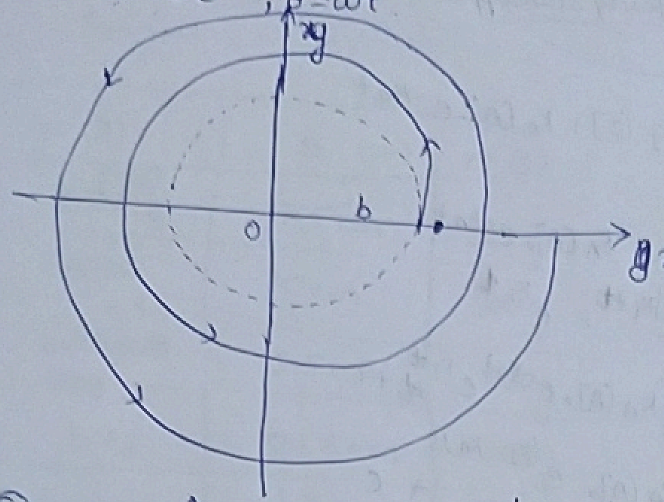
Physics - (Assignment - I)

① (i) $r = be^{(\omega t)}$, $\theta = \omega t$



At $\theta = 0\ (t=0)$: $r = b$

$\theta > 0 \rightarrow r > b$

$\theta \uparrow \rightarrow r \uparrow$

$\Rightarrow$ $\boxed{\text{Spiral with increasing } r}$

(ii) $\bar{r} = r\hat{r}$

$\bar{v} = \dot{r}\hat{r} + r\dot{\hat{r}}$

$= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

$= b\omega e^{\omega t}\hat{r} + be^{\omega t}\omega\hat{\theta}$

$\therefore \boxed{\bar{v} = b\omega e^{\omega t}(\hat{r} + \hat{\theta})}$

$r = be^{\omega t}$, $\theta = \omega t$

$\Rightarrow \dot{r} = b\omega e^{\omega t}$, $\dot{\theta} = \omega$

$\Rightarrow \ddot{r} = b\omega^2 e^{\omega t}$, $\ddot{\theta} = 0$

and, $\bar{a} = \dot{\bar{v}} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$

$= (\underline{b\omega^2 e^{\omega t} - be^{\omega t}.\omega^2})\hat{r} + (2b\omega e^{\omega t}.\omega + \cancel{be^{\omega t} \times 0})\hat{\theta}$

$\therefore \boxed{\bar{a} = (2b\omega^2 e^{\omega t})\hat{\theta}}$

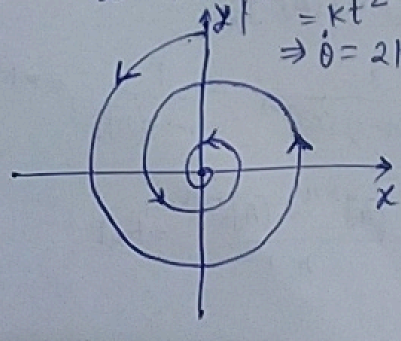
(iii) Angle b/w $\bar{v}$ and $\bar{a}$, $\cos\alpha = \dfrac{\bar{v}.\bar{a}}{|\bar{v}||\bar{a}|}$

$= \dfrac{2b^2\omega^3 e^{2\omega t}}{(\sqrt{2}b\omega e^{\omega t})(2b\omega^2 e^{\omega t})} = \dfrac{1}{\sqrt{2}}$

