INDIAN INSTITUTE OF SPACE SCIENCE AND TECHNOLOGY
THIRUVANANTHAPURAM 695 547

Backlog Examination - May 2016

B. Tech - II Semester

MA121 - Vector Calculus and Differential Equations

Date: 06/05/2016 **Time: 9.30 am - 12.30 pm** **Max. Marks: 100**

SECTION A (Attempt all 10 questions - 10x5= 50 marks.)

1. Find the general solution of $\dfrac{d^2y}{dx^2} - xf(x)\dfrac{dy}{dx} + f(x)y = 0$.

2. Find a particular solution of $y'' + y = \operatorname{cosec} x$, by using method of variation of parameters.

3. Find the general solution of of $(D-2)^3 y = e^{2y}\sin y$, where $D \equiv \dfrac{d}{dx}$.

4. Solve the differential equation $(e^y - 2xy)\dfrac{dy}{dx} = y^2$.

5. Check whether the sequence $f_n(x) = \dfrac{n \ln x}{x^n}$, $x \in [1, \infty)$ converges uniformly on the given interval.

6. Show that the series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n}$ converges uniformly on $[0,1]$.

7. Define directional derivative of a function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ at a point P_0 along a vector $\vec{v}$. Check whether the directional derivative of $f(x,y,z) = ze^{xy}$ at the point $P_0 = (1,1,1)$ along the vector $\vec{v} = (0,4,3)$ exists, and if exists find the value of the directional derivative. [2+1+2]

8. State Green's theorem over simply-connected regions. Let D be an elliptical region on xy-plane given by $D = \left\{ (x,y) \middle| \dfrac{x^2}{16} + \dfrac{y^2}{4} \leq 1 \right\}$. With explanation find the area of the region D using Green's theorem . [1+4]

9. Define arc length function of a continuously differentiable curve $\gamma : [a,b] \longrightarrow \mathbb{R}^3$. Find arc length function of the curve $C : y = 4x^2,\ z = 5,\ x \geq 0$ with initial point $(0,0,5)$. $[2+3]$

10. Let $\vec{F}$ be a continuous vector field such that $\vec{F} = \nabla f$ on $\mathbb{R}^3$. Let $C = \gamma : [a,b] \longrightarrow \mathbb{R}^3$ be a continuously differentiable curve. Show that $\displaystyle\int_C \vec{F} = f(\gamma(b)) - f(\gamma(a))$. [5]

SECTION B (Attempt any 5 questions - 5x10= 50 marks.)

11. Find the general solution of the following equation

$$2x^2\frac{d^2y}{dx^2} + x(2x+1)\frac{dy}{dx} - y = 0$$

12. (a) Show that $J_{-m}(x) = (-1)^m J_m(x)$.

    (b) Show that between any two positive zeros of $J_1(x)$ there is a zero of $J_0(x)$.

13. (a) Let $\{f_n(x)\}$ be a sequence of continuous functions defined on a finite interval $[a, b]$. Suppose that $\{f_n(x)\}$ converges uniformly to f on $[a, b]$. Then show that

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx.$$

    State whether the converse of this statement is true or false. [5]

    (b) Compute the following with appropriate justification: [5]

$$\lim_{n\to\infty} \int_0^1 \frac{\sin nx}{n + x^2} dx.$$

14. (a) Suppose that the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly to a function F on an interval I. Let each f_n be continuous on I. Then prove that F is also continuous on I. [3]

    (b) Consider the function $F(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{1 + x^{2n}}$, $x \in (0, 2/3)$. Show that F is continuous on the interval $(0, 2/3)$. [7]

15. (a) State Stoke's theorem for simply-connected surfaces. Verify Stoke's theorem for $\mathbf{F} = (\mathbf{x},\ \mathbf{y},\ \mathbf{0})$ over the surface $S\ :\ x^2 + y^2 + z^2 = 1,\ z \geq 0$ [5]

    (b) Let $\mathbf{F} = \left(\frac{-\mathbf{y}}{\mathbf{x^2 + y^2}}, \frac{\mathbf{x}}{\mathbf{x^2 + y^2}} \right)$. Find the domain of $\mathbf{F}$. Calculate $Curl(\mathbf{F})$. Using Green's theorem for multiply connected domains show that for any simple loop C around the point $(0, 0)$, we have $\int_C \mathbf{F} = \mathbf{2}\pi$. [1+ 1 + 3]

16. Define conservative vector field. Let

$$\vec{F}(x, y, z) = (y \exp{(x + y)} + xy \exp{(x + y)} + yz,\ x \exp{(x + y)} + xy \exp{(x + y)} + xz,\ xy)$$

for all $(x, y, z) \in \mathbb{R}^3$. Find the domain of F. Let $C = C_1 * C_2 * C_3$ be a curve where C_1 is the line segment from $(-2, -1, 5)$ to $(-1, 0, 5)$, C_2 is given by $x^2 + y^2 = 1$, $z = 5$, $y \leq 0$ and C_3 is the line segment from $(1, 0, 5)$ to $(2, -1, 5)$. Show that the vector field F is conservative. Find the gradient function of $\vec{F}$. Find $\int_C \vec{F}$, if exists. Is the integral path independent? Justify your answer. [1+2+2+3+2]

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