1. In class, we discussed the concept of homogeneous coordinates. In this example, we will confine ourselves to the real 2D plane. A point $(x, y)^T$ on the real 2D plane can be represented in homogeneous coordinates by a 3-vector $(wx, wy, w)^T$, where $w \neq 0$ is any real number. All values of $w \neq 0$ represent the same 2D point. Dividing out the third coordinate of a homogeneous point (x, y, z) converts it back to its 2D equivalent: $(x/z, y/z)^T$.

Consider a line in the 2D plane, whose equation is given by $ax+by+c = 0$. This can equivalently be written as $l^T x = 0$, where $\mathbf{l} = (a, b, c)^T$ and $\mathbf{x} = (x, y, 1)^T$. Noticing that $\mathbf{x}$ is a homogeneous representation of $(x, y)^T$, we define $\mathbf{l}$ as a homogeneous representation of the line $ax+by+c = 0$. Note that the line $(ka)x + (kb)y + (kc) = 0$ for $k \neq 0$ is the same as the line $ax + by + c = 0$, so the homogeneous representation of the line $ax + by + c = 0$ can be equivalently given by $(a, b, c)^T$ or $(ka, kb, kc)^T$ for any $k \neq 0$. All points $(x, y)^T$ that lie on the line $ax + by + c = 0$ satisfy the equation $\mathbf{l}^T\mathbf{x} = 0$, thus, we can say that a condition for a homogeneous point x to lie on the homogeneous line l is that their dot product is zero, that is, $\mathbf{l}^T\mathbf{x} = 0$. We note this down as a fact:

A point x in homogeneous coordinates lies on the homogeneous line l if and only if $\mathbf{x}^T\mathbf{l} = \mathbf{l}^T\mathbf{x} = 0$

Now let us solve a few simple examples:

(a) Give at least two homogeneous representations for the point $(3, 5)^T$ on the 2D plane, one with $w > 0$ and one with $w < 0$. (1 mark)

Choose $w = 1$ for $(3, 5, 1)^\top$ and say, $w = -1$ for $(-3, -5, -1)^\top$.

(b) What is the equation of the line passing through the points $(1, 1)^T$ and $(-1, 3)^T$ [in the usual Cartesian coordinates]. Now write down a 3-vector that is a homogeneous representation for this line. (1 mark)

Equation of line in usual Cartesian coordinates: $x + y - 2 = 0$.
Homogeneous representation: $(1, 1, -2)^\top$.

(c) Consider the two lines $x + y - 5 = 0$ and $4x - 5y + 7 = 0$. find their intersection in homogeneous coordinates. Convert this homogeneous point back to standard Cartesian coordinates. Is your answer compatible with what basic coordinate geometry suggests? (2 marks)

To find intersection point in homogeneous coordinates, first compute cross product of homogeneous lines $(1, 1, -5)^\top$ and $(4, -5, 7)^\top$:

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & -5 \\ 4 & -5 & 7 \end{vmatrix} = -18\hat{\mathbf{i}} - 27\hat{\mathbf{j}} - 9\hat{\mathbf{k}}$$

Thus, homogeneous point of intersection is $(-18, -27, -9)^\top$, or equivalently, $(2, 3, 1)^\top$ by dividing out the third coordinate.

To convert back to Cartesian, simply divide out the third coordinate and drop it, so Cartesian representation is $(2, 3)^\top$.

The coordinate geometry approach would be to solve the simultaneous system of equations:

$$\begin{aligned} x + y &= 5 \\ 4x - 5y &= -7 \end{aligned}$$

Solution is $x = 2$, $y = 3$. So, coordinate geometry gives the same point of intersection as projective geometry.

(d) Consider the two lines $x + 2y + 1 = 0$ and $3x + 6y - 2 = 0$. What is the special relationship between these two lines in the Euclidean plane? What is the interpretation of their intersection in standard Cartesian coordinates? (2 marks)

The two lines are parallel, both have slope $-\frac{1}{2}$.
They do not intersect at a finite point on the Euclidean plane.

(e) Write the homogeneous representations of the above two lines and compute their point of intersection in homogeneous coordinates. What is this point of intersection called in computer vision parlance? (2 marks)

Homogeneous representation: $(1,2,1)^\top$ and $(3,6,-2)^\top$.
Intersection point in homogeneous coordinates: $(1,2,1)^\top \times (3,6,-2)^\top = (-10,-5,0)^\top$.
This is called the vanishing point for the lines. A vanishing point is a point at infinity (or an ideal point).

