4. Find the solution of the following cauchy problems.

a) $yu_x + xu_y = 0$; $u(0,y) = e^{-y^2}$

b) $xu_x + yu_y = 2xy$; $u = 2$ on $y = x^2$

c) $u_x + xu_y = 0$; $u(0,y) = \sin y$.

d) $yu_x + xu_y = xy$; $x \geq 0, y \geq 0$

; with $u(0,y) = e^{-y^2}$; $y > 0$

; $u(x,0) = e^{-x^2}$; $x > 0$.

An: a) $yu_x + xu_y = 0$

Characterestic equation : $\frac{dx}{y} = \frac{dy}{x} = \frac{du}{0}$

$\frac{du}{0} = \frac{dx}{y} \Rightarrow u = c_1$

$\frac{dx}{y} = \frac{dy}{dx} \Rightarrow x^2 - y^2 = c_2$.

<u>one method</u>

Let $u = \phi(x^2 - y^2)$ — (a)

$u(0,y) = e^{-y^2}$ then eqn(a) becomes $e^{-y^2} = \phi(-y^2)$.

$\therefore$ $\phi(t) = e^t$.

so $u = \phi(x^2 - y^2) = e^{x^2 - y^2}$

$\therefore$ Solution is $u = e^{x^2 - y^2}$.

Another method

Let, $x^2 - y^2 = \psi(u)$ —— (b)

but $u(0,y) = e^{-y^2}$.

Hence (b) becomes $-y^2 = \psi(e^{-y^2})$

put $t = e^{-y^2} \Rightarrow \log t = -y^2$.

ie, $\psi(t) = \log t$

$\therefore x^2 - y^2 = \log(u) \Rightarrow u = e^{x^2 - y^2}$ is the solution.

(b). $xu_x + yu_y = 2xy$; $u = 2$ on $y = x^2$.

characterestics equation, $\frac{dx}{x} = \frac{dy}{y} = \frac{du}{2xy}$

$\frac{dx}{x} = \frac{dy}{y} \Rightarrow \frac{x}{y} = c_1$

$\frac{ydx + xdy}{2xy} = \frac{du}{2xy} \Rightarrow ydx + xdy = du$

$\Rightarrow d(xy) = du$

$\Rightarrow xy - u = c_2$

one method

$\frac{x}{y} = \phi(xy - u)$ —— (a)

$\because u = 2$ on $y = x^2$, (a) becomes $\frac{x}{x^2} = \phi(x \cdot x^2 - 2)$

$\Rightarrow 1/x = \phi(x^3 - 2)$.

$t = x^3 - 2 \Rightarrow t + 2 = x^3 \Rightarrow (t+2)^{1/3} = x$

$\Rightarrow (t+2)^{-1/3} = \phi(t)$

$$\frac{x}{y} = \phi(xy-u) = (xy-u+2)^{-1/3}$$

$$\Rightarrow \frac{y^3}{x^3} = (xy-u+2)$$

$$\Rightarrow u = xy + 2 - y^3/x^3$$ is the solution.

Second method

$$xy - u = \phi\left(\frac{x}{y}\right) \quad \text{— (b)}$$

$u = 2$ on $y = x^2$; (b) becomes, $x^3 - 2 = \phi(1/x)$

$t = 1/x \Rightarrow x = 1/t$

$$\therefore \phi(t) = \frac{1}{t^3} - 2$$

$$xy - u = \frac{1}{(x/y)^3} - 2 = y^3/x^3 - 2$$

$$\Rightarrow u = xy + 2 - y^3/x^3$$ is the solution.

c) $u_x + x u_y = 0$; $u(0, y) = \sin y$

characterestic equation, $\frac{dx}{1} = \frac{dy}{x} = \frac{du}{0}$.

$$\frac{du}{0} = dx \Rightarrow u = c_1$$

$$\frac{dx}{1} = \frac{dy}{x} \Rightarrow x^2 - 2y = c_2.$$

one method

$u = \phi(x^2 - 2y)$ —— (a).

we have $u(0,y) = \sin y$ then,

$\sin y = \phi(-2y)$

$t = -2y \Rightarrow y = -t/2$

$\phi(t) = \sin(-t/2)$

$u = \phi(x^2 - 2y) = \sin\left(-\frac{(x^2-2y)}{2}\right) = \sin\left(\frac{2y-x^2}{2}\right).$

$u = \sin\left(\frac{2y-x^2}{2}\right)$ is the solution.

second method

$x^2 - 2y = \psi(u)$ —— (b)

we have $u(0,y) = \sin y$, then

(b) becomes, $-2y = \psi(\sin y)$

put $t = \sin y$ then $y = \sin^{-1}(t)$.

$\therefore \psi(t) = -2\sin^{-1}(t)$.

