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PARTIAL DIFFERENTIAL EQUATIONS

10-01-2024

Partial Differential Equations

ODE: Ordinary Differential Equation

↳ Linear algebra + Real analysis

First Order ODE:

$$f\left(x, y, \frac{dy}{dx}\right) = 0$$

$$\Rightarrow \left.\begin{array}{l} \dfrac{dy}{dx} = g(x,y) \\ y(x_0) = y_0 \end{array}\right\} \text{IVP}$$

integrate

$$\Leftrightarrow y = y_0 + \int_{x_0}^{x} g(t, y(t))\,dt$$

equivalent if g is constant.

Second Order ODE:

$$y'' + p(x)y' + q(x)y = R(x), \quad x \in I$$

BVP in ODE:

Eg $y' = 0 \Rightarrow y = c$

In ODE, solution contains additive constant.

$$S = \left\{ \phi \in C'(I) : \frac{d\phi}{dx} = 0 \right\} = \mathbb{R}$$

$$\dim(S) = 1$$

Eg. $\frac{dz}{dx} = 0 \Rightarrow p = 0$

~~In PDE, Solution contains arbitrary functions.~~

$\cancel{z = f(y)}$

$\cancel{S = \left\{ \phi \in C^1(\Omega) \mid \frac{d\phi}{dx} = 0 \right\}}$

$\cancel{c_1\phi_1 + c_2\phi_2 \in S}$

~~$\dim(S)$ is infinite.~~

$\cancel{S = \{\log y, \sin y, \cos y, \dots\}}$

PDE: Partial Differential Equation

↳ Linear algebra + Real analysis + Complex analysis

First order PDE:

$f(x, y, z, p, q) = 0$ ⇝ general form

Notation: z or u : dependent variable

x, y : independent variable

↳ can be more than 2

$$z_x = \frac{\partial z}{\partial x} = p, \quad z_y = \frac{\partial z}{\partial y} = q$$

Second Order PDE: Same as ODE.

BVP in PDE: Same as ODE.

Eg. $\frac{dz}{dx} = 0 \Rightarrow z = f(y)$

Solutions contains only function.

$$S = \left\{ \phi \in C'(\Omega) \mid \frac{d\phi}{dx} = 0 \right\}.$$

$$c_1\phi_1 + c_2\phi_2 \in S$$

$\dim(S)$ = infinite.

$$S = \{\log y, \sin y, \cos y, \dots\}$$

Remark: Solution space of n^{th} order linear homogeneous ODE is a vector space in dimension 'n'.

Eg. $p+q=0$

$$\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=0 \Rightarrow \frac{\partial z}{\partial x}=-\frac{\partial z}{\partial y}$$

$$\Rightarrow z=x-y$$
$$z=\sin(x-y)$$
$$z=f(x-y)$$

$$z_x=f(x-y).(1);$$
$$z_y=f(x-y).(-1)$$

$$z_x+z_y=0$$
$$\Rightarrow p+q=0$$

Eg. $ap+bq=0$
$$z=f(bx-ay)$$

Eg. $ap+bq=c$

Eg. $zp^2+q=0$

WE ARE JUST GUESSING THE SOLUTION, NOT FINDING IT.

12-01-2024

→ $f(x,y,z,p,q)=0$

$$p=z_x=\frac{\partial z}{\partial x} \; ; \; q=z_y=\frac{\partial z}{\partial y}$$

$$az_x+bz_y=0 \Rightarrow z=f(bx-ay)$$

Construction of PDE

Eg. $x^2+y^2+(z-c)^2=a^2$, where a and c are arbitrary constants.

↳ family of spheres.

Diff. with x ⇒ $2x+2(z-c)p=0$ $[p=z_x, q=z_y]$

Diff. with y ⇒ $2y+2(z-c)q=0$

Eliminate constant

$$\Rightarrow \boxed{yp-xq=0}$$

Eg. $x^2+y^2=(z-c)^2\tan^2\alpha$ (c and α are arb. constants)

↳ Family of cones

Diff. with w.r.t. $x \Rightarrow 2x = 2(z-c)p\tan^2\alpha$

$\Rightarrow x = (z-c)p\tan^2\alpha$

Diff. w.r.t. $y \Rightarrow 2y = 2(z-c)q\tan^2\alpha$

$\Rightarrow y = (z-c)q\tan^2\alpha$

$\Rightarrow \boxed{yp-xq=0}$

Eg. $z=f(x^2+y^2)$

Diff. w.r.t. $x \Rightarrow p=f'(x^2+y^2).2x$

Diff. w.r.t. $y \Rightarrow q=f'(x^2+y^2).2y$

$\Rightarrow \boxed{yp-xq=0}$ → this could be a general solution

Eg. $(x-a)^2+(y-b)^2+z^2=1$ (a, b are arb. const.s)

Diff. w.r.t. $x \Rightarrow 2(x-a)+2zp=0$

$\Rightarrow (x-a)+zp=0$

Diff. w.r.t. $y \Rightarrow 2(y-b)+2zq=0$

$\Rightarrow (y-b)+zq=0$

$\Rightarrow \boxed{z^2(1+p^2+q^2)=1}$

Eg. $(y-mx-c)^2=(1+m^2)(1-z^2)$ (m and c are arb. const.s)

↳ PDE: $z^2(1+p^2+q^2)=1$

We can't write 4 in terms of 5, so there is no general sol^n.

Solving PDE, which solution we will arrive?

Here, we cannot talk about general solution.

$f(x,y,y')=0$

$y'=f(x,y)$

$y'^2+\sin y=0$

non linearity can't be derivative, otherwise we can't talk about general solution.

→ $f(\underbrace{x,y,z}_{\text{can have power}},p,q)=0$

$\rightarrow f(\phi,\psi)=0\,;\quad \phi=\phi(x,y,z)$
$\qquad\qquad\qquad\quad \psi=\psi(x,y,z)$

Diff. w.r.t. $x \Rightarrow \frac{\partial F}{\partial\phi}(\phi_x+\cancel{\phi_y(0)}^{0}+\phi_z p)+\frac{\partial F}{\partial\psi}(\psi_x+\psi_z p)=0 \quad \ldots ①$

Diff. w.r.t. $y \Rightarrow \frac{\partial F}{\partial\phi}(\phi_y+\phi_z q)+\frac{\partial F}{\partial\psi}(\psi_y+\psi_z q)=0 \quad \ldots ②$

$Ax=0,$

$$A=\begin{bmatrix}\phi_x+\phi_z p & \psi_x+\psi_z p\\ \phi_y+\phi_z q & \psi_y+\psi_z q\end{bmatrix}, \quad x=\begin{bmatrix}\partial F/\partial\phi\\ \partial F/\partial\psi\end{bmatrix}$$

For non-trivial solution,

$|A|=0$

$$\Rightarrow \begin{vmatrix}\phi_x+\phi_z p & \psi_x+\psi_z p\\ \phi_y+\phi_z q & \psi_y+\psi_z q\end{vmatrix}=0$$

$$\Rightarrow \boxed{p\frac{\partial(\phi,\psi)}{\partial(y,z)}+q\frac{\partial(\phi,\psi)}{\partial(z,x)}=\frac{\partial(\phi,\psi)}{\partial(x,y)}},$$

where $\frac{\partial(\phi,\psi)}{\partial(x,y)}=\begin{vmatrix}\phi_x & \phi_y\\ \psi_x & \psi_y\end{vmatrix}$

$\hookrightarrow$ For this, we can talk above general solution

$\Rightarrow \boxed{F(\phi,\psi)=0}$ ... general solution

$\rightarrow$ p and q should have power 1.

Eg. $\phi=x+y+z$
$\quad\ \psi=x^2+y^2+z^2$

→ $F(x, y, z, a, b) = 0$ ⇝ most general solution.

Diff. w.r.t. $x \Rightarrow \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} p = 0$

Diff. w.r.t. $y \Rightarrow \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} q = 0$

Eliminate a, b.

$f(x, y, z, p, q) = 0$

17-01-2024

Recall

$(x-a)^2 + (y-b)^2 + z^2 = 1$

no relⁿ ⇅

$(y - mx - c)^2 = (1+m^2)(1-z^2)$

PDE: $z^2(1+p^2+q^2) = 1$

① Complete Solution: Any solution involves two arbitrary constants.

$$f(x, y, z, a, b) = 0$$

② General Solution: $b = \phi(a)$ envelope of $f(x, y, z, a, \phi(a)) = 0$.

③ Particular solution: Choose a ϕ, $\boxed{b=a}$ (for example)

④ Singular Solution: Envelope $f(x, y, z, a, b) = 0$.

Envelope

$f(x, y, z, a) = 0$

Diff. w.r.t. a,

$\frac{df}{da}(x, y, z, a) = 0$

$f(x, y, z, a, b) = 0$

$\left.\begin{array}{l} \frac{\partial f}{\partial a} = 0 \\ \frac{\partial f}{\partial b} = 0 \end{array}\right\}$ Eliminate a & b

Envelope ↳ a particular curve; not a family

$p + q = 0$

solⁿ: $z = x - y$

Gen. solⁿ: $z = f(x-y)$

Eg. $z^2(1+p^2+q^2) = 1$

① Complete solⁿ: $(x-a)^2 + (y-b)^2 + z^2 = 1$.

② General solⁿ: $(x-a)^2 + (y - \phi(a))^2 + z^2 = 1$

Diff. w.r.t $a \Rightarrow -2(x-a) - 2(y - \phi(a))\phi'(a) = 0$.

To get particular solⁿs, choose the fⁿ $\phi(a)$.

③ Particular solⁿ: $\phi(a) = a$ (say).

$$\left.\begin{array}{l} (x-a)^2 + (y-b)^2 + z^2 = 1 \\ (x-a) + (y-b) = 0 \end{array}\right\}$$

$$\Rightarrow a = \frac{x+y}{2}$$

$$\Rightarrow \left(\frac{x-y}{2}\right)^2 + \left(\frac{x-y}{2}\right)^2 + z^2 = 1$$

$$\Rightarrow \frac{(x-y)^2}{2} + z^2 = 1$$

$$\Rightarrow (x-y)^2 + 2z^2 = 2.$$

Put $\phi(a) = \sin a$ or $\phi(a) = a^2$ to get another particular solⁿ.

④ Singular soln:

$$(x-a)^2+(y-b)^2+z^2=1$$

$$\left.\begin{aligned}(x-a)&=0\\(y-b)&=0\end{aligned}\right\} \Rightarrow z=\pm 1$$

Solution of First Order PDE

$$f(x,y,z,p,q)=0$$

① Linear : $a(x,y)\,z_x + b(x,y)\,z_y + c(x,y)\,z = d(x,y)$

② Semi-linear: $a(x,y)\,z_x + b(x,y)\,z_y = c(x,y,z)$

③ Quasi-linear: $a(x,y,z)\,z_x + b(x,y,z)\,z_y = c(x,y,z)$

④ Non-linear: $f(x,y,z,p,q)=0$.

Eg. $az_x + bz_y = 0$, $\underbrace{a^2+b^2\neq 0}$ (Both a & b ≠ 0 to have a PDE)

$\boxed{z=f(bx-ay)}$

$\dfrac{(a,b)}{\sqrt{a^2+b^2}}.\nabla z = 0 \quad (z\in C^1)$

Let $v = \dfrac{(a,b)}{\sqrt{a^2+b^2}}$

$\Rightarrow D_{\vec{v}}(z)=0$ (Dirn derivative is 0)

$\Rightarrow$ z is constant along the line $\parallel$ to $\vec{v}$.

Linear $\Rightarrow$ $\boxed{bx-ay=c}$

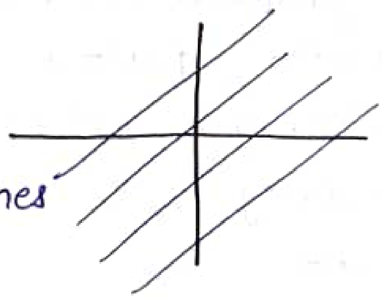
$\Rightarrow$ $\boxed{z=f(bx-ay)}$: Solution

$= f(c)$

$bx-ay=c$

$\Big\}$

Characteristic lines



$(a,b)\underbrace{(z_x, z_y)}_{\nabla z\,:\,\text{gradient}}$

$f:\mathbb{R}^2\to\mathbb{R}$

$\nabla f = (f_x, f_y)$

Line $\parallel$ to $(1,2)$: $2x-y=c$

18-01-2024

Recall, $aZ_x + bZ_y = 0$

↳ Solution : $Z = f(bx - ay)$

→ $Z_x + yZ_y = 0$

or more general,

$a(x,y)Z_x + b(x,y)Z_y$

$\Rightarrow$ Z is constant on a curve $\boxed{\phi(x,y) = c}$ for which the tangent vector at any point (x,y) is parallel to (a,b).

