DSP

06-08-2024

Consider

$$y = Ax$$

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_M \end{bmatrix}}_{\in \mathbb{R}^{M\times 1}} = \underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1N} \\ a_{21} & \cdots & a_{2N} \\ \vdots & \cdots & \vdots \\ a_{M1} & & a_{MN} \end{bmatrix}}_{\in \mathbb{R}^{M\times N}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}}_{\in \mathbb{R}^{N\times 1}}$$

Transformed

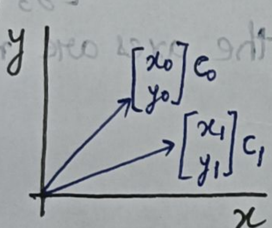
$$\begin{bmatrix} y_1 \\ \vdots \\ y_M \end{bmatrix} = \begin{bmatrix} a_{11} \\ \vdots \\ a_{M1} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ \vdots \\ a_{M2} \end{bmatrix} x_2 + \cdots + \begin{bmatrix} a_{1N} \\ \vdots \\ a_{MN} \end{bmatrix} x_N$$

$$Y = \underbrace{a_{1c}x_1 + a_{2c}x_2 + \cdots + a_{Nc}x_N}$$

∴ Y is a linear combination of the columns of A

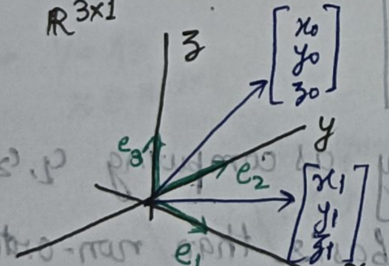
Each column vector $a_{ic} \in \mathbb{R}^{M\times 1}$.

Eg: $\mathbb{R}^{2\times 1}$



$$\underbrace{c_0 \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + c_1 \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}}_{\text{linear combination}} = \begin{bmatrix} x' \\ y' \end{bmatrix} \in \mathbb{R}^{2\times 1}$$

$\mathbb{R}^{3\times 1}$



linear combination $\in \mathbb{R}^{3\times 1}$

$$\left. e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \text{Basis vectors (orthogonal)}$$

$$\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = x_0 e_1 + y_0 e_2 + z_0 e_3$$

$$e_1 . e_2 = [1\ 0\ 0]\begin{bmatrix}0\\1\\0\end{bmatrix} = 0$$

$$e_1 . e_3 = [1\ 0\ 0]\begin{bmatrix}0\\0\\1\end{bmatrix} = 0$$

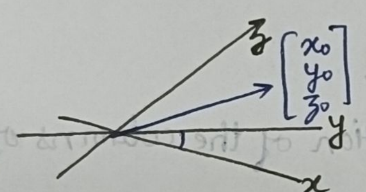
$$e_2 . e_3 = [0\ 1\ 0]\begin{bmatrix}0\\0\\1\end{bmatrix} = 0$$

$\left.\right\}$ $e_1 \perp e_2$, $e_2 \perp e_3$, $e_1 \perp e_3$

Properties:

① Any point in $\mathbb{R}^{3\times1}$ can be represented as a linear combination of e_1, e_2, e_3.

② The amount of each of the components can be represented by dropping $\perp^r$ from the vector to those axis.



→ Basis vectors are not orthogonal.

$$e_1 = \begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix},\ e_2 = \begin{bmatrix}x_2\\y_2\\z_2\end{bmatrix},\ e_3 = \begin{bmatrix}x_3\\y_3\\z_3\end{bmatrix}$$

↳ Properties 1 & 2 hold even when the axes are not $\perp^r$ to one another.

Here as well,

$$\begin{bmatrix}x_0\\y_0\\z_0\end{bmatrix} = d_1\begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix} + d_2\begin{bmatrix}x_2\\y_2\\z_2\end{bmatrix} + d_3\begin{bmatrix}x_3\\y_3\\z_3\end{bmatrix}$$

③ Computing d_1, d_2, d_3 is not as easy as computing c_1, c_2, c_3.

It is convenient to have orthogonal basis, than non-orthogonal basis while representing a vector

$Y = a_{1C}x_1 + a_{2C}x_2 + \dots + a_{NC}x_N \in \mathbb{R}^{M\times 1}$

$$Y = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1M} \\ a_{21} & a_{22} & \dots & a_{2M} \\ \vdots & & \dots\dots & \\ a_{N1} & a_{N2} & \dots & a_{NM} \end{bmatrix} X, \; x \in \mathbb{R}^{M\times 1}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_M \end{bmatrix} = \begin{bmatrix} a_{1R} \\ a_{2R} \\ \vdots \\ a_{MR} \end{bmatrix} x \Bigg\} \text{ Each } y_i = a_{iR}x = \langle a_{iR}, x\rangle$$

Suppose $Y = AX = 0 \longrightarrow \langle a_{iR}, x\rangle = 0 \;\forall i$

$\longrightarrow$ The linear ~~constants~~ combination of all column vectors is 0.

Suppose a_1, a_2, a_3 are non-zero such that

$$\left.\begin{aligned} a_1x_1 + a_2x_2 + a_3x_3 &= 0 \\ a_1 &= -\frac{1}{x_1}\left[x_2a_2 + x_3a_3\right] \end{aligned}\right\} \text{One of the vectors is dependent on the other vectors.}$$

$\Updownarrow$

a_1 is linear comb. of a_2 & a_3.

Suppose $a_1, a_2, a_3 \in \mathbb{R}^{3\times 1}$, non-zero vectors, such that

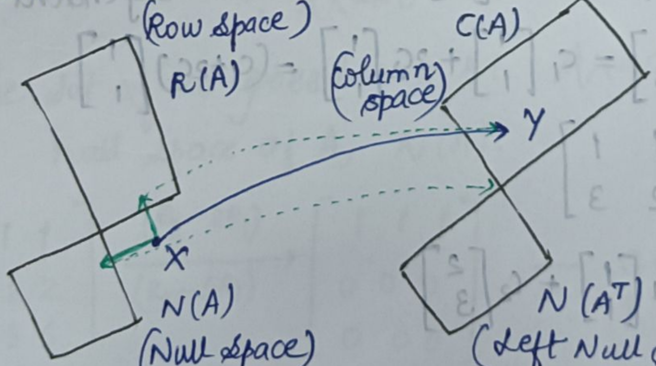
$$a_1x_1 + a_2x_2 + a_3x_3 = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \neq 0 \text{ for any } x.$$

$\Downarrow$

a_1, a_2, a_3 are independent.

$\rightarrow Y = AX$

$x \in \mathbb{R}^{N\times 1}$ $\qquad Y \in \mathbb{R}^{M\times 1}$

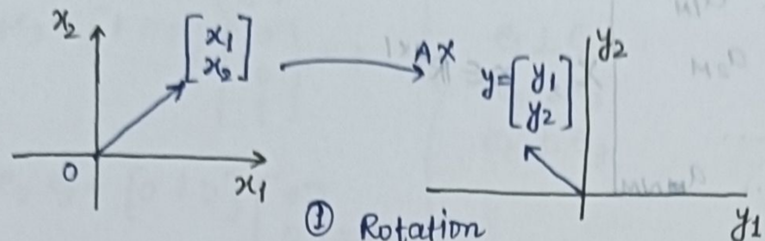


$x = x_R + x_{R\perp}$ $\qquad \left[R(A) \perp N(A)\right]$

$Ax = Ax_R + \underbrace{Ax_{R\perp}}_{0}$, $x_{R\perp}$: Null vector

09-08-2024

$Y = AX$



① Rotation
② scaling

$$A = \begin{bmatrix} a_{1r} \\ a_{2r} \\ \vdots \\ a_{Mr} \end{bmatrix} \in \mathbb{R}^{M \times N}$$

$$a_{1r}^T = \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1N-1} \\ a_N \end{bmatrix} \in \mathbb{R}^{N \times 1} \longrightarrow$$ M rows of $N \times 1$ vectors that form the rows of A.

$\Updownarrow$

<u>$M = N$</u>

will these set of N row vectors span $\mathbb{R}^{N \times 1}$?

$\Downarrow$

Can any arbitrary $X \in \mathbb{R}^{N \times 1}$ be represented as a linear combination of the row vectors of A?

Eg. Can any $x \in \mathbb{R}^{2 \times 1}$

Case-①: $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

$$x = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 2 \end{bmatrix} \rightarrow \text{linear dependent}$$

$$x_1 = \begin{bmatrix} 4 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (c_1 + 2c_2) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

<u>Case-2:</u> $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$

$$x = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

→ If the row vectors of A are linearly independent, then $Span(R(A)) = \mathbb{R}^{N\times 1}$

If rows of A are linearly dependent, then R(A) do not span $\mathbb{R}^{N\times 1}$.

Spanned by rows of A → Rowspace of A = R(A) (subspace of $\mathbb{R}^{N\times 1}$)

Not spanned by rows of A

M × N Matrix

M<N

M { (rows) } M indep. rows = M indep. columns = Rank(A)

N → M rows of A will not span $\mathbb{R}^{N\times 1}$

Consider an x ($x \neq 0$), such that

$$Y = Ax = 0$$

$$\begin{bmatrix} a_{1r} \\ a_{2r} \\ \vdots \\ a_{Mr} \end{bmatrix} x = 0 \Rightarrow \begin{array}{l} x \perp \{a_{1r}, a_{2r}, \ldots, a_{Mr}\} \\ x \perp L\{a_{1r}, \ldots, a_{Mr}\} \\ \underbrace{x \perp R\{A\}}_{\text{Any vector } x \perp R\{A\}} \end{array}$$

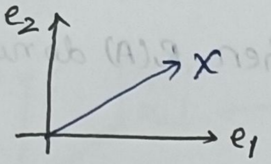
Null vector $\because Ax \rightarrow 0$

The set of all possible x s.t. $Ax = 0$ form the null space of A, N(A).

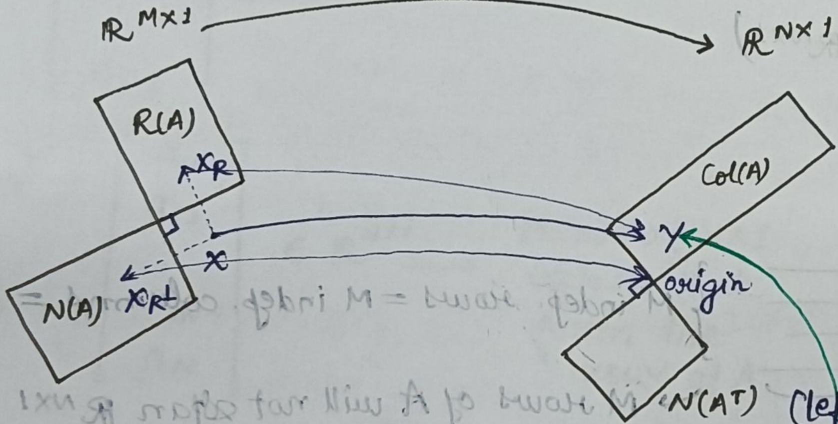
Eg. $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 5 & 5 & 5 \end{bmatrix} \xrightarrow[(R_3 - 5R_1)]{(R_2 - 2R_1)} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \underset{\substack{\uparrow \\ 0}}{c_1} a_{1r} + c_2 a_{2r} + c_3 a_{3r}$

⇒ Rank = 1

→ $Y = AX$, $X \in \mathbb{R}^{M\times 1}$



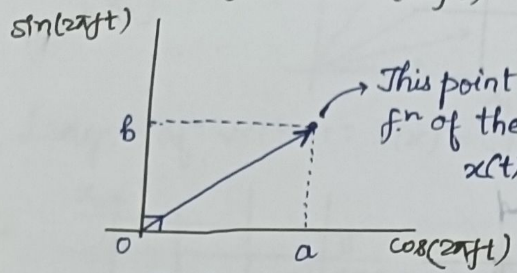
Any vector $x \in \mathbb{R}^{M\times 1}$ can be resolved into two vectors, x_R & $x_{R\perp}$,
where x_R lies in $R(A)$,
and $x_{R\perp}$ lies in $N(A)$,
as $R(A) \perp N(A)$.



(left null space of A)

$$\underbrace{Y = a_{1c}x_1 + a_{2c}x_2 + \dots + a_{Nc}x_N}_{\text{a linear comb. of col(A)}}$$

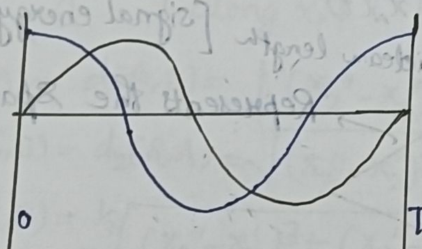
$x(t) = a\cos(2\pi f t) + b\sin(2\pi f t)$

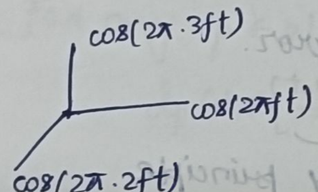


This point represents the f^n of the form
$x(t) = a\cos(\omega_0 t) + b\sin(\omega_0 t)$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\int_0^T \cos(2\pi f t)\sin(2\pi f t)\, dt$$





$$\int_0^T \cos(2\pi m f t).\cos(2\pi n f t)\, dt = 0, \; m \neq n$$

$\cos(2\pi m f t) \perp \cos(2\pi n f t)$ (over $0 \leq t \leq T$)

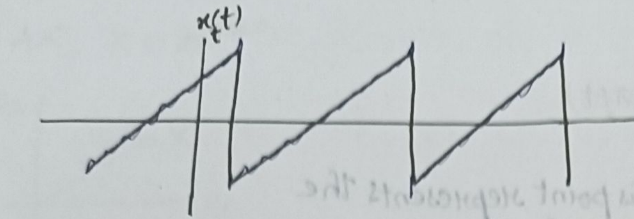
Basis f^ns : $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ | $\cos(n\omega_0 t)$ $\forall$ n : integer

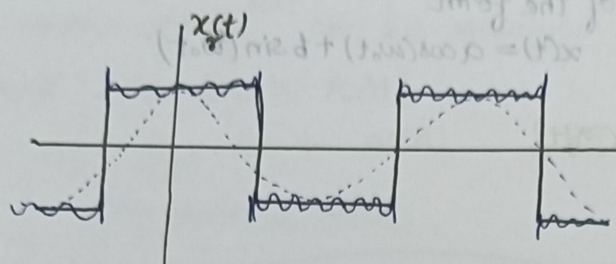
$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \right\}$$

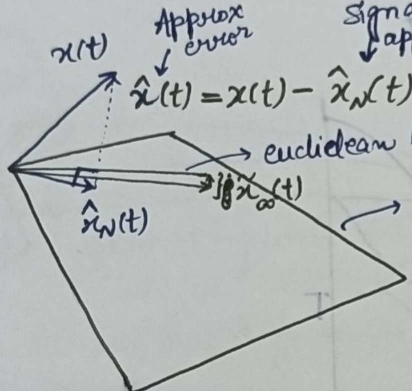
$$= \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \quad \Big| \quad e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$x(t)$ is a point in the infinite dimension vectors whose basis f^ns are harmonically related complex exponentials.

$$C_n = \frac{1}{T}\int_t^{t+T} x(t)\, e^{-jn\omega_0 t}\, dt$$







→ euclidean length [signal energy $= 0$ even if signal $\neq 0$]

→ Represents the span of $\{e^{jn\omega_0 t}\}$ $\forall$ n : integers

Pythogorean theorem : The approx error.

$$\underbrace{\tilde{x}_N(t) \perp \mathcal{L}\{e^{jn\omega_0 t}\}}_{\text{The orthogonality principle}}$$

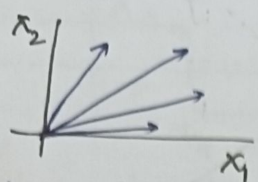
1. Norms — vectors/fn's
2. correlation

As $N \to \infty$, $x(t)$ gradually gets closer to the vector space, however for certain functions, it shall remains outside the v.s. The inevitable error in the approximation $\hat{x}_N(t)$, as $N \to \infty$ shows up as the Gibbs phenomenon.

12-08-2024

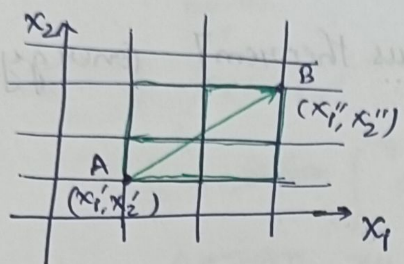
Norms

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$



Length of vector: $l(x) = \sqrt{x_1^2 + x_2^2}$

Eg.



money $\propto \left| \begin{matrix} \text{distance} \\ \text{along } x_1 \end{matrix} \right| + \left| \begin{matrix} \text{distance} \\ \text{along } x_2 \end{matrix} \right|$

$d_1(B,A) = d(A,B) = |(x_2'' - x_2')| + |(x_1'' - x_1')| \rightarrow l_1(x) = |x_1| + |x_2|$

$d_2(A,B) = d_2(B,A) = \sqrt{(x_2'' - x_2')^2 + (x_1'' - x_1')^2}$

$d_3(A,B) = \sqrt[1/3]{(x_2'' - x_2')^3 + (x_1'' - x_1')^3}$

For any $p \in N$, $p \geq 1$

$d_p(A,B) = \sqrt[1/p]{(x_2'' - x_2')^p + (x_1'' - x_1')^p}$

Distance measures: Different ways

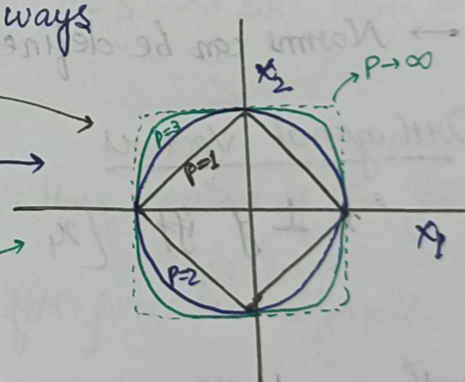
$l_1(x) = |x_1| + |x_2|$

$l_2(x) = (|x_1|^2 + |x_2|^2)^{1/2}$

$l_3(x) = (|x_1|^3 + |x_2|^3)^{1/3}$

$\vdots$

$l_p(x) = (|x_1|^p + |x_2|^p)^{1/p}$

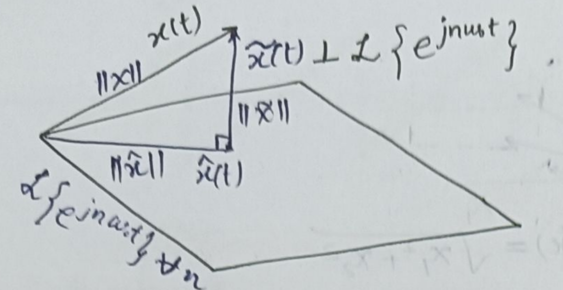


$f(x) = \|x\| \longrightarrow \alpha, \ \alpha \in R^+ =$ Norm

iff $\|x+y\| \leq \|x\| + \|y\|$ (Triangle inequality)

$\|\alpha x\| = |\alpha| \, \|x\|$, α: real quantity

$\|x\| \geq 0$



$\|x\|_2^2 = \|\tilde{x}\|_2^2 + \|\hat{x}\|_2^2$ [Pythagorus theorem] (energy)

As $n \to \infty$

$\|\tilde{x}_N\|_2^2 \to 0$,

$\|x\|_2^2 = \|\hat{x}\|_2^2$

$$\int_t^{t+T} |x(t)|^2 \, dt = \sum_{n=0}^{\infty} |a_n|^2 + |b_n|^2$$

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \xrightarrow{N \to \infty} x(t) \quad \begin{cases} l_1(x(t)) = \int_t |x(t)| \, dt \\ l_2(x(t)) = \sqrt{\int_t |x(t)|^2 dt} \\ \vdots \\ l_p(x(t)) = \left[\int_t |x(t)|^p dt\right]^{1/p} \end{cases}$$

Same defn holds for infinite dimension vector

$$l_p = \left(\sum_{i=1}^{N} |x_i|^p\right)^{1/p}$$

→ Norms can be defined for matrices also.

Orthogonal vectors

$x \perp y$ iff $\begin{bmatrix} x_1 & \dots & x_N \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} = 0$

Orthogonal Signals

$x(t) \perp y(t)$ iff $\int_t x(t) y(t) \, dt = 0$

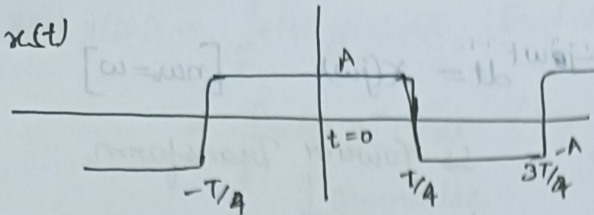
Notes: Two things are orthogonal iff their inner product $=0$.

