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ORDINARY DIFFERENTIAL EQUATIONS

DIFFERENTIAL EQUATIONS

3-04-2023

Function: $y : (a,b) \to \mathbb{R}$

$y = y(x),\ x \in (a,b)$

$x \to$ Independent variable

$y \to$ dependent variable

$\frac{dy}{dx}$ or $y'(x) \to$ Derivative

Function in two variables:

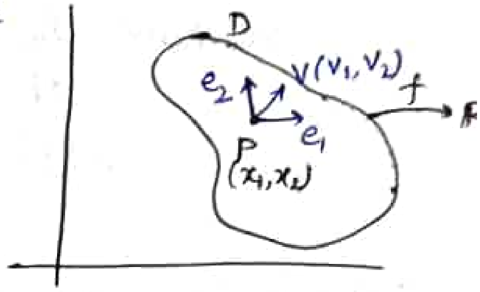
$y : D \to \mathbb{R},\ D \subset \mathbb{R}^2$

$y = y(x_1, x_2)$

$x_1, x_2 \to$ Independent variables

$y \to$ Dependent variable

$\frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}$ or $y_{x_1}, y_{x_2} \to$ Partial derivative



$$D_v f(p) = \lim_{t \to 0} \frac{f(p+v) - f(p)}{t}$$

$$(v_1, v_2) = v = v_1 e_1 + v_2 e_2$$
$$= v_1(1,0) + v_2(0,1)$$
$$= (v_1, v_2)$$

→ A differential equation is an equation involving a dependent variable and its derivaties with respect to the independent variabl

→ An ordinary differential equation is a differential eqn in which there is only one independent variable so that the derivatives involved are only ordinary derivatives.

→ A differential equation in which more than one independent variables are involved is called a partial differential equation. so that in which derivatives involved are all partial derivatives.

→ A general form of an ODE is

$$F(x, y, y', y'', \ldots) = 0$$

Eg. ① $y''(x) - y(x) = 4x$

② $y'' + y' + \cos x = 0$

Eg. Radioactive disintegration of a substance.

Rate of disintegration depends on the mass of the substance (i.e., the amount of substance present at that time).

Let m be the mass of the undisintegrated substance at time t.

$$m = m(t)$$

$$m : [0, \infty) \to \mathbb{R}$$

We have $\frac{dm}{dt} = -Km$, K is positive number.

$$m'(t) + Km(t) = 0$$

Order of differential equation:

Order of an ODE is the order of the highest derivative involved in the differential eqn.

Eg. ① $y' + 2x = \sin x$ ⟷ order = 1

② $(1+(y')^3)^{1/2} = y$ → Order = 1

③ $y'' = \sqrt{x} + \sqrt{y}$ → order = 2

Degree:

Degree of an ODE is the degree or power of the highest order derivative involved in the equation, after making the eqn free of radicals and fractions of derivatives.

Eg ① $(y')^3 + y = 0$ ⟶ degree = 3

② $(1 + (y')^3)^{1/4} = y \Rightarrow 1 + (y')^3 = y^4$ ⟶ degree = 3

③ $y'' = (y')^{1/2} \Rightarrow (y'')^2 = y'$ ⟶ degree = 2

FIRST ORDER ODE

we consider first, an ODE of first order.

General form is

$$F(x, y, y') = 0$$

or $y' = f(x, y)$

where $y : (a,b) \to \mathbb{R}$

$y = y(x)$

$\cancel{y'^2 = \cos x}$

Linear ODE

→ An ODE is said to be linear if the dependent variable and its derivatives have power not more than one and no product of dependent variable and its derivatives and of derivatives of any order.

→ An ODE which is not linear is said to be non-linear eq^n

Eg $\quad y'' + y' = 2xy \;\}$ linear

$$\left.\begin{array}{l} y'^2 = \cos x + y \\ y'' + yy' = 2 \\ y''y' + y = 2x \end{array}\right\} \text{not linear}$$

Solution :

Solution of a first order ODE

$$y' = f(x, y)$$

means a real or complex function $y = h(x)$ defined on an interval which is differentiable and satisfies $y' = f(x, y)$,

i.e., $y : (a,b) \to \mathbb{R}$ or $\mathbb{C}$

$x \to y = h(x)$

Eg ① $y' = y$

Soln: $y = e^x$

② $x^2 y' + 1 = 0 \longrightarrow (*)$

Soln : $y = \frac{1}{x}$ in $(1,3)$

or $xy = 1$ or $xy - 1 = 0$ is a soln of $(*)$

6-4-2023

First Order ODE

$F(x, y, y')=0$ or $y'=f(x,y)$

where $y:(a,b)\to\mathbb{R}$, $y=y(x)$

Solution: $\exists\, y=h(x)$ on (a,b) which is differential and it satisfies

$y'=f(x,y)$, i.e., $h'(x)=f(x,y)$.

Eg. ① $y'-y=0$ has soln $y(x)=e^x$ (explicit)

② $x^2y'+1=0$

$xy-1=0$ is a soln in the interval $(2,4)$.

Note: $y=ce^x$ is also soln of $y'-y=0$ for $c\in\mathbb{R}$, any constant.

→ A solution involving a constant is called a general solution of the first order ODE.

Geometrically, the solutions are curves.

If we put a specific value for 'c', we call it as a particular ~~sol~~ solution.

Separable Equations

A first order ODE of the form

$$g(y)\,y'=f(x) \quad ; \quad y:(a,b)\to\mathbb{R},\ y=y(x)$$

is called a separable ODE.

$$g(y)\frac{dy}{dx}=f(x) \Rightarrow g(y)\,dy=f(x)\,dx$$

Integrating w.r.t x,

$$\int g(y)\,dy=\int f(x)\,dx+c$$

If the integral exists, we get a solution.

Eg. ① Find a soln of $yy' + 25x = 0$

Soln: Integrating w.r.t. x

$$\int yy'\,dx + \int 25x\,dx = c$$

$$\Rightarrow \int y\,dy + \int 25x\,dx = c$$

$$\Rightarrow y^2 + 25x^2 = c'$$

② $xy' = x + y$

Soln: $x^2u' + xu = x + xu$

Put $u = y/x$

$\Rightarrow y = x\,u$

$\Rightarrow y' = xu' + u$

To find a soln in an interval that is not containing zero.

$\Rightarrow x^2u' = x$

$\Rightarrow u' = \frac{1}{x} \quad (x \neq 0)$

$\Rightarrow u = \log x + c$

i.e., $\frac{y}{x} = \log x + c$

$\Rightarrow y = x\log x + cx$

Exact Differential Equation

A first order ODE of the form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is said to be exact if (the differential form) $M(x,y)dx + N(x,y)\,dy$ is exact, i.e., $\exists$ a function $u(x,y)$ such that

$$du = M(x,y)\,dx + N(x,y)\,dy.$$

$$u : D \overset{\subset \mathbb{R}^3}{\longrightarrow} \mathbb{R}$$

$u(x,y)$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

i.e., $M = \frac{\partial u}{\partial x}$ & $N = \frac{\partial u}{\partial y}$.

$\therefore$ The ODE $M\,dx + N\,dy = 0$

becomes $du = 0$

So, the solution is $u(x,y) = c$.

Note: Suppose M and N have continuous partial derivatives.

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x}$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x \partial y}$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

→ The necessary and sufficient condition for $M dx + N dy = 0$ to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Necessary and sufficient condition means converse is also true.

Eg. Find soln of ① $2xy\,dx + x^2 dy = 0$

Soln: $M(x,y) = 2xy$, $N(x,y) = x^2$

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$$

So, it is exact.

Assume such a $u(x,y)$ exists.

Then $\frac{\partial u}{\partial x} = M(x,y)$.

Integrating wrt x,

$$u(x,y) = \int M(x,y)\,dx + K(y)$$

constant, may depend on y (as we've integrated wrt x)

$$= \boxed{x^2 y + K(y)} \xrightarrow{\text{diff}} =0$$

$$\frac{\partial u}{\partial y} = x^2 + K'(y) = N(x,y)$$
$$= x^2$$

$$\Rightarrow K'(y) = 0 \Rightarrow K(y) = c, \text{ constant}$$

So, soln is

$$u(x,y) = yx^2 + c.$$

② $x\,dy + (y - x^3)\,dx = 0$.

Soln: $M(x,y) = y - x^3$, $N(x,y) = x$.

10-04-23

Initial Value Problem

An ordinary differential eqⁿ with an initial condition is said to be an initial value problem (IVP).

In general, an IVP of first order is

$$f = f(x,y), \quad y(x_0) = y_0$$

where x_0, y_0 are given values.

The existence and Uniqueness of Solutions to an IVP:

There may be

(i) no solution other than trivial solⁿ

(ii) there is a unique solⁿ

(iii) infinitely many solutions.

(Q. type): For an IVP

(i) the conditions for existence of solⁿ

(ii) the conditions for unique solⁿ.

Theorem (Existence):

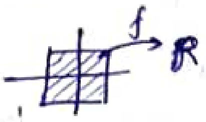
If $f(x,y)$ is continuous for all (x,y) in a rectangle $R : |x-x_0| < a$, $|y-y_0| < b$ and $f(x,y)$ is bounded in R, say $|f(x,y)| \le K$ for all $(x,y) \in R$ then the IVP $y' = f(x,y)$, $y(x_0) = y_0$ has at least one solution.

$D \subset \mathbb{R}^2$; $f: D \to \mathbb{R}$ is bounded means

$$|f(x,y)| \le K \text{ for all } (x,y) \in D, \ K > 0.$$

Eg. ① $f(x,y) = |x| + |y|$

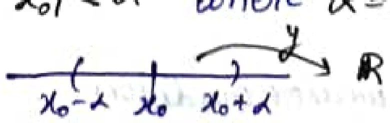
in the rectangle $|x| \le 1$, $|y| \le 1$.



② $f(x,y) = \dfrac{2y}{x}$ is not bounded in $|x| \le 1, |y| \le 1$.

↳ The solution $y = y(x)$ is defined for all x with

$$|x - x_0| < \alpha \quad \text{where } \alpha = \min\left\{a, \frac{b}{K}\right\}.$$



Theorem (Uniqueness):

If $f(x,y)$ and $\frac{\partial f}{\partial y}$ are continuous and bounded in a rectangle R: $|x-x_0|<a$, $|y-y_0|<b$, say $|f(x,y)|\le K$ and $\left|\frac{\partial f}{\partial y}\right| \le M$ for all $(x,y)\in R$,

then the IVP $y'=f(x,y)$, $y(x_0)=y_0$ has at most one solution.

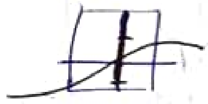
That is, the IVP has a unique solution (by existence theorem).

The unique solⁿ exists in $|x-x_0|<\alpha$ where $\alpha=\min\{a,\frac{b}{K}\}$.

Note: ① The conditions in both the theorems are only sufficient conditions. (i.e., conditions ⇒ solⁿ exist, but solⁿ can also exist in other cases).

② The conditions $\left|\frac{\partial f}{\partial y}\right|\le M$ in R can be replaced by a weaker ~~soluti~~ condition.

By MVT,



$$f(x,y_2)-f(x,y_1) = (y_2-y_1)\left.\frac{\partial f}{\partial y}\right|_{\tilde{y}}$$

where (x,y_1), (x,y_2) are points in R and $\tilde{y}$ is a point b/w y_1 & y_2.

$$\left|\frac{\partial f}{\partial y}\right| \le M \text{ for all } (x,y)\in R$$

$$\text{So, } \left|\left.\frac{\partial f}{\partial y}\right|_{\tilde{y}}\right| \le M$$

$$\therefore\ |f(x,y_1)-f(x,y_2)| \le M|y_2-y_1|$$

If $f(x,y)$ satisfies this condition, we say that f is Lipschitz function with respect to y.

Eg. $f(x,y)=x|y|$ where $|x|\le 1$, $|y|\le 1$:

$\frac{\partial f}{\partial y}$ does not exist throughout R

$$\frac{|f(0,y)-f(0,0)|}{|y|} \le M$$

So, Lipschitz condition is a weaker condition.

Q. Show that $xy'=4y$, $y(0)=1$ has no solution. Does this contradict the existence theorem?

Solⁿ $y' = 4\frac{y}{x}$

R: $|x|<a$, $|y-1|<b$

f is not continuous in a rectangle containing $(0,1)$.

$\frac{dy}{y}=\frac{4}{x}dx$

$\Rightarrow \log|y| = 4\log x + c_1$

$\Rightarrow y = cx^4$

c depends on y (treating)

$y' = f(x,y)$, $y(x_0) = y_0$

$|x - x_0| < a$, $|y - y_0| < b$

If $f(x,y)$ is continuous and bounded in R, then IVP has a soln.

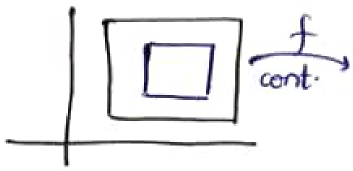
Then soln exists in $|x - x_0| < \alpha$

where $\alpha = \min\left(a, \frac{b}{K}\right)$

K is such that $|f(x,y)| \le K$ on R.

For uniqueness,

both f & $\frac{\partial f}{\partial y}$ are continuous and bounded in R, the unique soln exists.



Q Show that $y' = 3y^{2/3}$, $y(0) = 0$ has more than one solution for $|x| < 1$, $|y| < 1$.

Does it contradict the uniqueness theorem.

Soln: Note that $y(x) = 0$ is a solution.

Also $y(x) = x^3$ is a soln.

$$\frac{\partial f}{\partial y} = \frac{2}{y^{1/3}}$$

This is not continuous and bounded in the given rectangle. So, the unique soln doesn't exist which doesnot contradict the theorem.

Q Find the Largest possible interval in which the soln of

$y' = y^2 + \cos x$, $y(0) = 0$ exists.

Soln: Take $R: |x| < a$, $|y| < b$

$f(x,y) = y^2 + \cos x$ is continuous in R.

$|f(x,y)| = |y^2 + \cos x| < 1 + b^2$

$\frac{\partial f}{\partial y} = 2y$ is also continuous & bounded.

So, the soln exists.

Solutions exist in the interval

$|x| < \alpha$

where $\alpha = \min\left\{a, \frac{b}{k}\right\}$

$= \min\left\{a, \frac{b}{1+b^2}\right\}$

Note that $\frac{1}{2}$ is the maximum possible value α can take.

