Chapter 2

Directional Derivative and Differentiability

2.1 Introduction

Let G be an open set in R and $f : G \longrightarrow \mathbb{R}$ a real valued function of real variable. We call f to be differentiable at $x_0 \in G$ if $\lim\limits_{h \to 0} \dfrac{f(x_0 + h) - f(x_0)}{h}$ exists, and the limit is termed as $\frac{d}{dx}(f(x))|_{x_0}$ or $f'(x_0)$. $f'(x_0)$ represents the rate of change of f along x-axis at the point x_0. We already have learned that if f is differentiable at x_0, it also continuous at x_0.

If G be open in $\mathbb{R}^2$ and $f : G \longrightarrow \mathbb{R}$ a scalar field, then we defined the limits $\lim\limits_{h \to 0} \dfrac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$ and $\lim\limits_{h \to 0} \dfrac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$, if exist, partial derivatives of f with respect to x and y respectively at $(x_0, y_0) \in G$. We denoted the value of the limit as $\frac{\partial}{\partial x}(f)|_{(x_0,y_0)}$ and $\frac{\partial}{\partial y}(f)|_{(x_0,y_0)}$ respectively; and they represent the rate of change of f along x-axis and y-axis respectively. It is well known that existence of partial derivatives of a function can not ensure the continuity of the function. For example let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^4 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

One may check that $\frac{\partial}{\partial x}(f)|_{(0,0)}$ and $\frac{\partial}{\partial y}(f)|_{(0,0)}$ exist, but f is not continuous at $(0, 0)$.

This tells us the concept of partial differentiability for several variable functions is too weak to be defined as the equivalent concept of differentiability as we have for real variable function.

We further have learned that a scalar field $f : G \longrightarrow \mathbb{R}$ is said to be differentiable at $P_0 = (x_0, y_0) \in G$ if there exists $\alpha, \beta \in \mathbb{R}$ such that

$$\lim_{(h,k) \to (0,0)} \frac{f(x_0 + h, y_0 + k) - f(x_0, y_0) - \alpha h - \beta k}{\sqrt{h^2 + k^2}} = 0.$$

And, in that case it can be verified that $\alpha = \frac{\partial}{\partial x}(f)|_{P_0}$ and $\beta = \frac{\partial}{\partial y}(f)|_{P_0}$.

One should be careful that the mere existence of the partial derivatives at P_0 does not ensure differentiability of the given function. Further, if the partial derivatives of the function at the point P_0 do not exists, the function is not differentiable at P_0. One may check that that if f is differentiable at P_0, the it is continuous at P_0 (Prove it!).

Exercise 2.1.1. *1. Let* $f(x,y) = \begin{cases} x+y & \textit{if } x=0 \textit{ or } y=0 \\ 1 & \textit{otherwise} \end{cases}$. *Show that* f *has partial derivatives at* $(0,0)$, *but* f *is not continuous at* $(0,0)$.

2. *Show that the function* $f(x,y) = |x| + |y|$ *for all* $(x,y) \in \mathbb{R}^2$ *is continuous, but partial derivatives of* f *at* $(0,0)$ *does not exists, and therefore* f *is not differentiable at* $(0,0)$.

3. *Let* $f(x,y) = x^2 + y^2$ *for all* $(x,y) \in \mathbb{R}^2$. *Show that partial derivatives of* f *exists at any point of* $\mathbb{R}^2$. *Is* f *differentiable at every point of* $\mathbb{R}^2$*?*

4. *Let* $f(x,y) = \sqrt{x^2+y^2}$ *for all* $(x,y) \in \mathbb{R}^2$. *Show that partial derivatives of* f *does not exists at* $(0,0)$.

5. *Let* $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \textit{if } (x,y) \neq (0,0) \\ 0 & \textit{if } (x,y) = (0,0) \end{cases}$. *Show that* f *has partial derivatives at* $(0,0)$, *but* f *is not continuous at* $(0,0)$.

At this point we ask

Problem 2.1.2. *1. As partial differentiation represents the rate of change of a function along respective axes, can we talk of rate of change of function along any given straight line or, in general, along any given curve? If yes, whether it has any relation with the partial derivatives.*

2. *Under what condition the partial differentiability can replace the concept of differentiability of a scalar field?*

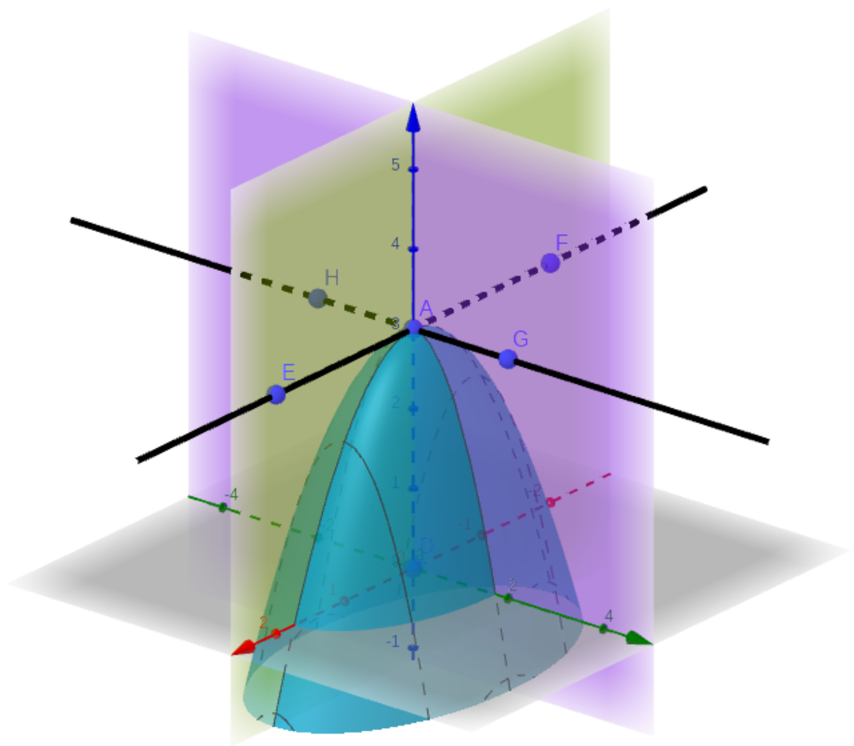
In this chapter we mainly try to get the answer to the above questions.