→ Two random variables $E[x, y] = 0$ iff

$$\int_X \int_Y xy \, f_{xy}(x, y) \, dx \, dy = 0.$$

19-08-2024

Q] $x(t)$



$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{j\omega_0 n t}$$

$$C_n = \frac{1}{T}\int_{t}^{t+T} x(t)\, e^{-j\omega_0 n t}\, dt$$

$$= \frac{1}{T} A \cancel{e^{-j\omega_0 n}} \int_{-T/4}^{T/4} dt \cdot e^{-j\omega_0 n t}$$

$$= \frac{A \cancel{e^{-j\omega_0 n}}}{T} \cancel{\tfrac{T/2}{4}} \left.\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\right|_{-T/4}^{T/4}$$

$$= \frac{A\cancel{e^{-j\omega_0 n}}}{T} \cdot \frac{1}{(-jn\omega_0)}\left[e^{-jn\omega_0 T/4} - e^{jn\omega_0 T/4}\right] = \frac{2A\cancel{e^{-j\omega_0 n}}}{n\omega_0 T} \sin(n\omega_0 T/4)$$

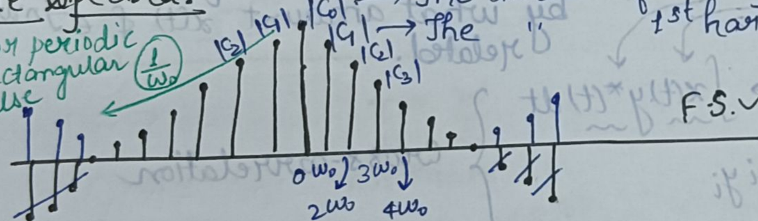
$$\therefore\ x(t) = \frac{2A}{\omega_0 T}\sum_{n=-\infty}^{\infty} e^{-jn\omega_0 t} \cdot \sin\left(n\omega_0 \frac{T}{4}\right)$$

Line Spectra:

$|C_0| \rightarrow$ The amount of DC component in the signal (periodic)

$|C_1| \rightarrow$ The " 1st harmonic "

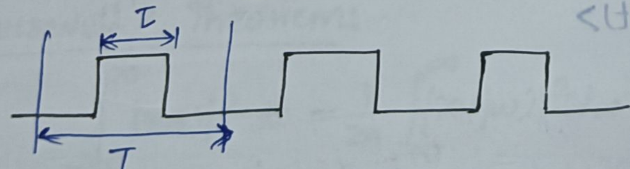
for periodic rectangular pulse



F.S. ⇝ like mass function

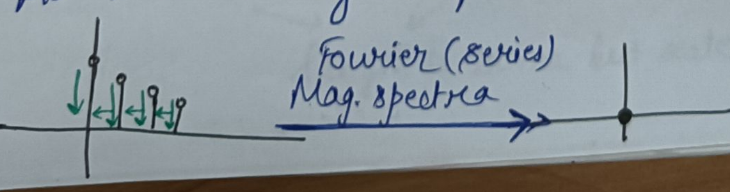
$\omega_0 = 2\pi f_0 = \frac{2\pi}{T}$, T: time period of the periodic signal.

→ You have only line spectra for periodic signals.



* τ remains constant.
* T increases.

As $T \rightarrow \infty \Rightarrow$ only one pulse remains.

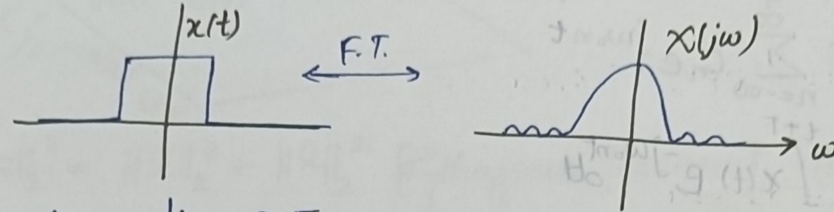


(Everything vanishes)

As $T \to \infty$, $C_n \to 0$

$$\lim_{T\to\infty} C_n T = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = X(j\omega) \qquad [n\omega_0 = \omega]$$

↳ Fourier Transform



$$X(j\omega) = \lim_{T\to\infty} C_n T$$

$$= \lim_{\substack{T\to\infty \\ \text{or} \\ \omega_0 \to 0}} \left(\frac{C_n}{\omega_0}\right) 2\pi$$

↳ a density function

→ There is zero amount of all frequency components contained in a pulse-type signal.

22-08-24

$$\omega_n = n\omega_0 \longrightarrow |C_n|$$

$$C_n = \frac{1}{T}\int_t^{t+T} x(t)\, e^{-jn\omega_0 t}\, dt$$: cross correlation formula that says by what amount $x(t)$ & $e^{-jn\omega_0 t}$ are related.

$$\langle x(t), y(t)\rangle \longrightarrow \int x(t)\, y^*(t)\, dt$$

$$\langle x, y\rangle = \sum_{i=1}^{N} x_i y_i$$

} cross-correlation

Given an arbitrary signal, $x(t)$, you need to find how much of a given template, $y(t)$, signal is contained in it.

$$\Downarrow$$

$$\langle x(t), y(t)\rangle$$

23-08-2024

$\langle x(t), y(t) \rangle = \int_t x(t)\, y(t)\, dt$ (Real signals)

$= \int_t x(t)\, y^*(t)\, dt$ (For complex signals)

} Cross-correlation b/w $x(t), y(t)$

↓ generalise

$\gamma_x(\tau) = \int_t x(t)\, x(t-\tau)\, dt$ → time-shift variable } Auto-correlation of lag τ

$\gamma_x(\tau) = \gamma_x(-\tau)$ [Even f^n]

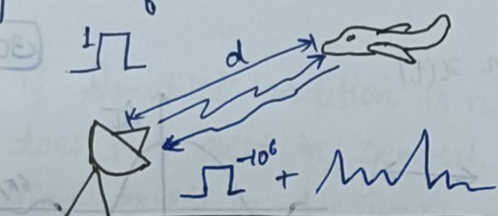
$\gamma_x(0) \geq \gamma_x(\tau)$

$\gamma_x(\tau) \xrightarrow[\tau \to \infty]{} 0$

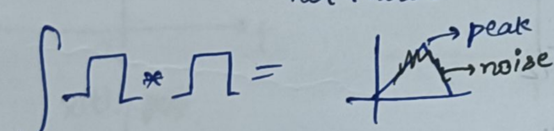
$\gamma_{xy}(\tau) = \int x(t)\, y(t-\tau)\, dt$ } Cross-correlation of lag τ

Cross-correlation + Orthogonality properties are the pillars of DSP.

Eg.



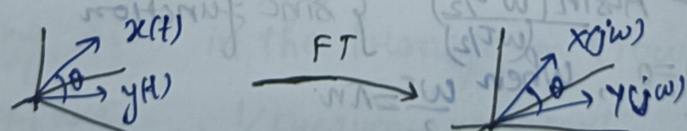
$\tau = \frac{2d}{c}$
↓
not known



Energy → 2nd order norm (corresponds to Euclidean dist.)

Parsevell's Theorem:

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2\, d\omega$$



length is same but scaled. ⇒ Information is preserved.

$$\int_{-\infty}^{\infty} x(t)\,y^*(t)\,dt = \frac{1}{2\pi}\int X(j\omega)\,Y^*(j\omega)\,d\omega$$ $\Rightarrow$ Correlation is ~~preserved~~ same on F.T.

$\Downarrow$

F.T. does not disturb correlation

$\Downarrow$

Angle is preserved.

Fourier matrix (F.T. put in the matrix) is orthogonal matrix

$\Downarrow$

Rows & columns are $\perp^r$.

Unitary matrix $\rightarrow$ extension of orthogonal matrix to complex.

$$X = Y = UX$$

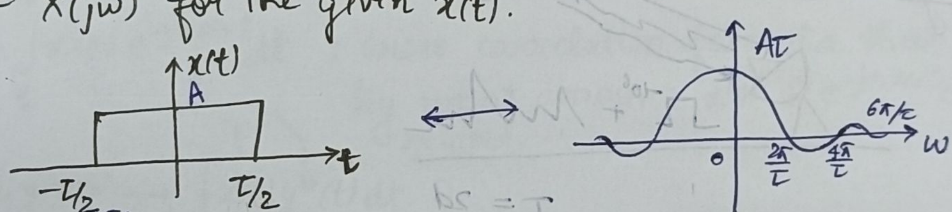
$$\underbrace{\|X\|_2^2 = \|Y\|_2^2}$$ when U is unitary/orthogonal.

Energy/length is preserved in a unitary transform.

$\therefore$ F.T. is a unitary transform.

30-08-2024

Eg. Compute $X(j\omega)$ for the given $x(t)$.



Soln: $$X(j\omega) = \int_{-\tau/2}^{\tau/2} A e^{-j\omega t}\,dt$$

$$= A \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-\tau/2}^{\tau/2}$$

$$= \frac{A}{j\omega}\left[e^{j\omega\tau/2} - e^{-j\omega\tau/2}\right]$$

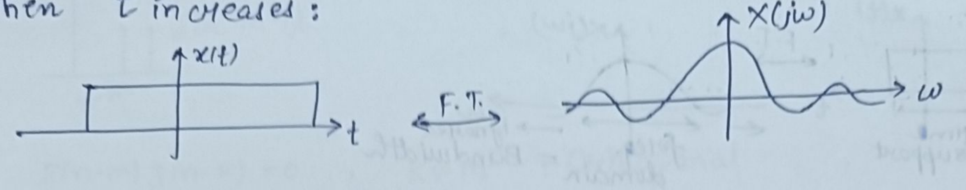
$$= \frac{2A\sin(\omega\tau/2)}{\omega}$$

$$= A\tau \frac{\sin(\omega\tau/2)}{(\omega\tau/2)} \Big\}\ \text{sinc function}$$

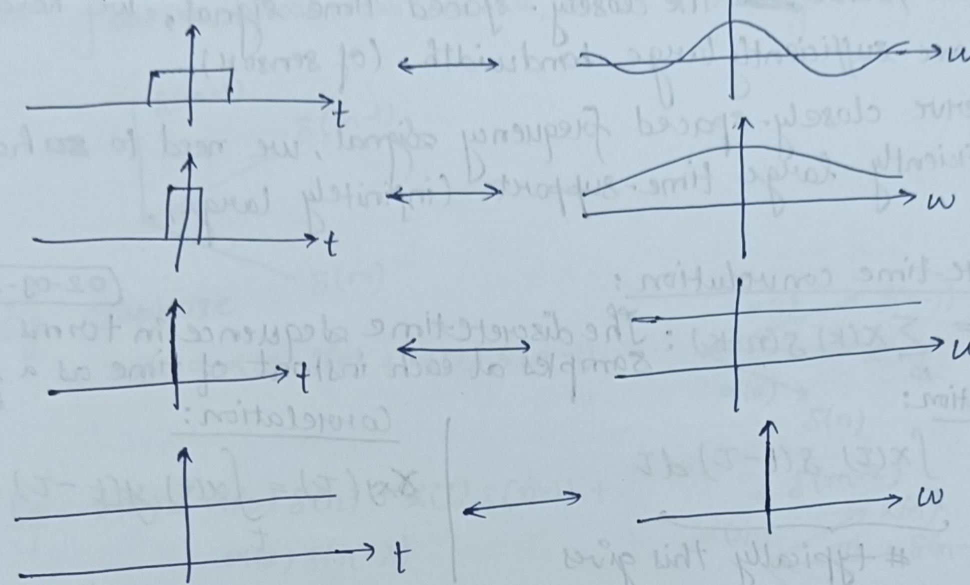
$\sin(\omega\tau/2) = 0$, when $\frac{\omega\tau}{2} = \pi n$

$$\Rightarrow \omega = \frac{2n\pi}{\tau}$$

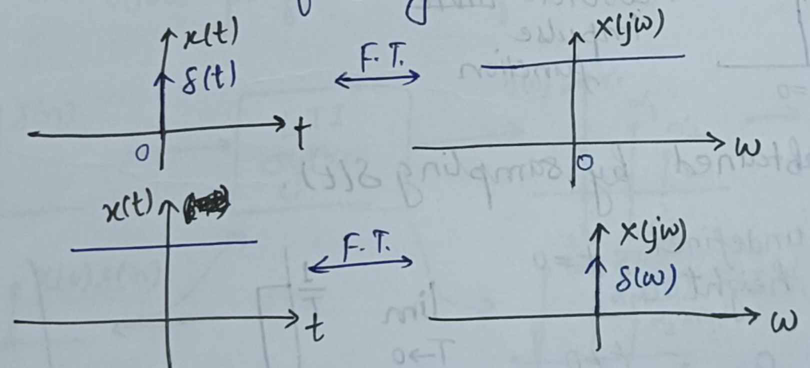
When τ increases :



τ smaller :



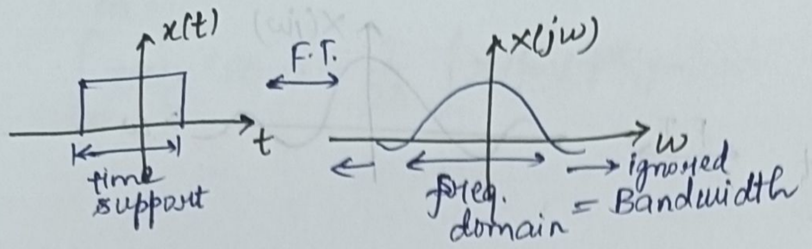
→ Even if Dirichlet condition is not satisfied, Fourier transform does not exist in general, but it may exist provided delta function exists in frequency domain.



When the signal is not absolutely integrable (violating Dirichlet's condition), still F.T. can be computed by including $\delta(\omega)$ in the transform (Fourier) domain

→ Time support of the signal $\propto$ 1/ Frequency support of the signal.

→ We cannot have small time & small freq. support for a signal simultaneously.



To observe/sense the closely-spaced time signal, we need to have sufficiently large bandwidth (of sensor).

To observe closely-spaced frequency signal, we need to have sufficiently large time-support (infinitely large).

<u>Discrete-time convolution:</u> $\boxed{02\text{-}09\text{-}2024}$

$x(n) = \sum_{k} x(k)\,\delta(n-k)$: The discrete-time sequence in terms of its samples at each instant of time as a linear sum.

<u>convolution:</u>

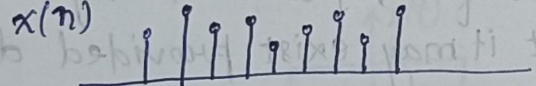
$x(t) = \int_{\tau} x(\tau)\,\delta(t-\tau)\,d\tau$

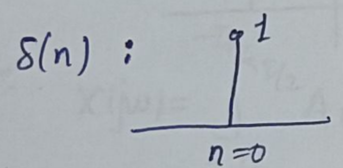
$x(t)$ ↑ value of the function

typically this gives an area

<u>Correlation:</u>

$\gamma_{xy}(\tau) = \int_{t} x(t)\,y(t-\tau)\,dt$

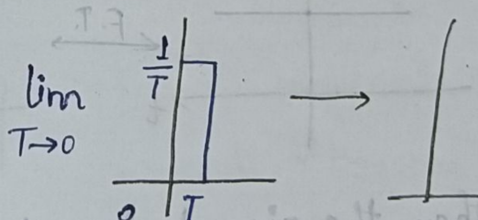


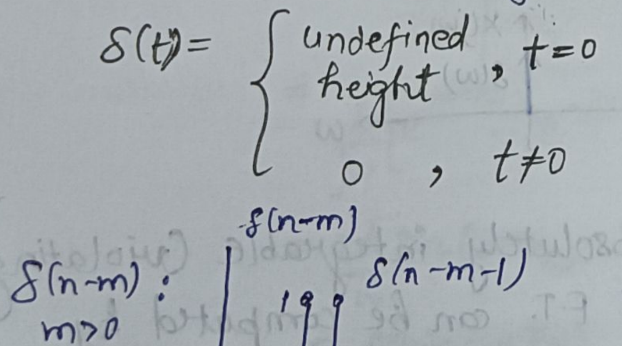


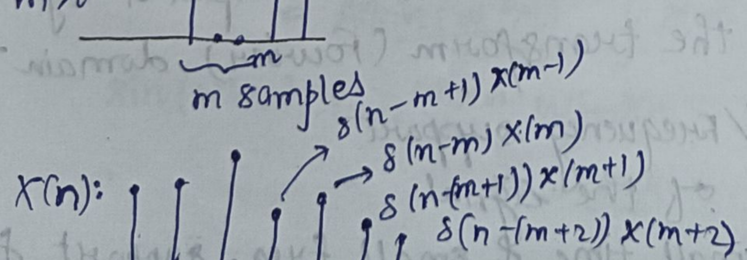
: discrete-~~time~~ impulse function

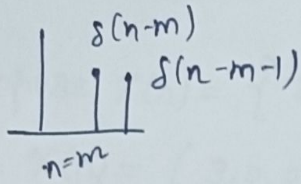
→ $\delta(n)$ is not obtained by sampling $\delta(t)$.

$$\delta(t) = \begin{cases} \text{undefined height}, & t=0 \\ 0, & t \neq 0 \end{cases}$$



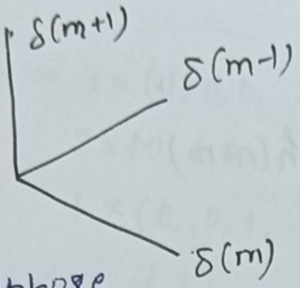






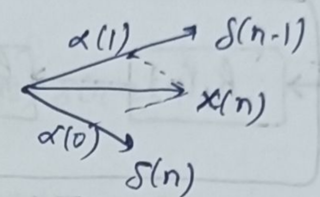
$\delta(n-m)\,\delta(n-K) = 0$, $K \neq m$: Orthogonal

$\int \cos(n\omega_0 t)\cos(m\omega_0 t)\,dt = 0$, $n \neq m$: Orthogonal

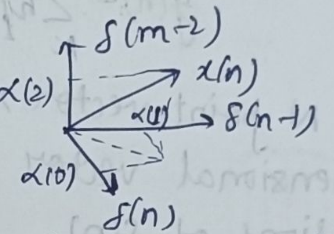


Eg. Suppose

$x(n) = \alpha(0)\,\delta(n) + \alpha(1)\,\delta(n-1)$

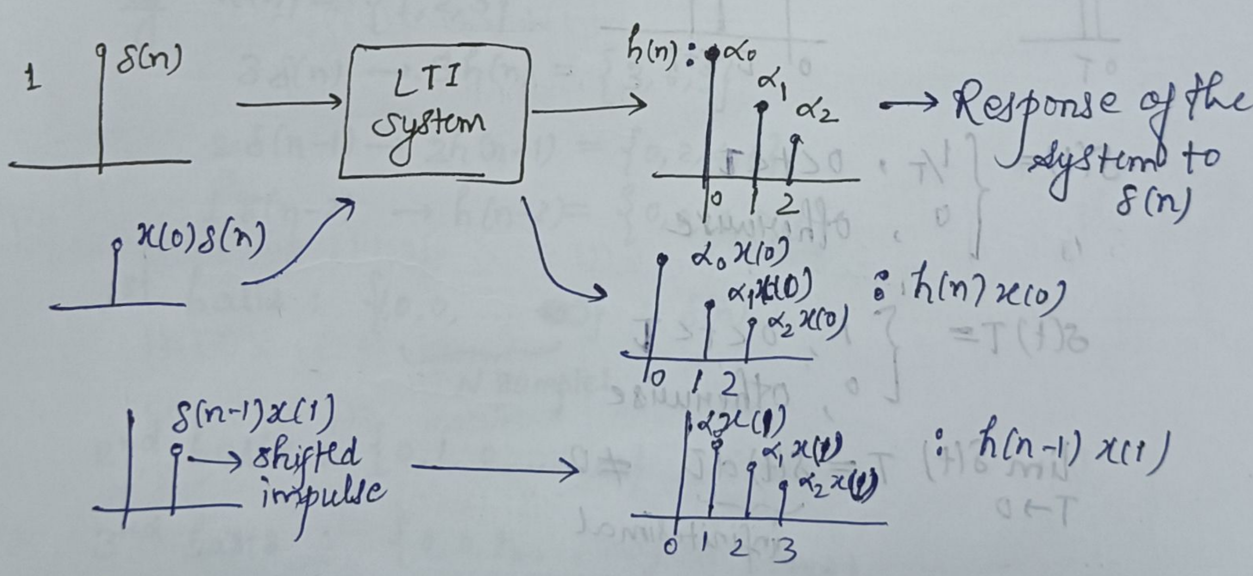


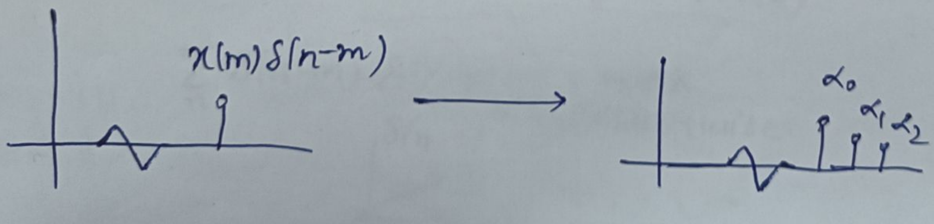
Eg. $x(n) = \alpha(0)\,\delta(n) + \alpha(1)\,\delta(n-1) + \alpha(2)\,\delta(n-2)$



Convolution: Gives the behaviour of an LTI system to any arbitrary input.

