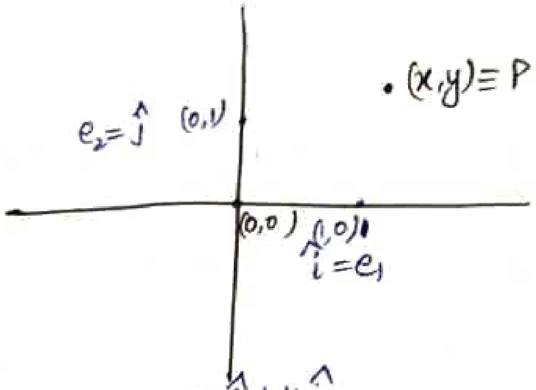
सदिशकलनम्

VECTOR CALCULUS

VECTOR



$x\hat{i} + y\hat{j}$

$x(1,0) = (x,0)$

$y(0,1) = (0,y)$

$(x,y) = x e_1 + y e_2$

$\|e_1\| = 1 = \|e_2\|$

Spatial structure

$d((0,0),(x,y)) = \sqrt{x^2+y^2}$

$= \|(x,y)\|$: Euclidean distance / norm

$\|.\|$ is called norm function.

$\|.\| : \mathbb{R}^2 \longrightarrow \mathbb{R}$

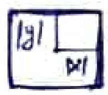
$\|\bar{v}\| = 0 \Leftrightarrow \bar{v} = 0$

$\|.\|_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$\|(x,y)\|_1 = |x| + |y|$

$\|\bar{v}\| < 1$

$\|(x,y)\|_\infty = \sup(|x|, |y|)$



Inner product:

$<(x_1,y_1),(x_2,y_2)> = x_1x_2 + y_1y_2$ ⇝ a fn

↳ Euclidean inner product

↓ gives Euclidean product

$< \cdot , \cdot > : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$

$<\bar{v},\bar{v}> = |\bar{v}|^2$

$\|\bar{v}\| = \sqrt{<\bar{v},\bar{v}>}$

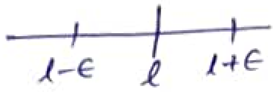
Euclidean Space: Space endowed with Euclidean norm.

Limit

→ $f: A \to \mathbb{R}$, $A \subset \mathbb{R}$, $x \in \mathbb{R}$,

$\lim_{x \to x_0} f(x) = \bar{l}$

or $f(x) \to l$ as $x \to x_0$.



Algebraic defn of zero:

$e + a = a \ \forall a \in \mathbb{R}$

$\Rightarrow e = 0$.

For any $\epsilon > 0$, there exists $\delta > 0$ such that

$|f(x) - l| < \epsilon$ when $|x - x_0| < \delta \equiv x \in (x_0 - \delta, x_0 + \delta)$.

$\equiv f(x) \in (l-\epsilon, l+\epsilon)$

$A \subset \mathbb{R}^2$

$f: A \to \mathbb{R}^2$

$p_0 \in \mathbb{R}^2$

$\lim_{p \to p_0} f(p) = \bar{l}$ → vector

$\|f(p) - \bar{l}\| < \epsilon$ whenever

$\|p - p_0\| < \delta$.

$f(p) \in B(\bar{l}, \epsilon)$

whenever $p \in B(p_0, \delta)$.

$\|\bar{x}\| = 1 \to$ circle

$\|\bar{x}\| \leq 1 \to$ disc

$\|\bar{x}\| < 1 \to$ open disc (open ball)

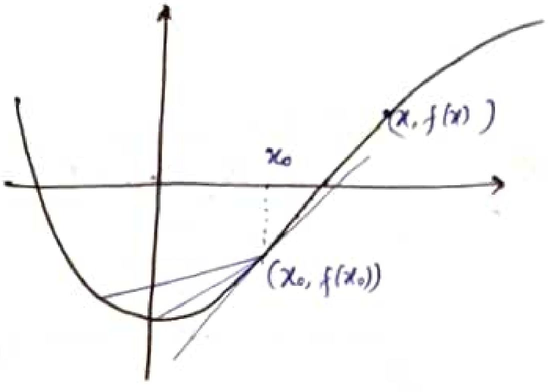
Ball ⇒ open ball (say)

Direction Derivative

$f:(a,b)\to\mathbb{R}$
$x_0\in(a,b)$

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=l$$

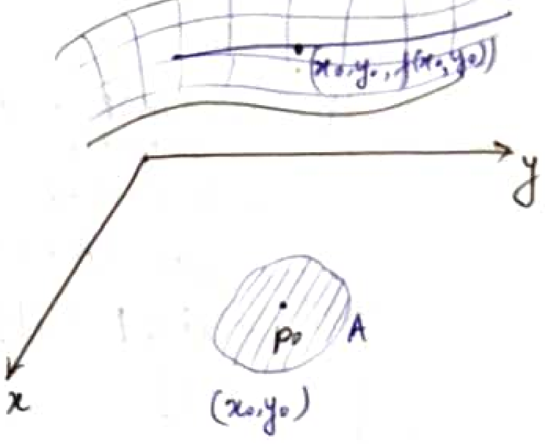
$$f'(x_0):=l$$



$A\subset\mathbb{R}^2$

$f: A\to\mathbb{R}$

$p_0\in A$ s.t. there is an open ball around p_0 in A.



2-06-2023

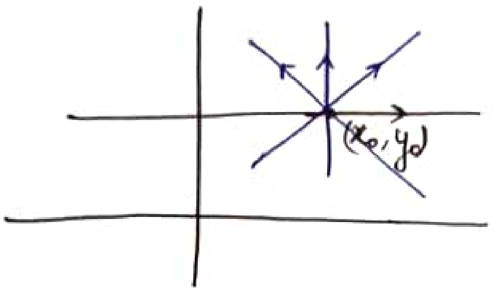
→ $f: G \to \mathbb{R}$, G open in $\mathbb{R}^2$

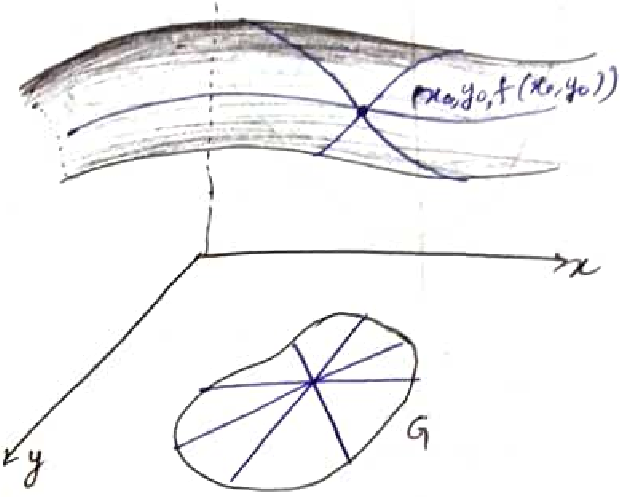
$P_0 \in G$.

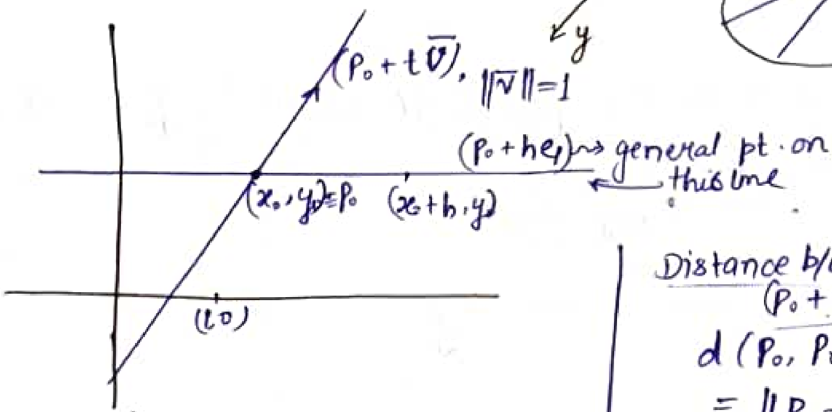
$$\lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

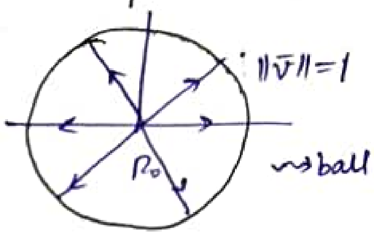
$P_0 = (x_0, y_0)$

$P = (x_0+h, y_0)$









Distance b/w $P_0 = (x_0, y_0)$ & $(P_0 + t\bar{v})$,

$d(P_0, P_0 + t\bar{v})$

$= \|P_0 - (P_0 + t\bar{v})\|$

$= \|t\bar{v}\| = |t| \, \|\bar{v}\|$

$= |t|$

→ $$\lim_{t \to 0} \frac{f(P_0 + t\bar{v}) - f(P_0)}{t},$$

if exists, is called the direction derivate of f at P_0 along $\bar{v}$ ($\|\bar{v}\| = 1$) and is denoted by $D_{\bar{v}}(f)\big|_{P_0}$.

