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COMPLEX ANALYSIS

Complex Numbers

Real Analysis

Analysis of

$f: \mathbb{R} \to \mathbb{R}$

$f: \mathbb{R}^2 \to \mathbb{R}$

$f: \mathbb{R}^2 \to \mathbb{R}^2$

$f: \mathbb{R}^n \to \mathbb{R}^m$

Analysis includes:
- Sequence & series
- Limit
- Continuity
- Differentiation
- Integration

Complex Analysis

$\mathbb{R}^2 : \{(x,y) \mid x, y \in \mathbb{R}\}$

Eg. $f: \mathbb{R}^2 \to \mathbb{R} \Rightarrow f(x,y) = x^2 + y^2$

$f: \mathbb{R}^2 \to \mathbb{R}^2 \Rightarrow f(x,y) = (x^2, y^2)$

Complex Number

$Z = x + iy, \quad x, y \in \mathbb{R}$

OR

$Z = (x, y)$

$x = Re(Z), \; y = Im(Z)$

Set of Complex Number

$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}$

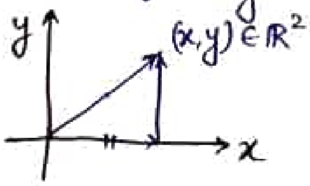
OR

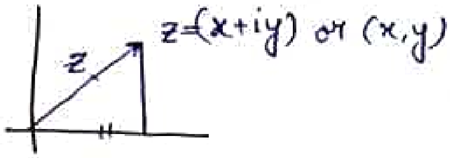
$\mathbb{C} = \{(x,y) \mid x, y \in \mathbb{R}\}$

Question: Are $\mathbb{R}^2$ and $\mathbb{C}$ ~~are~~ same?

↳ Looks same

Geometrically, they are same.





→ $\vec{x}, \vec{y} \in \mathbb{R}^2$

$\vec{x} \to (x_1, y_1)$, $\vec{y} \to (x_2, y_2)$

$\vec{x} \pm \vec{y} = (x_1 \pm x_2, y_1 \pm y_2)$

→ $z_1, z_2 \in \mathbb{C}$

$(x_1, y_1) \quad (x_2, y_2)$

$z_1 \pm z_2 = (x_1 \pm x_2, y_1 \pm y_2)$

→ If $\mathbb{R}^2 = \mathbb{C}$,

Real analysis of $f: \mathbb{R}^2 \to \mathbb{R}^2$ $\xrightarrow[\text{as}]{\text{same}}$ complex analysis of $f: \mathbb{C} \to \mathbb{C}$

(Limit, Cont., Diff.)

→ But, division is not defined for vectors ($\mathbb{R}^2$).

↳ Multiplication and division are not the same for $\mathbb{R}^2$ and $\mathbb{C}$.

→ R is not subset of $\mathbb{R}^2$.

$z_1 \cdot z_2 = x_1 x_2 + y_1 y_2$

$z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$

$= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$

↳ Definition

→ Algebraically, $\mathbb{R}^2$ and $\mathbb{C}$ are different, however geometrically they are the same.

If $z \neq 0$, $\frac{1}{z}$ is possible.

If $\bar{x} \neq 0$, $\frac{(1,0)}{\bar{x}}$ is not possible.

$z = (x, y)$

$z = (z_0)^{1/n}$

Eg. $i \quad -1$

$z^n = z_0$.

$z \neq 0$, $\frac{1}{z} = \frac{1}{x+iy} = \frac{1}{x+iy} \cdot \frac{(x-iy)}{(x-iy)} = \frac{x-iy}{x^2+y^2}$ $(x^2+y^2 \neq 0)$

$z_2 \neq 0$, $\frac{z_1}{z_2} = \frac{x_1+iy_1}{x_2+iy_2} \times \frac{x_2-iy_2}{x_2-iy_2} = \frac{z_1 \cdot \bar{z_2}}{x_2^2+y_2^2}$ $(\because \bar{z_2} = x_2 - iy_2)$

11-08-2023

Fundamental Theorem of Algebra

Every polynomial of degree 'n' with complex coefficient have exactly 'n' root in $\mathbb{C}$. These roots may not be distinct.

Eg. Solve $z^2 = z_0$.
- → Find z whose square is z_0
- → Find square root of z_0
- → Find root of $(z^2 - z_0) = 0$.

Soln: $(x+iy)^2 = x_0 + iy_0$

$$\Rightarrow \begin{cases} x^2 - y^2 = x_0 \\ 2xy = y_0 \end{cases} \Rightarrow \begin{cases} x^2 - y^2 = x_0 \\ x^2 + y^2 = |z_0| \end{cases} \Rightarrow x^2 = \frac{|z_0| + x_0}{2}, \quad y^2 = \frac{|z_0| - x_0}{2}, \quad |z_0| \geq x_0.$$

↳ 4 solutions; not possible

$$\Downarrow$$

$$z = \pm\left(\sqrt{\frac{|z_0| + x_0}{2}} + i\,\operatorname{sgn}(y_0)\sqrt{\frac{|z_0| - x_0}{2}}\right),$$

$$\operatorname{sgn}(y_0) = \begin{cases} 1, & y_0 \geq 0 \\ -1, & y_0 < 0 \end{cases}$$

Properties of Complex Numbers:

$$\bar{z} = x - iy$$

$$\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$$

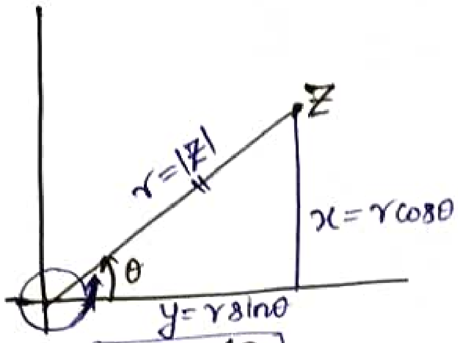
$$\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}, \quad z_2 \neq 0$$

$$|z_1 \pm z_2| \leq |z_1| + |z_2|$$

$$|z_1 \pm z_2| \geq \big||z_1| - |z_2|\big|$$

Polar Form of Complex Number



θ : Angle which Z makes from the ⊕ve x-axis.

$Z \neq 0$, $\boxed{Z = re^{i\theta}}$

Argument of Z:

$$\arg(Z) = \{\theta \mid x = r\cos\theta \text{ and } y = r\sin\theta\}$$

$$\tan^{-1}\left(\frac{y}{x}\right) = \left\{\theta \mid \tan\theta = \frac{y}{x}\right\}$$

$$\arg(Z) \neq \tan^{-1}\left(\frac{y}{x}\right)$$

Eg. $Z = -\sqrt{3} - i$

$$\arg(Z) = -\frac{5\pi}{6} + 2k\pi,\ k \in \mathbb{Z}$$

$$\tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{6} + k\pi,\ k \in \mathbb{Z}$$

Claim: $\exists$ unique $\theta \in (-\pi, \pi]$

such that $x = r\cos\theta$

$y = r\sin\theta$

$\theta = \underset{\text{principal argument}}{\mathrm{Arg}(Z)} \in \arg(Z)$

$\arg(Z) = \mathrm{Arg}(Z) + 2k\pi$, $k = 0, \pm 1, \pm 2, \ldots$ or $k \in \mathbb{Z}$

→ $Z_1 = Z_2$

$\Rightarrow \boxed{r_1 e^{i\theta_1} = r_2 e^{i\theta_2}}$

$\Updownarrow$

$\boxed{r_1 = r_2 \text{ and } \theta_1 = \theta_2 + 2k\pi,\ k \in \mathbb{Z}}$

$x_1 + iy_1 = x_2 + iy_2$

$\Updownarrow$

$x_1 = x_2$

$y_1 = y_2$

14-08-2023

Recall: $z \neq 0$

$z = re^{i\theta}$, where $r = |z|$ and $\theta \in arg(z)$

$arg(z) = \{\theta \mid x = r\cos\theta \text{ and } y = r\sin\theta\}$

$z_1, z_2 \neq 0$,

$$arg(z_1 \cdot z_2) = arg(z_1) + arg(z_2)$$

$$arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2)$$

$$arg\left(\frac{1}{z_1}\right) = -arg(z_1)$$

Remark: $Arg(z_1 \cdot z_2) \neq Arg(z_1) + Arg(z_2)$

$Arg(z_1 \cdot z_2) = Arg(z_1) + Arg(z_2)$, where $\begin{pmatrix} Re\, z_1 > 0 \\ Re\, z_2 > 0 \end{pmatrix}$.

Eg. $z_1 = -1$ and $z_2 = i$.

$Arg(z_1 \cdot z_2) = Arg(-i) = -\frac{\pi}{2}$

$Arg(z_1) + Arg(z_2) = \pi + \frac{\pi}{2}$

→ $arg(z) = Arg(z) + 2k\pi$, $k \in \mathbb{Z}$

→ $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$

$z_1 = z_2 \Leftrightarrow r_1 = r_2$ and $\theta_1 = \theta_2 + 2k\pi$, $k \in \mathbb{Z}$

→ $\boxed{z^n = r^n e^{in\theta}}$

Eg. Find z such that $z^n = z_0 \neq 0$. (n^{th} root of z_0)

$z_0 = r_0 e^{i\theta_0}$, $z = re^{i\theta}$ $\Rightarrow r^n e^{in\theta} = r_0 e^{i\theta_0}$

$\Rightarrow r^n = r_0$ and $n\theta = \theta_0 + 2k\pi$, $k \in \mathbb{Z}$

$$\Rightarrow r = \sqrt[n]{r_0}, \quad \theta = \frac{\theta_0 + 2k\pi}{n}, \quad \underbrace{k = 0, 1, 2, \ldots, n-1}_{n\text{-distinct roots}}$$

Eg Find the fourth root of $(\sqrt{3}+i)$.

or

$$z^4 = \sqrt{3}+i.$$

Soln: $z = re^{i\theta}$

$r=2,\ \theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) + 2k\pi,\ k \in \mathbb{Z}$

$\Rightarrow r=2,\ \theta_0 = \frac{\pi}{6}$

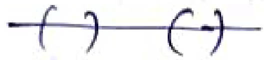
Complex Analysis

→ $f: D \subseteq \mathbb{C} \to \mathbb{C}$

open+connected (Domain)

\# $f: I \subseteq \mathbb{R} \to \mathbb{R}$

interval = $[a,b]$



$I_1 \quad I_2$

$I = I_1 \cup I_2$

Define $f(x) = \begin{cases} 1, & x \in I_1 \\ 2, & x \in I_2 \end{cases}$

$f'(x) = 0, \ \forall x \in I \Rightarrow f(x) = \text{constant}$ ✗

Defn: Neighbourhood of z:

$N_{z_0}^{\epsilon} = \{z \mid |z - z_0| < \epsilon\}$ ⇝ open ball

Defn: Interior point:

A point z_0 of a set S is said to be an interior point if there exists a neighbourhood of z_0 which is contained in S.

Defn: Exterior point:

A point z_0 of a set S is said to be an exterior point if there exists a n.b.h. of z_0 which doesn't contain points from S.

Defn: Boundary Point:

A point z_0 is said to be a boundary pt. if it is ~~neigh~~ neither an interior nor an exterior point.

Defn: Open set:

A set is open iff all of its points are interior points.

Defn: Closed set:

A set is said to be closed if it contains all of its boundary points (By default contain interior pts).

Eg. $S = \{z \in \mathbb{C} \mid |z| \leq 1\}$ → closed

$S_2 = \{ z \in \mathbb{C} \mid |z| < 1 \} \rightarrow$ Open

$S_3 = \{ z \in \mathbb{C} \mid 0 < |z| \le 1 \} \rightarrow$ neither open nor closed.

\# As geometrically $\mathbb{C} = \mathbb{R}^2$.

$\mathbb{C} \rightarrow$ open + closed

$\mathbb{R}^2 \rightarrow$ open + closed.

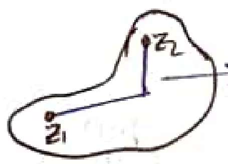
• If D doesn't have boundary, then by default it contains the boundary points.

$\Downarrow$

Open & closed

Connected set:

A set S is said to be connected if each pair of the points, say (z_1, z_2), can be joined by a line segment that lie entirely in S.



→ line segment (collection of lines).

Remark: The word 'line segment' can be replaced by 'path' or 'curve'.

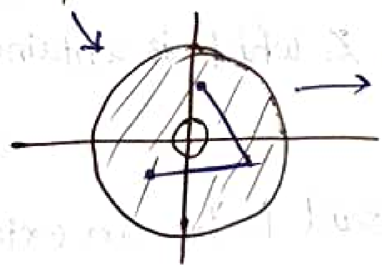
$\downarrow$

"Path-connected"

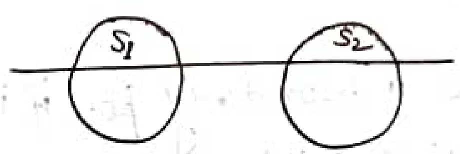
$\Downarrow$

connected.

Eg. $S = \{ z \in \mathbb{C} \mid 0 < |z| \le 1 \}$



→ Connected but not simply connected.



→ not connected.