So, the maximum possible interval is $|x|<\frac{1}{2}$.

Linear Differential Equations

First, we look at first order linear ODE.

A first ~~fo~~ order linear ODE is of the form

$$y' + p(x)y = r(x)$$

If $r(x) \equiv 0$, the equation is called a homogeneous equation; otherwise non-homogeneous.

→ Consider homogeneous case:

$$y' + p(x)y = 0 \qquad \left(\text{Assume } p(x) \text{ is continuous}\right)$$

$$\Rightarrow dy + p(x)y\,dx = 0$$

$$\Rightarrow \frac{dy}{y} = -p(x)\,dx \;, \quad y(x) \neq 0$$

$$\Rightarrow \log|y| = -\int p(x)\,dx + c_1$$

$$\Rightarrow \underline{y = c e^{-\int p(x)\,dx}}$$

→ consider non-homogeneous case:

$$y' + p(x)y = r(x)$$

$$\Rightarrow dy + (p(x)y - r(x))\,dx = 0$$

$M = p(x)y - r(x)$, $N = 1$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, so it is not exact.

By multiplying with a factor $\mu(x)$, we assume it becomes exact,

i.e, $\frac{\partial}{\partial y}\left(\mu(x)p(x)y - \mu(x)r(x)\right) = \frac{\partial}{\partial x}(\mu(x))$

$$\Rightarrow \mu(x)p(x) = \frac{d}{dx}(\mu(x))$$

$$\Rightarrow \frac{d\mu}{\mu} = p(x)\,dx$$

$$\Rightarrow \log|\mu| = \int p(x)\,dx + \cancel{c}$$

doesn't matter

$$\Rightarrow \mu = e^{\int p(x)\,dx}$$

So, the integral factor $\mu(x) = e^{\int p dx}$

$(p(x)y - r(x))dx + dy = 0$

$\Rightarrow e^{\int p dx}(p(x)y - r(x))dx + e^{\int p dx} dy = 0$

As $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, so it is exact.

Then, solutions are in the exact ODE case.

$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$

OR

From $\underline{e^{\int p(x)}(p(x)y - r(x))dx + e^{\int p dx}} dy = 0$,

$d(y e^{\int p dx}) = e^{\int p dx} r(x) dx$

$\Rightarrow y e^{\int p dx} = \int e^{\int p dx} r(x) dx$

$\Rightarrow y = e^{-\int p dx}\left[\int e^{\int p dx} r(x) dx\right]$

Q) Find the general solution of $y' + 4y = \cos x$.

Soln: $dy + (4y - \cos x)dx = 0$

Multiplying factor, $p(x) = e^{\int 4 dx} = e^{4x}$

$\Rightarrow p(x) = 4$

$e^{4x} dy + e^{4x}(4y - \cos x)dx = 0$ is exact.

$I.F. = e^{\int 4 dx} = e^{4x}$

$\therefore y e^{4x} = \int \cos x \, e^{4x} dx$

Let $I = \int \cos x \, e^{4x} dx = \cos x \frac{e^{4x}}{4} + \int \sin x \frac{e^{4x}}{4} dx$

$= \cos x \frac{e^{4x}}{4} + \frac{1}{4}\left[\sin x \frac{e^{4x}}{4} - \underbrace{\int \cos x \frac{e^{4x}}{4} dx}_{I}\right]$

$\Rightarrow \frac{17}{16} I = \cos x \frac{e^{4x}}{4} + \frac{1}{16} \sin x \, e^{4x} + c'$

$\Rightarrow I = \frac{4}{17} \cos x \, e^{4x} + \frac{1}{17} \sin x . e^{4x} + C$

$\therefore y = \frac{1}{17}\left[4\cos x + \sin x\right] + k$

24-04-2023

Q) Find the general solution of

$$y' + 3y = \sin x, \quad y(\pi/2) = 0.3$$

Soln: $dy + (3y - \sin x)dx = 0$

$IF = e^{\int 3\,dx} = e^{3x}$

$\therefore y e^{3x} = \int \sin x \, e^{3x} \, dx$

$I = \int \sin x \, e^{3x} = \sin x \frac{e^{3x}}{3} - \int \cos x \frac{e^{3x}}{3} dx$

$= \sin x \frac{e^{3x}}{3} - \frac{1}{3}\left[\cos x \frac{e^{3x}}{3} + \int \sin x \frac{e^{3x}}{3} dx\right] + c$ (I)

$\Rightarrow \frac{10 I}{9} = \sin x \cdot \frac{e^{3x}}{3} - \frac{\cos x \cdot e^{3x}}{9} + K' \Rightarrow I = \frac{3}{10} \sin x \, e^{3x} - \frac{\cos x \, e^{3x}}{10} + K''$

$\therefore y = \frac{1}{10}(3\sin x - \cos x) + K.$

SECOND ORDER LINEAR EQUATION

A second order linear equation is of the form

$$y'' + p(x)y' + q(x)y = r(x).$$

An equation which cannot be written in this form is called ~~is called~~ non-linear second order ODE.

If $r(x) \equiv 0$, the equation is called homogeneous, otherwise non-homogeneous.

Consider $y'' + 2y' + y = 0$.

$y_1 = e^{-x}$ is a solution.

Also, $y_2(x) = xe^{-x}$ is a solution.

Note that $y(x) = c_1 y_1(x) + c_2 y_2(x)$ is a solution where c_1, c_2 are constants.

The expression $c_1 y_1 + c_2 y_2$ is called a linear combination of $y_1(x)$ and $y_2(x)$.

For a linear second order homogeneous eqn, a linear combination of solutions on an interval I is again a solution on I.

In particular, sum of two solutions, a multiple of (constant multiple ~~or~~ a soln are again solutions.

But this is NOT true for non-homogeneous case.

Eg. $y'' + y = 1$

$y_1 = 1 + \cos x$ is a soln

$y_2 = 1 + \sin x$ is a soln.

But $(y_1 + y_2)$ is not a soln.

<u>Note</u>: In the first order, general soln's involve a constant. In the first order IVP there is only one initial condition

$y(x_0) = y_0$.

In the second order linear homogeneous case a general solution is of the form

$$c_1 y_1(x) + c_2 y_2(x).$$

In the second order case, an IVP involves two initial conditions

$y(x_0) = K_0$, $y'(x_0) = K_1$ where x_0, K_0, K_1 are given values.

<u>Defn</u>: Two solutions $y_1(x)$ and $y_2(x)$ of

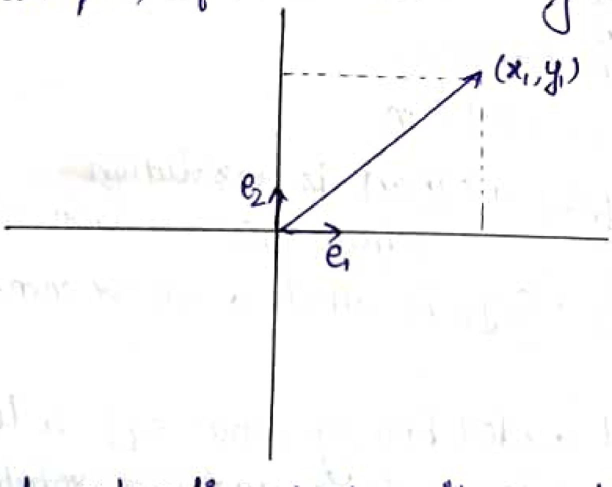
$y'' + p(x)y' + q(x)y = 0$ on an interval is said to be <u>linearly independent</u> if

$$k_1 y_1(x) + k_2 y_2(x) = 0 \text{ on } I \Rightarrow k_1 = 0, k_2 = 0,$$

(k_1, k_2: constants)

otherwise linearly dependent.

<u>Note:</u> If y_1 and y_2 are linearly dependent, then one of them is a multiple (constant multiple) of the other, or they are proportional.



$(x_1, y_1) = ke_1 + ke_2$

For a second order linear ODE, if y_1 and y_2 are linearly independent solutions,

then a linear combination $c_1 y_1 + c_2 y_2$ is a solution and $\{y_1(x), y_2(x)\}$ is said to be a basis,

or a fundamental system of set of solutions of $y'' + p(x)y' + q(x)y = 0$.

To find a basis of solution if one solution is known.

Let $y_1(x)$ be a solution of $y'' + p(x)y' + q(x)y = 0$.

Take $y_2(x) = u(x)\, y_1(x)$ for some function $u(x)$.

If $y_2(x)$ is a soln. we have

$$y_2'' + p(x)y_2' + q(x)y_2 = 0$$

$$y_2' = u'y_1 + uy_1'$$
$$y_2'' = u''y_1 + 2u'y_1' + uy_1''$$

i.e., $u''y_1 + 2u'y_1' + uy_1'' + p(x)(u'y_1 + uy_1') + q(x)uy_1 = 0$

$$\Rightarrow u''y_1 + u'(2y_1' + p(x)y_1) + u(\underbrace{y_1'' + p(x)y_1' + q(x)y_1}_{\to 0}) = 0$$

$$\Rightarrow u''y_1 + u'(2y_1' + p(x)y_1) = 0$$

Divide with y_1 and also put $u'' = U'$, $u' = U$.

$$U' + U\left(\frac{2y_1'}{y_1} + p(x)\right) = 0$$

$$\Rightarrow \frac{dU}{dx} = -\left(\frac{2y_1'}{y_1} + p(x)\right)U$$

$$\Rightarrow \frac{dU}{U} = -\left(\frac{2y_1'}{y_1} + p(x)\right)dx$$

$$\therefore \log|U(x)| = -2\log|y_1| - \int p(x)\,dx$$

$$\Rightarrow \boxed{U(x) = \frac{1}{y_1^2}\, e^{-\int p(x)dx}}$$

$u(x) = \int U(x)\,dx$, $y_2(x) = u(x)\,y_1(x)$

$\{y_1, y_2\}$ is linearly indep.

$\because y_2/y_1$ is not a constant.

Q. Find a basis of solutions of

$$x^2y'' - 5xy' + 9y = 0$$

given $y_1(x) = x^3$ is a soln.

Soln: $x^2y'' - 5xy' + 9y = 0$

$$\Rightarrow y'' - \frac{5}{x}y' + \frac{9}{x^2}y = 0$$

$$p(x) = -\frac{5}{x}$$

$$U(x) = \frac{1}{x^6}\, e^{\int \frac{5}{x}dx} = \frac{1}{x}$$

$$u(x) = \int U(x)\,dx = \int \frac{1}{x}dx = \log x$$

$$y_2(x) = x^3 \log x$$

$\{x^3, x^3\log x\}$ is a basis.

28-04-23

Q) Find a basis of solutions of

$$(1-x^2)y'' - 2xy' + 2y = 0.$$

Given a solution $y_1(x) = x$.

Soln: $y'' - \frac{2x}{1-x^2} y' + \frac{2}{1-x^2} y = 0 \qquad (x \neq \pm 1)$

$$U(x) = \frac{1}{y_1^2(x)} e^{-\int p\,dx} = \frac{1}{x^2} e^{-\ln|1-x^2|} = \frac{1}{x^2(1-x^2)}$$

$$u(x) = \int U(x)\,dx = \int \frac{dx}{x^2(1-x^2)}$$

$$\boxed{y_2(x) = u(x)\, y_1(x)}$$

$$I = \int \frac{dx}{x^2(1-x^2)} = \int \left(\frac{A}{x^2} + \frac{B}{1-x} + \frac{C}{1+x} \right) dx$$

$$\frac{1}{x^2(1-x^2)} = \frac{A(1-x^2) + B(1+x)x^2 + C\,x^2(1-x)}{x^2(1-x^2)}$$

$$\Rightarrow \frac{1}{x^2(1-x^2)} = \frac{x^3(B-C) + x^2(-A+B+C) + A}{x^2(1-x^2)}$$

$\Rightarrow A = 1,\ B + C = 1,\ B = C$

$\Rightarrow 2B = 1 \Rightarrow B = \frac{1}{2} = C$

$$\therefore I = \int \left(\frac{1}{x^2} + \frac{-1}{2(1-x)} + \frac{1}{2(1+x)} \right) dx$$

$$\Rightarrow u(x) = -\frac{1}{x} - \frac{1}{2}\ln(1-x) + \frac{1}{2}\ln(1+x)$$

$$\therefore y_2 = u(x)\,y_1(x) = -1 - \frac{x}{2}\ln(1-x) + \frac{x}{2}\ln(1+x)$$

$$\therefore \text{Basis: } \left\{ x,\ -1, -\frac{x}{2}\ln(1-x) + \frac{x}{2}\ln(1+x) \right\}.$$

Second Order Linear Homogeneous Equation with Constant Coefficient

$y'' + ay' + by = 0 \quad -(1)$, a, b: constant

Try with $y = e^{\lambda x}$, λ: constant

So (1) becomes

$\lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + b e^{\lambda x} = 0$

$e^{\lambda x}(\lambda^2 + a\lambda + b) = 0$

Then, $y = e^{\lambda x}$ is a soln of (1) if $\lambda^2 + a\lambda + b = 0$.

The equation $\lambda^2 + a\lambda + b = 0$ is called the characteristic equation or auxiliary ~~arbitrary~~ equation associated to the given differential equation.

If λ_1, λ_2 are the roots, solutions are

$y_1 = e^{\lambda_1 x}$; $y_2 = e^{\lambda_2 x}$

Roots $= \dfrac{-a \pm \sqrt{a^2 - 4b}}{2}$

Case-(1) Roots of $\lambda^2 + a\lambda + b = 0$ are simple roots, i.e, distinct real roots.

This is, if $(a^2 - 4b) > 0$

In this case we have two roots λ_1, λ_2.

So, the solutions are $y_1(x) = e^{\lambda_1 x}$, $y_2(x) = e^{\lambda_2 x}$

$\{y_1, y_2\}$ forms a basis of solutions of the given eqn.

Q1 Find the general solutions of $4y'' + 4y' - 3y = 0$

Soln: $y'' + y' - \frac{3}{4}y = 0$

Char. eqn: $\lambda^2 + \lambda - \frac{3}{4} = 0$

$\Rightarrow \lambda = \frac{1}{2}, -\frac{3}{2}$

$y_1(x) = e^{\frac{1}{2}x}$, $y_2(x) = e^{-\frac{3}{2}x}$

$\therefore y(x) = c_1 e^{\frac{1}{2}x} + c_2 e^{-\frac{3}{2}x}$ is a general soln.