(b) $\Rightarrow x^2 - 2y = -2\sin^{-1}(u)$

$\Rightarrow \frac{2y - x^2}{2} = \sin^{-1}(u)$

$\Rightarrow u = \sin\left(\frac{2y - x^2}{2}\right)$ is the solution.

d) $yu_x + xu_y = xy$

$$\frac{dx}{y} = \frac{dy}{x} = \frac{du}{xy}$$

$$\frac{dx}{y} = \frac{dy}{x} \Rightarrow x^2 - y^2 = c_1$$

$$\frac{dx}{y} = \frac{du}{xy} \Rightarrow x^2 - 2u = c_2$$

* $u(0,y) = e^{-y^2}$; $y > 0$

<u>one method</u>

Let $x^2 - y^2 = \phi(x^2 - 2u)$

By applying the initial condition,

$$-y^2 = \phi(-2e^{-y^2})$$

Put $t = -2e^{-y^2} \Rightarrow -t/2 = e^{-y^2}$

$$\Rightarrow \log(-t/2) = -y^2$$

$\therefore$ $\phi(t) = \log(-t/2)$

then $x^2 - y^2 = \phi(x^2 - 2u) = \log\left(\frac{2u - x^2}{2}\right)$

$$\Rightarrow \frac{2u - x^2}{2} = e^{x^2 - y^2} \Rightarrow u = \frac{1}{2}\left(x^2 + 2e^{x^2 - y^2}\right)$$

$u = \frac{x^2}{2} + e^{x^2 - y^2}$ is the solution.

Another ~~so~~ method

$$x^2 - 2u = \psi(x^2 - y^2)$$

$$u(0,y) = e^{-y^2} \; ; \; y > 0$$

$$\Rightarrow \quad -2e^{-y^2} = \psi(-y^2)$$

Put $t = -y^2$ then $\psi(t) = -2e^{t}$

so $x^2 - 2u = \psi(x^2 - y^2) = -2e^{x^2 - y^2}$

$\Rightarrow \quad u = \frac{x^2}{2} + e^{x^2 - y^2}$ is the solution.

* $u(x,0) = e^{-x^2} \; ; \; x > 0$. — ⓐ

One method

Let $x^2 - y^2 = \phi(x^2 - 2u)$

From ⓐ, $\quad x^2 = \phi(x^2 - 2e^{-x^2})$

Put $\quad t = x^2 - 2e^{-x^2}$

$-(t - x^2)e^{x^2} = 2 \Rightarrow (x^2 - t)e^{x^2} = 2$

$\Rightarrow (x^2 - t)\, e^{x^2 - t} = 2e^{-t}$

$t = x^2 - 2e^{-x^2}$
$t - x^2 = -2e^{-x^2}$
$(x^2 - t)e^{x^2} = 2$
$(x^2 - t)e^{x^2 - t} = 2e^{-t}$
$w((t+y^2)e^{y^2+t}) = w(2e^{t})$
$t + y^2 = 2$

We have Lambert function which has the property that it is the solution of the equation $ye^y = x$ for complex numbers x and y. ie, $w(x) = y$.

Here apply Lambert function, then we get $w((x^2 - t)e^{x^2 - t}) = w(2e^{-t})$

$$\Rightarrow (x^2 - t) = w(2e^{-t}) \quad \Rightarrow \quad x^2 = t + w(2e^{-t}).$$

$\therefore \ \phi(t) = t + \omega(2e^{-t})$

$\Rightarrow \ x^2 - y^2 = \phi(x^2 - 2u) = x^2 - 2u + \omega(2e^{2u - x^2})$

$\Rightarrow \ 2u - y^2 = \omega(2e^{2u - x^2})$

$\Rightarrow \ \omega^{-1}(2u - y^2) = 2e^{2u - x^2}$

$\Rightarrow \ (2u - y^2)e^{2u - y^2} = 2e^{2u - x^2}$

$\Rightarrow \ \cancel{2}(2u - y^2)e^{x^2 - y^2} = 2$

$\Rightarrow \ u = \frac{1}{2}(y^2 + 2e^{y^2 - x^2})$ is the solution.

<u>Another method</u>

$x^2 - 2u = \phi(x^2 - y^2)$

From ⓐ, $x^2 - 2e^{-x^2} = \phi(x^2)$

put $t = x^2$ then $\phi(t) = t - 2e^{-t}$

$\therefore \ x^2 - 2u = \phi(x^2 - y^2) = x^2 - y^2 - 2e^{y^2 - x^2}$

$\Rightarrow \ u = \frac{1}{2}(y^2 + 2e^{y^2 - x^2})$ is the solution.

5. Find the solution of the cauchy problem

$$2xyu_x + (x^2+y^2)u_y = 0 \quad \text{with} \quad u = \exp\left(\frac{x}{x-y}\right) \text{ on } x+y=1.$$

An: Chara eqn is $\dfrac{dx}{2xy} = \dfrac{dy}{x^2+y^2} = \dfrac{du}{0}$

$\dfrac{du}{0}$ will give $u = c_1$.

$$\frac{dx+dy}{x^2+y^2+2xy} = \frac{dy-dx}{x^2+y^2-2xy} \Rightarrow \frac{dx+dy}{(x+y)^2} = \frac{dy-dx}{(x-y)^2} = \frac{dy-dx}{(y-x)^2}$$

$$\Rightarrow \frac{-1}{(x+y)} + \frac{-1}{x-y} = -c_2 \quad \Rightarrow \frac{1}{x+y} + \frac{1}{x-y} = c_2$$

$$\Rightarrow \frac{2x}{x^2-y^2} = c_2$$

One method

Let $\quad u = \phi\left(\dfrac{2x}{x^2-y^2}\right)$ —— ⓐ

We have $x+y=1$, $u = e^{\frac{x}{x-y}}$

ie, $\quad y = 1-x$ and $u = e^{\frac{x}{x-(1-x)}} = e^{\frac{x}{2x-1}}$

So eqn ⓐ becomes $\quad e^{\frac{x}{2x-1}} = \cancel{\phi\left(\frac{2(1-x)}{x^2-(1-x)^2}\right)}\ \phi\left(\dfrac{2x}{2x-1}\right)$

put $t = \frac{2x}{2x-1} \Rightarrow t/2 = \frac{x}{2x-1}$

then $e^{t/2} = \phi(t)$.