Observations:

① $\frac{\partial}{\partial x}(f)\big|_{P_0} = D_{e_1}(f)\big|_{P_0}$.

② $\phi(t) = f(P_0 + t\bar{v})$

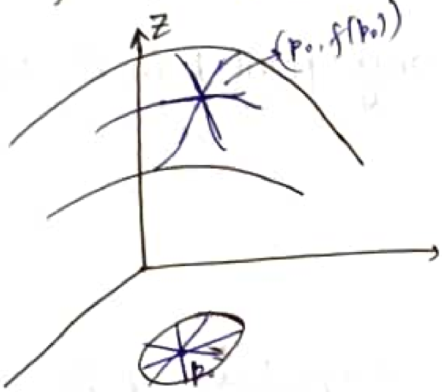
$$\frac{d}{dt}\phi(t)\Big|_{t=0} = \lim_{t\to 0} \frac{\phi(t)-\phi(0)}{t-0}$$

$$= \lim_{t\to 0} \frac{f(P_0+tv)-f(P_0)}{t}$$

$$= D_{\bar{v}}(f)\Big|_{P_0}$$

5-06-2023

→ $f : G \to \mathbb{R}$, $G \subset \mathbb{R}^2$

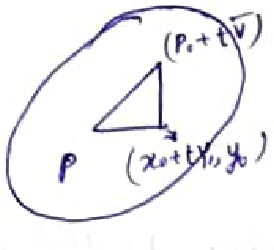


$$\lim_{t\to 0} \frac{f(P_0+t\bar{v})-f(P_0)}{t}$$

$$= \lim_{t\to 0} \frac{f(x_0+tv_1, y_0+tv_2) - f(x_0+tv, y_0)}{tv_2} \times v_2 + \frac{f(x_0+tv_1, y_0) - f(x_0, y_0)}{tv_1} \cdot v_1$$

$$= \frac{\partial}{\partial y}(f)\Big|_{(x_0+t\bar{v}, y_0)} \times v_2$$

$$+ \frac{\partial}{\partial x}(f)\Big|_{(x_0, y_0)} \times v_1$$



$P_0 \equiv (x_0, y_0)$

$P_0 + t\bar{v} = (x_0+tv_1, y_0+tv_2)$

$\bar{v} = (v_1, v_2)$

$\|\bar{v}\| \Leftrightarrow v_1^2 + v_2^2 = 1$

$v_1 \neq 0 \neq v_2$

$$\frac{f(x_0, y_0) + t(1,0)}{t}$$

Conclusions

① Partial derivatives exists in a neighbourhood of P_0.

② One of the partial derivatives is continuous at P_0.

$$\lim_{t\to 0} \phi(x_0+tv_1, y_0) = \phi(x_0, y_0)$$

$$\lim_{p \to p_0} f(p) = f(p_0)$$

$$= \frac{\partial}{\partial y}(f)\Big|_{(x_0, y_0)} \times v_2 + \frac{\partial}{\partial x}(f)\Big|_{(x_0, y_0)} \times v_1$$

$$= \langle \nabla f|_{P_0}, \bar{v} \rangle$$

Theorem:

$P_0 \in G \subset \mathbb{R}^3$,
G is open

[A subset G of $\mathbb{R}^n$ is called **open** if for any point P in it, one can find an open ball around P in G.]

$$f : G \to \mathbb{R}$$

Suppose that

① the partial derivative of f exists in a neighbour of P_0.
② two of the three partial derivatives are continuous at P_0.

Then $\begin{cases} \text{Ⓐ } D_{\bar{v}}(f)|_{P_0} \text{ exists } \forall \bar{v}. \\ \text{Ⓑ } D_{\bar{v}}(f)|_{P_0} = \langle \nabla f|_{P_0}, \bar{v} \rangle. \end{cases}$

Corollary: $f : G \to \mathbb{R}$, $G \subset \mathbb{R}^3$, open.
If f is c^1-type, then
Ⓐ $D_{\bar{v}}(f)|_{P_0}$ exists $\forall \bar{v}$.
Ⓑ $D_{\bar{v}}(f)|_{P_0} = \langle \nabla|_{P_0}, \bar{v} \rangle$.

Defn: A function is called c^1-type if its 1^{st} order partial derivati exist and they are continuous.

Q If $f(x, y, z) = \sin(x^2 + y^2) + e^{xyz}$.
Prove $D_{\bar{v}}(f)$ exists when $\bar{v} = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ & $P_0 = (0, 1, 0)$.

Soln: $\sin(x^2+y^2)$ is c^1-type f^n as it is a composition of two c^1-type functions (x^2+y^2) which is a polynomial function and $\sin(x)$ which is trigonometric function.

e^{xyz} is also c^1-type f^n, being composition of exp. f^n and xyz f^n, both of which are c^1-type.

Since f is c^1-type, by the theorem, $D_{\bar{v}}(f)|_{P_0}$ exists $\forall$ dir. $\bar{v}$ and for any pt. P_0.

and hence being specific,

$D_{\bar{v}}(f)|_{P_0}$ exists when $\bar{v} = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$.

and. $D_{\bar{v}}(f)|_{P_0} = \langle \nabla f|_{P_0}, \bar{v} \rangle$

$$= \frac{1}{\sqrt{3}}\left[\cos(x^2+y^2).2x + yz\,e^{xyz}\right]\Big|_{(0,1,0)} + \frac{1}{\sqrt{3}}\left[\cos(x^2+y^2).2y + xz\,e^{xyz}\right]\Big|_{(0,1,0)} + \frac{1}{\sqrt{3}}\left[xy\,e^{xyz}\right]\Big|_{(0,1,0)}$$

$$= 0 + 0 + 0 = 0$$

$$\Rightarrow D_{\bar{v}}(f)|_{P_0} = 0$$

Q] Find directions $\bar{v}$ for which $D_{\bar{v}}(f)|_{(0,0)}$ exists.

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & \text{otherwise.} \end{cases}$$

Soln: $\lim\limits_{t \to 0} \dfrac{f(P_0 + t\bar{v}) - f(P_0)}{t}$

$$\left[\begin{array}{l} P_0 \equiv (0,0) \\ \bar{v} = (v_1, v_2),\ \|\bar{v}\| = 1 \Leftrightarrow = v_1^2 + v_2^2 = 1 \end{array}\right]$$

$\left.\begin{array}{l} v_1 = 0, v_2 = 1 \\ v_1 = 1, v_2 = 0 \\ v_1 \neq 0, v_2 \neq 0. \end{array}\right\}$ never zero together

$$= \lim_{t \to 0} \frac{f(tv_1, tv_2), f(0,0)}{t}$$

$$= \lim_{t \to 0} \frac{\dfrac{t^2 v_1 v_2}{t^2(v_1^2+v_2^2)} - 0}{t}$$

$$= \lim_{t \to 0} \frac{v_1 v_2}{t}$$

$\xrightarrow{v_1 v_2 = 0}$ exists value is 0

$\xrightarrow{v_1 v_2 \neq 0}$ doesnot exit.