$\Rightarrow \boxed{\alpha = 45^\circ}$

② $r = b - ct \Rightarrow \dot{r} = -c$

$\omega = kt$ | $\theta = \omega t$
$= kt^2$
$\Rightarrow \dot{\theta} = 2kt$



$\bar{v} = \dot{r}\hat{r} + r\dot{\hat{r}}$

$= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

$= -c\hat{r} + (b-ct)2kt\hat{\theta}$

$\Rightarrow \boxed{\bar{v} = -c\hat{r} + 2kt(b-ct)\hat{\theta}}$

Speed, $|\bar{v}| = \sqrt{c^2 + 4k^2t^2(b-ct)^2}$

③ $r = \frac{bt}{\tau^2}(2\tau - t) \Rightarrow r = \frac{2b}{\tau}t - \frac{b}{\tau^2}t^2 \Rightarrow \dot{r} = \frac{2b}{\tau} - \frac{2b}{\tau^2}t \Rightarrow \ddot{r} = -\frac{2b}{\tau^2}$

$\theta = \frac{t}{\tau} \Rightarrow \dot{\theta} = \frac{1}{\tau} \Rightarrow \ddot{\theta} = 0$ $\qquad \hookrightarrow = \frac{2b}{\tau^2}(\tau - t)$

$\bar{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

$= \left(\frac{2b}{\tau^2}(\tau - t)\right)\hat{r} + \left(\frac{bt}{\tau^2}(2\tau - t)\cdot\frac{1}{\tau}\right)\hat{\theta}$

$= \left[\frac{2b}{\tau^2}(\tau - t)\right]\hat{r} + \left[\frac{bt}{\tau^3}(2\tau - t)\right]\hat{\theta}$

Speed, $|\bar{v}| = \frac{2b}{\tau^2}\left[4(\tau - t)^2 + \frac{t^2}{\tau^2}(2\tau - t)^2\right]$

$= \frac{b}{\tau^2}\left[\underbrace{4\tau^2} + 4t^2 - 8t\tau + 4t^2 + \frac{t^4}{\tau^2} - 4\frac{t^3}{\tau}\right]$

$\frac{d|\bar{v}|}{dt} = 0 \Rightarrow 8t - 8\tau + 8t + \frac{4}{\tau^2}t^3 - 12\frac{t^2}{\tau} = 0$

$\Rightarrow 2\tau^2 t - 2\tau^3 + 2\tau^2 t + t^3 - 3t^2\tau = 0$

$\Rightarrow 4\tau^2 t - 3t^2\tau + t^3 - 2\tau^3 = 0$

For $\boxed{t = \tau}$, $\frac{d|\bar{v}|}{dt} = 0$

Now,

$\bar{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + \cancel{r\ddot{\theta}}^{0})\hat{\theta}$

$= \left(-\frac{2b}{\tau^2} - \frac{bt}{\tau^2}(2\tau - t)\cdot\frac{1}{\tau^2}\right)\hat{r} + \left(2\left(\frac{2b}{\tau} - \frac{2b}{\tau^2}t\right)\cdot\frac{1}{\tau} + 0\right)\hat{\theta}$

At $t = \tau$,

$\bar{a} = \left(-\frac{2b}{\tau^2} - \frac{b}{\tau^2}\right)\hat{r} + 0\,\hat{\theta}$ $\leftarrow$ $\cancel{\boxed{-b\left(\frac{2}{\tau^2}+1\right)\hat{r}}}$

$= \boxed{-\frac{3b}{\tau^2}\hat{r}}$

$$\dot{\theta}$$

$$\Rightarrow b\frac{\ddot{\theta}}{\dot{\theta}} r + b\dot{r} - r\dot{\theta} = \frac{\ddot{\theta}}{\dot{\theta}} r$$

$$\Rightarrow -2b^2\dot{\theta} r + b^2\dot{\theta} r - r\dot{\theta} = -2b\dot{\theta} r$$

$$-b^2 = -2b$$

④ $r = ae^{b\theta} \Rightarrow \dot{r} = ab\dot{\theta} r$

$a_\theta = 0 \Rightarrow 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$

$$\Rightarrow 2b\dot{\theta} r \dot{\theta} = -r \cancel{\omega} \frac{d\omega}{d\theta}$$

$$\cancel{\Rightarrow 2b\dot{\theta}^2 a e^{b\theta} = 0}$$

$$\Rightarrow -2b\int_{\theta_0}^{\theta} d\theta = \int_{\omega_0}^{\omega} \frac{d\omega}{\omega} = \ln\frac{\omega}{\omega_0}$$