So (a) becomes $u = \phi\left(\frac{2x}{x^2-y^2}\right) = e^{\frac{x}{x^2-y^2}}$.

Hence $u = e^{\frac{x}{x^2-y^2}}$ is the solution.

Another method

$$\frac{2x}{x^2-y^2} = \phi(u)$$

then using $x+y=1$, $u = e^{\frac{x}{x-y}}$, we get

$$\frac{2x}{2x-1} = \phi\left(e^{\frac{x}{2x-1}}\right)$$

Put $t = e^{\frac{x}{2x-1}}$ then $\log t = \frac{x}{2x-1}$.

Then $\phi(t) = 2\log t \Rightarrow \frac{2x}{x^2-y^2} = \phi(u) = 2\log u$

$\Rightarrow \log u = \frac{x}{x^2-y^2} \Rightarrow u = e^{\frac{x}{x^2-y^2}}$.

$\therefore$ $u = e^{\frac{x}{x^2-y^2}}$ is the solution.

6) Solve the equation $u_x + x u_y = y$ with the cauchy data,

a) $u(0,y) = y^2$

b) $u(1,y) = 2y$.

An: $u_x + x u_y = y \Rightarrow \frac{dx}{1} = \frac{dy}{x} = \frac{du}{y}$

$\frac{dx}{1} = \frac{dy}{x} \Rightarrow x^2 - 2y = c_1$

$\frac{dx}{1} = \frac{du}{y}$ but $x^2 - 2y = c_1 \Rightarrow y = \frac{x^2 - c_1}{2}$

$\Rightarrow \frac{x^2 - c_1}{2} dx = du$

Hence $\frac{x^3}{6} - \frac{c_1 x}{2} - u = c_2$

or $\frac{x^3}{6} - \frac{(x^2 - 2y)x}{2} - u = c_2$.

a) <u>One method</u>

Let $x^2 - 2y = \phi\left(\frac{x^3}{6} - \frac{(x^2-2y)x}{2} - u\right)$ — ①

using $u(0,y) = y^2$, we get,

$-2y = \phi(-y^2)$

or $-2(y^2)^{1/2} = \phi(-y^2)$

Put $t = -y^2 \Rightarrow -t = y^2$ then $-2(-t)^{1/2} = \phi(t)$

$\therefore \phi(t) = -2\sqrt{-t}$

ie, $x^2 - 2y = -2\sqrt{u - \frac{x^3}{6} + \frac{(x^2-2y)x}{2}}$ from ① or $\dfrac{(x^2-2y)^2}{4} = u - \dfrac{x^3}{6} + \dfrac{(x^2-2y)x}{2}$

$\Rightarrow u = \dfrac{(x^2-2y)^2}{4} - \dfrac{(x^2-2y)x}{2} + \dfrac{x^3}{6}$ is the solution.

Another method

Let $\dfrac{x^3}{6} - \dfrac{(x^2-2y)x}{2} - u = \varphi(x^2-2y)$. —— (2)

with $u(0,y) = y^2$

then (2) $\Rightarrow -y^2 = \varphi(-2y)$.

$t = -2y \Rightarrow y = -t/2$

$\Rightarrow \varphi(t) = -\dfrac{t^2}{4}$

ie, $\dfrac{x^3}{6} - \dfrac{(x^2-2y)x}{2} - u = \dfrac{-(x^2-2y)^2}{4}$

$\therefore u = \dfrac{(x^2-2y)^2}{4} - \dfrac{(x^2-2y)x}{2} + \dfrac{x^3}{6}$ is the solution.

b) One method

Let $x^2 - 2y = \phi\left(\dfrac{x^3}{6} - \dfrac{(x^2-2y)x}{2} - u\right)$ —— (3)

using $u(1,y) = 2y$, we get

$\frac{1}{6} - \frac{1}{2} + y - 2y$

$-\frac{1}{3} - y$

$1 - 2y = \phi\left(\frac{1}{6} - \dfrac{(1-2y)}{2} - 2y\right) = \phi(-\frac{1}{3} - y)$

$t = -\frac{1}{3} - y \Rightarrow -y = t + \frac{1}{3}$.

Hence $\phi(t) = 1 + 2(t + \frac{1}{3}) = 2t + \frac{5}{3}$

$$x^2-2y = 2\left(\frac{x^3}{6} - \frac{(x^2-2y)x}{2} - u\right) + 5/3$$

$$\Rightarrow \tfrac{1}{2}\left(x^2-2y-5/3\right) = \frac{x^3}{6} - \frac{(x^2-2y)x}{2} - u$$

$$\Rightarrow \quad u = \frac{x^3}{6} - \frac{(x^2-2y)x}{2} - \frac{x^2-2y}{2} + 5/6 \quad \text{is the solution.}$$

Another method

Let $\dfrac{x^3}{6} - \dfrac{(x^2-2y)x}{2} - u = \psi(x^2-2y)$ —— (4)