$v_1 v_2 = 0$

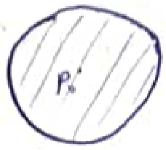
- $v_1 = 0, v_2 = 1 \longrightarrow \bar{v} = e_2 \longrightarrow \frac{\partial}{\partial y}$
- $v_1 = 1, v_2 = 0 \longrightarrow \bar{v} = e_1 \longrightarrow \frac{\partial}{\partial x}$

8-06-2023

→ ~~$P \in G \subset \mathbb{R}^2$ open~~ $P_0 \in G \subset \mathbb{R}^2$ open.

$f : G \to \mathbb{R}$

If f diff. at P_0?

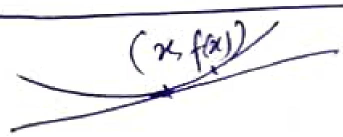


$$\lim_{(h,k)\to(0,0)} \frac{f(x_0+h,\, y_0+h) - f(x_0,y_0) - \alpha h - \beta k}{\sqrt{h^2+k^2}} = 0$$

~~Case:/ can see that~~

→ $\boxed{\begin{array}{l} \text{Existence of tangent plane at } P_0 \\ \text{In general No! } \Uparrow ? \quad \Downarrow ? \to \text{yes} \\ \text{Existence of tangent lines at } P_0 \\ \qquad\qquad ||| \\ \text{Differentiability of the } f^n\ f \text{ at } P_0 \\ \text{No, in general } \Uparrow \quad \Downarrow \to \text{yes} \\ \text{Existence of dir. der. of } f^n \text{ at } P_0 \text{ for all dir}^n \end{array}}$

→



$f(x) = f(a) + (x-a) f_1(x)$

$\lim_{x\to a} f(x) = f(a)$

$\Leftrightarrow \lim_{x\to a} (f(x) - f(a)) = 0$

$\Rightarrow$ when $x \approx a$, then $f(x) \approx f(a)$.

$f(x) = f(a) + (x-a) f_1(x)$ $\qquad\qquad | f'(a) = \alpha = f_1(a)$

when $x \approx a$

$f(x) \approx f(a) + (x-a) f_1(a)$

$= f(x) + (x-a)\alpha$

As if $f(x) = f(a) + (x-a)\alpha$.

9-06-2023

Q $f(x,y) = \begin{cases} \dfrac{x^2y}{x^4+y^2}, & (x,y)\neq(0,0) \\ 0, & \text{otherwise} \end{cases}$

Show that f is not differentiable at $(0,0)$.

Soln: $\lim_{t\to 0} \dfrac{f(P_0+t\bar{v}) - f(P_0)}{t}$, $\quad P_0=(0,0)$, $\bar{v}=(v_1,v_2)$

$= \lim_{t\to 0} \dfrac{f(t\bar{v}_1, t\bar{v}_2) - f(0,0)}{t}$, $\quad |\bar{v}|=1,\ v_1^2+v_2^2=1$

$= \lim_{t\to 0} \dfrac{\dfrac{\cancel{t^2} v_1^2 \cancel{t} v_2}{\cancel{t^2}(t^2v_1^4+v_2^2)}}{\cancel{t}} = 0$

$= \lim_{t\to 0} \dfrac{v_1^2 v_2}{t^2v_1^2+v_2^2}$

$= \begin{cases} 0, & \bar{v}=(1,0)\to e_1 \rightsquigarrow \dfrac{\partial f}{\partial x} \\ 0, & \bar{v}=(0,1)\to e_2 \rightsquigarrow \dfrac{\partial f}{\partial y} \\ \dfrac{v_1^2}{v_2}, & \bar{v}=(v_1,v_2),\ v_1\neq 0,\ v_2\neq 0 \to \text{arbitrary dirn} \end{cases}$

$D_v f|_{(0,0)}$ exists for all direction $\bar{v}$.

$\langle \nabla f|_{(0,0)}, \bar{v} \rangle = 0 \neq \dfrac{v_1^2}{v_2} = D_{\bar{v}}(f)|_{P_0}$ when $v_1\neq 0, v_2\neq 0$

$\langle \dfrac{\partial}{\partial x} f|_{(0,0)}, \dfrac{\partial}{\partial y} f|_{(0,0)}, \bar{v} \rangle$

$= \langle (0,0) \bar{v} \rangle$

$\Rightarrow f$ is not differentiable.

Q $f(x,y,z) = z^2 + \sin(e^{x^2+y^2})$. Is it diff. at $(0,0,0)$?

Soln:

If f is C_1-type, then f is differentiable.

z^2, x^2+y^2, exponent, sin are C_1-type.

$\therefore$ Composition is C_1 type $\Rightarrow f$ is C_1 type.

$\Rightarrow f$ is diff.

Remark:

$$A: \begin{cases} A_1: \text{ all the partial derivatives of } f \text{ exist in open ball around } P_0. \\ A_2: \text{ two of 3 partial derivatives of } f \text{ are continuous at } P_0. \end{cases}$$

$$\Downarrow$$

$$B: \{ f \text{ is differentiable at } P_0 \}$$

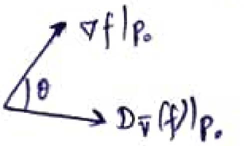
$$\Downarrow$$

$$C: \begin{cases} C_1: D_v(f)|_{P_0} \text{ exist for all dir}^n \\ C_2: D_v(f)|_{P_0} = \langle \nabla f|_{P_0}, \bar{v} \rangle \end{cases}$$

→ Suppose f is diff / C_1-type.

(i) $D_v(f)|_{P_0}$ exists $\forall\, \bar{v}$.

(ii) $D_v(f)|_{P_0} = \langle \nabla f|_{P_0}, \bar{v} \rangle$



$$= \|\nabla f|_{P_0}\| \cdot \|\cancel{\bar{v}}\|^{1} \cos\theta$$

$$= \|\nabla f|_{P_0}\| \cos\theta$$

$$\min\left[D_{\bar{v}}(f)|_{P_0} \right] = 0, \quad \theta = \pi/2$$

$$\max\left[D_{\bar{v}}(f)|_{P_0} \right] = \|\nabla f|_{P_0}\|, \quad \theta = 0, \pi \ ; \quad \bar{v} = \pm \nabla f|_{P_0}$$

→ **Corollary:**

If (A) $\begin{cases} (A_1)\ D_{\bar{v}}(f)|_{P_0} \text{ exists } \forall\, \bar{v} \\ (A_2)\ D_{\bar{v}}(f)|_{P_0} = \langle \nabla f|_{P_0}, \bar{v} \rangle \end{cases}$

does not hold, then

f is not differentiable at P_0.

$B \Rightarrow A \; (= A_1 \,\&\, A_2)$

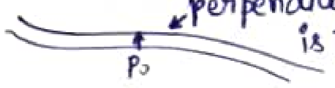
$\sim B \Leftarrow \sim A \; (= \sim A_1 \,\&\, A_2)$

$(= \sim A_1 \sim\!\& \sim A_2)$

$(= \sim A_1 \text{ or } \sim A_2)$

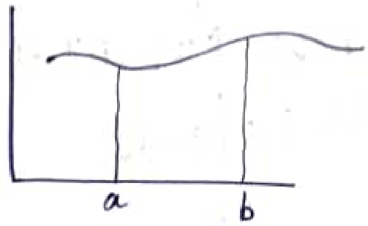
ARC LENGTH FUNCTION

12-06-2023

→ 

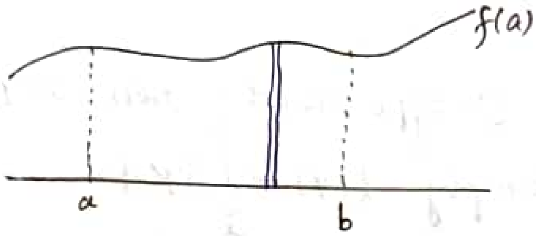
$\nabla f|_{P_0}$ → gives perpendicular dirⁿ

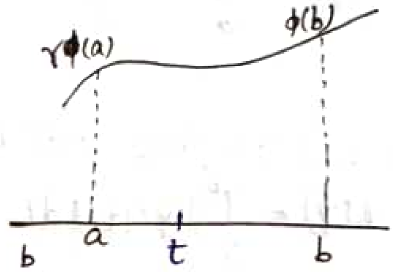
→ $\int_a^b f(x)dx$



weighted length ~~of~~ in the fⁿ of interval $[a,b]$

→ Parametric equations give ~~disto~~ directed / oriented curve.