2.2 Directional Derivative and Differentiability

Let G be open in $\mathbb{R}^3$, $P_0 \in G$, and $f : G \longrightarrow \mathbb{R}$ a scalar field. Suppose $\frac{\partial}{\partial x}(f)|_{P_0}$ exists. If we look at the definition of partial derivatives carefully we shall see that it is the rate of change of f along the line passing though P_0 and parallel to the x-axis, i.e, the vector $e_1 = (1,0,0)$. Note that the parametric form of line $L_{e_1}(P_0)$ is $P_0 + he_1$ where $h \in \mathbb{R}$; and

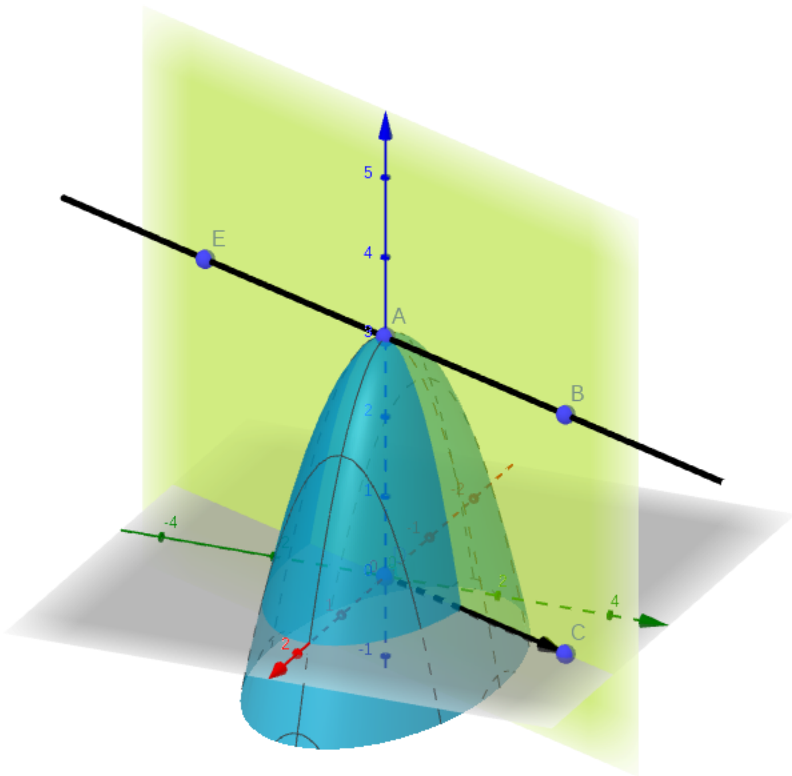
$$\begin{aligned} \frac{\partial}{\partial x}(f)|_{P_0} &= \lim_{h\to 0} \frac{f(x_0+h, y_0, z_0) - f(x_0,y_0,z_0)}{h} \\ &= \lim_{t\to 0} \frac{f((x_0,y_0,z_0) + t(1,0,0)) - f((x_0,y_0,z_0))}{t} \\ &= \lim_{t\to 0} \frac{f(P_0+te_1) - f(P_0)}{t} \end{aligned}$$

If the domain of f, i.e., G is a subset of $\mathbb{R}^2$, then the points $(x, y, f(x, y))$ will generate a surface which we call the surface induced from f. In that case the existence of $\frac{\partial}{\partial x}(f)|_{P_0}$ can be seen, geometrically, to be equivalent to the existence of the tangent line to the curve generated as a intersection of the plane induced from f and the plane parallel to xz-plane at the point $P_0 = (x_0, y_0) \in G$.



Now, imitating this we define the rate of change of f along any line passing though P_0 and parallel to a unit vector $\bar{v}$ (i.e., $\|\bar{v}\| = 1$), i.e., along the line $L_{\bar{v}}(P_0)$ as $\lim\limits_{t\to 0} \dfrac{f(P_0 + t\bar{v}) - f(P_0)}{t}$, if the limit exists. We define this quantity the **directional derivative of f at P_0 along the unit vector $\bar{v}$** and is denoted by $D_{\bar{v}}(f)|_{P_0}$, i.e., $D_{\bar{v}}(f)|_{P_0} = \lim\limits_{t\to 0} \dfrac{f(P_0 + t\bar{v}) - f(P_0)}{t}$, if the limit exists.

To understand the geometric equivalence of the existence of directional derivative assume that the domain of f, i.e., G is a subset of $\mathbb{R}^2$. In that case the existence of $D_{\bar{v}}(f)|_{P_0}$ can be seen, geometrically, to be equivalent to the existence of the tangent line to the curve generated as a intersection of the plane induced from f and the plane perpendicular to xy-plane and passing though the line $L_{\bar{v}}(P_0)$ i.e., $P_0 + t\bar{v}$, $t \in \mathbb{R}$, which is the line passing through the point P_0 and parallel to the vector $\bar{v}$.



Exercise 2.2.1. *1. Show that* $D_{\bar{v}}(f)|_{P_0} = \frac{d}{dt}(f(P_0 + t\bar{v}))|_{t=0}$.

2. For any direction $\bar{v}$*, find the directional derivative* $D_{\bar{v}}(f)|_{(0,0)}$ *where* $f(x, y) = x^2 + y^2$.

3. Let $f(x, y) = \begin{cases} x + y & \textit{if } x = 0 \textit{ or } y = 0 \\ 1 & \textit{otherwise} \end{cases}$. *Find directions* $\bar{v}$ *such that* $D_{\bar{v}}(f)|_{(0,0)}$ *exists.*

4. Let $f(x, y) = \sqrt{x^2 + y^2}$ *for all* $(x, y) \in \mathbb{R}^2$*. Show that* $D_{\bar{v}}(f)|_{(x_0,y_0)}$ *exists for all direction* $\bar{v}$ *if* $(x_0, y_0) \neq (0, 0)$.