$$\Rightarrow 2b(\theta_0 - \theta) = \ln\frac{\omega}{\omega_0}$$

$$\Rightarrow 2b\theta_0 - 2b\theta = \ln\frac{\omega}{\omega_0}$$

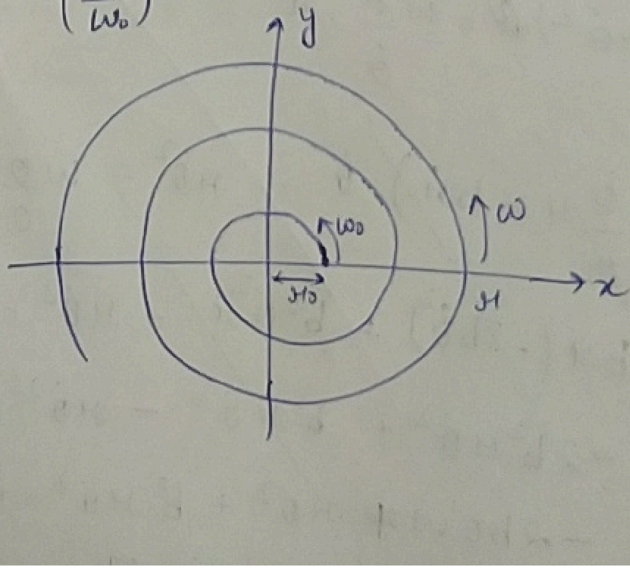
$$\Rightarrow \theta = \frac{2b\theta_0 - \ln\frac{\omega}{\omega_0}}{2b} = \theta_0 - \frac{1}{2b}\ln\frac{\omega}{\omega_0}$$

$$\therefore r = ae^{b\theta} = a e^{b\left(\theta_0 - \frac{1}{2b}\ln\frac{\omega}{\omega_0}\right)}$$

$$= a\frac{e^{b\theta_0}}{e^{\ln\left(\frac{\omega}{\omega_0}\right)^{1/2}}} = \frac{ae^{b\theta_0}}{\left(\frac{\omega}{\omega_0}\right)^{1/2}}$$

$$\Rightarrow r^2\left(\frac{\omega}{\omega_0}\right) = a^2 e^{2b\theta_0}$$

$$\Rightarrow \boxed{\omega = \omega_0\left(\frac{r_0}{r}\right)^2}$$



(5) Given that $r = Ke^{\alpha\theta}$ $\Rightarrow \dot{r} = K\alpha\dot{\theta}e^{\alpha\theta} \Rightarrow \ddot{r} = K\alpha\ddot{\theta}e^{\alpha\theta} + K\alpha^2\dot{\theta}^2e^{\alpha\theta}$

$a_\theta = 0$ $\qquad = \alpha r\ddot{\theta} + \alpha^2\dot{\theta}^2 r$

$\Rightarrow 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$

$\Rightarrow 2\alpha Ke^{\alpha\theta}\dot{\theta}^2 + Ke^{\alpha\theta}\ddot{\theta} = 0$

$\Rightarrow 2\alpha\omega^2 = -\frac{d\omega}{dt}$

$\Rightarrow \int_{\omega_0}^{\omega}\frac{d\omega}{\omega^2} = 2\alpha\int_0^t dt$

$\Rightarrow -\left[\frac{1}{\omega}\right]_{\omega_0}^{\omega} = 2\alpha t$

$\Rightarrow -\frac{1}{\omega} + \frac{1}{\omega_0} = 2\alpha t$

$\Rightarrow \frac{1}{\omega} = \frac{1}{\omega_0} - 2\alpha t = \frac{1 - 2\omega_0\alpha t}{\omega_0}$