↳ Gives performance of the system.





Input signal	Output signal

individual values
$$\left\{\begin{aligned} &x(0)\,\delta(n) \longrightarrow x(0)\,h(n)\\ &x(1)\,\delta(n-1) \longrightarrow x(1)\,h(n-1)\\ &x(m)\,\delta(n-m) \longrightarrow x(m)\,h(n-m)\end{aligned}\right.$$

sequence

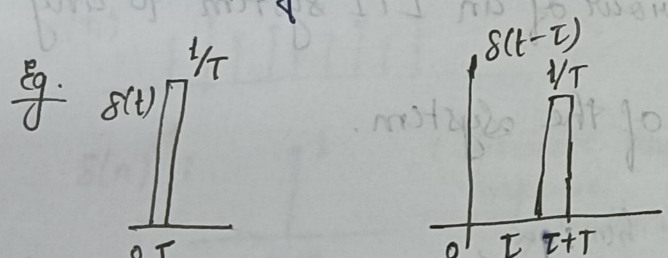
$$\underbrace{\sum_m x(m)\,\delta(n-m)}_{\text{conv. sum at i/p}} \longrightarrow \sum_m x(m)\,h(n-m)$$

i/p → $h_1(n)$ → $h_2(n)$ → o/p

$$g(n) = \sum_K h_1(K)\,h_2(n-K)$$

→ For N point vector x, ~~then~~ it can be represented in an N-dimensional vector space whose basis are the shifted versions of $\delta(n)$.

Eg.



$$\delta(t) = \begin{cases} 1/T, & 0<t<T \\ 0, & \text{otherwise}\end{cases}$$

$$\delta(t)\,T = \begin{cases} 1, & 0<t<T \\ 0, & \text{otherwise}\end{cases}$$

$$\lim_{T\to 0} \delta(t)\,T = \underbrace{\delta(t)\,d\tau}_{\text{infinitesimal}} \neq 0$$

06-09-2024

Suppose $x(n) = \{3, 2, 1, 0, 4, \dots\}$

$$
\begin{aligned}
x(n) = &\ (3, 0, 0, \dots, 0) + \\
&(0, 2, 0, \dots, 0) + \\
&(0, 0, 1, \dots, 0) + \\
&(0, 0, 0, 0, \dots, 0) + \\
&\ \vdots \\
= &\ 3 \times (1, 0, 0, \dots, 0) + \\
&2 \times (0, 1, 0, \dots, 0) + \\
&1 \times (0, 0, 1, \dots, 0) + \\
&\ \vdots \\
= &\ x(0)\delta(n) + x(1)\delta(n-1) + x(2)\delta(n-2) + \dots
\end{aligned}
$$

$$\therefore \boxed{x(n) = \sum_k x(k)\,\delta(n-k)}$$

$\delta(n) \longrightarrow h(n)$

$\alpha\delta(n) \longrightarrow \alpha h(n)$

$\delta(n-m) \longrightarrow h(n-m)$

$$y(n) = \sum_k x(k)\,h(n-k)$$

↳ Output sequence

Eg. $h(n) = \{1, 2, 3\}$

$3\delta(n) \rightarrow 3h(n) = \{3, 6, 9\}$

$2\delta(n-1) \rightarrow 2h(n-1) = \{0, 2, 4, 6\}$

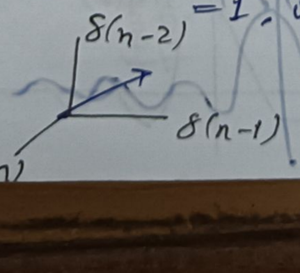
$1.\delta(n-2) \rightarrow h(n-2) = \{0, 0, 1, 2, 3\}$

1st basis : $\{\underbrace{1, 0, 0, \dots \infty}_{N \text{ samples}}\} = \delta(n)$

2nd basis : $\{0, 1, 0, \dots \infty\} = \delta(n-1)$

3rd basis : $\{0, 0, 1, \dots \infty\} = \delta(n-2)$

$$\sum_n \delta(n-m)\,\delta(n-k) = 0, \quad m \neq k$$
$$= 1, \text{ otherwise.}$$



$$x(n) = \sum \left(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right)$$

$$x(n) = \sum_{k} x(k)\,\delta(n-k)$$
$$x(t) = \sum_{n} C_n e^{-jn\omega_0 t}$$
} vector space Representation

$$C_n = \frac{1}{T}\int x(t)\, e^{jn\omega_0 t}\, dt$$

↑

FC : cross-correlation b/w (Fourier coeff.) signals & corresponding Basis f^n

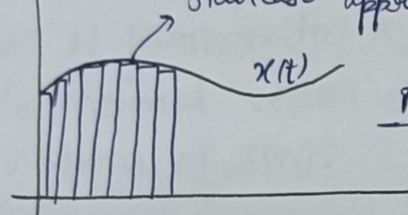
$$x(k) = \sum_{n} x(n)\,\delta(n-k) \quad \ldots \text{ FS formula}$$

$$x(n) = \{3, 2, 1, 0, 4, \ldots\}$$
↑ 5th value

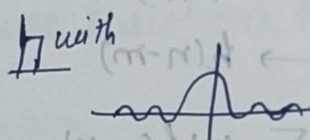
$x(k=5) = ?$

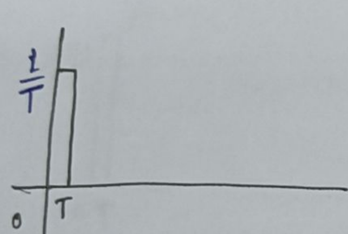
$x(n)\,\delta(n-5) = x(5)\,\delta(0) = 4.$

staircase approx. is $\hat{x}(t)$

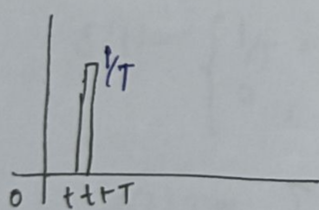


Replace → ⊓ with





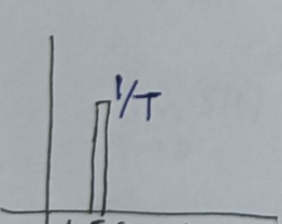
$$\hat{x}(t) = \{\, \text{rect}(t) \times x(0T) + \text{rect}(t-T).\, x(T) + \text{rect}(t-2T).\, x(2T) + \ldots \}\, T$$

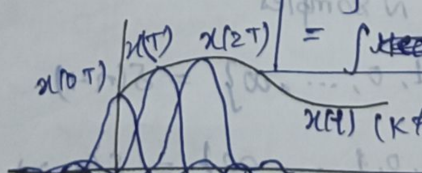


F.C.

F.S. expansion formula

$$\lim_{T\to 0} \hat{x}(t) = \sum_{k}^{\infty} \text{rect}(t-kT)\, x(kT)\, T$$
$$= \int \text{rect}(t-\tau)\, x(\tau)\, dt$$
$$= \int \delta(t-\tau)\, x(\tau)\, dt$$



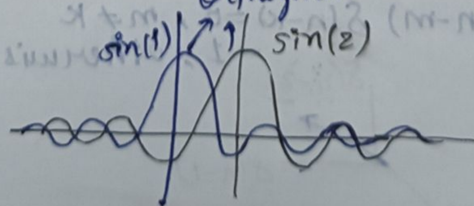


$(k \neq n)$ | rect $(t-kT) \perp$ rect $(t-nT)$

$$x(t) = \sum x(kT)\, \frac{\sin(t-kT)}{(t-kT)}$$

orthogonal Basis f^n

orthogonal



$$\hat{x}(t) = \sum C_n \phi_k(t)$$

Prob. : $\sum_t |x(t) - \hat{x}(t)|^2$

[ϕ_k: basis f^n]

$\phi_k(t) = e^{-jk\omega_0 t}$

Fourier Series

$$x(t) = \sum_n C_n e^{-jn\omega_0 t}$$

$$C_n = \frac{1}{T}\int x(t)\, e^{jn\omega_0 t}\, dt$$

$T \to \infty$

$\phi_k(t) = e^{j\omega t}$

Fourier Transform

$$X(j\Omega) = \int x(t)\, e^{-j\Omega t}\, dt$$

$$x(t) = \int X(j\Omega)\, e^{j\Omega t}\, d\Omega$$

$\sigma = 0$

$\phi(t) = e^{j s t}$

$$X(s) = \int x(t)\, e^{-st}\, dt$$

$x(t)$: Inversion method using contour integrals

$\phi_k(t) = \delta(n-k)$

$$x(n) = \sum x(k)\,\delta(n-k)$$

$$x(k) = \sum_n x(n)\,\delta(n-k)$$

$$\lim_{T \to 0} x(t) = \sum x(kT)\, \mathrm{rect}(t-nT) = \int x(\tau)\,\delta(t-\tau)\, d\tau$$

→ Staircase approx.

CT convolution

$$x(t) = \sum_k x(k)\, \mathrm{sinc}(t-kT)$$

$$(\overline{\overline{k}}\, x(kT))$$

→ Shannon Nyquist sampling Theorem

09-09-2024

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

If $x(n) = A\cos(\omega_0 n) \longrightarrow Ae^{j\omega_0 n}$

$$y(n) = \sum_k x(k)\, h(n-k)$$
$$= \sum_k h(k)\, x(n-k)$$
$$= \sum h(k)\, e^{j\omega_0 (n-k)}$$
$$= \left[\sum_k h(k) e^{-j\omega_0 k}\right] e^{j\omega_0 n}$$

$$\boxed{y(n) = \underbrace{H(e^{j\omega})\Big|_{\omega=\omega_0}}_{\text{Transfer } f^n \text{ evaluated at } \omega=\omega_0.} e^{j\omega_0 n} = H(e^{j\omega_0})\, x(n)}$$

x is an eigenvector of H

$$Hx = \underbrace{\lambda}_{\text{scalar}} x$$

(x is not rotated / only scale by a scalar)

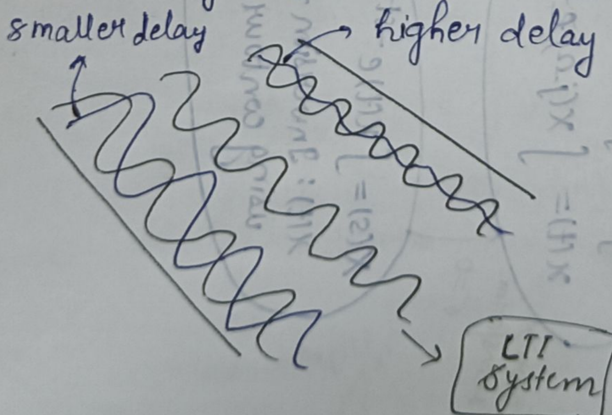
→ LTI systems do not change the frequency of the i/p signal.

→ $e^{j\omega n}$ is called an eigen function of LTI systems.

→ Since $H(e^{j\omega_0}) \in \mathbb{C}$,

$$y(n) = \underbrace{|H(e^{j\omega_0})|}_{\text{amplitude scaling by}} e^{(j\omega_0 n + \overset{\uparrow \text{ a phase shift}}{\angle H(e^{j\omega_0})})}$$

→ F.S. / F.T. is important because it deals with eigen functions as i/p to LTI systems.



→ If a composite signal which is composed of sinusoids of different frequencies, each of the components undergo different:

(i) amplification, and
(ii) suffer different amounts of delay.

Eg. ① $h(n) = \delta(n)$
$$H(e^{j\omega}) = 1 \;\forall\, \omega$$

② $h(n) = \delta(n-m)$
$$H(e^{j\omega}) = e^{-j\omega m}$$
→ Mag. $= 1 \;\forall\, \omega$
→ $\measuredangle = -\omega m \Rightarrow$ Linear fn of ω

Suppose
$$x(n) = r^n e^{j\omega_0 n}$$

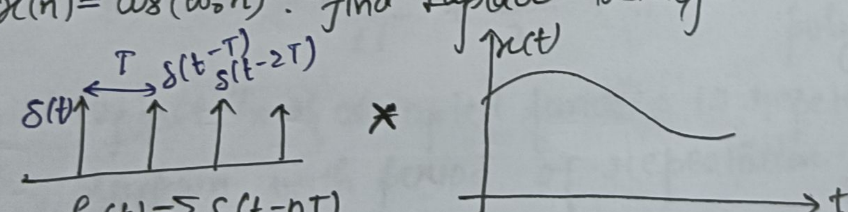
$$y(n) = \sum x(n-k)\,h(k)$$
$$= \sum_k h(k)\, r^{(n-k)}\, e^{j\omega_0 (n-k)}$$

$$y(n) = \left[\sum h(k)\, r^{-k} e^{-j\omega_0 k}\right] r^n e^{j\omega_0 n}$$
$$= H(z)\Big|_{z = re^{j\omega_0}} \cdot z_0$$
where, $z_0 = re^{j\omega_0}$

→ z (of z-transform) is also an eigen fn of DT LTI Systems.

Eg. $x(n) = \cos(\omega_0 n)$. Find Laplace transform

\#



$$P_T(t) = \sum \delta(t - nT)$$

$$x(nT) = x(t) \cdot \sum_n \delta(t - nT)$$
(sampled signal)

$$x(t) = \cos(\omega_0 t) \sum_n \delta(t - nT)$$

$$X(s) = \int_{t=0}^{\infty} \cos(\omega_0 t) \sum_n \delta(t - nT)\, e^{-st}\, dt$$

$$= \int_0^\infty \left(\frac{e^{\omega_0 t} + e^{-\omega_0 t}}{2}\right) e^{-st} \sum_n \delta(t-nT).\, dt$$

$$= \frac{1}{2} \int_0^\infty \left(e^{-(s-\omega_0)t} + e^{-(s+\omega_0)t}\right) \sum_n \delta(t-nT)\, dt$$

$$= \frac{1}{2} \int_0^\infty \left[\sum_n e^{-(s-\omega_0)(t-nT)} + e^{-(s+\omega_0)(t-nT)}\right] dt$$

$$= \sum_{n=0}^{\infty} \cos(n\omega_0 T)\, e^{-snT}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} e^{(j\omega_0 - s)nT} + \frac{1}{2} \sum_{n=0}^{\infty} e^{-(j\omega_0 + s)nT}$$

12-09-2024

Analog Signal

$x(t) = \cos(\omega_0 t)$

$X(s) = \dfrac{s}{s^2+\omega_0^2}$

↳ * Ratio of polynomials in 's'

* Rational function
* The roots of these Nr/Dr polynomials can be used to analyse the signal/systems stability and design filters/controllers, etc.

Zeros : $s=0$

Poles : $s = \pm j\omega_0$

Sampled Analog Signal

$x_s(t) = x(t) \sum_n \delta(t-nT)$

$x(t) = \cos(\omega_0 t), \; t \geq 0$

$x(nT) \equiv x(n) = \cos(\omega_0 n T)$

T: sampling duration

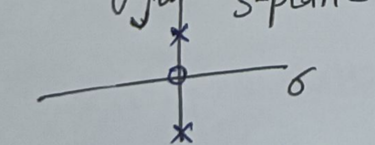
$$X_s(s) = \frac{1 - e^{-sT}\cos(\omega_0 T)}{1-2e^{-sT}\cos(\omega_0 T) + e^{-2sT}}$$

* Ratio containing terms e^{-st}

$$e^{-sT} = 1 - \frac{sT}{1!} + \frac{s^2T^2}{2!} + \ldots \equiv \infty\text{-order polynomial}$$

* Laplace Tx of sampled function is repetitive in frequency domain with period of repeatation $\frac{2\pi}{T}$.

Eg. Analog signal



Pole-zero plot

Sampled Analog signal:

Suppose zero: $s = s_0 = \sigma_z + j\omega_z$

Pole: $p = p_0 = \sigma_p + j\omega_p$

$$X_s(s) = \frac{1 - e^{-(\sigma_z + j\omega_z)T} \cos(\omega_z T)}{1 - 2e^{-(\sigma_p + j\omega_p)T} \cos(\omega_0 T) + e^{-2(\sigma_p + j\omega_0)T}}$$

Suppose $s \to s + j2\pi/T$

$$= \frac{1 - e^{-(\sigma + j\omega + j\frac{2\pi}{T})T} \cos(\omega_0 T)}{1 - 2e^{-(\sigma_p + j\omega_p + j\frac{2\pi}{T})T} \cos(\omega_0 T) + e^{-2(\sigma_p + j\omega_p + j\frac{2\pi}{T})T}}$$

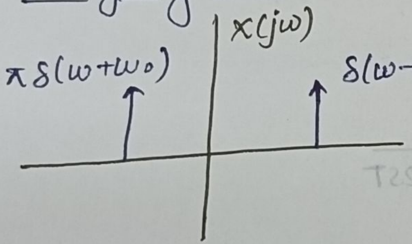
↳ Infinitely many repeating poles / zeros

[FT from LT]

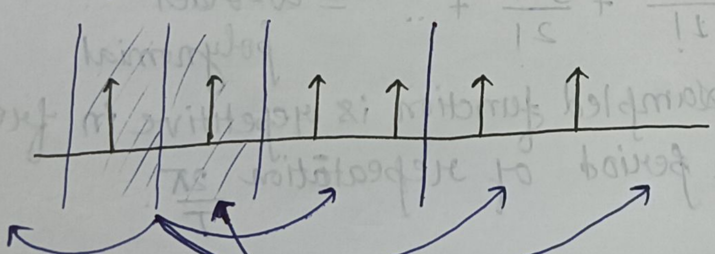
$\therefore X_s(s) = X_s(s + j\frac{2\pi}{T}k)$, k is integer.

$\therefore X_s(s)$ is periodic with period $\frac{2\pi}{T}$.