$u(1,y) = 2y \Rightarrow$ (4) become $\tfrac{1}{6} - \dfrac{(1-2y)}{2} - 2y = \psi(1-2y)$

Put $t = 1-2y \Rightarrow y = \dfrac{1-t}{2}$

Hence $\tfrac{1}{6} - \tfrac{t}{2} - (1-t) = \psi(t)$

$$\Rightarrow \psi(t) = t/2 - 5/6$$

So $\dfrac{x^3}{6} - \dfrac{(x^2-2y)x}{2} - u = \dfrac{(x^2-2y)}{2} - 5/6$

$$\Rightarrow \quad u = \frac{x^3}{6} - \frac{(x^2-2y)x}{2} - \frac{(x^2-2y)}{2} + 5/6 \quad \text{is the solution.}$$

8. Solve the cauchy problem $(y+u)u_x + y u_y = x-y$ with $u = 1+x$ on $y=1$.

An:
$$\frac{dx}{y+u} = \frac{dy}{y} = \frac{du}{x-y}$$

$$\frac{dx-dy}{y+u-y} = \frac{du}{x-y} \Rightarrow \frac{dx-dy}{u} = \frac{du}{x-y}$$

$$\Rightarrow (x-y)\,d(x-y) = u\,du$$

$$\Rightarrow (x-y)^2 - u^2 = C_1$$

$$\frac{dx+du}{x+u} = \frac{dy}{y} \Rightarrow \frac{1}{x+u}\,d(x+u) = \frac{1}{y}\,d(y).$$; Apply integral then ~~[illegible]~~ remove logarithm.

$$\Rightarrow \frac{x+u}{y} = C_2.$$

<u>one method</u>

Let $(x-y)^2 - u^2 = \phi\left(\frac{x+u}{y}\right)$ —— ①

$u = 1+x,\ y=1 \Rightarrow$ ① become $(x-1)^2 - (1+x)^2 = \phi(2x+1)$

$$\Rightarrow \phi(2x+1) = -4x$$

Put $t = 2x+1$ then $t-1 = 2x$

$$\Rightarrow \phi(t) = -2(t-1)$$

① $\Rightarrow (x-y)^2 - u^2 = \phi\left(\frac{x+u}{y}\right) = -2\left(\frac{x+u}{y} - 1\right)$

$\Rightarrow (x-y)^2 - u^2 = \dfrac{2(y-x-u)}{y}$ is the solution.

Another solution

Let $\frac{x+u}{y} = \psi\left((x-y)^2 - u^2\right)$ — ②

$u = 1+x$; $y = 1$

② ~~be come~~ become , $2x+1 = \psi\left((x-1)^2 - (1+x)^2\right)$
$= \psi(-4x)$.

Put $t = -4x \Rightarrow -t/2 = 2x$

$\Rightarrow -t/2 + 1 = \psi(t)$.

$\psi(t) = -t/2 + 1$

so ② $\Rightarrow \frac{x+u}{y} = \frac{u^2 - (x-y)^2}{2} + 1$

$\Rightarrow (x-y)^2 - u^2 = \bar{2}\left(\frac{x+y}{y} - 1\right)$
$= \bar{2}\left(\frac{x+u-y}{y}\right)$
$= \frac{2(y-x-u)}{y}$

So $(x-y)^2 - u^2 = \frac{2(y-x-u)}{y}$ is the solution.

9). Show that the solution of the equation $yu_x - xu_y = 0$ containing the curve $x^2 + y^2 = a^2$, $u = y$ does not exist.

9 An: Here r is $(t, \sqrt{a^2-t^2}, \sqrt{a^2-t^2})$

Here $yu_x - xu_y = 0$ so $a = y$, $b = -x$

$$y_0 = \sqrt{a^2-t^2} \Rightarrow y_0' = \frac{-t}{\sqrt{a^2-t^2}}$$

$$x_0 = t \Rightarrow x_0' = 1$$

$$a = y = \sqrt{a^2-t^2}$$

$$b = -x = -t.$$

$$\frac{b}{a} = \frac{-t}{\sqrt{a^2-t^2}} \text{ and } \frac{y_0'}{x_0'} = \frac{-t}{\sqrt{a^2-t^2}}$$

Hence initial data curve and characterestic curves are parallel hence there doesnot exist a solution for this cauchy problem.

10. Find the solution surface of the equation

$(u^2-y^2)u_x + xy\,u_y + xu = 0$ with $u(x,x) = x$; $x>0$.

An: characterestic equation : $\frac{dx}{u^2-y^2} = \frac{dy}{xy} = \frac{du}{-xu}$

$$\frac{dx}{u^2-y^2} = \frac{dy+du}{x(y-u)} \Rightarrow -x\,dx = (u+y)\,d(u+y)$$

$$\Rightarrow x^2 + (u+y)^2 = c_1$$

$$\frac{dy}{xy} = \frac{du}{-xu} \Rightarrow \frac{dy}{y} = \frac{-du}{u} \Rightarrow uy = c_2$$

one method

Let $x^2 + (u+y)^2 = \phi(uy)$

we have $u(x,x) = x \quad ; x > 0$

$\Rightarrow \quad x^2 + 4x^2 = \phi(x^2)$

Put $t = x^2 \quad \Rightarrow \phi(t) = t + 4t = 5t$

Hence $x^2 + (u+y)^2 = 5uy$. is the solution.