Case-②: $\lambda^2+a\lambda+b=0$ has repeated real roots

if $a^2-4b=0$.

Roots are $-a/2, -a/2$.

$y_1(x)=e^{-a/2\, x}$ is a soln.

$y_2(x)=u(x)y_1(x)$

Substituting in $y''+ay'+by=0$

$u''y_1+2u'y_1'+uy_1''+a(uy_1'+u'y_1)+buy_1=0$

$\Rightarrow u''y_1+u(\cancel{y_1''+ay_1'+by_1})^{0}+u'(\cancel{2y_1'+ay_1})^{0}=0$

$$\#\begin{cases} y_1=e^{-a/2\, x} \\ \Rightarrow y_1'=-\frac{a}{2}e^{-a/2\, x} \\ \Rightarrow 2y_1'=-ay_1 \Rightarrow 2y_1'+ay_1=0 \end{cases}$$

$\Rightarrow u''y_1=0 \qquad (y_1\neq 0)$

$\Rightarrow u''=0 \Rightarrow u(x)=cx+d$

$y_2=u(x)y_1(x)$
$=(cx+d)y_1(x)$

To get linearly independent sol^n's, it is enough to take $u(x)=x$.

So the basis of solution is

$\{y_1(x), xy_1(x)\}$

i.e., $\{e^{-a/2\, x}, xe^{-a/2\, x}\}$

$c_1y_1+c_2y_2$
$=c_1dy_1+c_2y_1$
$=c_3y_1$
$\therefore c_1y_2+c_3y_1$

Q) Find general solutions of

$$y''+2ky'+k^2y=0$$

Soln: Char. eqn: $\lambda^2+2k\lambda+k^2=0 \Rightarrow \lambda=-k,-k$

$a^2-4b=0$

Basis of soln set is

$\{e^{-kx}, xe^{-kx}\}$.

Case-③: $\lambda^2 + a\lambda + b = 0$ has complex roots

if $(a^2 - 4b) < 0$

Roots are $\lambda_1 = \frac{-a + \sqrt{a^2-4b}}{2}$, $\lambda_2 = \frac{-a - \sqrt{a^2-4b}}{2}$

$\lambda_1 = -\frac{a}{2} + i\sqrt{b - \frac{1}{4}a^2}$, $\lambda_2 = -\frac{a}{2} - i\sqrt{b - \frac{1}{4}a^2}$

$\lambda_1 = -\frac{a}{2} + i\omega$, $\lambda_2 = -\frac{a}{2} - i\omega$, where $\omega = \sqrt{b - \frac{1}{4}a^2}$

Solutions are

$y_1 = e^{\left(-\frac{a}{2} + i\omega\right)x}$, $y_2 = e^{\left(-\frac{a}{2} - i\omega\right)x}$

$y_1 = e^{-\frac{a}{2}x}(\cos\omega x + i\sin\omega x)$

$y_2 = e^{-\frac{a}{2}x}(\cos\omega x - i\sin\omega x)$

Consider $\frac{y_1 + y_2}{2} = e^{-\frac{a}{2}x}\cos\omega x$

$\frac{y_1 - y_2}{2} = e^{-\frac{a}{2}x}\sin\omega x$

$\left\{e^{-\frac{a}{2}x}\cos\omega x,\ e^{-\frac{a}{2}x}\sin\omega x\right\}$ is a basis with functions.

Q) Solve

$y'' + 0.4y' + 0.29y = 0$

Soln: $y(0) = 1$, $y'(0) = -1.2$

ch. eqn: $\lambda^2 + 0.4\lambda + 0.29 = 0$

$a^2 - 4b = 0.16 - 1.16$

$= -1 < 0$

$\omega = \sqrt{b - \frac{1}{4}a^2}$

$= \sqrt{0.29 - 0.04}$

$= \sqrt{0.25}$

$= 0.5$

Solns are

$y_1 = e^{-0.2x}\cos 0.5x$

$y_2 = e^{-0.2x}\sin 0.5x$

Second Order Linear Equation

A linear equation is of the form

$$y'' + p(x)y' + q(x)y = r(x)$$

otherwise non-linear.

Consider $y'' + 2y' + y = 0$

$y_1(x) = e^{-x}$ is a sol^n

$y_2(x) = xe^{-x}$

Note: $y'' + p(x)y' + q(x)y = 0$.

This eq^n has a general sol^n (on I) if $p(x)$ and $q(x)$ are continuous on some interval I.

1-05-2023

Boundary Value Problem (BVP)

A differential equation together with the conditions $y(p_1) = k_1$, $y(p_2) = k_2$ where p_1, p_2 are endpoints of an interval I and k_1, k_2 are given values is called BVP.

$$y'' + p(x)y' + q(x)y = 0$$

$$y = c_1 y_1 + c_2 y_2$$

Q) Solve $y'' + 2y' + 2y = 0$.

$y(0) = 1,\ y(\pi/2) = 0$

Soln: Ch. eq^n: $\lambda^2 + 2\lambda + 2 = 0$

$a^2 - 4b = -4 < 0,\ \omega = \sqrt{b^2 - \frac{1}{4}a^2} = 1.$

$y(x) = e^{-x}(A\cos x + B\sin x)$

$y(0) = 1 \Rightarrow A = 1,\ y(\pi/2) = 0 \Rightarrow e^{-\pi/2}B = 0$

$\Rightarrow B = 0$

$\therefore y(x) = e^{-x}\cos x$

Note: If $p(x)$ and $q(x)$ are continuous on an interval I and $x_0 \in I$, then the IVP

$$y'' + p(x)y' + q(x)y = 0$$

$$y(x_0) = k_0,\ y'(x_0) = k_1$$

has a unique solution.

② $y_1(x)$ and $y_2(x)$ are linearly independent on I.

if $k_1 y_1(x) + k_2 y_2(x) = 0$ on $I \Rightarrow k_1 = 0, k_2 = 0$.

→ For functions $y_1(x), y_2(x)$ defined on I, the Wronskian of y_1, y_2 at $x \in I$

$W(y_1, y_2)|_x$ is defined as $(y_1, y_2 : I \to \mathbb{R})$

$$W(y_1, y_2)|_x = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}.$$

$$= \underbrace{y_1(x) y_2'(x) - y_2(x) y_1'(x)}_{} \rightarrow \text{real no.} \quad W(y_1, y_2) : I \to \mathbb{R}$$

→ The Wronskian function $W(y_1, y_2)$ is neither zero on I or identically zero on I (i.e., if at all $W(y_1, y_2)$ is zero at a point on I, then it is zero for all points in I).

Defn: The functions y_1, y_2 are linearly independent on I if there exists a point $x_1 \in I$ such that the Wronskian of y_1, y_2 at x_1 is not zero, i.e.,

$$W(y_1, y_2)|_{x_1} \neq 0.$$

Theorem: If $p(x)$ and $q(x)$ are continuous on I, then

$$y'' + p(x) y' + q(x) y = 0 \quad —①$$ has a general solution.

Proof: Consider the IVP

$$y'' + p(x) y' + q(x) y = 0, \; y(x_0) = 1, \; y'(x_0) = 0, \text{ for } x_0 \in I$$

Since $p(x)$ and $q(x)$ are continuous, this IVP has a unique solution say $y_1(x)$ on I.

Consider the IVP

$$y'' + p(x) y' + q(x) y = 0,$$
$$y(x_0) = 0, \; y'(x_0) = 1.$$

Let $y_2(x)$ be the unique solution of this IVP.

$$W(y_1, y_2)|_{x_0} = y_1(x_0) y_2'(x_0) - y_2(x_0) y_1'(x_0)$$
$$= 1 \neq 0$$

$\Rightarrow y_1(x)$ & $y_2(x)$ are linear independent on I.

∴ General soln of ① is $y(x) = c_1 y_1(x) + c_2 y_2(x)$.

Note: A solution of $y''+p(x)y'+q(x)y=0$ which is not obtained from the general solution is called a singular solution.

If $p(x)$ and $q(x)$ are continuous, then there does not exist singular solution.

Second Order Linear Non-Homogeneous Equations

$$y''+p(x)y'+q(x)y=r(x) \quad -① \qquad (\text{where } r(x)\neq 0)$$

Consider the homogeneous part

$$y''+p(x)y'+q(x)y=0 \quad -②$$

Theorems: ⓐ Difference of two solutions of eqⁿ ① on I is a solution of equation ②.

ⓑ Sum of a solution of ① and a solⁿ of ② on I is a solⁿ of ①.

Proof: ⓐ Let y_1, y_2 be two solutions of ①.

Eqⁿ ① may be written as $L[y]=r(x)$

where $L=\frac{\partial^2}{\partial x^2}+p(x)\frac{\partial}{\partial x}+q(x)$

y_1, y_2 are solⁿs of ①, so $L[y_1]=r(x)$, $L[y_2]=r(x)$.

$$L[y_1-y_2]=L[y_1]-L[y_2]$$
$$=r(x)-r(x)$$
$$=0$$

$$(y_1-y_2)''=y_1''-y_2''$$
$$(y_1-y_2)'=y_1'-y_2'$$

$\Rightarrow y_1-y_2$ is a solⁿ of eqⁿ ②.

ⓑ Let y be a solⁿ of ① and y^* be a solⁿ of ②.

$$L[y]=r(x), \quad L[y^*]=0$$
$$L[y+y^*]=L[y]+L[y^*]=r(x)$$

$\Rightarrow y+y^*$ is a solⁿ of eqⁿ ①.

Defn: A general solution of

$$y'' + p(x)y' + q(x)y = r(x) \quad -①$$

is a solution of the form $y(x) = y_h(x) + y_p(x)$
where $y_h(x)$ is a general soln of $y'' + p(x)y' + q(x)y = 0$
and $y_p(x)$ is a solution of ① which is <u>not</u> involving any constant.

$$(y_h(x) = c_1 y_1 + c_2 y_2)$$

A particular solution of ① is obtained by assigning values to the constants c_1 and c_2.

Q Find the general solution of

$$y'' - y = 8e^{-3x} \text{ and } y_p(x) = e^{-3x}.$$

Soln: Consider $y'' - y = 0$

Char. eqn : $\lambda^2 - 1 = 0$
$\Rightarrow \lambda = \pm 1$

$$y_h(x) = c_1 e^x + c_2 e^{-x}$$

General soln: $y(x) = c_1 e^x + c_2 e^{-x} + e^{-3x}$.

Q Solve $y'' + y = 2x$

$y(0) = -1$, $y'(0) = 8$, $y_p(x) = 2x$.

Soln: Consider $y'' + y = 0$

char. eqn : $\lambda^2 + 1 = 0$
$\Rightarrow \lambda = \pm i$

$y_h(x) = A\cos x + B\sin x$; $y(x) = A\cos x + B\sin x + 2x$

$y(0) = -1 \Rightarrow -1 = A$

$y'(0) = 8 \Rightarrow B = 6$

$\therefore$ General solution : $-\cos x + 6\sin x + 2x = y(x)$.

Method of Undetermined Coefficients

This is a procedure to find $y_p(x)$ when the differential equation of the form $y'' + ay' + by = r(x)$ — ①

where $r(x)$ is a function of the form

① an exponential,
② sine or cosine or their linear combination,
③ a polynomial,
④ a combination of above type of functions.

Polynomial $a_0 + a_1x + \dots + a_nx^n$
$a_n \in \mathbb{R}$
$x \to$ variable
eg. $1+2x+3x^2$; $\mathbb{R} \to \mathbb{R}$

Case ①: $r(x)$ is e^{mx} or a multiple of e^{mx}.

In this case, try with $y_p(x) = Ae^{mx}$.

Eqn ① becomes

$$m^2Ae^{mx} + amAe^{mx} + bAe^{mx} = e^{mx}$$

$y_p' = mAe^{mx}$
$y_p'' = m^2Ae^{mx}$

$$\Rightarrow A(m^2+am+b)e^{mx} = e^{mx}$$

$$\Rightarrow A(m^2+am+b) = 1$$

$$\Rightarrow A = \frac{1}{m^2+am+b} \quad \text{if } m^2+am+b \neq 0.$$

Char. eqn: $\lambda^2 + a\lambda + b = 0$

So, $A = \dfrac{1}{m^2+am+b}$ if m is NOT a root of the char. eqn.

If m is a root of the char. eqn

Take $y_p(x) = Axe^{mx}$

$y_p' = Ae^{mx} + Amxe^{mx}$
$y_p'' = Ame^{mx} + Ame^{mx} + Am^2xe^{mx}$

Then eqn $y'' + ay' + by = e^{mx}$ becomes

$$Am^2xe^{mx} + 2Ame^{mx} + a(Ae^{mx} + Amxe^{mx}) + bAxe^{mx} = e^{mx}$$

$$\Rightarrow A(\overset{0}{\overbrace{m^2+am+b}})xe^{mx} + (2m+a)Ae^{mx} = e^{mx}$$

$$\Rightarrow A = \frac{1}{2m+a}, \text{ provided } 2m+a \neq 0, \text{ i.e., } m \neq -\frac{a}{2}.$$

$\lambda^2 + a\lambda + b = 0$, $-a/2$ is the repeated root.

$y_p(x) = Axe^{mx}$ (with $A = \frac{1}{2m+a}$), if m is not a repeated root of the characteristic equation, i.e., m is a simple root of the char. eqn.

If m is a repeated root of the char. eqn,
take $y_p(x) = Ax^2 e^{mx}$.

Thus, if $r(x)$ is e^{mx} or a constant multiple of e^{mx}.

We have (i) $y_p(x) = Ae^{mx}$, if m is not a root of the char. eqn.

(ii) $y_p(x) = Axe^{mx}$, if m is a simple root of the char. eqn.

(iii) $y_p(x) = Ax^2 e^{mx}$, if m is a repeated root of the char. eqn.

Q) Find a general solution of

$$y'' + 2y' + y = 6e^x.$$

Soln: $y_p(x) = Ae^x$

$y_p' = Ae^x$, $y_p'' = Ae^x$

$\therefore Ae^x + 2Ae^x + Ae^x = 6e^x$

$\Rightarrow A = 3/2$

$\therefore y_p(x) = \frac{3}{2}e^x$, $y_h(x) = (c_1 + c_2 x)e^{-x}$

$r(x) = 6e^x$
$m = 1$
$\lambda^2 + 2\lambda + 1 = 0$
$\lambda = -1, -1$

$y'' + 2y' + y = 0$.

