FILTER DESIGN – FIRST ORDER

Let us consider a RC circuit powered by a battery shown as the source $v(t)$ in FIG 1.

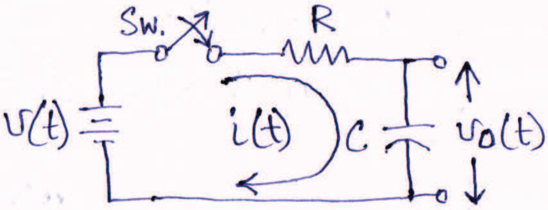


FIG.1

The load is capacitive, it can be thought of as ~~another~~ a basic model of another battery which has to be charged by the primary source $v(t)$. Charging can be controlled by the switch Sw. Once the switch is closed, applying KVL yields the following

$$v(t) = Ri(t) + v_o(t). \quad \text{where} \quad v_o(t) = \frac{1}{C}\int_0^t i(t)\,dt.$$

If the source $v(t) = E_B$ then,

$$v_o(t) = E_B(1 - e^{-t/\tau}) \quad \text{where} \quad \tau = RC.$$

The load voltage $v_o(t)$ can also be analyzed in Laplace domain or 's' domain where s is the complex frequency,

$$s = \sigma + j\omega. \longrightarrow (1)$$

~~In s domain,~~ We had in time domain, $v(t) = Ri(t) + \frac{1}{C}\int_0^t i(t)\,dt$

~~In s domain, $v(s) = Ri(s) + \frac{1}{sC}$~~ $\quad = RC\frac{dv_o(t)}{dt} + v_o(t)$

$$\therefore \quad v(s) = RC\left[s v_o(s) - v_o(0)\right] + v_o(s)$$

For filter design as well as for frequency response analysis which would be discussed shortly, the initial energy storage is considered ~~by~~ to be zero. $\therefore$ in our example, $v_o(0) = 0$ and hence, $v(s) = (1 + sRC)v_o(s)$

$$\Rightarrow \frac{v_o(s)}{v(s)} = \frac{1}{1 + s\tau}$$

The ratio of the output variable to the input variable under the condition that all initial energy storage are zero is known as the 'Transfer function' in s-domain. Denoting the transfer function as $G(s)$, for the input-output definition in our example,

$$G(s) = \frac{v_0(s)}{v(s)} = \frac{1}{1+s\tau} \longrightarrow (2)$$

It is very important to note that $G(s)$ is defined by the definition of input and output variables and is not unique for a given circuit. For example, some batteries can be charged in constant current or in constant voltage mode. In constant current mode, ~~the~~ monitoring and controlling $i(s)$ will be the focus and hence $i(s)$ may be defined as the output variable. Consequently the transfer function ~~was~~ becomes,

$$G(s) = \frac{i(s)}{v(s)} = \frac{\cancel{sx}\, sC}{1+s\tau}$$

In frequency response analysis, steady state performance is the ~~only concern~~ main concern for filter design. So ~~for~~ the complex frequency s, we are interested in the behaviour of the system for variation in ω. Here, ω is the frequency of the input which is the forcing function. ∴ replacing s by $j\omega$ in equation(2) the following is obtained,

$$G(j\omega) = \frac{v_0(j\omega)}{v(j\omega)} = \frac{1}{1+j\omega\tau} \Rightarrow |G(j\omega)| = \frac{1}{[1+\omega^2\tau^2]^{1/2}}$$

$$\& \ \angle G(j\omega) = \angle -\tan^{-1}\omega$$

$$\left.\begin{array}{l} \text{For } \omega \ll 1/\tau, \ |G(j\omega)| \simeq 1 \\ \& \text{ for } \omega \gg 1/\tau, \ |G(j\omega)| \to 0 \\ \omega = 0, \ |G(j\omega)| = 1 \end{array}\right\}$$

Unity gain for DC and at low frequencies.