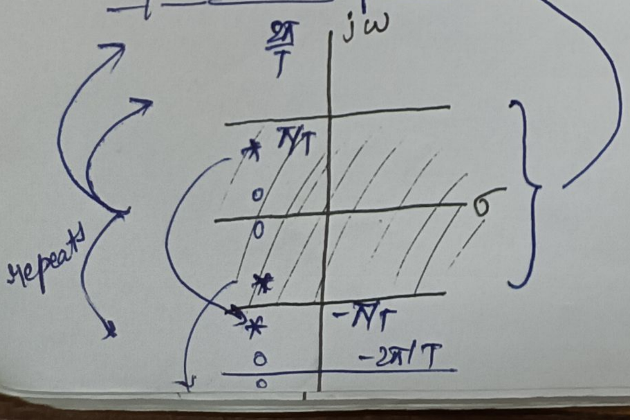
Eg. Analog signal



Sampled Analog signal

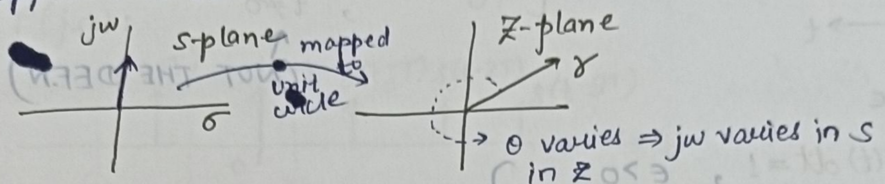


Spectrum repeats:



$$Z = e^{sT} = e^{(\sigma + j\omega)T} = e^{\sigma T} \cdot e^{j\omega T}$$
$$= r \cdot e^{j\theta}$$

① suppose σ is varied (in s)



② jw is varied
jw axis → mapped to unit circle

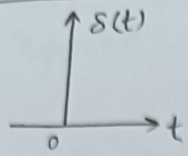
$$X_s(s) = \frac{1 - e^{-sT}\cos(\omega_0 T)}{1 - 2e^{-sT}\cos(\omega_0 T) + e^{-2sT}}$$

$$= \frac{1 - Z^{-1}\cos(\omega_0 T)}{1 - 2Z^{-1}\cos(\omega_0 T) + Z^{-2}}$$

↳ Ratios of polynomials in Z
⇓
Z-plane

19-09-2024

Delta Function:



$$\delta(t) = \begin{cases} \text{undefined}, & t=0 \\ 0, & t \neq 0 \end{cases}$$

(NOT THE DEF.N)

① $\int_{0-\epsilon}^{0+\epsilon} \delta(t)\,dt = 1 , \quad \epsilon > 0$

② $\int_{-\infty}^{\infty} \delta(t-\tau)\, g(t)\, dt = g(\tau)$

Apply these formulae

$\delta(t-\tau)g(t) = ?$

Sampling (Mathematically)

$$x_s(t) = x_a(t) \sum_n \delta(t-nT)$$

$$= x_a(t)\left[\ldots + \delta(t-2T) + \delta(t-T) + \delta(t) + \delta(t+T) + \ldots\right]$$

$$\int_t x_a(t) \sum_n \delta(t-nT)\, dt = \sum_n \int x_a(t)\delta(t-nT)\, dt$$

$$= \underbrace{\sum x_a(nT)}$$

Sum of the values of the sequel $x_a(t)$ sampled

Forever Undecided Book

$$X_s(t) = x_a(t) \sum_n \delta(t-nT)$$

$$X_s(j\Omega) = \int x_a(t) \sum_n \delta(t-nT)\, e^{-j\Omega t}\, dt$$

(CTFT of $x(t)$)

$$= \sum_n x_a(nT) e^{-j\Omega T n}$$

= DTFT of a discrete sequence

$$X(e^{j\omega}) = \sum_n x(n) e^{-j\omega n}$$

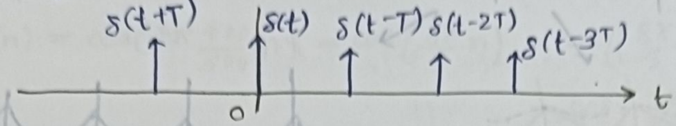
$$\boxed{\omega = nT}$$

T: sampling duration

$$x_s(t) \xleftrightarrow{FT} X_s(j\Omega) = X_a(j\Omega) * P(j\Omega)$$

where, $P(t) = \sum_n \delta(t-nT)$

$P(j\Omega)$:



FT of Periodic Signals:

① Compute F.S. expansion of $x(t)$.

$$x(t) = \sum_n C_n e^{-jn\Omega_0 t}$$

$$F.T.\left[x(t)\right] = FT\left[\sum C_n e^{-jn\Omega_0 t}\right]$$

$$= \sum_n C_n \, F.T.\left[e^{-jn\Omega_0 t}\right]$$

$$F.T.\left[e^{-jn\Omega_0 T}\right] = \delta(\omega - \Omega_0) \cdot 2\pi$$

$$\therefore \boxed{X(j\Omega) = \sum_n 2\pi C_n \, \delta(\omega - n\Omega_0)}$$

F.T. is computed using the F.S. coefficients.

For $P(j\Omega)$:

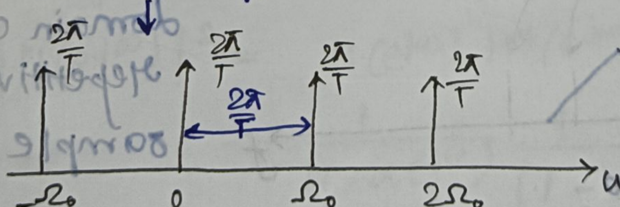
$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{jn\omega_0 t} dt = \frac{1}{T}$$

$$\therefore x(t) = \frac{1}{T}\sum_n e^{-jn\omega_0 t}$$

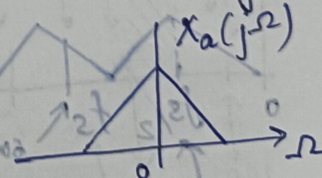
$$\downarrow F.T.$$

$$X(j\Omega) = \frac{2\pi}{T}\sum_n \delta(\Omega - n\Omega_0)$$

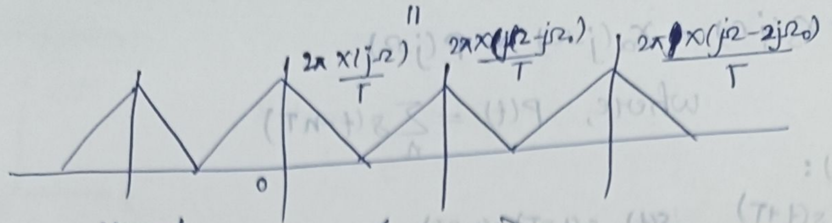
For impulse train

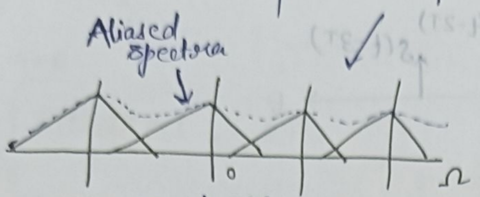


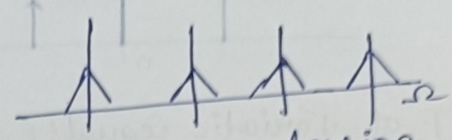
F.T. of analog signal



[F.T. of periodic discrete signal is an periodic discrete signal with period T]







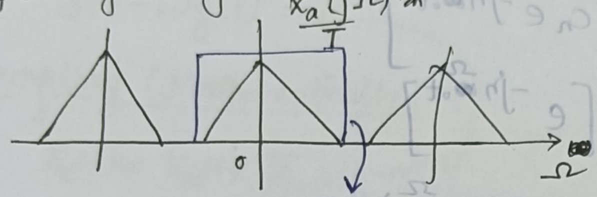
overlapping

Non-overlapping

⇓

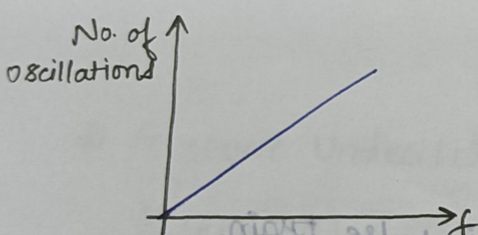
Aliasing

To get original signal:

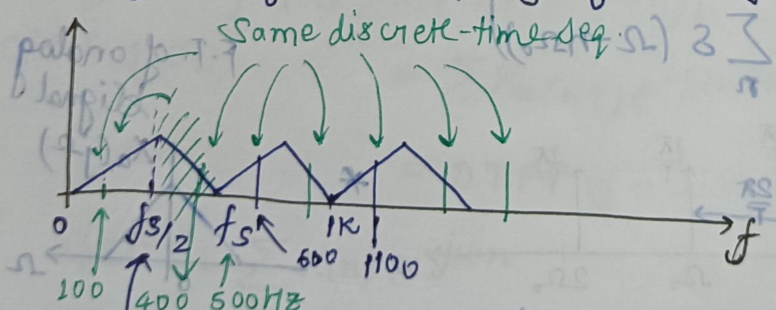


$$\boxed{\begin{aligned} X_a(t) &= X(nT) * \text{ideal LPF} \\ &= \sum X(nT) \cdot \text{sinc}(t - nT) \end{aligned}}$$

20-09-2024



For a given fixed sampling frequency, f_s,



→ The structure of freq. domain changes becomes repetitive when you sample the T.D. signal.

Max freq. that is ~~...~~ unambiguously representable

||

$x_1(t) = \cos(2\pi\, 100\, t) \longrightarrow$ sampled at 500 Hz

$x_1(n) = \cos\left(2\pi \frac{100}{500} n\right) = \cos\left(\frac{2\pi}{5} n\right)$

$x_2(t) = \cos(2\pi\, 400 t)$

$x_2(n) = \cos\left(2\pi \frac{400}{500} n\right) = \cos\left(\frac{8\pi}{5} n\right) = \cos\left(\left(\frac{8\pi}{5} - 2\pi\right) n\right)$

$= \cos\left(\frac{2\pi}{5} n\right)$

$$\left.\begin{array}{l} x_3(t) = \cos(2\pi\, 200\, t) \\ x_4(t) = \cos(2\pi\, 300\, t) \end{array}\right\} f_s = 500\ Hz$$

$x_5(t) = \cos(2\pi\, 600 t)$, $f_s = 500\ Hz$

$x_5(n) = \cos\left(2\pi \frac{600}{500} n\right) = \cos\left(\frac{12\pi n}{5}\right)$

$= \cos\left(\frac{12 n\pi}{5} - 2n\pi\right) = \cos\left(\frac{2n\pi}{5}\right)$

All sinusoids of frequencies

$$\underbrace{\begin{array}{l} f, f+f_s, f+2f_s, \dots, f+kf_s \quad \forall\ k \\ f-f_s, f-2f_s, \dots, f-kf_s \end{array}}_{\text{will result in the same DT sequence when sampled at } f_s.}$$

Considering the DT system/signal

For e.g., $y(n) = \alpha y(n-1) + x(n)$

$Y(e^{j\omega}) = \alpha$ $\quad [\alpha : \text{Real}]$

$\Rightarrow Y(e^{j\omega}) = \alpha Y(e^{j\omega}) e^{-j\omega} + X(e^{j\omega})$

$\Rightarrow Y(e^{j\omega})\left[1 - \alpha e^{-j\omega}\right] = X(e^{j\omega})$

$\Rightarrow H(e^{j\omega}) = \dfrac{Y(e^{j\omega})}{X(e^{j\omega})} = \dfrac{1}{1 - \alpha e^{-j\omega}}$

$|H(e^{j\omega})|^2 = H(e^{j\omega})\, H^*(e^{j\omega})$

$= \dfrac{1}{(1 - \alpha e^{-j\omega})} \dfrac{1}{(1 - \alpha e^{j\omega})}$

$= \dfrac{1}{1 - \alpha(e^{j\omega} + e^{-j\omega}) + \alpha^2} = \dfrac{1}{1 - 2\alpha \cos(\omega) + \alpha^2}$

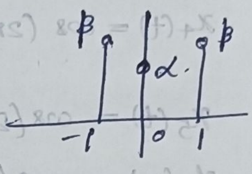
$$\measuredangle H(e^{j\omega}) = \tan^{-1}\left[\frac{Im(H(e^{j\omega}))}{Re(H(e^{j\omega}))}\right]$$

$$H(e^{j\omega}) = \frac{1}{(1-\alpha e^{-j\omega})} \times \frac{(1-\alpha e^{-j\omega})^*}{(1-\alpha e^{-j\omega})^*}$$

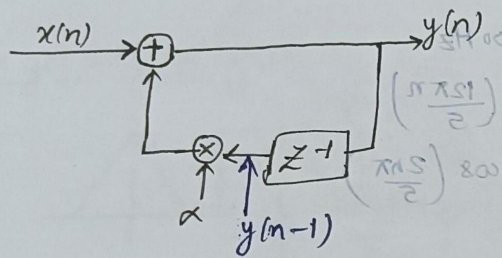
$$= \frac{1-\alpha\cos(\omega) + j\alpha\sin(\omega)}{|1-\alpha e^{-j\omega}|^2}$$

$$\cancel{[H(e^{j\omega})]^2 = H(e^{j\omega})H^*(e^{j\omega}) = \frac{1}{(1-\alpha e^{-j\omega})(1-\alpha e^{j\omega})} \neq \frac{1}{1-\alpha}}$$

$$\therefore \measuredangle H(e^{j\omega}) = \tan^{-1}\left[\frac{\alpha\sin(\omega)}{1-\alpha\cos(\omega)}\right]$$

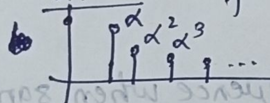


: $h(n)$



① Feedback system
② Infinite Impulse Response (IIR) Filter
③ AR process (Auto Regressive)

$h(n)$: (Infinite memory)



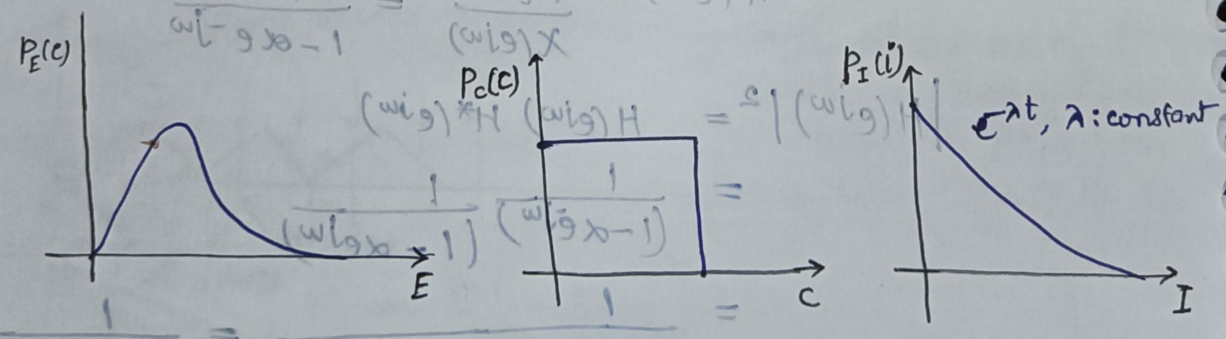
$$H(Z) = \frac{1}{1-\alpha Z^{-1}}$$

$$h(n) = [1, \alpha, \alpha^2, \alpha^3, ...]$$

3-10-2024

$$\frac{dL(t)}{dt} = KECIL(t),$$

E, C, I and L(t) are all random quantities.



$$\int \frac{dL(t)}{L(t)} = \int KECI\, dt$$

$$\Rightarrow \ln(L(t)) = KECIt$$

$$\Rightarrow L(t) = L(t_0)\, e^{-KECIt}$$

→ Discretization

① $\frac{L(n) - L(n-1)}{T} = KECI\, L(n)$

$\Rightarrow (1 - KECI\,T)\, L(n) = L(n-1)$

② $\frac{L(n+1) - L(n)}{T} = KECI\, L(n)$

$\Rightarrow ((KECI)T + 1)\, L(n) = L(n+1)$

Book:
Digital control system
↳ By M. Gopal / Ogata / Benjamin Kuo.

Forward and Backward Equations:

$L_{s_1}(t) = L_{s_1}(t_0)\, e^{-\beta_1 t}$ [$\beta = KECI$: Random quantity]

$L_{s_2}(t) = L_{s_2}(t_0)\, e^{-\beta_{s_2} t}$

$\vdots$

$L_{s_n}(t) = L_{s_n}(t_0)\, e^{-\beta_{s_n} t}$

Assuming that $L_{si}(t_0)$ = same, $\forall i$,

$L_{si}(t)$ is a random quantity.

$E[L_{si}]$ = Mean learning (in the class)

$\sigma^2_{L_{si}}$ = Variance.

$$E[L_{si}] = L_s(t_0)\, E[e^{-\beta t}]$$

$$= L_s(t_0) \int_{\beta} P_{\beta}(\beta)\, e^{-\beta t}\, dt$$

β will be Gaussian / Normal distribution.

Transform Domain Analysis of Linear Time Invariant Systems

Stability : ROC should contain the unit circle.

— Closely related to Fourier Transformability.

$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$

$b_0 y(n) + b_1 y(n-1) + b_2 y(n-2) + \dots + b_M y(n-M) = a_0 x(n) + a_1 x(n-1) + \dots + a_N x(n-N)$

↳ obtained from appropriate differential eqns.

↳ Used to model LTI/LTV systems.

$b_0 Y(Z) + b_1 Y(Z) Z^{-1} + b_2 Y(Z) Z^{-2} + \dots + b_M Y(Z) Z^{-M}$
$= a_0 X(Z) + a_1 X(Z) Z^{-1} + \dots + a_N X(Z) Z^{-N}$

$$\Rightarrow \frac{Y(Z)}{X(Z)} = \frac{a_0 + a_1 Z^{-1} + a_2 Z^{-2} + \dots + a_N Z^{-N}}{b_0 + b_1 Z^{-1} + b_2 Z^{-2} + \dots + b_M Z^{-M}}$$

$$\Rightarrow \left(\frac{\alpha_0}{\beta_0}\right) \frac{\prod_{i=1}^{N} (1 - \alpha_i z^{-i})}{\prod_{i=1}^{M} (1 - \beta_i z^{-i})}$$

; At each α_i, Nr polynomial goes to zero
↳ Zero of system

At each β_i, Dr polynomial goes to zero
↳ Poles of the system

$$H(Z)\Big|_{Z=e^{j\omega}} = \left(\frac{\alpha_0}{\beta_0}\right) \frac{\prod_{i=1}^{N} (1 - \alpha_i e^{-j\omega})}{\prod_{i=1}^{M} (1 - \beta_i e^{-j\omega})} \begin{cases} \rightarrow \text{Mag.} \\ \rightarrow \text{Phase} \end{cases}$$

$= H(e^{j\omega})$
$= H(e^{j(\omega t + 2\pi t)})$

$f_0 \quad |H(f_0)| \quad \measuredangle H(f_0)$

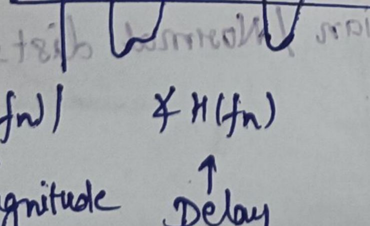
$f_1 \quad |H(f_1)| \quad \measuredangle H(f_1)$

$f_2 \quad |H(f_2)| \quad \measuredangle H(f_2)$

$\vdots$

$f_N \quad |H(f_N)| \quad \measuredangle H(f_N)$

$h(t) \approx$



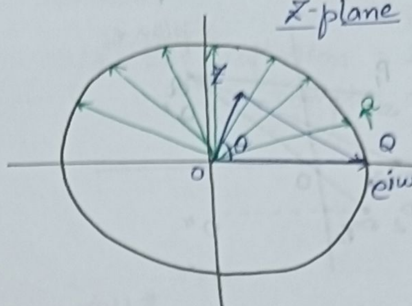
$|H(f_N)|$ $\quad$ $\measuredangle H(f_n)$

↑ Magnitude $\quad$ ↑ Delay

07-10-2024

$$H(z) = (1 - a z^{-1}) = \left(\frac{z-a}{z}\right)$$

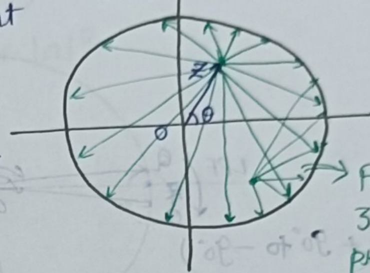
Z-plane



$$H(e^{j\omega}) = \frac{e^{j\omega} - a}{e^{j\omega}}$$

$$|H(e^{j\omega})| = \frac{|e^{j\omega} - a|}{|e^{j\omega}|} = \frac{|ZQ|}{|OQ|}$$ ⇝ remains constant

Phasor always rotates in anti-clockwise dirn.

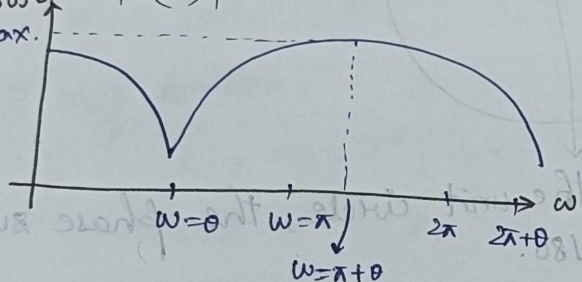


→ For multiple zeros, take product of the lengths

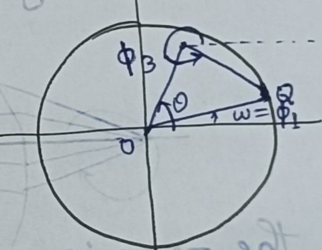
$OQ \parallel OZ$, $\omega = \theta$

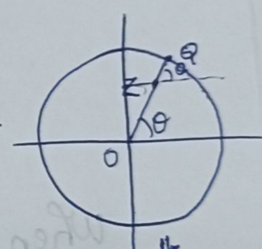
$\Rightarrow |ZQ|$ is smallest

Length of phasor : $|H(e^{j\omega})|$



$$\measuredangle H(e^{j\omega}) = \measuredangle [e^{j\omega} - a] - \measuredangle e^{j\omega}$$
$$= \measuredangle QZ - \measuredangle QO$$
$$= \phi_3 - \phi_1$$

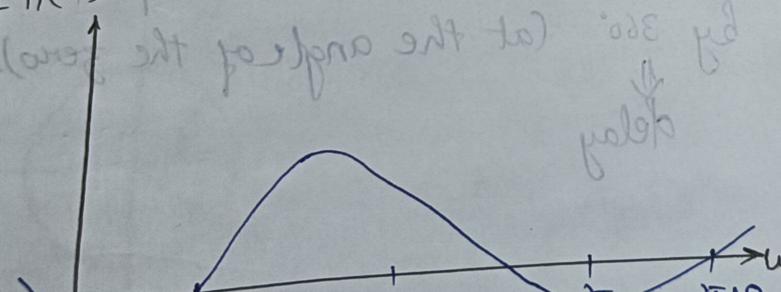




$\Downarrow$

$\measuredangle H(e^{j\omega}) = 0$

$\measuredangle H(e^{j\omega})$

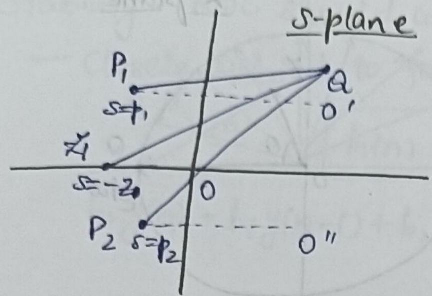


Inference: At $\omega = \theta$, (i) the phase goes to 0.