(f) Give (with justification) an expression for the homogeneous representation of the line passing through two homogeneous points x1 and x2. (2 marks)

For a line $\mathbf{l}$ to pass through $\mathbf{x}_1$ and $\mathbf{x}_2$, both $\mathbf{x}_1$ and $\mathbf{x}_2$ must lie on the line $\mathbf{l}$. Thus, from **Fact 1**, $\mathbf{x}_1^\top \mathbf{l} = 0$ and $\mathbf{x}_2^\top \mathbf{l} = 0$. The vector $\mathbf{l}$ must be orthogonal to both $\mathbf{x}_1$ and $\mathbf{x}_2$, such a vector is $\mathbf{x}_1 \times \mathbf{x}_2$. So, the line passing through points $\mathbf{x}_1$ and $\mathbf{x}_2$ is $\mathbf{l} = \mathbf{x}_1 \times \mathbf{x}_2$.

(g) Find the point of intersection of the 3D line $x = \lambda d + x_0$ with the 3D plane $p^T x = 0$. The parameters have the following (2 marks)
$d = [1,2,0]^T$,$x_0 = [3,4,5]^T$, $p = [1,2,0,7]^T$

$$\lambda \mathbf{p}_{1:3}^\top \mathbf{d} + \mathbf{p}_{1:3}^\top \mathbf{x}_0 + p_4 = 0$$

Solve for λ.

$$\lambda = -\frac{\mathbf{p}_{1:3}^\top \mathbf{x}_0 + p_4}{\mathbf{p}_{1:3}^\top \mathbf{d}}$$

Point of intersection is

$$\mathbf{x} = \lambda \mathbf{d} + \mathbf{x}_0$$

2. Consider a world coordinate system W, centered at the origin $(0,0,0)$, with axes given by unit vectors $\hat{\mathbf{i}} = (1,0,0)^T$, $\hat{\mathbf{j}} = (0,1,0)^T$ and $\hat{\mathbf{k}} = (0,0,1)^T$. Recall our notation where boldfaces stand for a vector and a hat above a boldface letter stands for a unit vector.

(a) Consider another coordinate system, with unit vectors along two of the orthogonal axes given by $\hat{\mathbf{i}}' = (0.9, 0.4, 0.1\sqrt{3})$ and $\hat{\mathbf{j}}' = (-0.41833, 0.90427, 0.08539)$ Find the unit vector, $\hat{\mathbf{k}}'$, along the third axis orthogonal to both $\hat{\mathbf{i}}'$ and $\hat{\mathbf{j}}'$. Is there a unique unit vector orthogonal to both $\hat{\mathbf{i}}'$ and $\hat{\mathbf{j}}'$? If not, choose the one that makes an acute angle with $\hat{\mathbf{k}}'$. (2 marks)

There are two vectors orthogonal to $\hat{\mathbf{i}}'$ and $\hat{\mathbf{j}}'$. One of them is $\hat{\mathbf{i}}' \times \hat{\mathbf{j}}'$ and the other is $\hat{\mathbf{j}}' \times \hat{\mathbf{i}}'$. Let $\mathbf{a} = \hat{\mathbf{i}}' \times \hat{\mathbf{j}}'$ and $\mathbf{b} = \hat{\mathbf{j}}' \times \hat{\mathbf{i}}' = -\mathbf{a}$. Computing the cross-product, $\mathbf{a} = (-0.1225, -0.1493, 0.9812)^\top$.
Then, $\mathbf{b} = -\mathbf{a} = (0.1225, 0.1493, -0.9812)^\top$.

Let θ be the angle between $\mathbf{a}$ and $\widehat{\mathbf{k}}$. Let θ_b be the angle between $\mathbf{b}$ and $\widehat{\mathbf{k}}$. Then,

$$\begin{aligned}\cos\theta_a &= \mathbf{a}\cdot\widehat{\mathbf{k}} \\ &= (-0.1225, -0.1493, 0.9812)^\top \cdot (0,0,1)^\top \\ &= 0.9812\end{aligned}$$

Since $\cos\theta_a > 0$, $|\theta_a| < 90^\circ$, so we choose $\widehat{\mathbf{k}}' = \mathbf{a}$. (Note that $\|\mathbf{a}\| = 1$ already.)