Equation of such curve : $\frac{dx}{a} = \frac{dy}{b}$

$a, b : \mathbb{R}^2 \to \mathbb{R}$

$\Rightarrow \frac{dx}{1} = \frac{dy}{y}$

$\Rightarrow y = e^x c$

$\Rightarrow \boxed{\underset{\phi}{ye^{-x}} = c}$

$Z = f(ye^{-x})$

<u>Working method:</u>

$a(x,y)Z_x + b(x,y)Z_y = 0$

Write characteric eqn : $\frac{dx}{a} = \frac{dy}{b}$

Then solve $\boxed{\phi(x,y) = c}$

$\therefore \boxed{Z = f(\phi(x,y))}$

$\frac{dx}{a} = \frac{dy}{b} \Rightarrow \underbrace{bx - ay}_{\phi} = c$

$\Rightarrow \boxed{Z = f(bx - ay)}$

$Z_x + x = xy$

Integrate w.r.t. x,

$\frac{dy}{dx} + P(x)y = Q(x,y)$

Recall linear,

$aZ_x + bZ_y + cZ = d$, $a \neq 0$

→ $aZ_x + bZ_y = c$, $c \neq 0$

Introduce new variables

$(\alpha, \beta) \xleftrightarrow{\text{related to}} (x,y)$

$\alpha = \alpha(x,y)$

$\beta = \beta(x,y)$

$Z \to (\alpha, \beta)$

Conditions: ① Z_α or Z_β must disappear.

② Jacobian, $J = \begin{vmatrix} \alpha_x & \alpha_y \\ \beta_x & \beta_y \end{vmatrix} \neq 0$

[To get invertible ren $x = x(\alpha,\beta)$, $y = y(\alpha,\beta)$ Inverse theorem]

$$\left.\begin{aligned} z_x &= z_\alpha \alpha_x + z_\beta \beta_x \\ z_y &= z_\alpha \alpha_y + z_\beta \beta_y \end{aligned}\right\} \quad \text{(Chain rule)}$$

$$\Rightarrow a(z_\alpha \alpha_x + z_\beta \beta_x) + b(z_\alpha \alpha_y + z_\beta \beta_y) = c$$

$$\Rightarrow (a\alpha_x + b\alpha_y) z_\alpha + (a\beta_x + b\beta_y) z_\beta = c \qquad [a, b \text{ are fn of } x, y]$$

Choose β such that $a\beta_x + b\beta_y = 0$.

Characteristic eqn : $\frac{dx}{a} = \frac{dy}{b} \Rightarrow \phi(x,y) = c$

$$\boxed{\beta = f(\phi(x,y)) = \phi(x,y)} \quad (\beta_y \neq 0)$$

Choose α such that

$$\begin{vmatrix} \alpha_x & \alpha_y \\ \beta_x & \beta_y \end{vmatrix} \neq 0 \quad [\text{Any } \alpha \text{ satisfying this will work.}]$$

Take $\boxed{\alpha = x}$.

$$J = \begin{vmatrix} 1 & 0 \\ \beta_x & \beta_y \end{vmatrix} = \beta_y \neq 0$$

Eg. Solve $xz_x - yz_y + y^2 z = y^2 \quad (x \neq 0, y \neq 0)$

\# $z_\alpha + cz = d$ $\quad \Rightarrow c = f(\beta)$

$\Downarrow$

$$\frac{dy}{dx} + Py = Q$$

$$\Rightarrow \underbrace{e^{\int P dx} \frac{dy}{dx} + Py\, e^{\int P dx}} = Q e^{\int P dx}$$

$$\Rightarrow \frac{d}{dx}\left(y e^{\int P dx}\right) = Q e^{\int P dx}$$

$$\Rightarrow y = e^{-\int P dx} \int Q e^{\int P dx} dx + c_1$$

α, β variables :

$\beta \to x\beta_x - y\beta_y = 0$

ch. eqn : $\frac{dx}{x} = \frac{dy}{-y} \Rightarrow xy = c$

$\beta = f(x,y) = xy$ (choice)

$\alpha = x$

$\Rightarrow x = \alpha$

$y = \beta/\alpha$

$$\therefore \alpha Z_\alpha + (\beta/\alpha)^2 Z = (\beta/\alpha)^2$$

$$Z_\alpha + (\beta^2/\alpha^3) Z = \beta^2/\alpha^3$$

$$\Rightarrow Z = f(x, y) + e^{y^2/2} + 1.$$

19-01-2024

<u>Recall,</u>

$$a(x,y) Z_x + b(x,y) Z_y + c(x,y) Z = d(x,y)$$

$$\downarrow$$

$$\frac{dy}{dx} + Py = Q$$

$(\alpha, \beta) \longleftrightarrow (x, y)$

$$J = \begin{vmatrix} \alpha_x & \alpha_y \\ \beta_x & \beta_y \end{vmatrix} \neq 0$$

$a \neq 0$, $\alpha = x$

If $b \neq 0$, $\beta = y$

$$\beta \rightarrow a\beta_x + b\beta_y = 0$$

$$\frac{dx}{a} = \frac{dy}{b} \rightarrow \beta$$

$$\frac{dy}{dx} = \frac{b}{a}$$

$$\alpha \rightarrow \frac{dy}{dx} = -\frac{a}{b}$$

$$\frac{b}{a} \neq -\frac{a}{b}$$

$$a^2 + b^2 \neq 0.$$

$\alpha_x \beta_y \neq \alpha_y \beta_x$

$\Rightarrow \frac{\alpha_x}{\alpha_y} \neq \frac{\beta_x}{\beta_y}$

\# (α, β) such that (i) $J = \begin{vmatrix} \alpha_x & \alpha_y \\ \beta_x & \beta_y \end{vmatrix} \neq 0$

(ii) Z_α or Z_β disappear.

Semi-Linear

$$a(x,y) Z_x + b(x,y) Z_y = c(x, y, Z)$$

$$\downarrow$$

$$\boxed{\frac{dy}{dx} = f(x, y)}$$

$\left.\begin{array}{l} \beta \rightarrow a\beta_x + b\beta_y = 0 \\ \alpha = x \end{array}\right\}$ of the form $Z_\alpha = f(\alpha, \beta, Z)$

Quasi Linear

$$a(x,y,z)z_x + b(x,y,z)z_y = c(x,y,z) \quad \ldots \text{(A)}$$

Theorem: Let $a, b, c \in C^1$. Then, the general solution of (A) is given by $f(\phi, \psi) = 0$, where

$$\phi = \phi(x,y,z)$$
$$\psi = \psi(x,y,z)$$

f is an arbitrary smooth function,

and $\phi(x,y,z) = c_1$ and $\psi(x,y,z) = c_2$ are two solutions of

$$\frac{dx}{a} = \frac{dy}{b} = \frac{dz}{c}.$$

Method:

Step-1: Write the characteristic equation, $\frac{dx}{a} = \frac{dy}{b} = \frac{dz}{c} \ldots$ (B).

Step-2: Find two (2nd) solutions of $\frac{dx}{a} = \frac{dy}{b} = \frac{dz}{c}$.

Let $\phi(x,y,z) = c_1$ and $\psi(x,y,z) = c_2$ are two solutions of (B).

Step-3: The general solution of (A) is

$\boxed{f(\phi, \psi) = 0}$, where f is an arbitrary function.

Proof:

$$\phi = c_1 \Rightarrow d\phi = \phi_x dx + \phi_y dy + \phi_z dz = 0 \quad \text{(Chain rule)}$$
$$\psi = c_2 \Rightarrow d\psi = \psi_x dx + \psi_y dy + \psi_z dz = 0.$$

Given $\frac{dx}{a} = \frac{dy}{b} = \frac{dz}{c}$.

$$\Rightarrow \left.\begin{aligned} a\phi_x + b\phi_y + c\phi_z &= 0 \\ a\psi_x + b\psi_y + c\psi_z &= 0 \end{aligned}\right\}$$

Take $a = $ some α
$\alpha\phi_x = \ldots$
$\alpha\psi_x = \ldots$

$$\Rightarrow \frac{a}{\frac{\partial(\phi,\psi)}{\partial(y,z)}} = \frac{b}{\frac{\partial(\phi,\psi)}{\partial(z,x)}} = \frac{c}{\frac{\partial(\phi,\psi)}{\partial(x,y)}}$$

Recall, we have proved that

$$f(\phi,\psi) = 0 \Rightarrow \frac{\partial(\phi,\psi)}{\partial(y,z)} z_x + \frac{\partial(\phi,\psi)}{\partial(z,x)} z_y = \frac{\partial(\phi,\psi)}{\partial(x,y)}.$$

So, the solution is $f(\phi, \psi) = 0$.

Eg. Solve $M\,dx + N\,dy = 0$.

Soln: If $\exists f(x,y)$ such that $f_x = M$ and $f_y = N$,

$$f_x\,dx + f_y\,dy = 0$$
$$\Rightarrow d(f) = 0$$
$$\Rightarrow \boxed{f = c}$$

Eg. $M\,dx + N\,dy + R\,dz = 0$

$\exists f$ such that $f_x = M, f_y = N, f_z = R$.

$$d(f) = 0 \Rightarrow \boxed{f = c}$$

24-01-2024

→ To find the solution of $\frac{dx}{a} = \frac{dy}{b} = \frac{dz}{c}$.

Convert it in the form $M\,dx + N\,dy + S\,dz = 0$.

Method-1: Look for P, Q, R such that

(i) $\underbrace{P\,dx + Q\,dy + R\,dz}_{d(f)}$ is exact.

(ii) $aP + bQ + cR = 0$

$$\frac{dx}{a} = \frac{dy}{b} = \frac{dz}{c} = \frac{P\,dx + Q\,dy + R\,dz}{aP + bQ + cR = 0}$$

$$\Rightarrow P\,dx + Q\,dy + R\,dz = 0$$
$$\Rightarrow d(\phi) = 0 \Rightarrow \phi = c_1.$$

Method-2: Look for P_1, Q_1, R_1 and P_2, Q_2, R_2 such that

$$\underset{(I)}{\frac{dx}{a}} = \underset{(II)}{\frac{dy}{b}} = \underset{(III)}{\frac{dz}{c}} = \underset{\substack{(IV) \\ \downarrow \\ d(w_1)}}{\frac{P_1\,dx + Q_1\,dy + R_1\,dz}{P_1 a + Q_1 b + R_1 c}} = \underset{\substack{(V) \\ \downarrow \\ d(w_2)}}{\frac{P_2\,dx + Q_2\,dy + R_2\,dz}{P_2 a + Q_2 b + R_2 c}}$$

$$\Rightarrow d(w_1) = d(w_2)$$
$$\Rightarrow d(w_1 - w_2) = 0$$
$$\Rightarrow w_1 - w_2 = c \Rightarrow \phi = c$$

Method-3:

$$\frac{dx}{a(x,y)} = \frac{dy}{b(x,y)} = \frac{dz}{c(x,y,z)}$$
(I) (II) (III)

I and II $\Rightarrow f(x,y)=c_0$

$\Rightarrow y = c_1 g(x)$

I and III

$$\Rightarrow \frac{dx}{a(x,y)} = \frac{dz}{c(x,y,z)}$$

Eliminate y using $y = c_1 g(x)$.

$\phi(x, z, c_0) = c_2$

$\Rightarrow \phi(x, z, f(x,y)) = c_1$

Eg Solve $(y-z)z_x + (z-x)z_y = x-y$.

Sdn: ch. eqn: $\dfrac{dx}{(y-z)} = \dfrac{dy}{(z-x)} = \dfrac{dz}{(x-y)}$ $\Rightarrow \dfrac{x\,dx}{x(y-z)} = \dfrac{y\,dy}{y(z-x)} = \dfrac{z\,dz}{z(x-y)}$

or

$\dfrac{1.dx}{1.(y-z)} = \dfrac{1.dy}{1.(z-x)} = \dfrac{1.dz}{1.(x-y)}$

$$= \underbrace{\frac{x\,dx + y\,dy + z\,dz}{}}_{0}$$

Here $P=x$, $Q=y$, $R=z$.