It is obvious that if $D_{P_0}(f)|_{\bar{v}}$ exists for all direction $\bar{v}$ (unit vector), then all the partial derivatives of f exists at P_0, as $D_{e_1}(f)|_{P_0} = \frac{\partial}{\partial x}(f)|_{P_0}$, $D_{e_2}(f)|_{P_0} = \frac{\partial}{\partial y}(f)|_{P_0}$, and $D_{e_3}(f)|_{P_0} = \frac{\partial}{\partial z}(f)|_{P_0}$. So, we ask

Problem 2.2.2. *If all the partial derivatives of* f *exists at* P_0*, does* $D_{\bar{v}}(f)|_{P_0}$ *exit for all direction* $\bar{v}$*?*

This question may come to many naturally as we know that any vector $\bar{v}$ can be written as linear combinations of e_1, e_2 and e_3 in $\mathbb{R}^3$, i.e., in $\mathbb{R}^3$ we can reach any point $\bar{v}$ from $\bar{0}$ by moving along x-axis, y-axis and z-axis.

However, if we translate this question geometrically, assuming the domain of f, i.e., G as subset of $\mathbb{R}^2$, we shall see that it asks whether the existence of two tangent lines on the two specific curves generated as intersections of the plane induced from f and the plane parallel to xz-plane and yz-plane at the point $P_0 = (x_0, y_0) \in G$, ensures the tangent lines on the all possible curves generated as intersection of the plane induced from f and the planes perpendicular to the xy-plane and passing though the lines $L_{\bar{v}}(P_0)$ i.e., $P_0 + t\bar{v}$, $t \in \mathbb{R}$, which are the lines passing through the point P_0 and parallel to the vector $\bar{v}$, where $\|\bar{v}\| = 1$. We do not see any reason why existence of tangent of two curves at a point will unconditionally ensure existence of tangents to other curves passing though the same point. In reality, we shall see that this will be ensured by some extra conditions. The following discussion is towards the same.

Let $P_0 = (x_0, y_0, z_0)$ and $\bar{v} = (v_1, v_2, v_3)$ where $\|\bar{v}\| = 1$. Note that $v_i \neq 0$ for $i = 1, 2, 3$ as $\|\bar{v}\| = 1$. Now, Write $f(P_0 + t\bar{v}) - f(P_0)$ as

$$\begin{aligned} f(P_0 + t(v_1, v_2, v_3)) - f(P_0) = \; & f(P_0 + t(v_1, v_2, v_3)) - f(P_0 + t(v_1, v_2, 0)) + \\ & f(P_0 + t(v_1, v_2, 0)) - f(P_0 + t(v_1, 0, 0)) \\ & + f(P_0 + t(v_1, 0, 0)) - f(P_0 + t(0, 0, 0)). \\ \\ = \; & f([P_0 + t(v_1, v_2, 0)] + tv_3 e_1) - f(P_0 + t(v_1, v_2, 0)) + \\ & f([P_0 + t(v_1, 0, 0)] + tv_2 e_2) - f(P_0 + t(v_1, 0, 0)) \\ & + f([P_0 + t(0, 0, 0)] + tv_1 e_1) - f(P_0 + t(0, 0, 0)). \end{aligned}$$

And, therefore,

$$\begin{aligned} & \lim_{t \to 0} \frac{f(P_0 + t\bar{v}) - f(P_0)}{t} \\ = & \lim_{t \to 0} \frac{f(P_0 + t(v_1, v_2, 0) + tv_3 e_3) - f(P_0 + t(v_1, v_2, 0))}{t} + \\ & \lim_{t \to 0} \frac{f(P_0 + t(v_1, 0, 0) + tv_2 e_2) - f(P_0 + t(v_1, 0, 0))}{t} + \\ & \lim_{t \to 0} \frac{f(P_0 + t(0, 0, 0) + tv_1 e_1) - f(P_0 + t(0, 0, 0))}{t} \\ = & \lim_{t \to 0} \frac{f(P_0 + t(v_1, v_2, 0) + tv_3 e_3) - f(P_0 + t(v_1, v_2, 0))}{t} \frac{v_3}{v_3} + \\ & \lim_{t \to 0} \frac{f(P_0 + t(v_1, 0, 0) + tv_2 e_2) - f(P_0 + t(v_1, 0, 0))}{t} \frac{v_2}{v_2} + \\ & \lim_{t \to 0} \frac{f(P_0 + t(0, 0, 0) + tv_1 e_1) - f(P_0 + t(0, 0, 0))}{t} \frac{v_1}{v_1} \end{aligned}$$

Now, if partial derivative of f with respect to x exists at P_0, with respect to y at $P_0 + t(v_1, 0, 0)$ and with respect to z at $P_0 + t(v_1, v_2, 0)$, then we can calculate the above limit and we get the following.

$$\lim_{t \to 0} \frac{f(P_0 + t\bar{v}) - f(P_0)}{t} = v_3 \frac{\partial}{\partial z}(f)|_{P_0 + t(v_1, v_2, 0)} + v_2 \frac{\partial}{\partial y}(f)|_{P_0 + t(v_1, 0, 0)} + v_1 \frac{\partial}{\partial x}(f)|_{P_0}.$$