(ii) the magnitude dips.

(iii) phase continuously varies with 'ω'.

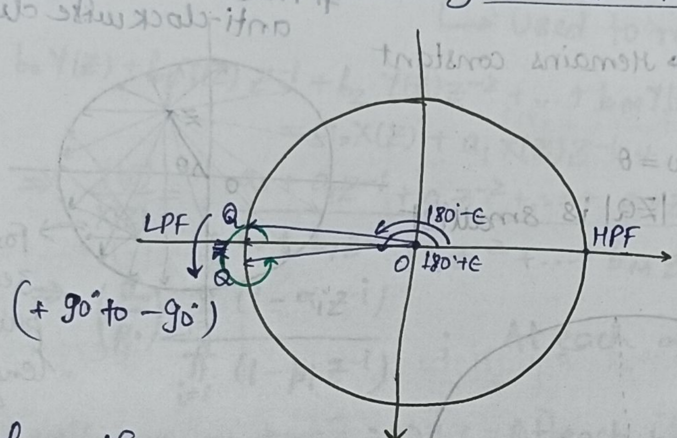
Consider $H(s) = \dfrac{K(s+z)}{(s+p_1)(s+p_2)}$.



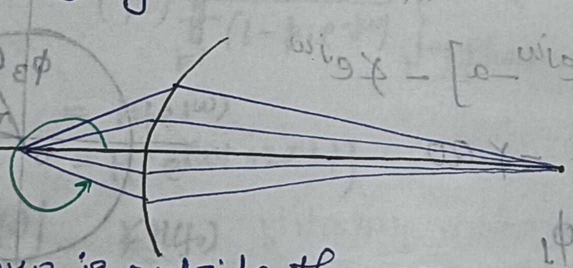
$$|H(z)| = \frac{|K||QZ_1|}{|QP_1||QP_2|}$$

$$\measuredangle H(z) = \angle QZ_1O - (\angle QP_1O' + \angle QP_2O'')$$

Consider the case (DT) when zero is on the unit-circle.



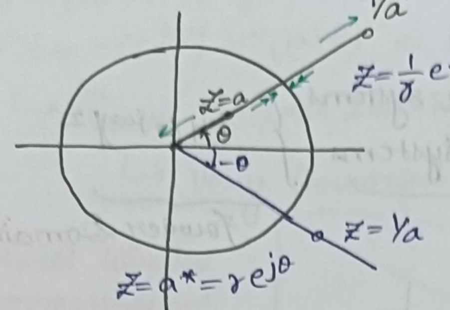
When the zero is on the unit circle, the phase suffers a discontinuity by 180°.



When the zero is outside the unit circle, the phase suffers a discontinuity by 360° (at the angle of the zero).

$\Downarrow$

delay



linear reflection : complex conjugate
spherical reflection : 1 by complex conjugate

$$H(z) = (1 - az^{-1})$$

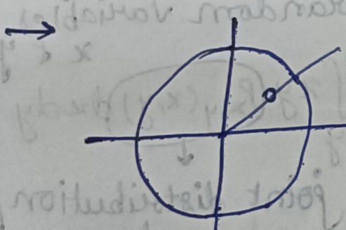
$$|H(e^{j\omega})|^2 = (1-ae^{-j\omega})(1-ae^{-j\omega})^*$$
$$= (1-ae^{-j\omega})(1-a^*e^{j\omega})$$
$$= 1 - a^*e^{j\omega} - ae^{-j\omega} + |a|^2$$

Suppose $a = re^{j\theta}$

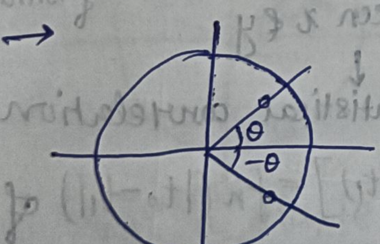
$$\frac{1}{a} = \frac{1}{re^{j\theta}} = \frac{1}{r} e^{-j\theta}$$

Z-mag. function :

$$C(z) = H(z) H^*(1/z^*) \quad \text{[complex system]}$$
$$= H(z) H(1/z) \quad \text{[Real System]}$$



$H(z) = (1 - az^{-1})$
↓
complex system
↓
Impulse response (IR) has complex coeff. ∵ $IR \equiv [1, -a]$ [a : complex]



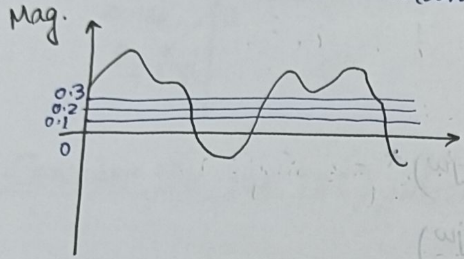
$H(z) = (1 - az^{-1})(1 - a^* z^{-1})$
↓
Real system
↓
IR has real coeff.

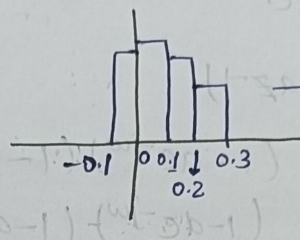
10-10-2024

The Magnitude Function

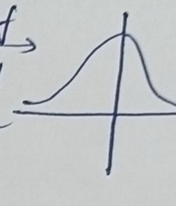
$C(z) = H(z)H^*(1/z^*)$ for complex systems
$= H(z)H(1/z)$ for real systems

$\underbrace{}_{\text{Z-Domain}}$ $\quad |H(e^{j\omega})|^2$ — Fourier Domain

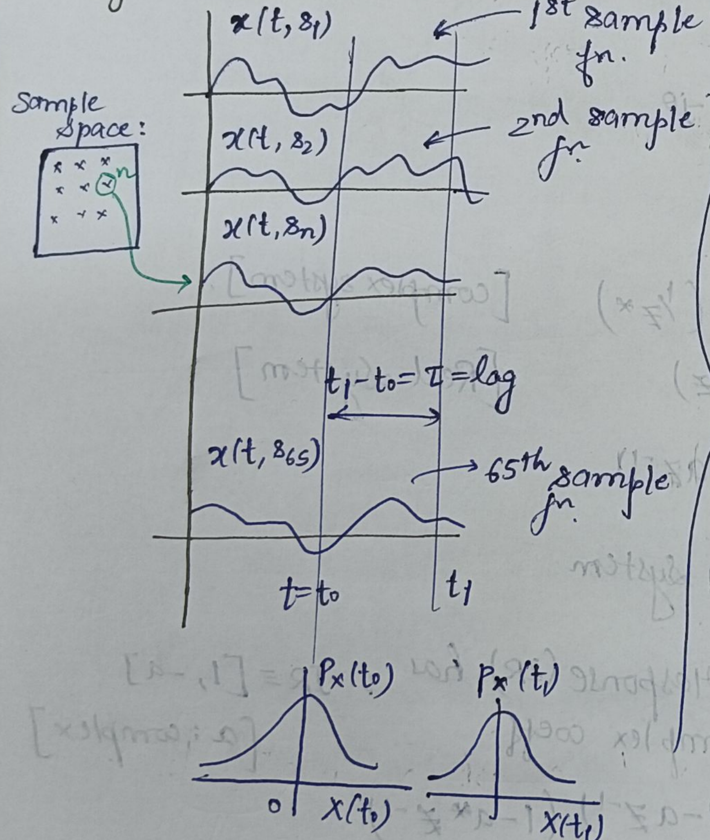




converges to pdf of the signal amplitude



Eg. 65 Radios



1st sample fn. → Correlation of each of the sample fn.s

Collection of sample functions is called a random process (R.P.)

Given a signal $x(t)$. [Temporal correlation]

$$\gamma_x(\tau) = \int x(t)\,x(t-\tau)\,dt$$

τ → lag

Given two random variables x & y

$$E(x,y) = \iint_{x\,y} x\,y\,P_{x,y}(x,y)\,dx\,dy$$

$P_{x,y}(x,y)$ → joint distribution/density

↓

Statistical measure of similarity between x & y

↓

Statistical correlation

$E[x(t_0)\,x(t_1)] = fn(|t_0 - t_1|)$ of x & y.

ACF $\xleftrightarrow{FT}$ PSD : WK theorem

Eg. $x(t) \rightarrow \boxed{h(t)} \rightarrow y(t) = x(t) * h(t)$

$x(t)$ ↓ ACF $= \gamma_x(\tau)$

Calculate $\gamma_y(\tau)$.

14-10-2024

$C(z) = H(z)\,H^*(1/z^*)$ → for complex.

$C(e^{j\omega}) = H(e^{j\omega})\,H^*(e^{j\omega}) = |H(e^{j\omega})|^2$ → The magnitude f^n of System response (DSP)

↳ Power Spectral Density → Probability & Random process.

↳ Very powerful industrial diagnostic tool.

- Model wireless comm. channel
- Modelling jitter in digital ckts.
- Atmospheric Turbulence (when aircraft takes off)
- Atmospheric Turbulence (Optical comm.)

$|H(e^{j\omega})|^2 = (1 - a e^{-j\omega})(1 - a^* e^{j\omega})$ → Real qty

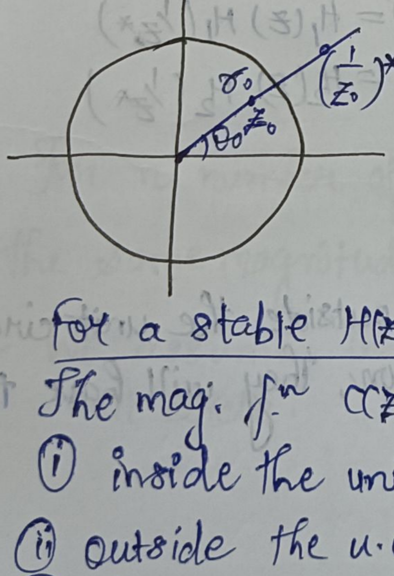
$$H(z)\,H^*(z) = (1 - a(re^{j\omega})^{-1})\,(1 - a^*(re^{j\omega}))$$

$$= \left[1 - \frac{a e^{-j\omega}}{r}\right]\left[1 - a^* r e^{j\omega}\right]$$

Magnitude f^n. : $\underbrace{H(z)\,H^*(1/z^*)}$
(z-Spectral density)

If the system $H(z)$ has a zero at $z = z_0$, then $C(z)$ has zeros at $z = z_0$, $z = 1/z_0^*$

$C(z) = H(z)\,H^*(1/z^*)$



$z_0 = re^{j\theta_0}$

$\frac{1}{z_0} = \frac{1}{re^{j\theta_0}}$

$\therefore \frac{1}{z_0^*} = \frac{1}{re^{-j\theta_0}} = \frac{1}{r}e^{j\theta_0}$

For a stable $H(z)$:

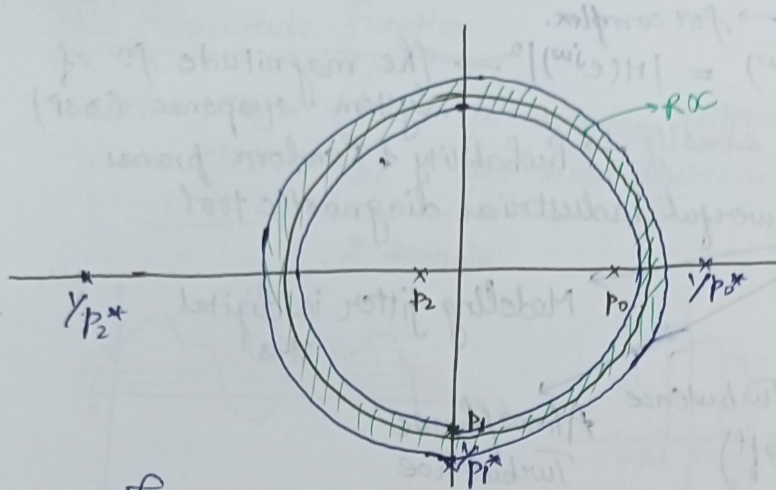
The mag. f^n $C(z)$ has a zero/pole

(i) inside the unit circle.

(ii) outside the u.c. spherically reflected position.

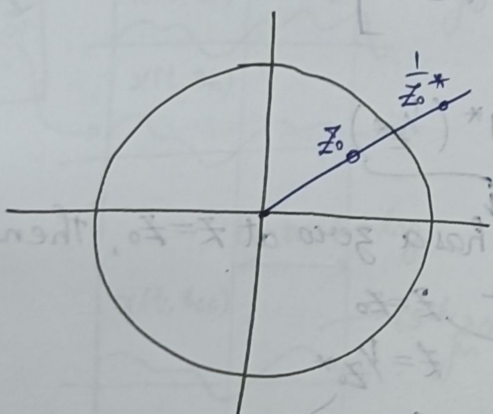
(iii) the ROC of $C(z)$ will include u.c.

(iv) The u.c. outside the outmost pole & inside the innermost pole.



→ For a stable causal system $H(z)$, the ROC is outside the outermost pole.

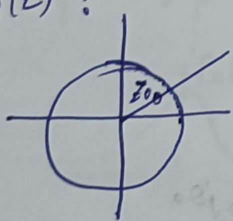
→ The ROC of magnitude function of stable system always includes the unit circle.



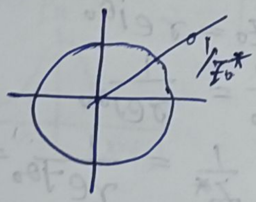
$$C(z) = (1 - z_0 z^{-1})\left(1 - \frac{1}{z_0} z^{-1}\right)$$

← The pole-zero plot of the mag. f^n

eg: $H_1(z)$:



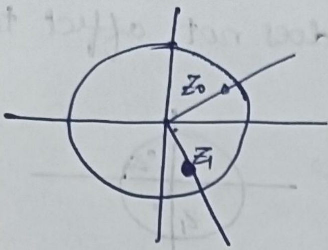
$H_2(z)$:



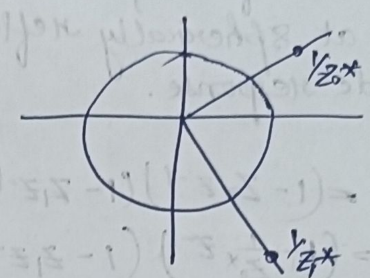
$$C(z) = H_1(z)\, H_1(1/z^*)$$
$$= H_2(z)\, H_2(1/z^*)$$

NOTE: ~~If~~ ~~&~~ Whether the zero is inside ~~or~~ outside the unit circle, if they are at spherically reflected position, they will have the same magnitude response.

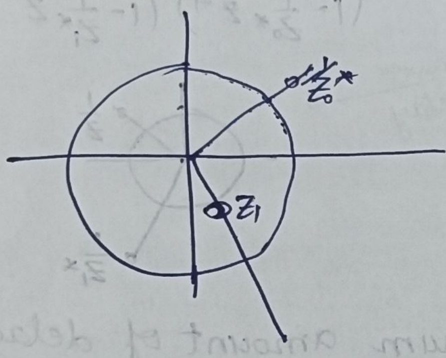
Eg. $H_1(z) = (1 - z_0 z^{-1})(1 - z_1 z^{-1})$



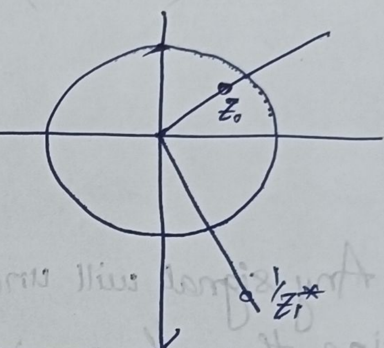
$H_2(z) = \left(1 - \frac{1}{z_0^*} z^{-1}\right)\left(1 - \frac{1}{z_1^*} z^{-1}\right)$



$H_3(z) = \left(1 - \frac{1}{z_0^*} z^{-1}\right)(1 - z_1 z^{-1})$



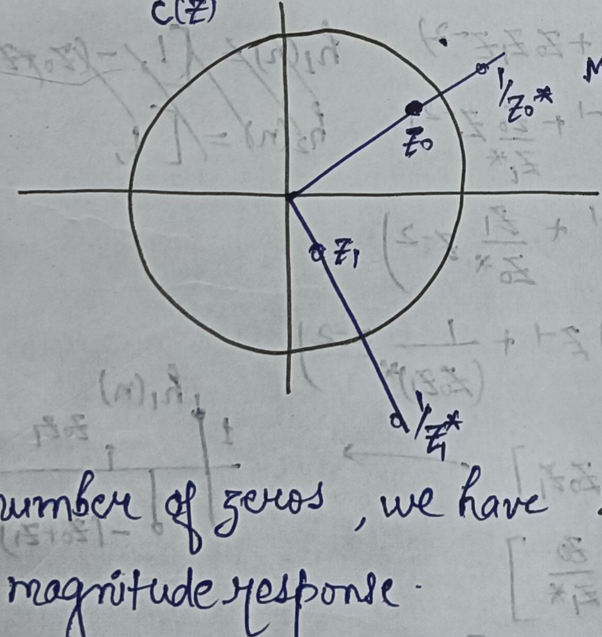
$H_4(z) = (1 - z_0 z^{-1})\left(1 - \frac{1}{z_1^*} z^{-1}\right)$



Mag. response

$\Downarrow$

C(Z)



→ For n number of zeros, we have 2^n no. of system having the same magnitude response.

We assume throughout that poles are inside the unit circle.

17-10-2024

→ The location of the pole whether inside the unit circle or outside at spherically reflected point does not affect the magnitude response.

$$H_1(z) = (1 - z_0 z^{-1})(1 - z_1 z^{-1})$$

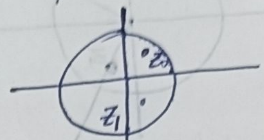
Mixed phase:

$$H_2(z) = \left(1 - \frac{1}{z_0^*} z^{-1}\right)(1 - z_1 z^{-1})$$

$$H_3(z) = (1 - z_0 z^{-1})\left(1 - \frac{1}{z_1^*} z^{-1}\right)$$

$$H_4(z) = \left(1 - \frac{1}{z_0^*} z^{-1}\right)\left(1 - \frac{1}{z_1^*} z^{-1}\right)$$

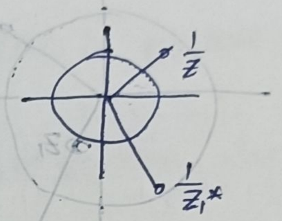
min^m delay



Minimum phase

$$C(z) = (1 - z_0 z^{-1})(1 - z_1 z^{-1})\left(1 - \frac{1}{z_0^*} z^{-1}\right)\left(1 - \frac{1}{z_1^*} z^{-1}\right)$$

max^m delay



→ Any signal will undergo minimum amount of delay when passing through a minimum phase system and max^m delay when passing through a max^m phase system.