$\Rightarrow x\,dx + y\,dy + z\,dz = 0$

$\Rightarrow d\left(\dfrac{x^2+y^2+z^2}{2}\right) = 0$

$\Rightarrow \underbrace{x^2+y^2+z^2}_{\phi} = c_1$

and,

$\underbrace{\dfrac{dx+dy+dz}{}}_{0} \Rightarrow dx+dy+dz=0$

$\Rightarrow \underbrace{x+y+z}_{\psi} = c_2$

$\therefore$ solution: $f(x+y+z,\ x^2+y^2+z^2)=0$ [ϕ and ψ need not be unique.]

Q] Solve $x z_x - y z_y = y^2 - x^2$.

Soln: Char. eqn: $\underset{(I)}{\dfrac{dx}{x}} = \underset{(II)}{\dfrac{dy}{-y}} = \underset{(III)}{\dfrac{dz}{y^2-x^2}} = \underset{(IV)}{\dfrac{\overbrace{x\,dx + y\,dy + dz}}{0}}$

(I) & (II) $\Rightarrow \underbrace{yx}_{\phi} = c_1$

(I) & (IV) $\Rightarrow x\,dx + y\,dy + dz = 0$

$\Rightarrow d\left(\frac{x^2}{2} + \frac{y^2}{2} + z\right) = 0$

$\Rightarrow \underbrace{\frac{x^2}{2} + \frac{y^2}{2} + z}_{\psi} = c_2$

$\therefore$ Solution: $f\left(yx, \frac{x^2}{2} + \frac{y^2}{2} + z\right) = 0$

Q] $(y(x+y) + az) z_x + (x(x+y) - az) z_y = z(x+y)$, $a \in \mathbb{R}$.

Soln: ch. eqn: $\dfrac{dx}{y(x+y)+az} = \dfrac{dy}{x(x+y)-az} = \dfrac{dz}{x+y}$

$= \dfrac{\overbrace{x\,dx - y\,dy - a\,dz}}{0}$

$\Rightarrow \underbrace{x^2 - y^2 - 2az}_{\phi} = c_1$

$\dfrac{dx+dy}{(x+y)^2} = \dfrac{dz}{x+y}$

$\dfrac{dx+dy}{x+y} = \dfrac{dz}{z}$

and,

$\dfrac{\overbrace{\frac{1}{z}dx + \frac{1}{z}dy - \frac{(x+y)}{z^2}dy}}{0}$

$\Rightarrow \frac{1}{z}dx + \frac{1}{z}dy - \frac{(x+y)}{z^2}dz = 0$

$\Rightarrow d\left(\frac{x+y}{z}\right) = 0 \Rightarrow \underbrace{\frac{x+y}{z}}_{\psi} = c_2$

$\therefore$ Solution: $f\left(x^2 - y^2 - 2az, \frac{x+y}{z}\right) = 0$.

Q) Solve $(x^2-y^2-z^2) z_x + 2xy\, z_y = 2xz$.

Soln: ch. eqn : $\dfrac{dx}{x^2-y^2-z^2} = \dfrac{dy}{2xy} = \dfrac{dz}{2xz} = \dfrac{x\,dx + y\,dy + z\,dz}{x(x^2+y^2+z^2)}$
(I) (II) (III) (IV)

(II) & (III) $\Rightarrow \boxed{\dfrac{y}{z} = c_1}$

(III) & (IV) $\Rightarrow d\left(\ln(x^2+y^2+z^2)\right) = \ln z$

$\Rightarrow \boxed{\dfrac{x^2+y^2+z^2}{z} = c_2}$

$\therefore f\left(\dfrac{x^2+y^2+z^2}{z}, \dfrac{y}{z}\right) = 0.$

Q) Solve $p-q = \ln(x+y)$.

Soln: ch. eqn : $\underbrace{\dfrac{dx}{1} = \dfrac{dy}{-1}}_{x+y=c_1} = \dfrac{dz}{\ln(x+y)}$

and, $\dfrac{dx}{1} = \dfrac{dz}{\ln(x+y)}$

$\Rightarrow dx \ln c_1 = dz$

$\Rightarrow x \ln c_1 = z + c$

$\Rightarrow x \ln(x+y) = z + c$

$\Rightarrow z - x\ln(x+y) = c_2$

$\therefore f(x+y,\ z - x\ln(x+y)) = 0.$

Q) If $f_1(\phi, \psi) = 0$, then $\phi = g_1(\psi)$ or $\psi = h_1(\phi)$, where g & h are arbitrary?

↳ Yes!

↳ Special type of explicit function theorem.

Q. Solve $y^2p - xyq = x(z-2y)$.

Soln: $$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

$$\Rightarrow x^2+y^2 = c_1$$

and, $$\frac{dy}{-y} = \frac{dz}{z-2y} = \frac{dy-dz}{-(y-z)}$$

$$\Rightarrow \frac{dy}{y} = \frac{dy-dz}{y-z}$$

$$\Rightarrow \frac{y-z}{y} = c_2$$

$\therefore$ Soln: $f\left(x^2+y^2, \frac{y-z}{y}\right) = 0.$

Q. Solve $px - qy = y^2 - x^2$.

Soln: $$\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{y^2-x^2}$$

$$\Rightarrow \overbrace{yx}^{\phi} = c_1$$

and, $$\frac{dx}{x} = \frac{dz}{\frac{c_1^2}{x^2} - x^2}$$

$$\Rightarrow \left(\frac{c_1^2}{x^3} - x\right) dx = dz$$

$$\Rightarrow \frac{c_1^2}{-2x^2} - \frac{x^2}{2} = z + c_2'$$

$$\Rightarrow \underbrace{\frac{x^2}{2} + \frac{y^2}{2}}_{\psi} + z = c_2$$

$\therefore$ Solution: $f\left(xy, \frac{x^2}{2} + \frac{y^2}{2} + z\right) = 0.$

Check whether ODE

$$\frac{dy}{dx} = f(x,y) \xrightarrow{\int} \text{constant} \quad (x \in I)$$

$$y(x_0) = y_0 \xrightarrow{\text{find } c} \text{has a solution.}$$

~~But~~ ↳ By finding solution.
↳ But finding the solⁿ is hard.
↳ Try checking without solution.

<u>Picard Theorem</u>: Uniqueness

$$y = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt \rightarrow \text{sol}^n : \psi$$

Sequence $y_n = y_0 + \int_{x_0}^{x}$

↳ should converge to the solⁿ ψ, to get unique solⁿ.

<u>Quasi-linear PDE</u>:

$$\left.\begin{aligned} a z_x + b z_y &= c. \\ z(\Gamma_0) &= z_0(t) \end{aligned}\right\} \text{cauchy problem}$$

$\Gamma_0 - (x_0(t), y_0(t))$.

31-01-2023

Cauchy Problem

$$\left.\begin{aligned} f(x,y,z,p,q) &= 0 \\ z(\Gamma_0) &= z_0(t) \end{aligned}\right\}$$

$\Gamma_0 : (x_0(t), y_0(t)), \quad a \le t \le b$

$\Gamma_0 : [a,b] \to \mathbb{R}^2$

Eg. $a z_x + b z_y = 0 \Rightarrow z = f(bx - ay)$

$\Gamma_0 : y = x$

$z(x,x) = x^2 \longrightarrow x^2 = f(bx - ax)$

$$\Downarrow \begin{bmatrix} bx - ax = t \\ x = \frac{t}{b-a} \end{bmatrix}$$

$f(t) = \left(\frac{t}{b-a}\right)^2$

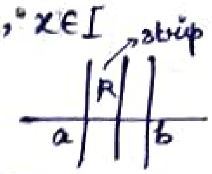
Soln:

$z = \left(\frac{bx-ay}{b-a}\right)^2$

ODE:

$\frac{dy}{dx} = f(x,y), \; x \in I$

$y(x_0) = y_0$



Picard Theorem:

$R = \{(x,y) \mid |x - x_0| \le a, |y - y_0| \le b\}$

f is cont. on R, $|f(x,y)| \le L$

f is Lib. cont. in y on R,

i.e., $\frac{\partial f}{\partial y}$ is cont. on R.

Unique soln: $|x - x_0| \le h$

$h = \min\left(a, \frac{b}{L}\right)$.

$$\rightarrow \left.\begin{aligned} a(x,y,z) z_x + b(x,y,z) z_y &= c(x,y,z) \\ z(\Gamma_0) &= z_0(t) \end{aligned}\right\}$$

Eg. $\begin{cases} 2z_x + 3z_y + 8z = 0. \longrightarrow z = e^{-4x} f(3x - 2y) \\ z(x,0) = \sin x. \longrightarrow \sin x = e^{-4x} f(3x) \end{cases}$

$\Rightarrow e^{4x} \sin x = f(\underbrace{3x}_{t})$

$\Rightarrow f(t) = e^{4t/3} \sin(t/3)$

$\therefore z = e^{-4x} e^{\frac{4(3x-2y)}{3}} \sin\left(\frac{3x-2y}{3}\right) \rightsquigarrow$ Unique solution.

Eg. $\begin{cases} 2z_x + 3z_y + 8z = 0 \rightarrow z = e^{-4x} f(3x - 2y) \\ z(\Gamma_0) = \ln x \longrightarrow \ln x = e^{-4x} f(1) \\ \Gamma_0 : 3x - 2y = 1 \quad \Rightarrow f(1) = e^{4x} \ln x \rightsquigarrow \text{not possible.} \end{cases}$

$z\left(x, \frac{3x-1}{2}\right) = \ln x.$

$\therefore$ No solution.

\# For any $\Gamma_0 : 3x - 2y = c$ (characteristic lines), there will be no solution.

Eg. $\begin{cases} 2z_x + 3z_y + 8z = 0 \\ f(\Gamma_0) = e^{-4x} \\ \Gamma_0 : 3x - 2y = 1 \end{cases}$ $\rightarrow z = e^{-4x} f(3x-2y)$

$\rightarrow e^{-4x} = e^{-4x} f(1)$

$\Rightarrow \boxed{f(1) = 1} \rightsquigarrow$ there are infinitely many such functions.

$\therefore$ Infinitely many solutions.

For any $\Gamma_0 : 3x - 2y = c$, $c \in \mathbb{R}$, there will be infinitely many solutions.

$\hookrightarrow \underbrace{\frac{dx}{2} = \frac{dy}{3}} = \frac{dz}{-8}$

$\Rightarrow 3x - 2y = c$: characteristic curves.

For $\Gamma_0 \parallel$ to char. curve, there will be infinitely many sol^{n}s, or no sol^n.

Observation

① Γ_0 is parallel to the characteristic lines.

$\Rightarrow$ Infinitely many solution or no solution.

② $\frac{y_0'(t)}{x_0'(t)} \neq \frac{b(x_0(t), y_0(t), z_0(t))}{a(x_0(t), y_0(t), z_0(t))} \Rightarrow$ Unique solution. $\Big|$ $\frac{dy}{dx} \neq \frac{b}{a}$.

Cauchy Theorem

$D \subseteq \mathbb{R}^3$ - Open + Connected.

- $\Gamma : (x_0(t), y_0(t), z_0(t)) \in \mathbb{R}^3 \rightsquigarrow$ curve in XYZ plane
- D contains Γ
- $a, b, c \in C^1(D)$. $\frac{y_0'(t)}{x_0'(t)} \neq \frac{b(x_0(t), y_0(t), z_0(t))}{a(x_0(t), y_0(t), z_0(t))}$.

Then $\begin{cases} a(x,y,z)\, z_x + b(x,y,z)\, z_y = c(x,y,z) \\ z(\Gamma_0) = z_0(t). \end{cases}$

has a unique solution in some neighbourhood of Γ_0.

Recall $\begin{cases} a(x,y,z)z_x + b(x,y,z)z_y = c(x,y,z) \\ z(\Gamma_0) = z_0(t) \end{cases}$ } Quasi-linear

$\Gamma - (x_0(t), y_0(t), z_0(t)) \rightarrow xyz$ plane

$\Gamma \in D \subseteq \mathbb{R}^3$, $a, b, c \in C'(D)$ and $\frac{y_0'(t)}{x_0'(t)} \neq \frac{b(x_0(t), y_0(t), z_0(t))}{a(x_0(t), y_0(t), z_0(t))}$

$\hookrightarrow$ Unique soln in some neighbourhood of Γ.