Limits

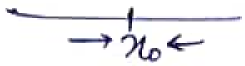
Def: Let f is defined in some $D_\epsilon^{z_0} = \{ z \in \mathbb{C} \mid 0 < |z - z_0| < \epsilon \}$

we say $\lim_{z \to z_0} f(z) = w_0$, if $\forall\ \epsilon > 0, \exists\ \delta$ such that

$$|f(z) - w_0| < \epsilon, \text{ whenever } 0 < |z - z_0| < \delta.$$

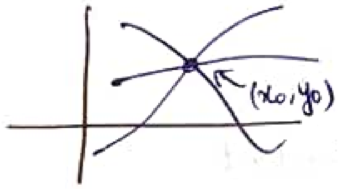
→ $f : I \subseteq \mathbb{R} \to \mathbb{R}$.

$\lim_{x \to x_0} f(x)$



$f : \mathbb{R}^2 \to \mathbb{R}$

$\lim_{(x,y) \to (x_0,y_0)} f(x,y)$



→ If the limit is path-dependent, then $\lim_{(x,y) \to (x_0,y_0)} f(x,y)$ does not exist,

or limit along any path doesn't exist, then $\lim_{(x,y) \to (x_0,y_0)} f(x,y)$ d.n.e.

$\lim_{(x,y) \to (x_0,y_0)} f(x,y)$ exists $\Rightarrow$ (limit along any path exists.)
$\not\Leftarrow$

Note: Given any f^n f of z, $f(z) = u(x,y) + iv(x,y)$,
where $\boxed{u, v : \mathbb{R}^2 \to \mathbb{R}}$.

Eg. $f(z) = z^2 = \underbrace{(x^2 - y^2)}_{u} + i\,\underbrace{2xy}_{v}$

$f(z) = u(r,\theta) + iv(r,\theta)$.

Eg. $f(z) = z^2 = r^2 e^{i2\theta} = r^2 \cos 2\theta + r^2 \sin 2\theta$.

Theorem: Let $f(z) = u + iv$ defined in $D_\epsilon^{z_0}$. Then,

$$\lim_{z \to z_0} f(z) = w_0 = u_0 + iv_0 \iff \lim_{(x,y) \to (x_0,y_0)} u(x,y) = u_0 \text{ and } \lim_{(x,y) \to (x_0,y_0)} v(x,y) = v_0.$$

Eg. $f_1(z) = \begin{cases} z^2, & z \neq 1 \\ 10, & z = 1 \end{cases}$ or ~~$f_1(z) = z^2$~~

$\lim_{z \to 1} f_1(z) = 1 = \lim_{z \to 1} z^2$

Eg. $f_2(z) = \begin{cases} z^2, & z \neq 1 \\ \infty, & z = 1 \end{cases}$

$\lim_{z \to 1} f_2(z) = 1.$

Eg. $f(z) = \frac{xy}{x^2+y^2}$

↳ Limit d.n.e. at $(0,0)$.

$u(x,y) = \frac{xy}{x^2+y^2}$

Take $y = mx$.

$\lim_{(x,y)\to(0,0)} u(x,y)$ ⇝ depends on 'm'

↓

path-dependent

↓

limit does not exist.

$\frac{(x)(mx)}{x^2+m^2x^2} = \frac{mx^2}{x^2(1+m^2)} = \frac{m}{1+m^2}$

Eg. $f(z) = \frac{2x^3}{x^2+y^2}$.

Take $y = mx$.

$f(z) = \frac{2x}{1+m^2}$.

$\lim_{z \to 0} f(z) = 0.$

$\frac{2x^3}{x^2(1+m^2)} = \frac{2x}{1+m^2}$

Eg. $f(z) = \lim_{z\to 0} \frac{z}{\bar{z}}$

↳ Limit does not exist.

Continuity

$f: D \subseteq \mathbb{C} \to \mathbb{C}$

Defn: $\lim_{z \to z_0} f(z) = f(z_0)$.

or

$\forall \epsilon > 0, \exists \delta$ such that $|f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$.

Theorem: Let $f(z) = u + iv$ defined on $N_\epsilon^{z_0} = \{z \mid |z - z_0| < \epsilon\}$.
Then f is continuous at $z_0 \Leftrightarrow u$ and v are continuous at (x_0, y_0).

Eg. $f(z) = \frac{xy}{x^2+y^2}$

↳ Not continuous at $(0,0)$.

$f(x) = \frac{2x^3}{x^2+y^2}$

↳ Not cont. at $(0,0)$, as not defined at $(0,0)$.

But, ~~f(z)~~ $f(z) = \begin{cases} \frac{2x^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$

↳ continuous at $(0,0)$.

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→ $f: D \subseteq \mathbb{C} \to \mathbb{C}$

$\lim_{z \to z_0} f(z)$ | $f(z) = u + iv$, $u, v \in \mathbb{R}^2 \to \mathbb{R}$

If $\lim_{z \to z_0} f(z) = A$ and $\lim_{z \to z_0} g(z) = B$,

then $\lim_{z \to z_0} (f \pm g) z = A \pm B$

$\lim_{z \to z_0} f(z) g(z) = AB$

$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{A}{B}$, $B \neq 0$.

Theorem: Let f and g are continuous at z_0.

Then $f \pm g$, $f \cdot g$, $\frac{f}{g}$ $(g \neq 0)$ are also continuous at $z = z_0$.

~~eg~~ $(f \pm g) z = f(z) \pm g(z)$

$(f \cdot g) z = f(z) g(z)$

$(f \circ g) z = f(g(z))$ is continuous at z_0, provided f is continuous at $g(z_0)$.

Remark: Let f be a continuous function $f(z_0) \neq 0$, then $\exists N_\epsilon^{z_0}$ such that $f(z) \neq 0$, $\forall z \in N_\epsilon^{z_0}$.

Differentiability

Recall $f : I \subseteq \mathbb{R} \to \mathbb{R}$

If $\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0)$ exists,

then f is differentiable at x_0 and its derivative is $f'(x_0)$.

$\rightarrow f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$

$\bar{x}_0 \in D$.

$\lim_{\bar{h} \to 0} \frac{f(\bar{x}_0 + \bar{h}) - f(\bar{x}_0)}{\bar{h}}$ $\times$ $\rightarrow \frac{1}{\bar{h}}$ is ~~def~~ not defined.

$\lim_{(h_1, h_2) \to (0,0)} \frac{f(\bar{x}_0 + \bar{h}) - f(\bar{x}_0)}{\sqrt{h_1^2 + h_2^2}}$ $\times$ $\rightarrow$ as it gives real value.

If $\lim_{(h,k) \to (0,0)} \frac{f(x_0 + h, y_0 + k) - h f_x(x_0, y_0) - k f_y(x_0, y_0) - f(x_0, y_0)}{\sqrt{h^2 + k^2}} = 0$,

then f is diff. at (x_0, y_0) and its derivatives is $\nabla f \big|_{(x_0, y_0)}$.

$\rightarrow f : D \subseteq \mathbb{C} \to \mathbb{C}$

$f(z) = u + iv$, $u, v \in \mathbb{R}^2 \to \mathbb{R}$.

If u & v are diff., we can't claim that $f(z)$ is diff.

If $\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$ ~~exists~~ $= f'(z_0)$ exists $\rightarrow \mathbb{C}$

then f is differentiable at z_0, and its derivative is $f'(z_0)$.

Eg. $f(z) = z^2$ is diff. $\forall z \in \mathbb{C}$.

$$f'(z) = 2z.$$

Eg. $f(z) = z^n$ is diff. $\forall z \in \mathbb{C}$

and $f'(z) = n z^{n-1}$.

Eg. $f(z) = \bar{z} \rightarrow$ Nowhere differentiable.

For any $z_0 \in z$,

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(\overline{z_0 + \Delta z}) - \bar{z_0}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} \rightsquigarrow \text{does not exist.}$$

Approach through x-axis $(\Delta y = 0)$

$$\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = 1$$

Approach through y-axis $(\Delta x = 0)$

$$\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = -1.$$

↳ Limits are not equal.

Eg. $f(z) = |z|^2 = z\bar{z}$

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)(\bar{z_0} + \overline{\Delta z}) - z_0\bar{z_0}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left(z_0 \frac{\overline{\Delta z}}{\Delta z} + \bar{z_0} + \overline{\Delta z} \right)$$

If $\lim f(x)$ & $\lim g(x)$ ~~exist~~ d.n.e., then $\lim (f(x) \pm g(x))$ may exist.

Through x-axis $(\Delta y = 0)$

$$\lim_{\Delta z \to 0} \left(z_0 \frac{\overline{\Delta z}}{\Delta z} + \bar{z_0} + \overline{\Delta z} \right) = z_0 + \bar{z_0}$$

Through y-axis $(\Delta x = 0)$

$$\lim_{\Delta z \to 0} \left(z_0 \frac{\overline{\Delta z}}{\Delta z} + \bar{z_0} + \overline{\Delta z} \right) = \bar{z_0} - z_0$$

Now, for $|z|^2$ to be diff.,

$$z_0 + \bar{z_0} = \bar{z_0} - z_0 \Leftrightarrow z_0 = 0$$ $\rightsquigarrow$ NOT diff. for $\boxed{z_0 \neq 0}$.

↳ Eg. Show that $f(z) = |z|^2$ is diff. at $z=0$.

Eg $f(z) = \bar{z} = \underbrace{x}_{u} - i\underbrace{y}_{v}$

Remark: u and v are diff. at $(x_0, y_0) \not\Rightarrow f$ is diff. at $z_0 = (x_0, y_0)$.

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→ f is diff. at $z_0 \underset{\not\Leftarrow}{\Rightarrow} f$ is continuous at z_0.

Remark: Continuity is a necessary condition for differentiability.

Eg $\bar{z} = x - iy$

→ $f(z) = u + iv$

$$u_x(x_0, y_0) = \lim_{h \to 0} \frac{u(x_0 + h, y_0) - u(x_0, y_0)}{h}$$ ⇝ 1 variable limit

$\left.\begin{matrix} u_y \\ v_x \\ v_y \end{matrix}\right\}$

$$\Delta w = f(z_0 + \Delta z) - f(z_0) = \Delta u + i\Delta v$$
$$\Delta u = u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)$$
$$\Delta v = v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)$$

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Delta w}{\Delta x + i\Delta y}$$

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{[u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)] + i[v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)]}{\Delta x + i\Delta y}$$

Approach through x-axis $(\Delta y = 0)$,

→ $u_x + iv_x$

Approach through y-axis $(\Delta x = 0)$,

$-iu_y + v_y$ (Multiplied N^r & D^r by i)

If f is diff. at z_0,

$$\boxed{\begin{matrix} u_x = v_y \\ \text{and, } u_y = -v_x \quad \text{at } z_0 = (x_0, y_0) \end{matrix}}$$

⇝ necessary condition for diff.
⇝ involves single variables.

→ $\boxed{\text{Cauchy-Reimann (C-R) Equation}}$

no relation ↔ $\boxed{\text{Continuity}}$ → also necessary condition

$f'(z_0) = u_x + i v_x$ at $z_0 = (x_0, y_0)$.

Eg. $f: \mathbb{R}^2 \to \mathbb{R}$

$f(x,y) = |x|$

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$f(z) = f(x+iy)$
$= f(x,y)$
$= u(x,y) + iv(x,y)$

$\longrightarrow f(z) = u + iv$

Check C.R. eq^n; $u_x = v_y$
$u_y = -v_x$

if NOT $\Rightarrow$ NOT differentiable.

- Suppose the CR eq^n is satisfied, then can we say that the function is differentiable?

<u>Remark</u>: CR eq^{n}s are only necessary condition for differentiabiity, not sufficient condition.

Eg. $$f(z) = \begin{cases} \dfrac{(\bar{z})^2}{z}, & z \neq 0 \\ 0, & z = 0. \end{cases}$$

Check diff. at $z=0$. $\times$ | Check C-R eq^n at $z=0$. ✓

(Check $\displaystyle\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$)

$$f(z) = \begin{cases} \dfrac{(\bar{z})^2}{z} = \dfrac{(\bar{z})^3}{|z|^2} = \dfrac{(x-iy)^3}{x^2+y^2} = i\dfrac{y^3-3x^2y}{x^2+y^2} + \dfrac{x^3-3xy^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

$$v = \begin{cases} \dfrac{y^3 - 3x^2y}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

$$u = \begin{cases} \dfrac{x^3 - 3xy^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

$$u_x(0,0) = \lim_{h\to 0} \frac{u(h,0) - u(0,0)}{h} = 1$$

$$u_y(0,0) = 0$$

$$v_x(0,0) = 0$$

$$v_y(0,0) = 1$$

$\therefore$ CR-eq^n $\begin{pmatrix} u_x = v_y \\ u_y = -v_x \end{pmatrix}$ is satisfied.

Now, check the limit

$$\lim_{\Delta z\to 0} \frac{f(\Delta z) - f(0)}{\Delta z}$$

Approach through x-axis ($\Delta y = 0$),

$$\text{limit} = \lim_{\Delta x\to 0} \frac{\Delta x}{\Delta x} = 1.$$

Through y-axis ($\Delta x = 0$),

$$\text{limit} = 1$$

Through $\Delta y = \Delta x$,

$$\text{limit} = \lim_{\Delta x\to 0} (-1) = -1$$

$\therefore$ $\displaystyle \text{Limit}_{\Delta z\to 0} \frac{f(\Delta z) - f(0)}{\Delta z}$ does not exist.

Recall: C-R eq^{n}s $u_x = v_y$
$u_y = -v_x$

are the necessary condition for $f(z) = u + iv$ to be differentiable.

If C.R holds $\not\Rightarrow$ f is diff.

$f: \mathbb{R}^2 \to \mathbb{R}$
$f_x, f_y \Rightarrow f$ is diff.
$+$ cont.

