AYD 641 - Computer Vision
Assignment 1

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SC22B146

① Prove that in homogeneous coordinates, the intersection point $\hat{x}$ of two lines $\hat{l}_1$ and $\hat{l}_2$ is given by $\hat{x} = \hat{l}_1 \times \hat{l}_2$.

Soln: Let $\hat{l}_1 : a_1 x_0 + b_1 y_0 + c_1 = 0$

$\hat{l}_2 : a_2 x_0 + b_2 y + c_2 = 0$

$$\Rightarrow a_1 b_2 x + \cancel{b_1 b_2 y} + c_1 b_2 = 0$$
$$a_2 b_1 x + b_1 \cancel{b_2} y + c_2 b_1 = 0$$
$$- \quad - \quad -$$
$$(a_1 b_2 - a_2 b_1) x + (c_1 b_2 - c_2 b_1) = 0$$

$$\Rightarrow x = \frac{c_2 b_1 - c_1 b_2}{a_1 b_2 - a_2 b_1}$$

$$\therefore y = -\frac{c_1}{b_1} - \frac{a_1}{b_1} x$$
$$= -\frac{c_1}{b_1} - \frac{a_1}{b_1}\left(\frac{c_2 b_1 - c_1 b_2}{a_1 b_2 - a_2 b_1}\right)$$
$$= \frac{-c_1(\cancel{a_1 b_2} - a_2 b_1) - a_1 b_1 c_2 + \cancel{a_1 b_2 c_1}}{b_1(a_1 b_2 - a_2 b_1)}$$
$$= \frac{a_2 c_1 - a_1 c_2}{a_1 b_2 - a_2 b_1}$$

Point of intersection: $\left(\frac{c_2 b_1 - c_1 b_2}{a_1 b_2 - a_2 b_1}, \frac{a_2 c_1 - a_1 c_2}{a_1 b_2 - a_2 b_1}\right) \equiv \hat{x}$

Now $$\hat{l}_1 \times \hat{l}_2 = [\hat{l}_1]_\times \hat{l}_2 = \begin{pmatrix} 0 & -c_1 & b_1 \\ c_1 & 0 & -a_1 \\ -b_1 & a_1 & 0 \end{pmatrix}_{3\times 3} \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}_{3\times 1}$$

$$= \begin{pmatrix} -c_1 b_2 + b_1 c_2 \\ c_1 a_2 - c_2 a_1 \\ -a_2 b_1 + b_2 a_1 \end{pmatrix}_{3\times 1}$$

$$\equiv \begin{pmatrix} \frac{b_1 c_2 - c_1 b_2}{a_1 b_2 - a_2 b_1} \\ \frac{a_2 c_1 - a_1 c_2}{a_1 b_2 - a_2 b_1} \\ 1 \end{pmatrix} = \hat{x} \qquad \therefore \boxed{\hat{l}_1 \times \hat{l}_2 = \hat{x}}.$$

② Prove that the line that joins two points $\hat{x}_1$ and $\hat{x}_2$ is given by $\hat{l} = \hat{x}_1 \times \hat{x}_2$.

Soln: Let $\left.\begin{array}{l} \hat{x}_1 : (a_1, b_1) \\ \hat{x}_2 : (a_2, b_2) \end{array}\right\} \in \mathbb{R}^2$

Line joining $\hat{x}_1$ and $\hat{x}_2$:

$$y - b_1 = \frac{b_2 - b_1}{a_2 - a_1}(x - a_1)$$

$$\Rightarrow (a_2 - a_1)y + (b_1 - b_2)x + (-b_1 a_2 + \cancel{b_1 a_1} + a_1 b_2 - \cancel{a_1 b_1}) = 0$$

$$\Rightarrow (b_1 - b_2)x + (a_2 - a_1)x + (a_1 b_2 - a_2 b_1) = 0$$

$$\hookrightarrow \hat{l}\ (\in \mathbb{P}^2) \equiv \begin{pmatrix} b_1 - b_2 \\ a_2 - a_1 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

Now,

$$\hat{x}_1 \times \hat{x}_2 = [\hat{x}_1]_\times \hat{x}_2 = \begin{pmatrix} 0 & -1 & b_1 \\ 1 & 0 & -a_1 \\ -b_1 & a_1 & 0 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} b_1 - b_2 \\ a_2 - a_1 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} = \hat{l}$$

$$\therefore \boxed{\hat{l} = \hat{x}_1 \times \hat{x}_2}.$$

③ Given the following two lines in $\mathbb{R}^3$:

$$l_1 = \{(x, y)^T \in \mathbb{R}^3 \mid x + y + 3 = 0\},$$

$$l_2 = \{(x, y)^T \in \mathbb{R}^3 \mid -x - 2y + 7 = 0\},$$

perform the following tasks:

ⓘ Find the intersection point of the two lines by solving the corresponding system of linear equations.