General soln: $y(x) = (c_1 + c_2 x)e^{-x} + \frac{3}{2}e^x$.

Case-(ii): $r(x) = \sin kx$; $\boxed{y'' + ay' + by = \sin kx}$ — (1) (a, b, k are given values).

Try with $y_p(x) = A\sin kx + B\cos kx$ — (2)

$-Ak^2 \sin kx - Bk^2 \cos kx + a(Ak\cos kx - Bk\sin kx)$
$+ b(A\sin kx + B\cos kx) = \sin kx$

Solve for A and B.

If (2) happens to be a solution of the homogeneous eqn:

$$y'' + ay' + by = 0$$

then take $y_p(x) = x(A\sin kx + B\cos kx)$.

Note: Same is the case with $r(x) = \cos kx$ or combination of sin & cos.

Q) Find a particular soln of

(a) $y'' + 4y = \sin 3x$

(b) $y'' + y = 2\cos x$.

Soln: (a) $r(x) = \sin 3x$, $y_p = A\sin 3x + B\cos 3x$.

$-9A\sin 3x - 9B\cos 3x + 4A\sin 3x + 4B\cos 3x = \sin 3x$.

$\Rightarrow A = -\frac{1}{5}$, $B = 0$

$\therefore y_p(x) = -\frac{1}{5}\sin 3x$.

(b) $y'' + y = 2\cos x$

$y_p(x) = A\sin x + B\cos x$

~~$x\sin x$~~.

~~$y_p(x) = x(A\sin x + B\cos x)$~~.

$\Rightarrow y_p'(x) = -A\sin x + B\cos x$,

$y''_p(x) = -A\cos x - B\sin x$

$\because y_p'' + y_p = 0$

Thus $y_p(x) \neq A\sin x + B\cos x$

Take $y_p(x) = x(A\cos x + B\sin x)$

$y_p' = A\cos x - Ax\sin x + B\sin x + xB\cos x$,

$y_p'' = -A\sin x - A\sin x - Ax\cos x + B\cos x + B\cos x - xB\sin x$.

$= -2A\sin x + 2B\cos x - xA\cos x - xB\sin x$.

$\because y_p''(x) + y_p(x) = -2A\sin x + 2B\cos x = 2\cos x$

$\Rightarrow B = 1,\ A = 0$

$\therefore y_p(x) = \cos x$.

Case-(iii): $r(x)$ is a polynomial

$$y'' + ay' + by = a_0 + a_1 x + \dots + a_n x^n \quad -(1)$$

Take $y_p(x) = A_0 + A_1 x + \dots + A_n x^n$

Substitute in (1) and equate coefficients to find $A_0, A_1, \dots, A_n$.

If $b=0$ in eq^n (1), take $y_p = x(A_0 + A_1 x + \dots + A_n x^n)$.

Q) $y'' - y' - 2y = 4x^2$. Find the general solution.

Sol^n:

$y'' - y' - 2y = 0$

char. eq^n: $\lambda^2 - \lambda - 2 = 0 \Rightarrow \lambda = -1, 2$

$y_h(x) = c_1 e^{-x} + c_2 e^{2x}$

Take $y_p(x) = A_0 + A_1 x + A_2 x^2$

$$2A_2 - A_1 - 2A_2 x - 2A_0 - 2A_1 x - 2A_2 x^2 = 4x^2$$

$$\Rightarrow \begin{cases} 2A_2 - A_1 - 2A_0 = 0 \\ -2A_2 - 2A_1 = 0 \\ -2A_2 = 4 \Rightarrow A_2 = -2 \end{cases}$$

$\Rightarrow A_1 = 2,\ A_0 = -3$

$\therefore y(x) = c_1 e^{-x} + c_2 e^{2x} - 3 + 2x - 2x^2.$

Case-(iv): $r(x)$ is sum of the functions of above type.

Q) Find the general solution of $y'' - 2y' + 2y = e^x + 4x^2$ -(1)

Sol^n: $y'' - 2y' + 2y = 0$

char. eq^n: $\lambda^2 - 2\lambda + 2 = 0$

$\Rightarrow \lambda = 1+i,\ 1-i$

$e^x \cos x,\ e^x \sin x$

$y_h(x) = e^x(c_1 \cos x + c_2 \sin x)$

Take $y_p(x) = Ae^x + A_0 + A_1 x + A_2 x^2$

So (1) becomes

$$Ae^x + 2A_2 - 2Ae^x - 2A_1 - 4A_2 x + 2Ae^x + 2A_0 + 2A_1 x + 2A_2 x^2 = e^x + 4x^2.$$

$\Rightarrow A = 1$

$2A_2 - 2A_1 + 2A_0 = 0 \Rightarrow A_0 = 2$

$-4A_2 + 2A_1 = 0 \Rightarrow A_1 = 4$

$A_2 = 2$

$\therefore y(x) = (c_1 \cos x + c_2 \sin x)e^x + 2 + 4x + 2x^2 + e^x.$

Method of Variation of Parameters

The demerits of the method of undetermined coefficients are :

(i) Eq^n with constant coefficients

(ii) $r(x)$ is of specific type

Variation of parameter works for functions $p(x)$, $q(x)$ and $r(x)$ in

$$y'' + p(x)y' + q(x)y = r(x), \quad -(1)$$

provided the general solution $y_h(x) = c_1y_1 + c_2y_2$ exists for the homogeneous equation.

In this case, take $y_p(x) = u(x)\,y_1(x) + v(x)\,y_2(x)$ on I,

for some functions $u(x), v(x)$ satisfying

$$u'y_1 + v'y_2 = 0 \quad \text{on } I.$$

Eq^n (1) becomes

$$uy_1'' + u'y_1' + vy_2'' + v'y_2' + p(x)(uy_1' + vy_2') + q(x)(uy_1 + vy_2) = r(x)$$

$$y_p' = \cancel{u'y_1} + uy_1' + \cancel{v'y_2} + vy_2' = uy_1' + vy_2'$$

$$y_p'' = uy_1'' + u'y_1' + vy_2'' + v'y_2'$$

$$\Rightarrow uy_1'' + vy_2'' + u'y_1' + v'y_2' + u(p(x)y_1' + q(x)y_1) + v(p(x)y_2' + q(x)y_2) = r(x)$$

$$\Rightarrow u(y_1'' + p(x)y' + q(x)y_1) + v(\underbrace{y_2'' + p(x)y_2' + q(x)y_2}_{0}) + u'y_1' + v'y_2' = r(x)$$

$$\Rightarrow u'y_1' + v'y_2' = r(x) \quad -(2)$$

$$u'y_1 + v'y_2 = 0 \quad -(3)$$

$$(3)y_2' - (2)y_2 \Rightarrow u'(y_1y_2' - y_2y_1') = -y_2r(x)$$

$$\Rightarrow u'W(y_1, y_2) = -y_2r(x)$$

$$\Rightarrow u' = \frac{-y_2r(x)}{W}$$

[$\{y_1, y_2\}$ basis for sol^{n}s of the homogeneous eq^n $\Rightarrow W(y_1, y_2) \neq 0$]

$$\Rightarrow u(x) = \int \frac{-y_2r(x)}{W}\,dx$$

[$W(y_1, y_2) : I \to R$
$x \to W(y_1, y_2)|_x$]

$$(2)y_1 - (3)y_1' \Rightarrow v'(y_1y_2' - y_2y_1') = y_1r(x)$$

$$\Rightarrow v' = \frac{y_1r(x)}{W}$$

$$\Rightarrow v(x) = \int \frac{y_1r(x)}{W}\,dx$$

Thus, $y_p(x) = y_1 \int \frac{-y_2 r(x)}{W} dx + y_2 \int \frac{y_1 r(x)}{W} dx$

Q) Find the general solution of

$$y'' - 4y' + 4y = \frac{e^{2x}}{x}$$

Soln: $y'' - 4y' + 4y = 0$

$\lambda^2 - 4\lambda + 4 = 0 \Rightarrow \lambda = 2, 2$

$y_1(x) = e^{2x}, \; y_2(x) = xe^{2x}$.

$$W(y_1, y_2) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x}+2xe^{2x} \end{vmatrix} = e^{4x}$$

$$u(x) = \int \frac{-y_2 r(x)}{W} dx = -\int \frac{xe^{2x}e^{2x}}{e^{4x}.x} dx = -x$$

$$v(x) = \int \frac{y_1 r(x)}{W} dx = \int \frac{e^{2x}e^{2x}}{e^{4x}x} dx = \ln|x|.$$

$$y_p(x) = -xe^{2x} + xe^{2x}\ln x$$

$$\therefore y(x) = (c_1 + c_2x)e^{2x} + xe^{2x}(\ln x - 1).$$

HIGHER ORDER ODE

An nth order ODE is of the form

$$F(x, y, y', \ldots, y^{(n)}) = 0$$

where $y: I \to R$, x is the independent variable.

A linear n^{th} order equation is of the form

$$y^{(n)} + p_{n-1}(x) y^{(n-1)} + \ldots + p_1(x) y' + p_0(x) y = r(x)$$

where $p_n(x)$ are functions of x.

If $r(x) \equiv 0$ on I, eq^n is said to be homogeneous.

For homogeneous eq^n,

$$y^{(n)} + p_{n-1}(x) y^{(n-1)} + \ldots + p_0(x) y = 0,$$

there are 'n' linearly independent solutions $y_1, y_2, \ldots, y_n$ and the general solution is $y(x) = c_1 y_1(x) + \ldots + c_n y_n(x)$.

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Def^n: The functions $y_1, y_2, \ldots, y_n$ defined on I is said to be linearly independent if there exists a point $x_1 \in I$ with

$$W(y_1, \ldots, y_n)\Big|_{x_1} = \begin{vmatrix} y_1(x_1) & \cdots & y_n(x_1) \\ y_1'(x_1) & \cdots & y_n'(x_1) \\ \cdots & \cdots & \cdots \\ y_1^{(n-1)}(x_1) & \cdots & y_n^{(n-1)}(x_1) \end{vmatrix} \neq 0,$$

otherwise linearly dependent.

→ The general solution of

$$y^{(n)} + p_{n-1}(x) y^{(n-1)} + \ldots + p_0(x) y = 0$$

is of the form $y(x) = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$

where $\{y_1, \ldots, y_n\}$ forms a basis for the solutions.

IVP: $y^{(n)} + p_{n-1}(x)y^{(n-1)} + \ldots + p_0(x)y = r(x)$

$y(x_0) = K_0,\ y'(x_0) = K_1, \ldots y^{(n-1)}(x_0) = K_{n-1}$

where $x_0, K_0, \ldots, K_{n-1}$ are given values.

Higher Order Homogeneous Equations with Constant Coefficients

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0, \quad a_0, a_1, \ldots, a_{n-1} : \text{constants}$$

Characteristic equation: $\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 = 0$

case-①: All the n roots $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ are real and distinct.

Basis is $\{e^{\lambda_1 x}, e^{\lambda_2 x}, \ldots, e^{\lambda_n x}\}$

case-②: Root λ repeats $\underline{m}$ times, then the corresponding solutions are $e^{\lambda x}, xe^{\lambda x}, \ldots, x^{m-1}e^{\lambda x}$.

case-③: Simple complex root

If $(\gamma + i\omega)$ is a root of the characteristic equation, then $(\gamma - i\omega)$ is also a root. $(\gamma, \omega \in \mathbb{R})$

The corresponding solutions are

$$e^{\gamma x}\cos\omega x,\ e^{\gamma x}\sin\omega x.$$

case-④: ~~$(\gamma+i\omega)$ is~~ Multiple complex root.

If $(\gamma + i\omega)$ is a double root (repeated two ~~roots~~ times), then $(\gamma - i\omega)$ is also a double root.

$$e^{\gamma x}\cos\omega x,\ e^{\gamma x}\sin\omega x,\ xe^{\gamma x}\cos\omega x,\ xe^{\gamma x}\sin\omega x.$$

If it is a triple root, two more solutions:

$$x^2e^{\gamma x}\cos\omega x,\ x^2e^{\gamma x}\sin\omega x, \text{ and so on..}$$

Q Find the general solution of:

ⓐ $y''' - 2y'' - y' + 2y = 0$

ⓑ $y^{(iv)} + 5y'' + 4y = 0$

ⓒ $y^{(v)} - 3y^{(iv)} + 3y''' - y'' = 0$

ⓓ $y^{(iv)} + 2y'' + y = 0$.

Soln: (a) Char. eqn: $\lambda^3-2\lambda^2-\lambda+2=0 \Rightarrow (\lambda^2-1)(\lambda-2)=0$

$\Rightarrow \lambda = 1, 2, -1$

$y(x) = c_1e^{x} + c_2e^{2x} + c_3e^{-x} \to$ general soln

(b) char. eqn: $\lambda^4+5\lambda^2+4=0 \Rightarrow (\lambda^2+4)(\lambda^2+1)=0$

$\Rightarrow \lambda = \underbrace{\pm i}_{r=0,\, \omega=1},\ \underbrace{\pm 2i}_{r=0,\, \omega=2}$

$\therefore y(x) = c_1\cos x + c_2\sin x + c_3\cos 2x + c_4\sin 2x.$

(c) char. eqn: $\lambda^5-3\lambda^4+3\lambda^3-\lambda^2=0 \Rightarrow \lambda^2(\lambda-1)^3=0$

$\Rightarrow \lambda = 0, 0, 1, 1, 1$

$\therefore d_1 + d_2x + (c_1+c_2x+c_3x^2)e^{x}$

(d) char. eqn: $\lambda^4+2\lambda^2+1=0$

$\Rightarrow \lambda = \pm i$ (double root)

$= 0 \pm 1.i$ (double root)

$\therefore y(x) = c_1\cos x + c_2\sin x + c_3x\cos x + c_4x\sin x.$

Method of Undetermined Coefficients

Same as in the case of second order.