Sufficient Condition (Theorem):

Let f is defined in $N_\epsilon^{z_0}$.

- u_x, u_y, v_x and v_y exist on $N_\epsilon^{z_0}$.
- $u_x = v_y$ and $u_y = -v_x$ at z_0.
- u_x, u_y, v_x and v_y are continuous at z_0.

Then, f is diff. at z_0 and $f'(z_0) = u_x + iv_x$ at z_0. [or $f'(z_0) = v_y - iu_y$]

Proof: $\lim_{\Delta z \to 0} \dfrac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$

Choose Δz such that

$$0 < |\Delta z| < \epsilon.$$

$f(z_0 + \Delta z) - f(z_0) = \Delta u + i\Delta v$

$\Delta u = u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)$

$\Delta v = v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)$

$\Delta z = \Delta x + i\Delta y$

(By Taylor series, u_x, u_y are cont.)

$$\Delta u = \Delta x u_x + \Delta y u_y + \epsilon_1 \sqrt{\Delta x^2 + \Delta y^2}$$
$$\Delta v = \Delta x v_x + \Delta y v_y + \epsilon_2 \sqrt{\Delta x^2 + \Delta y^2}$$

where $\epsilon_1, \epsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$.

[$\because u(x_0 + \Delta x, y_0 + \Delta y) = u(x_0, y_0) + \Delta x\, u_x(x_0, y_0) + \Delta y\, u_y(x_0, y_0) + \epsilon_1 \sqrt{\Delta x^2 + \Delta y^2}$]

$$\therefore \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$
$$= \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{(\Delta x u_x + \Delta y u_y) + i(\Delta x v_x + \Delta y v_y) + (\epsilon_1 + i\epsilon_2)\sqrt{\Delta x^2 + \Delta y^2}}{\Delta x + i\Delta y}$$
$$= \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{(\Delta x + i\Delta y)u_x + i(\Delta x + i\Delta y)v_x + (\epsilon_1 + i\epsilon_2)\sqrt{\Delta x^2 + \Delta y^2}}{\Delta x + i\Delta y}$$

$(\because u_x = v_y,\ u_y = -v_x)$

$$= \lim_{(\Delta x, \Delta y) \to (0,0)} u_x + iv_x + \frac{(\epsilon_1 + i\epsilon_2)\sqrt{\Delta x^2 + \Delta y^2}}{\Delta x + i\Delta y}$$

Since $\left| \dfrac{\sqrt{\Delta x^2+\Delta y^2}}{\Delta x+i\Delta y} \right| \leq 1,$

$$(\epsilon_1+i\epsilon_2)\underbrace{\frac{\sqrt{\Delta x^2+\Delta y^2}}{\Delta x+i\Delta y}}_{\text{Bounded}} \to 0$$

as $\epsilon_1, \epsilon_2 \to 0$

$$\therefore \text{limit} = \lim_{(\Delta x,\Delta y)\to(0,0)} u_x + iv_x$$

$$= u_x + iv_x$$

$$= u_x(x_0,y_0) + i v_x(x_0,y_0).$$

Q) Prove or disprove the following.

If u_x, u_y, v_x, v_y are not cont. at a point $z_0 = (x_0,y_0)$, then $f = u+iv$ is not differentiable at z_0.

Can we have necessary and sufficient condition?

↳ Yes, we have!

Polar form of C-R eq^n

$$\left.\begin{aligned} r u_r &= v_\theta \\ \text{and, } u_\theta &= -r v_r \end{aligned}\right\} \text{C-R eq}^n\text{s}$$

for $f(z) = u(r,\theta) + i v(r,\theta)$.

Theorem: Let f be defined in $N_\epsilon^{z_0}$.

- u_r, u_θ, v_r and v_θ exist in $N_\epsilon^{z_0}$.
- C-R eq^{n}s $\Rightarrow r u_r = v_\theta$ and $u_\theta = -r v_r$ hold.
- u_r, u_θ, v_r and v_θ are continuous at z_0.

Then, f is differentiable at z_0.

$$\boxed{f'(z_0) = e^{-i\theta}(u_r + i v_r) \text{ at } z_0}$$

Eg. $f(z) = \frac{1}{z}$, $z \neq 0$.

$\Rightarrow f(z) = \frac{1}{r}(\cos\theta - i\sin\theta)$

$u = \frac{\cos\theta}{r}$, $v = -\frac{\sin\theta}{r}$

$$\left.u_r = -\frac{\cos\theta}{r^2},\quad v_r = \frac{\sin\theta}{r^2}\right\} \text{cont. for } r \neq 0.$$

By the theorem, f^n $f(z) = \frac{1}{z}$ is diff. at any non-zero point.

$$\begin{aligned} f'(z) &= e^{-i\theta}(u_r + i v_r) \\ &= -\frac{e^{-i\theta}}{r^2}(\cos\theta - i\sin\theta) \\ &= -\frac{e^{-2i\theta}}{r^2} = -\frac{1}{z^2} \end{aligned}$$

Eg. $f(z) = z^{1/2}$, $z \neq 0$. $\longrightarrow$ not a $f^n \rightarrow$ 2 outputs for single input

$\hookrightarrow$ cont., diff. not defined.

4-09-2023

$= r^{1/2} e^{i\theta/2}$, $\theta \in \arg(z) = \operatorname{Arg}(z) + 2k\pi$, $k \in \mathbb{Z}$

$f_1(z) = r^{1/2} e^{i\operatorname{Arg} z/2}$, $\theta = \operatorname{Arg} z + 2\pi$

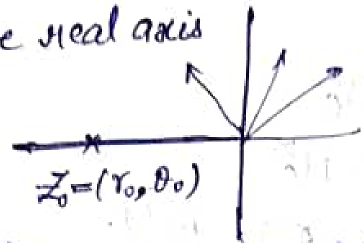
$$\begin{aligned} f_2(z) &= r^{1/2} e^{i\left(\frac{\operatorname{Arg} z + 2\pi}{2}\right)} \\ &= r^{1/2} e^{i(\operatorname{Arg} z/2 + \pi)} \\ &= -f_1(z) \end{aligned}$$

Discuss the diff. of $f_1(z)$.

$f_1(z) = r^{1/2} \cdot e^{i\theta/2}$ $\quad (r>0, -\pi < \theta \le \pi)$

Claim: $f_1(z)$ is discontinuous.

Along -ve real axis



$\lim_{z \to z_0} f_1(z)$ does not exist.

In upper half plain,

$$\lim_{z\to z_0} f_1(z) = \lim_{(r,\theta)\to(r_0,\pi)} r^{1/2} \cdot e^{i\theta/2}$$
$$= i r_0^{1/2}.$$

In lower half plain,

$$\lim_{z\to z_0} f_1(z) = \lim_{(r,\theta)\to(r_0,-\pi)} r^{1/2} \cdot e^{i\theta/2}$$
$$= -i r_0^{1/2} \quad (r_0 \neq 0)$$

As the limits are not the same, $\lim_{z\to z_0} f_1(z)$ d.n.e.

Claim: $f_1(z)$ is diff. except -ve real axis.

$$f_1(z) = \underbrace{r^{1/2}\cos\theta/2}_{u} + i\underbrace{r^{1/2}\sin\theta/2}_{v}$$

Check CR eqs $r u_r = v_\theta$ and $u_\theta = -r v_r$

$$f'(z) = e^{-i\theta}(u_r + i v_r)$$
$$= \frac{1}{2}\frac{1}{r^{1/2}\cdot e^{i\theta/2}}$$
$$= \frac{1}{2} z^{-1/2}, \quad z \neq 0.$$

Eg., $f(z) = z^{1/3}$, $z \neq 0$ ⇝ not a function

$(r>0, \alpha < \theta \le \alpha + 2\pi)$

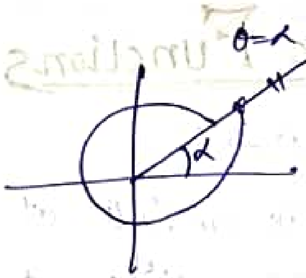
For each α, we get a f^n.

$\alpha \to f_\alpha$ - function

$\boxed{\alpha = -\pi}$

Discuss the diff. of f_α.

Eg. Show that f_α is discont. along $\boxed{\theta=\alpha}$



$\lim_{z\to z_0} f_\alpha(z)$ does NOT exist.

Claim: f_α is diff. except at $\theta=\alpha$.

Check C-R eq^ns. ✓

$u = r^{1/3}\cos\theta/3$

$v = r^{1/3}\sin\theta/3$.

$$f'(z) = e^{-i\theta}(u_r + i v_r)$$

$$= e^{-i\theta}\left(\frac{1}{3} r^{-2/3}\cos\theta/3 + \frac{1}{3} r^{-2/3}\sin\theta/3\right)$$

$$= \frac{1}{3} z^{-2/3}, \quad z \neq 0.$$

Analytic Functions

$f: D \subseteq \mathbb{C} \to \mathbb{C}$

We say f is analytic at z_0 if

(i) f is differentiable at z_0.

(ii) f is also differentiable on $N_{z_0}^{\epsilon}$.

Eg. $f(z) = z \rightsquigarrow$ analytic on $\mathbb{C}$

$f(z) = \bar{z} \rightsquigarrow$ nowhere analytic.

$f(z) = |z|^2 \rightsquigarrow$ diff. only at $z=0$

↳ NOT analytic at $z=0$

↳ nowhere analytic.

→ we can't have a function which is analytic only at a point.

→ open set $-D$

f is diff. on $D \Longleftrightarrow f$ is analytic on D.

Remark: On open set, differentiability and analyticity are the same.

$\boxed{7\text{-}09\text{-}2023}$

Singular Point

A point z_0 is said to be singular point of $f: D \subseteq \mathbb{C} \to \mathbb{C}$ if

(i) f fails to be analytic at z_0.

(ii) f is analytic at some points of every neighbourhood of z_0.

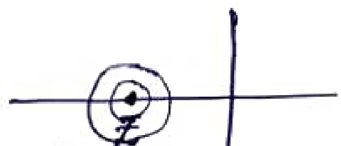
Eg. $f(z) = \frac{1}{z} \to z=0$ is a singular point.

$f(z) = z \to$ No singular point.

$\left.\begin{array}{l} f(z) = \bar{z} \\ f(z) = |z|^2 \end{array}\right\}$ No singular point.

$f(z) = z^{1/2} \quad (r=0, -\pi < \theta \leq \pi)$

$= r^{1/2} e^{i\,\mathrm{Arg}(z)/2}$ ⟶



All the pts. on -ve real axis are singular pts.

$f(z) = \frac{1}{(z-1)(z-2)} \to$ Two singular pts. $(z=1, z=2)$

→ If f is analytic at z_0, then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

- $f(z) \approx P_n(z)$

Properties of Analytic function:

D - domain = open + connected.

Theorem: If $f'(z)=0$ on D, then f is constant.

Proof: $f'(z) = u_x + i v_x = 0$

$$\Rightarrow \left| \begin{array}{l} u_x = 0 \text{ and } v_x = 0. \\ u_x = v_y \text{ and } u_y = -v_x \quad (CR) \end{array} \right.$$

$$\Rightarrow \left. \begin{array}{l} u_x = 0 \\ u_y = 0 \end{array} \right\} \Rightarrow u = \text{constant}$$

$$\left. \begin{array}{l} v_x = 0 \\ v_y = 0 \end{array} \right\} \Rightarrow v = \text{constant}$$

$\therefore f = u + iv \longrightarrow$ constant.

Eg. $f(z) = \begin{cases} 1, & |z| < 1 \\ 2, & |z| > 2 \end{cases} \longrightarrow$ Not constant

↳ not connected

Theorem: $f(z)$ and $\overline{f(z)}$ are analytic on D, then f is constant on D.

$f(z) = u + iv \longrightarrow u_x = v_y$, $u_y = -v_x$

$\overline{f(z)} = u - iv \longrightarrow u_x = -v_y$, $u_y = v_x$

$$\Rightarrow \{ u_x = 0 \text{ and } v_x = 0$$

$$\Rightarrow f'(z) = \underbrace{u_x}_{=0} + i v_x$$

$\Rightarrow f$ is constant.

How to Construct an analytic function?

Define Harmonic function:

$\phi : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$

we say ϕ is harmonic on Ω if

(i) $\phi \in C^2(\Omega) \Rightarrow \phi, \phi_x, \phi_y, \phi_{xx}, \phi_{yy}, \phi_{xy}$ are continuous on Ω.

(ii) $\phi_{xx} + \phi_{yy} = 0$ on Ω $(\Delta\phi = 0)$.

Eg. $\phi = xy$ or $x^2 - y^2 \to$ harmonic.

08-09-2023

Theorem: $f(z) = u + iv$ is analytic on D, $(u, v \in C^2(D))$.

$\Downarrow$

u and v are harmonic

Proof: $u_x = v_y$ and $u_y = -v_x$

$\Downarrow \qquad\qquad \Downarrow$

$u_{xx} = v_{yx} \qquad u_{yy} = -v_{xy}$

$\Downarrow$

$u_{xx} + u_{yy} = 0$

Remark:

If a function is analytic at a point, then all of its derivatives exist and are also continuous.

That means,

All partial derivatives of u and v are continuous.

Remark:

Another necessary condition for analyticity:

~~Wha~~ if f is analytic, u and v are to be harmonic.