Low gain at higher frequenc

So the RC circuit works as a lowpass filter. Another RC network is shown in Fig. 2. where the load is resistive and the output of this network or system is $v_o(t)$. Referring the transfer function or gain of the plant as $G_2(s)$,

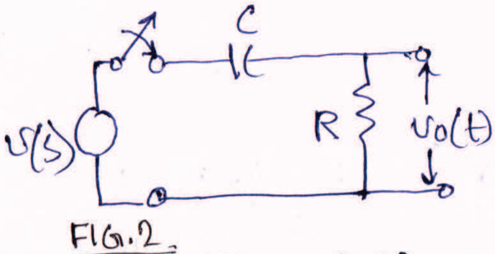


FIG. 2

$$G_2(s) = \frac{v_o(s)}{v(s)} = \frac{sRC}{1+sRC} = \frac{s\tau}{1+s\tau} \Rightarrow |G_2(j\omega)| = \frac{\omega\tau}{\sqrt{1+\omega^2\tau^2}}$$

$\therefore$ $\cancel{|G_2(j\omega)|}$ as $\omega \to 0$, $|G_2(j\omega)| \to 0$
$\omega \to \infty$, $|G_2(j\omega)| \approx 1$ } Blocks DC component & passes high frequency component

$\therefore$ ~~it is a~~ w

$\therefore$ this network behaves as a High pass filter.

How to design a filter? The first basic requirements are the knowledge of which frequency band has to be passed and which one is to be stopped/attenuated. These form the design inputs. Besides this, since electronic & electrical filters are mostly designed in frequency domain, knowledge of frequency response is a necessary prerequisite.

Any real system ~~can be represented~~ in s domain is composed of either the following factors or a combination of them

$$\frac{K}{s},\ Ks,\ \frac{K}{s\tau+1},\ K(s\tau+1),\ \frac{K}{[as^2+bs+c]^{\pm 1}}$$ [K, a, b & c are constants]

$\therefore$ any physical system can be represented by the generic transfer function,

$$G(s) = \frac{\Pi(s a_1+1)\,\Pi(s^2+b_1 s+c_1)}{s^m\,\Pi(s d_1+1)\,\Pi(s^2+e_1 s+f_1)}$$ [normalized form]

$$= K\frac{\Pi(s+z_1)\,\Pi(s^2+b_1 s+c_1)}{s^m\,\Pi(s+p_1)\,\Pi(s^2+e_1 s+f_1)}$$

$\therefore \; G(s) = g_1(s)\, g_2(s) \cdots\cdots g_n(s)$

where $g_1(s)$ to $g_n(s)$ are formed of the basic first order or quadratic factors.

$$\therefore \; G(j\omega) = \prod_{i=1}^{n} g_i(j\omega) \Rightarrow \log\{G(j\omega)\} = \log g_1(j\omega) + \cdots \log g_n(j\omega)$$

$$\Rightarrow 20\log|G(j\omega)| = \underbrace{G_{dB}}_{\text{gain magnitude in decibels}} = \sum_{i=1}^{n} 20\log|g_i(j\omega)|$$

* This is one of the major advantage of frequency response analysis. The frequency response of the individual factors can be studied and then simply added to arrive at the frequency response of the transfer function $G(s)$. So, from the frequency response of basic first and second order factors, the an entire system's frequency response can be derived. Let us look at the frequency response of the following $\frac{K}{j\omega}$, $j\omega$, $\frac{1}{1+j\omega\tau}$, & $1+j\omega\tau$)

Form (i) ~~$\frac{K}{j\omega}$~~ $G(j\omega) = \frac{K}{j\omega} \Rightarrow G(j\omega) = \frac{K}{\omega} \angle -90^\circ$

$$= K \cdot \frac{1}{\omega} \angle -90^\circ$$

$$20\log|G(j\omega)| = |G|_{dB} = 20\log K + 20\log\frac{1}{\omega}$$

$\rightarrow 20\log K$ dB for $\omega = 1$

$\rightarrow \alpha$ for $\omega \ll 1$

$\rightarrow -\alpha$ for $\omega \gg 1$

$= 0$ dB for $\omega = K$

For $\omega = K/10$, $|G|_{dB} = 20$dB

$= 10K$, $|G|_{dB} = -20$dB

The response has a negative slope of -20dB. (Pls refer to the plot)