Another method

Let $uy = \psi(x^2 + (u+y)^2)$

$u(x,x) = x \quad \Rightarrow x^2 = \psi(5x^2)$.

Put $t = 5x^2 \Rightarrow t/5 = x^2$

then $\psi(t) = t/5$

$\Rightarrow uy = \psi(x^2 + (u+y)^2) = \dfrac{x^2 + (u+y)^2}{5}$

$\Rightarrow 5uy = x^2 + (u+y)^2$ is the solution.

13). Find the solution of the equation $yu_x - 2xyu_y = 2xu$ with the condition $u(0,y) = y^3$.

An: Chara eqn : $\dfrac{dx}{y} = \dfrac{dy}{-2xy} = \dfrac{du}{2xu}$.

$\dfrac{dx}{y} = \dfrac{dy}{-2xy} \Rightarrow 2x\,dx = -dy \Rightarrow x^2 + y = c_1$

$\dfrac{dy}{-2xy} = \dfrac{du}{2xu} \Rightarrow \dfrac{-dy}{y} = \dfrac{du}{u} \Rightarrow uy = c_2$.

<u>One method</u>

Let $x^2+y = \phi(uy)$.

$u(0,y) = y^3 \Rightarrow y = \phi(y^4)$

Put $t = y^4 \Rightarrow t^{1/4} = y$

then $\phi(t) = t^{1/4}$

$\Rightarrow x^2+y = (uy)^{1/4}$

$\Rightarrow uy = (x^2+y)^4$ is the solution.

<u>Another method</u>

Let $uy = \varphi(x^2+y)$

$u(0,y) = y^3 \Rightarrow y^4 = \varphi(y)$

$\Rightarrow \varphi(t) = t^4$

$\Rightarrow uy = \varphi(x^2+y) = (x^2+y)^4$.

$\Rightarrow uy = (x^2+y)^4$ is the solution.

14). obtain the general solution of the equation.
$(x+y+5z)p + 4zq + (x+y+z) = 0$ and find the particular solution which passes through the circle, $z=0,\ x^2+y^2=a^2$.

An: Chara eqn,
$$\frac{dx}{x+y+5z} = \frac{dy}{4z} = \frac{dz}{-(x+y+z)}.$$

$$\frac{dx-dy+dz}{0} = \frac{dy}{4z} \Rightarrow x-y+z = c_1$$

$$\frac{dx}{x+y+5z} = \frac{dy}{4z}$$

$$\because \quad x-y+z=c_1 \qquad z = c_1 - x + y$$

Hence $$\frac{dx}{x+y+5c_1-5x+5y} = \frac{dy}{4c_1-4x+4y}$$

$$\Rightarrow \quad \frac{dx}{5c_1-4x+6y} = \frac{dy}{4c_1-4x+4y}$$

$$\Rightarrow \quad 4c_1\,dx - 4x\,dx + 4y\,dx = 5c_1\,dy - 4x\,dy + 6y\,dy$$

$$\Rightarrow \quad (4c_1-4x)\,dx + 4(y\,dx + x\,dy) = (5c_1+6y)\,dy$$

$$\Rightarrow \quad 4c_1x - 2x^2 + 4xy = 5c_1y + 3y^2 + c_2$$

$$\Rightarrow \quad 4(x-y+z)x - 2x^2 + 4xy = 5(x-y+z)y + 3y^2 + c_2$$

$$\Rightarrow \quad 4x^2 - 4xy + 4xz - 2x^2 + 4xy = 5xy - 5y^2 + 5yz + 3y^2 + c_2$$

$$\Rightarrow \quad 2x^2 + 4xz - 5xy - 5yz + 2y^2 = c_2$$

$$\Rightarrow \quad 2(x^2+y^2) + 4xz - 5xy - 5yz = c_2.$$

So general solution is $f(x-y+z,\ 2(x^2+y^2)+4xz-5xy-5yz) = 0$

one method

Let $x-y+z = \phi\left(2(x^2+y^2)+4xz-5xy-5yz\right)$

$z=0,\ x^2+y^2=a^2$

then $x-\sqrt{a^2-x^2} = \phi\left(2a^2-5x\sqrt{a^2-x^2}\right)$

$t = 2a^2-5x\sqrt{a^2-x^2} \Rightarrow \frac{2a^2-t}{5} = x\sqrt{a^2-x^2}$

But $\left(x-\sqrt{a^2-x^2}\right)^2 = x^2+(a^2-x^2)-2x\sqrt{a^2-x^2}$

$= a^2-2x\sqrt{a^2-x^2}$

$= a^2-\frac{2(2a^2-t)}{5} = \frac{5a^2-4a^2+2t}{5}$

$= \frac{a^2+2t}{5}$

$\Rightarrow x-\sqrt{a^2-x^2} = \sqrt{\frac{a^2+2t}{5}}$

i.e, $\phi(t) = \sqrt{\frac{a^2+2t}{5}}$

$\Rightarrow x-y+z = \sqrt{\frac{a^2+2\left(2(x^2+y^2)+4xz-5xy-5yz\right)}{5}}$

$\Rightarrow 5(x-y+z)^2 = a^2+4(x^2+y^2)+8xz-10xy-10yz$

$\Rightarrow$ $x^2+y^2+5z^2+2xz = a^2$ is the solution.