Q) Find the general solution of :

(a) $y''' + 3y'' + 3y' + y = 8e^{x} + x + 3.$

(b) $y''' - 4y' = 10\cos x + 5\sin x.$

Soln: (a) $y''' + 3y'' + 3y' + y = 0$

Char. eqn: $\lambda^3+3\lambda^2+3\lambda+1=0$

$\Rightarrow \lambda = -1, -1, -1$

$\therefore y_h(x) = (c_1+c_2x+c_3x^2)e^{-x}$

$y_p(x) = Ae^{x} + A_0x + A_1$ (m=1, e^x)

$\Rightarrow y_p' = Ae^{x} + A_0,\ y_p'' = Ae^{x},\ y_p''' = Ae^{x}$

$\therefore Ae^{x} + 3Ae^{x} + 3Ae^{x} + 3A_0 + Ae^{x} + A_0x + A_1 = 8e^{x} + x + 3$

$\Rightarrow A=1,\ 3A_0 + A_1 = 3,\ A_0 = 1$

General soln: $y(x) = (c_1+c_2x+c_3x^2)e^{-x} + e^{x} + x.$

(b) $y''' - 4y' = 0$

Char. eqn: $\lambda^3 - 4\lambda = 0$

$\Rightarrow \lambda = 0, -2, 2$

$y_h(x) = c_1 + c_2 e^{-2x} + c_3 e^{2x}$

$y_p(x) = A\cos x + B\sin x$

$\Rightarrow y_p' = -A\sin x + B\cos x,\ y_p'' = -A\cos x - B\sin x,$

$y_p''' = A\sin x - B\cos x$

$\therefore\ A\sin x - B\cos x + 4A\sin x - 4B\cos x = 10\cos x + 5\sin x$

$\Rightarrow A = 1,\ B = -2$

General soln: $c_1 + c_2 e^{-2x} + c_3 e^{2x} + \cos x - 2\sin x = y(x)$.

Variation of Parameters

Solution of $y_p(x)$ of $y^{(n)} + p_{n-1}(x) y^{(n-1)} + \dots + p(x) y = r(x)$.

is of the form

$$y_p(x) = y_1 \int \frac{W_1}{W} r(x)\,dx + y_2 \int \frac{W_2}{W} r(x)\,dx + \dots + y_n \int \frac{W_n}{W} r(x)\,dx$$

where $y_1, y_2, \dots, y_n$ are linearly independent solutions of the homogeneous part, W is the Wronskian, and

W_j, for $j = 1, 2, \dots, n$, is obtained from W by replacing the jth column by $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$.

Second order:

$y_1,\ y_2$

$uy_1 + vy_2,\ v = \int \frac{y_1}{W} r(x)\,dx,\ u = -\int \frac{y_2}{W} r(x)\,dx$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$u(x) = \int \frac{W_1}{W} r(x)\,dx$

$W_1 = \begin{vmatrix} 0 & y_2 \\ 1 & y_2' \end{vmatrix} = -y_2$

$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & 1 \end{vmatrix} = y_1$

Q) Find the general solution of:

$$y''' - 6y'' + 11y' - 6y = e^{-x}.$$

Soln Char. eqn: $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$\Rightarrow \lambda = 1, 2, 3.$

$y_1(x) = e^{x}, \; y_2(x) = e^{2x}, \; y_3(x) = e^{3x}$

$$W = \begin{vmatrix} e^{x} & e^{2x} & e^{3x} \\ e^{x} & 2e^{2x} & 3e^{3x} \\ e^{x} & 4e^{2x} & 9e^{3x} \end{vmatrix} = 2e^{6x}$$

$$W_1 = \begin{vmatrix} 0 & e^{2x} & e^{3x} \\ 0 & 2e^{2x} & 3e^{3x} \\ 1 & 4e^{2x} & 9e^{3x} \end{vmatrix} = e^{5x}$$

$$W_2 = \begin{vmatrix} e^{x} & 0 & e^{3x} \\ e^{x} & 0 & 3e^{3x} \\ e^{x} & 1 & 9e^{3x} \end{vmatrix} = -2e^{4x}$$

$$W_3 = \begin{vmatrix} e^{x} & e^{2x} & 0 \\ e^{x} & 2e^{2x} & 0 \\ e^{x} & 4e^{2x} & 1 \end{vmatrix} = e^{3x}$$

$$\therefore y_p(x) = y_1 \int \frac{W_1}{W} r(x)\,dx + y_2 \int \frac{W_2}{W} r(x)\,dx + y_3 \int \frac{W_3}{W} r(x)\,dx.$$

$$= e^{x} \int \frac{e^{5x}}{2e^{6x}} \cdot e^{-x}\,dx + e^{2x} \int \frac{-2e^{4x}}{2e^{6x}} e^{-x}\,dx + e^{3x} \int \frac{e^{3x}}{2e^{6x}} e^{-x}\,dx$$

$$= -\frac{1}{4}e^{-x} + \frac{1}{3}e^{-x} - \frac{1}{8}e^{-x}$$

$$= -\frac{1}{24}e^{-x}$$

$$\therefore y(x) = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x} - \frac{1}{24}e^{-x}.$$

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Non-Homogeneous Euler-Cauchy Equation

An equation of the form

$$x^n y^{(n)} + x^{n-1} a_{n-1} y^{(n-1)} + \dots + x a_1 y' + a_0 y = r(x).$$

is Euler-Cauchy equation.

For repeated roots k, k, take, $y_1 = x^k$, $x^k \ln x = y_2$

Q) Find the general solution of

$$x^3 y''' + x^2 y'' - 2xy' + 2y = x^2$$

Soln: Consider $x^3 y''' + x^2 y'' - 2xy' + 2y = 0$.

Put $y = x^m$ in the equation

$$x^m[m(m-1)(m-2) + m(m-1) - 2m + 2] = 0$$

$$\Rightarrow m(m-1)(m-2) + m(m-1) - 2m + 2 = 0$$

$$\Rightarrow m = 1, -1, 2$$

Then take $y_1 = x$, $y_2 = x^{-1}$, $y_3 = x^2$.

$$y_h(x) = c_1 x + c_2 x^{-1} + c_3 x^2$$

$$W = \begin{vmatrix} x & x^{-1} & x^2 \\ 1 & -x^{-2} & 2x \\ 0 & 2x^{-3} & 2 \end{vmatrix} = x(-2x^{-2} - 4x^{-2}) - x^{-1}(2-0) + x^2(2x^{-3} - 0)$$

$$= -6x^{-1} - 2x^{-1} + 2x^{-1} = -\frac{6}{x}$$

$$W_1 = \begin{vmatrix} 0 & x^{-1} & x^2 \\ 0 & -x^{-2} & 2x \\ 1 & 2x^{-3} & 2 \end{vmatrix} = 0 - x^{-1}(0 - 2x) + x^2(0 + x^{-2})$$

$$= 3$$

$$W_2 = \begin{vmatrix} x & 0 & x^2 \\ 1 & 0 & 2x \\ 0 & 1 & 2 \end{vmatrix} = -x^2$$

$$W_3 = \begin{vmatrix} x & x^{-1} & 0 \\ 1 & -x^{-2} & 0 \\ 0 & 2x^{-3} & 1 \end{vmatrix} = -2x^{-1}$$

$$\therefore y''' + \frac{1}{x} y'' - \frac{2}{x^2} y' + \frac{2}{x^3} y = x^{-5}$$

$$\Rightarrow \boxed{r(x) = x^{-5}}$$

→ If $m = a + ib$ a complex no, take

soln $y = x^{a+ib} = x^a (e^{\ln x})^{ib} = x^a e^{i(b \ln x)}$

which is

$y = x^a[\cos(b \ln x) + i\sin(b \ln x)]$.

Similarly for $a - ib$, we have

$y = x^a[\cos(b \ln x) - i\sin(b \ln x)]$.

Therefore, two linearly independent solutions can be

$y_1 = x^a[\cos(b \ln x)]$

and $y_2 = x^a[\sin(b \ln x)]$.

$$y_p(x) = y_1 \int \frac{W_1}{W} r(x)\,dx + y_2 \int \frac{W_2}{W} r(x)\,dx + y_3 \int \frac{W_3}{W} r(x)\,dx$$

$$= x \int \frac{3}{-6/x} x^{-5}\,dx + x^{-1} \int \frac{-x^2}{-6/x} x^{-5}\,dx + x^2 \int \frac{-2x^{-1}}{-6/x} x^{-5}\,dx$$

$$= -\frac{1}{12} x^{-2}.$$

$\therefore$ General soln: $y(x) = c_1 x + c_2 x^{-1} + c_3 x^2 - \frac{1}{12} x^{-2}$.

Q Find the general soln of:

$$x^3 y''' - 3x^2 y'' + 6xy' - 6y = x^4 \ln x, \quad (x>0).$$

Soln: Put $y = x^m$ in the homog. eqn

$$x^m \left[m(m-1)(m-2) - 3m(m-1) + 6m - 6 \right] = 0.$$

$$\Rightarrow m = 1, 2, 3. \quad \Rightarrow y_h(x) = c_1 x + c_2 x^2 + c_3 x^3.$$

Take $y_1 = x$, $y_2 = x^2$, $y_3 = x^3$. ~~$y_h(x) = x$~~

$$W = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = x(12x^2 - 6x^2) - x^2(6x) + x^3(6x)$$

$$= 6x^3 - 6x^3 + 6x^4$$

$$= 6x^4$$

$$W_1 = \begin{vmatrix} 0 & x^2 & x^3 \\ 0 & 2x & 3x^2 \\ 1 & 2 & 6x \end{vmatrix} = -x^2(0 - 3x^2) + x^3(-2x)$$

$$= 3x^4 - 2x^4 = x^4$$

$$W_2 = \begin{vmatrix} x & 0 & x^3 \\ 1 & 0 & 3x^2 \\ 0 & 1 & 6x \end{vmatrix} = x(-3x^2) + x^3(1) = -3x^3 + x^3$$

$$= -2x^3$$

$$W_3 = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix} = x(2x) - x^2(1) = 2x^2 - x^2 = x^2$$

As $r(x) = \frac{x^4 \ln x}{x^3} = x \ln x$.

$$\therefore y_p(x) = y_1 \int \frac{W_1}{W} r(x)\,dx + y_2 \int \frac{W_2}{W} r(x)\,dx + y_3 \int \frac{W_3}{W} r(x)\,dx$$

$$= x \int \frac{x^4\, x \ln x}{6x^4}\,dx + x^2 \int \frac{-2x^3 \cdot x \ln x}{6x^4}\,dx + x^3 \int \frac{x^2 \cdot x \ln x}{6x^4}\,dx$$

$$= \frac{x}{6}\left[x \cdot x \ln x - x - x \ln x + x\right] - \frac{x^2}{3}\left[x^2 - x\right] \ln x + \underbrace{\frac{x^3}{6}\left[(\ln x)^2 - \frac{\ln x}{x}\right]}_{I = (\ln x)^2 - I \;\Rightarrow\; I = (\ln x)}$$

$$= \frac{(x-1)x^2 \ln x}{3}\left[\frac{1}{2} - x\right] + \frac{(\ln x)^2}{2}$$

$$\therefore y(x) = c_1 x + c_2 x^2 + c_3 x^3 + \frac{(x-1)x^2 \ln x}{3}\left(\frac{1}{2} - x\right) + \frac{(\ln x)^2}{2}$$

POWER SERIES SOLUTION

An infinite series of the form

$$a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots$$

or $$\sum_{m=0}^{\infty} a_m (x-x_0)^m$$

where the centre $x_0, a_0, a_1, \dots$ are real numbers and x is a variable.

When $x_0 = 0$, we have $\sum_{m=0}^{\infty} a_m x^m$.

$$R = \frac{1}{\lim_{m\to\infty} \sqrt[m]{a_m}} \text{ or } \frac{1}{\lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right|}$$, if it exists, is called the radius of convergence.

The power series converges in $|x-x_0| < R$.

$$\underbrace{\sum a_m (x-x_0)^m}_{s(x) \text{ in } I_1}, \quad \underbrace{\sum b_m (x-x_0)^m}_{t(x) \text{ in } I_2}$$

$$\underbrace{\sum (a_m+b_m)(x-x_0)^m}_{s(x)+t(x)} \text{ converges in } I_1 \cap I_2.$$

If $\sum_{m=0}^{\infty} a_m (x-x_0)^m = y(x)$ in $|x-x_0| < R$.

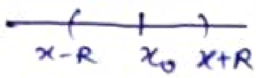
$$a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots + a_m(x-x_0)^m + \dots$$

$$a_1 + 2a_2 x + \dots + m a_m (x-x_0)^{m-1} + \dots$$

$\sum_{m=1}^{\infty} m a_m (x-x_0)^{m-1}$ converges to $y'(x)$ in $|x-x_0| < R$.

Defn: A real valued function $f(x)$ is said to be ~~multiple~~ analytic at x_0 if it can be expressed as a Taylor series (power series) about the point x_0.

$$\underbrace{f(x_0)}_{a_0} + \underbrace{f'(x_0)}_{a_1}(x-x_0) + \underbrace{\frac{f''(x_0)}{2!}}_{a_2}(x-x_0)^2 + \dots + \underbrace{\frac{f^{(n)}(x_0)}{n!}}_{a_n}(x-x_0)^n + \dots = f(x)$$



Power ~~Soluti~~ Series Solution

We can find power series solution $(x-x_0)$ for

$$y'' + p(x)y' + q(x)y = r(x) \quad (1)$$ if

$p(x), q(x)$ and $r(x)$ are polynomials or can be expressed as power series about x_0 (are analytic at x_0),

then solution $y(x) = \sum_{m=0}^{\infty} a_m (x-x_0)^m$ in $|x-x_0| < R$.

Note: That is, $p(x), q(x)$ and $r(x)$ are analytic at x_0, then power series solution exists.

Q/ Get power series solution about $x_0 = 0$,

for $y'' + xy = 0$.