Remark:

The cauchy theorem is not applicable for non-linear problems in some neighbourhood.

Eg. $p^2+q^2=1$, $z=0$ on $x+y=1$. $\rightsquigarrow$ Non-linear

Soln: $z = \pm\frac{1}{\sqrt{2}}(x+y-1)$.

Eg. Solve $z(x+y)z_x + z(x-y)z_y = x^2+y^2$

$z=0$ on $y=2x$.

Soln: Ch. eqn: $\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)} = \frac{dz}{x^2+y^2} = \frac{ydx+xdy-zdz}{0} = \frac{xdx-ydy-zdz}{0}$

$ydx+xdy-zdz=0$

$\Rightarrow \underbrace{\boxed{2xy-z^2=c_1}}_{\phi}$

$\underbrace{\boxed{x^2-y^2-z^2=c_2}}_{\psi}$

The general soln is:

$$f(2xy-z^2, z^2-x^2+y^2)=0$$

$$\Rightarrow f(4x^2, 3x^2)=0$$

$f(\phi,\psi)=0 \Rightarrow \phi=h(\psi)$ or $\psi=g(\phi)$

$\Rightarrow 2xy-z^2 = h(x^2+y^2+z^2)$

$\Rightarrow 4x^2 = h(\underbrace{3t^2}_{t})$

$\Rightarrow h(t) = \frac{4}{3}t$

$\therefore$ soln: $\boxed{2xy-z^2 = \frac{4}{3}(z^2-x^2+y^2)}$

or, with $\psi=g(\phi)$

$z^2+y^2-x^2 = g(2xy-z^2)$

$\Rightarrow 3x^2 = g(4x^2)$

$\Rightarrow g = \frac{3}{4}t.$

OR Using $f(\underbrace{4x^2}_{\alpha}, \underbrace{3x^2}_{\beta})=0$

$4\beta-3\alpha=0$ | $f(\alpha,\beta)=4\beta-3\alpha$

From general soln,

$$\boxed{4(z^2-x^2+y^2)=3(2xy-z^2)}$$

OR From $2xy-z^2=c_1,\ z^2-x^2-y^2=c_2$

$\Rightarrow 4x^2=c_1,\ 3x^2=c_2$

$\Rightarrow \frac{4}{3}=\frac{c_1}{c_2}$

$\Rightarrow \frac{4}{3}=\frac{2xy-z^2}{z^2-x^2-y^2}$

Eg. $xz_x+yz_y=xe^{-z}$

$z=0$ on $y=x^2$.

Soln: ch. eqn: $\underbrace{\frac{dx}{x}=\frac{dy}{y}}=\frac{dz}{xe^{-z}}$

$\Rightarrow \frac{y}{x}=c_1 \ : \phi$

& $e^z-x=c_2 \ : \psi$

$f\left(\frac{y}{x}, e^z-x\right)=0 \Rightarrow e^z-x=g\left(\frac{y}{x}\right)$

$\Rightarrow 1-x=g(x)$

soln: $\boxed{e^z-x=1-\frac{y}{x}}$

OR $x=c_1$ and $1-x=c_2$

$\Rightarrow c_1+c_2=1 \Rightarrow \boxed{\frac{y}{x}+e^z-x=1}$

Non-linear Problem

Solve $f(x, y, z, p, q) = 0.$ —①

↳ Solving means finding a complete solution.

↳ Any solution involves two arbitrary constants.

Charpet Method:

$$g(x, y, z, p, q) = 0 \quad \text{—②}$$

Defn: I and II are compatible if they have common solution.

Remark: (SNEDDON BOOK)

~~Every solution of I will be a solution of II and vice-versa.~~

Eg.
$$xp - yq - x = 0 \quad \text{—ⓐ}$$
$$x^2p + q - xq = 0 \quad \text{—ⓑ}$$

Claim: $z = x + c(1+xy)$, c is arbitrary,
is a soln of ⓐ and ⓑ

But $z = x(y+1)$ is a soln of ⓐ, not ⓑ.

∴ ⓐ and ⓑ are compatible.

Idea: Solve I and II for p and q.

$$p = \phi(x, y, z), \quad q = \psi(x, y, z).$$

Then, common solve is given by $\boxed{dq = \phi dx + \psi dy}$.

Theorem:

I and II are compatible if

(i) $\begin{vmatrix} f_p & f_q \\ g_p & g_q \end{vmatrix} \neq 0 \Rightarrow$ I and II can be solved for p and q.
$$\begin{bmatrix} p = \phi(x, y, z) \\ q = \psi(x, y, z) \end{bmatrix}$$

(ii) $dz = \phi dx + \psi dy$ is integrable.

Eg. $xp - yq = 0 \quad -ⓐ$

$z(xp + yq) = 2xy, \quad -ⓑ$

$x, y, z \neq 0$

Soln: Check if ⓐ and ⓑ are compatible.

$f = xp - yq$

$g = z(xp + yq) - 2xy$

1st condn: $\begin{vmatrix} f_p & f_q \\ g_p & g_q \end{vmatrix} = \begin{vmatrix} x & -y \\ zx & zy \end{vmatrix} = xyz + xyz$

$= 2xyz$

$\neq 0$

ⓐ and ⓑ $\Rightarrow p = \frac{y}{z}, \; q = \frac{x}{z}$

2nd condn: $\boxed{dz = \frac{y}{z}dx + \frac{x}{z}dy}$ ⇝ Integrable

∴ ⓐ and ⓑ are compatible.

$\Rightarrow \boxed{z^2 = 2xy + c} \rightarrow$ Solution

Remark: (Sneddon)

$Pdx + Qdy + Rdz = 0$

2 variables: $Mdx + Ndy = 0$

$X = (P, Q, R), \; d\vec{r} = (dx, dy, dz)$

$\Rightarrow X \cdot d\vec{r} = 0$

$X \cdot dr$ is integrable $\Leftrightarrow \boxed{X \cdot \text{Curl}(X) = 0}$.

Theorem:

$f(x, y, z, p, q) = 0 \quad - (I)$

$g(x, y, z, p, q) = 0 \quad - (II)$

Let $\begin{vmatrix} f_p & f_q \\ g_p & g_q \end{vmatrix} \neq 0$.

Then, (I) and (II) are compatible.

$\Updownarrow$

$[f, g] = 0$.

where, $[f, g] = \frac{\partial(f,g)}{\partial(x,p)} + \frac{\partial(f,g)}{\partial(y,q)} + p\frac{\partial(f,g)}{\partial(z,p)} + q\frac{\partial(f,g)}{\partial(z,q)}$.

$\frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} f_x & f_p \\ g_x & g_p \end{vmatrix}$

Eg. $xp - yq = 0$ — (a)

and $z(xp + yq) = 2xy$ — (b)

Check if $[f, g] = 0$.

Soln: $[f, g] = \begin{vmatrix} f_x & f_p \\ g_x & g_p \end{vmatrix} + \begin{vmatrix} f_y & f_q \\ g_y & g_q \end{vmatrix} + p\begin{vmatrix} f_z & f_p \\ g_z & g_p \end{vmatrix} + q\begin{vmatrix} f_z & f_q \\ g_z & g_q \end{vmatrix}$

$$= \begin{vmatrix} p & x \\ (zp - 2y) & zx \end{vmatrix} + \begin{vmatrix} -q & -y \\ (zq - 2x) & zy \end{vmatrix} + p\begin{vmatrix} 0 & x \\ (xp + yq) & zx \end{vmatrix} + q\begin{vmatrix} 0 & -y \\ (xp + yq) & zy \end{vmatrix}$$

$$= (\cancel{pzx} - \cancel{pzx} + \cancel{2xy}) + (-\cancel{qyz} + \cancel{qzy} - \cancel{2xy}) + (-p^2x^2 - \cancel{pqxy}) + (\cancel{pqxy} + q^2y^2)$$

$$= q^2y^2 - p^2x^2$$

$$= \underbrace{(yq - xp)}_{0}(yq + xp)$$

$$= 0$$

02-02-2024

Charpit Method:

$$f(x,y,z,p,q)=0 \quad -\text{(I)}$$

Look for $g(x,y,z,p,q,a)=0$ $-\text{(II)}$

(I) and (II) are compatible.

$\Rightarrow [f,g]=0$

$$\Rightarrow \frac{\partial(f,g)}{\partial(x,p)}+\frac{\partial(f,g)}{\partial(y,q)}+p\frac{\partial(f,g)}{\partial(z,p)}+q\frac{\partial(f,g)}{\partial(z,q)}=0$$

$$\Rightarrow f_p\frac{\partial g}{\partial x}+f_q\frac{\partial g}{\partial y}+(pf_p+qf_q)\frac{\partial g}{\partial z}-(f_x+pf_z)\frac{\partial g}{\partial p}-(f_y-qf_z)\frac{\partial g}{\partial q}=0 \quad -\text{(III)}$$

(III) is a linear PDE in g in 5 variables.

$$\Rightarrow \frac{\partial x}{f_p}=\frac{\partial y}{f_q}=\frac{dz}{pf_p+qf_q}=\frac{dp}{-(f_x+pf_z)}=\frac{dq}{-(f_y+qf_z)} \quad -\text{(IV)}$$

From these ODEs, find a 'g' which involves p and q.
Also, we can solve from (I).

Equation (IV) : charpit equation.

Eg. Solve $p^2x+q^2y=z$ (a) ⇝ (Find the complete solution)

↳ $f=p^2x+q^2y-z$

Charpit eqn:

$$\underset{\text{(I)}}{\frac{dx}{2px}}=\underset{\text{(II)}}{\frac{dy}{2qy}}=\underset{\text{(III)}}{\frac{dz}{2p^2x+2q^2y}}=\underset{\text{(IV)}}{\frac{dp}{-(p^2-p)}}=\underset{\text{(V)}}{\frac{dq}{-(q^2-q)}}$$

\# can be solve to get g; but solving it with the eqn (a) is complicated.

$$\# \quad \frac{pdx+qdy}{2p^2x+2q^2y}=\frac{dz}{2p^2x+2q^2y}$$

↳ But $pdx+qdy$ cannot be integrated

(I) & (IV), (II) & (V) $\Rightarrow$
$$\frac{p^2x+2pxdp}{p^2x}=\frac{q^2dy+2qydq}{q^2y}$$

$$\Rightarrow \boxed{p^2x=aq^2y}$$

$$\# \quad \frac{d(p^2x)}{p^2x}=\frac{d(q^2y)}{q^2y}$$

$$\Rightarrow p^2x=aq^2y$$

$\Rightarrow g \equiv p^2x - aq^2y \quad \ldots (b)$

(a) and (b) $\Rightarrow p = \underbrace{\left[\frac{az}{(1+a)x}\right]^{1/2}}_{\phi}, \quad q = \underbrace{\left[\frac{z}{(1+a)y}\right]^{1/2}}_{\psi}$

$$\therefore dz = \left[\frac{az}{(1+a)x}\right]^{1/2} dx + \left[\frac{z}{(1+a)y}\right]^{1/2} dy$$

$\Rightarrow \boxed{[(1+a)z]^{1/2} = (ax)^{1/2} + y^{1/2} + b}$ ⇝ complete solution for (a) as well as (b).

Particular cases :

Case-I: $f(p,q) = 0$. ⇝ doesn't depend on x, y, z.

Charpit eqn : $\ldots = \frac{dp}{0} = \frac{dq}{0}$

$$\Rightarrow dp = 0$$
$$\Rightarrow p = a \text{ (const.)} \rightarrow (g)$$
$$\Rightarrow q = g(a)$$

$\Bigg\} \; dz = a\,dx + g(a)\,dy \Rightarrow \boxed{z = ax + g(a)y + b}$

Say, $pq = 1 \Rightarrow p = a, q = \frac{1}{a}$

Solution : $dz = a\,dx + \frac{1}{a}dy$

$$\Rightarrow \boxed{z = ax + \frac{y}{a} + b}$$

Case-II: $f(p, q, z) = 0$.