Another method

Let $2(x^2+y^2)+4xz-5xy-5yz = \psi(x-y+z)$

$z=0,\ x^2+y^2=a^2$

Hence $\quad 2a^2+0-5x\sqrt{a^2-x^2}-0 = \psi(x-\sqrt{a^2-x^2})$

$t = x-\sqrt{a^2-x^2} \Rightarrow t^2 = a^2-2x\sqrt{a^2-x^2} \Rightarrow \frac{a^2-t^2}{2} = x\sqrt{a^2-x^2}$

$\therefore\ 2a^2-5x\sqrt{a^2-x^2} = \psi(x-\sqrt{a^2-x^2}) \Rightarrow \psi(t) = 2a^2-5\left(\frac{a^2-t^2}{2}\right)$

$\Rightarrow \psi(t) = 2a^2 - \frac{5a^2-5t^2}{2} = \frac{4a^2-5a^2+5t^2}{2} = \frac{5t^2-a^2}{2}.$

$\therefore\ \psi(t) = \frac{5t^2-a^2}{2}$

So $\quad 2(x^2+y^2)+4xz-5xy-5yz = \frac{5(x-y+z)^2-a^2}{2}$

$\Rightarrow 4x^2+4y^2+8xz-10xy-10yz = 5x^2+5y^2+5z^2-10xy+10xz$
$\qquad -10yz-a^2$

$\Rightarrow x^2+y^2+5z^2+2xz = a^2$ is the solution.

15) Obtain the general solution of the equation

$$(z^2-2yz-y^2)p + x(y+z)q = x(y-z)$$

Find the integral surfaces of this equation passing through

a) the x-axis $\quad$ b) the y-axis and $\quad$ (c) the z-axis.

An: chara eqn : $\dfrac{dx}{z^2-2yz-y^2} = \dfrac{dy}{x(y+z)} = \dfrac{dz}{x(y-z)}$

$$\frac{x\,dx + y\,dy + z\,dz}{xz^2 - 2xyz - xy^2 + xy^2 + xyz + xyz - xz^2} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

$$\Rightarrow x^2 + y^2 + z^2 = c_1$$

$$\frac{dy}{x(y+z)} = \frac{dz}{x(y-z)} \Rightarrow \frac{dy}{y+z} = \frac{dz}{y-z} \Rightarrow y\,dy - z\,dy = y\,dz + z\,dz$$

$$\Rightarrow y\,dy = z\,dy + y\,dz + z\,dz$$

$$\Rightarrow y^2 - 2yz - z^2 = c_2$$

So general solution is $f(x^2+y^2+z^2,\ y^2-2yz-z^2) = 0$.

a) Here solution passing through x-axis. ie, $y=0, z=0$.

<u>onemethod</u>

Let $y^2 - 2yz - z^2 = \phi(x^2+y^2+z^2)$

$\Rightarrow \quad 0 = \phi(x^2)$. (Put $y=0, z=0$).

$\Rightarrow \phi(t) = 0$ then $y^2 - 2yz - z^2 = 0$. is the solution

<u>Another method</u>

Let. $x^2 + y^2 + z^2 = \phi(y^2 - 2yz - z^2)$

ie, $x^2 = \phi(0)$

So we cannot proceed in this way.

$$x^2+y^2+z^2=c_1$$
$$y^2-2yz-z^2=c_2$$

b) <u>one method</u>

Let $x^2+y^2+z^2=\phi(y^2-2yz-z^2)$

here $x=0, z=0$,

then $0+y^2+0=\phi(y^2-0-0)$

$\phi(y^2)=y^2 \Rightarrow \phi(t)=t$.

so $x^2+y^2+z^2=y^2-2yz-z^2 \Rightarrow x^2+2z^2+2yz=0$ is the solution

<u>Another method</u>

Let $y^2-2yz-z^2=\psi(x^2+y^2+z^2)$

$\because$ y axis, $x=0, z=0$

$\Rightarrow y^2=\psi(y^2)$.

So $y^2-2yz-z^2=x^2+y^2+z^2$

$\Rightarrow x^2+2z^2+2yz=0$ is the solution.

c) <u>One method</u>

Let $x^2+y^2+z^2=\phi(y^2-2yz-z^2)$

z-axis $\Rightarrow x=0, y=0$

$\therefore z^2=\phi(-z^2) \Rightarrow \phi(t)=-t$

so $x^2+y^2+z^2=-y^2+2yz+z^2 \Rightarrow x^2+2y^2=2yz$ is the solution.

Another method

Let $y^2 - 2yz - z^2 = \psi(x^2+y^2+z^2)$ —①

~~on~~ z-axis $\Rightarrow x=0, y=0$

So ① $\Rightarrow -z^2 = \psi(z^2)$

$\Rightarrow \psi(t) = -t$

Hence $y^2-2yz-z^2 = -(x^2+y^2+z^2) \Rightarrow x^2+2y^2 = 2yz$ is the solution.

16) Solve the Cauchy problem

$(x+y)u_x + (x-y)u_y = 1, \quad u(1,y) = \frac{1}{\sqrt{2}}$.

