases}$$

(4) $D = 5$.

$x_d(n) = x(5n)$

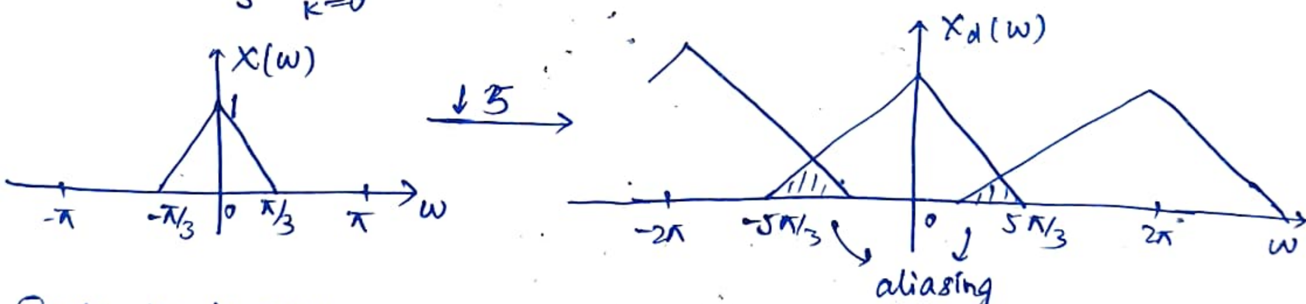


$$\Rightarrow X_d(Z) = \frac{1}{5} \sum_{K=0}^{4} X(Z^{1/5} \cdot W_5^{-k})$$

$$= \frac{1}{5} \sum_{K=0}^{4} X(Z^{1/5} \cdot e^{-j\frac{2\pi}{5}k})$$

$$X_d(w) = \frac{1}{5} \sum_{K=0}^{4} X(e^{-jw/5} \cdot e^{-j2\pi/5 K})$$

$$= \frac{1}{5} \sum_{K=0}^{4} X(e^{-\frac{j}{5}(w - 2\pi k)})$$



Output spectrum results in aliasing, which cannot be reconstructed. To reconstruct, we have to avoid aliasing, we should apply low-pass filtering before decimation.

$$\longrightarrow \boxed{H(Z)} \longrightarrow \boxed{\downarrow 5}$$

↳ LPF of $|w_c| \leq \frac{\pi}{5}$.

5 $\begin{bmatrix} H_0(z) & H_1(z) & H_2(z) \end{bmatrix} = \begin{bmatrix} 1 & z^{-1} & z^{-2} \end{bmatrix} \begin{bmatrix} 2 & 4 & 1 \\ -1 & 4 & -2 \\ 2 & -1 & 2 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} H_0(z) \\ H_1(z) \\ H_2(z) \end{bmatrix} = \begin{bmatrix} 2 & -1 & 2 \\ 4 & 4 & -1 \\ 1 & -2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \\ z^{-2} \end{bmatrix}$$

$$= P(z^3)\,a(z)$$

$$P(z^3) = \begin{bmatrix} 2 & -1 & 2 \\ 4 & 4 & -1 \\ 1 & -2 & 2 \end{bmatrix}$$

$Q(z^3) = c\,[P(z^3)]^{-1}$, choose $c = 1 \times \det(P(z^3))$.

$$= \operatorname{adj}(P(z^3))$$

$$= \begin{bmatrix} 6 & -2 & -7 \\ -9 & 2 & 10 \\ -12 & 3 & 12 \end{bmatrix}$$

Synthesis filter:

$$G(z) = z^{-2}\,Q(z^3)\,a(z^{-1})$$

$$= z^{-2} \begin{bmatrix} 6 & -2 & -7 \\ -9 & 2 & 10 \\ -12 & 3 & 12 \end{bmatrix} \begin{bmatrix} 1 \\ z \\ z^2 \end{bmatrix}$$

$$= \cancel{z^{-2}} \begin{bmatrix} 6 & -2 & -7 \\ -9 & 2 & 10 \\ -12 & 3 & 12 \end{bmatrix} \begin{bmatrix} z^{-2} \\ z^{-1} \\ 1 \end{bmatrix}$$

6 Downsampler noble property:

$$x(n) \to \underbrace{\boxed{\downarrow D}}_{\text{System}} \to \boxed{H(z)} \to y(n) \equiv x(n) \to \boxed{H(z^D)} \to \underbrace{\boxed{\downarrow D}}_{\text{System}} \to y(n)$$

But for time-invariant system:

$$x(n) \to \boxed{z^{-n}} \to \boxed{\text{system}} \to y(n) \equiv x(n) \to \boxed{\text{system}} \to \boxed{z^{-n}} \to y(n)$$

which are not the same as downsampler property, hence downsampler is a time-varying system.

7. For half-band filter,

$$H(z) + H(-z) = 2\alpha = \text{constant}.$$

For $H(z) = -3 + 19z^{-2} + 32z^{-3} + 19z^{-4} - 39z^{-6}$

$$H(z) + H(-z) = -6 + 38z^{-2} + 38z^{-4} - 78z^{-6} \neq 2\alpha.$$

$\hookrightarrow$ Not a half-band filter

For $H(z) = 3 - 25z^{-2} + 150z^{-4} + 256z^{-5} + 150z^{-6} - 25z^{-8} + 3z^{-10}$

$$H(z) + H(-z) = 6 - 50z^{-2} + 300z^{-6} + 300z^{-6} - 50z^{-8} + 6z^{-10} \neq 2\alpha.$$

$\hookrightarrow$ Not a half-band filter.

8.
$$\begin{bmatrix} H_0(z) \\ H_1(z) \\ H_2(z) \\ H_3(z) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -2 & 3 & -1 \\ 2 & 1 & 1 & 0 \\ 0 & 3 & -1 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix}}_{P(z^4)} \begin{bmatrix} 1 \\ z^{-1} \\ z^{-2} \\ z^{-3} \end{bmatrix}$$

$$Q(z^4) = c\,[P(z^4)]^{-1}$$

$$= \frac{c}{\cancel{\det(P(z^4))}} \begin{bmatrix} -2/3 & 1 & -2/3 & -1/3 \\ 1 & -1 & 1 & 1 \\ 1/3 & 0 & 1/3 & -1/3 \\ -8/3 & 3 & -5/3 & -10/3 \end{bmatrix}$$

Choose $c = 3.\cancel{\det(P(z^4))}$.

$$\therefore Q(z^4) = \begin{bmatrix} -2 & 3 & -2 & -1 \\ 3 & -3 & 3 & 3 \\ 1 & 0 & 1 & -1 \\ -8 & 9 & -5 & -10 \end{bmatrix}$$

Synthesis filters:

$$G(z) = \begin{bmatrix} G_0(z) \\ G_1(z) \\ G_2(z) \\ G_3(z) \end{bmatrix} = z^{-3} Q(z^4)\, a(z^{-1})$$

$$= \begin{bmatrix} -2 & 3 & -2 & -1 \\ 3 & -3 & 3 & 3 \\ 1 & 0 & 1 & -1 \\ -8 & 9 & -5 & -10 \end{bmatrix} \begin{bmatrix} z^{-3} \\ z^{-2} \\ z^{-1} \\ z^{0} \end{bmatrix}.$$