Charpit eqn : $\ldots = \frac{dp}{-pf_z} = \frac{dq}{-qf_z} \Rightarrow p = aq \rightarrow (g)$

$\vdots$

Case-III: $g(x,p) = h(y,q)$

$$f = g(x,p) - h(y,q)$$

Charpit eqn : $\frac{dx}{g_p} = \frac{dy}{-h_q} = \frac{dp}{-g_x} = \frac{dq}{h_y}$

$$\Rightarrow \frac{dp}{dx} = F(x,p) \longrightarrow (g)$$

Case-IV: $z = px + qy + f(p,q)$ ... Clairaut's form

Complete solution: $\boxed{z = ax + by + F(a,b)}$

Charpit eqⁿ : $\frac{dp}{0} = \frac{dq}{0} \Rightarrow \boxed{\begin{matrix} p=a \\ q=b \end{matrix}}$

Eg: $z = px + qy + \sin(pq)$

$\Rightarrow z = ax + by + \sin(ab).$

07-02-2023

Second Order Linear PDE

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = h(x,y), \quad x,y \in \Omega \subset \mathbb{R}^2 \quad -①$$

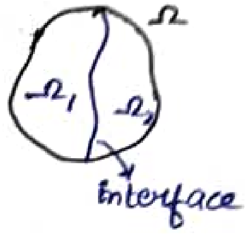
Canonical form (Simple form) | # $A, B, C = f(x,y)$

Classification of ① :

$B^2 - 4AC > 0$, ① is called hyperbolic.

$B^2 - 4AC = 0$, ① is called parabolic.

$B^2 - 4AC < 0$, ① is called elliptic.



Replace $(x,y) \longleftrightarrow (\alpha, \beta)$ | $u \in C^2(\bar{\Omega}) \Rightarrow u_{xy} = u_{yx}$

$$J = \begin{vmatrix} \alpha_x & \alpha_y \\ \beta_x & \beta_y \end{vmatrix} \neq 0$$

$$u_x = u_\alpha \alpha_x + u_\beta \beta_x, \quad u_y = u_\alpha \alpha_x + u_\beta \beta_y$$

$$\begin{aligned} u_{xx} = (u_x)_x &= (u_\alpha \alpha_x)_x + (u_\beta \beta_x)_x \\ &= \underline{(u_\alpha)_x \alpha_x + u_\alpha \alpha_{xx}} + (u_\beta)_x \beta_x + u_\beta \beta_{xx} \\ &= (u_{\alpha\alpha} \alpha_x + u_{\alpha\beta} \beta_x) \alpha_x + u_\alpha \alpha_{xx} + \\ &\quad (u_{\beta\alpha} \alpha_x + u_{\beta\beta} \beta_x) \beta_x + u_\beta \beta_{xx} \\ &= \alpha_x^2 u_{\alpha\alpha} + \beta_x^2 u_{\beta\beta} + 2\alpha_x \beta_x u_{\alpha\beta} + \\ &\quad u_\alpha \alpha_{xx} + u_\beta \beta_{xx}. \end{aligned}$$

Similarly,

$(u_x)_y = \ldots$

$(u_y)_y = \ldots$

$$① \Rightarrow A^* u_{\alpha\alpha} + B^* u_{\alpha\beta} + C^* u_{\beta\beta} + D^* u_\alpha + E^* u_\beta + F^* u = h^*(\alpha, \beta) \quad -②$$

$$A^* = A\alpha_x^2 + B\alpha_x \alpha_y + C\alpha_y^2$$

$$B^* = 2A\alpha_x \beta_x + B(\alpha_x \beta_y + \beta_x \alpha_y) + 2C\alpha_y \beta_y \quad \text{(check!)}$$

$$C^* = A\beta_x^2 + B\beta_x \beta_y + C\beta_y^2$$

Remark: The nature of the PDE does not change under this transformation.

If ① is hyperbolic, ② is also hyperbolic.

Eg. Show that $B^{*2} - 4A^*C^* = J^2(\underbrace{B^2-4AC}_{>0})$

Case-I: ① is hyperbolic, $B^2 - 4AC > 0$

Given $B^2 - 4AC > 0 \Rightarrow B^{*2} - 4A^*C^* > 0$

Choose α and β such that $A^* = 0$ and $C^* = 0$.

$$A^* = 0 \Rightarrow A\alpha_x^2 + B\alpha_x\alpha_y + C\alpha_y^2 = 0$$

$$\Rightarrow A\left(\frac{\alpha_x}{\alpha_y}\right)^2 + B\left(\frac{\alpha_x}{\alpha_y}\right) + C = 0$$

$$\Rightarrow \frac{\alpha_x}{\alpha_y} = \frac{-B \pm \sqrt{B^2-4AC}}{2A}$$

Take + sign $\Rightarrow \dfrac{\alpha_x}{\alpha_y} = \dfrac{-B + \sqrt{B^2-4AC}}{2A}$

$$\Rightarrow (2A)\alpha_x + (B - \sqrt{B^2-4AC})\alpha_y = 0$$

$\alpha \rightarrow$ ch. eqn $\Rightarrow \dfrac{dx}{2A} = \dfrac{dy}{B - \sqrt{B^2-4AC}} \Rightarrow \phi = c$

$\boxed{\alpha = \phi}$

To find β, use $C^* = 0$:

$$C^* = 0 \Rightarrow A\beta_x^2 + B\beta_x\beta_y + C\beta_y^2 = 0$$

$$\Rightarrow A\left(\frac{\beta_x}{\beta_y}\right)^2 + B\left(\frac{\beta_x}{\beta_y}\right) + C = 0$$

$$\Rightarrow \frac{\beta_x}{\beta_y} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

Take $-$ sign $\Rightarrow \dfrac{\beta_x}{\beta_y} = \dfrac{-B - \sqrt{B^2-4AC}}{2A}$

$$\Rightarrow (2A)\beta_x + (B - \sqrt{B^2-4AC})\beta_y = 0$$

$\beta \to$ ch. eq^n $\Rightarrow \dfrac{dx}{2A} = \dfrac{dy}{B - \sqrt{B^2-4AC}} \Rightarrow \psi = C$

$\therefore \boxed{\beta = \psi}$.

Canonical form:

$$u_{\alpha\beta} = f(\alpha, \beta, u, u_\alpha, u_\beta) \quad [\text{from } ②]$$

08-02-2024

Eg Find a canonical form.

$$y^2 u_{xx} - x^2 u_{yy} = 0, \quad x, y \neq 0.$$

Soln: $B^2 - 4AC > 0$

$A = y^2$
$B = 0$
$C = -x^2$

$\alpha : \dfrac{\alpha_x}{\alpha_y} = \dfrac{2xy}{2y^2}$

$$\Rightarrow \frac{\alpha_x}{\alpha_y} = \frac{x}{y} \Rightarrow y\alpha_x - x\alpha_y = 0$$

ch. eq^n : $\dfrac{dx}{y} = \dfrac{dy}{-x} \Rightarrow x^2 + y^2 = c_1$

$\therefore \boxed{\alpha = y^2 + x^2}$

Similarly, $\boxed{\beta = y^2 - x^2}$

$$u_x = u_\alpha \alpha_x + u_\beta \beta_x = 2x u_\alpha - 2x u_\beta$$

$$\Rightarrow u_{xx} = 2u_\alpha - 2u_\beta$$

$$u_y = u_\alpha \alpha_y + u_\beta \beta_y = 2y u_\alpha + 2y u_\beta$$

$$\Rightarrow u_{yy} = 2u_\alpha + 2u_\beta$$

$$\Rightarrow u_{\alpha\beta} = \frac{\beta}{2(\alpha^2-\beta^2)} u_\alpha - \frac{\alpha}{2(\alpha^2-\beta^2)} u_\beta$$

<u>Case-II:</u> $B^2-4AC=0 \quad (A\neq 0)$

$$\Rightarrow B^{*2}-4A^*C^*=0$$

Choose α and β such that $A^*=0 \Rightarrow B^*=0 \Rightarrow \frac{\alpha_x}{\alpha_y} = -\frac{B}{2A}$.

β such that $J = \begin{vmatrix} \alpha_x & \alpha_y \\ \beta_x & \beta_y \end{vmatrix} \neq 0 \Rightarrow \frac{\alpha_x}{\alpha_y} \neq \frac{\beta_x}{\beta_y}$.

Take $\boxed{\beta=y}$ or $\boxed{\frac{\beta_x}{\beta_y} = \frac{2A}{B}}$ ⇝ makes $J\neq 0$.

As $B^2+4A \neq 0$ (as $B>0$ and $A\neq 0$)

Canonical form:

$$u_{\beta\beta} = f(\alpha, \beta, u_\alpha, u_\beta, u).$$

<u>OR</u> choose α and β such that $C^*=0$.

$$\Rightarrow \frac{\beta_x}{\beta_y} = -\frac{B}{2A}, \quad \boxed{\alpha=x}$$

or

$$\boxed{\frac{\alpha_x}{\alpha_y} = \frac{2A}{B}}$$

Canonical form:

$$u_{\alpha\alpha} = f(\alpha, \beta, u_\alpha, u_\beta, u).$$

Eg. $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = 0, \; x\neq 0.$

As $B^2-4AC=0$

$$\frac{\alpha_x}{\alpha_y} = -\frac{2xy}{2x^2} = -\frac{y}{x}$$

$$\Rightarrow x\alpha_x + y\alpha_y = 0.$$

Ch. eq.^{n} : $\frac{dx}{x} = \frac{dy}{y}$.

$$\Rightarrow \frac{y}{x} = c_1$$

$$\therefore \boxed{\alpha = \frac{y}{x}}$$

Take $\boxed{\beta=y}$. [Any β for which $J\neq 0$ will work.]

$$\therefore \boxed{u_{\beta\beta}=0}.$$

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$\rightarrow A u_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + F u = G$:

$A^* u_{\alpha\alpha} + B^* u_{\alpha\beta} + C^* u_{\beta\beta} + D^* u_\alpha + E^* u_\beta + F^* u = G(\alpha, \beta)$

$A^* = A\alpha_x^2 + B\alpha_x\alpha_y + C\alpha_y^2$

$B^* = 2A\alpha_x\beta_x + B(\alpha_x\alpha_y + \beta_x\beta_y) + 2C\alpha_y\beta_y$.

$C^* = A\beta_x^2 + B\beta_x\beta_y + C\beta_y^2$

$\rightarrow B^{*2} - 4A^*C^* = J^2(B^2 - 4AC)$, $J = \begin{vmatrix} \alpha_x & \alpha_y \\ \beta_x & \beta_y \end{vmatrix}$.

Case-I: $B^2 - 4AC > 0$

Case-II: $B^2 - 4AC = 0$

Case-III: $B^2 - 4AC < 0$

We have $B^* - 4A^*C^* < 0$

Choose α and β such that $B^* = 0$, $A^* = C^*$.

$A^* = C^* \Rightarrow A(\alpha_x^2 - \beta_x^2) + B(\alpha_x\alpha_y - \beta_x\beta_y) + C(\alpha_y^2 - \beta_y^2) = 0$ ...(I)

$B^* = 0 \Rightarrow i2A\alpha_x\beta_x + iB(\alpha_x\beta_y + \beta_x\alpha_y) + i2C\alpha_y\beta_y = 0$ ...(II)

coupled

Define $\phi = \alpha + i\beta \Rightarrow \phi_x = \alpha_x + i\beta_x$, $\phi_y = \alpha_y + i\beta_y$

(I) and (II) $\Rightarrow A\phi_x^2 + B\phi_x\phi_y + C\phi_y^2 = 0$

$\Rightarrow A\left(\frac{\phi_x}{\phi_y}\right)^2 + B\left(\frac{\phi_x}{\phi_y}\right) + C = 0$

$\Rightarrow \frac{\phi_x}{\phi_y} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$

Take + sign, $\frac{\phi_x}{\phi_y} = \frac{-B + i\sqrt{4AC - B^2}}{2A}$.

$\Rightarrow 2A\phi_x + (B - i\sqrt{4AC - B^2})\phi_y = 0$.

ch. eqn: $\frac{dx}{2A} = \frac{dy}{B - i\sqrt{4AC - B^2}} \Rightarrow \psi = c \rightarrow \phi \rightsquigarrow$ Imaginary

$\therefore \alpha = Re(\phi)$

$\beta = Im(\phi)$.

Canonical form : $u_{\alpha\alpha} + u_{\beta\beta} = f(\alpha, \beta, u_\alpha, u_\beta, u)$.