Soln: $l_1 : x + y + 3 = 0$

$l_2 : -x - 2y + 7 = 0$

$-y + 10 = 0 \Rightarrow \boxed{y = 10} \Rightarrow x = -y - 3 \Rightarrow \boxed{x = -13} \quad \therefore \hat{x}_1 = \begin{pmatrix} -13 \\ 10 \\ 1 \end{pmatrix}$

(ii) Rewrite the lines using homogeneous coordinates, and calculate their intersection point using the cross product of their homogeneous representations.

Soln: $l_1 \rightarrow \hat{l}_1 : \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$, $l_2 \rightarrow \hat{l}_2 : \begin{pmatrix} -1 \\ -2 \\ 7 \end{pmatrix}$

$\downarrow$ $\qquad\qquad$ $\downarrow$

$(1\ 1\ 3)\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$ $\qquad$ $(-1\ -2\ 7)\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$

$$\therefore \hat{x}_2 = \hat{l}_1 \times \hat{l}_2$$
$$= [\hat{l}_1]_\times l_2$$
$$= \begin{pmatrix} 0 & -3 & 1 \\ 3 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ -2 \\ 7 \end{pmatrix}$$
$$= \begin{pmatrix} 13 \\ -10 \\ -1 \end{pmatrix} \equiv \begin{pmatrix} -13 \\ 10 \\ 1 \end{pmatrix}.$$

(iii) Verify whether the intersection point obtained in homogeneous coordinates is the same as the one obtained from solving the system of equations.

Soln: Cartesian coordinate : $x = -13,\ y = 10$

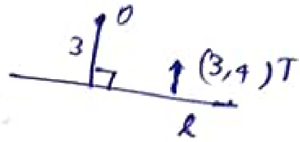
$$\Rightarrow \hat{x}_1 \equiv \begin{pmatrix} -13 \\ 10 \\ 1 \end{pmatrix}$$

Homogeneous coordinate : $\hat{x}_2 = \begin{pmatrix} 13 \\ -10 \\ -1 \end{pmatrix} = \hat{l}_1 \times \hat{l}_2$

which represent the same point as $\hat{x}_1 = \begin{pmatrix} -13 \\ 10 \\ 1 \end{pmatrix} (= \hat{l}_2 \times \hat{l}_1)$ in the projective space, as in projective space, the points that differ by only a non-zero scalar factor are considered the same.

4. Write down the equation of the line whose normal vector is in the direction $(3,4)^T$ and which is at a distance of 3 units from the origin.

Soln: Let line $l: ax+by+c=0$



As $(3,4)^T \perp \hat{l} \Rightarrow l: 3x+4y+c=0$

Also, $\perp^r$ distance from $(0,0)$ is 3.

$$\Rightarrow \frac{|c|}{\sqrt{9+16}} = 3 \Rightarrow |c| = 15$$
$$\Rightarrow c = \pm 15$$

$$\therefore l \equiv 3x+4y \pm 15 = 0$$

In homogeneous coordinate,

$$\hat{l}: \begin{pmatrix} 3 \\ 4 \\ \pm 15 \end{pmatrix}^T \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0$$

$$\text{or } \hat{l} = \begin{pmatrix} 3 \\ 4 \\ \pm 15 \end{pmatrix}.$$

5. Determine the distance from the origin and the normalised normal vector for the homogeneous line $\hat{l} = \begin{pmatrix} 2 \\ 5 \\ \sqrt{29}/5 \end{pmatrix}$.

Soln: $\hat{l} = \begin{pmatrix} 2 \\ 5 \\ \frac{\sqrt{29}}{5} \end{pmatrix} \longrightarrow l: \; 2x+5y+\frac{\sqrt{29}}{5} = 0$

$$\text{Distance from origin} = \frac{\left|\frac{\sqrt{29}}{5}\right|}{\sqrt{4+25}} = \frac{\frac{\sqrt{29}}{5}}{\sqrt{29}} = \frac{1}{5}.$$

$$\text{Normalised normal vector} = \frac{(2,5)}{\|(2,5)\|} = \frac{(2,5)}{\sqrt{29}} = \left(\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\right)$$

$$= \begin{pmatrix} \frac{2}{\sqrt{29}} \\ \frac{5}{\sqrt{29}} \\ 1 \end{pmatrix} \text{ in } \mathbb{P}^2.$$

⑥ Write down the 2x3 translation matrix that maps the point $(1,2)^T$ to $(0,3)^T$.