$\to$ $f(z) = u + iv$

we say v is a harmonic conjugate of u on D if

$\left.\begin{array}{l} \text{(i) } u \text{ and } v \text{ are harmonic.} \\ \text{(ii) } u_x = v_y \text{ and } u_y = -v_x. \end{array}\right\}$ on D.

Theorem: $f(z) = u + iv$ is analytic on D.

$\Downarrow \Uparrow$

v is harmonic conjugate of u.

Remark: If v is HC. of $u \not\Rightarrow u$ is a H.C. of v.

Eg. $f(z) = \underbrace{(x^2-y^2)}_{u} + i\underbrace{(2xy)}_{v} = z^2 \rightsquigarrow$ analytic

$\hookrightarrow$ v is HC of u

Let $g(z) = 2xy + i(x^2-y^2)$

$\hookrightarrow$ CR eq^{n}s are not satisfied

$\hookrightarrow$ $g(z)$ is not analytic

$\hookrightarrow$ (x^2-y^2) is not HC of $(2xy)$.

Remark: If u and v are HC of each other, then f has to be a constant function.

Eg. $u = y^3 - 3x^2y$. Find an analytic function.

Soln: Find v such that $u_x = v_y$ and $u_y = -v_x$.

$\Rightarrow -6xy = v_y$

$\Rightarrow -3xy^2 + h(x) = v$

$-3y^2 + h'(x) = -3y^2 + 3x^2$

$\Rightarrow h'(x) = 3x^2$

$\Rightarrow h(x) = x^3 + \underset{\rightsquigarrow \text{ any value}}{c}$

$\therefore v = -3xy^2 + x^3$ (for $c=0$)

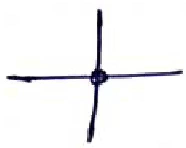
$f(z) = (y^3 - 3x^2y) + i(-3xy^2 + x^3)$.

Q) Given a harmonic function u, can we always find its HC?

Ans: No!

Eg. $u = \ln\sqrt{x^2+y^2}$

$\hookrightarrow$ Domain, $D = \mathbb{R}^2/\{0\}$



Check u is harmonic on D.

But u does not have HC on D.

$$\left.\begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}\right\}$$

→ Connected

↙ ↘

Simply   Multiply.

<u>Simply connected</u>:

↳ A connected domain is said to be simply connected if it does not have holes

<u>OR</u> every close curve in the domain consists of points within the domain.

<u>Theorem</u>:

Let u be harmonic on a simply connected domain D, then u has a HC on D.

<u>Theorem (calculus)</u>:

- Let D be a simply connected domain.
- $f, g \in C'(D)$
- $f_y = g_x$

Then, $\exists\ h \in C^2(D)$ such that $h_x = f$ and $h_y = g$.

<u>Proof</u>: Given $u_{xx} + u_{yy} = 0$, $u \in C^2(D)$.

$$f = -u_y \text{ and } g = u_x$$

$$\Rightarrow \exists\ v \in C^2(D) \text{ such that } v_x = -u_y$$
$$v_y = u_x$$

HC. $\boxed{v_{xx} + u_{yy} = 0}$.

↳ v is HC.

Elementary Function

exp: $\mathbb{C} \to \mathbb{C}/\{0\}$

$$e^z \text{ or } \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} > 0$$

$$e^z = e^{x+iy} = e^x(\cos y + i\sin y)$$

<u>Entire Function:</u> A function which is analytic throughout $\mathbb{C}$ (everywhere analytic).

Eg. Show that e^z is entire function.

<u>Hint:</u> $e^z = \underbrace{e^x\cos y}_{u} + i\underbrace{e^x \sin y}_{v}$

$$\frac{d}{dz} e^z = e^z$$

Eg. $e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$

$e^{z+2\pi i} = e^z e^{2\pi i} = e^z$

$\exists\, z$ such that $e^z = -1$ $\rightsquigarrow$ e^z can be negative (but $e^x > 0$)

- $|e^z| = e^x$
- $\arg(e^z) = y + 2k\pi,\ k \in \mathbb{Z}$
- $e^z \neq 0 \ \forall z \in \mathbb{C}$

Log

$\ln: \mathbb{R}^+ \to \mathbb{R}$

$$\ln(x) = \int_1^x \frac{1}{t}\,dt$$

<u>Idea:</u> Find w such that $e^w = z$.

Let $w = u + iv$, $z = re^{i\theta}$, $\theta \in \arg(z)$

$\therefore\ e^{u+iv} = re^{i\theta}$

$\Rightarrow e^u \cdot e^{iv} = re^{i\theta}$

$\Rightarrow e^u = r$ and $v = \theta + 2k\pi,\ k \in \mathbb{Z}$

$\Downarrow$

$\Downarrow$

$u = \ln r, \; v = \theta + 2k\pi$

$u = \ln|z|, \; v = \arg(z).$

Defn: $\log : \mathbb{C}/\{0\} \to \mathbb{C}$

$\boxed{\log z = \ln|z| + i\arg(z)}$ ⇝ not a function (arg z : set)

$\mathrm{Log}\,(z) = \ln|z| + i\,\mathrm{Arg}\,(z)$ | $\mathrm{Arg}\, z \in \arg z.$

Eg. $\log(-1-i)$
$= \ln\sqrt{2} + i\left(-\frac{3\pi}{4} + 2k\pi\right), \; k \in \mathbb{Z}$

→ $\log(e^z) = \ln e^x + i(y + 2k\pi)$
$= \underbrace{(x+iy)}_{z} + i\,2k\pi$
$\neq z.$

| But $\ln(e^x) = x$

| $\frac{d}{dz}\log z = \frac{1}{z} \to \times$
$\log(e^z) = z \to \times$

→ For a fixed $\alpha \in \mathbb{R}$,

$\log_\alpha(z) = \ln r + i\theta \quad (r>0, \; \alpha < \theta \leq \alpha + 2\pi)$

not base ⇝ $\alpha \to \log_\alpha$

If $\alpha = -\pi$,
$\log_{-\pi} \to \mathrm{Log}\, z$

→ $\log(z_1 \cdot z_2) = \log z_1 + \log z_2$
$\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2.$

| $\mathrm{Log} = \ln$, when $\arg = 0$ & $r > 0$

$\mathrm{Log}\,(z_1 \cdot z_2) \neq \mathrm{Log}\,(z_1) + \mathrm{Log}\,(z_2) \quad \forall z_1, z_2 \in \mathbb{C}$

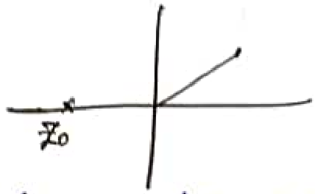
→ $\mathrm{Log}\,(z_1 \cdot z_2) = \mathrm{Log}\,(z_1) + \mathrm{Log}\,(z_2),$
if $Re(z_1) > 0$
$Re(z_2) > 0$

as $\mathrm{Arg}\,(z_1 \cdot z_2) = \mathrm{Arg}(z_1) + \mathrm{Arg}\,(z_2)$ when $Re(z_1), Re(z_2) > 0$

Continuity and Differentiability of $\log_\alpha z$.

Claim: $\operatorname{Log} z$ is discontinuous along $\ominus$ve real axis.

$$\operatorname{Log} z = \ln r + i\theta \quad (r>0,\ -\pi < \theta \leq \pi)$$



$\lim\limits_{z \to z_0} \operatorname{Log} z$ does not exist.

Claim: $\operatorname{Log} z$ is differentiable except $\ominus$ve real axis.

$$\operatorname{Log} z = \underbrace{\ln r}_{u} + i\underbrace{\theta}_{v}$$

Check $r u_r = v_\theta$ and $u_\theta = -r v_r$

$$\frac{d}{dz}\operatorname{Log}(z) = e^{-i\theta}(u_r + i v_r)$$
$$= e^{-i\theta}\left(\frac{1}{r}\right) = \frac{1}{re^{i\theta}} = \frac{1}{z}.$$

Claim: $\log_\alpha(z)$ is differentiable except at $\theta = \alpha$.

$$\frac{d}{dz}\log_\alpha(z) = \frac{1}{z}$$

We have proved $\frac{d}{dz}\log_\alpha(z) = \frac{1}{z}$ $\underbrace{(r>0,\ \alpha<\theta<\alpha+2\pi)}_{D_\alpha \text{ : domain}}$

15-09-2023

Recall,

$\log_\alpha z = \ln r + i\theta \qquad (\underbrace{r>0, \ \alpha < \theta < \alpha + 2\pi}_{D_\alpha})$

$\frac{d}{dz} \log_\alpha z = \frac{1}{z}$ on D_α.

$z^c, \ z \neq 0$

$z^c = e^{c \log z},$

$z_\alpha^c = e^{c \log_\alpha z} \qquad (r>0, \ \alpha < \theta < \alpha+2\pi).$

Eg. find all values $(-2i)^i$.

$(-2i)^i = e^{i \log(-2i)}$

Eg. find principle values of $(-2i)^i$.

$(-2i)^i = e^{i \operatorname{Log}(-2i)}$

$\rightarrow \frac{d}{dz} z_\alpha^c = \frac{d}{dz}(e^{c \log_\alpha z})$

$= e^{c \log_\alpha z} \cdot \frac{c}{z}$

$= e^{c \log_\alpha z} \cdot \frac{c}{e^{\log_\alpha z}}$

$= e^{(c-1)\log_\alpha z} \cdot c$

$= z_\alpha^{c-1} \cdot c$

$\frac{d}{dz} z^c = c z^{c-1}$

$\frac{d}{dz} \log z = \frac{1}{z}$

Defn: $c^z = e^{z \log c}$

$\frac{d}{dz} c^z = \frac{d}{dz}(e^{z \log c})$

$= e^{z \log c} \cdot \log c$

$\therefore \boxed{\frac{d}{dz} c^z = c^z \log c}$

Trigonometric Functions

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$ ⇝ function

$\sin(z_1+z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$

$\sin(iy) = i\sinh y,$

$\cos(iy) = \cosh y.$

$$\sin(x+iy) = \underbrace{\sin x \cosh y}_{u} + i\underbrace{\cos x \sinh y}_{v}$$

→ Hyperbolic functions:

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Eg. Show $\sin z$ is entire function.

↳ Show that CR equations are satisfied and the partial derivatives are continuous.

→ $\frac{d}{dz} \sin z = \cos z$

→ $\sin z$ is bounded? → No!

→ $\cos z = \frac{e^{iz} + e^{-iz}}{2}$

$\frac{d}{dz} \cos z = -\sin z$

→ $\tan z = \frac{\sin z}{\cos z}$, $\cot z = \frac{\cos z}{\sin z}$.

$\frac{d}{dz} \tan z = \sec^2 z$, $z \neq \pi/2 + 2k\pi$, $k \in \mathbb{Z}$

Inverse Function

Eg. Find w such that $\sin w = z$.

$$\Rightarrow \frac{e^{iw} - e^{-iw}}{2i} = z$$

$$\Rightarrow (e^{iw})^2 - (2iz)e^{iw} - 1 = 0$$

$$\Rightarrow e^{iw} = iz + (1-z^2)^{1/2}$$

$$\Rightarrow \log(e^{iw}) = \log[iz + (1-z^2)^{1/2}]$$

$$\Rightarrow iw = \log[iz + (1-z^2)^{1/2}]$$

$$\Rightarrow w = -i\log[iz + (1-z^2)^{1/2}]$$

$$\sin^{-1} z = -i\log[iz + \underbrace{(1-z^2)^{1/2}}_{2 \text{ values}}]$$

for principal value: take Log; Principal root (with +ve)

$az^2 + bz + c = 0$

$\log(e^z) \neq z$

$z^{1/2} \downarrow$ 2 values

Eg. Find all the values of $\sin^{-1}(1-i)$.

Sol^n: $\sin^{-1}(1-i) = -i\log[i(1-i) + (1-(1-i)^2)^{1/2}]$

$z = [1-(1-i)^2]^{1/2}$

$z^2 = 1 - (1-1-2i)$

$(a+ib)^2 = 1 + 2i = a^2 - b^2 + 2abi$

$\Rightarrow \left.\begin{array}{l} a^2 - b^2 = 1 \\ 2ab = 2 \end{array}\right\} \Rightarrow \begin{array}{l} a^2 = 1 + b^2 \\ ab = 1 \Rightarrow a = \frac{1}{b} \end{array}$

$\Rightarrow \frac{1}{b^2} = 1 + b^2$

$\Rightarrow 1 = b^2 + b^4 \Rightarrow b^4 + b^2 - 1 = 0$

$\Rightarrow b^2 = \frac{-1 \pm \sqrt{5}}{2}$

$\Rightarrow b^2 = \frac{\sqrt{5}-1}{2} \Rightarrow a^2 = \frac{\sqrt{5}+1}{2}$

$\therefore z = \pm \frac{\sqrt{\sqrt{5}+1} + i\sqrt{\sqrt{5}-1}}{\sqrt{2}}$

$$\rightarrow \frac{d}{dz}\sin^{-1}(z) = \frac{1}{\sqrt{1-z^2}}$$

$$\frac{d}{dz}\cos^{-1}(z) = -\frac{1}{\sqrt{1-z^2}}$$

$$\frac{d}{dz}\tan^{-1}(z) = \frac{1}{1+z^2}$$

18-09-2023

Complex Integral

→ $f:[a,b]\to\mathbb{R}$

$$\int_a^b f(x)\,dx = F(b)-F(a)$$

Look for $F:[a,b]\to\mathbb{R}$
$F'(x)=f(x)\ \forall\ x\in[a,b]$ (using anti-derivative)

- If f is continuous $[a,b]$, F exists. } sufficient condition & but necessary (can get F even if f is not cont.)

$$F(x)=\int_a^x f(t)\,dt \Rightarrow F'(x)=f(x)$$

→ $f:\mathbb{R}^2\to\mathbb{R}$

$C:\gamma\ [a,b]\to\mathbb{R}^2$
↳ Map

$\gamma(t),\quad a\le t\le b$

$$\int_a^b f(\gamma(t))\,[\gamma'(t)]\,dt = \int_C f(x,y)\,dx.$$

Curve:

$z:[a,b]\longrightarrow\mathbb{C}$ or $\mathbb{R}^2$

$z(t)=x(t)+iy(t),\quad a\le t\le b$
cont. on $[a,b]$

Smooth curve:

$z'(t)$ is continuous on $a\le t\le b$,
and $z'(t)\ne 0$, $a<t<b$

Contour: Piecewise smooth curve.