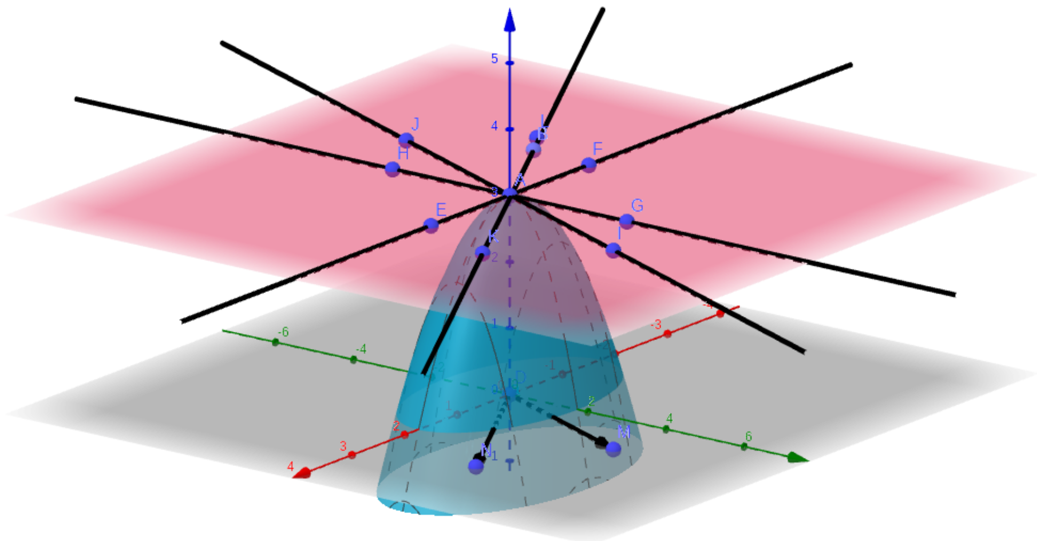
Which will be equal to $v_3 \frac{\partial}{\partial z}(f)|_{P_0} + v_2 \frac{\partial}{\partial y}(f)|_{P_0} + v_1 \frac{\partial}{\partial x}(f)|_{P_0}$ if $\frac{\partial}{\partial z}(f)|_{(x,y,z)}$ and $\frac{\partial}{\partial y}(f)|_{(x,y,z)}$ are continuous at P_0.

Therefore, we have the following:

Theorem 2.2.3. *Let G be open in $\mathbb{R}^3$, $P_0 \in G$ and $f : G \longrightarrow \mathbb{R}$ an scalar field. Suppose partial derivatives of f exists in an open ball around P_0 and, two of the three partial derivatives are continuous at P_0, then for any unit vector $\bar{v}$ in $\mathbb{R}^3$,* *(I)* $D_{\bar{v}}(f)|_{P_0}$ *exists and*

(II) $D_{\bar{v}}(f)|_{P_0} =< \nabla f|_{P_0}, \bar{v} >= v_1 \frac{\partial}{\partial x}(f)|_{P_0} + v_2 \frac{\partial}{\partial y}(f)|_{P_0} + v_3 \frac{\partial}{\partial z}(f)|_{P_0}$.

It is well known that if f is differentiable at P_0, then all the partial derivatives exist at P_0. It is worth exploring whether $D_{\bar{v}}(f)|_{P_0}$ exists for all direction $\bar{v}$ if f is differentiable at P_0. Geometrically, it is something very trivial to understand, however, to prove it we need some tools. before going into the detailed discussion, we first see the geometric view towards it. Assume that G, the domain of f is a subset of $\mathbb{R}^2$. Then, the differentiability of f at $P_0 \in G$ will mean the existence of the tangent plane at the point $(P_0, f(P_0))$ on the surface induced by f. Clearly, the existence of tangent plane means the existence of the tangents ot the all possible curves lying on that surface at the point $(P_0, f(P_0))$; and hence in specific, the existence of the tangent lines on the all possible curves generated as intersection of the plane induced from f and the planes perpendicular to the xy-plane and passing though the lines $L_{\bar{v}}(P_0)$ where $\|\bar{v}\| = 1$, which essentially implies the existence of directional derivatives of f along all possible direction at the point P_0. Therefore, differentiability should imply directional differentiability along all direction.



Now to tackle the situation we need tool which converts the differentiability to something a geometric object. For this we first revisit the definition of differentiability of a real valued fucntion $g : G \longrightarrow \mathbb{R}$ where $G \subset R$ is open. Suppose $a \in G$. We say that g is differentiable at a if

$\lim_{x \to a} \dfrac{g(x) - g(a)}{x - a}$ exists. If this limit exists we call this value to be $g'(a)$, i.e., $g'(a) = \lim_{x \to a} \dfrac{g(x) - g(a)}{x - a}$. Now define a function $g_1 : G \longrightarrow \mathbb{R}$ by

$$g_1(x) = \begin{cases} \frac{f(x)-f(a)}{x-a} & \text{if } x \neq a \\ \alpha & \text{if } x = a \end{cases}$$

Note that if we set $\alpha = g'(a)$, then g_1 is continuous at $x = a$, as g is differentiable at $x = a$. Now suppose it is not known whether g is differentiable at a. If we assume that g_1 is continuous at a, then it is easy to see that g is differentiable at a, and $\alpha = f'(a)$. Thus, we have

Theorem 2.2.4 (Caratheodory's Lemma)**.** *Let $G \subset \mathbb{R}$ be open, $a \in G$, and $f : G \longrightarrow \mathbb{R}$ a function. Then, f is differentiable at a if and only if there exists a function $f_1 : G \longrightarrow \mathbb{R}$ such that*

(I) f_1 is continuous at a, and

(II) $f(x) - f(a) = (x - a)f_1(x)$ for all $x \in G$.

Specifically, $f_1(x) = \dfrac{f(x) - f(a)}{x - a}$ *for* $x \in G\backslash\{a\}$*; and* $f_1(a) = \frac{d}{dx}(f)|_a = f'(a)$.

It can be seen that the above result also holds true for functions of several variables.