Soln: Take $y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \dots$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \dots$$

$$y'' = 2a_2 + 6a_3 x + 12 a_4 x^2 + 20 a_5 x^3 + \dots$$

$y'' + xy = 0$ becomes

$$(2a_2 + 6a_3 x + 12 a_4 x^2 + 20 a_5 x^3 + \dots) + (a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + a_4 x^5 + \dots) = 0.$$

$$2a_2 = 0 \Rightarrow a_2 = 0$$

$$a_0 + 6a_3 = 0 \Rightarrow a_3 = -\tfrac{1}{6} a_0$$

$$a_1 + 12 a_4 = 0 \Rightarrow a_4 = -\tfrac{1}{12} a_1$$

$$a_2 + 20 a_5 = 0 \Rightarrow a_5 = 0$$

$$a_3 + 30 a_6 = 0 \Rightarrow a_6 = -\frac{1}{30} a_3 = \frac{1}{180} a_0$$

$$a_4 + 42 a_7 = 0 \Rightarrow a_7 = -\frac{1}{42} a_4 = \frac{1}{504} a_1$$

$$y(x) = a_0 + a_1 x - \frac{1}{6} a_0 x^3 - \frac{1}{12} a_1 x^4 + \frac{1}{180} a_0 x^6 + \frac{1}{504} a_1 x^7 + \dots$$

$$= a_0 \left(1 - \frac{1}{6} x^3 + \frac{1}{108} x^6 + \dots\right) + a_1 \left(x - \frac{1}{12} x^4 + \frac{1}{504} x^7 + \dots\right)$$

12-05-23

Q) Find power series solution about $x_0=0$ for $y''+9y=0$. — (1)

Soln: Take $y(x)=\sum a_m x^m$

Eq (1) becomes

$$\sum m(m-1)a_m x^{m-2}+9\sum a_m x^m=0$$

Coeff. of $x^0 \rightarrow 2\cdot1\, a_2+9a_0=0$

$x^1 \rightarrow 3\cdot2\, a_3+9a_1=0$

$x^2 \rightarrow 4\cdot3\, a_4+9a_2=0$

$x^m \rightarrow m(m-1)a_m+9a_{m-2}=0$ (Recursive formula)

$$a_m=\frac{-9a_{m-2}}{m(m-1)}$$

a_0, a_1

$a_2=-\frac{9a_0}{2\cdot1}=\frac{-9a_0}{2}$

$a_3=-\frac{9a_1}{6}$

$a_4=-\frac{9a_2}{12}=\frac{81a_0}{24}$

$a_5=-\frac{9a_3}{20}=\frac{81a_1}{120}$

$a_6=\frac{-9a_4}{30}=\frac{-729}{720}a_0$

$$\therefore y(x)=a_0+a_1x-\frac{9a_0}{2}x^2-\frac{9a_1}{6}x^3+\frac{81}{24}a_0x^4+\frac{81}{120}a_1x^5-\frac{729}{720}a_0x^6+\dots$$

$$=a_0\left(1-\frac{9}{2!}x^2+\frac{81}{4!}x^4-\frac{729}{6!}x^6+\dots\right)$$

$$+a_1\left(x-\frac{9}{3!}x^3+\frac{81}{5!}x^5-\dots\right)$$

$$=a_0\cos 3x+\frac{1}{3}a_1\sin 3x.$$

Defn: x_0 is said to be a regular point of $y''+p(x)y'+q(x)y=0$ if both $p(x)$ and $q(x)$ are analytic at x_0.

Note: Power series solution (about $(x-x_0)$) exists for $y''+p(x)y'+q(x)y=0$ if x_0 is a regular point of the equation.

→ x_0 is a singular point if both $p(x), q(x)$ or one of them fails to be analytic at x_0.

Defn: x_0 is said to be a regular singular point if $(x-x_0)p(x)$, $(x-x_0)^2 q(x)$ are analytic at x_0.

Eg. $x^2y'' + xy' + y = 0$

$\Rightarrow y'' + \frac{1}{x}y' + \frac{1}{x^2}y = 0$

$p(x) = \frac{1}{x} \to$ not analytic at $x=0$. $\quad | \; x\,p(x) = 1$

$x=0$ is a regular singular point.

Frobenius Method

This is an extension of power series method.
Using this method, we can ~~method~~ find a basis for solutions of eqn ① even if x_0 is a regular singular point of ①.

\# $\{(a,b) \mid a \in \mathbb{R}, b \in \mathbb{R}\} = \mathbb{C} = \{a+ib \mid a, b \in \mathbb{R}\}$

$(a,b)(c,d) = (ac-bd,\ ad+bc) \quad -(*)$

$(0,1)(0,1) = -1 = (-1,0)$

$z = a+ib \Rightarrow z = (a,b)$

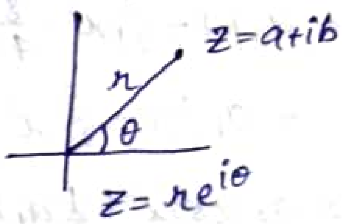
$F_1 = (0,1) = 0 + i1 = i$

$\mathbb{C} = \{(a,b) \mid a,b \in \mathbb{R}$, usual $+$, ~~usual~~ multiply $*\}$

$\mathbb{R}^2 = \{(a,b) \mid a,b \in \mathbb{R}\}$

$(a,b) + (c,d) =$

$r(a,b) = (ra + rb)$



\# $e^z = -3$

$\Rightarrow e^{a+ib} = -3$

$\Rightarrow e^a \cdot e^{ib} = -3$

$\Rightarrow e^a(\cos b + i \sin b) = -3$

$\Rightarrow a = \ln 3,\ b = \pi$

15-05-2023

Consider $x^2y'' + xp(x)y' + q(x)y = 0$ —①

Case: $p(x)$ and $q(x)$ are analytic at $x=0$.

$y'' + \frac{1}{x}p(x)y' + \frac{1}{x^2}q(x)y = 0$

$x=0$ is a regular singular point.

$(x-0)\frac{1}{x}p(x) = p(x)$, $(x-0)^2\frac{1}{x^2}q(x) = q(x)$ } analytic

Take $\boxed{y(x) = x^r \sum a_m x^m, \text{ with } a_0 \neq 0, \; r \in \mathbb{R} \text{ or } \mathbb{C}}$

$y(x) = \sum a_m x^{m+r}$

$y'(x) = \sum (m+r) a_m x^{m+r-1}$

$y''(x) = \sum (m+r)(m+r-1) a_m x^{m+r-2}$

Eq^n ① becomes

$\sum (m+r)(m+r-1) a_m x^{m+r} + (b_0 + b_1x + b_2x^2 + \dots)\sum (m+r)a_m x^{m+r} + (c_0 + c_1x + c_2x^2 + \dots)\sum a_m x^{m+r} = 0$

Since $p(x)$ & $q(x)$ are analytic at $x=0$, we have the series expansion:

$p(x) = b_0 + b_1x + b_2x^2 + \dots$

$q(x) = c_0 + c_1x + c_2x^2 + \dots$

$\Rightarrow x^r\left[\sum (m+r)(m+r-1) a_m x^m + (b_0 + b_1x + \dots)\sum (m+r)a_m x^m + (c_0 + c_1x + \dots)\sum a_m x^m\right] = 0$

Equating coeff. of x^r to zero, $\longrightarrow (m=0)$

$r(r-1)a_0 + rb_0a_0 + c_0a_0 = 0$

we have $a_0 \neq 0$.

$\Rightarrow \boxed{r(r-1) + rb_0 + c_0 = 0}$

This equation is said to be Indicial equation.

Let the roots of this eq^n be $r = r_1, r_2$.

Take one solution as

$$y_1(x) = x^{r_1}\sum a_m x^m$$

The second solution depends on three cases.

Case-①: Both the roots r_1, r_2 are distinct and their difference is NOT an integer $1, 2, \dots$.

Take $y_2 = x^{r_2}\sum A_m x^m$.

Case-(ii): $r_1 = r_2$ (say 'r').

Take $y_1 = x^{r} \sum_{m=0}^{\infty} a_m x^m$.

$$y_2 = y_1 \ln x + x^{r} \sum_{m=1}^{\infty} A_m x^m.$$

Case-(iii): r_1 and r_2 differ by an integer.

Then, denote the roots by r_1, r_2 such that $r_1 - r_2 > 0$.

$$y_1(x) = x^{r_1} \sum a_m x^m$$
$$y_2(x) = K y_1 \ln x + x^{r_2} \sum_{m=0}^{\infty} A_m x^m.$$

Note: If r_1 is a complex root, then second root is

$\bar{r}_1 = r_1 - \bar{r}_1 = 2i\, \mathrm{Im}\, r_1$, which is not an integer,

therefore we apply case (i).

$$\begin{array}{r} a+ib \\ -\ a-ib \\ \hline 2ib \end{array}$$

Q) Solve by Frobenius method.

$$xy'' + \frac{1}{2}y' - y = 0 \quad \text{---(1)}$$

Soln: $y'' + \frac{1}{2x}y' - \frac{1}{x}y = 0$

$x=0$ is a regular sing. pt.

Take $y(x) = x^{r} \sum a_m x^m$ with $a_0 \neq 0$.

$= \sum a_m x^{m+r}$

Eqn (1) becomes

$$\sum (m+r)(m+r-1) a_m x^{m+r-1} + \frac{1}{2}\sum (m+r) a_m x^{m+r-1} - \sum a_m x^{m+r} = 0.$$

$$\cancel{x^{r}}\left[\sum (m+r)(m+r-1) a_m x^{m-1} + \frac{1}{2}\sum (m+r) a_m x^{m-1} - \sum a_m x^m\right] = 0.$$

Equate coeff. of $x^0, x^1, x^2, \ldots$ to zero.

$$\left[\begin{array}{l}\text{Coeff. of} \\ x^0 \rightarrow r(r+1) + \frac{1}{2}(r+1)a_1 - a_0 = 0 \quad \text{(Indicial eqn)}\end{array}\right] \rightarrow \times$$

$$x^{r-1}\left[\sum (m+r)(m+r-1) a_m x^m + \frac{1}{2}\sum (m+r) a_m x^m - \sum a_m x^{m+1}\right] = 0$$

Coeff. of

$x^0 \rightarrow r(r-1)a_0 + \frac{1}{2} r a_0 = 0$

$\Rightarrow r(r-1) + \frac{1}{2}r = 0 \quad (\because a_0 \neq 0)$

$\Rightarrow r = 0, \frac{1}{2}$.

Solutions are

$$y_1(x) = x^0 \sum a_m x^m$$

$$y_2(x) = x^{1/2} \sum A_m x^m$$

$$y_1 = \sum a_m x^m$$

Eq^n ① becomes

$$\sum m(m-1) a_m x^{m-1} + \frac{1}{2} \sum m a_m x^{m-1} - \sum a_m x^m = 0.$$

Equating coefficients of $x^0, x^1, x^2, \ldots$ to zero.

coeff. of

$$x^0 \rightarrow \frac{1}{2} a_1 - a_0 = 0$$

$$x^1 \rightarrow 2 \cdot a_2 + \frac{1}{2} \cdot 2a_2 - a_1 = 0$$

$$x^2 \rightarrow 3 \cdot 2\, a_3 + \frac{1}{2} 3a_3 - a_2 = 0$$

....

$$x^m \rightarrow m(m-1) a_m + \frac{1}{2} m \cdot a_m - a_{m-1} = 0$$

$$\Rightarrow \left(m^2 - m + \frac{1}{2}m\right) a_m = a_{m-1}$$

$$\Rightarrow a_m = \frac{a_{m-1}}{m(m-1/2)}$$

$$a_1 = \frac{a_0}{1/2} = 2a_0$$

$$a_2 = \frac{a_1}{3} = \frac{2}{3} a_0$$

$$a_3 = \frac{a_0}{3 \cdot \frac{5}{2}} = \frac{2 \cdot 2}{3\, 3 \cdot 5} a_0 = \frac{4}{45} a_0$$

....

$$y_1(x) = a_0 + 2a_0 x + \frac{2}{3} a_0 x^2 + \frac{4}{45} a_0 x^3 + \ldots$$

$$= a_0 \left[1 + 2x + \frac{2}{3} x^2 + \frac{4}{45} x^3 + \ldots\right]$$

$$y_2(x) = x^{1/2} \cdot \sum A_m x^m$$

$$y_2' = x^{1/2} \sum m A_m x^{m-1} + \frac{1}{2} x^{-1/2} \sum A_m x^{m-1}$$

$$y_2'' = x^{1/2} \sum m(m-1) A_m x^{m-2} + \underline{\frac{1}{2} x^{-1/2} \sum m A_m x^{m-1}}$$

$$\underline{+ \frac{1}{2} m^{-1/2} \sum m A_m x^{m-1}} - \frac{1}{4} x^{-3/2} \sum A_m x^m.$$

$xy'' + \frac{1}{2}y' - y = 0$ becomes

$$x^{1/2}\sum m(m-1)A_m x^{m-1} + x^{1/2}\sum m A_m x^{m-1} - \cancel{\frac{1}{4}x^{-1/2}\sum A_m x^m}$$
$$+\frac{1}{2}x^{1/2}\sum m A_m x^{m-1} + \cancel{\frac{1}{4}x^{-1/2}\sum A_m x^m} - x^{1/2}\sum A_m x^m = 0$$

$$\Rightarrow x^{1/2}\left[\sum m(m-1)A_m x^{m-1} + \frac{3}{2}\sum m A_m x^{m-1} - \sum A_m x^m\right] = 0.$$

Equating coeff. of $x^0, x^1, x^2, \ldots$ to zero,

$\frac{3}{2}A_1 - A_0 = 0 \Rightarrow A_1 = \frac{2}{3}A_0$

$2 \cdot 1\, A_2 + \frac{3}{2} \cdot 2\, A_2 - A_1 = 0$

$3 \cdot 2\, A_3 + \frac{3}{2} \cdot 3\, A_3 - A_2 = 0$

$m^2 - m + \frac{3}{2}m$

$m(m-1)A_m + \frac{3}{2}m A_m - A_{m-1} = 0$

$\Rightarrow \boxed{A_m = \frac{A_{m-1}}{m(m+\frac{1}{2})}}$, $A_2 = \frac{A_1}{5} = \frac{2}{15}A_0.$

Q) Solve by Frobenious method. 18-05-2023

$$xy'' + y' - y = 0 \quad -①$$

Soln: $y'' + \frac{1}{x}y' - \frac{1}{x}y = 0$

$x=0$ is a reg. singular point.

Take $y(x) = \sum a_m x^{m+r}$, $a_0 \neq 0$.

Eqn ① Becomes

$$\sum (m+r)(m+r-1) a_m x^{m+r-1} + \sum (m+r) a_m x^{m+r-1} - \sum a_m x^{m+r} = 0$$

$$x^{r-1}\left[\sum (m+r)(m+r-1) a_m x^m + \sum (m+r) a_m x^m - \sum_0^\infty a_m x^{m+1}\right] = 0$$

Equating coefficients of $x^0, x^1, \ldots$ to zero.