Ans: chara eqn: $\dfrac{dx}{x+y} = \dfrac{dy}{x-y} = \dfrac{du}{1}$

$\dfrac{dx}{x+y} = \dfrac{dy}{x-y} \Rightarrow x\,dx - y\,dx = x\,dy + y\,dy$

$\Rightarrow x\,dx = x\,dy + y\,dx + y\,dy$

$\Rightarrow x^2 - 2xy - y^2 = c_1$

So $x^2 - 2yx - (y^2+c_1) = 0$. Consider this as quadratic equation in x. Then

$$x = \frac{2y \pm \sqrt{4y^2+4y^2+4c_1}}{2} = y \pm \sqrt{2y^2+c_1}$$

Consider $\dfrac{dy}{x-y} = \dfrac{du}{1} \Rightarrow \dfrac{dy}{y \pm \sqrt{2y^2+c_1} - y} = \dfrac{du}{1}$

$$\Rightarrow \frac{dy}{\pm\sqrt{2y^2+c_1}} = du$$

so $x = y \pm \sqrt{2y^2+c_1}$ $\Rightarrow$ $\pm \dfrac{dy}{\sqrt{2y^2+c_1}} = du$

i.e., $x-y>0$ $\Rightarrow$ $\dfrac{dy}{\sqrt{2y^2+c_1}} = du$

$x-y<0$ $\Rightarrow$ $\dfrac{-dy}{\sqrt{2y^2+c_1}} = du$

$\dfrac{1}{\sqrt{2}} \dfrac{1}{\sqrt{y^2+\left(\frac{\sqrt{c_1}}{\sqrt{2}}\right)^2}}$ $\sqrt{\dfrac{c_1}{2}}$

$\dfrac{dy}{\sqrt{2y^2+c_1}} = du$ $\Rightarrow$ $\dfrac{1}{\sqrt{2}} \log\left| y + \sqrt{y^2+\frac{c_1}{2}}\right| = u + c_2.$

or $u - \dfrac{1}{\sqrt{2}} \log\left| y + \dfrac{1}{\sqrt{2}}\sqrt{2y^2+c_1}\right| = c_2.$

If $x-y>0$ then $u - \dfrac{1}{\sqrt{2}} \log\left| y + \dfrac{1}{\sqrt{2}}\sqrt{2y^2+c_1}\right| = c_2$

$x-y<0$ then $u + \dfrac{1}{\sqrt{2}} \log\left| y + \dfrac{1}{\sqrt{2}}\sqrt{2y^2+c_1}\right| = c_2$

$\therefore$ $u - \dfrac{sgn(x-y)}{\sqrt{2}} \log\left| y + \dfrac{1}{\sqrt{2}}\sqrt{2y^2+c_1}\right| = c_2$

or $u - \dfrac{sgn(x-y)}{\sqrt{2}} \log\left| y + \dfrac{1}{\sqrt{2}}\sqrt{x^2-2xy+y^2}\right| = c_2$

$\Rightarrow$ $u - \dfrac{sgn(x-y)}{\sqrt{2}} \log\left| y + \dfrac{1}{\sqrt{2}}|x-y|\right| = c_2$

Now we have $x^2-2xy-y^2 = c_1$ and

$u - \dfrac{sgn(x-y)}{\sqrt{2}} \log\left| y + \dfrac{1}{\sqrt{2}}|x-y|\right| = c_2.$

One method

Let $u - \dfrac{sgn(x-y)}{\sqrt{2}} \log\left| y + \dfrac{1}{\sqrt{2}}|x-y|\right| = \phi\left(x^2-2xy-y^2\right)$ — (1)

with $u(1,y) = \dfrac{1}{\sqrt{2}}$.

$\Rightarrow$ $\dfrac{1}{\sqrt{2}} - \dfrac{sgn(1-y)}{\sqrt{2}} \log\left| y + \dfrac{1}{\sqrt{2}}|1-y|\right| = \phi\left(1-2y-y^2\right)$ — (2)

put $t = 1-2y-y^2$

$$-t = y^2+2y-1$$
$$= y^2+2y+1-2$$
$$= (y+1)^2-2$$

$$\Rightarrow 2-t = (y+1)^2$$

$$\Rightarrow |y+1| = \sqrt{2-t} \quad ; \quad 2-t \geq 0.$$

$$y = \sqrt{2-t}-1 \quad ; \quad y \geq -1$$

$$y = -\sqrt{2-t}-1 \quad ; \quad y \leq -1$$

$$\Rightarrow \text{② becomes, } \frac{1}{\sqrt{2}} - \frac{\operatorname{sgn}(2 \mp \sqrt{2-t})}{\sqrt{2}} \log\left| \pm\sqrt{2-t} - 1 + \frac{1}{\sqrt{2}}\left|2 \mp \sqrt{2-t}\right|\right|$$
$$= \phi(t)$$

$\therefore$ ① become

$$u - \frac{\operatorname{sgn}(x-y)}{\sqrt{2}} \log\left|y + \frac{1}{\sqrt{2}}|x-y|\right| = \frac{1}{\sqrt{2}} - \frac{\operatorname{sgn}(2 \mp \sqrt{2+y^2+2xy-x^2})}{\sqrt{2}} \times$$
$$\log\left|\pm\sqrt{2+y^2+2xy-x^2} - 1 + \frac{1}{\sqrt{2}}\left|2 \mp \sqrt{2+y^2+2xy-x^2}\right|\right|$$

$$\Rightarrow u = \frac{\operatorname{sgn}(x-y)}{\sqrt{2}} \log\left|y + \frac{1}{\sqrt{2}}|x-y|\right| + \frac{1}{\sqrt{2}}$$
$$- \frac{\operatorname{sgn}(2 \mp \sqrt{2+y^2+2xy-x^2})}{\sqrt{2}} \log\left|-1 \pm \sqrt{2+y^2+2xy-x^2} + \frac{1}{\sqrt{2}}\left|2 \mp \sqrt{2+y^2+2xy-x^2}\right|\right|$$

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Solution of Tutorial I.