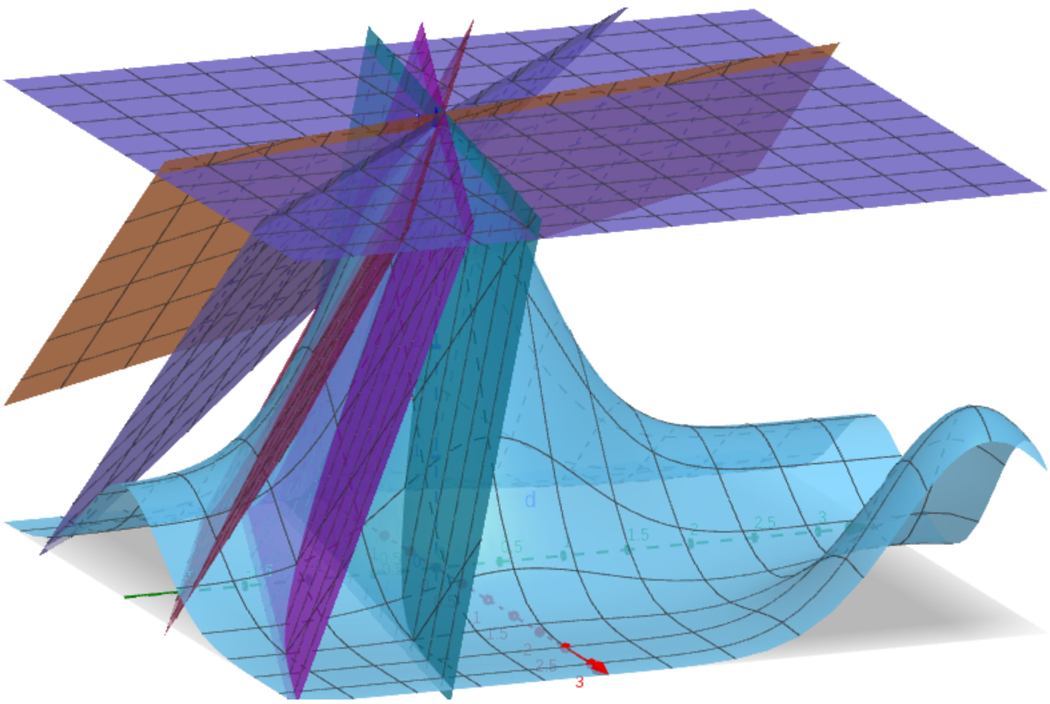
Theorem 2.2.5 (Caratheodory)**.** *Let $G \subset \mathbb{R}^3$ be open, $\bar{a} \in G$, and $f : G \longrightarrow \mathbb{R}$ a function. Then, f is differentiable at $\bar{a}$ if and only if there exists three functions $f_1, f_2, f_3 : G \longrightarrow \mathbb{R}$ such that*

(I) f_1, f_2 and f_3 are continuous at $\bar{a}$, and

(II) $f(x, y, z) - f(a_1, a_2, a_3) = (x - a_1)f_1(x, y, z) + (y - a_2)f_2(x, y, z) + (z - a_3)f_3(x, y, z)$ for all $(x, y, z) \in G$.

Further, it can be seen that $f_1(\bar{a}) = \frac{\partial}{\partial x}(f)|_{\bar{a}}$, $f_2(\bar{a}) = \frac{\partial}{\partial y}(f)|_{\bar{a}}$, *and,* $f_3(\bar{a}) = \frac{\partial}{\partial z}(f)|_{\bar{a}}$.

Proof. Assignment! □



Now, we assume that $f : G \longrightarrow \mathbb{R}$ be differentiable at P_0 where $G \subset \mathbb{R}^3$ is open and $P_0 \in G$. Then by Theorem 2.2.5 there exists functions $f_1, f_2, f_3 : G \longrightarrow \mathbb{R}$ continuous at P_0 such that

$$f(x, y, z) - f(x_0, y_0, z_0) = (x-x_0)f_1(x, y, z) + (y-y_0)f_2(x, y, z) + (z-z_0)f_3(x, y, z)$$

for all $(x, y, z) \in G$. Now, let $\bar{v}$ be any unit vector in $\mathbb{R}^3$ and we choose the points $(x, y, z) \in G$ such that they lie on the straight line $P_0 + t\bar{v}$ where $t \in \mathbb{R}$. Then we get

$$\begin{aligned} f(P_0 + t\bar{v}) \; - \; f(P_0) = \; & ((x_0 + tv_1) \; - \; x_0) \; f_1(P_0 + t\bar{v}) \; + \\ & ((y_0 + tv_2) \; - \; y_0) \; f_2(P_0 + t\bar{v}) \; + \\ & ((z_0 + tv_3) \; - \; z_0) \; f_3(P_0 + t\bar{v}) \end{aligned}$$

for all $t \in \mathbb{R}$, i.e.,

$$\begin{aligned} \frac{f(P_0 + t\bar{v}) - f(P_0)}{t} = \; & \frac{((x_0 + tv_1) \; - \; x_0)}{t} \; f_1(P_0 + t\bar{v}) \; + \\ & \frac{((y_0 + tv_2) \; - \; y_0)}{t} \; f_2(P_0 + t\bar{v}) \; + \\ & \frac{((z_0 + tv_3) \; - \; z_0)}{t} \; f_3(P_0 + t\bar{v}) \end{aligned}$$

for all $t \in \mathbb{R}\backslash\{0\}$.

Since f_1, f_2 and f_3 are continuous at P_0, as $t \longrightarrow 0$, we see the limits in the right side exists and therefore, the limit in the left side exists, i.e., $D_{\bar{v}}(f)|_{P_0}$ exist. Since $\bar{v}$ was arbitrary, we see that $D_{\bar{v}}(f)|_{P_0}$ exists for all direction $\bar{v}$. Further, from the above equality we get, under limit,

$$D_{\bar{v}}(f)|_{P_0} = v_1 \frac{\partial}{\partial x}(f)|_{P_0} + v_2 \frac{\partial}{\partial y}(f)|_{P_0} + v_3 \frac{\partial}{\partial z}(f)|_{P_0} = < \nabla f|_{P_0}, \bar{v} > .$$

From this we conclude the following

Theorem 2.2.6. *Let* $G \subset \mathbb{R}^3$ *be open,* $P_0 \in G$ *and* $f : G \longrightarrow \mathbb{R}$ *differentiable at* P_0*, then*
(I) $D_{\bar{v}}(f)|_{P_0}$ *exists for all direction* $\bar{v}$*, and*
(II) $D_{\bar{v}}(f)|_{P_0} = < \nabla f|_{P_0}, \bar{v} >$.

Now, if we follow the statements of Theorem 2.2.3 and Theorem 2.2.6, we shall see that two different "hypotheses" gives the same result. So, naturally one asks whether those two "hypotheses" in Theorem 2.2.3 and Theorem 2.2.6 have any relation. Clearly, mere differentiability of a function can not ensure continuity of partial derivatives or even the existence of partial derivatives in neighbourhood points. So, we should check whether the "conditions" of partial differentiations in Theorem 2.2.3 ensure differentiability of the function in the corresponding point. The answer is, in fact, "Yes"! From the technique applied in Theorem 2.2.3, and the technique of forming the increment function f_1 in Theorem 2.2.4 we see a scope/ we sense our intuition of proving differentiability of a scalar field under the condition the partial derivatives are continuous.