Soln: Translation matrix is given by

$$T = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \end{pmatrix}$$

$$\therefore \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\Rightarrow 1 + t_x = 0 \Rightarrow t_x = -1$$

$$2 + t_y = 3 \Rightarrow t_y = 1$$

$$\therefore T = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

⑦ Assume you are given N correspondence pairs in 2D:

$$(x_i, y_i) = \left(\begin{pmatrix} x_1^i \\ x_2^i \end{pmatrix}, \begin{pmatrix} y_1^i \\ y_2^i \end{pmatrix} \right), \quad i = 1, 2, \dots, N.$$

Find the 2x3 translation matrix T that maps x_i onto y_i, which is optimal in the least-squares sense.

Soln: Translation matrix, $T = \begin{pmatrix} 1 & 0 & t_1 \\ 0 & 1 & t_2 \end{pmatrix}$

$$T.x_i = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \end{pmatrix} \cdot \begin{pmatrix} x_{1i} \\ x_{2i} \\ 1 \end{pmatrix} = \begin{pmatrix} x_{1i} + t_1 \\ x_{2i} + t_2 \end{pmatrix}$$

Define a cost function as

$$E(T) = \sum_{i=1}^{N} \| Tx_i - y_i \|_2^2$$

$$= \sum_{i=1}^{N} \left((x_{1i} + t_1 - y_{1i})^2 + (x_{2i} + t_2 - y_{2i})^2 \right)$$

Find the optimal T^* that minimizes E as

$$T^* = \arg\min_T E(T).$$

Find T by calculating the Jacobian J_E of E and setting it to 0^T:

$$J_E = \left[\frac{\partial E}{\partial t_1}, \frac{\partial E}{\partial t_2}\right] = 0^T = (0,0).$$

$$\Rightarrow \frac{\partial E}{\partial t_1} = 2\sum_{i=1}^{N}(x_{1i} + t_1 - y_{1i}) = 0,$$

$$\frac{\partial E}{\partial t_2} = 2\sum_{i=1}^{N}(x_{2i} + t_2 - y_{2i}) = 0$$

$$\Rightarrow \sum_{i=1}^{N} x_{1i} + N t_1 - \sum_{i=1}^{N} y_1^i = 0,$$

$$\sum_{i=1}^{N} x_2^i + N t_2 - \sum_{i=1}^{N} y_2^i = 0,$$

$$\Rightarrow t_1 = \frac{\sum_{i=1}^{N} y_1^i - \sum_{i=1}^{N} x_1^i}{N}, \quad t_2 = \frac{\sum_{i=1}^{N} y_{2i} - \sum_{i=1}^{N} x_2^i}{N}.$$

$$\therefore T^* = \begin{pmatrix} 1 & 0 & t_1 \\ 0 & 1 & t_2 \end{pmatrix}, \text{ where } t_1, t_2.$$

Explanation: T^* is computing the average shift required to align points x_i to points y_i, by t_1 doing it in x-direction and t_2 in y-direction.

(8) You are given the following three correspondence pairs:

$$\left(\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}3\\-5\end{pmatrix}\right), \left(\begin{pmatrix}5\\7\end{pmatrix}, \begin{pmatrix}7\\6\end{pmatrix}\right), \left(\begin{pmatrix}4\\1\end{pmatrix}, \begin{pmatrix}5\\-4\end{pmatrix}\right).$$

Using the equation you derived for T^*, calculate the optimal 2×3 translation matrix T^*.

Soln: $$t_1 = \frac{1}{N}\left[\sum_1^N y_1^i - \sum_1^N x_1^i\right] = \frac{1}{3}\left[(3+7+5) - (0+5+4)\right] = \frac{6}{3} = 2.$$

$$t_2 = \frac{1}{N}\left[\sum_{i=1}^{N} y_2^i - \sum_{i=1}^{N} x_2^i\right] = \frac{1}{3}\left[(-5+6+(-4)) - (1+7+1)\right] = -\frac{12}{3} = -4.$$

$$\therefore T^* = \begin{pmatrix} 1 & 0 & t_1 \\ 0 & 1 & t_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -4 \end{pmatrix}.$$