INDIAN INSTITUTE OF SPACE SCIENCE AND TECHNOLOGY
THIRUVANANTHAPURAM 695 547

Summer Supplementary Examination - June 2015

B. Tech - II Semester

MA121 - Vector Calculus and Differential Equations

Date: 22/06/2015 **Time: 9.30 am - 12.30 pm** **Max. Marks: 100**

SECTION A (Attempt all 10 questions - 10x5= 50 marks.)

1. Let a surface S be given by $x^2 + y^2 = 4z^2$ where $z \in [0, 1]$. Find the parametric representation of S. Find $\int_S f$, if exists, where $f(x, y, z) = x^2 + y^2 + z^2$. [2+3]

2. State Green's theorem over simply-connected regions. Let D be an elliptical region on xy-plane given by $D = \left\{ (x, y) \in \mathbb{R}^2 \middle| \frac{x^2}{16} + \frac{y^2}{4} \leq 1 \right\}$. With explanation find the area of the region D using Green's theorem . [1+4]

3. Define directional derivative. If exists, find the directional derivative of $f(x, y, z) = xye^z + z^2$ at the point $(1, 1, 0)$ along the vector $\vec{v} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$. If exists, with explanation, find the direction along which the rate of change of f is maximum. [1+2+2]

4. Let $\vec{F}$ be a conservative vector field with gradient function f defined on an open set in $\mathbb{R}^3$ containing a curve $C = \gamma : [a, b] \longrightarrow \mathbb{R}^2$. Show that $\int_C \vec{F} = f(\gamma(b)) - f(\gamma(a))$ [5]

5. Find the general solution of the following differential equations [5]

$$x^3 \frac{d^3y}{dx^3} - 4x^2 \frac{d^2y}{dx^2} + 8x \frac{dy}{dx} - 8y = 4 \ln x.$$

6. Discuss the existence and uniqueness of the solution for the following differential equations [5]

   (a) $\frac{dy}{dx} = y^{1/3}, \quad y(0) = 0,$

   (b) $\frac{dy}{dx} = |y|, \quad y(0) = 0.$

7. Find two linearly independent solutions of the following differential equations. Also, write the general solution. $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - 4y = 0.$ [5]

8. Using method of variation of parameter, find a particular solution of $\frac{d^2y}{dx^2} + y = f(x)$, where f is a continuous function on $\mathbb{R}$. [5]

9. Discuss the point wise and uniform convergence of the sequence $f_n(x) = \frac{n^2 \ln x \ \sin nx}{x^n}$, when $x \in [2, +\infty)$. [5]

10. Check whether the series $\sum_{n=1}^{\infty} \frac{xe^{-nx}}{n^2}$ converges uniformly on $(0, +\infty)$. [5]

SECTION B (Attempt any 5 questions - 5x10= 50 marks.)

11. (a) State Stoke's theorem for simply-connected surfaces. Verify Stoke's theorem for $\vec{F}(x,y,z) = (x^2,\ 0,\ 3y^2)$ over the surface $S\ :\ x^2+y^2+z^2=1,\ z \geq 0$ [2+4]

    (b) Let $\vec{F} = \left(\dfrac{-y}{x^2+y^2},\ \dfrac{x}{x^2+y^2},\ e^z\right)$. Find the domain of $\vec{F}$. Calculate $Curl(\vec{F})$. Show that for any loop C around z-axis $\displaystyle\int_C \vec{F} = 2\pi$. [0.5+ 0.5 + 3]

12. (a) Define arc length function of a curve $\gamma : [a,b] \longrightarrow \mathbb{R}^3$. Find arc length function of the curve $C : y = 4x^2,\ z = 5,\ x \geq 0$ with initial point $(0,0,5)$. [2+2]

    (b) Let $\vec{F}(x,y,z) = (2xe^{yz}+2x,\ x^2ze^{yz}+2y,\ x^2ye^yz+2z)$. Find the domain of F. Let $C = C_1 * C_2 * C_3$ be a curve where C_1 is the line segment from $(-2,-1,5)$ to $(-1,0,5)$, C_2 is given by $x^2+y^2=1,\ z=5,\ y \leq 0$ and C_3 is the line segment from $(1,0,5)$ to $(2,-1,5)$. Find $\displaystyle\int_C \vec{F}$, if exists. Is the integral path independent? Justify your answer. [4+2]

13. Find the general solution of the following differential equations: [10]

    (a) $\dfrac{d^4y}{dx^4} - 4\dfrac{d^3y}{dx^3} + 14\dfrac{d^2y}{dx^2} - 20\dfrac{dy}{dx} + 25y = 0$

    (b) $(D-2)^3y = e^{2x}x^4$, here $D = \dfrac{d}{dx}$

14. Find all the eigen (characteristics) values and the eigen functions of the following differential equations [10]

$$\frac{d}{dx}\left[x\frac{dy}{dx}\right] + \frac{\lambda}{x}y = 0, \quad \lambda \geq 0.$$
$$y'(1) = 0, \quad y'(e^{2\pi}) = 0.$$

15. Let $\{f_n\}$ be a sequence of functions defined on an interval I. Under what conditions the following can be justified: $\dfrac{d}{dx}\left(\lim\limits_{n\to\infty} f_n(x)\right) = \lim\limits_{n\to\infty}\left(\dfrac{d}{dx}f_n(x)\right)$. Examine this relation for the sequence $f_n(x) = \dfrac{x}{1+nx^2}$, $x \in \mathbb{R}$. [10]

16. State the Weierstrass test for uniform convergence of a series $\displaystyle\sum_{n=1}^{\infty} f_n(x)$ on an interval I. Check whether the function $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{\sin nx}{2^n}$ is differentiable on $(-\infty,+\infty)$. Justify your answer. [10]

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