Let $G \subset \mathbb{R}^n$ be open and $P_0 = (x_0, y_0, z_0) \in G$. Suppose $f : G \longrightarrow \mathbb{R}$ a scalar field. Then,

$$\begin{aligned} f(x,y,z) - f(x_0,y_0,z_0) = \; & f(x,y,z) - f(x,y,z_0) + \\ & f(x,y,z_0) - f(x,y_0,z_0) + \\ & f(x,y_0,z_0) - f(x_0,y_0,z_0) \end{aligned}$$

Therefore, if we define

$$f_1(x,y,z) = \begin{cases} \frac{f(x,y,z)-f(x,y,z_0)}{x-x_0} & \text{if } z \neq z_0 \\ \frac{\partial}{\partial z}(f)|_{(x,y,z_0)} & \text{if } z = z_0 \end{cases},$$

$$f_2(x,y,z) = \begin{cases} \frac{f(x,y,z_0)-f(x,y_0,z_0)}{y-y_0} & \text{if } y \neq y_0 \\ \frac{\partial}{\partial y}(f)|_{(x,y_0,z_0)} & \text{if } y = y_0 \end{cases} \text{ and}$$

$$f_3(x,y,z) = \begin{cases} \frac{f(x,y_0,z_0)-f(x_0,y_0,z_0)}{x-x_0} & \text{if } x \neq x_0 \\ \frac{\partial}{\partial x}(f)|_{(x_0,y_0,z_0)} & \text{if } x = x_0 \end{cases},$$

provided $\frac{\partial}{\partial z}(f)|_{(x,y,z_0)}$, $\frac{\partial}{\partial y}(f)|_{(x,y_0,z_0)}$ and $\frac{\partial}{\partial x}(f)|_{(x_0,y_0,z_0)}$ exist.

If we further assume that $\frac{\partial}{\partial z}(f)|_{(x,y,z)}$, and $\frac{\partial}{\partial y}(f)|_{(x,y,z)}$ are continuous at P_0, then we see that the functions f_1, f_2 and f_3 are continuous at P_0. And, it is easy to check that

$$f(x,y,z) - f(x_0,y_0,z_0) = (z-z_0)f_1(x,y,z) + (y-y_0)f_2(x,y,z) + (x-x_0)f_1(x,y,z)$$

for all $(x,y,z) \in G$. Therefore, by Theorem 2.2.5, we have f is differentiable at P_0. This observation leads to the following

Theorem 2.2.7. *Let $G \subset \mathbb{R}^3$ be open and $P_0 \in G$. Suppose $f : G \longrightarrow \mathbb{R}$ a scalar field such that all the partial derivatives exists in an open ball around P_0, and two of the three partial derivatives are continuous at P_0, then f is differentiable at P_0.*

Thus, in view of Theorem 2.2.7 and Theorem 2.2.6, we see that Theorem 2.2.3 follows as a corollary; and combining those two results we can write the following for $G \subset \mathbb{R}^3$ open, $P_0 \in G$ and $f : G \longrightarrow \mathbb{R}$:

Remark 2.2.8. (I)

$$A : \left\{ \begin{array}{ll} A1: & \textit{all the partial derivatives of } f \textit{ exist in an open ball araound } P_0 \\ A2: & \textit{two of the three partail derivatibes of } f \textit{are continuous at } P_0 \end{array} \right\}$$

$$\Downarrow$$

$$B : \left\{ \; f \textit{ is differentiable at } P_0 \; \right\}$$

$$\Downarrow$$

$$C : \left\{ \begin{array}{ll} C1: & D_{\bar{v}}(f)|_{P_0} \textit{ exists for all direction } \bar{v} \\ C2: & D_{\bar{v}}(f)|_{P_0} = < \nabla f|_{P_0}, \bar{v} > \end{array} \right\}$$

(II) *If we assume that f is oc $\mathcal{C}^1$-type, then both the conditions $A1$ and $A2$ are satisfied.*

(III) *Since $A \Rightarrow B \Rightarrow C$ follows, contrapositively, we have $(\sim C) \Rightarrow (\sim B) \Rightarrow (\sim A)$. These results will be useful to prove or disprove a scalar field is differentiable.*

Example 2.2.9. *1. Let $f(x, y) = \exp(x + y)$ for all $(x, y) \in \mathbb{R}^2$. Since f is a composition of $\mathcal{C}^\infty$-type function, f is a $\mathcal{C}^\infty$-type function on $\mathbb{R}^2$ (and therefore it satisfies all the conditions of "A" in Remark 2.2.8 or the hypothesis of Theorem 2.2.7), we see that f is differentiable on $\mathbb{R}^2$.*

2. Let $f(x, y) = \sqrt{x^2 + y^2}$ for all $(x, y) \in \mathbb{R}^2$. It can be checked that the partial derivatives of f do not exist at $(0, 0)$, and therefore ("C1" fails to hold in Remark 2.2.8) f is not differentiable at $(0, 0)$.

3. Let $f(x, y) = \begin{cases} \frac{x^2 y}{x^4+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$. It can be shown that for any direction $\bar{v}$, $D_{\bar{v}}(f)|_{(0,0)}$ exits; but $D_{\bar{v}}(f)|_{(0,0)} \neq < \nabla(f)|_{(0,0)}, \bar{v} >$ (i.e., "C2" of Remark 2.2.8 fails to hold) and therefore f is not differentiable at $(0, 0)$.

4. Let $f(x, y) = |x| + |y|$ for all $(x, y) \in \mathbb{R}^2$. It can be seen that $D_{\bar{v}}(f)|_{(0,0)}$ does not exist for any direction $\bar{v}$, and therefore f is not differentiable at $(0, 0)$.

Exercise 2.2.10. *1. $f(x, y) = \begin{cases} \frac{x^2 y^2}{x^4+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$. Show f_x and f_y exists in a neighborhood $(0, 0)$, f_x is continuous at $(0, 0)$, f_y is not continuous at $(0, 0)$. Conclude that f is differentiable at $(0, 0)$.*

2. Let $f(x, y) = |xy|$ for all $(x, y \in \mathbb{R}^2)$. Using definition show that f is differentiable at $(0, 0)$.

3. let $f(x, y) = \sqrt{|xy|}$ for all $(x, y) \in \mathbb{R}^2$. Show that f is not differentiable at $(0, 0)$. [Hint: Use contrapositive part of Theorem 2.2.6]

4. $f(x, y) = \begin{cases} \frac{x^3 y}{x^4+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$. Using definition show that f fails to be differentiable at $(0, 0)$, though $D_{\bar{v}}(f)|_{(0,0)} =< \nabla(f)|_{(0,0)}, \bar{v} >$ for any direction $\bar{v}$.