Eg. Find a canonical form : $u_{xx} + x^2 u_{yy} = 0$, $x \neq 0$.

Soln: Ch. eqn: $\frac{dx}{2} = \frac{dy}{-i2x} \Rightarrow \underbrace{x^2 - i2y}_{\phi} = c$ | $A=1$, $B=0$, $C=x^2$

$\therefore \alpha = x^2, \beta = -2y$

$$\Rightarrow \boxed{u_{\alpha\alpha} + u_{\beta\beta} = -\frac{1}{2\beta} u_\beta}.$$

Second Order PDE with Constant Coefficients

\# $y'' + py' + qy = r(x)$ ....(N-H)

$y'' + py' + qy = 0$ ....(H), $x \in I$ → | Define $D = \frac{d}{dx}$

$(D^2 + pD + q)y = 0$

$(D - m_1)(D - m_2)y = 0$

$y_1 \rightarrow (D - m_1)y = 0$

$y_2 \rightarrow (D - m_2)y = 0$

$y_H \rightarrow c_1 y_1 + c_2 y_2$

$S = \{ y \in C^2(I) \mid y'' + py' + qy = 0 \}$

$\boxed{\dim(S) = 2}$

$\boxed{y_H = c_1 y_1 + c_2 y_2}$

$\boxed{y_{NH} = y_H + y_P}$

\# $\boxed{a u_{xx} + b u_{xy} + c u_{yy} + d u_x + e u_y + f u = g(x,y)}$

Non-homogeneous:

y_{NH} (Direct)

$(D - m_1)(D - m_2)y = r(x)$

Let $z = (D - m_2)y$

$\Rightarrow (D - m_1)z = r(x) \rightarrow z$

$\Rightarrow (D - m_2)y = z$

Solving this is not easy!

Homogeneous:

$(D - m)^2 y = 0$

$(D - m)(D - m)y = 0$

Let $z = (D - m)y$

$\Rightarrow (D - m)z = 0$

$\Rightarrow z = c_1 e^{mx}$

$(D - m)y = c_1 e^{mx}$

$y e^{-mx} = c_1 x + c_2$

$\Rightarrow \boxed{y = x c_1 e^{mx} + c_2 e^{mx}}$

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→ $au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = g(x,y)$ ... (NH)

If $g(x,y) = 0$ ... (H)

$u_{NH} = U_H + U_P$

General Solution:

How to find u_H (general solution)?

→ If a solution of homogeneous function involves two arbitrary functions, it is a general solution..

Let $D \equiv \frac{\partial}{\partial x}$, $D' \equiv \frac{\partial}{\partial y}$

$$F(D,D')u = g(x,y) \quad \ldots (NH)$$

$$F(D,D')u = 0 \quad \ldots (H)$$

Theorem: Let $u_1, u_2, \ldots$ are the solutions of H. Then, $\sum_{i=1}^{n} c_i u_i$ is also a solution of H.

Theorem: Let u_H be a general solution of H, and u_p is a paraticular solution of NH.

Then, $u_H + u_P$ is the general solution of NH.

$$F(D,D')(c_1u_1 + c_2u_2 + \ldots + c_nu_n) = 0$$

Proof: $F(D,D')(u_H + u_P) = g(x,y)$.

To find u_H:

Assumption: $F(D,D') \equiv (\alpha_1 D + \beta_1 D' + \lambda_1)(\alpha_2 D + \beta_2 D' + \lambda_2)$
↳ linearly factorable

Eg. $F(D,D') = D^2 - D'^2 = (D + D')(D - D')$

However, $F(D,D') = D^2 - D'$ ✗

If $F(D,D')$ can be linearly factored, it is called reducible otherwise, irreducible.

$$\Rightarrow (\alpha_1 D + \beta_1 D' + \lambda_1)(\alpha_2 D + \beta_2 D' + \lambda_2)\, u = 0 \quad \dots (H)$$

$$u_1 \rightarrow (\alpha_1 D + \beta_1 D' + \lambda_1) u = 0$$

$$u_2 \rightarrow (\alpha_2 D + \beta_2 D' + \lambda_2) u = 0$$

$$\therefore \boxed{u_H = u_1 + u_2}$$

⇝ involves two arbitrary f^ns

u_1: involves one arbitrary f^n; u_2: involves another arbitrary f^n

Let $(\alpha D + \beta D' + \lambda) u = 0$

$$\frac{dx}{\alpha} = \frac{dy}{\beta} = \frac{du}{-\lambda u}$$

$$\Rightarrow \quad u = e^{-\frac{\lambda x}{\alpha}} \phi(\beta x - \cancel{\alpha y}) \quad , \text{ if } \alpha \neq 0$$

$$u = \left[e^{-\frac{\lambda y}{\beta}}\right] \phi(\beta x) \quad , \text{ if } \alpha = 0$$

→ If $F(D, D') = (\alpha D + \beta D' + \lambda)^2$

<u>To find u_H:</u>

$$\Rightarrow (\alpha D + \beta D' + \lambda)(\alpha D + \beta D' + \lambda) u = 0 \quad \dots (H)$$

Let $Z = (\alpha D + \beta D' + \lambda) u$

$$\Rightarrow (\alpha D + \beta D' + \lambda) Z = 0 \Rightarrow Z = e^{\left(\frac{-\lambda x}{\alpha}\right)} \phi(\beta x - \alpha y)$$

$$\Rightarrow (\alpha D + \beta D' + \lambda) u = e^{-\frac{\lambda x}{\alpha}} \phi(\beta x - 2y)$$

<u>Ch. eq^n:</u> $\underbrace{\frac{dx}{\alpha} = \frac{dy}{\beta}}_{\Rightarrow\ \beta x - 2y = c} = \dfrac{du}{e^{-\frac{\lambda x}{\alpha}} \phi(\beta x - \alpha y) - \lambda u}$

and, $$\frac{dx}{\alpha} = \frac{du}{e^{-\lambda x/\alpha} \phi(\beta x - \alpha y) - \lambda u}$$

$$\Rightarrow \frac{dx}{\alpha} = \frac{du}{e^{-\lambda x/\alpha} \phi(c) - \lambda u}$$

$$\Rightarrow \frac{du}{dx} + \left(\frac{\lambda}{\alpha}\right) u = \phi(c)\, e^{-\frac{\lambda x}{\alpha}}$$

$\Rightarrow u e^{\lambda x/\alpha} = \phi(c) x + c_1$

$\Rightarrow u e^{\frac{\lambda x}{\alpha}} - \phi(\beta x - \alpha y) x = c_2$

$\Rightarrow u e^{\lambda x/\alpha} - \phi(\beta x - \alpha y) x = \psi(\beta x - \alpha y)$

$\therefore u_H = e^{-\frac{\lambda x}{\alpha}} \left[\psi(\beta x - \alpha y) + x \phi(\beta x - \alpha y) \right]$

$\rightarrow$ If $\alpha = 0 \Rightarrow (\beta D' + \lambda)^2 u = 0 \quad \dots (H)$

$\therefore u_H = e^{-\frac{\lambda y}{\beta}} \left[\phi_1(\beta x) + y \, \phi_2(\beta x) \right]$

$\boxed{16\text{-}02\text{-}2024}$

Recall

$F(D, D') u = f(x, y) \quad \dots (NH)$

$F(D, D') u = 0 \quad \dots (H)$

$u_{NH} = u_P + u_H$

$F(D, D') = \underbrace{(\alpha_1 D + \beta_1 D' + \gamma_1)}_{u_1} \underbrace{(\alpha_2 D + \beta_2 D' + \gamma_2)}_{u_2}$

$u_H = u_1 + u_2$

To find u_P:

$u_P = \frac{1}{F(D, D')} = \cancel{\frac{1}{(\alpha_1 D + \beta D')(\alpha}}$

$= \frac{1}{(\alpha_1 D + \beta D' + \gamma_1)(\alpha_2 D + \beta_2 D' + \gamma_2)} . f(x, y)$

$\frac{1}{\alpha D + \beta D' + \gamma} f(x, y)$

$\Rightarrow (\alpha D + \beta D' + \gamma) u = f(x, y)$

$\underbrace{y'' + p y' + q y}_{P(D) y} = r(x)$

$P(D) y = r(x)$

$y_P = \frac{1}{P(D)} r(x)$

$= \frac{1}{(D - m_1)(D - m_2)} r(x)$

\# $\frac{1}{D - m_1} r(x) = e^{mx} \int r(x) e^{-mx} dx$

\# $\frac{1}{D} r(x) = \int r(x)$

$\downarrow$

$D y = r(x)$

\# $(D - m) y = r(x)$

$y e^{-mx} = \int r(x) e^{-mx} dx$

$y = e^{mx} \int r(x) e^{-mx}$

Eg. Solve $u_{xx} - u_{yy} = (x - y)$

$\underbrace{(D^2 - D'^2)}_{F(D, D')} u = (x - y)$

$\Rightarrow (D - D')(D + D') u = x - y$

$u_H = \phi_1(y + x) + \phi_2(y - x)$

$$(D-D')(D+D')u = x-y$$

Let $z=(D+D')u$

$$\Rightarrow (D-D')z = x-y \Rightarrow z = \frac{1}{4}(x-y)^2 + \underbrace{f(x+y)}_{=0}$$ (say) or any other function

$$= \frac{1}{4}(x-y)^2$$

$$\Rightarrow (D+D')u = \frac{1}{4}(x-y)^2$$

$$\Rightarrow \boxed{u_p = \frac{1}{4}x(x-y)^2}$$

Higher Order PDEs

Let $F(D,D') = \prod_{r=1}^{n} (\alpha_r D + \beta_r D' + \gamma_r)^{m_r}$

To find u_H:

u_r: Solution of r^{th} factor

$$u_r = e^{-\frac{\gamma_r x}{\alpha_r}} \phi_r(\beta_r x - \alpha_r y), \quad \alpha_r \neq 0$$

$$u_r = e^{-\frac{\gamma_r y}{\beta_r}} \phi_r(\beta_r x), \quad \alpha_r = 0$$

\# $(\alpha_r D + \beta_r D' + \gamma_r)^{m_r} u = 0$

$$\Rightarrow u_r = e^{-\frac{\gamma_r x}{\alpha_r}} \sum_{s=1}^{m_r} x^{s-1} \phi_{rs}(\beta_r x - \alpha_r y), \quad \alpha_r \neq 0.$$

$$u_r = e^{-\frac{\gamma_r y}{\beta_r}} \sum_{s=1}^{m_r} y^{s-1} \phi_{rs}(\beta_r x), \quad \alpha_r = 0.$$

$$\therefore \boxed{u_H = \sum_{r=1}^{n} e^{-\frac{\gamma_r x}{\alpha_r}} \sum_{s=1}^{m_r} x^{(s-1)} \phi_{rs}(\beta_r x - \alpha_r y)}$$

where ϕ_{rs} are arbitrary functions.

Eg. $F(D,D') = (D-D')^4 (D+D')^3$ | $n=7=4+3$

$F(D,D')u=0$

$u_H = \phi_1(y+x) + x\phi_2(y+x) + x^2\phi_3(y+x) + x^3\phi_4(y+x)$

$+ \psi_1(y-x) + x\psi_2(y-x) + x^2\psi_3(y-x).$

To find u_p:

Solve $\dfrac{\partial^4 u}{\partial x^4} + \dfrac{\partial^4 u}{\partial y^4} - 2\dfrac{\partial^4 u}{\partial x^2 \partial y^2} = f(x,y)$.

$\Rightarrow (D+D')^2 (D-D')^2 u = f(x,y)$

$\Rightarrow (D+D') \underbrace{(D+D')(D-D')(D-D')u}_{z_1} = f(x,y)$

Let $z_1 = (D+D')(D-D')(D-D')u$

$\Rightarrow (D+D')z_1 = f(x,y) \rightsquigarrow z_1$

Now, $z_1 = (D+D')\underbrace{(D-D')(D-D')u}_{z_2}$

$\Rightarrow z_2 = (D-D')(D-D')u$

$\Rightarrow z_1 = (D+D')z_2 \rightsquigarrow z_2$

Now, $z_2 = (D-D')\underbrace{(D-D')u}_{z_3}$

$\Rightarrow z_2 = (D-D')z_3 \rightsquigarrow z_3$

$\therefore \boxed{z_3 = (D-D')u}$.

u_p for special cases

$$F(D, D')u = f(x,y)$$

Case-I: $f(x,y) = e^{ax+by}$

$$u_p = \frac{1}{F(D,D')} e^{ax+by} = \frac{1}{F(a,b)} e^{ax+by}, \quad [F(a,b) \neq 0]$$

Let $f(x,y) = e^{ax+by} g(x,y)$

$$u_p = \frac{1}{F(D,D')}\left[e^{ax+by} g(x,y)\right]$$

$$= e^{ax+by} \frac{1}{F(D+a, D'+b)} g(x,y)$$

→ For trigonometric functions, write them in terms of exponentials

Case-II: $f(x,y)$ is a polynomial.