Eg.,

piecewise smooth

→ Let C be a contour.

- f is piecewise continuous C

or

- $f[z(t)]$ is piecewise continuous on $a \le t \le b$.

$$\int_C f(z)\,dz = \underbrace{\int_a^b f[z(t)].\, z'(t)\,dt}_{g(t)}, \quad z'(t) = x'(t) + iy'(t)$$

* $z(t)$ is the parameterisation of C

→ $f: \mathbb{R} \to \mathbb{C}$

$f: [a,b] \to \mathbb{C}$

$f(t) = u(t) + iv(t)$

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

<u>OR</u>

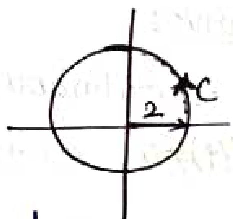
$$\int_a^b f(t)\,dt = F(t)\Big|_a^b \Rightarrow \boxed{F'(t) = f(t)}$$

$F = U(t) + iV(t)$

$\Rightarrow F'(t) = U'(t) + iV'(t)$

Eg., $\displaystyle\int_0^1 (1+it)^2\,dt$

Eg., $\displaystyle\int_C \bar{z}\,dz$



$z(t) = 2e^{it},\ 0 \le t \le 2\pi$

$$\int_C \bar{z}\,dz = \int_0^{2\pi} 2e^{-it}.\, 2ie^{it}\,dt = 8\pi i$$

$$\int_C \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{2e^{it}}.\, 2ie^{it}\,dt = 2\pi i$$

$$\int_C z\,dz = \int_0^{2\pi} 2e^{it}.\, 2ie^{it}\,dt = 0$$

$$\int_C f(x,y)\,dz$$

$$= \int_C f(x,y)\,(dx + i\,dy)$$

$$= \int_C f(x,y)\,dx + i\int_C f(x,y)\,dy.$$

↓

X

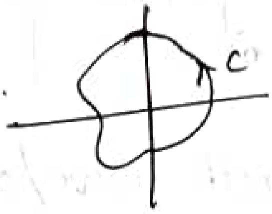
as $f(x,y)$ is not scalar

$(f: D \subset \mathbb{C} \to \mathbb{C})$

$$\int_C \frac{1}{z^2}\,dz = \int_0^{2\pi} \frac{1}{4e^{2it}} \cdot 2ie^{it}\,dt = 0$$

$$\int_C z^2\,dz = 0$$

$\hookrightarrow$ $\int_C z^n\,dz = 0$, for any closed curve C.



$$\int_C e^z\,dz = 0$$

$$\int_C \frac{1}{z^n}\,dz = 0, \text{ for } n \geq 2.$$

$$\int_C \frac{1}{z}\,dz = 2\pi i$$

21-09-2023

Recall:

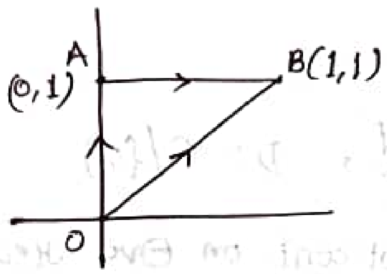
$C: \; z(t) = x(t) + iy(t), \quad a \leq t \leq b$

$$\int_C f\,dz = \int_a^b \underbrace{f[z(t)]\, z'(t)}_{g(t)}\,dt$$

$$\rightarrow \int_C f\,dz = \int_{z_1}^{z_2} f(z)\,dz$$

$\hookrightarrow$ Path-independant

Eg., $f(z) = y - x - 3ix^2$



$$\int_{OAB} f(z)\,dz = \int_{OA} f(z)\,dz + \int_{AB} f(z)\,dz$$

OA: $0 + it$, $0 \leq t \leq 1$

$$\int_0^1 t\,dt$$

AB: $t + i$, $0 \leq t \leq 1$

$$\int_{AB} f(z)\,dz = \int_0^1 (1 - t - 3it^2)\,dt$$

$\underline{OB}$: $z(t) = t + it$, $0 \le t \le 1$.

$$\int_{OB} f(z)\,dz = \int_0^1 -3it^2(1+i)\,dt$$

$$\therefore \boxed{\int_{OAB} f(z)\,dz = \int_{OB} f(z)\,dz}.$$

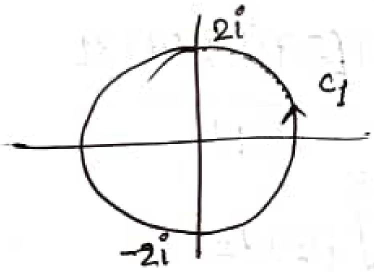
→ Let c be a smooth curve/contour.

$z(t)$; $a \le t \le b$.

$$\int_C z\,dz = \int_a^b z(t) z'(t)\,dt = \left.\frac{[z(t)]^2}{2}\right|_a^b$$

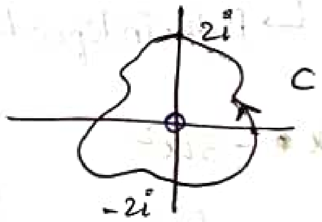
$$\rightarrow \int_C z\,dz = \int_{z_1}^{z_2} z\,dz = \left.\frac{z^2}{2}\right|_{z_1}^{z_2}$$

Eg. $\int_{C_1} \frac{1}{z}\,dz = 2\pi i$



$$\int_{C_1} \frac{1}{z^2}\,dz = 0$$

$$\int_C \frac{1}{z}\,dz = 2\pi i$$



$\frac{d}{dz} \underset{\sim}{\text{Log}}\, z = \frac{1}{z}$, $D = \mathbb{C}/\{0\}$

↳ not cont. on −ve real axis.

Defn: $f : D \subseteq \mathbb{C} \to \mathbb{C}$

$F : D \subseteq \mathbb{C} \to \mathbb{C}$

$F'(z) = f(z)$, $\forall z \in D$.

then F is called anti-derivative of f on D.

$$F(z) = \int_{z_0}^{z_1} f(z)\,dt$$

Theorem:

$f : D \subseteq \mathbb{C} \to \mathbb{C}$, $\exists\ F : D \subseteq \mathbb{C} \to \mathbb{C}$ such that $F'(z) = f(z)$.

f is continuous on D. Then, the following are equivalent:

(i) f has an anti-derivative on D.

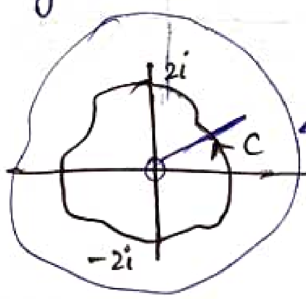
(ii) $\int_C f(z)\,dz = \int_{z_1}^{z_2} f(z)\,dz$ (C lying in domain)

OR Integral is path independant.

(iii) Integral along any closed curve lies in D is zero.

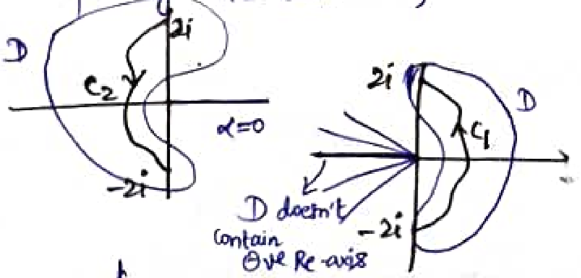
$\boxed{29\text{-}09\text{-}2023}$

Eg. $\int_C \frac{1}{z}\,dz = 2\pi i$



$\log z = \ln r + i\theta$, $\boxed{\theta = \alpha}$ $(r > 0,\ \alpha < \theta \le \alpha + 2\pi)$

Decomposing the curve C,



$F(z) = \log z$ ✓

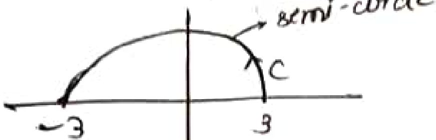
$\therefore$ $\int_C \frac{1}{z}\,dz = \int_{C_1} \frac{1}{z}\,dz + \int_{C_2} \frac{1}{z}\,dz$.

$$\int_{C_1} \frac{1}{z}\,dz = \log z \Big|_{-2i}^{2i}$$

$$\int_{C_2} \frac{1}{z}\,dz$$

$$\int_C \frac{1}{z}\,dz = 0$$

Eg. $\displaystyle\int_C z^{1/2}\,dz$



$f(z) = z^{1/2}\ (r>0,\ 0<\theta\leq 2\pi) \rightarrow$ for $\alpha = 0$.

$f_0(z) = z^{1/2}\ (r>0,\ 0<\theta<2\pi)$

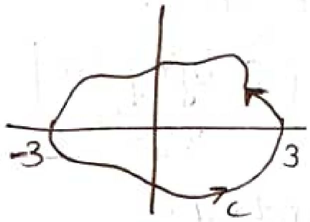
↳ continuous; not defined at 3

$z(\theta) = 3e^{i\theta},\ 0\leq\theta\leq\pi.$

$$\int_C z^{1/2}\,dz = \int_0^{\pi} \sqrt{3}\,e^{i\theta/2}\, 3e^{i\theta} i\,d\theta$$

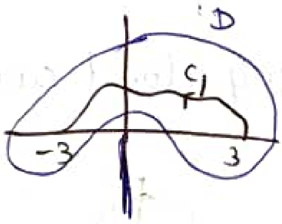
$$= -2\sqrt{3}(1+i)$$

Eg $\displaystyle\int_C z^{1/2}\,dz$



$f_1(z) = z^{1/2}$

$(r>0,\ -\pi/2 < \theta \leq 3\pi/2)$



$F_1(z) = \frac{2}{3} z^{3/2}.$

$(r>0,\ \frac{-\pi}{2} < \theta \leq \frac{3\pi}{2})$

$$\int_{C_1} f_1(z)\,dz = F_1(z) \Big|_{3}^{-3}$$

$$\int_{C_1} f(z) = \int_{C_1} f(z)\,dz.$$

$z^c : e^{c \log z}$

05-10-2023

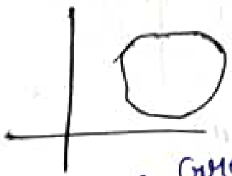
Cauchy Goursat Theorem

f is analytic on and inside of a simple closed contour.
Then, $\int_C f(z)\,dz = 0$.

Eg. $\int_C e^z\,dz = 0$

$\int_C \sin z\,dz = 0$

$\int_C \frac{1}{z}\,dz = 0$, $z \neq 0$.



Proof:

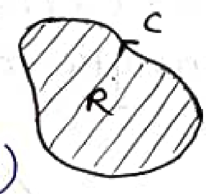
used in Green's Theorem

Assumption: f' is continuous. (can be proved).

$$\int_C f(z)\,dz = \int_C (u+iv).(dx+idy)$$
$$= \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy$$

Idea: Green's Theorem: $[P,Q \in C^1(R)]$

$$\int_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dA$$



(Line integral $\rightarrow$ Area Integral)

we have $u_x = v_y$ and $u_y = -v_x$ on R

$$\therefore \int_C f(z)\,dz = \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy$$
$$= 0.$$

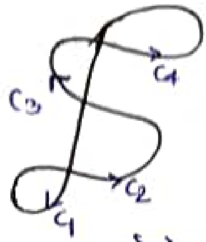
Cauchy-Goursat Theorem (Simply Connected domain):

Let D be a simply connected domain.
f is analytic on D.
Then, $\int_C f(z)\,dz = 0$, for any closed contour lines.

Q. Can we decompose every closed contour into simply connected?
Ans: No!

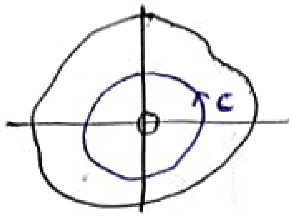
Remark: This theorem is not true for multiply connected domain.

$$\int_C = \underbrace{\int_{C_1}}_{=0} + \underbrace{\int_{C_2}}_{=0} + \underbrace{\int_{C_3}}_{=0} + \underbrace{\int_{C_4}}_{=0}$$



→ contour

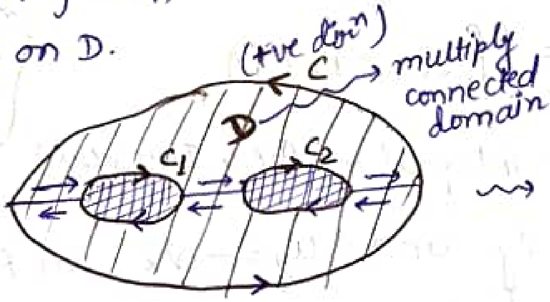
Eg. $\int_C \frac{1}{z} dz = 2\pi i$



→ D - simply connected domain.
f is analytic on D
$\Rightarrow$ f has an antiderivative on D.
$\Leftarrow$ ✓

Cauchy Goursat Theorem (Multiply Connected domain):

Let C be a simple closed contour in +ve direction.
$C_1, C_2, \dots, C_n$ are simple closed contours in −ve direction, lying inside C. $(C_i \cap C_j = \phi)$
f is analytic on D.