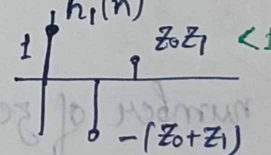
$$H_1(z) = (1 - (z_0 + z_1) z^{-1} + z_0 z_1 z^{-2})$$ $\cancel{h_1(n) = [1, -(z_0+z_1), z_0 z_1]}$

$$H_2(z) = \left(1 - \left(z_0 + \frac{1}{z_1^*}\right) z^{-1} + \frac{z_0}{z_1^*} z^{-2}\right)$$ $\cancel{h_2(n) = [1,}$

$$H_3(z) = \left(1 - \left(\frac{1}{z_0^*} + z_1\right) z^{-1} + \frac{z_1}{z_0^*} z^{-2}\right)$$

$$H_4(z) = \left(1 - \left(\frac{1}{z_0^*} + \frac{1}{z_1^*}\right) z^{-1} + \frac{1}{(z_0 z_1)^*} z^{-2}\right)$$

$$h_1(n) = \left[1, -(z_0 + z_1), z_0 z_1\right] \longrightarrow$$

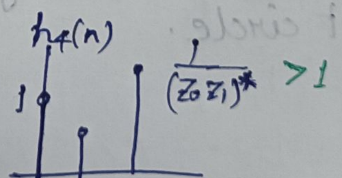


→ Tap mag. decreases

$$h_2(n) = \left[1, -\left(z_0 + \frac{1}{z_1^*}\right), \frac{z_0}{z_1^*}\right]$$

$$h_3(n) = \left[1, -\left(\frac{1}{z_0^*} + z_1\right), \frac{z_1}{z_0^*}\right]$$

$$h_4(n) = \left[1, -\left(\frac{1}{z_0^*} + \frac{1}{z_1^*}\right), \frac{1}{(z_0 z_1)^*}\right] \longrightarrow$$



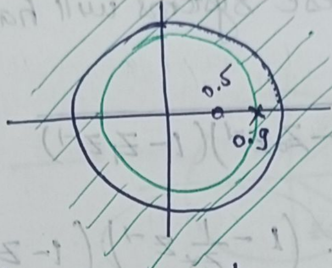
Inverse System:

$x(n) \rightarrow \boxed{H(Z)} \rightarrow \boxed{G(Z)} \rightarrow$

$H(Z)\,G(Z) = 1$

→ Maximum phase system does not have stable inverse, while minimum phase system have stable inverse.

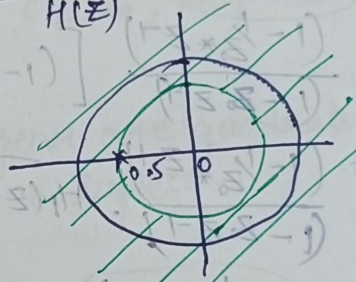
$H(Z) = \dfrac{1-0.5Z^{-1}}{1-0.9Z^{-1}}$



$h(n) = (0.9)^n u(n) - (0.5)(0.9)^{n-1} u(n-1)$

Suppose $H(Z)\,G(Z) = 1$, $G(Z) = \dfrac{1}{H(Z)}$

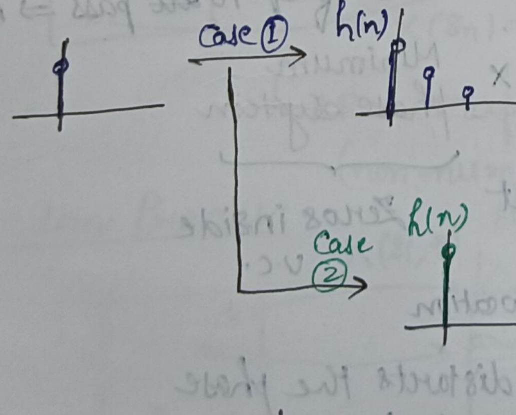
then $G(Z) = \dfrac{1-0.9Z^{-1}}{1-0.5Z^{-1}}$



$x(n) \rightarrow \boxed{H(Z)} \rightarrow \boxed{G(Z)} \rightarrow$

- → Inverse System
- → Deconvolver
- → Equilizer

→ Poles of the system become zeros of the inverse system and vice-versa.

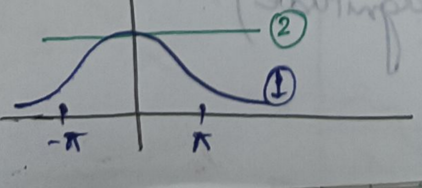


→ Lower BW

→ Spread ⇒ Gains memory

→ Higher BW

Full BW from $(-\pi, \pi)$



→ Convolution introduces memory

→ Inverse system kills the memory introduced by $H(Z)$.

→ For mixed phase system, for zeros inside the uc, we can get full compensation for zeros outside u.c.; for spherically mirror zeros inside the uc. but phase does not get compensated. $\boxed{18\text{-}10\text{-}2024}$

$H_1(Z) = (1 - Z_0 Z^{-1})(1 - Z_1 Z^{-1})$ $\quad |Z_1||Z_0| < 1 \to$ Min. phase

$H_2(Z) = (1 - \frac{1}{Z_0^*} Z^{-1})(1 - Z_1 Z^{-1})$
$H_3(Z) = (1 - Z_0 Z^{-1})(1 - \frac{1}{Z_1^*} Z^{-1})$ } Mixed phase
$H_4(Z) = (1 - \frac{1}{Z_0^*} Z^{-1})(1 - \frac{1}{Z_1^*} Z^{-1})$ – Max phase

Zeros outside the u.c.

→ The inverses of these systems will have poles outside the u.c.

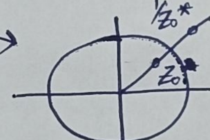
$$H_{1,inv.} = \frac{1}{(1 - Z_0 Z^{-1})(1 - Z_1 Z^{-1})}$$ → causal, stable inverse

$$H_2(Z) = \frac{(1 - Z_0 Z^{-1})}{(1 - Z_0 Z^{-1})} \cdot \left(1 - \frac{1}{Z_0^*} Z^{-1}\right)(1 - Z_1 Z^{-1})$$

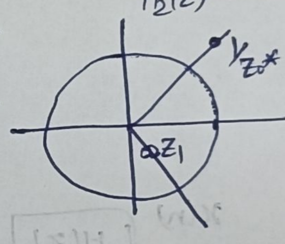
$$= \frac{(1 - 1/Z_0^* \, Z^{-1})}{(1 - Z_0 Z^{-1})} \left[(1 - Z_0 Z^{-1})(1 - Z_1 Z^{-1})\right]$$

All pass system

$$H_2(Z) = \frac{(1 - 1/Z_0^* \, Z^{-1})}{(1 - Z_0 Z^{-1})} \cdot H_1(Z)$$



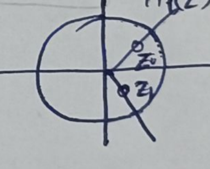
H2(Z)



$$H_3(Z) = \frac{(1 - 1/Z_1^* \, Z^{-1})}{(1 - Z_1 Z^{-1})} \cdot H_1(Z)$$

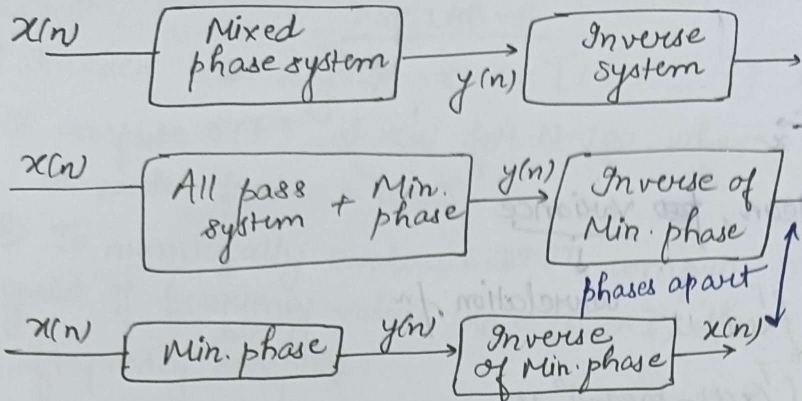
→ Mag. Resp. of Nr = Mag. Resp. of Dr
⇓
Combined Resp. = 1
⇓
Allows all freq. to ~~att~~ pass ⇒ A.P.S.

H1(Z)

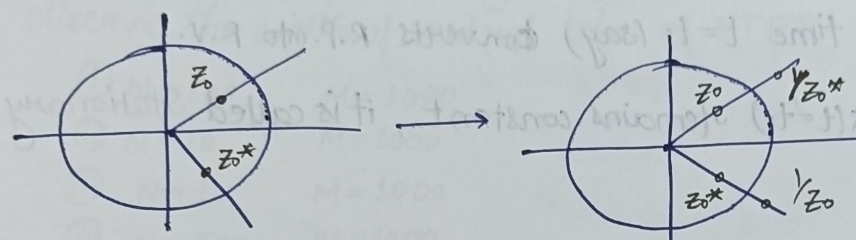


$$\underbrace{\text{Mixed phase system function}}_{\text{Zeros both inside/ outside u.c.}} = \underbrace{\left[\text{All pass system}\right]}_{\text{Zeros/poles at spherically reflected location}} * \underbrace{\text{Minimum phase system}}_{\text{Zeros inside u.c.}}$$

↳ Only distorts the phase (not magnitude)



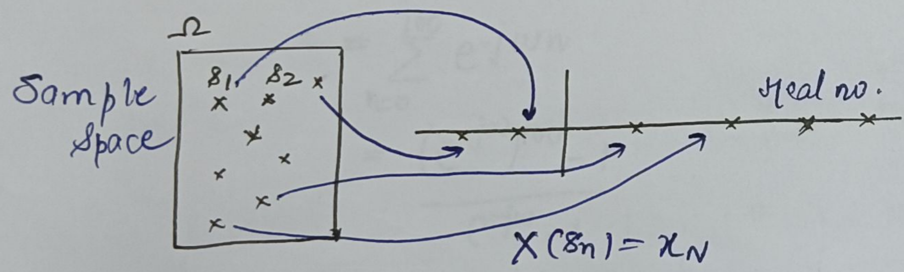
∴ The phase distortion introduced by mixed phase system cannot be completely compansated, where mag. distortion can be corrected.



H(Z) (of a real system) C(Z) : Mag. response

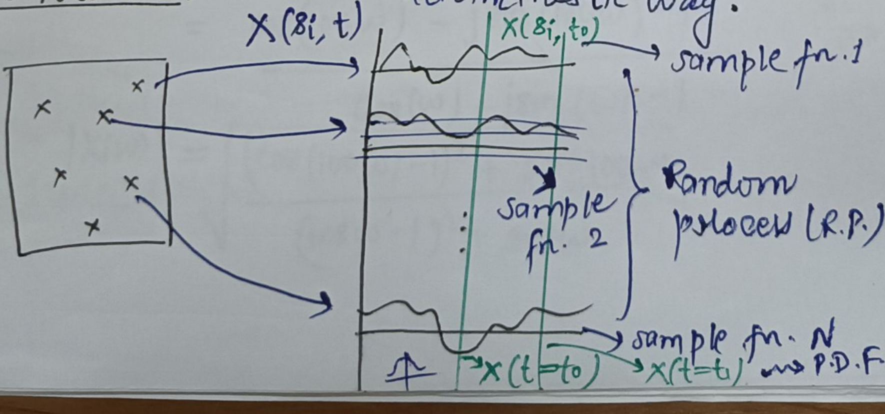
→ The zeros of real system appear are quadrats, whereas for complex system, they appear as duplets.

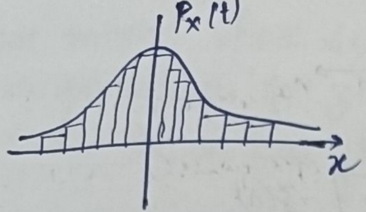
Random Variable:



↳ R.V. maps sample space to a real value (non-numerical value to numerical value) in a deterministic way.

Random Process:





↳ statistics : Mean, ~~for~~ $\underbrace{\text{Variance}}$

↓

correlation f^n

$$\text{Mean} = \frac{1}{T}\int_0^T x(t)\,dt$$

$$\text{Var.} = \int_0^T (x(t) - \text{mean})^2\,dt$$

→ Freezing the time $t = t_0$ (say) converts R.P. into R.V.

→ If the PDF $X(t=t_0)$ remains constant, it is called Stationary R.P..

DSP LAB-5

(1) Consider a DT signal $x(n) = [1\ 1\ 1\ \dots\ 1]$ (A sequence of N ones)

(a) Compute DTFT of $x(n)$ for $N=100$, where frequency variable ω varies from 0 to 2π.

(b) To numerically evaluate DTFT, compute DTFT over a dense grid of frequency values $\omega_k \in [0, 2\pi]$, where $M=1000$ (no. of frequency samples).

(c) Repeat DTFT computation for different values of N and observe the width of central lobe of DTFT.

(i) $N=100, \quad M=1000$

(ii) $N=10, \quad M=1000$

(iii) $N=1, \quad M=1000$

(iv) $N=500, \quad M=1000.$

(d) Plot the results using MATLAB subplot function.

(e) Write the observations.

Soln: (1) (a)

$$x(n) = \begin{cases} 1, & 0 \le n \le N=100 \\ 0, & \text{otherwise} \end{cases}$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$= \sum_{n=0}^{100} e^{-j\omega n}$$

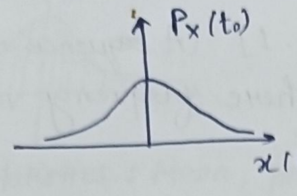
$$= \frac{(e^{-j\omega})^{100} - 1}{e^{-j\omega} - 1}$$

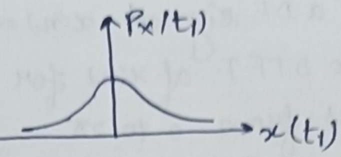
$$= \frac{e^{-j100\omega} - 1}{e^{-j\omega} - 1}$$

$$= \frac{\cos(100\omega) - j\sin(100\omega) - 1}{\cos(\omega) - j\sin(\omega) - 1}$$

$$|X(\omega)| = \sqrt{\frac{(\cos(100\omega) - 1)^2 + \sin^2 100\omega}{(\cos\omega - 1)^2 + \sin^2\omega}}$$

24-10-2024





Ensemble states

$$E[x(t_0)x(t_1)] = \int_{x(t_0)}\int_{x(t_1)} x(t_0)\,x(t_1)\,p[x(t_0), x(t_1)]\,dx(t_0)\,dx(t_1)$$

$$= R_x(t_0, t_1)$$

$$E[x(t_1)] = \int_{x(t_1)} x(t_1)\,P_{x(t_1)}(x(t_1))\,dx(t_1)$$

$$\bar{x}(t) = \lim_{T\to\infty} \frac{1}{T}\int_0^T x(t)\,dt$$

$$\gamma_x(\tau) = \int_0^\infty x(t)\cdot x(t-\tau)\,dt$$

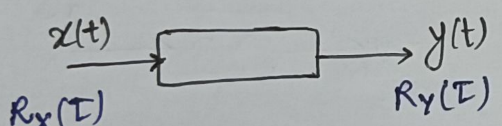
Time statistics

If time statics ≡ Esemble states ⇒ Ergodic R.P.

WSSRP : ~~White~~ Wide sense Stationary Random Process

$E(x) = c$

$R_x(t_0, t_1) = f_n(|t_0 - t_1|)$



$$R_y(\tau) = R_x(\tau) * h(\tau) * h(-\tau)$$

↓ O/P ACF, ↓ I/P ACF, ↓ IR, ↓ IR (time flipped)

FT | ZT

$$S_y(e^{j\omega}) = S_x(e^{j\omega})\,|H(e^{j\omega})|^2$$

$\sigma^2 H(Z) H^*(1/Z^*)$

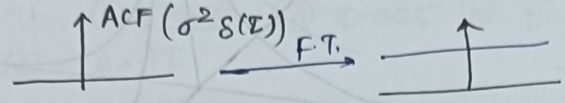
$$S_y(Z) = S_x(Z)\,H(Z)\,H^*(1/Z^*) \;:\; \text{complex}$$

$$= S_x(Z)\,H(Z)\,H(1/Z) \;:\; \text{Real}$$

$\sigma^2 H(Z) H(1/Z)$

→ we cannot take FT of RP

→ compute ACF and then compute FT, this computed FT is called ~~power density~~ spectral density.

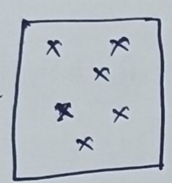


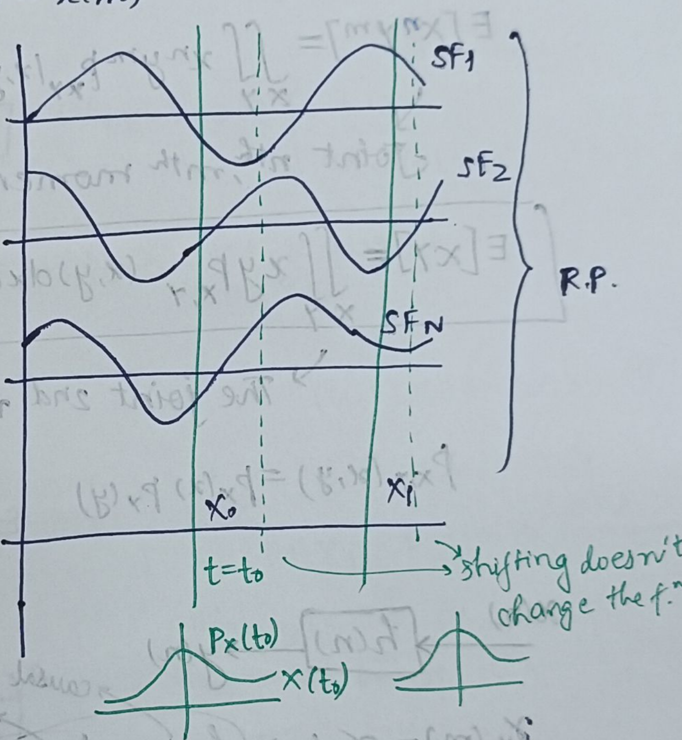
→ Concept of phase is not present in the analysis of R.P., only magnitude.

25-10-2024

Suppose $y(m) = h(m) * h(-m) * x(m)$

S_K





$x(t) = A\cos(\omega_0 t + \phi_0)$

$\phi_0 = fn(S_K)$

$\phi_0 = \pi/2$

$S_K \longrightarrow \phi_0 \sim U(0, 2\pi)$

$\bar{x} = \lim_{T \to 0} \frac{1}{T} \int_0^T x(t)\,dt$: 1st order

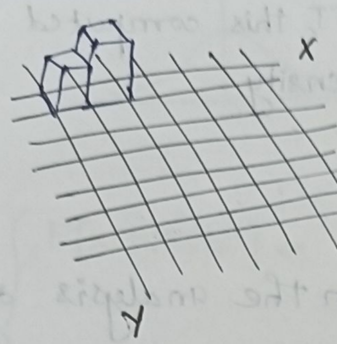
$\gamma_x(\tau) = \int x(t)\, x(t-\tau)\,dt$: 2nd order

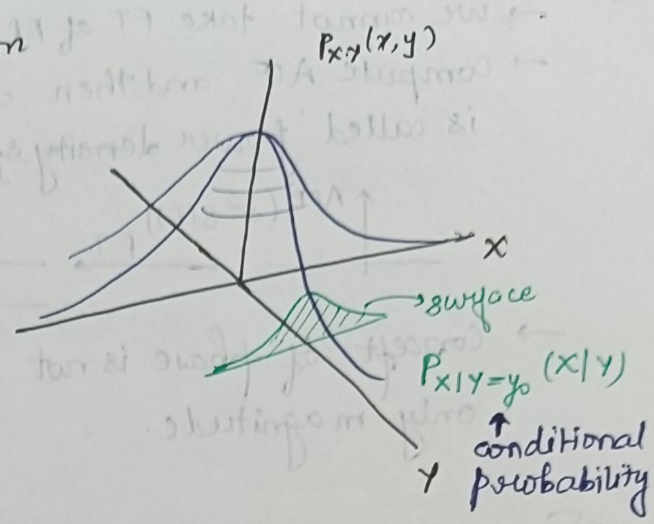
Time Statistics

f^n of τ fixed

$m_{x_0} = \int_{x(t_0)} x_0\, p_{x_0}(x_0)\, dx_0$: sample mean

2 RVs X, Y : Joint Distribution





$E[X^n] = \int_X x^n p_X(x)\, dx$

$E[X^n Y^m] = \iint_{X\,Y} x^n y^m p_{XY}(x,y)\, dx\, dy$

→ Joint n^{th}, m^{th} moment of X, Y

$$\boxed{E[XY] = \iint_{X\,Y} x y\, p_{X,Y}(x,y)\, dx\, dy} = E(X)E(Y)$$

→ The joint 2nd moment of X, Y : Correlation b/w X & Y.