$\Rightarrow \frac{d\theta}{dt} = \frac{\omega_0}{1 - 2\omega_0\alpha t}$

$\Rightarrow \int_0^\theta d\theta = \int_0^t \frac{\omega_0\,dt}{1 - 2\omega_0\alpha t}$

$\Rightarrow \theta = \frac{\omega_0}{-2\omega_0\alpha}\ln(1 - 2\omega_0\alpha t)$

$\Rightarrow \theta = -\frac{1}{2\alpha}\ln((1) - (2\omega_0\alpha t))$

$\therefore r = ke^{\alpha\left(-\frac{1}{2\alpha}\ln(1 - 2\omega_0\alpha t)\right)}$

$\Rightarrow r = ke^{-\frac{1}{2}\ln(1 - 2\omega_0\alpha t)}$

Central force,

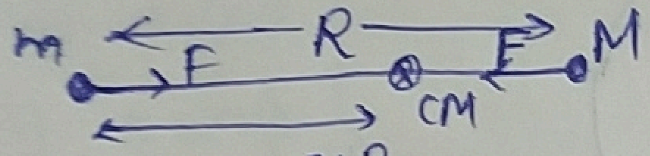
$F = m(\ddot{r} - r\dot{\theta}^2)$

$= m\left(\ddot{r} - r\frac{a^2}{b^2r^2}\right)$

$$= \frac{\int_0^a (x + x^3)\,dx}{\int_0^a (1+x^2)\,dx} = \frac{\frac{a^2}{2} + \frac{a^4}{4}}{a + \frac{a^3}{3}} = \frac{\frac{2a^2 + a^4}{4}}{\frac{3a + a^3}{3}}$$

$$\therefore \vec{r}_{CM} = \frac{3}{4}\left(\frac{2a + a^3}{3 + a^2}\right)\hat{i} + \frac{a}{2}\hat{j}$$

(6)



$$x_{CM} = \frac{MR}{m+M}$$

Net force on m,

$$F = m\omega^2 x_{CM}$$

$$= m\omega^2 \left(\frac{MR}{m+M}\right)$$

$$= \frac{mM}{m+M}\omega^2 R = \mu \omega^2 R$$

$$\Rightarrow \boxed{R = \frac{F}{\mu\omega^2}} \quad \text{where } \mu = \frac{mM}{m+M}$$

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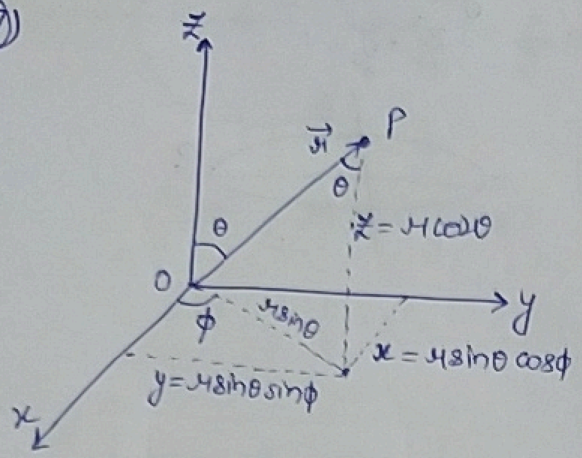


(a)

$$\left.\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}\right\} \quad \vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$\Rightarrow \hat{r} = \frac{\vec{r}}{r} = \frac{\hat{x}+\hat{y}+\hat{z}}{\sqrt{x^2+y^2+z^2}} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