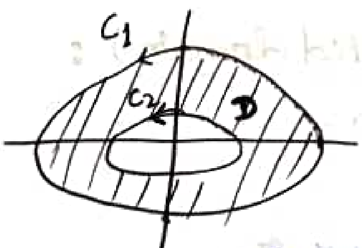


⇝ Contribution from upper part + lower part = 0 (from line = 0)

Then,

$$\int_C f(z)dz + \sum_{k=1}^{n} \int_{C_k} f(z)dz = 0.$$

Eg.

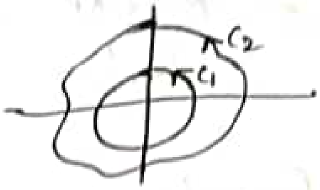


$$\int_{C_1} f(z)dz - \int_{C_2} f(z)dz = 0$$

↓ C_1, C_2 dirⁿs are same.

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

Eg $\displaystyle\int_{C_1} \frac{1}{z}\,dz = \int_{C_2} \frac{1}{z}\,dz.$



ML-Inequality

$$\left| \int_C f(z)\,dz \right|$$

M: Upper bound of function on the curve

L: Length of the contour.

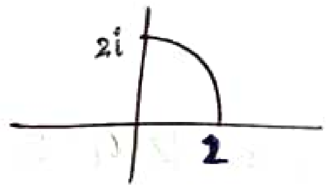
C: contour with length L

$|f(z)| \leq M \;\; \forall\, z \in C.$

$$\Rightarrow \left| \int_C f(z)\,dz \right| \leq ML$$

Proof:

Eg $\displaystyle\int_C \frac{z+4}{z^3-1}\,dz$



$L = \pi$

$f(z) = \dfrac{z+4}{z^3-1}$

$|z+4| \leq 6$

$|z^3-1| \geq \big|\, |z|^3 - 1 \,\big| = 7$

$\Rightarrow M = 6/7$

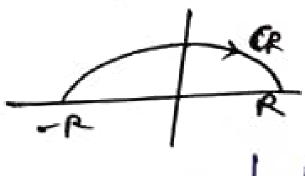
$$\therefore \left| \int_C \frac{z+4}{z^3-1}\,dz \right| \leq 6\pi/7$$

Eg $\displaystyle\int_{C_R} \frac{z^{1/2}}{z^2+1}\,dz$

$z^{1/2} = r^{1/2} e^{i\theta/2} \quad (r > 0,\; -\pi/2 < \theta \leq 5\pi/2)$

Show that

$$\lim_{R\to\infty} \int_{C_R} \frac{z^{1/2}}{z^2+1}\,dz = 0$$



$f(z) = \dfrac{z^{1/2}}{z^2+1}$, $\quad M_R = \dfrac{R^{1/2}}{R^2 - 1}$, $\quad L = \pi R$

$|z^2+1| \geq \big|\, |z|^2 - 1 \,\big|$

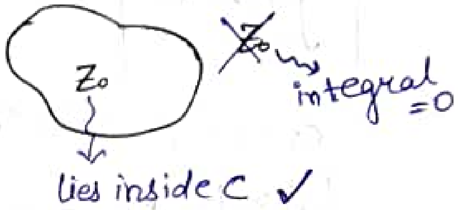
$$\therefore \left| \int_{C_R} \frac{z^{1/2}}{z^2+1}\, dz \right| \le \frac{R^{1/2} \times \pi R}{R^2-1} \longrightarrow 0 \text{ as } R \to \infty.$$

Cauchy Integral Formula

Theorem: Let C be a simple closed contour in +ve direction.

- f is analytic inside and on C.
- z_0 is an ~~integer~~ interior point of C.

Then, $$\int_C \frac{f(z)}{(z-z_0)}\, dz = f(z_0) \times 2\pi i$$



Eg. $\displaystyle\int_C \frac{z^2+5}{z-3}\, dz, \quad C: |z|=4$

$$\begin{bmatrix} z_0 = 3 \to \text{lies inside } C \\ f(z) = z^2+5 \end{bmatrix}$$

$= 2\pi i (9+5)$

$= 28\pi i$

Eg. $\displaystyle\int_C \frac{e^z}{z^2+4}\, dz, \quad C: |z-i| = 2$

$$= \int \frac{e^z}{(z-2i)(z+2i)}\, dz$$

$$= 2\pi i \left(\frac{e^{2i}}{4i} \right)$$

$-2i \to$ outside C

$2i \to$ inside C

$f(z) = \dfrac{e^z}{z+2i}$

$\displaystyle\int_C \frac{e^z}{z+2i}\, dz = 0$

$\to -2i$ lies outside C

Eg. $\displaystyle\int_C \frac{1}{z}\, dz = 2\pi i$

→ If both point lies inside C, separate the integral

$$\int \frac{f(z)}{(z-a)(z-b)}\, dz = \int \frac{A}{z-a}\, dz + \int \frac{B}{z-b}\, dz$$

Theorem: Let C be a simple closed contour in +ve direction.

- f is analytic inside and on C.
- Z is any interior point of C.

Then,

$$f^{n}(z) = \frac{n!}{2\pi i}\int_C \frac{f(s)}{(s-z)^{n+1}}\,ds = \left.\frac{d^n f(z)}{dz^n}\right|_{z=z_0}$$

(n ← derivative)

OR

$$f^{n}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$ ⇝ More general formula

Eg. $$\int_C \frac{z^2+5}{(z-3)^{3}}\,dz \quad (\text{Here, } n=2)$$ ($3 = n+1$)

$$= \frac{2\pi i}{2!}(2)$$

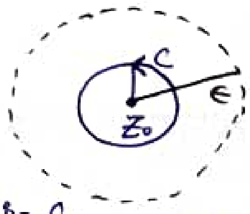
Eg. $$\int_C \frac{\sin z}{(z-\pi/6)}\,dz = \frac{2\pi i}{0!}\sin^{(0)}(\pi/6) = \frac{2\pi i}{1}\sin\left(\frac{\pi}{6}\right) = \pi i \qquad [\because f^{(0)}(a) = f(a)]$$

(Here, n=0)

Theorem: If f is analytic at a point z_0, then all of its derivatives exist and analytic at z_0.

Proof: $\exists\ N_{z_0}(\epsilon)$ on which f is analytic.

$C: |z-z_0| = \epsilon/2$



Cauchy Integral formula:

$$f''(z) = \frac{1}{\pi i}\int_C \frac{f(s)}{(s-z)}\,ds \quad \forall z \text{ lying in } C. \quad (n=2)$$

f'' exists inside C. ⇒ f' is analytic.

Theorem (E. Morera):

- f is continuous on a domain D.
- $\int_C f(z)dz = 0$ for any closed contour lying in D.

Then, f is analytic on D.

$\Rightarrow \exists$ F on D such that $F'(z) = f(z)$ $\forall z \in D$.

$\Rightarrow$ F is analytic on D. (F' exists)

$\Rightarrow F''(z)$ exist and $f''(z) = f'(z)$ $\forall z \in D$.

Remark: D: ~~Simply~~ simply connected domain.

f is analytic on $D \Leftrightarrow f$ has an antiderivative.

Major Application of Complex Analysis:

To Compute ~~Calculate~~ $\int_{-\infty}^{\infty} f(x)\,dx$.

$\hookrightarrow$ Improper integral

06-10-2023

Cauchy - Inequality

- CR: $|z-z_0|=R$
- f is analytic inside and on C_R, and $|f(z)| \le M_R$.

$$\left|f^n(z_0)\right| \le n! \frac{M_R}{R^n} \Rightarrow \text{For } n=1, \ \left|f'(z_0)\right| \le \frac{M_R}{R}.$$

- By Cauchy-Integral formula:

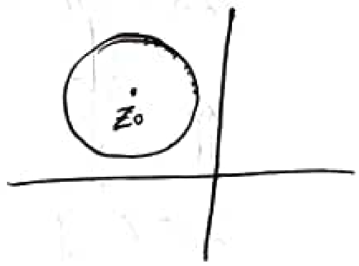
$$\left|f^n(z_0)\right| = \left|\frac{n!}{2\pi i}\int_{C_R} \frac{f(z)}{(z-z_0)^{n+1}}\right|$$

$$\le \frac{n!}{2\pi} \times \frac{M_R}{R^{n+1}} \times 2\pi R = \frac{n! \times M_R}{R^n}$$

Theorem: (Liouvelle's)

f is entire and bounded $\Rightarrow$ f is constant.

$|f(z)| \le M \ \forall \ z \in \mathbb{C}$.



Proof: $|f'(z_0)| \le \frac{M_R}{R}$

$\Rightarrow |f'(z)| \le \frac{M}{R} \to 0$ as $R \to \infty$

$\Rightarrow f'(z)=0 \ \forall \ z \in \mathbb{C}$

$\Rightarrow f$ is constant.

Fundamental Theory of Algebra:

A polynomial of degree greater than, equal to 1, has atleast one root in $\mathbb{C}$.

Polynomial : $P(z)$

No zero $\Rightarrow P(z) \neq 0 \ \forall \ z \in \mathbb{C}$

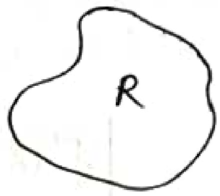
$\Rightarrow \frac{1}{P(z)} \to$ entire + bounded $\Rightarrow$ constant : Contradiction

$P(z) \to \infty$ as $z \to \infty$ (as $P(z)$ is polynomial)

Maximum Principle

(Theorem)
- f is continuous on a closed region R.
- f is analytic and non-constant inside R.

Then, $|f(z)|$ has maximum value occurring on the boundary of R.



$$\int_C \frac{f(z)}{(z-z_0)^m} \rightsquigarrow \text{cannot apply Cauchy-Integral formula.}$$

Sequence and Series of Numbers

$\{x_n\}$, $\sum x_n$

$\{z_n\}$

$$z_n = x_n + i y_n$$

$$x_n \xrightarrow{\text{converges to}} x, \quad y_n \to y$$

$\Updownarrow$

$$z_n = x + iy$$

$$\sum z_n = \underbrace{\sum x_n}_{S_1} + i \underbrace{\sum y_n}_{S_2}$$

$$\sum z_n = S_1 + i S_2$$

Taylor Series

f is analytic on $|z-z_0| < R$: $D_R(z_0)$.
↳ Radius of convergence

Then,

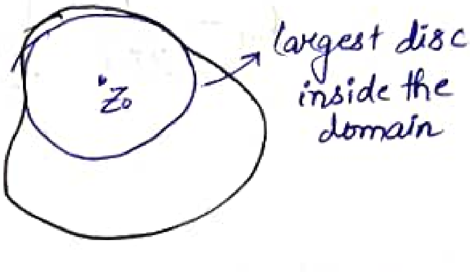
$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \forall z \in D_R(z_0)$$

where $a_n = \dfrac{f^n(z_0)}{n!}$

OR

f is analytic on D.

$$f(z) = \sum a_n (z-z_0)^n$$



Eg: $e^z = \sum_{n=0}^{\infty} \dfrac{z^n}{n!}$, $|z| < \infty$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \quad |z| < \infty$$

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$$

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$$

$$\frac{1}{1-z} = \sum z^n, \quad |z| < 1.$$

Eg $f(z) = \dfrac{1+2z^2}{z^3+z^5}$ → NOT analytic at $z=0$.

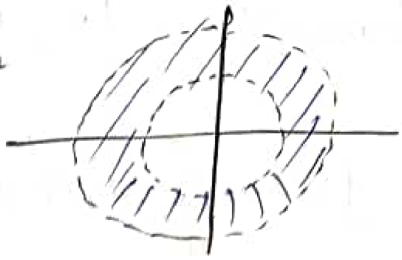
$= \dfrac{1}{z^3}\left[\dfrac{1+2z^2}{1+z^2}\right]$ → Taylor Series not possible at $z=0$.

Remark: If f is not analytic at z_0, then we cannot expect Taylor series around z_0.

$D_{R_1R_2}(z_0)$ (punctured disc)

<u>Theorem:</u> f is analytic on $R_1 < |z-z_0| < R_2$ $(R_1, R_2 > 0)$.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$



(Laurent's Series)

where $a_n = \dfrac{1}{2\pi i} \displaystyle\int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz$

$b_n = \dfrac{1}{2\pi i} \displaystyle\int_C \frac{f(z)}{(z-z_0)^{-n+1}}\, dz$, where C is any simple closed contour in $D_{R_1R_2}(z_0)$.