Weighted length of interval $[a,b] = \int_a f(x)dx$

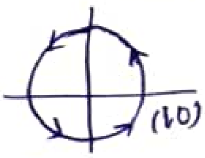
$\gamma\ \phi\ [a,b] \xrightarrow{\text{continuous}} \mathbb{R}^2$ (or $\mathbb{R}^3$)

We shall call γ to be parametric curve.

eg. $\gamma_1(t) = \cos t, \sin t$, $t = (0, 2\pi]$

$\gamma_2(t) = (\cos 2\pi s, \sin 2\pi s)$, $s \in (0,1)$

$\gamma_3(t) = (t, 0)$, $t \in (a,b)$



oriented / directed curve

$$l_n = \sum_{i=1}^{n} \| \gamma(t_i) - \gamma(t_{i-1}) \|$$

If $\lim_{n\to\infty} l_n = l$ exists, l → length of γ

$$\lim_{\substack{\Delta t \to 0 \\ n\to\infty}} \sum_{t=1}^{\infty} \gamma(t_i)\Delta t$$

Q) (i) $\gamma(t) = \cos t, \sin t$, $t \in [0, 2\pi]$.

(ii) $\phi(t) = (x, 0)$, $x \in [a, b]$.

Sol^n: γ is C_1-f^n of sin & cos

ϕ is C_1-f^n.

$\Rightarrow l(\gamma)$ exists & $l(\phi) = \int_0^{2\pi} \| \gamma'(t) \| dt$

$\gamma'(t) = (\sin t, \cos t)$, $t \in [0, 2\pi]$.

$\| \gamma'(t) \| = 1$

$l(\gamma) = \int_a^b \| \gamma'(t) \| dt = \int_0^{l_n(\gamma)}$

$\int_a^b f(x)\, dx = \int_a^b dx = \int_0^{b-a} dx$

<u>Theorem</u>: Let $\gamma : [a,b] \to \mathbb{R}^3$ be a C'-type curve. Then, $l(\gamma)$ exists, and $l(\gamma) = \int_a^b \|\gamma'(t)\| dt$ or simply $l(\gamma) = \int_a^b \|\gamma'\|$.

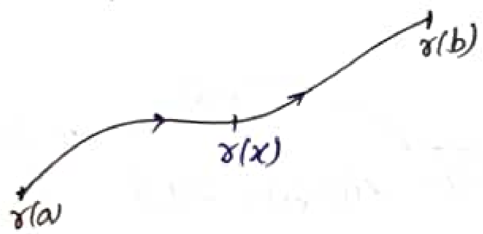
Length of Curves

ARC LENGTH FUNCTION

[15-06-23]

$\gamma : [a,b] \xrightarrow{c^1\text{-type}} \mathbb{R}^2$ (or $\mathbb{R}^3$)

$$\underset{\text{exists}}{l(\gamma)} = \int_a^b \|\gamma'(t)\|\,dt = \int_0^{l(\gamma)} (\ ?\ )\,d(?)$$



$$l([a,b]) = b-a = \int_a^b 1.dx = \int_0^{b-a} 1.dx$$

$$s(x) = \int_a^x \|\gamma'(t)\|\,dt.$$

$\rightarrow$ $\gamma(t) = (t,0)$, $t \in [a,b]$

is c^1-type

$\gamma'(t) = (1,0)$.

$\Rightarrow$ $l(\gamma)$ exists and

$$l(\gamma) = \int_a^b \|\gamma'(t)\|\,dt$$

$$= b-a$$

$$s(x) = \int_a^x \|\gamma'(t)\|\,dt = x-a$$

- Aim is to study $s(x)$ now.

① $\text{dom}(s) = \text{dom}(\gamma)$ ↘ domain

② s is increasing

<u>Proof:</u> $x_1 < x_2$ (say)

$s(x_1) \leq s(x_2)$

$\Rightarrow s(x_2) - s(x_1) \geq 0$

$\Rightarrow \int_a^{x_2} \|\gamma'(t)\|dt - \int_a^{x_1} \|\gamma'(t)\|dt \geq 0$

$\Rightarrow \int_{x_1}^{x_2} \underbrace{\|\gamma'(t)\|}_{\geq 0} dt \geq 0$

<u>Fundamental Theorem of Integral Calculus</u>

$f : [a,b] \xrightarrow[\text{cont}]{} \mathbb{R}$

then f has an anti-derivativ[e] and is given by

$$F(x) = \int_a^x f(t)\,dt$$

i.e., $\frac{d}{dx}F(x) = f(x)$

<u>Corollary:</u> $\int_a^b f(t)\,dt = F(b) - F(a)$

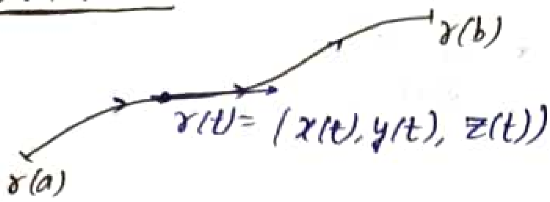
③ γ is c^1-type $\Rightarrow$ γ' is continuous $\Rightarrow$ $\|\gamma'\|$ is continuous

$\Rightarrow$ s is diff $\Rightarrow$ s is continuous.

$\Rightarrow$ $\boxed{s \text{ is } c^1\text{-type}}$.

$\frac{d}{dx} s(x) = \|\gamma'(x)\|$ ← continuous.

Smoothness :



γ is c^1-type. (say)

$$\gamma'(t) = (x'(t), y'(t), z'(t))$$

Tangent vector at 't',

$$T(t) = \gamma'(t)$$

If $\gamma'(t) \neq 0 \ \forall \ t$, then we call γ

Smoothness $\Leftrightarrow$ c'-type + non-zero tangent

$$\mathbb{T}(t) = \frac{T(t)}{\|T(t)\|}$$ is called unit tangent at the point 't'.

c'-type $\Rightarrow$ Tangent is continuous.

Smoothness $\Rightarrow$ non-zero continuous tangent at any pt.

$$s(x) = \int_a^x \|\gamma'(t)\| dt$$

γ is smooth $\Rightarrow$ ① $s \uparrow$ strictly
② s is smooth.

→ $l(\gamma) = \int_a^b \|\gamma'(t)\| dt$

$$s(x) = \int_a^x \|\gamma'(t)\| dt$$

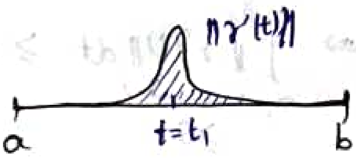
$$\boxed{\frac{d}{dx} s(x) = \|\gamma'(x)\|}$$

and if $ds = \|\gamma'(t)\| dt$

$$l(\gamma) = \int_0^{l(\gamma)} 1 . ds$$

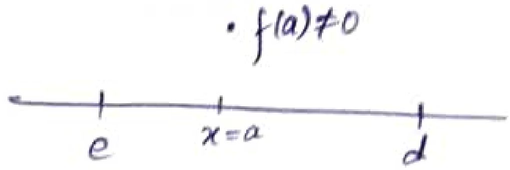
$s(a) = 0$

$s(b) = l(\gamma)$.



→ If a continuous fn takes non-zero value at a point, then it takes non-zero values ~~at~~ in an interval around that point.

#



If there is no interval

then $f \equiv 0$ on $[e,d] \setminus \{a\}$

~~reads~~ reads 'minus'.

$$\lim_{x\to a} \underset{\overset{\shortparallel}{0}}{f(x)} = \underset{\overset{\shortparallel}{0}}{f(a)}$$

$$\lim_{\substack{x\to a\\ x\neq a}} f(x) = \lim_{x\to a} 0$$
$$= 0$$
$$\neq f(a)$$

This contradicts the assumption of $f(x)$ is continuous.

