Indian Institute of Space Science and Technology
Thiruvananthapuram-695 547

B.Tech Summer Examination - July 2012

MA121 - Vector Calculus and Differential Equations (2011 Batch)

Date : 3^{rd}July, 2012 **Time: 9.30 am to 12.30 pm** **Max. Marks: 100**

SECTION A (Attempt all 10 questions - 10x5= 50 marks.)

1. Does the sequence $f_n(x) = \dfrac{n^2x^2}{1+n^3x^2}$, $x \in (0, \infty)$ converges uniformly?

2. Show that the series $\sum u_n(x) = \sum \dfrac{\sin nx}{n^3}$ converges pointwise on [0,1] to a limit 'u'. Examine the validity of the relation $u'(x) = \displaystyle\sum_{n=1}^{\infty} u_n'(x)$.

3. Let $h(x, y) = 2e^{-(x^2+y^2)}$ denote the height on a mountain at position (x, y). In what direction from (2,2) should one begin to walk in order to climb fastest? Explain your answer.

4. Evaluate $\displaystyle\int_C (xy + yz + zx + x^2 + y^2 + z^2)ds$ where C is the curve defined by $C : \{x^2 = y, z = 3\}$ from the point (0,0,3) to (2,4,3) in the first octant of xyz space.

5. Find the area of the surface $(z-1)^2 = x^2 + y^2,\ 2 \leq z \leq 7$ in the first octant using surface integration.

6. Verify **Green's theorem** for the vector field $\vec{F} = (x^2, xy)$ in the region $D : x^2 + y^2 \leq 1$.

7. Verify whether the pair of functions $\{f(x) = x^5,\ g(x) = |x|^5\}$ are linearly independent on $\mathbb{R}$ and also compute their Wronskian. Determine whether they can be solutions of the differential equation $y'' + p(x)y' + q(x)y = 0$ with p and q continuous on $[-a, a]$, $a > 0$.

8. Solve $e^x dx + \big(e^x \cot y + 2y\, cosec\, y\big)dy = 0$, by finding appropriate integrating factor if necessary.

9. Using **Picard's successive approximation method**, find first two approximations $\phi_1(x)$,$\phi_2(x)$ of the initial value problem $\dfrac{dy}{dx} = y + y^2, \quad y(0) = 1$. Also compare the first three terms of $\phi_2(x)$ with the exact solution.

10. Find the possible general solutions of the differential equation

$$\lambda^2 \frac{d^2y}{dt^2} + \frac{\lambda^2}{t}\frac{dy}{dt} + \left(\lambda^4 - \frac{p^2}{t^2}\right)y = 0, \quad t > 0$$

in terms of Bessel's functions, using using the substitution $z = \lambda t$, where λ and p are positive integers.

[P.T.O.]

SECTION B (Attempt any 5 questions - 5x10= 50 marks.)

11. Show that the sequence $f_n(x) = nxe^{-nx^2}$ converges pointwise on [0,1]. Examine whether the relation $\lim_{n\to\infty} \int_0^1 \lim_{n\to\infty} f_n(x)dx$ holds and also whether the convergence of $\{f_n\}$ is uniform.

12. (a) Let C be a curve in the first octant of the xyz space given by $C : \{y = x^2, z^2 = x^6\}$. Find the length of the curve from the point (0,0,0) to (2,4,8).

    (b) Using definition of directional derivative find the directional derivative of the function $f(x, y, z) = x + y + e^z$ at a point $P_0 = (0, 0, 1)$ along the vector $\vec{v} = (1, 0, 0)$.

13. (a) Let $\vec{F} = (e^x, y, z)$ be a vector field and C be the curve given by $x^2 + y^2 = 1,\ z = 5$. Find the line integral $\int_C \vec{F}.d\vec{r}$. Is the line integral path independent?

    (b) Find the flux of the vector field $\vec{F}(x, y, z) = (x, y, z)$ out of the unit sphere by

       i. using **surface integration**.

       ii. using **Gauss divergence theorem**.

14. (a) Find the largest interval $|x| \le h$, $h > 0$ on which **Picard's theorem** guarantee the existence of a unique solution of the initial value problem $\dfrac{dy}{dx} = y^3, \quad y(0) = 1.$

    (b) Solve the equation $y^2(y - px) = x^4p^2$, $p = \dfrac{dy}{dx}$, by reducing to Clairaut's form using the substitution $X = \dfrac{1}{x}, Y = \dfrac{1}{y}$.

15. (a) Find the power series expansions of $\frac{1}{1 - x + \pi}$ and $\frac{1}{9(x - \pi)^2 + 4}$ about the point $x = \pi$. Also determine the interval of convergence of the corresponding series solution of

$$(x - \pi)y'' + \frac{1}{2\pi - 2\pi x + 2\pi^2}y' + \frac{1}{\left(\frac{3}{2}(x - \pi)^3 + \frac{2}{3}(x - \pi)\right)}y = 0,$$

    about that point. (Hint:Use the power series expansion: $1 + x + x^2 + \ldots$ of $\frac{1}{1-x}$.)

    (b) Find the real-valued general solution of $\mathbf{L}(y) \equiv y^{vi} - y^{iv} - 2y'' = 0$. Also find the correct linear combination to obtain particular integral of $\mathbf{L}(y) = x\sinh(\sqrt{2}\,x)$ by the method of undetermined coefficients. (Hint: $\sinh x = (e^x - e^{-x})/2$.)

16. (a) Using Rodrigues' formula $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}(x^2 - 1)^n$, for the Legendre polynomial $P_n(x)$ of degree n, find <u>first three terms</u> of the Legendre series of

$$f(x) = \begin{cases} 0, \text{ if } -1 \le x < 0, \\ x, \text{ if } 0 \le x \le 1. \end{cases}$$

    (b) Find the eigenvalues and eigenfunctions of the boundary value problem

$$(xy')' + (\lambda/x)y = 0, \quad 1 < x < l, \quad y(1) = 0,\ y(l) = 0, \quad \lambda \in \mathbb{R}.$$

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