<u>Remark:</u> Taylor series is a particular case of Laurent series.

b_n: $b_1 = \dfrac{1}{2\pi i} \displaystyle\int_C f(z)\, dz = 0$ (for $n=1$) (by Cauchy-Goursat theorem)

$b_2 = \dfrac{1}{2\pi i} \displaystyle\int_C f(z)(z-z_0)\, dz = 0$ (for $n=2$)

$\therefore b_n = 0$

By Cauchy-Integral formula,

$$a_n = \frac{f^{n}(z_0)}{n!}$$

<u>For $n=1$</u>

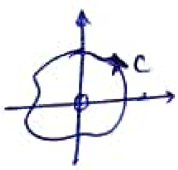
$b_1 = \dfrac{1}{2\pi i} \displaystyle\int_C f(z)\, dz \Rightarrow \int_C f(z)\, dz = 2\pi i \times b_1$

(b_1 is known $\Rightarrow \int_C f(z)dz$ can be computed)

$\hookrightarrow$ coefficient of $\dfrac{1}{z-z_0}$.

Eg. Prove : $\frac{1}{1-z} = \sum z^n, \; |z|<1.$

Eg. $\int_C e^{1/z}\, dz,$ → Cauchy-Integral formula not applicable where c is a closed contour around origin.



$\int_C e^{1/z}\, dz = 2\pi i \times 1$
$= 2\pi i$

[Here, $z=0$
$\Rightarrow b_1 =$ coeff. of $\frac{1}{z}$]

$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z}\right)^n$

$0 < |z| < \infty$

$\Rightarrow$ coefficient of $\frac{1}{z} = 1$

Eg $\int_C e^{1/z^2}\, dz = 0 \quad (\text{as } b_1 = 0)$

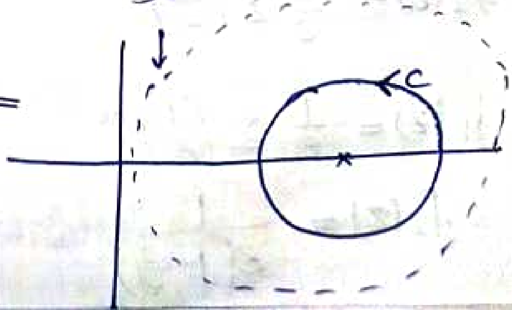
$e^{1/z^2} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z^2}\right)^n$

$\Rightarrow b_1 =$ coeff. of $\frac{1}{z} = 0.$

Eg. $\int_C \frac{dz}{z(z-2)^4}\, dz, \quad c : |z-2| = 1.$

$0 < |z-2| < 2$
↳ domain

Don't include origin



$$f(z) = \frac{1}{z(z-2)^4}$$

$$= \frac{1}{(z-2+2)(z-2)^4}$$ ⤳ Must contain $(z-z_0) = (z-2)$ term.

$$= \frac{1}{2} \times \frac{1}{(z-2)^4} \frac{1}{\left(1+\frac{z-2}{2}\right)}$$

$\left[\because \frac{1}{1+z} = \sum (-1)^n z^n, \ |z|<1 \right]$

$$= \frac{1}{2} \times \frac{1}{(z-2)^4} \times \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-2}{2}\right)^n \Rightarrow b_1 = -\frac{1}{16}$$

$\uparrow$ coeff. of $\frac{1}{(z-2)}$.

$$\therefore \int_C \frac{dz}{z(z-2)} dz = 2\pi i \times \left(-\frac{1}{16}\right)$$

$$= -\frac{\pi i}{8}$$

Eg. $f(z) = \frac{-1}{(z-1)(z-2)} = \underbrace{\frac{1}{z-1}}_{f_1} - \underbrace{\frac{1}{z-2}}_{f_2}$

(i) $|z|<1$

(ii) $1<|z|<2$

(iii) $2<|z|<\infty$.

Soln: (i) $f_1 = \frac{1}{z-1} = -\frac{1}{1-z} = -\sum z^n$

$f_2 = -\frac{1}{z-2} = \frac{1}{2}\left(\frac{1}{1-z/2}\right) = \frac{1}{2}\sum\left(\frac{z}{2}\right)^n$

(ii) $f_1 = \frac{1}{z\left(1-\frac{1}{z}\right)} = \frac{1}{z}\sum\left(\frac{1}{z}\right)^n$

$f_2(z) = \frac{1}{2}\sum\left(\frac{z}{2}\right)^n$

(iii) $f_1(z) = \frac{1}{z}\sum\left(\frac{1}{z}\right)^n$

$f_2(z) = \frac{1}{z\left(1-\frac{2}{z}\right)} = -\frac{1}{z}\sum\left(\frac{2}{z}\right)^n.$

Eg. $f(z) = ze^{2z} \to$ expand around $z=-1$.

$$= (z+1-1)e^{(2z+2-2)}$$

$$= (z+1-1)\, e^{2(z+1)} \cdot e^{-2}$$

$$= \frac{1}{e^2}\left[(z+1)e^{2(z+1)} - e^{2(z+1)}\right] \cdot \sum \frac{[2(z+1)]^n}{n!}.$$

Q) If we donot have Laurent's series expansion, how to find b_1?

$b_1 =$ coefficient of $\frac{1}{z-z_0}$ = Residue at $z=z_0$ of $f(z)$

$$= \operatorname*{Res}_{z=z_0}\left[f(z)\right]$$

Residue
↳ not analytic (singular)

How to find b_1?

Isolated singular Point: A singular point z_0 of a function $f(z)$ is said to isolated if f is analytic for

$$D_\delta(z_0) = \{z \in \mathbb{C} \mid 0 < |z-z_0| < \epsilon\}.$$

↳ Deleted neighbourhood of z_0.

Eg. $f(z) = \frac{1}{(z-1)(z-2)} \longrightarrow z=1, 2$ are isolated singular points.

$f(z) = \text{Log}\, z \longrightarrow z=0$ is NOT isolated.

Note: Finite number of singular points are always isolated.

Eg. $f(z) = \frac{1}{\sin(\pi/z)}$

↳ Singular points: $z=0$, $z=\frac{1}{n}$, $n \in \mathbb{Z}$

↳ $z=0$ is not isolated as we can always choose $1/n < \epsilon$ st. $0 < |z| \leq \frac{1}{n} < \epsilon$.

↳ $z=\frac{1}{n}$ are isolated points.

$\rightarrow \int_C f(z)\,dz = 2\pi i \times b_1$,

$b_1 = \text{coeff. of } \frac{1}{z-z_0} = \underset{z=z_0}{\text{Res}}\,[f(z)]$

If $z=z_0$ is an isolated singular point of $f(z)$.

$\Rightarrow f(z)$ is analytic on $\boxed{\underset{R_1}{0} < |z-z_0| < \underset{R_2}{\epsilon}}$

$$f(z) = \sum a_n (z-z_0)^n + \underbrace{\sum \frac{b_n}{(z-z_0)^n}}_{\text{Principle Part (PP)}}$$

Isolated singular pt. exists

$\Downarrow$

b_1 exists

Classification:

(i) $z=z_0$ is said to be removable singular point if PP has no term. $(b_1 = 0)$

Eg. $f(z) = \frac{1-\cos z}{z^2}$, $z \neq 0$

$= \frac{1}{z^2}\left[\cancel{1}-\cancel{1}+\frac{z^2}{2!}+\ldots\right]$

$= \frac{1}{2}+\ldots$

$f(z) = \frac{1}{z}$, $z=0$

(ii) If PP has finite no. of terms, then $z=z_0$ is called pole.

If $b_m \neq 0$, $b_{m+1} = 0 = b_{m+2} = b_{m+3} = \ldots$

then $z=z_0$ is called a pole of order m.

Eg. $f(z) = \frac{\sinh z}{z^3}$

$= \frac{z+\frac{z^3}{3!}+\frac{z^5}{5!}+\ldots}{z^3}$

$= \frac{1}{z^2}\left[1+\frac{z^2}{3!}+\frac{z^4}{5!}+\ldots\right]$

$\Rightarrow z=0$ is a pole of order 2.

(iii) PP has infinitely many terms,

then $z=z_0$ is called essential singular point.

Eg. $f(z) = e^{1/z}$

$= \sum \frac{1}{n!}\left(\frac{1}{z}\right)^n$

Remark: We can always find residue at pole.

Residue at Pole:

Theorem: An isolated singular point $z=z_0$ of $f(z)$ is a pole of order m.

$$\Updownarrow$$

$$f(z) = \frac{\phi(z)}{(z-z_0)^m}, \text{ where } \phi(z_0) \neq 0,$$

ϕ is analytic at z_0.

Moreover,

$$\underset{z=z_0}{\text{Res}}\left[f(z)\right] = \frac{\phi^{m-1}(z_0)}{(m-1)!}, \quad m \geq 2$$

derivative

$$= \phi(z_0), \quad \text{if } m=1.$$

Remark: For $m=1$, we call $z=z_0$ as a simple pole.

Eg. $f(z) = \frac{\sinh(z)}{z^3}$

$\phi(z) \neq \sinh(z)$

$\phi(z) \neq \frac{\sinh(z)}{z}$ → not analytic

$$\phi(z) = \begin{cases} \frac{\sinh(z)}{z}, & z \neq 0 \\ 1, & z = 0 \end{cases} \quad \text{or } \phi(z) = 1 + \frac{z^2}{3!} + \frac{z^4}{3!} + \dots$$

(analytic as it is T.S expansion)

↳ Pole of order 2.

$$\therefore \underset{z=z_0}{\text{Res}} f(z) = \frac{\phi'(z_0)}{1!} = 0.$$

Eg $f(z) = \dfrac{e^z}{(z-1)^{100}}$

$$\underset{z=1}{\text{Res}}\left[f(z)\right] = \frac{e}{99!}$$

Eg. $f(z) = \dfrac{1}{z(e^z-1)}$

$$= \frac{1}{z\left[z + \frac{z^2}{2!} + \dots\right]}$$

$$= \frac{1}{z^2} \frac{\overset{1 \to P}{}}{\underbrace{\left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \dots\right)}_{Q}} \longrightarrow \frac{P}{Q} \text{ form} \quad \left.\begin{array}{l} P \to \text{analytic} \\ Q \to \text{T.S.} \to \text{analytic} \end{array}\right\} \downarrow \text{analytic}$$

$\therefore$ $z=0$ is a pole of order 2.

$$\underset{z=0}{\text{Res}} = \phi'(0) = -\frac{1}{2}.$$

Zeroes of a Function

Defn: Let f is analytic at z_0 if

$$f(z_0) = f'(z_0) = \dots = f^m(z_0) \neq 0,$$

then $z = z_0$ is called a zero of order m.

Theorem: $z = z_0$ is a zero of order m.

$$\Updownarrow$$

$\exists$ $g(z)$ such that $f(z) = (z-z_0)^m g(z)$,

$g(z_0) \neq 0$ and g is analytic at z_0.

Eg. $f(z) = z(e^z - 1)$

$f(0) = 0$, $f'(0) = 0$, $f''(0) \neq 0$.

$$g(z) = \begin{cases} \dfrac{e^z - 1}{z}, & z \neq 0 \\ 1, & z = 0. \end{cases} \qquad \Big| \; f(z) = z^2 g(z).$$

Eg. $f(z) = \tan z$

$z = \frac{\pi}{2} \longrightarrow$ pole

$\hookrightarrow$ Residue cannot be determined.

Theorem: $f(z) = \frac{p(z)}{q(z)}$

- p and q are analytic at z_0
- $p(z_0) \neq 0$.
- $z = z_0$ is a zero of order m for q.

Then, $z = z_0$ is a pole of order m for f.

$$f(z) = \frac{p(z)}{q(z)} = \frac{p(z)}{(z-z_0)^m g(z)} \quad \phi$$

For $m=1$,

$$\operatorname*{Res}_{z=z_0} [f(z)] = \frac{p(z_0)}{q'(z_0)}$$

Proof:

$$q(z) = (z-z_0)\, g(z)$$

$$f(z) = \frac{p(z)}{(z-z_0)\, g(z)}$$

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{p(z_0)}{g(z_0)} = \frac{p(z_0)}{q'(z_0)} \qquad [\because g(z_0) = q'(z_0)]$$

$\therefore$ $f(z) = \tan z = \frac{\sin z}{\cos z}$

$$\Rightarrow \operatorname*{Res}_{z=\frac{\pi}{2}} [f(z)] = -1.$$

18-11-2023

Recall $\quad b_1 = \text{coeff. of } \frac{1}{z-z_0} = \underset{z=z_0}{\text{Res}} (f(z))$

$$\int_c f(z)\,dz = 2\pi i \times b_1$$

$\rightarrow$ z_0 is a pole of order $m \Leftrightarrow f(z) = \frac{\phi(z)}{(z-z_0)^m}$,

$\phi(z_0) \neq 0$ and ϕ is analytic at z_0.

$$\underset{z=z_0}{\text{Res}} [f(z)] = \frac{\phi^{m-1}}{(m-1)!} (z_0)$$

$\rightarrow f(z) = \frac{p(z)}{q(z)}$, $p(z) \neq 0$, $q(z) = g(z)(z-z_0)^m$

$\downarrow$ z_0 is a zero of order m.

$\Rightarrow z_0$ is a pole of order m.

$\rightarrow f(z) = \frac{p(z)\,/\,g(z)}{(z-z_0)^m}$ ϕ

For $m=1$,

$$\phi = \frac{p(z_0)}{q'(z_0)}$$

$\rightarrow f(z) = \frac{p(z)}{g(z)(z-z_0)}$

$q(z) = g(z)(z-z_0) \Rightarrow q'(z_0) = g(z_0)$

$$\underset{z=z_0}{\text{Res}} f(z) = \frac{p(z_0)}{q'(z_0)}$$

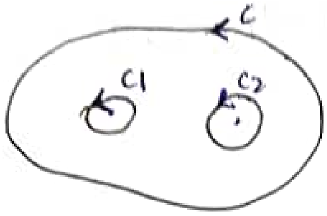
Eg. $\int_c \tan z\,dz$, $c: |z| = 2$

Eg. $\int_c \frac{e^z}{(z+1)^3}\,dz$, $c: |z| = 2$

Eg. $\int \frac{z^2}{(z-1)(z+3)}\,dz$, $c: |z| = 4$

Eg. $\displaystyle\int \frac{z^2}{(z-1)^3(z+3)}\,dz$, $c : |z| = 4$.