1. Done in the class, try yourself.

2. Find the pde arising from each of the following surfaces.

a) $Z = x + y + f(xy)$

$Z_x = 1 + f'(xy)\, y$

$Z_y = 1 + f'(xy)\, x$

ie, $p - 1 = f'(xy)\, y$

$q - 1 = f'(xy)\, x$

$\Rightarrow \dfrac{p-1}{q-1} = \dfrac{y}{x}$

$\Rightarrow xp - x = yq - y$

$\Rightarrow xp - yq + y - x = 0$ is a pde arising from given z.

Similarly do b, c and d.

3. Find the general solution of each of the following equations.

b) $(y+u)\, u_x + y\, u_y = x - y$.

Chara eqn : $\dfrac{dx}{y+u} = \dfrac{dy}{y} = \dfrac{du}{x-y}$

Consider $\dfrac{dx - dy}{u} = \dfrac{du}{x-y} \Rightarrow (x-y)^2 - u^2 = c_1$

$\dfrac{dx + du}{x+u} = \dfrac{dy}{y} \Rightarrow \dfrac{(x+u)}{y} = c_2$.

So $f\left((x-y)^2 - u^2, \dfrac{x+u}{y}\right) = 0$ is general solution.

7. $(u_x+u_y)^2 - u^2 = 0$

$u = e^x \Rightarrow u_x = e^x; \ u_y = 0,$

$\Rightarrow (u_x+u_y)^2 - u^2 = (e^x+0)^2 - (e^x)^2 = 0$ ✓

$u = e^{-y} \Rightarrow u_x = 0; \ u_y = -e^{-y}$

$\Rightarrow (u_x+u_y)^2 - u^2 = (0-e^{-y})^2 - (e^{-y})^2 = 0$ ✓

$u = e^x + e^{-y} \Rightarrow u_x = e^x; \ u_y = -e^{-y}$

$\Rightarrow (u_x+u_y)^2 - u^2 = (e^x - e^{-y})^2 - (e^x + e^{-y})^2$

$\Rightarrow = e^{2x} + e^{-2y} - 2e^{x-y} - (e^{2x} + e^{-2y} + 2e^{x-y})$

$= -4e^{x-y} \neq 0 \quad \forall \, x, y \in \mathbb{R}.$

$\therefore$ $e^x + e^{-y}$ is not a sln.

11. solve the following equations.

a) $(y+u)u_x + (x+u)u_y = x+y$

Chara eqn : $\frac{dx}{y+u} = \frac{dy}{x+u} = \frac{du}{x+y}$

Consider $\frac{dx-dy}{y-x} = \frac{dx-du}{u-x} \Rightarrow \frac{x-y}{x-u} = C_1$

$\frac{dx-dy}{y-x} = \frac{dy-du}{u-y} \Rightarrow \frac{x-y}{u-y} = C_2$.

Solution is $f\left(\frac{x-y}{x-u}, \frac{x-y}{u-y}\right) = 0$, where f is an arbitrary function.

Do (b) and (c) yourself.

19. Find a complete integral of the equation

$(p^2+q^2)x = pz$.

$f(x,y,z,p,q) = (p^2+q^2)x - pz = 0$.

$f_z = -p$

Charpits equations

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{pf_z + f_x} = \frac{dq}{qf_z + f_y}$$

$$\Rightarrow \frac{dx}{-(2px-z)} = \frac{dy}{-(2qx)} = \frac{dz}{-2p^2x - pz - 2q^2x} = \frac{dp}{-p^2+p^2+q^2} = \frac{dq}{-pq}$$

Consider $\frac{dp}{+q^2} = \frac{dq}{-pq} \Rightarrow \frac{dp}{q} = \frac{-dq}{p} \Rightarrow p^2+q^2 = c^2$

but $(p^2+q^2)x = pz \Rightarrow c^2 x = pz$

$\Rightarrow p = \frac{c^2 x}{z}$

$\Rightarrow q = \sqrt{c^2 - \left(\frac{cx}{z}\right)^2}$

$= \frac{c}{z}\sqrt{z^2 - c^2x^2}$

$\therefore dz = p\,dx + q\,dy$

$= \frac{c^2 x}{z}\,dx + \frac{\sqrt{z^2-c^2x^2}\,dy}{z}$

$\Rightarrow \frac{z\,dz - c^2 x\,dx}{\sqrt{z^2-c^2x^2}} = a\,dy$

put $z^2 - c^2x^2 = t$ so that $2(z\,dz - c^2x\,dx) = dt$

$\Rightarrow \frac{1}{2\sqrt{t}}\,dt = a\,dy$

$\Rightarrow \sqrt{t} = ay + b$

$\Rightarrow \sqrt{z^2 - c^2x^2} = cy + b$

$\Rightarrow z^2 - c^2x^2 = (cy+b)^2$

$\Rightarrow z^2 = c^2x^2 + (cy+b)^2$. is the complete solution.

Note Refer solution pdf II. Rest of the problem try yourself. If you are facing any problem discuss with me.