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Problem 1. *Show that an affine transformation can map a circle to an ellipse, but cannot map an ellipse to a hyperbola or parabola.*

Solution. Consider the general quadratic equation of a conic section:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0. \tag{1}$$

An affine transformation is given by:

$$\begin{bmatrix} x \\ y \end{bmatrix} = M \begin{bmatrix} x' \\ y' \end{bmatrix} + \mathbf{b}, \tag{2}$$

where M is a non-singular 2×2 matrix:

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{3}$$

and the translation vector is

$$\mathbf{b} = \begin{bmatrix} e \\ f \end{bmatrix}. \tag{4}$$

This gives,

$$x = ax' + by' + e, \quad y = cx' + dy' + f. \tag{5}$$

Substituting these expressions into the general quadratic equation:

$$A(ax'+by'+e)^2+B(ax'+by'+e)(cx'+dy'+f)+C(cx'+dy'+f)^2+D(ax'+by'+e)+E(cx'+dy'+f)+F = 0. \tag{6}$$

Solving, we get a transformed conic equation:

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0, \tag{7}$$

where the new coefficients A', B', C' depend on the original coefficients A, B, C and the transformation matrix M.

For the transformed conic, the discriminant is

$$\Delta' = B'^2 - 4A'C'. \tag{8}$$

and the new coefficients are:

$$A' = Aa^2 + Bac + Cc^2, \tag{9}$$
$$B' = 2Aab + B(ad + bc) + 2Ccd, \tag{10}$$
$$C' = Ab^2 + Bbd + Cd^2. \tag{11}$$

This gives,

$$\Delta' = (B^2 - 4AC)(ad - bc)^2. \tag{12}$$

Since $\det(M) = ad - bc \neq 0$ for an affine transformation (which is non-singular), we get:

$$\Delta' = (ad - bc)^2 \Delta. \tag{13}$$

Since $(ad - bc)^2 > 0$ for a non-singular transformation, the sign of Δ remains unchanged. This means that for a circle or an ellipse, $\Delta < 0$ (ellipse), then $\Delta' < 0$, hence it will remain an ellipse. It cannot change an ellipse ($\Delta < 0$) into a hyperbola ($\Delta > 0$) or a parabola ($\Delta = 0$).

□

Problem 2. *Prove that under an affine transformation, the ratio of lengths on parallel line segments is an invariant, but that the ratio of two lengths that are not parallel is not.*

Solution. Consider two parallel line segments AB and CD, with lengths $|AB|$ and $|CD|$. The ratio of these lengths is:

$$r = \frac{|AB|}{|CD|} \tag{14}$$

In affine transformation, each point $\mathbf{x}$ is mapped to a new point $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$. Since affine transformations preserve parallelism, the images of AB and CD remain parallel.

Affine transformations comprises of a linear transformation followed by a translation. The translation does not affect lengths, while the linear transformation A may scale distances differently in different directions.

Since the transformation is linear, it acts uniformly along parallel lines. That is, there exists a constant scale factor k along the direction of these parallel segments such that:

$$|A'B'| = k|AB|, \quad |C'D'| = k|CD| \tag{15}$$

Thus, the new ratio after transformation is:

$$\frac{|A'B'|}{|C'D'|} = \frac{k|AB|}{k|CD|} = \frac{|AB|}{|CD|} \tag{16}$$

which shows that the ratio of lengths along parallel lines is invariant under an affine transformation.

Now, consider two segments AB and PQ that are not parallel. The affine transformation may scale them by different factors because affine transformations scales differently in different directions.

Let the scale factor along AB be k_1 and along PQ be k_2. Then, after transformation:

$$|A'B'| = k_1|AB|, \quad |P'Q'| = k_2|PQ| \tag{17}$$

where $k_1 \neq k_2$ in general.

The new ratio is:

$$\frac{|A'B'|}{|P'Q'|} = \frac{k_1|AB|}{k_2|PQ|} \tag{18}$$

Since k_1 and k_2 are not equal, the ratio is not preserved in general.

□