$r(r-1)a_0 + r a_0 = 0$ (Indicial eqn)

$a_0 \neq 0$

$\therefore r^2 = 0 \Rightarrow r = 0, 0$

$(r+1) r a_1 + (r+1) a_1 - a_0 = 0$

$(r+2)(r+1) a_2 + (r+2) a_2 - a_1 = 0$

. . . .

$(r+m)(r+m-1) a_m + (r+m) a_m - a_{m-1} = 0$

$$\boxed{a_m = \frac{a_{m-1}}{(r+m)^2}}$$

$a_0 \neq 0$, $r = 0$

$\left[y_1 = x^0 \sum a_m x^m = \sum a_m x^m\right]$

$a_1 = a_0$

$a_2 = \frac{a_1}{4} = \frac{a_0}{4}$

$a_3 = \frac{a_2}{9} = \frac{a_0}{36}$

. . . .

$y_1(x) = a_0\left[1 + x + \frac{1}{4}x^2 + \frac{1}{36}x^3 + \ldots\right]$

$r_2 = r_1 = 0$

$y_2(x) = y_1 \ln x + x^0 \sum_{m=1}^\infty A_m x^m$

$y_2' = \frac{y_1}{x} + y_1' \ln x + \sum m A_m x^{m-1}$

$y_2'' = -\frac{y_1}{x^2} + \frac{2y_1'}{x} + y_1'' \ln x + \sum m(m-1) A_m x^{m-2}$

The eqn $xy''+y'-y=0$ becomes

$$\ln x\left[xy_1''+y_1'-y_1\right]-\frac{y_1}{x}+2y_1'+\frac{y_1}{x}+\sum m(m-1)A_m x^{m-1}$$
$$+\sum mA_m x^{m-1}-\sum A_m x^m=0$$

$$\Rightarrow \boxed{2y_1'}+\sum m(m-1)A_m x^{m-1}+\sum mA_m x^{m-1}-\sum mA_m x^m=0$$

$$\Rightarrow \sum m(m-1)A_m x^{m-1}+\sum mA_m x^{m-1}-\sum mA_m x^m= -2\underbrace{\left[1+\frac{1}{2}x+\frac{1}{12}x^2+\dots\right]}_{y_1'}$$

[Take $a_0=1$ in y_1]

Equating coeff.s of $x^0, x^1, x^2, \dots$, we can find $A_0, A_1, \dots$ (we can take any non-zero constant a_0)

Q Solve $x^2y''-xy'-\left(x^2+\frac{5}{4}\right)y=0$, (1) using frobenius method.

(Need not compute the values of the coefficients in the series expression.)

Soln: $y''-\frac{y'}{x}-y\;\frac{5}{4x^2}y=0$

$x=0$ is a reg. singular point.

Take $y(x)=\sum a_m x^{m+r}$, $a_0\neq 0$.

Eqn (1) becomes

$$\sum (m+r)(m+r-1)a_m x^{m+r}-\sum(m+r)a_m x^{m+r}-\sum a_m x^{m+r+2}$$
$$-\frac{5}{4}\sum a_m x^{m+r}=0.$$

$$r(r-1)a_0-ra_0-\frac{5}{4}a_0=0.$$

$a_0\neq 0$.

So, $r(r-1)-r-\frac{5}{4}=0$

$$\Rightarrow r^2-2r-\frac{5}{4}=0$$
$$\Rightarrow r=\frac{5}{2},-\frac{1}{2}$$

Since difference is an integer,

take r_1, r_2 such that $r_1-r_2>0$.

$r_1=\frac{5}{2}$, $r_2=-\frac{1}{2}$

$y_1=x^{5/2}\sum a_m x^m$.

$y_2=Ky_1\ln x+x^{-1/2}\sum_{m=0}^{\infty}A_m x^m$.

Special Functions

① Legendre Polynomial

Consider the Legendre eq^n

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad -①$$

If $x=x_0$ is reg.
$y(x) = \sum a_m (x-x_0)^m$.

where n is a given constant (real number).

$$y'' - \frac{2x}{1-x^2} y' + \frac{n(n+1)}{1-x^2} y = 0$$

→ not regular at $x = \pm 1$.

$x=0$ is regular point of ①.

Take $y(x) = \sum_{m=0}^{\infty} a_m x^m$

$$a_m = \frac{(m-2)(m-1) - n(n+1)}{m(m-1)} a_{m-2} \quad \rightarrow (*)$$

Substitute y, y', y'' in ① and equating coeff. of x^0, x^1, x^2 get the recursive formula

The general solution is

$$y(x) = a_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 + \dots\right]$$
$$+ a_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 + \dots\right]$$
$$= a_0 y_1(x) + a_1 y_2(x).$$

Both the series converge for $|x| < 1$.

Note that y_1 is of even powers of x and y_2 involves odd powers of x. So, they are linearly independent.

$\therefore \{y_1, y_2\}$ is a basis for solutions of ①.

Note that if 'n' is an even integer, then y_1 becomes a polynomial of degree n.

If n is an odd integer, $y_2(x)$ becomes a polynomial of degree 'n'.

19-05-2023

Legendre Polynomial:

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad -①$$

$x=0$ is a regular point.

If n is an even integer, $y_1(x)$ reduces to a polynomial of degree 'n' (involves only even power of x).

If n is odd, y_2 involves only odd power of x and y_2 is a polynomial of degree 'n'.

$$y(x) = a_0\underbrace{\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 + \dots\right]}_{y_1} + a_1\underbrace{\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \dots\right]}_{y_2}$$

$\therefore \{y_1, y_2\}$ is a basis of solution set.

These polynomials multiplied by some constants are called Legendre polynomial and are denoted by $P_n(x)$.

Choose $a_n = \dfrac{(2n)!}{2^n(n!)^2}$ as the coefficient of highest power of x^n.

Legendary poly. of degree 'n':

$$P_n(x) = \begin{cases} a_0 + a_2x^2 + a_4x^4 + \dots + a_nx^n, & \text{if } n \text{ is even} \\ a_1x + a_3x^3 + \dots + a_nx^n, & \text{if } n \text{ is odd.} \end{cases}$$

$$a_n = \frac{1.3.5\dots(2n-1)}{n!}$$

$a_2 = \dfrac{4!}{4(2!)^2} = \dfrac{\cancel{}1.2.3.4}{4(2\times1)2!} = \dfrac{1.3}{2!}$

Use the recursive formula

$$a_m = \frac{(m-2)(m-1) - n(n+1)}{m} a_{m-2}$$

$$\Rightarrow a_{m-2} = \frac{m(m-1)}{(m-2)(m-1) - n(n+1)} a_m$$

$$a_{n-2} = \frac{n(n-1)}{(n-2)(n-1) - n(n+1)} a_n$$

$$\Rightarrow \boxed{a_{n-2} = -\frac{n(n-1)}{2(2n-1)} a_n}$$

$$P_4 = a_0 - a_2x^2 + a_4x^4 = \sum_{n=0}^{2} a_{2n}x^{2n}(-1)^n$$

Use the recursive formula

$$a_{n-2} = \frac{-n(n-1)}{2(2n-1)} a_n$$ to find the other coefficients.

$P_0(x) = a_0 = 1$

$P_1(x) = a_1 x = x$

$P_2(x) = \boxed{a_0 - a_2 x^2}$

$= -\frac{1}{2} - \frac{3}{2} x^2$

$= -\frac{1}{2}\left[3x^2 + 1\right]$

$P_3(x) = a_1 x - a_3 x^3$

$= -\frac{1}{2}\left[5x^3 + 3x\right]$

$a_n = \frac{(2n)!}{2^n (n!)^2}$

$a_1 = \frac{2!}{2(1!)^2} = 1$

$a_2 = \frac{4!}{4(2!)^2} = \frac{3}{2}$

$a_{n-2} = \frac{-n(n-1)}{2(2n-1)} a_n$

$a_0 = \frac{-2}{2.3} a_2 = -\frac{1}{2}$

$a_3 = \frac{6!}{8(3!)^2} = \frac{5}{2}$

$a_1 = -\frac{3.2}{2.5} \cdot \frac{5}{2} = -\frac{3}{2}$

22-05-2023

Bessel Equation

Consider the Bessel eqⁿ

$$x^2y'' + xy' + (x^2-k^2)y = 0 \quad -(1)$$

where k is a non-negative real no.

Note that $x=0$ is a regular singular point of eqⁿ (1).

Using Frobenius method,

take $y(x) = x^r \sum_0^\infty a_m x^m$, $a_0 \neq 0$.

Eqⁿ (1) becomes

$$\sum (m+r)(m+r-1)a_m x^{m+r} + \sum (m-r)a_m x^{m+r} +$$
$$\sum a_m x^{m+r+2} - k^2 \sum a_m x^{m+r} = 0.$$

Equating coeff. of x^r to zero,

$$r(r-1)a_0 + ra_0 - k^2 a_0 = 0$$

Since $a_0 \neq 0$, we get $r^2 - k^2 = 0$

$$r = k, -k$$

For $r=k$

$$y_1(x) = x^k \sum_0^\infty a_m x^m$$

Substitute in (1) and find out a_m, $m=1,2,\ldots$

$$a_1 = 0,\ a_3 = 0,\ a_5 = 0, \ldots$$

$$a_{2m} = -\frac{a_{2m-2}}{2^2 m(m+k)}$$

$$\therefore a_2 = -\frac{a_0}{2^2(k+1)}$$

$$a_4 = -\frac{a_2}{2^2 \cdot 2(k+2)} = \frac{a_0}{2^4 \cdot 2!(k+1)(k+2)}$$

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m! (k+1)(k+2)\ldots(k+m)} \rightarrow (*), \text{ for } m=1,2,\ldots$$

Bessel Functions $J_k(x)$ for integer $k=n$:

For $k=n$ an integer, eqⁿ (*) becomes

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m! (n+1)(n+2)\ldots(n+m)}$$

Choose $a_0 = \frac{1}{2^n n!}$.

Then $a_{2m} = \frac{(-1)^m}{2^{2m+n} m! \, (n+m)!}$

So, for $r = K = n$ an integer, a particular soln of the Bessel eqn, denoted by $J_n(x)$, is

$$\boxed{\begin{aligned} J_n(x) &= x^n \sum_{m=0}^{\infty} a_{2m} x^{2m} \\ &= \sum_{0}^{\infty} \frac{(-1)^m x^{2m+n}}{2^{2m+n} m! \cdot (m+n)!} \end{aligned}}$$

$y_1 = x^K \sum_{m=0}^{\infty} a_m x^m$
where $K = n$,
$0 = a_1 = a_3 = a_5 = \dots$

This is called Bessel ~~eqn~~ function of the first kind of order n.

$$J_0(x) = \sum_{0}^{\infty} \frac{(-1)^m x^{2m}}{2^m (m!)^2}$$

$$= 1 - \frac{x^2}{2^2 (1!)^2} + \frac{x^4}{2^4 (2!)^2} - \frac{x^6}{2^6 (3!)^2} + \dots$$

$\boxed{r = K}, -K$

$r = K$, ~~where~~ when $K = n$ is an integer,
the particular soln (Bessel fn of order n) are $J_n(x)$.

<u>Bessel function $J_K(x)$ for any $K \geq 0$:</u>

<u>Gamma function:</u>

Defined as $\Gamma(K) = \int_0^{\infty} e^{-t} t^{K-1} dt$, $K > 0$.

$$\Gamma(K+1) = \int_0^{\infty} e^{-t} t^K dt$$

$$= \left[\cancel{t^K (-e^{-t})} \right]_0^{\infty} - \int K t^{K-1} (-e^{-t}) dt$$

$$= K \int e^{-t} t^{K-1} dt$$

$$= K \Gamma(K), \quad K > 0$$

$$\Gamma(1) = \int_0^{\infty} e^{-t} dt = \left[-e^{-t} \right]_0^{\infty} = 1$$

$$= 1!$$

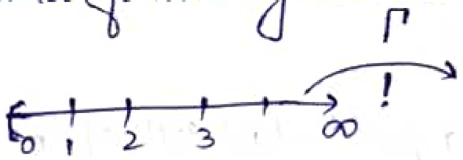
$\Gamma(2) = 1 \, \Gamma(1) = 1!$

$\Gamma(3) = 2 \, \Gamma(2) = 2!$

In general,

$$\Gamma(n+1) = n! \quad , n=1,2,\dots$$

So, gamma function generalises the factorial.



In the Bessel function, we have

$$a_0 = \frac{1}{2^n . n!}$$

$$= \frac{1}{2^n \Gamma(n+1)}$$

So, for any $k \geq 0$ real no.,

choose $a_0 = \dfrac{1}{2^k \Gamma(k+1)}$.

Then, $$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m! (k+1)(k+2)\dots(k+m)}$$

$$= \frac{(-1)^m}{2^{2m+k} m! (k+1)(k+2)\dots(k+m)\Gamma(k+1)}$$

$$(k+1)\Gamma(k+1) = \Gamma(k+2)$$

$$(k+2)\Gamma(k+2) = \Gamma(k+3)$$

$$(k+1)(k+2)\dots(k+m)\Gamma(k+1) = \Gamma(k+m+1)$$

$$\therefore a_{2m} = \frac{(-1)^m}{2^{m+k} m! \Gamma(k+m+1)}$$

So, for $r = k$, where $k \geq 0$ real, a particular soln $J_k(x)$ is given by

$$J_k(x) = x^k \sum a_{2m} x^{2m}$$

$$= \sum \frac{(-1)^m x^{2m+k}}{2^{2m+k} m! \Gamma(k+m+1)} \quad -②$$

called Bessel fn of the first kind of order k.

Using ratio test, we can show that $J_K(x)$ converges for all x.

We have $r = K, -K$ and for $r = K \geq 0$, sol^{n}s are $J_K(x)$.

Now to find $J_K(x)$,

we find the particular solutions (Bessel f^n) $J_K(x)$ for K not an integer.

We get $J_{-K}(x)$, if K is not an integer, by replacing K with $-K$ in eq^n ②.

$$\therefore J_{-K}(x) = \sum \frac{(-1)^m x^{2m-K}}{2^{2m-K} \cdot m! \, \Gamma(m-K+1)}, \text{ K is NOT an integer.}$$

For $r = K$, we have $y_1 = J_K(x)$.

$r = -K$, we have $y_2 = J_{-K}(x)$, K is NOT an integer.