5. $f(x, y) = \begin{cases} \frac{x^2 y}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$. Show that f is continuous at $(0, 0)$; $D_{\bar{v}}(f)|_{P_0}$ exists for all direction $\bar{v}$; but f is not differentiable at $(0, 0)$. [Hint: Use contrapositive part of Theorem 2.2.6]

Remark 2.2.11. *Let $G \subset \mathbb{R}^2$ be open, $P_0 \in G$, and $f : G \longrightarrow \mathbb{R}$ a scalar field. Let f be such that for any direction $\bar{v}$ we have*

$$D_{\bar{v}}(f)|_{P_0} =< \nabla(f)|_{P_0}, \bar{v} >= ||\nabla(f)|_{P_0}||\ ||\bar{v}||\ \cos(\theta) = ||\nabla(f)|_{P_0}||\ \cos(\theta).$$

Where θ is the angle inscribed by $\nabla(f)|_{P_0}$ and $\bar{v}$ with $\bar{0}$. Then we have

1. $D_{\bar{v}}(f)|_{P_0}$ is maximum if $\cos(\theta) = 1$, i.e., if $\bar{v} = \frac{\nabla(f)|_{P_0}}{||\nabla(f)|_{P_0}||}$.

2. $D_{\bar{v}}(f)|_{P_0}$ is mimimum if $\cos(\theta) = -1$, i.e., $\bar{v} = -\frac{\nabla(f)|_{P_0}}{||\nabla(f)|_{P_0}||}$.

3. $D_{\bar{v}}(f)|_{P_0} = 0$ if $\cos(\theta) = \pi/2$, i.e., $\bar{v} = \pm(\frac{\partial}{\partial y}(f)|_{P_0}, -\frac{\partial}{\partial x}(f)|_{P_0})$.

We now list down all the pathological examples.

Example 2.2.12. *1. Partial differentiability at a point does not imply continuity at that point:*

Let $f(x,y) = \begin{cases} x+y & \text{if } x=0 \text{ or } y=0 \\ 1 & \text{otherwise} \end{cases}$

f *has partial derivatives at* $(0,0)$*, but* f *is not continuous at* $(0,0)$*.*

2. *Continuity at a point does not imply partial differentiability at that point, and therefore, does not imply differentiability at that point:*

   The function $f(x,y) = |x|+|y|$ *for all* $(x,y) \in \mathbb{R}^2$ *is continuous, but partial derivatives of* f *at* $(0,0)$ *does not exists, and therefore* f *is not differentiable at* $(0,0)$*. One may consider the following function too.* $f(x,y) = \sqrt{x^2+y^2}$ *for all* $(x,y) \in \mathbb{R}^2$ *(check at* $(0,0)$*).*

3. *Existence of partial derivatives at a point do not guarantee existence of all the directional derivative at the same point:*

   Let $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$.

   f *has partial derivatives at* $(0,0)$*, at the same time none of directional derivative at* $(0,0)$ *exist except for* e_1 *and* e_2*. Here* f *is also not continuous at* $(0,0)$*.*

4. *Existence of all the directional derivative at a point neither imply that the directional derivatives at that point can be written as a linear combination of partial derivatives at that point, nor imply continuity at that point:*

   Let $f(x,y) = \begin{cases} \frac{x^2y}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$.

   For any direction $\bar{v}$*,* $D_{\bar{v}}(f)|_{(0,0)}$ *exits; but* $D_{\bar{v}}(f)|_{(0,0)} \neq < \nabla(f)|_{(0,0)}, \bar{v} >$ *and therefore* f *is not differentiable at* $(0,0)$*. One may check that* f *is not continuous at* $(0,0)$

5. *Existence of all the directional derivatives at a point along with continuity at that point does not imply that directional derivatives at that point can be written as a linear combination of partial derivatives at that point, and consequently differentiability at that point is also not implied:*

   Let $f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$.

   f *is continuous at* $(0,0)$ *and* $D_{\bar{v}}(f)|_{P_0}$ *exists for all direction* $\bar{v}$*,* $D_{\bar{v}}(f)|_{(0,0)} \neq < \nabla(f)|_{(0,0)}, \bar{v} >$ *and therefore* f *is not differentiable at* $(0,0)$*.*

6. *Existence of all the directional derivatives at a point along with the expression directional derivatives at that point in terms of a linear combination of partial derivatives at that point does not imply differentiability at that point:*

   Let $f(x,y) = \begin{cases} \frac{x^3y}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

f fails to be differentiable at $(0,0)$, though $D_{\bar{v}}(f)|_{(0,0)}$ exists for all direction $\bar{v}$ and $D_{\bar{v}}(f)|_{(0,0)} =< \nabla(f)|_{(0,0)}, \bar{v} >$ for any direction $\bar{v}$.

7. *A function which differntiable at a point and having partial derivatives in a neighborhood of the point: However the partial derivatives are not continuous at that point:*

   $$\text{Let } f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

   f is differentiable at $(0,0)$ and f_x, f_y exist in a neighborhood of $(0,0)$. However, f_x and f_y are not continuous at $(0,0)$.

8. *A function which differntiable at a point and having partial derivatives in a neighborhood of the point with the property that only one of the partial partial derivatives is continuous at that point:*

   $$\text{Let } f(x,y) = \begin{cases} \frac{x^2y^2}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$

   f_x and f_y exists in a neighborhood $(0,0)$. Though f_y is not continuous at $(0,0)$, f_x is continuous at $(0,0)$, and therefore, f is differentiable at $(0,0)$.