$P(D)y = r(K)$

$(1+D^5)y = x^4$

$$y_p = \frac{x^4}{1+D^5} = (1-D^5)x^4$$

$$\left[\because \frac{1}{1+T} = 1 - T + T^2 \ldots\right]$$

$P(D)y = r(x)$

Eg $(1+D^5)y = x^4$

$$y_p = \frac{1}{1+D^5} x^4$$

$$= (1 - D^5 + \cancel{D^{10}} \ldots)x^4$$

$$= (1-D^5)x^4$$

$\frac{1}{1+T} = 1 - T + T^2 \ldots$

$F(D,D')u = f(x,y)$, f is a polynomial

operator

$\boxed{22-02-2024}$

Eg $F(D,D') = D^2 - 3DD' + D'^2$

$f(x,y) = x+y$

$\frac{1}{D}$: integral w.r.t. x

D' : diff w.r.t x

$$u_p = \frac{1}{D^2 - 3DD' + D'^2}(x+y)$$

$$= \frac{1}{D^2} \cdot \frac{1}{\left[1 - \left[\frac{3D'}{D} - \left(\frac{D'}{D}\right)^2\right]\right]}(x+y) \quad \left[\because \frac{D'^2}{D^2}(x+y) = 0\right]$$

$$= \frac{1}{D^2}\left(1 + \frac{3D'}{D}\right)(x+y)$$

$$= \frac{1}{D^2}\left[(x+y+3x\right] = \frac{1}{D^2}(4x+y)$$

$\boxed{F(D,D')u=0}$ ...(1)

Let $F(D,D') = a_0 D^n + a_1 D^{n-1} D' + \dots + a_n D'^n$

Idea: Let $u = \phi(y+mx)$ be a soln of (1).

$D^i u = m^i \phi^i (y+mx)$, $D'^j u = \phi^j (y+mx)$

$D^i D'^j u = m^i \phi^{i+j} (y+mx)$

$\rightarrow (a_0 m^n + a_1 m^{n-1} + \dots + a_n) \phi^n (y+mx) = 0$

$\Rightarrow a_0 m^n + a_1 m^{n-1} + \dots + a_n = 0 \longrightarrow$ Auxiliary eqn

Case-(1): $m_1, m_2, \dots, m_n$ are roots.

$u_H = \phi_1(y+m_1x) + \phi_2(y+m_2x) + \dots + \phi_n(y+m_nx)$

Case-(2): If roots are repeated, multiply by x.

$u_H = \phi_1(y+m_1x) + x\phi_2(y+mx) + \dots$

To find Particular solution:

$F(D,D') = a_0 D^n + a_1 D^{n-1} D' + \dots + a_n D'^n$

$= \underbrace{(D-m_1D')}_{\phi(y+mx)} (D-m_2D') \dots (D-m_nD')$

$\alpha D + \beta D' + \lambda$

$\hookrightarrow e^{-\frac{\lambda x}{\alpha}} \phi(\beta x - \alpha y)$

To find $u_p = \frac{1}{(D-mD')} f(x,y)$

$u_p = \frac{1}{F(D,D')} = \frac{f(x,y)}{(D-m_1D')(D-m_2D')\dots(D-m_nD')}$

$\Rightarrow (D-mD')u_p = f(x,y)$: Linear PDE

Characteristic eqn:

$\frac{dx}{1} = \frac{dy}{-m} = \frac{du_p}{f(x,y)} \Rightarrow \frac{dx}{1} = \frac{du_p}{f(x,y)} \Rightarrow \frac{dx}{1} = \frac{du_p}{f(x, c-mx)}$

$\Rightarrow y+mx=c$

$\Rightarrow u_p = \int f(x, c-mx)\,dx$

$\therefore \boxed{\frac{1}{D-mD'} f(x,y) = \int f(x, c-mx)\,dx}$, where $\boxed{y+mx=c}$

Eg Solve $(D^2 - 4D'D + 4D'^2)u = e^{2x+y}$

$$\lambda^2 - 4\lambda + 4 = 0 \Rightarrow (\lambda-2)^2 = 0$$
$$\Rightarrow \lambda = 2, 2$$

$$u_H = \phi_1(y+2x) + x\phi_2(y+2x)$$

$$u_p = \frac{1}{(D-2D')(D-2D')} e^{2x+y}$$

$$\Rightarrow \frac{1}{D-2D'} e^{2x+y} = \int e^{(2x+c-2x)} dx$$
$$= x e^{y+2x}$$

$$\frac{1}{D-2D'} x e^{2x+y} = \int x e^{c-2x+2x} dx$$
$$= \int x e^c dx$$
$$= \frac{x^2}{2} e^{y+2x}$$

$$\therefore u = u_H + u_p$$
$$\Rightarrow u = \phi_1(y+2x) + x\phi_2(y+2x) + \frac{x^2}{2} e^{y+2x}$$

Eg. Solve $(x^2D^2 + 2xy\,DD' + y^2D'^2)u = x^2 + y^2$

$$\alpha = \ln x, \quad \beta = \ln y$$
$$u_x = u_\alpha \alpha_x + u_\beta \beta_y$$

Eg $x^2D^2 = D(D-1)$, $xy\,DD' = DD'$

$$y^2D'^2 = D'(D'-1)$$
$$[D(D-1) + 2DD' + D'(D'-1)]u = e^{2\alpha} + e^{2\beta}$$

23-02-2024

Boundary Value Problem

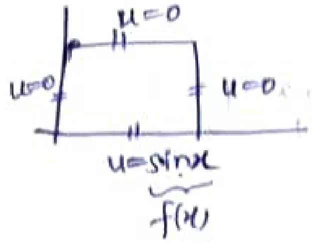
$$y'' + py' + qy = r(x), \quad a < x < b$$

$y(a) = \alpha$, $y'(a) = \beta$ → initial value problem

$y(a) = \alpha$, $y(b) = \beta$ → boundary value problem.

Eg. Find u such that $u_{xx} + u_{yy} = 0$ on $\Omega \subseteq \mathbb{R}^2$, $\underbrace{u|_{\partial\Omega}}_{}= f(x)$.

↳ value of u on boundaries.



Hyperbolic / wave Equation

Eg. Find u such that (1A) $u_{tt} - c^2 u_{xx} = 0$, $0 < x < L$, $t > 0$,

(1B) $u(0,t) = 0$, $u(L,t) = 0$, $t > 0$

(1C) $u(x,0) = f(x)$, $u_t(x,0) = g(x)$, $0 < x < L$

continuous

"String vibration"

Method of separation of variable

Let $u(x,t) = X(x)\,T(t)$

(1A) $\Rightarrow$ $\dfrac{X''}{X} = \dfrac{1}{c^2}\dfrac{T''}{T} = K$ (constant)

$\Rightarrow X'' - KX = 0$ and $T'' - c^2 K T = 0$

(1B) $\Rightarrow$ $u(0,t) = 0 \Rightarrow X(0)\,T(t) = 0$

$\Rightarrow X(0) = 0$ $(\because T(t) \neq 0)$

$u(L,t) = 0 \Rightarrow X(L) = 0$

$\Rightarrow X'' - KX = 0$ with $X(0) = 0$ and $X(L) = 0$, $(T - \lambda I)Y = 0$.

(i) $K > 0$ (ii) $K = 0$ (iii) $K < 0$.

Exercise $K \geq 0 \Rightarrow x = 0$.

If $K < 0$, let $K = -p^2$

$X = A\cos px + B\sin px$ $(\because X'' - KX = 0)$

$X(0) = 0 \Rightarrow A = 0$

$X(L) = 0 \Rightarrow B\sin pL = 0$

$\Rightarrow p = \dfrac{n\pi}{L}$

$\therefore K_n = -\left(\dfrac{n\pi}{L}\right)^2$ → eigen values

$\boxed{X_n = B'_n \sin \dfrac{n\pi}{L}}$ → eigen function

Solving $T'' - c^2 K T = 0$

for $T^2 + \left(\frac{cn\pi}{L}\right)^2 T = 0$

$$\Rightarrow \boxed{T_n = C_n \cos\left(\frac{n\pi c}{L} t\right) + D_n \sin\left(\frac{n\pi c}{L} t\right)}$$

$$\Rightarrow u_n(x,t) = X_n T_n$$

$$= \left[A_n \cos \frac{n\pi c}{L} t + B_n \sin \frac{n\pi c}{L} t\right] \sin \frac{n\pi}{L} x$$

Since (IA) is homogeneous equation,

$\sum_{n=1}^{\infty} u_n(x,t)$ is also a solⁿ of (IA). [Principle of superposition]

Assume $\sum_{n=1}^{\infty} u_n(x,t)$ converges uniformly to $u(x,t)$.

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi c}{L} t + B_n \sin \frac{n\pi c}{L} t\right) \cdot \left(\sin \frac{n\pi}{L} x\right)$$

↳ Find A_n, B_n.

(IC) $\Rightarrow$ $u(x,0) = f(x)$, $u_t(x,0) = g(x)$, $0 < x < L$

$$u(x,0) = f(x) \Rightarrow f(x) = \sum_{n=1}^{\infty} \underbrace{A_n}_{} \sin \frac{n\pi}{L} x$$

Fourier series coefficient

$$\therefore \boxed{A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx}, \quad f(x): \text{continuous}$$

(IC) $\Rightarrow$ $u_t(x,0) = g(x)$

$$\Rightarrow g(x) = \sum \underbrace{B_n \frac{n\pi c}{L}}_{} \sin\left(\frac{n\pi}{L} x\right)$$

Fourier sine coefficient.

$$\therefore B_n \frac{n\pi c}{L} = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L} dx$$

$$\Rightarrow \boxed{B_n = \frac{2}{n\pi c} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx}.$$

Solution of (IA):

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi c}{L} t\right) + B_n \sin\left(\frac{n\pi c}{L} t\right)\right] \sin\left(\frac{n\pi x}{L}\right),$$

where $A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx$

$B_n = \frac{2}{n\pi c} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx.$

Parabolic / Heat Problem

Eg Find u such that

(1A) $u_t - c\,u_{xx} = 0$, $0<x<L$, $t>0$

(1B) $u(0,t)=0$, $u(L,t)=0$, $t>0$

(1C) $u(x,0)=f(x)$. → continuous

Sol: Method of separation of variable:

Let $u(x,t) = X(x)T(t)$

(1A) $\Rightarrow \dfrac{X''}{X} = \dfrac{1}{c}\dfrac{T'}{T} = k$ (constant)

$\Rightarrow X'' - kX = 0$, $T' - ckT = 0$

(1B) $\Rightarrow X(0)=0$, $X(L)=0$

ODE: $X'' - kX = 0$, with $X(0)=0$, $X(L)=0$

for $k \geq 0 \Rightarrow X=0$

For $k<0 \Rightarrow$ ~~$k=p^2$~~ $k=-p^2$

$X_n = B_n \sin \dfrac{n\pi x}{L}$

$p_n = \dfrac{n\pi}{L}$, $k_n = -\left(\dfrac{n\pi}{L}\right)^2$

$T_n = C_n e^{\left(\frac{n\pi}{L}\right)^2 ct}$

$u_n = X_n T_n = A_n e^{\left(\frac{n\pi}{L}\right)^2 ct} \sin\left(\frac{n\pi}{L}\right)x$

$\Rightarrow u(x,t) = \sum u_n(x,t)$

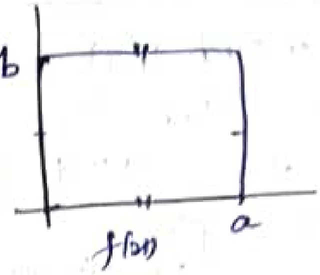
$\Rightarrow \boxed{u(x,t) = \sum A_n e^{\left(\frac{n\pi}{L}\right)^2 ct} \sin\left(\frac{n\pi}{L}\right)x.}$

(1C) $\Rightarrow u(x,0)=f(x)$

$\Rightarrow f(x) = \sum A_n \sin\left(\frac{n\pi}{L}\right)x$

- $A_n = \dfrac{2}{L}\displaystyle\int_0^L f(x) \sin \frac{n\pi}{L} x$, A_n: Fourier sine coefficient
  $f(x)$: continuous.