The frequency response plots are,

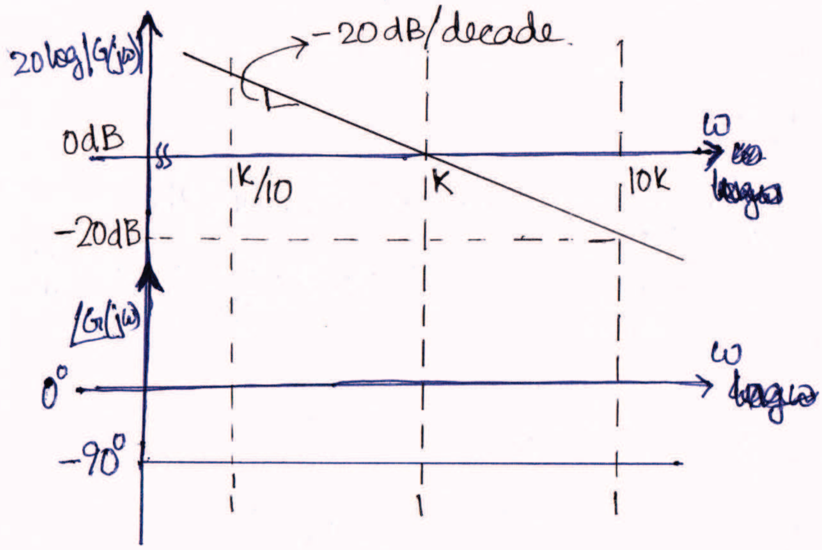


FIG.3a

In frequency response plots, the frequency ratios are expressed in terms of octaves or decades.

Octave is a frequency band from ω to 2ω.

Decade is the frequency band from ω to 10ω.

- ω axis is plotted in ~~linear~~ log scale while both the gain $|G(j\omega)|_{dB}$ and phase $\angle G(j\omega)$ are plot in linear scale.
- In this example, the magnitude/gain plot i.e. $|G(j\omega)|_{dB}$ plot has a slope of -20dB/decade i.e. it reduces by 20 dB over every decade.

Form

ii) $G(j\omega) = j\omega$ or say $G(j\omega) = jK\omega \rightarrow$ the original transfer fn. $G(s) = Ks$.

$= \underbrace{K\omega\angle 90^\circ}$ → Phase is constant $\forall\omega$

$$|G(j\omega)|_{dB} = 20\log|G(j\omega)| = 1 \text{ for } \omega = 1/K$$
$$= 20\text{dB for } \omega = 10/K$$
$$= -20\text{dB for } \omega = \frac{1}{10K}$$

∴ the gain plot has a slope of 20dB/decade

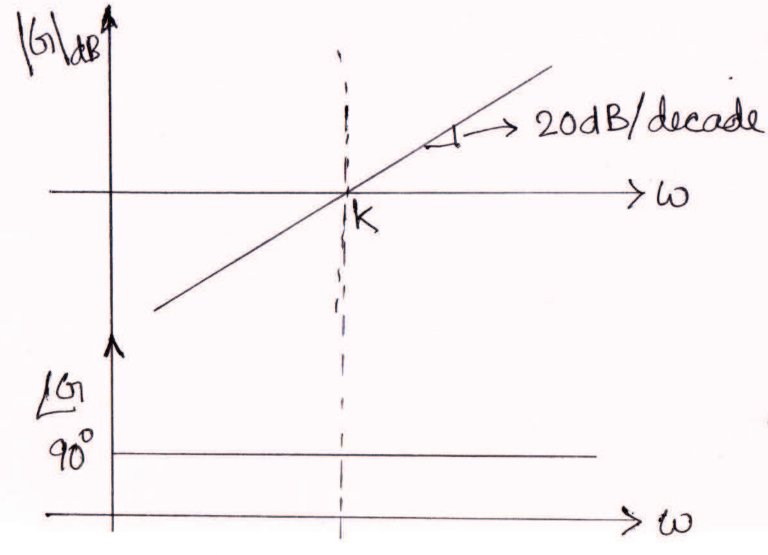


FIG.3b

(iii) $G(s) = \frac{1}{1+s\tau} \rightarrow$ The plant/transferfunction can also be in the form, $G_1(s) = \frac{K_1}{s+a}$ where K_1 & a are constants.

Transfer it to $G_1(s) = \frac{K_1}{a} \cdot \underbrace{\frac{1}{1+s\tau}}_{G(s)}$ $\therefore$ $20\log|G_1(j\omega)|$

$= 20\log\frac{K_1}{a} + 20\log|G(j\omega)|$

$$G(j\omega) = \frac{1}{1+j\omega\tau} = \frac{1}{[1+\omega^2\tau^2]^{1/2}} \cdot \angle -\tan^{-1}\omega\tau$$

$$|G(j\omega)| = \frac{1}{[1+\omega^2\tau^2]^{1/2}} \Rightarrow |G|_{dB} = 20\log[1+\omega^2\tau^2]^{-1/2}$$