2.3 Application: Level sets

Let $U \subset \mathbb{R}^n$, $P_0 \in U$, and $f : U \longrightarrow \mathbb{R}$ a scalar field. For each $c \in \mathbb{R}$, the level set of f of the level c is donoted by $S_c(f)$ (or by $L_c(f)$), and is defined by

$$S_c(f) := \{P \in U \mid f(P) = c\}.$$

When $n = 2$, we call level sets to be level curves, as geometrically those represent a curves; and when $n = 3$, we call level sets to be level surfaces as geometrically those represent surfaces.

Example 2.3.1. 1. *Let $f(x,y) = x^2 + y^2$ for all $(x,y) \in \mathbb{R}^2$. Then for each $c > 0$, the level curve $S_c(f)$ is the circle $x^2 + y^2 = c$.*

2. *Let $f(x,y,z) = x^2 + y^2 + z^2$ for all $(x,y,z) \in \mathbb{R}^3$. Then for each $c > 0$, the level surface $S_c(f)$ is the sphere $x^2 + y^2 + z^2 = c$.*

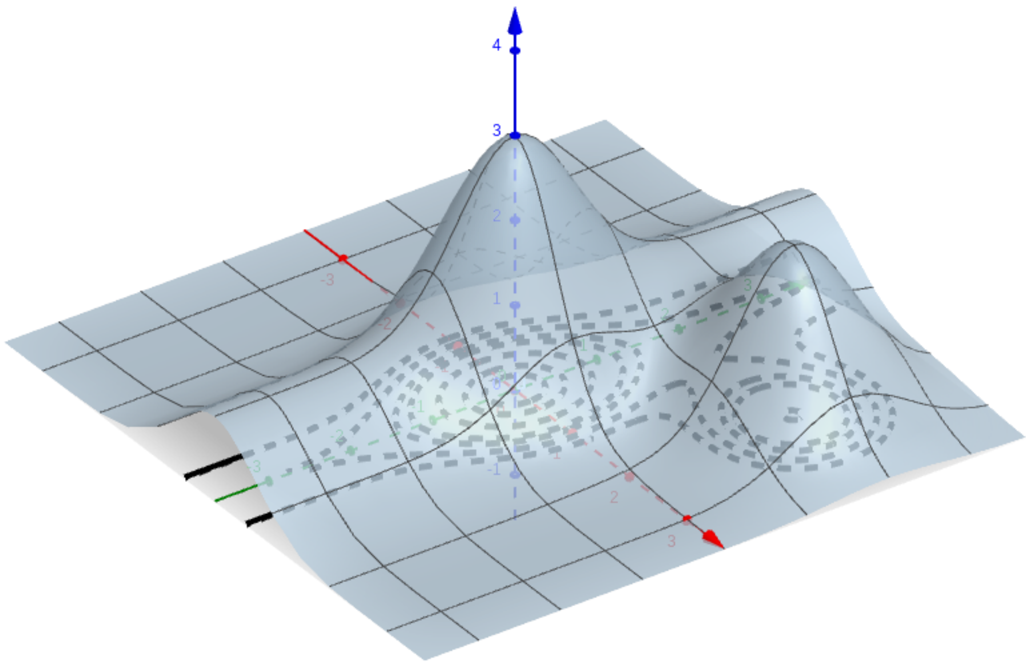
Let $n = 3$, and suppose that the scalar field f is smooth. Consider the level surface $S_c(f)$. Since $S_c(f)$ is a surface, it is union of curves. Let $\gamma(t) = (x(t), y(t), z(t))$ for all $t \in I \subset \mathbb{R}$ be any smooth curve on $S_c(f)$ passing though the point P_0. Then $f(\gamma(t)) = c$. From which, on differentiating with respect to t, we get

$$\frac{\partial}{\partial x}(f)|_{P_0}\frac{d}{dt}(x)|_{t_0} + \frac{\partial}{\partial y}(f)|_{P_0}\frac{d}{dt}(y)|_{t_0} + \frac{\partial}{\partial z}(f)|_{P_0}\frac{d}{dt}(z)|_{t_0} = 0 \text{ , i.e.,}$$

$$< \nabla(f)|_{P_0}, \gamma'(t)|_{t_0} >= 0,$$

where $\gamma(t_0) = P_0$, $\gamma'(t)|_{t=t_0} = (x'(t), y'(t), z'(t))_{t=t_0}$ is tangent to the curve γ at the point P_0.

This shows that $\nabla(f)|_{P_0}$ is normal to the curve γ at the point P_0. Since P_0 is arbitrary, we see that **at any point, the gradient of a scalar field is normal to its corresponding level set at that point, i.e.,** $\nabla(f)|_{P_0} \perp S_c(f)$ **at** P_0.



Exercise 2.3.2. *1. Draw the level surfaces of* $f(x,y,z) = x^2 + y^2 - z^2$ *for all* $(x,y,z) \in \mathbb{R}^3$.

2. *Draw the level curves of* $f(x,y) = x^2 - y^2$ *for all* $(x,y) \in \mathbb{R}^2$.

3. *Let* $h(x,y) = 2e^{-x^2} + e^{-3y^2}$ *denote the height on a mountain at position* (x,y). *In what direction from* $(1,0)$ *should one begin walking in order to climb the fastest?*

4. *Find a unit normal vector to the following surfaces at the specified point (i)* $x^2 + y^2 + z^2 = 9$ *at* $(0, \sqrt{3}, \sqrt{3})$, *(ii)* $x^3y^3 + y - z + 2 = 0$ *at* $(0,0,2)$, *(iii)* $z = 1/(x^2 + y^2)$ *at* $(1,1,1/2)$.

5. *Suppose that a particle is ejected from the surface* $x^2 + y^2 - z^2 = -1$ *at the point* $(1,1,\sqrt{3})$ *along the normal directed toward the* xy *plane to the surface at time* $t = 0$ *with a speed of* 10 *units per second. When and where does it cross the* xy *plane?*

6. *Starting from* $(1, 1)$, *in which direction should one travel in order to obtain the most rapid rate of decrease of the function* $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *defined by* $f(x, y) := (x + y - 2)^2 + (3x - y - 6)^2$?

7. *About how much will the function* $f(x, y) := ln\sqrt{x^2 + y^2}$ *change if the point* (x, y) *is moved from* $(3, 4)$ *a distance* 0.1 *unit straight toward* $(3, 6)$?