→



$$\int_c f(z)\,dz = 2\pi i \times (\text{sum of the residues})$$

Cauchy Residue Theorem

f is analytic inside and on c.

- c is a simple closed contour in +ve direction.
- f has finite no. of singular points $(z_0, z_1, \dots, z_n)$ inside c.

Then, $\displaystyle\int_c f(z)\,dz = 2\pi i \left(\sum_{k=0}^{n} \operatorname{Res}_{z=z_k} (f(z)) \right)$

Eg. $\displaystyle\int_c \tan z\,dz = 2\pi i(-1-1) = -4\pi i$

$c : |z| = 2$

$f(z) = \tan z = \dfrac{\sin z}{\cos z}$

Zeroes: $z = \left(\dfrac{2n+1}{2}\right)\pi$

Eg. $\displaystyle\int_c \frac{e^{z}}{(z+1)^3}\,dz = 2\pi i \left(\frac{e^{-1}}{2!} \right)$ ϕ

$c : |z| = 2$

(singular pt. : $z = -1$)

Eg. $\displaystyle\int \frac{z^2}{(z-1)(z+3)}\,dz = 2\pi i \operatorname{Res}_{z=1} f(z)$ ϕ

$c : |z| = 4$

$= 2\pi i \left(\frac{1}{4}\right) = \frac{\pi i}{2}$

Eg. $\displaystyle\int \frac{z^2}{(z-1)^3(z+3)}\,dz$ ϕ

$c : |z| = 4$

Improper Integral

Defn: $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R_1\to\infty}\int_{-R_1}^{0} f(x)\,dx + \lim_{R_2\to\infty}\int_{0}^{R_2} f(x)\,dx$ ... ①

Principal value : $(f : \mathbb{R}\to\mathbb{R})$
(P.V.)

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx \quad \ldots ②$$

① $\Rightarrow$ ② but ② $\not\Rightarrow$ ①

Eg. $f(x)=x \Rightarrow$ ① does not exist
② $=0$.

Eg $f(x)=x^2$
If f is even,
$f(-x)=f(x)\ \forall x$
Then, ① $\Leftrightarrow$ ②

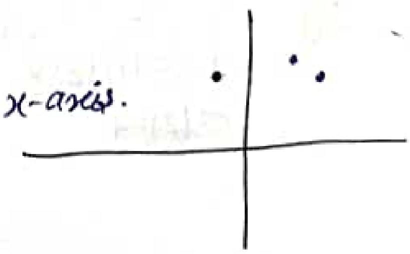
$$\int_{-\infty}^{\infty} f(x)\,dx = \text{P.V.}\int_{-\infty}^{\infty} f(x)\,dx = 2\int_{0}^{\infty} f(x)\,dx.$$

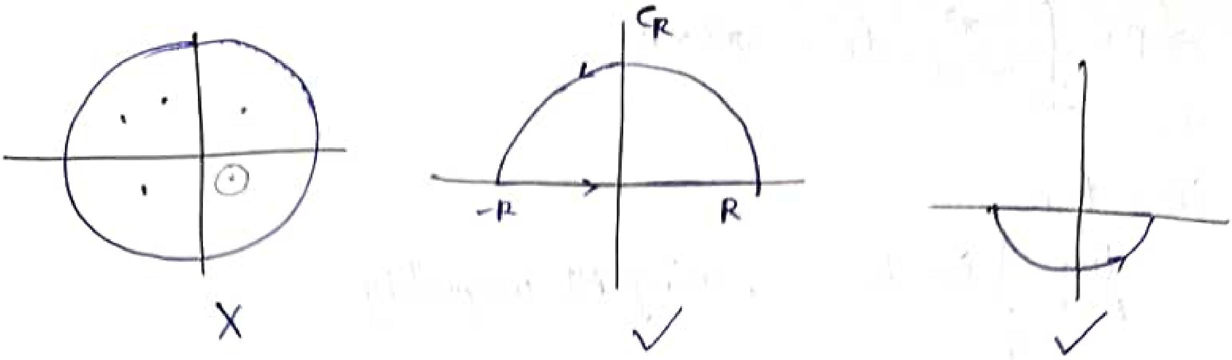
We can find P.V. $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$.

Assumption : $f(x) = \dfrac{p(x)}{q(x)}$

- p and q are polynomial with real coefficients and have no common root.
- q does NOT have a real root.

Step-1: $f(z) = \dfrac{p(z)}{q(z)}$, find zeroes of $q(z)$,
say $z_0, z_1, \ldots, z_n$ lie above x-axis.





Step-2: (Cauchy-Residue Theorem)

$$\int_{C_R} f(z)\,dz + \int_{-R}^{R} f(z)\,dz = 2\pi i\ (A)$$

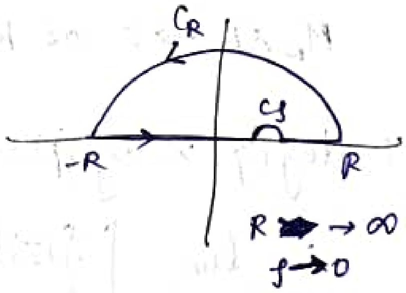
Sum of the residues

Step-3: $$\lim_{R\to\infty}\int_{C_R} f(z)\,dz + \underbrace{\lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx}_{P.V.} = 2\pi i(A)$$

$$\Rightarrow \lim_{R\to\infty}\int_{C_R} f(z)\,dz + P.V.\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \times A$$

Step-4: Show that

$$\lim_{R\to\infty}\int_{C_R} f(z)\,dz = 0.$$



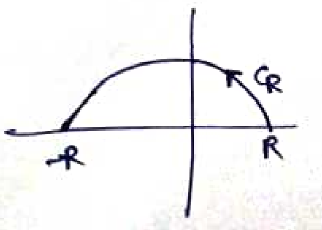
Eg. Find $\displaystyle\int_0^\infty \underbrace{\frac{x^2}{x^6+1}}_{f(x)}\,dx$

$$P.V.\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx = 2\int_0^\infty f(x)\,dx$$

Step-1: $f(z) = \dfrac{z^2}{z^6+1}$, Roots: $z =$

Step-2 & 3: $R > 1$

$$\lim_{R\to\infty}\int_{C_R} f(z)\,dz + \underbrace{\lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx}_{P.V.} = 2\pi i\ (A)$$



$$\operatorname*{Res}_{z=z_k} f(z) = \frac{z_k^2}{6z_k^5}$$

$$\Rightarrow P.V. \int_{-\infty}^{\infty} \frac{x^2}{x^6+1}\, dx = 2\pi i \times A$$

Step-4:

Show that

$$\lim_{R\to\infty} \int_{C_R} f(z)\, dz = 0$$, using ML inequality.

$$f(z) = \frac{z^2}{z^6+1}$$

ML Inequality: $$\left| \int_{C_R} f(z)\, dz \right| \le \boxed{M_R \times \underbrace{\pi R}_{L}} \to 0$$

$$C_R : |z| = R$$

$$M_R = \frac{R^2}{R^6-1}$$

$$|z|^2 \le R^2$$

$$|z^6+1| \ge \left| |z|^6 - 1 \right|$$

$$= R^6 - 1$$

$$M_R \cdot \pi R \to 0 \text{ as } R \to \infty$$

Remark: $\deg(q) \ge \deg(p) + 2$

$$\therefore \lim_{R\to\infty} \int_{C_R} f(z)\, dz = 0.$$

Eg. P.V. $\int_{-\infty}^{\infty} \underbrace{f(x)}_{p/q} \sin ax\, dx$ or $\int_{-\infty}^{\infty} f(x) \cos ax\, dx$, $a > 0$

$$F(z) = f(z) \underbrace{\sin az}_{\text{not bounded}}$$

$$F(z) = f(z)\, e^{iaz}, \quad |e^{iaz}| \le 1$$

Eg. $\displaystyle\int_0^\infty \frac{\cos 3x}{(x^2+1)^2}\,dx$

$$F(z) = \underbrace{\frac{1}{(z^2+1)^2}}_{f(z)} e^{i3z} = \frac{e^{i3z}}{(z+i)^2(z-i)^2} \quad \phi$$

$\boxed{R>1}$

$$\int_{C_R} F(z)\,dz + \int_{-R}^{R} F(z)\,dz = 2\pi i \times A$$

on real line $z = x$.

where $A = \operatorname{Res}_{z=i} \left(F(z) \right)$

$$\int_{C_R} F(z)\,dz + \int_{-R}^{R} \frac{1}{(x^2+1)^2} e^{i3x}\,dx = 2\pi i \times A$$

Take real values.

$$\operatorname{Re} \int_{C_R} F(z)\,dz + \operatorname{Re} \int_{-R}^{R} \frac{1}{(x^2+1)^2} e^{i3x}\,dx = \operatorname{Re}(2\pi i \times A)$$

$$\Rightarrow \operatorname{Re} \int_{C_R} F(z)\,dz + \int_{-R}^{R} \frac{\cos 3x}{(x^2+1)^2}\,dx = \operatorname{Re}(2\pi i \times A)$$

Take limit $R \to \infty$.

$$\lim_{R\to\infty} \operatorname{Re} \int_{C_R} F(z)\,dz + \lim_{R\to\infty} \int_{-R}^{R} \frac{\cos 3x}{(x^2+1)^2}\,dx = \operatorname{Re}(2\pi i \times A)$$

Show that $\displaystyle\lim_{R\to\infty} \int_{C_R} F(z)\,dz = 0$

$$|F(z)| \le \underbrace{\frac{1}{(R^2+1)^2}}_{M_R} \quad ; \quad \boxed{\deg q \ge \deg(p) + 2}$$

$M_R \times \pi R \to 0$ as $R \to \infty$

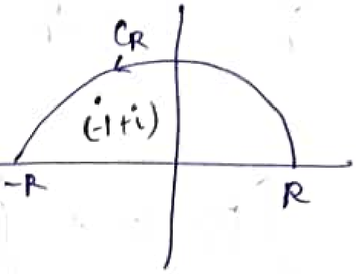
$$\therefore \lim_{R\to\infty} \int_{C_R} F(z)\,dz = 0.$$

Eg. Find P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+2x+2}\,dx$

Soln: $F(z) = \dfrac{z}{z^2+2z+2}\, e^{iz}$

Step-1: $z^2+2z+2=0 \Rightarrow$ Roots: $z = \underbrace{-1+i}_{\text{lies above } x\text{-axis}},\ -1-i.$

Step-2: $R > \sqrt{2}$



Step-3: $\displaystyle\int_{C_R} F(z)\,dz + \int_{-R}^{R} F(z)\,dz = 2\pi i \times \operatorname*{Res}_{z=-1+i}(F(z))$

Take imaginary part.

$$\operatorname{Im}\int_{C_R} F(z)\,dz + \int_{-R}^{R} \frac{x\sin x}{x^2+2x+2}\,dx = \operatorname{Im}(2\pi i \times A)$$

Take $R\to\infty$.

$$\lim_{R\to\infty} \operatorname{Im}\int_{C_R} F(z)\,dz + \text{P.V.}\int_{-R}^{R} \frac{x\sin x}{x^2+2x+2}\,dx = \operatorname{Im}(2\pi i \times A)$$

Step-4: Show that $\displaystyle\lim_{R\to\infty} \operatorname{Im}\int_{C_R} F(z)\,dz = 0.$

$$\int_{C_R} F(z)\,dz \le M_R \times \pi R$$

$$|z^2+2z+2| = |z-z_0|\,|z-\bar{z}_0|$$

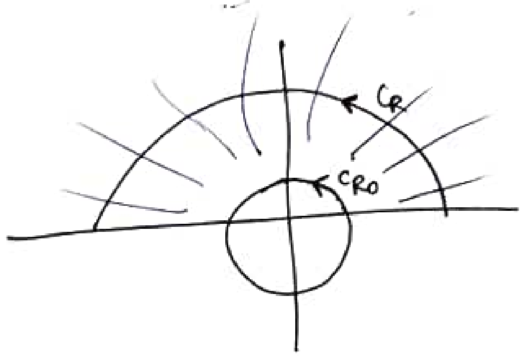
$$|z-z_0| \ge \big||z|-\sqrt{2}\big| = R-\sqrt{2}$$

$$M_R = \frac{R}{(R-\sqrt{2})^2},\quad M_R.\pi R \to 0 \text{ as } R\to\infty.$$

$|f(z)| \le \dfrac{R}{(R-\sqrt{2})^2} \to 0$ as $R\to\infty$ $\Rightarrow \therefore \displaystyle\lim_{R\to\infty}\int_{C_R} F(z)\,dz = 0$, by Jordan lemma.

Jordan Lemma

f is analytic in the upper half plane which is exterior to the circle C_{R_0}.



- M_R : $|f(z)| \le M_R$ on C_R.

If $\lim\limits_{R\to\infty} M_R = 0$, then

$$\lim_{R\to\infty} \int_{C_R} \underbrace{f(z)\, e^{iaz}}_{F(z)}\, dx = 0 \quad , \quad a>0$$