Laplace / Elliptic (Dirichlet):



Eg. find u such that:

(1A) $u_{xx} + u_{yy} = 0$ } → For applying superposition theorem

(1B) $u(0,y) = 0$, $u(a,y) = 0$ } → Sturm Liouville Problem

(1C) $u(x,b) = 0$, $u(x,0) = f(x)$.

Method of Separation of variable:

Let $u(x,y) = X(x)Y(y)$.

(1A) $\Rightarrow \dfrac{X''}{X} = -\dfrac{Y''}{Y} = K$

$\Rightarrow x'' - Kx = 0 , \; y'' + Ky = 0$

(1B) $\Rightarrow u(0,y) = x(0)y(y) = 0 \Rightarrow x(0) = 0$

$u(a,y) = x(a)y(y) = 0 \Rightarrow x(a) = 0$

$K < 0 \Rightarrow K = -p^2 , \; p_n = \dfrac{n\pi}{a}$

$\therefore \boxed{X_n = B_n' \sin \dfrac{n\pi}{a} x}$

For y: $y'' - \left(\dfrac{n\pi}{a}\right)^2 y = 0$

$\Rightarrow \boxed{y_n = C_n e^{\frac{n\pi}{a} y} + D_n e^{-\frac{n\pi}{a} y}}$.

$\therefore \boxed{u_n(x,y) = \left[A_n e^{n\pi/a \cdot y} + B_n e^{-\frac{n\pi}{a} y}\right] \sin \dfrac{n\pi}{a} x.}$

Now,

$$u(x,y) = \sum_{n=1}^{\infty} u_n(x,y).$$

(1C) $\Rightarrow u(x,b) = 0 \Rightarrow 0 = \displaystyle\sum_{n=1}^{\infty} \left[A_n e^{\frac{n\pi}{a} b} + B_n e^{-\frac{n\pi}{a} b}\right] \sin \dfrac{n\pi}{a} x.$

Now, here $\sin \frac{n\pi}{a} x$ can not be zero, as it will make corresponding $u_n(x,y) = 0$.

$\therefore A_n e^{\frac{n\pi}{a} b} + B_n e^{-\frac{n\pi}{a} b} = 0$

$\Rightarrow B_n = -\dfrac{A_n e^{n\pi/a \cdot b}}{e^{-\frac{n\pi}{a} \cdot b}}$

(1C) $\Rightarrow u(x,0) = f(x)$.

$$f(x) = \sum_{n=1}^{\infty}\left[A_n e^{\frac{n\pi}{a}y} + B_n e^{-\frac{n\pi}{a}y}\right]\sin\frac{n\pi}{a}x$$

$$= \sum_{n=1}^{\infty}[A_n + B_n]\sin\frac{n\pi}{a}x$$

$$= \sum_{n=1}^{\infty}\left[A_n - A_n\frac{e^{\frac{n\pi}{a}b}}{e^{-\frac{n\pi}{a}b}}\right]\sin\frac{n\pi}{a}x$$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} 2e^{\frac{n\pi b}{a}} A_n \frac{\left[e^{-\frac{n\pi b}{a}} - e^{\frac{n\pi b}{a}}\right]}{2} \cdot \sin\frac{n\pi}{a}x$$

$$= \sum_{n=1}^{\infty} \underbrace{-\left(\sinh\frac{n\pi b}{a} \times 2e^{\frac{n\pi b}{a}} A_n\right)}_{\text{Fourier sine coefficient } (E_n)} \sin\frac{n\pi}{a}x$$

$$E_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi}{a}x\,dx.$$

$$\therefore \boxed{-2\sinh\left(\frac{n\pi b}{a}\right)\cdot e^{\frac{n\pi b}{a}}A_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi}{a}x\,dx}.$$

↳ Find A_n.

Now,

$$u(x,y) = \sum_{n=1}^{\infty}\left[A_n e^{\frac{n\pi}{a}y} + B_n e^{-\frac{n\pi}{a}y}\right]\sin\left(\frac{n\pi}{a}x\right)$$

$$= \sum_{n=1}^{\infty}\left[A_n e^{\frac{n\pi}{a}y} - A_n\frac{e^{\frac{n\pi}{a}b}}{e^{-\frac{n\pi}{a}b}}\cdot e^{-\frac{n\pi}{a}y}\right]\sin\frac{n\pi}{a}y.$$

$$u(x,y) = \sum 2A_n e^{-\frac{n\pi b}{a}} \cdot \sinh\frac{n\pi}{a}(y-b)\cdot\sin\left(\frac{n\pi}{a}x\right).$$

Neuman Solution

(1A) $u_{xx} + u_{yy} = 0$

(1B) $u_x(0,y) = 0, \quad u_x(a,y) = 0$

(1C) $u_y(x,0) = 0, \quad u_y(x,b) = f(x).$

↳ It won't have unique soln. It's unique upto a additive constant.

$$\text{(1A)} \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = K$$

$$\text{(1B)} \Rightarrow X'(0) = 0,\ X'(a) = 0$$

$$\Rightarrow \begin{array}{l} X'' - KX = 0 \\ X'(0) = 0, \\ X'(a) = 0 \end{array} \Rightarrow X_n = A_n\cos\frac{n\pi}{a}x.$$

(i) $K>0 \Rightarrow X=0$

(ii) $K=0 \Rightarrow X=c_1+c_2x$

(iii) $K<0 \Rightarrow X=A\cos px + B\sin px.$

$K=-p^2 x,$

$px = \frac{n\pi}{a}, n=0,1,2,\dots$

$Y_n = C_n e^{\frac{n\pi}{a}y} + D_n e^{-\frac{n\pi}{a}y}$

$\therefore \boxed{u_n(x,y) = \left[A_n e^{\frac{n\pi y}{a}} + B_n e^{-\frac{n\pi}{a}y}\right] \cos \frac{n\pi}{a}x}.$

Putting $u_y(x,0)=0$

$\Rightarrow \left[\frac{n\pi}{a}A_n - \frac{n\pi}{a}B_n\right] \cos \frac{n\pi}{a}x = 0$

$\Rightarrow A_n = B_n$

$u(x,y) = \sum_{n=0}^{\infty} \left[A_n e^{\frac{n\pi y}{a}} + A_n e^{-\frac{n\pi y}{a}}\right] \cos \frac{n\pi}{a}x.$

$= \sum_{n=0}^{\infty} 2A_n \cosh\left(\frac{n\pi y}{a}\right) \cos\left(\frac{n\pi}{a}x\right).$

$\therefore \boxed{u(x,y) = A_0 + \sum_{n=1}^{\infty} 2A_n \cosh\left(\frac{n\pi}{a}y\right) \cdot \cos\left(\frac{n\pi}{a}x\right)}. \quad -(1)$

$u_y(x,b) = f(x)$

$f(x) = \sum_{n=1}^{\infty} \underbrace{2A_n \sinh \frac{n\pi}{a}y \cdot \frac{n\pi}{a}}_{\text{Fourier cosine}} \cos \frac{n\pi}{a}x.$

$\Rightarrow f(x) = 2A \sinh \frac{n\pi b}{a} \cdot \frac{n\pi}{a} = \frac{2}{a}\int_0^a f(x) \cos \frac{n\pi}{a}x.$

$\Rightarrow \boxed{A_n = \frac{1}{n\pi \sinh\left(\frac{n\pi}{a}b\right)} \int_0^a f(x) \cos \frac{n\pi}{a}x\, dx}$

Non-Homogeneous Hyperbolic

Eg. Find u such that $u_{tt} - c^2 u_{xx} = F(x,t)$. (1A)

$u(0,t)=0, \quad u(L,t)=0$ (1B)

$u(x,0)=f(x), \quad u_t(x,0)=g(x)$ (1C)

↳ we cannot apply method of supperposition.

Idea: Let $u(x,t) = \sum_{n=1}^{\infty} \phi_n(t) \underbrace{\sin \frac{n\pi}{L} x}_{\text{satisfies (1B)}}$ be a solution.

(1A) $\Rightarrow \sum \left[\phi_n''(t) + \left(\frac{n\pi c}{L}\right)^2 \phi_n(t) \right] \sin \frac{n\pi}{L} x = F(x,t)$.

[we know that

$$\int_0^L \sin \frac{n\pi}{L} x \cdot \sin \frac{m\pi}{L} x \, dx = 0, \text{ if } m \neq n$$
$$= L/2, \text{ if } m=n \quad]$$

Multiply by $\sin \frac{k\pi}{L} x$ and integrate from 0 to L.

$$\sum_{n=1}^{\infty} \left[\phi_n''(t) + \left(\frac{n\pi c}{L}\right)^2 \phi_n(t) \right] \sin \frac{n\pi}{L} x \cdot \sin \frac{k\pi}{L} x = F(x,t) \cdot \sin \frac{k\pi}{L} x.$$

↳ if $n \neq k$, then it will be zero.

Finding $n=k \Rightarrow \left(\phi_k''(t) + \omega_k^2 \phi_k(t)\right) \frac{L}{2} = \int_0^L F(x) \sin \frac{k\pi}{L} x$

$\left[\text{Let } \omega_n = \frac{n\pi c}{L} \right]$

$\Rightarrow \phi_k''(t) + \omega_k^2 \phi_k(t) = \bar{F}_k(t)$.

$\phi_k(t) = \phi_k^H + \phi_k^P(t)$

$\phi_k^H(t) = A_k \cos \omega_k(t) + B_k \sin \omega_k(t)$

Eg. Show that $\phi_k^P = \frac{1}{\omega_n} \int_0^t \bar{F}_k(\xi) \sin \omega_k (t-\xi) \, d\xi$.

$$\left[\bar{F}_k(t) = \frac{2}{L} \int_0^L F(x,t) \sin\left(\frac{k\pi}{L} x\right) dx \right]$$

$$\phi_k(t) = A_k \cos \omega_k(t) + B_k \sin \omega_k(t) + \frac{1}{\omega_k} \int_0^t \bar{F}_k(\xi) \sin \omega_k (t-\xi) \, d\xi.$$

$$\therefore u(x,t) = \sum \left[A_K \cos \omega_K(t) + B_n \sin \omega_K(t) + \frac{1}{\omega_K} \int_0^t F_n(\xi) \sin \omega_n (t-\xi) d\xi \right] \sin \frac{n\pi}{L} x.$$

Now, $u(x,0) = f(x)$

$$\Rightarrow \boxed{A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx}$$

$u(x,0) = g(x)$

$$\Rightarrow \boxed{B_n = \frac{2}{\omega_n L} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx}$$

Eg Find u such that

$$u_{tt} - c^2 u_{xx} = F(x,t)$$

$u(0,t) = p(t)$, $u(L,t) = q(t)$ → Try to make it homogeneous.

$u(x,0) = f(x)$, $u_t(x,0) = g(x)$.

Sol Let $u = v + w$ be a solution.

$$\Rightarrow v_{tt} - c^2 v_{xx} = \underbrace{F(x,t) - (w_{tt} - c^2 w_{xx})}_{G(x,t)}$$

Now,

$v(0,t) = p(t) - w(0,t)$., $w(L,t) = q(t) - w(L,t)$

~~v(0,t)~~

$v(x,0) = \underbrace{f(x) - w(x,0)}_{f_1(x)}$, $v_t(x,0) = \underbrace{g(x) - w_t(x,0)}_{g_1(x)}$.

choose w such that $w(0,t) = p(t)$, $w(L,t) = q(t)$.

$\Rightarrow v_{tt} - c^2 v_{xx} = G(x,t)$

$v(0,t) = 0$, $v(L,t) = 0$

$v(x,0) = f(x)$, $v_t(x,0) = g_1(x)$

↳ Find v, then $u = v + w$

Now,

for w,

let $\boxed{w(x,t) = p(t) + \frac{x}{L} [q(t) - p(t)]}$.