<u>Theorem:</u> Let $s:[a,b]\to\mathbb{R}$ be the arc length function of c^1-type curve $\gamma:[a,b]\to\mathbb{R}^3$. Then, the following holds

(i) $s(x)\geq 0 \ \forall\ x\in[a,b]$, $s(a)=0$, and $s(b)=l(\gamma)$.

(ii) s is non-decreasing

(iii) $s\not\equiv 0$ iff $\gamma'(t)\neq\bar{0}$ for some $t\in[a,b]$, i.e., <u>$s\equiv 0$ iff $\gamma'=\bar{0}$ on $[a,b]$</u>.

(iv) If $\gamma'(t)\neq\bar{0}\ \forall\ t\in[a,b]$ (i.e., γ is smooth), then $s(x)=0$ iff $x=a$.

(v) If $\gamma'(t)\neq\bar{0}\ \forall\ t\in[a,b]$ (i.e., γ is smooth), then s is strictly increasing on $[a,b]$.

(vi) s is continuous on $[a,b]$.

(vii) s is differentiable on $[a,b]$.

(viii) s is C^1-type on $[a,b]$, and $\frac{d}{dx}s(x)=\|\gamma'(x)\|\ \forall\ x\in[a,b]$.

(ix) s is smooth if γ is smooth.

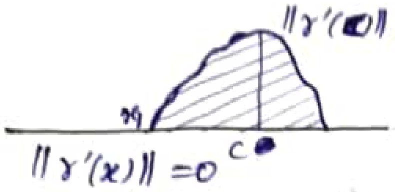
16-06-2023

Intermediate Value Theorem

$f:[a,b] \xrightarrow{\text{cont}} \mathbb{R}$

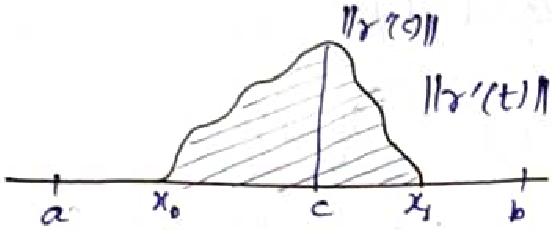
$x_1 < x_2 \in [a,b]$

Then f attains all the values between $f(x_1)$ and $f(x_2)$ in $[x_1, x_2]$.

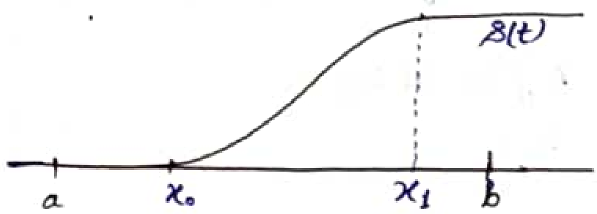


- $s \neq 0$, if $\exists\, t \in [a,b]$ such that $r'(t) \neq \bar{0}$.

$\left(\begin{array}{l} \text{i.e., } \exists\, c \in [a,b] \\ \text{s.t. } s(c) \neq 0 \end{array}\right)$



$$s(c) = \int_a^c \|r'(t)\|\,dt = \int_a^{x_0} \cancel{\|r'(t)\|}^{\,0}\,dt + \underbrace{\int_{x_0}^{c} \|r'(t)\|\,dt}_{>0}$$



Q] $c: x^2 = y^3,\ 9z^2 = 4y$. Initial point is $(0,0,0)$ & final pt. is $(8, 9, 4/3)$.

Parametrize the curve.

Check if $l(c)$ exists. Find arc-length f^n of c, if exists, and hence find $l(c)$, if possible.

Soln: $r(t) = (x(t), y(t), z(t))$ $\qquad t \in \left(0, \frac{4}{3}\right)$

$z^2 = \frac{4}{9} y$ & $x^2 = y^3$

$z = t,\ y = \frac{9}{4} t^2,\ x = \sqrt{\left(\frac{9}{4}\right)^3 t^6} = \pm \frac{27}{8} t^3$

Initial point is $(0,0,0)$ & final pt. is $(8, 9, 4/3)$.

$\gamma(t) = \left(\frac{27}{8}t^3, \frac{9}{4}t^2, t\right)$, $t \in [0, 4/3]$

$t = 0 \longrightarrow \gamma(0) = (0,0,0)$

$t = 4/3 \longrightarrow \gamma(4/3) = (8, 9, 4/3)$

$\gamma'(t) \neq \bar{0}$

$\ell(c)$ exists if c is c^1-type.

Since 'c' is given by $\gamma(t)$ and the component f^ns of γ are c^1-type, γ is c^1-type and hence $\ell(\gamma)$ exists and

$$\ell(\gamma) = \int_0^{4/3} \|\gamma(t)\| \, dt$$

$$\gamma'(t) = \left(\frac{3\times 27}{8}t^2, \frac{9\times 2}{4}t, 1\right)$$

$$\|\gamma'(t)\| = \sqrt{\left(\frac{3\times 27}{8}t^2\right)^2 + \left(\frac{9\times 2}{4}t\right)^2 + 1}$$

Since γ is c'-type, the arc lengths f^n

$s : [0, 4/3] \to \mathbb{R}$ exists and is given by

$$s(x) = \int_0^x \|\gamma'(t)\| \, dt, \quad x \in [0, 4/3] \quad = \ldots$$

$$\underset{\ell(c)}{\underset{\shortparallel}{\ell(\gamma)}} = s(4/3)$$

Opposite Orientation

$\rightarrow \gamma : [a,b] \xrightarrow{cont} \mathbb{R}^2$

$\gamma(a) \rightarrow \gamma(b) \rightarrow \gamma(c)$

$\delta(t) = \gamma(a+b-t)$, $t \in [a,b]$

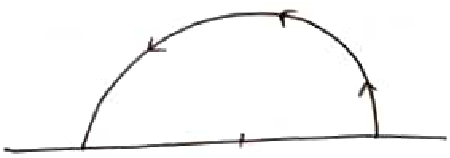
↳ will be opposite oriented.

$\phi(t) = (a+b-t)$, $t \in [a,b]$

$(-\gamma)$ or $\gamma^{-1} = \gamma(a+b-t)$, $t \in [a,b]$

is called inverse of γ.

Q) $\gamma(t) = (\cos t, \sin t)$, $t \in [0,\pi]$



Find γ^{-1}.

Soln: $\gamma^{-1}(t) = \gamma(0+\pi-t)$

$= (\cos(\pi-t), \sin(\pi-t))$, $t \in [0,\pi]$.

$= (-\cos t, \sin t)$.

- If γ is c^1-type, γ^{-1} is c^1-type

$\Rightarrow l(\gamma^{-1})$ exists.

$$l(\gamma^{-1}) = \int_a^b \|\gamma^{-1}(t)\| dt$$

Aim: To show $l(\gamma) = l(\gamma^{-1})$.

$\delta(t) = \gamma^{-1}(t) \quad \forall t \in [a,b]$

$= \gamma(a+b-t) \quad \forall t \in [a,b]$

$$l(\gamma^{-1}) = l(\delta) = \int_a^b \|\delta'(t)\| dt$$

$\delta'(t)$

$\|$

$$\frac{d}{dt}\delta(t) = \frac{d}{dt}(\gamma(a+b-t))$$

$$= \frac{d}{ds}\gamma(s) \cdot \frac{ds}{dt} = \gamma'(s) \cdot x - 1 \quad , \quad s = a+b-t$$

$$l(\delta) = \int_a^b \|-\gamma'(s\| \, ds = \int_a^b \|\gamma'(s)\| ds$$

$$= l(\gamma)$$

~~$\gamma(t) = (\cos t, \sin t) \,|\, t \in [0, 2\pi]$,~~

$$\int_a^b \|-\gamma'(s)\| \, dt$$

$s = a+b-t$

$\|$ as if $ds = -dt$

$a \xrightarrow{s} b$

as if $b \xrightarrow{s} a$

$$\int_a^b \|-\gamma'(s)\| \times -ds$$

$$= \int_a^b \|\gamma'(s)\| ds.$$

(19-09-23)

LINE INTEGRATION

→ $f : G \xrightarrow{\text{Cont.}} \mathbb{R}$

$\gamma : [a,b] \xrightarrow{C^1\text{-type}} \mathbb{R}^3$

$\{\gamma\} \subset G$ (to talk about $f(\gamma(t))$)