Note that y_1, y_2 are linearly independent.

So general sol^n is

$$y(x) = c_1 J_K(x) + c_2 J_{-K}(x), \text{ K is NOT an integer.}$$

Bessel Function

$$J_k(x) = x^k \sum \frac{(-1)^m x^{2m}}{2^{2m+k} m! \Gamma(m+k+1)}$$

$$J_{-k}(x) = x^{-k} \sum \frac{(-1)^m x^{2m}}{2^{2m-k} m! \Gamma(m-k+1)}$$, k is NOT an integer

Properties:

① For $k=n$ an integer, $n=1,2,\ldots$

$$J_{-n}(x) = (-1)^n J_n(x)$$

(J_{-n} and J_n are linearly dependent)

$$J_{-k}(x) = x^{-k} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m-k} m! \Gamma(m-k+1)},$$

As k approaches 'n',

then note that $\Gamma(m-k+1)$ is infinity for $m=0,1,2,\ldots,n-1$. $\quad |(-2)! = \infty$

The value of the f^n becomes zero for $m=0,1,2,\ldots,n-1$.

$$\therefore J_{-n}(x) = x^{-n} \sum_{m=n}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m-n} m! \Gamma(m-n+1)}$$

Put $m=n+s$.

$$J_{-n}(x) = x^{-n} \sum_{s=0}^{\infty} \frac{(-1)^{n+s} x^{2n+2s}}{2^{2s+n} (n+s)! \Gamma(s+1)}$$

$$= (-1)^m x^n \sum_{s=0}^{\infty} \frac{(-1)^s x^{2s}}{2^{2s+n} (n+s)! s!}$$

$$= (-1)^n J_n(x)$$

② $$\frac{d}{dx}[x^k J_k(x)] = x^k J_{k-1}(x)$$

$$\frac{d}{dx}\left[\sum_0^{\infty} \frac{(-1)^m x^{2m+2k}}{2^{2m+k} m! \Gamma(m+k+1)!}\right]$$

$$= \sum \frac{(-1)^m (2m+2k) x^{2m+2k-1}}{2^{2m-k} m! (m+k) \Gamma(m+k)}$$

$$= x^k \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+k-1}}{2^{2m+k-1} m! . \Gamma(m+1+k+1)}$$

$$= x^k J_{k-1}(x)$$

③ $\frac{d}{dx}\left[x^{-k} J_k(x)\right] = -x^{-k} J_{k+1}(x)$

$$\text{LHS} = \frac{d}{dx}\left[\sum \frac{(-1)^m x^{2m}}{2^{2m+k}\, m!\, \Gamma(m+k+1)}\right] = \sum_0^\infty \frac{(-1)^m x^{2m-1}}{2^{2m+k-1}\, (m-1)!\, \Gamma(m+k+1)}$$

Take $m = s+1$.

$$= \sum_{s=0}^\infty \frac{(-1)^{s+1} x^{2s+1}}{2^{2s+k+1}\, s!\, \Gamma(s+k+2)}$$

$$= -x^{-k} \sum_0^\infty \frac{(-1)^{s} x^{2s+k+1}}{2^{2s+k+1}\, s!\, \Gamma(s+k+1+1)}$$

$$= -x^{-k} J_{k+1}(x).$$

④ $J_{k-1}(x) + J_{k+1}(x) = \frac{2k}{x} J_k(x)$

$$J_k = x^k \sum_0^\infty \frac{(-1)^m x^{2m}}{2^{2m+k}\, m!\, \Gamma(m+k+1)}$$

We have from ②,

$$\frac{d}{dx}\left[x^k J_k(x)\right] = x^k J_{k-1}(x)$$

$$x^k J_k'(x) + k x^{k-1} J_k(x) = x^k J_{k-1}(x) \quad \text{—(i)}$$

From ③,

$$x^{-k} J_k'(x) - k x^{-k-1} J_k(x) = -x^{-k} J_{k+1}(x) \quad \text{—(ii)}$$

(i) becomes $J_k'(x) + \frac{k}{x}(x) = J_{k-1}(x)$ —(iii)

(ii) becomes $J_k'(x) - \frac{k}{x} J_k(x) = -J_{k+1}(x)$ —(iv)

(iii) − (iv)

$$\Rightarrow J_{k-1}(x) + J_{k+1}(x) = \frac{2k}{x} J_k(x)$$

⑤ $J_{k-1}(x) - J_{k+1}(x) = 2 J_k'(x)$

This is by (iii) + (iv).

<u>Note:</u> From property ④,

$$J_{k+1}(x) = \frac{2k}{x} J_k(x) - J_{k-1}(x)$$

This is a Recursive formula.

Sturm-Liouville Problem

Sturm-Liouville equation is

$$[p(x)\, y']' + [q(x) + \lambda\, r(x)]\, y = 0 \quad -(1)$$

with the boundary conditions

$$\left.\begin{aligned} k_1\, y(a) + k_2\, y'(a) &= 0 \\ l_1\, y(b) + l_2\, y'(b) &= 0 \end{aligned}\right\} -(2)$$

Solving (1) with the boundary conditions (2) is called Sturm-Liouville problem.

λ is a parameter.

We assume that $p(x)$, $q(x)$, $r(x)$ and $p'(x)$ are continuous on $[a,b]$ and $r(x) > 0$ on $[a,b]$.

k_1, k_2 are given constants, not both zero.

l_1, l_2 are ,, ,,

Note: Legendre equation and Bessel eq^n are Sturm-Liouville equations.

We have the Legendre eq^n

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

$$\Rightarrow [(1-x^2)y']' + \lambda y = 0$$

where $\lambda = n(n+1)$

This is a S-L eq^n with $p(x) = 1-x^2$, $q(x) = 0$, $r(x) = 1$, $\lambda = n(n+1)$.

Consider Bessel eq^n

$$x^2 y'' + xy' + (x^2 - k^2)y = 0$$

Change the variable x to $\mu x = \tilde{x}$

The eq^n becomes

$$\tilde{x}^2 \ddot{y} + \tilde{x}\dot{y} + (\tilde{x}^2 - k^2)y = 0$$

$$\Rightarrow \mu^2 x^2 \frac{y''}{\mu^2} + \mu x \frac{y'}{\mu} + (\mu^2 x^2 - k^2)y = 0$$

$$\Rightarrow x^2 y'' + xy' + (\mu^2 x^2 - k^2)y = 0$$

$$\Rightarrow xy'' + y' + \mu^2 xy + \frac{k^2}{x} y = 0$$

$$\Rightarrow (xy')' + \left[\frac{-k^2}{x} + \mu^2 x\right] y = 0$$

$$\dot{y} = \frac{dy}{d\tilde{x}} = \frac{dy}{dx} \cdot \frac{dx}{d\tilde{x}} = \frac{y'}{\mu}$$

$$\ddot{y} = \frac{y''}{\mu^2}$$

$p(x) = x,\ q(x) = \frac{-k^2}{x}$

$\lambda = \mu^2;\ r(x) = x.$

This is an S-L equation.

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Sturm-Liouville Problem

S-L equation

$$[p(x)y']' + [q(x) + \lambda r(x)]y = 0 \quad -①$$

with boundary conditions

$$\left.\begin{array}{l} k_1 y(a) + k_2 y'(a) = 0 \\ l_1 y(b) + l_2 y'(b) = 0 \end{array}\right\} - ②$$

where is a parameter $(\lambda \in \mathbb{R})$

$$y : [a, b] \longrightarrow \mathbb{R}$$

Note that $y(x) \equiv 0$ is a solution of the S-L problem.

We want to have non-trivial solⁿ.

A non-trivial solution of S-L problem is called an eigenfunction.

The values of λ corresponding to which eigenfunctions exist are called eigenvalue.

Q1 Find the eigenvalues and eigenfunction of

$$y' + \lambda y = 0.$$

$$y(0) = 0,\ y(\pi) = 0.$$

Solⁿ: $y' + \lambda y = 0$

char. eqⁿ: $m^2 + \lambda = 0.$

Case ①: $\lambda = 0$

Then $m^2 = 0 \Rightarrow m = 0, 0$

$y(x) = c_1 + c_2 x$

$y(0) = 0 \Rightarrow c_1 = 0$

$\therefore y(x) = c_2 x$

$y(\pi) = 0 \Rightarrow c_2 \pi = 0 \Rightarrow c_2 = 0$

$\therefore y(x) \equiv 0$ is the solution in this case.

Case-(ii): $\lambda<0$

Then $m = \pm\sqrt{-\lambda}_{\rightarrow \text{+ve}}$

$$\begin{aligned} m^2-\lambda&=0\\ m^2&=-\lambda \end{aligned}$$

We have the distinct real values.

General soln:

$$y(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$$

$$y(0)=0 \Rightarrow c_1 + c_2 = 0$$
$$y(\pi)=0 \Rightarrow c_1 e^{\sqrt{-\lambda}\pi} + c_2 e^{-\sqrt{-\lambda}\pi} = 0$$

$$\begin{bmatrix} 1 & 1 \\ e^{\sqrt{-\lambda}\pi} & e^{-\sqrt{-\lambda}\pi} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0$$

For non-trivial soln, we need the determinant

$$\begin{vmatrix} 1 & 1 \\ e^{\sqrt{-\lambda}\pi} & e^{-\sqrt{-\lambda}\pi} \end{vmatrix} = 0.$$

ie, $e^{-\sqrt{-\lambda}\pi} = e^{\sqrt{-\lambda}\pi}$

$\Rightarrow \lambda = 0$. But we have taken $\lambda<0$.

$\therefore c_1 = c_2 = 0$.

So, in this case, $\lambda<0$, also, only trivial soln.

Case-(iii): $\lambda>0$

Char. eqn: $m^2+\lambda=0$

$$\Rightarrow m = \pm\sqrt{-\lambda}$$
$$= \pm i\sqrt{\lambda}$$

$$y(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$$

$$y(0)=0 \Rightarrow c_1 = 0$$

$$\therefore y(x) = c_2 \sin\sqrt{\lambda}x$$

$$y(\pi)=0 \Rightarrow c_2 \sin(\sqrt{\lambda}\pi) = 0$$

For non-trivial solution

$$\sin(\pi\sqrt{\lambda}) = 0 \Rightarrow \sqrt{\lambda} = n$$
$$\text{or } \lambda = n^2$$

So solution:

$y(x) = c_2 \sin\sqrt{\lambda}x$, where c_2 is arbitrary.

Eigenvalues are

$$\lambda = 1, 4, 9, \ldots$$

corresponding eigenfunctions:

$a_1 \sin x,\ a_2 \sin 2x,\ a_3 \sin 3x, \ldots$, a_1, a_2 are arbitrary constants.

Q. Find the eigenvalues and eigen f^n of

$$\frac{d}{dx}[xy'] + \frac{\lambda}{x}y = 0 \quad \text{with } \lambda \geq 0,$$

$$y'(1)=0, \quad y'(e^{2\pi})=0.$$

$y: [1, e^{2\pi}] \to \mathbb{R}$

Soln: case (i): $\lambda=0 \Rightarrow$ The eq^n becomes

$$\frac{d}{dx}[xy']=0 \Rightarrow xy'=c \text{ constant}$$

$$\Rightarrow y' = \frac{c}{x}$$

$$\Rightarrow y = c\ln|x| + c_1$$

$$y'(1)=0 \Rightarrow c=0$$

$\therefore y(x) = c_1$, c_1 is arbitrary condition. is a sol^n for the given problem.

Eigenf^n is $y=c_1$.

Eigen value is $\lambda=0$.

case (ii): $\lambda>0$

Given equation is

$$xy'' + y' + \frac{\lambda}{x}y = 0$$

$$\Rightarrow x^2y'' + xy' + \lambda y = 0 \quad \text{(Euler-Cauchy eq}^n\text{)}$$

Take $y=x^m$

$$m(m-1)x^m + mx^m + \lambda x^m = 0$$

$$m^2+\lambda=0$$

$$\Rightarrow m = \pm i\sqrt{\lambda}$$

General sol^n:

$$y(x) = \boxed{c_1\cos(\sqrt{\lambda}\ln x)} + \boxed{c_2\sin(\sqrt{\lambda}\ln x)}$$

$$\Rightarrow y'(x) = c_1\frac{\sqrt{\lambda}}{x}\sin(\sqrt{\lambda}\ln x) + c_2\frac{\sqrt{\lambda}}{x}\cos(\sqrt{\lambda}\ln x)$$

$$y'(1)=0 \Rightarrow c_2\sqrt{\lambda}=0 \Rightarrow c_2=0$$

$$y'(e^{2\pi})=0 \Rightarrow -c_1\sqrt{\lambda}e^{-2\pi}\sin(\sqrt{\lambda}\,2\pi)=0$$

For non-trivial sol^n, take

$$\sin(2\pi\sqrt{\lambda})=0 \Rightarrow 2\sqrt{\lambda}=n$$

$$\Rightarrow 4\lambda=n^2$$

$$\Rightarrow \lambda=\frac{n^2}{4}$$

Solution is $y(x) = c_1\cos(\sqrt{\lambda}\ln x)$, $\lambda=\frac{1}{4}, 1, \frac{9}{4}, \ldots$

Soln: $a_1 \cos(\frac{1}{2}\ln x)$, $a_2 \cos(\ln x)$, $a_3 \cos(\frac{3}{2}\ln x)$, ...

For the given eqn,

Eigenvalues are $\lambda = 0, \frac{1}{4}, 1, \frac{9}{4}, \ldots$

Eigenfn are a_0; $a_1 \cos(\frac{1}{2}\ln x)$, $a_2 \cos(\ln x)$, $a_3 \cos(\frac{3}{2}\ln x)$, ...,

where $a_0, a_1, \ldots$ are non-zero arbitrary constants.

Summary:

$$y : [a,b] \to \mathbb{R}$$
$$x \to y(x)$$

$$y^{(n)} + p_{n-1} y^{(n-1)} + \ldots + p_0 y = r(x) \text{ (for non-homo. eqn)}$$
$$= 0 \text{ (for homogeneous eqn)}$$

$$\mathcal{F} = \{ f : [a,b] \to \mathbb{R} \}$$

S = soln set $\subset \mathcal{F}$ [soln 'y' in S.]

For homogeneous eqn ($r(x) = 0$), if y_1 & y_2 are solns, $c_1 y_1 + c_2 y_2$ is also soln.