$p_{X,Y}(x,y) = p_X(x)\, p_Y(y)$

Book: Prob. & R.P. for Electrical & Computer Engg.
— Leon Garcia
— Goodman & Roy Yates

$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$

causal

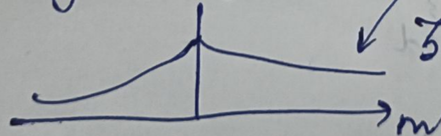
anti-causal

$\gamma_y(m) = \gamma_x(m) * h(m) * h(-m)$

$S_Y(e^{j\omega}) = S_X(e^{j\omega})\, |H(e^{j\omega})|^2$

$g(m) = g(-m)$

Zero-phase



DSP LAB-6 Assignment

① Consider a system with impulse response, $h(t) = Ae^{-t\tau}$ or $h(n) = Ae^{-n\tau}$.
Compute the -3 dB bandwidth of the system for
(i) $\tau = 1$, (ii) $\tau = 0.5$ (iii) $\tau = 0.25$ (iv) $\tau = 0.1$ (v) $\tau = 0.01$

Soln: $h(t) = Ae^{-t\tau}$

$$\Rightarrow H(\omega) = \int_{-\infty}^{\infty} Ae^{-t\tau} e^{-j\omega t}\, dt$$

$$= A\int_{0}^{\infty} e^{-t(\tau + j\omega)}\, dt \text{, for causal system}$$

$$= A \left. \frac{e^{-t(\tau+j\omega)}}{-(\tau + j\omega)} \right|_{0}^{\infty}$$

$$= \frac{-A}{\tau + j\omega}\left[0 - 1\right]$$

$$= \frac{A}{\tau + j\omega}$$

$$|H(\omega)| = \frac{A}{\sqrt{\tau^2 + \omega^2}}$$

For -3 dB bandwidth,

$$20\log(|H(\omega_c)|) = -3$$

$$\Rightarrow \frac{A}{\sqrt{\tau^2 + \omega_c^2}} = e^{-\frac{3}{20}}$$

$$\Rightarrow \tau^2 + \omega_c^2 = \left(\frac{A}{0.86}\right)^2$$

$$\Rightarrow \omega_c = \sqrt{1.35 A^2 - \tau^2}$$

For $A = 1 \Rightarrow \omega_c = \sqrt{1.35 - \tau^2}$

(i) $\tau = 1 \Rightarrow \omega_c = \sqrt{1.35 - 1} = 0.59$ rad/s

(ii) $\tau = 0.5 \Rightarrow \omega_c = \sqrt{1.35 - (0.5)^2} = 1.049$ rad/s

(iii) $\tau = 0.25 \Rightarrow \omega_c = \sqrt{1.35 - (0.25)^2} = 1.13$ rad/s

(iv) $\tau = 0.1 \Rightarrow \omega_c = \sqrt{1.35 - 0.01} = 1.16$ rad/s

(v) $\tau = 0.01 \Rightarrow \omega_c = \sqrt{1.35 - 0.0001} = 1.16$ rad/s

② $x(n) = A\cos\left(2\pi \frac{f_1}{f_s} n\right) + B\cos\left(2\pi \frac{f_2}{f_s} n\right)$, $f_1 = 10\,Hz$, $f_2 = 11\,Hz$, $f_s = 66\,Hz$, $n = 1, 2, \ldots, N$.

Measure the central lobe width for each of the cases for $N = 100, 200, 500, 1000$.

Soln: From the DTFT plot in MATLAB $[A=1, B=1]$,

<u>N=100</u>

4 out of 100 samples ⇒ width $= \frac{2\pi(4)}{100} = 0.08\pi$ rad/s.

<u>N=200</u>

6 out of 200 samples ⇒ width $= \frac{2\pi(6)}{200} = 0.06\pi$ rad/s

<u>N=500</u>

8 out of 500 samples ⇒ width $= \frac{2\pi(8)}{500} = 0.032\pi$ rad/s

<u>N=1000</u>

14 out of 1000 samples ⇒ width $= \frac{2\pi(14)}{1000} = 0.028\pi$ rad/s

③ $x(n) = \text{ones}(1, N)$

Derive the DTFT of the signal and simplify the expression.

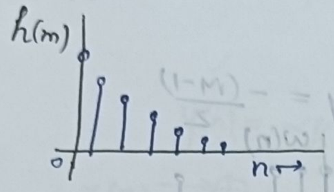
Soln:
$$X(\omega) = FT\{x(n)\}$$
$$= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$
$$= \sum_{n=0}^{N-1} 1 \cdot e^{-j\omega n} = \frac{1(1-e^{-j\omega N})}{1-e^{-j\omega}}$$
$$= \frac{1-e^{-j\omega N}}{1-e^{-j\omega}}$$

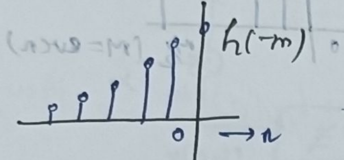
28-10-2024

$$r_y(m) = r_x(m) * h(m) * h(-m)$$

$\downarrow$ F.T.

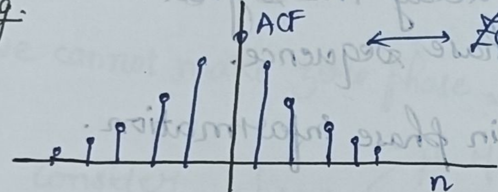
$$S_y(e^{j\omega}) = S_x(e^{j\omega})\,|H(e^{j\omega})|^2$$



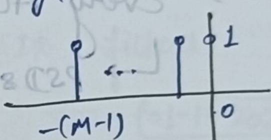


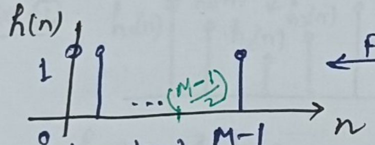
Even symmetry sequence

Eg.



ACF $\longleftrightarrow$ Zero phase ; $g(n) = h(-n)$





(causal system)

$$h(n) \xleftrightarrow{F.T.} H(e^{j\omega}) = \sum_{n=0}^{M-1} 1 \cdot e^{j\omega n}$$

$$\Rightarrow \angle H(e^{j\omega}) = -\tan^{-1}\left[\frac{\text{Img. part}}{\text{Real part}}\right] = \left(\frac{M-1}{2}\right)\omega$$

$$\Rightarrow H(e^{j\omega}) = \frac{1-e^{-j\omega M}}{1-e^{-j\omega}} = \frac{e^{-\frac{j\omega M}{2}}\left(e^{+\frac{j\omega M}{2}} - e^{-\frac{j\omega M}{2}}\right)}{e^{-\frac{j\omega}{2}}\left(e^{\frac{j\omega}{2}} - e^{-\frac{j\omega}{2}}\right)} = \underbrace{e^{-j\omega\frac{(M-1)}{2}}}_{\text{Phase (linear)}} \boxed{\frac{\sin\left(\frac{\omega M}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}}$$

fn of freq.

$$\Rightarrow \text{Phase delay} = \frac{\angle H(e^{j\omega})}{\omega} = \left(\frac{M-1}{2}\right)$$

(in no. of samples)

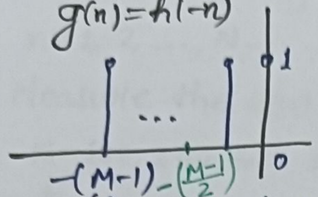
← No. of samples delay experienced by any i/p signal (at the output)

independent of the i/p signal frequency

→ **Important property of linear phase system:** All signals i/p to it will undergo the same delay - in terms of the no. of samples.

↳ constant delay = mildest form of phase distortion

$g(n) = h(-n)$



(Anti-causal system)

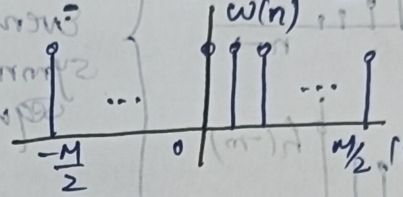
$$G(e^{j\omega}) \xleftrightarrow{F.T.} g(n), \qquad G(e^{j\omega}) = e^{\frac{j\omega(M-1)}{2}} \frac{\sin\left(\frac{\omega M}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$$

$$\Downarrow$$

$$\text{Phase delay} = -\frac{(M-1)}{2}$$

$$w(e^{j\omega}) = \frac{\sin\left(\frac{\omega M}{2}\right)}{\sin(\omega/2)} \xleftrightarrow{F.T.} w(n)$$

(Zero phase)



(M = even)

→ All causal systems will delay the signal.

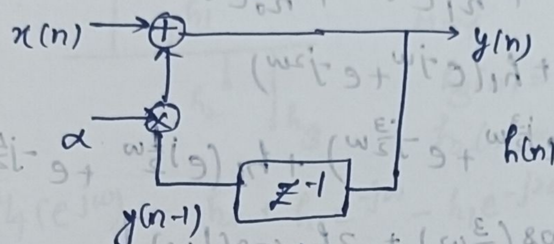
→ Every ACF is a zero-phase sequence.

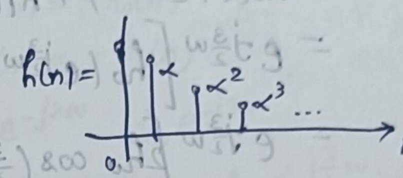
↳ PSDs do not contain phase information.

FILTERS

- FIR (Finite Impulse Response) → Feed-forward n/w
- IIR (Infinite Impulse Response)
  - Infinite memory
    - Feedback n/w.

$$y(n) = \alpha y(n-1) + x(n)$$



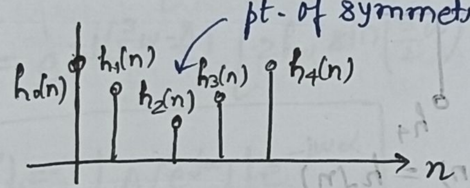


We cannot make zero phase system from IIR filters.

01-11-2024

Eg. Consider

Case (1)



$$h_0(n) = h_4(n)$$
$$h_1(n) = h_3(n)$$

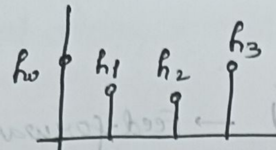
$$H_1(e^{j\omega}) = h_0 + h_1 e^{-j\omega} + h_2 e^{-j2\omega} + h_3 e^{-j3\omega} + h_4 e^{-j4\omega}$$
$$= h_0 + h_1 e^{-j\omega} + h_2 e^{-j2\omega} + h_1 e^{-j3\omega} + h_0 e^{-j4\omega}$$
$$= h_0(1 + e^{-j4\omega}) + h_1(e^{-j\omega} + e^{-j3\omega}) + h_2 e^{-j2\omega}$$
$$= e^{-j2\omega}\left[h_0(e^{j2\omega} + e^{-j2\omega}) + h_1(e^{+j\omega} + e^{-j\omega}) + h_2\right]$$
$$= \underbrace{e^{-j2\omega}}_{\text{phase}} \underbrace{\left[2h_0 \cos(2\omega) + 2h_1 \cos(\omega) + h_2\right]}_{\text{Real function}}$$

Real function → Magnitude

M = length of the FIR system

M = 5.

Case ②:



$M = 4$ tap

$h_1(n) = h_2(n)$

$h_0(n) = h_3(n)$

$$H_2(e^{j\omega}) = h_0 + h_1 e^{-j\omega} + h_2 e^{-j2\omega} + h_3 e^{-j3\omega}$$

$$= h_0 + h_1 e^{-j\omega} + h_1 e^{-j2\omega} + h_0 e^{-j3\omega}$$

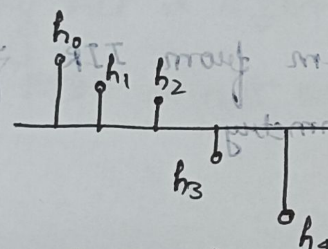
$$= h_0(1 + e^{-j3\omega}) + h_1(e^{-j\omega} + e^{-j2\omega})$$

$$= e^{-j\frac{3}{2}\omega}\left[h_0(e^{j\frac{3}{2}\omega} + e^{-j\frac{3}{2}\omega}) + h_1(e^{j\frac{1}{2}\omega} + e^{-j\frac{1}{2}\omega})\right]$$

$$= e^{-j\frac{3}{2}\omega}\left[2h_0 \cos\left(\tfrac{3}{2}\omega\right) + 2h_1 \cos\left(\tfrac{1}{2}\omega\right)\right]$$

↳ Phase delay $= \frac{3}{2}$

Case-③:



where $h_3(n) = -h_2(n)$

$h_4(n) = -h_1(n)$

$$H_3(e^{j\omega}) = h_0 + h_1 e^{-j\omega} + h_2 e^{-j2\omega} - h_2 e^{-j3\omega} - h_1 e^{-j4\omega}$$

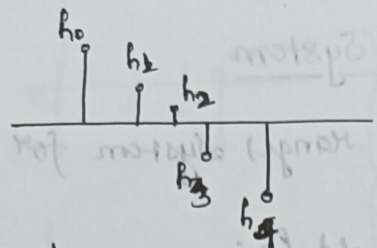
$$= h_0 + h_1(e^{-j\omega} - e^{-j4\omega}) + h_2(e^{-j2\omega} - e^{-j3\omega})$$

$$= e^{-j\frac{5}{2}\omega}\left[h_0 e^{j\frac{5}{2}\omega} + h_1(e^{j\frac{3}{2}\omega} - e^{-j\frac{3}{2}\omega}) + h_2(e^{j\frac{1}{2}\omega} - e^{-j\frac{1}{2}\omega})\right]$$

$$= h_0 + e^{-j\frac{5}{2}\omega}\left[2j\sin\left(\tfrac{3}{2}\omega\right) + 2j\sin\left(\tfrac{\omega}{2}\right)\right]$$

$$= h_0 + e^{-j\left(\frac{5}{2}\omega - \frac{\pi}{2}\right)}\left[2\sin\left(\tfrac{3}{2}\omega\right) + 2\sin\left(\tfrac{\omega}{2}\right)\right]$$

Case 3:

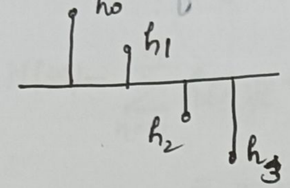


$h_3(n) = -h_1(n)$

$h_4(n) = -h_0(n)$

$$H_3'(e^{j\omega}) = e^{-j2\omega}\left[h_2 + h_0\, j2\sin(2\omega) + h_1\, 2j\sin(\omega)\right]$$

Case 4 :



$M=4$

$h_3(n) = -h_0(n)$

$h_2(n) = -h_1(n)$

$$H_4(e^{j\omega}) = h_0 + h_1 e^{-j\omega} - h_1 e^{-j2\omega} - h_0 e^{-j3\omega}$$

$$= e^{-j\frac{3}{2}\omega}\left[h_0 e^{j\frac{3}{2}\omega} - h_0 e^{-j\frac{3}{2}\omega} + h_1 e^{j\frac{\omega}{2}} - h_1 e^{-j\frac{\omega}{2}}\right]$$

$$= e^{-j\frac{3}{2}\omega}\left[2h_0 j\sin\left(\frac{3\omega}{2}\right) + 2h_1 j\sin\left(\frac{\omega}{2}\right)\right]$$

$$= e^{-j\left(\frac{3}{2}\omega - \frac{\pi}{2}\right)}\left[2h_0\sin\left(\frac{3}{2}\omega\right) + 2h_1\sin\left(\frac{\omega}{2}\right)\right]$$

$$\boxed{H(e^{j\omega}) = |A(e^{j\omega})|\, e^{-j\omega\alpha}}$$

$\angle H(e^{j\omega}) = -\alpha\omega$ → Linear phase

→ Phase delay $= \dfrac{\text{phase}}{\omega}$

→ Only FIR systems can have a point of symmetry, and therefore, a linear phase; IIR systems cannot have linear phase.

Case ③, ④ → High pass system [sin terms = 0 for $\omega \to 0$]

Case ①, ② ⇒ Low pass system.

↳ FIR has all the poles at zero (origin).

→ IIR ~~has~~ can have pole anywhere due to feedback and therefore stable.

04-11-2024

Generalised Linear Phase System

↳ Symmetric → FIR (long range) system for linear phase response.

→ The impulse response should be:

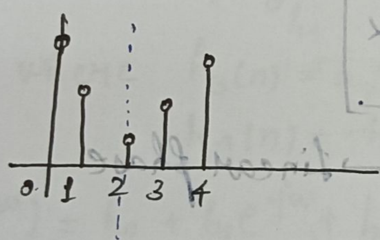
(i) Symmetric Even
(ii) odd symmetry
} ⇒ FIR system

Order of the filter } Even odd

→ Four types:

(i) Even symmetry, even no. of taps.
(ii) Even symmetry, odd no. of taps.
(iii) odd symmetry, even no. of taps.
(iv) odd symmetry, odd no. of taps.

Type-I:

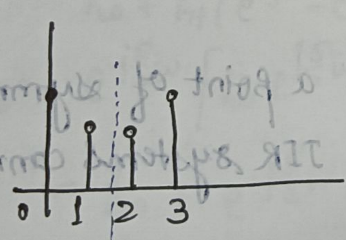


$$H_1(Z) = h(0) + h(1)Z^{-1} + h(2)Z^{-2} + h(3)Z^{-3} + h(4)Z^{-4}$$

$$= \sum_{n=0}^{M} h(n)Z^{-n}, \quad M=4$$

$$h(n) = h(M-n)$$

Type-II:



$$H_2(Z) = h(0) + h(1)Z^{-1} + h(2)Z^{-2} + h(3)Z^{-3}$$

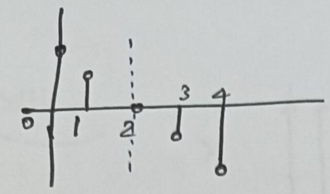
$$= \sum_{n=0}^{M} h(n).Z^{-n}, \quad M=3$$

$$H(Z) = \sum_{n=0}^{M} h(M-n)Z^{-n} = \sum_{K=M}^{0} h(K)\,Z^{-(M-K)} \quad [K=M-n]$$

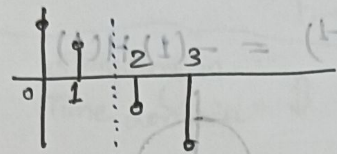
$$= Z^{-M}\sum_{K=0}^{M} h(K)Z^{K} = Z^{-M}H(Z^{-1})$$

$$\boxed{H(Z) = Z^{-M}H(Z^{-1})} \quad \text{[Even symmetry]}$$

Type-III :



Type-IV :



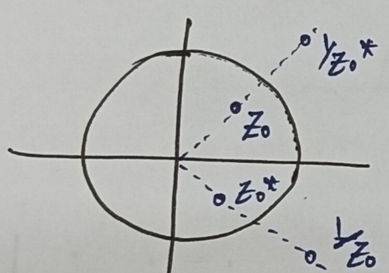
$$h(n) = -h(M-n)$$

$$\therefore H(z) = \sum_{n=0}^{M} h(n) z^{-n} = -\sum_{n=0}^{M} h(M-n) z^{-n}$$

$$= -\sum_{k=M}^{0} h(k) z^{-(M-k)} = -z^{-M} \sum_{k=0}^{M} h(k) z^{k} = -z^{-M} H(z^{-1})$$

$$\boxed{H(z) = -z^{-M} H(z^{-1})}$$

[odd symmetry]



→ If z_0 were a zero, then $1/z_0$ is also zero.