{Points of γ in G}

$$l(\gamma) = \int_a^b \|\gamma'(t)\| dt = \int_a^b 1 \cdot dx$$

We can talk about line integration of f along the length of γ (i.e. weighted length of γ & the weight is by the f^n f)

denoted by $\int_\gamma f$ and,

given by

$$\int_\gamma f = \int_a^b f(\gamma(t))\, ds$$

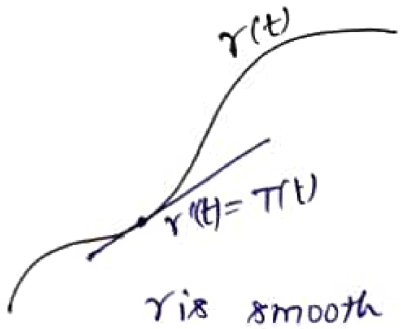
$$= \int_a^b \underbrace{f(\gamma(t)) \|\gamma'(t)\|}_{\text{Cont.}} dt$$

as if $ds = \|\gamma'(t)\| dt$

$$\frac{d}{dx} s(x) = \|\gamma'(x)\|$$

→ Field → $F : G \longrightarrow \mathbb{R}^2$ (or $\mathbb{R}^3$)

$$\int_a^b \frac{d}{dt}(g(t))\, dt = g(b) - g(a)$$



γ is smooth

$$\mathbb{T}(t) = \frac{T(t)}{\|T(t)\|}$$

$\int_\gamma \langle \vec{F}, \vec{v} \rangle$; $\vec{F} = \nabla f$

$$\int_\gamma \langle \vec{F}, \vec{\mathbb{T}}_\gamma(t) \rangle$$

$$= \int_a^b \left\langle \vec{F}(\gamma(t)), \frac{\gamma'(t)}{\|\gamma'(t)\|} \right\rangle \|\gamma'(t)\|\, dt$$

$$= \int_a^b \langle \vec{F}(\gamma(t)), \gamma'(t) \rangle\, dt$$

$\langle \nabla f(\gamma(t)), \gamma'(t) \rangle$

$= \langle (\frac{\partial}{\partial x}(f(\gamma(t)), \frac{\partial}{\partial y}(f)(\gamma(t)), (x'(t), y'(t)) \rangle$

$= \frac{\partial}{\partial t} f(\gamma(t)) \cdot x'(t) + \frac{d}{dy} f(\gamma(t)) \cdot y'(t)$

$= \frac{d}{dt} f(\gamma(t))$

$\rightarrow G \subset \mathbb{R}^3$

$\bar{F} : G \xrightarrow{\text{cont.}} \mathbb{R}^3$ (or $\mathbb{R}^3$)

$\gamma : [a,b] \xrightarrow{\text{smooth}} \mathbb{R}^3$ s.t. $\{\gamma\} \subset G$

Then, the line integration of $\bar{F}$ along γ is denoted by

$\int_\gamma \bar{F}$ (or $\int_\gamma \bar{F} \cdot d\bar{s}$) and is given by

$$\int_\gamma F = \int_a^b \langle \bar{F}(\gamma(t)), T(t) \rangle \, ds$$

$$= \int_a^b \langle \bar{F}(\gamma(t)), \frac{\gamma'(t)}{\|\gamma'(t)\|} \rangle \|\gamma'(t)\| \, dt$$

$$= \int_a^b \langle \bar{F}(\gamma(t)), \gamma'(t) \rangle \, dt$$

Theorem (Fundamental Theorem of Line Integral)

$G \subset \mathbb{R}^3$

$\bar{F} : G \xrightarrow{\text{continuous}} \mathbb{R}^2 \ (\text{or } \mathbb{R}^3)$

$\gamma : [a,b] \xrightarrow{\text{smooth}} \mathbb{R}^3$

If $\bar{F} = \nabla f$ on G, then

$$\int_\gamma \bar{F} = f(\gamma(b)) - f(\gamma(a))$$

Defn: A vector field $\bar{F}$ is called a conservative vector field (CVF) if $\bar{F} = \nabla f$ on $\text{dom}(\bar{F})$.

Suppose $\text{dom}(F) = G$, then $f : G \to \mathbb{R}$ s.t. $\nabla f(p) = F(p)$, $p \in G$.

Corollary: If F is a C.V.F., then

① $\int_{\gamma_1} \bar{F} = \int_{\gamma_2} \bar{F}$ if γ_1 & γ_2 are true path in its same initial & final pts.

② $\int_\gamma F = 0 \quad \forall$ loop γ.

(Assumption: when talking about line int. of V.F., the path/curve is always smooth or piece-wise smooth.)

Q) Show that:

① $\int_\gamma \bar{F} = -\int_{\gamma^{-1}} \bar{F}$

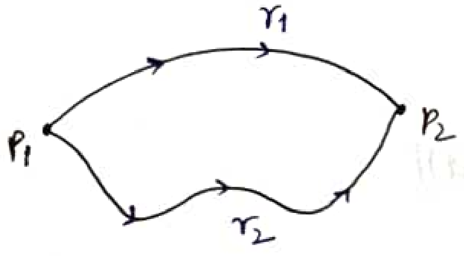
② $\int_\gamma f = \int_{\gamma^{-1}} f$

Joining Two Curves

γ_1, γ_2 are parametric curves s.t. end p.t. of γ_1 = beginning pt. of γ_2 then we can make a new curve by joining γ_1 & γ_2 s.t. the orientation is respected; the new curve is denoted by γ_1 and γ_2.

Defn: $\int\limits_{\gamma_1 * \gamma_2} f = \int\limits_{\gamma_1} f_1 + \int\limits_{\gamma_2} f$

$$\int\limits_{\gamma_1 * \gamma_2} \bar{F} = \int\limits_{\gamma_1} \bar{F} + \int\limits_{\gamma_2} \bar{F}$$



$\gamma_1 * \gamma_2$ ✗

$\gamma_1 * \gamma_2^{-1}$ ✓

$$\int\limits_{\gamma_1 * \gamma_2^{-1}} \bar{F} = 0$$

$$\Rightarrow \int\limits_{\gamma_1} \bar{F} + \int\limits_{\gamma_2^{-1}} \bar{F} = 0$$

$$\Rightarrow \int\limits_{\gamma_1} \bar{F} - \int\limits_{\gamma_2} \bar{F} = 0 \quad \text{(By property)}$$

$$\Rightarrow \int\limits_{\gamma_1} \bar{F} = \int\limits_{\gamma_2} \bar{F}$$

22-06-2023

Corollary:

If $\vec{F}$ is $\overbrace{\text{C.V.F}}^{C}$, then

(A) $\int_{\gamma_1} \bar{F} = \int_{\gamma_2} \bar{F}$ for any two paths γ_1 and γ_2 with same initial and final points.

(B) $\int_{\gamma} \bar{F} = 0 \ \forall$ loops γ.

Theorem:

If $\bar{F}$ is a continuous V.F., then the following are equivalent:

(A) $\int_{\gamma_1} \bar{F} = \int_{\gamma_2} \bar{F}$ for path γ_1 & γ_2 of same initial & final pts.

(B) $\int_{\gamma} \bar{F} = 0 \ \forall$ loops γ.

Proof: $B \Leftrightarrow$ (A)

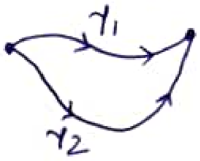
$\Uparrow$ C $\nearrow$ $\rightsquigarrow$ F is conservative vector field

Assume (B) holds.

Aim: To show (A) holds,

i.e, to show $\int_{\gamma_1} \bar{F} = \int_{\gamma_2} \bar{F}$ where ...

Let γ_1 & γ_2 be two paths. ~~then γ is a loop~~



Define $\gamma = \gamma_1 * \gamma_2$,

then γ is a loop.

$\int \bar{F} = 0$ by hypothesis (B)

$$\int_{\gamma_1 * \gamma_2'} \bar{F} = \int_{\gamma_1} \bar{F} + \int_{\gamma_2'} \bar{F} = \int_{\gamma_1} \bar{F} - \int_{\gamma_2} \bar{F}$$

$$\Rightarrow \int_{\gamma_1} \bar{F} = \int_{\gamma_2} \bar{F}$$

Now, aim: To prove Ⓐ $\Rightarrow$ Ⓑ.