For $\omega \ll \frac{1}{\tau}$, $|G(j\omega)| \approx 1 \Rightarrow |G|_{dB} = 0\,dB$

$\omega \gg \frac{1}{\tau}$, $|G(j\omega)| \approx \frac{1}{\omega\tau} \Rightarrow |G|_{dB} = -20\log\omega\tau \rightarrow -\infty$ as $\omega\uparrow$

$\omega = \frac{1}{\tau}$, $|G(j\omega)| = 0.707 \Rightarrow |G|_{dB} = -3\,dB$

$\omega = \frac{10}{\tau}$, $|G(j\omega)| \approx 0.1 \Rightarrow |G|_{dB} = -20\,dB$

$\angle G(j\omega) = -\tan^{-1}\omega\tau \rightarrow 0^\circ$ for $\omega\tau \ll 1$

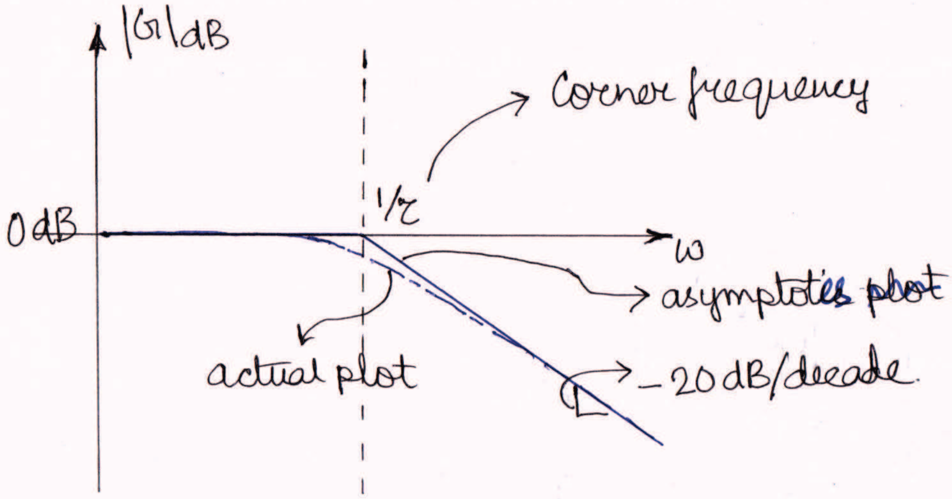
$= -45^\circ$ for $\omega\tau = 1$

$\rightarrow -90^\circ$ for $\omega\tau \gg 1$

$\approx -5^\circ$ for $\omega\tau = 0.1$

$\approx -85^\circ$ for $\omega\tau = 10$

Frequency response of $\frac{1}{1+j\omega\tau}$



$\frac{1}{\tau}$ is known as 'corner frequency'.

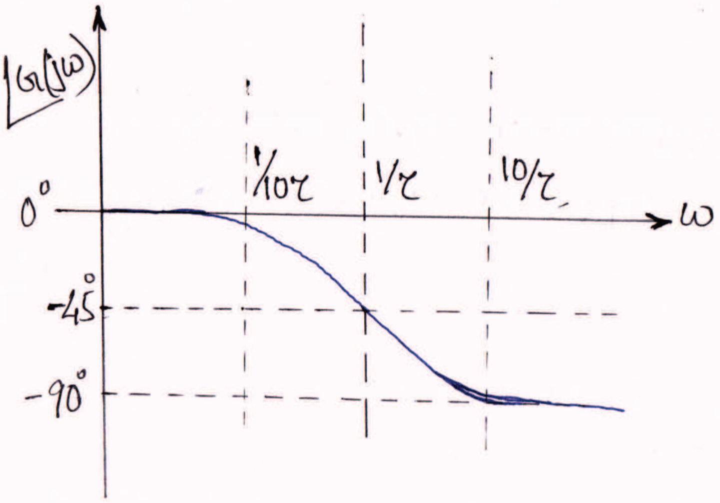


FIG.4

<u>Note</u>: Usually for the first design the asymptotes plot is sufficient.. Also, it is generally assumed for the 1st design that,

$$\angle G(j\omega) \approx 0^\circ \text{ for } \omega\tau \le 1/10$$
$$\approx -90^\circ \text{ for } \omega\tau = 10$$

iv) $G(s) = 1 + s\tau$.

The response would be dual of $\frac{1}{1+s\tau}$. ~~Work the~~ Work out this one in the same manner as done for $\frac{1}{1+s\tau}$.