(b) Find the rotation matrix that rotates any vector in the $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ coordinate system to the $\hat{\mathbf{i}}$', $\hat{\mathbf{j}}$' and $\hat{\mathbf{k}}$' coordinate system. (2 marks)

$$R = \begin{bmatrix} \widehat{\mathbf{i}}'\cdot\widehat{\mathbf{i}} & \widehat{\mathbf{i}}'\cdot\widehat{\mathbf{j}} & \widehat{\mathbf{i}}'\cdot\widehat{\mathbf{k}} \\ \widehat{\mathbf{j}}'\cdot\widehat{\mathbf{i}} & \widehat{\mathbf{j}}'\cdot\widehat{\mathbf{j}} & \widehat{\mathbf{j}}'\cdot\widehat{\mathbf{k}} \\ \widehat{\mathbf{k}}'\cdot\widehat{\mathbf{i}} & \widehat{\mathbf{k}}'\cdot\widehat{\mathbf{j}} & \widehat{\mathbf{k}}'\cdot\widehat{\mathbf{k}} \end{bmatrix} = \begin{bmatrix} 0.9 & 0.4 & 0.1732 \\ -0.4183 & 0.9043 & 0.0854 \\ -0.1225 & -0.1493 & 0.9812 \end{bmatrix}.$$

(c) What is the extrinsic parameter matrix for a camera that is located at a displacement $(-1,-2,-3)$ from the origin of W and oriented such that its principal axis coincides with $\hat{\mathbf{k}}$', the x-axis of its image plane coincides with $\hat{\mathbf{i}}$' and the y-axis of the image plane coincides with $\hat{\mathbf{j}}$'? (2 marks)

Center of camera in world coordinates, $\mathbf{C}_w = (-1,-2,-3)^\top$. Extrinsic parameter matrix is

$$P_E = \begin{bmatrix} R & -R\mathbf{C}_w \\ \mathbf{0}^\top & 1 \end{bmatrix} = \begin{bmatrix} 0.9 & 0.4 & 0.1732 & 2.2196 \\ -0.4183 & 0.9043 & 0.0854 & 1.6464 \\ -0.1225 & -0.1493 & 0.9812 & 2.5224 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(d) What is the intrinsic parameter matrix for this camera, if its focal length in the x-direction is 1050 pixels, aspect ratio(Aspect ratio is primarily dictated by the size of your camera's sensor, taken from the width and height of an image (W:H). For instance, if your camera sensor is 36mm wide and 24mm high, its aspect ratio would be 3:2.) is 1.0606, pixels deviate from rectangular by 0.573 degrees and principal point is offset from the center $(0,0)^T$ of the image plane to the location $(10,-5)^T$. (2 marks)

$f_x = 1050$ and aspect ratio = 1.0606 $\Rightarrow f_y = \dfrac{1050}{1.0606} = 990.$

Skew = $\cot(90^\circ - 0.573^\circ) = \cot\left((90-0.573)\times\dfrac{\pi}{180}\text{rad}\right) = 0.01.$

Position of principal point in new coordinates: $(p_x, p_y)^\top = (-10, 5)^\top$.

Intrinsic parameter matrix,

$$K = \begin{bmatrix} 1050 & 0.01 & -10 \\ 0 & 990 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

(e) Write down the projection matrix for the camera described by the configuration in parts (c) and (d). (2 marks)

Projection matrix, $P = K\,[\,I\,|\,\mathbf{0}\,]\,P_E$, where

$$[\,I\,|\,\mathbf{0}\,] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Computing the matrix product, we find

$$P = \begin{bmatrix} 946.2 & 421.5 & 172.1 & 2305.4 \\ -414.8 & 894.5 & 89.4 & 1642.5 \\ -0.1225 & -0.1493 & 0.9812 & 2.5224 \end{bmatrix}.$$

5. Which of the following factor does not aect the intrinsic parameters of camera model? (1 mark)

(a) Focal length

(b) Offset of optical center

(c) Exposure

(d) Image resolution

Exposure

4. Assuming the camera coordinate system is the same as the world coordinate system, the intrinsic and extrinsic parameters of the camera can map any point in homogenous world coordinates to a unique point in the image plane.(True or false, give a suitable reasoning) (1 mark)

Solution (False), In this situation, the vector $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ spans the null space of the camera matrix, and represents the camera origin, and the projective line in this case is ambiguous.