Assume Ⓐ holds to show Ⓑ holds

Let γ be a loop.

$\int_{\gamma} \bar{F}$

Let $\gamma_1 = P_1 \xrightarrow{\gamma} P_2$

$\gamma_2 = P_1 \xrightarrow{\gamma^{-1}} P_2$

Note that $\gamma_1 = \gamma_1 \star \gamma_2^{-1}$

By hyp., $\int_{\gamma_1} \bar{F} = \int_{\gamma_2} \bar{F}$

$\Rightarrow \int_{\gamma_1} \bar{F} - \int_{\gamma_2} F = 0$

$\Rightarrow \int_{\gamma_1} \bar{F} + \int_{\gamma_2} \bar{F} = 0$

$\Rightarrow \int_{\gamma_1 \star \gamma^{-1}} \bar{F} = 0 \Rightarrow \int_{\gamma} F = 0$

$\therefore$ Ⓐ $\Leftrightarrow$ Ⓑ

$\nwarrow$ Ⓒ $\nearrow$

First Structural Theorem of CVF:

Let $G \subset \mathbb{R}^3$ (or $\mathbb{R}^2$) open & $\bar{F}: G \to \mathbb{R}^3$ (or $\mathbb{R}^2$) a continuous V.F. Suppose that G is path connected, then the following are equivalent.

(A) ...

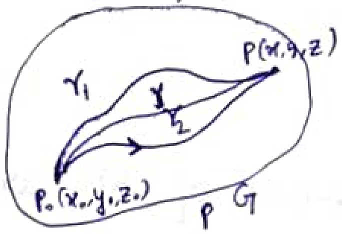
(B) ...

(C) $\bar{F}$ is a C.V.F., i.e., $\exists\ f: G \to \mathbb{R}$ such that $\nabla f = \bar{F}$ on G.

Proof: (A) $\Leftrightarrow$ (B) already proved
no need of condition that G is path connected.

(C) $\Rightarrow$ (B)
$\Downarrow$ (A) already done.

For (A) $\Rightarrow$ (C), assume (A) holds.



→ for a given
$g: [a,b] \xrightarrow{\text{cont.}} \mathbb{R}$
$h(x) = \int_a^x g(t)\,dt$

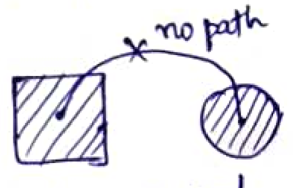
} Fundamental theorem of calculus

$$f(x,y,z) = \int_{P_0,\ \gamma}^{P} F$$

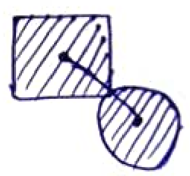
Since G is path connected, there exists a path from P_0 to P and take one of them (say γ).

Fact: $\nabla f = \bar{F}$ on G.

$G \subset \mathbb{R}^3$ (or $\mathbb{R}^2$) is called path connected if for any two points P_1 and P_2 in G, there exists a path /curve joining P_1 & P_2 and lying on G.



Not connected spaces



connected

$A \Leftrightarrow B \Leftrightarrow C$

$\sim A \Leftrightarrow \sim B \Leftrightarrow \sim C$

Q Let $F(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right) \quad \forall (x,y) \in \mathbb{R}^2 - \{0,0\}$.

Check conservativeness.

Soln: F is rational function $\Rightarrow$ continuous $\forall (x,y) \in \mathbb{R}^2 - \{0,0\}$.

$\gamma(t) = (r\cos t, r\sin t)$, $t \in [0, 2\pi]$

(claim: $\int_\gamma F = 2\pi \neq 0$)

$\gamma'(t) = (-r\sin t, r\cos t) \quad \forall t \in [0, 2\pi) \neq (0,0)$

$\Rightarrow \gamma'$ is continuous fn.

Hence. $\int_\gamma F$ makes sense.

$\int_0^{2\pi} \langle F(\gamma(t)), \gamma'(t) \rangle \, dt$ (F is cont. & γ is smooth.

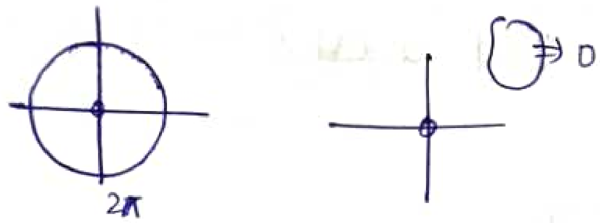
$= \int_0^{2\pi} \left\langle \left(-\frac{r\sin t}{r^2}, \frac{r\cos t}{r^2} \right), (-r\sin t, r\cos t) \right\rangle dt$

$= \int_0^{2\pi} 1 \cdot dt = 2\pi \neq 0.$

& Hence from the 1st structural theorem of CVF, we see that F is not CVF in G.

23-06-23

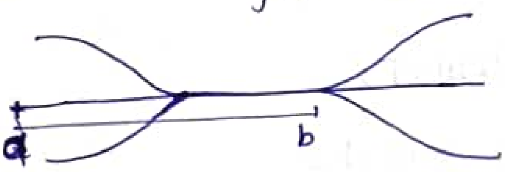
→ #

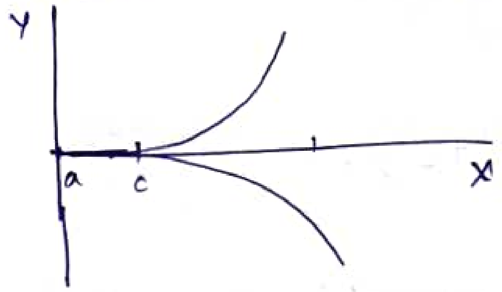


$\gamma(t) = (x(t), y(t))$

$\gamma'(t) = (x'(t), y'(t))$

$= T(t)$ $\quad\quad \int \|T(t)\|$





→ Check if $\bar{F} = \nabla f$ on dom(F).

Assume $\exists$ F such that $\bar{F} = \nabla f$ on dom(F) be defined by

$$\bar{F}(x,y,z) = \nabla f(x,y,z) \quad . \quad \text{Find } f.$$

Suppose $\bar{F} : G \to \mathbb{R}^3$

$$\bar{F} : (F_1, F_2, F_3) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$\frac{\partial f}{\partial x} = F_1 \Rightarrow f = \int F_1 \, dx + c_1(y,z)$$

$$F_2 = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\int F_1 \, dx\right) + \frac{\partial}{\partial y} c_1$$

$$\Rightarrow \frac{\partial}{\partial y} c_1 = F_2 - \frac{\partial}{\partial y}\left(\int F_1 \, dx\right)$$

$$\Rightarrow c_1 = \int \left(F_2 - \frac{\partial}{\partial y}\left(\int F_1 \, dx\right)\right) \partial y + c_2(z)$$

Q) $\bar{F}(x,y,z) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}, e^z \right)$. Is F a CVF?

Soln: Assume that $\bar{F}$ is CVF.

We shall try to find f such that

$\nabla f = F$ on $dom(F) = R^3 \setminus \underbrace{z\text{-axis}}$

$f = \frac{1}{2}\underbrace{\ln(x^2+y^2)}_{\text{diff. wrt. } x} + e^z \quad \left(R^3 - \{z\text{-axis}\} \right)$

$dom(f) = R^3 \setminus z\text{-axis} = dom(\bar{F})$

We see that $\nabla f = \bar{F}$ on $dom(\bar{F})$.

$\Rightarrow \bar{F}$ is CVF.

Q) $\bar{F}(x,y,z) = (\sin(z)e^x, y^2, \cos(z)e^x + z^3)$.

Check if CVF.

Soln: Assume that $\exists f$, $\nabla f = \bar{F}$ on $dom(F)$.