For real systems, z_0^*, $1/z_0^*$ are also zero.

$$H(z)\big|_{z=1} \Rightarrow H(1) = (1)^M H(1)$$

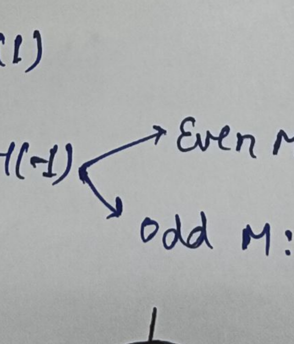
$$H(z)\big|_{z=-1} \Rightarrow H(-1) = (-1)^M H(-1)$$

→ Even M : $H(-1) = H(-1)$

→ Odd M : $H(-1) = -H(-1)$

⇓

only when $H(-1) = 0$



Symmetric, M = odd

↳ Type-II system

Odd symmetric systems :

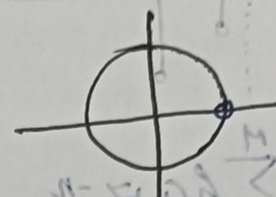
Type-III & Type-IV

$$\boxed{H(z) = -z^{-M} H(z^{-1})}$$

$$z=1 \Rightarrow H(1) = -(1)^{-M} H(1^{-1}) = -(1)\,H(1)$$

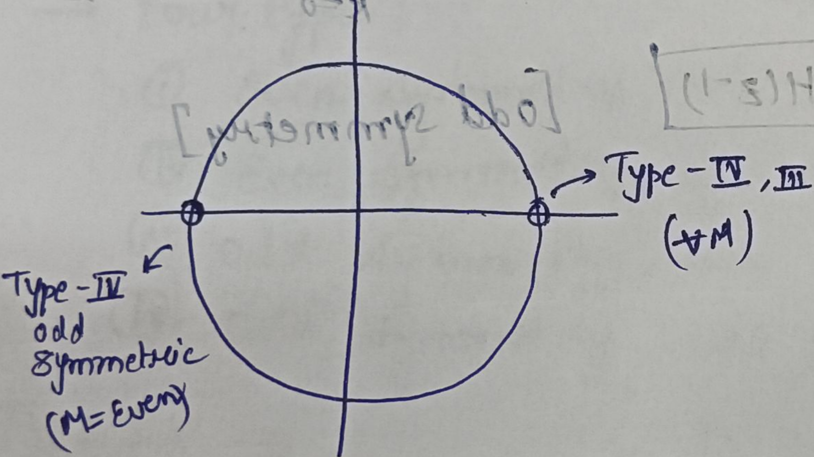
$$\Rightarrow H(1) = -H(1)$$

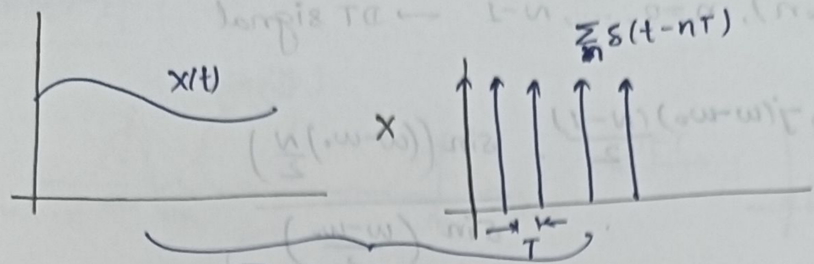
$$\Rightarrow H(1) = 0$$



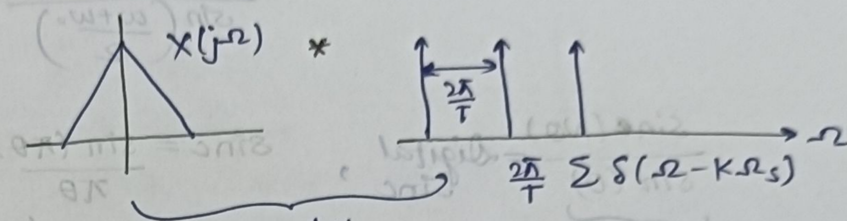
$$z=-1 \Rightarrow H(-1) = -(-1)^{-(M)}\, H(-1^{-1})$$

$$= -H(-1)\cdot(-1)^{M}$$

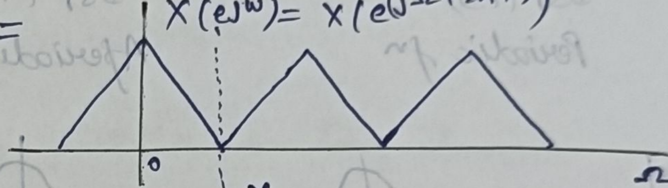


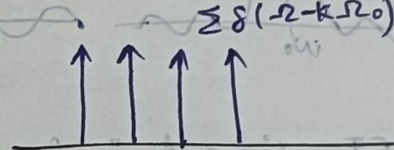


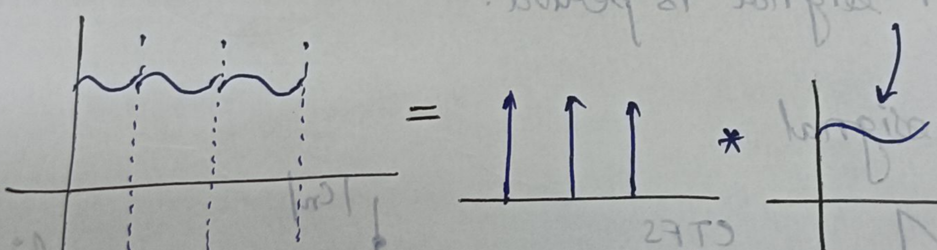
Sampling in Time domain ⇕



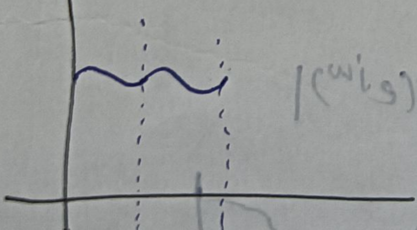
convolution







Aliasing can occur in time domain



18-11-2024

$x(n) = A\cos(\omega_0 n)$, $n = 0, \ldots, N-1$ → DT signal

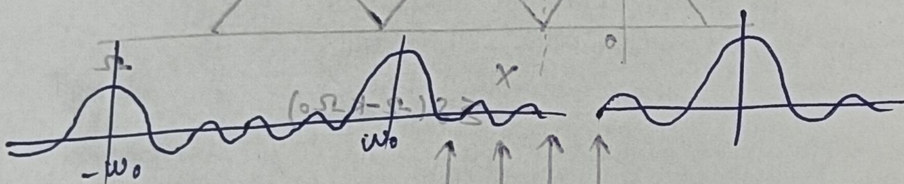
↓ DTFT

$$X(e^{j\omega}) = \frac{1}{2} e^{-j(\omega-\omega_0)\frac{(N-1)}{2}} \cdot \frac{\sin\left((\omega-\omega_0)\frac{N}{2}\right)}{\sin\left(\frac{\omega-\omega_0}{2}\right)}$$

$$+ \frac{1}{2} e^{-j(\omega+\omega_0)\frac{(N-1)}{2}} \cdot \frac{\sin\left((\omega+\omega_0)\frac{N}{2}\right)}{\sin\left(\frac{\omega+\omega_0}{2}\right)}$$

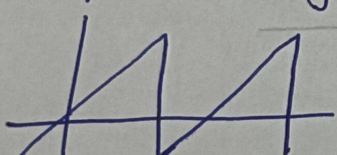
$\underbrace{\frac{\sin(N\theta)}{\sin(\theta)}}_{\text{DSP (FD)}}$ = Digital sinc, $\underbrace{\text{sinc} = \frac{\sin(\pi\theta)}{\pi\theta}}_{\text{DSP (TD)}}$, $\underbrace{\text{sinc}(\theta) = \frac{\sin(\theta)}{\theta}}_{\text{Maths}}$

Periodic fn — Aperiodic fn

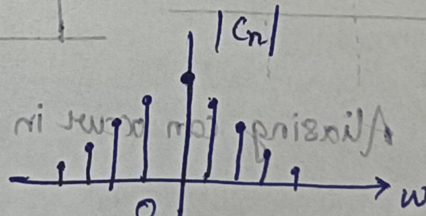


→ Every DTFT signal is period.

CT periodic signal



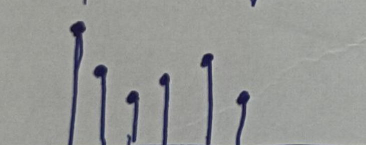
CTFS →



⇝ line spectra

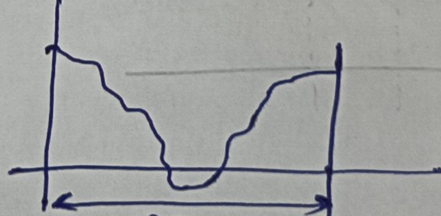
↕ Dual

sampled signal

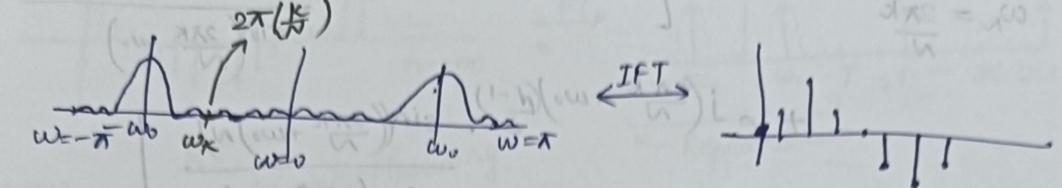


~~line spectra~~

⟷ DTFT



$|X(e^{j\omega})|$:



Sample the F.D. also :

$\omega \rightarrow [0, 2\pi], [-\pi, \pi]$

$\omega_k = 2\pi\left(\frac{k}{N}\right)$, k : integer

$k = 0, \dots N-1$

DFT :

$$X(\omega_k) = X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\omega_k n}$$

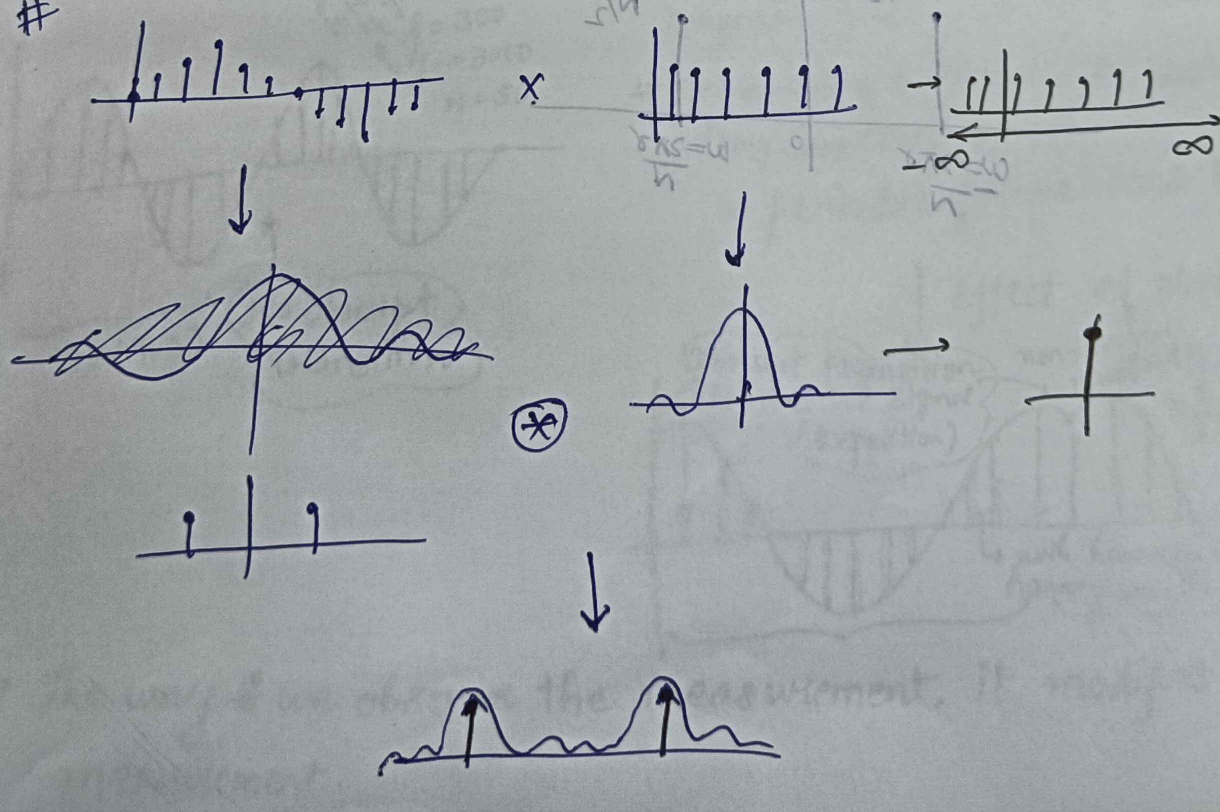
$$\boxed{X(k) = \sum_{n=0}^{N-1} x(n) e^{-j 2\pi \frac{k}{N} n}} \leftarrow \text{freq. domain}$$

↳ FFT/DFT command in MATLAB.

MATLAB gives samples of $X(e^{j\omega})$.

$$x(n) = \frac{1}{N} \sum_{n=0}^{N-1} X(k) e^{j\frac{2\pi k n}{N}} \longrightarrow \text{Reverse Transform}$$

#



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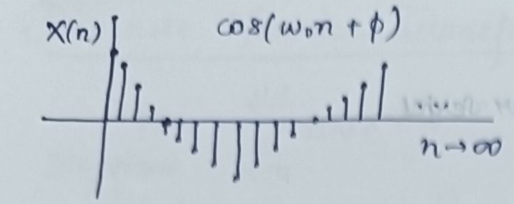
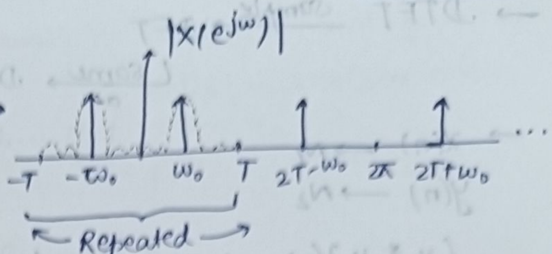


FT ⟷



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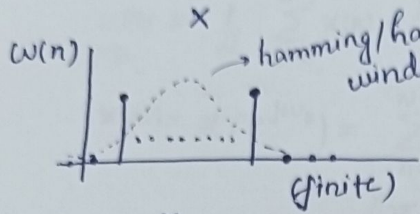

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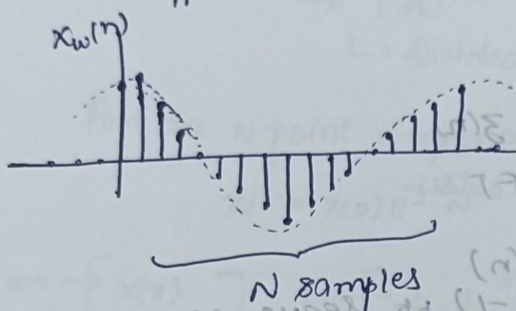
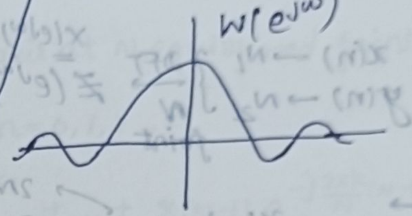
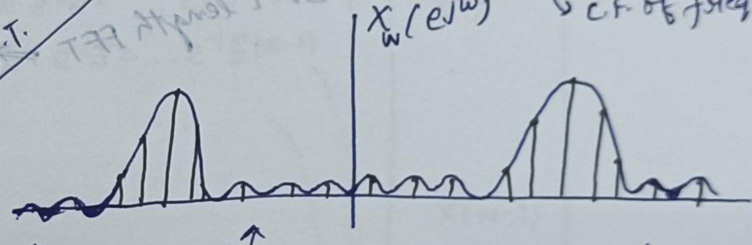


FT ⟶



$X_w(e^{jw})$ → DTFT of $x_w(n)$ → C.F. of freq.



F.T.

Sample by multiplying it with an impulse train in freq. domain

= Convolving in T.D. with another impulse train

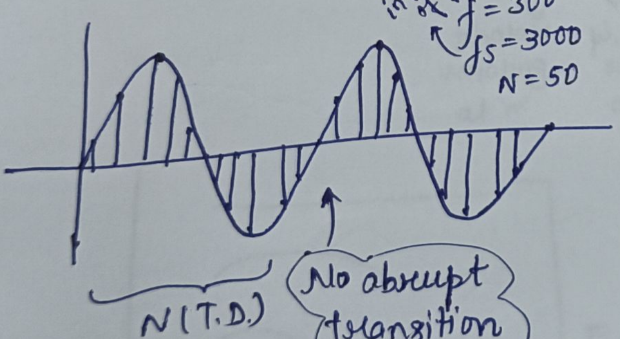
= periodic repetitions in T.D.

integral multiple of freq.

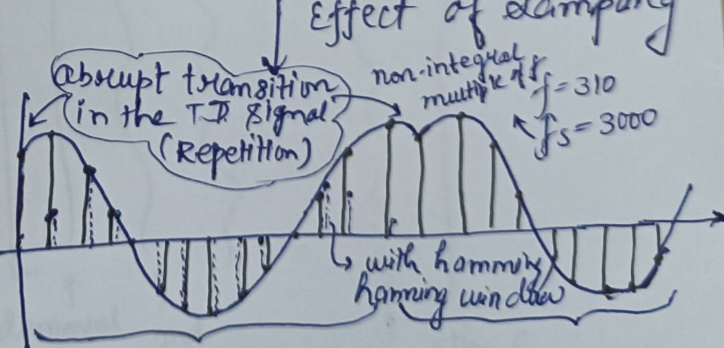
$f = 300$

$f_s = 3000$

$N = 50$



Effect of sampling



→ The way we observe the measurement, it modifies the measurement.

25-11-2024

→ DTFT $\xrightarrow{\text{sample}}$ DFT

└ same as → Discrete Fourier series

→ $x(n) \rightarrow N_1$

$y(n) \rightarrow N_2$

$(x * y)(n) \rightarrow N_1 + N_2 - 1$ length.

→ CT → infinite harmonics

DT → n harmonics

→ $\left.\begin{matrix} x(n) \rightarrow N_1 \\ y(n) \rightarrow N_2 \end{matrix}\right\} \xrightarrow[N \text{ point}]{DFT} \underset{=}{Z}(e^{j\omega}) \xrightarrow{IFT} z(n)$

$x(e^{j\omega}) \cdot Y(e^{j\omega})$

→ $x(n) \rightarrow N$ → 2N-1 length FFT

$y(n) \rightarrow N$ → 2N-1 length FFT

$\} \rightarrow \circledast \rightarrow z(n)$

(2N-1) pt. sequence

2N-1 length FFT ⟹ To get correct convolution.

22-11-2024

Discrete Fourier Transform

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi k}{N} n}, \quad n = 0, 1, \dots, N-1$$

Discretized F.D.

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi k}{N} n}, \quad k = 0, 1, \dots, N-1$$

T.D.

$$X(k) = X(e^{j\omega_k}) = \sum_{n=0}^{N-1} x(n) e^{-j\omega_k n}$$

$$\omega_k = \left(\frac{2\pi}{N}\right) k \rightarrow k\text{th point}$$

↳ Divided frequency axis at N equispaced points.

For an N-point sequence $\{x(n)\}$; $n = 0, 1, 2, \dots$

$$X(k) = x(0) e^{-j\frac{2\pi k}{N} 0} + x(1) e^{-j\frac{2\pi k}{N} 1} + \dots + x(N-1) e^{-j\frac{2\pi k}{N}(N-1)}$$

joint time & freq. function

is a phasor rotating at rate 'r'

$$\begin{array}{l} k=0 \rightarrow \\ k=1 \rightarrow \\ \\ k=N-1 \rightarrow \end{array} \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{array}{l} k=0 \rightarrow \\ k=1 \rightarrow \\ k \downarrow \\ \\ \\ \end{array} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & e^{-j\frac{2\pi}{N}} & e^{-j\frac{2\pi}{N}2} & \dots & & e^{-j\frac{2\pi}{N}(N-1)} \\ 1 & e^{-j\frac{2\pi 2}{N}1} & e^{-j\frac{2\pi}{N}2} & \dots & & e^{-j\frac{2\pi}{N}(N-1)} \\ 1 & & & & & \\ \vdots & & & & & \\ 1 & & & & & \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

$n=0$ ↓ phasor rotating at 'm'

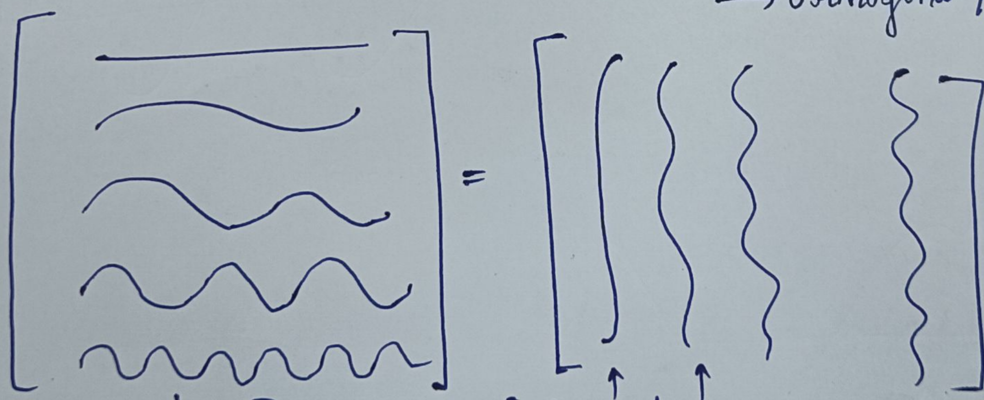
$\rightarrow n$ ↓ phasor rotating at '2m'

↓ 3m

phasor rotating at '2r'

3r

↳ Orthogonal / Unitary Matrix



↓ Ⓕ

fundamental

2nd harmonic

$(N-1)^{th}$ harmonic

The entries of the DFT matrix are samples of harmonically related complex sinusoids.

→ Any two columns or rows are orthogonal.
This matrix is called

$$FF^{H} = F(F^{T})^{*} = \begin{bmatrix} N & 0 & 0 & 0 & \cdots & 0 \\ 0 & N & 0 & 0 & \cdots & 0 \\ 0 & & & & & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & & \cdots & N \end{bmatrix} = NI$$

$H \rightarrow$ Hermitian

$$\boxed{F^{H} = NF^{-1}}$$

↳ Fourier matrix inverse can be found.