Shall

$f = \sin(z)e^x + \frac{y^3}{3} + \frac{z^4}{9} + c$

$\frac{\partial}{\partial y} c_1 = y^2$, $c_1 = y^2 dy + c_2(z)$

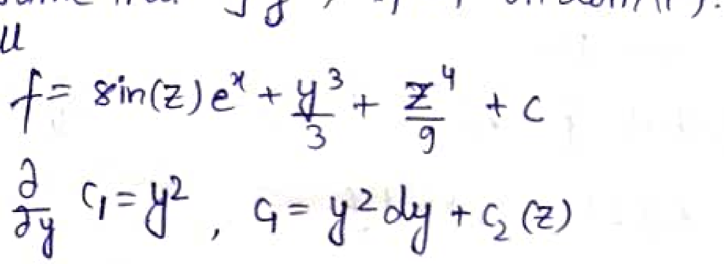
$\frac{\partial}{\partial z} c_2(x) = F_3 - \frac{\partial}{\partial x} f$

$= z^3$

$\therefore$ CVF

Q) $\bar{F}(x,y) = (\sqrt{\sin x}, y)$. check if CVF.

Soln: dom(F):



$\int \sqrt{\sin x}\, dx$

$[0,\pi] \times R$

$f = \int_0^x \sqrt{\sin t}\, dt + \frac{y^2}{2}$ in $[0,\pi] \times R$

$\nabla f_{\pi} = \bar{F}$ in $[0, \pi]$

$\nabla f_{2\pi} = \bar{F}$ in $[0, 2\pi]$

Q. $F(x,y) = \left(\dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \right)$, $G \subset R^2 \setminus \{0,0\}$.

Soln: $\text{dom}(F) = R^2 \setminus \{0,0\}$

Try finding f s.t.

$$\nabla f = \bar{F} \text{ in } \text{dom}(\bar{F})$$

$$f_1(x,y) = -\tan^{-1}\left(\frac{x}{y}\right)$$

$$f_2(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$$

$\nabla f_1 = \bar{F}$ on $\mathbb{R} \setminus x$-axis $\neq \text{dom}(F)$

$\nabla f_2 = \bar{F}$ on $\mathbb{R} \setminus y$-axis $\neq \text{dom}(F)$

Q. $F(x,y) = \left(\dfrac{x}{x^2+y^2}, \dfrac{y}{x^2+y^2}, e^z \right)$.

Soln: $C_1 : x^2+y^2 = 1$, oriented ⊕vely.

$C_2 : (x-5)^2 + (y-5)^2 = 25$, oriented ⊕vely.

$C_3 :$ $(2,5,3)$ — $(1,3,0)$

$$\int_{C_1} F \pm \int_{C_2} F = \int_{C_3} F = f(1,3,5) - f(2,5,3)$$

→ Suppose F is C.V.F.

$$(F_1, F_2) = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

$$F_1 = \frac{\partial}{\partial x} f, \quad \frac{\partial f}{\partial y} = F_2.$$

$$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f\right) \; ; \quad \frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f\right)$$

if F is c^1-type.

curl of F is zero,
for F is conservative.

Q. $\bar{F}(x,y) = (xy, y^2)$. Is $\bar{F}$ a CVF?

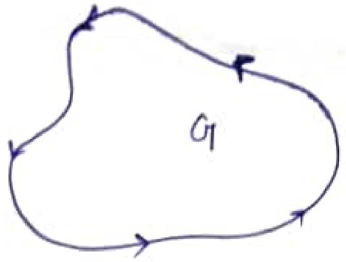
Soln: $F_1 = \frac{\partial}{\partial x} f, \quad F_2 = \frac{\partial}{\partial y} f$

$F_1 = y$

$F_2 = 2y$

$\frac{\partial F_1}{\partial y} \neq \frac{\partial}{\partial x} F_2$ & F is c^1-type

$\therefore$ F is not CVF.

→ 

G is simply x-y connected.

$\bar{F} : G \to \mathbb{R}^2$

$\bar{F} : (F_1, F_2)$

$F = c^1$-type

G has boundary ∂G oriented (anti-clockwise).

$$\iint_G \left(\frac{\partial}{\partial x} F_2 - \frac{\partial}{\partial y} F_1\right) = \int_{\partial G} F$$

$$\int_{\partial G} F = \int_{\partial G} \langle (F_1(\gamma(t)), F_2(\gamma(t))), (x'(t), y'(t)) \rangle \, dt$$

$$= \int_{\partial G} (F_1 \, dx + F_2 \, dy)$$

Q) $\int_C (2x^3 - y^3) dx + (x^3 + y^3) dy$.

Sol^n: 

$C : x^2 + y^2 = 1$ (Integrate using Green's theorem)

$\partial G = \gamma(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$

$$\int_{\partial G} (2x^3 - y^3) dx + (x^3 + y^3) dy$$

$$\int_0^{2\pi} \langle F(\gamma(t)), \gamma'(t) \rangle dt$$

$F(x, y) = (2x^3 - y^3, x^3 - y^3)$

G is simply connected, $\bar{F}$ is C_1-type, hence we can apply Green's theorem:

$$\iint \left(\frac{\partial F_2}{\partial x} - \frac{\partial}{\partial y} F_1\right)$$

$$RHS = \iint_{x^2+y^2=1} (3x^2+3y^2)\, dx\, dy$$

$$\cancel{J\left(\frac{r,\theta}{x,y}\right) = J_{(x,y)}(r,\theta)}$$

$$\cancel{\begin{vmatrix} \frac{\partial}{\partial x} r & \frac{\partial}{\partial x}\theta \\ \frac{\partial}{\partial y} r & \frac{\partial}{\partial y}\theta \end{vmatrix}}$$

GREEN'S THEOREM

$$\iint_G \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) = \int_{\partial G} \bar{F}$$

$$J\left(\frac{r,\theta}{x,y}\right) = J_{(x,y)}(r,\theta)$$

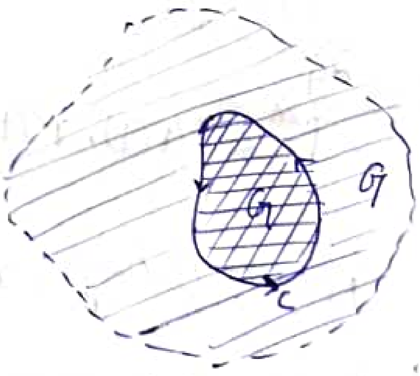
$$\begin{vmatrix} \frac{\partial}{\partial x} r & \frac{\partial}{\partial x}\theta \\ \frac{\partial}{\partial y} r & \frac{\partial}{\partial y}\theta \end{vmatrix}$$

$G \subset \mathbb{R}^2$ is open and simply connected.

$\bar{F}: G \xrightarrow{C_1\text{-type}} \mathbb{R}^2$
$= (F_1, F_2)$

Suppose $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0 \Rightarrow F$ is CVF

$$\bar{F} \text{ is CVF} \Leftrightarrow \int_\gamma F = 0$$



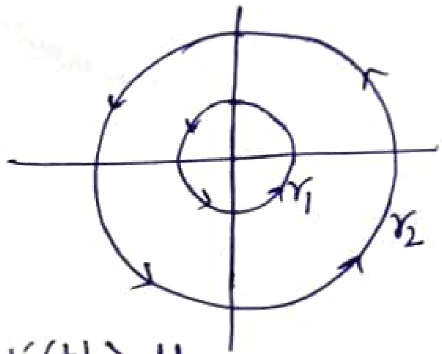
$$\iint \underbrace{\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)}_{\Downarrow \, 0} dx\, dy = \int_C F$$

$\therefore$ F is CVF.

$$\iint \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy = \int_{C_1} \bar{F} - \int_{C_2} \bar{F} - \int_{C_3} \bar{F}$$

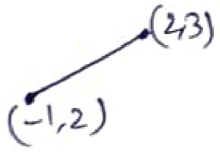
$$\int \underbrace{\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)}_{\hookrightarrow 0} dx\, dy = \int_{\gamma_2} \bar{F} - \int_{\gamma_1} F$$

$\therefore \int F_{\gamma_1} = \int F_{\gamma_2}$



Parametric eq^n is

$(1-t)P_1 + tP_2$



$$= \int_0^1 \langle F(L(t)), L'(t) \rangle \, dt$$