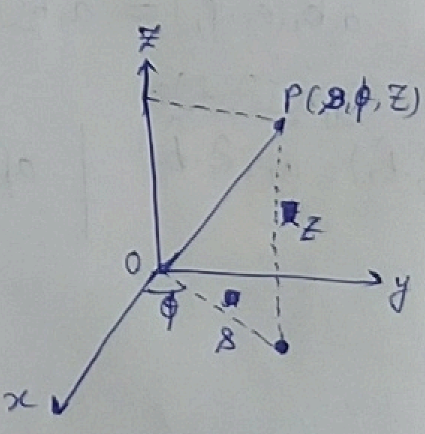
$$\hat{\theta} = \frac{\frac{\partial\vec{r}}{\partial\theta}}{|\partial\vec{r}/\partial\theta|} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$$

$$\hat{\phi} = \frac{\partial\vec{r}/\partial\phi}{|\partial\vec{r}/\partial\phi|} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

$$\Rightarrow \begin{aligned} \hat{x} &= \sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi} \\ \hat{y} &= \sin\theta\sin\phi\,\hat{r} + \cos\theta\sin\phi\,\hat{\theta} + \cos\phi\,\hat{\phi} \\ \hat{z} &= \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta} \end{aligned}$$

(b)

$$\left.\begin{aligned} x &= s\cos\phi \\ y &= s\sin\phi \\ z &= z \end{aligned}\right\}$$



$$\Rightarrow \begin{aligned} \hat{s} &= \cos\phi\,\hat{x} + \sin\phi\,\hat{y} \\ \hat{\phi} &= -\sin\phi\,\hat{x} + \hat{y}\cos\phi \\ \hat{z} &= \hat{z} \end{aligned}$$

$$\Rightarrow \begin{aligned} \hat{x} &= \cos\phi\,\hat{s} - \sin\phi\,\hat{\phi} \\ \hat{y} &= \sin\phi\,\hat{s} + \cos\phi\,\hat{\phi} \\ \hat{z} &= \hat{z} \end{aligned}$$

(8) $\hat{r} = \left[\sin\theta \cos\phi\, \hat{x} + \sin\theta \sin\phi\, \hat{y} + \cos\theta\, \hat{z}\right]$

$\vec{r} = r\hat{r}$

$\vec{v} = \dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\hat{r}}$

$= \dot{r}\hat{r} + (\dot{\theta}\hat{\theta} + \sin\theta.\dot{\phi}\hat{\phi})r$

$\Rightarrow \dot{\hat{r}} = [\cos\theta\cos\phi.\dot{\theta} - \sin\theta\sin\phi.\dot{\phi}]\hat{x}$

$+ [\cos\theta\sin\phi.\dot{\theta} + \sin\theta\cos\phi.\dot{\phi}]\hat{y}$

$+ (-\sin\theta.\dot{\theta})\hat{z}$

$= \dot{\theta}\hat{\theta} + \sin\theta.\dot{\phi}\hat{\phi}$

$KE = \frac{1}{2}m|\vec{v}|^2$

$= \frac{1}{2}m\sqrt{(\dot{r})^2 + (r\dot{\theta})^2 + (r\sin\theta\,\dot{\phi})^2}$

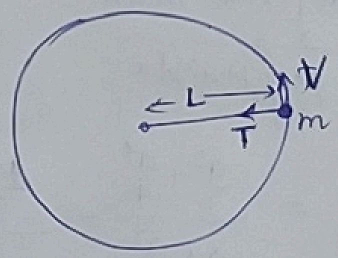
$= \frac{1}{2}m\sqrt{\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2}$

As $\vec{v} = \dot{r}\hat{r} + \dot{\theta}r\hat{\theta} + r\dot{\phi}\sin\theta\,\hat{\phi}$

$= v_r\hat{r} + v_\theta\hat{\theta} + v_\phi\hat{\phi}$

$\Rightarrow v_r = \dot{r},\ v_\theta = r\dot{\theta},\ v_\phi = r\sin\theta\,\dot{\phi}$

(9) (a) Net centripetal force,

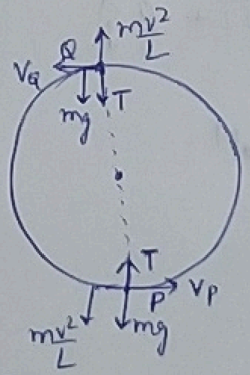


$\frac{mv^2}{L} = T$

$\Rightarrow \frac{mv_{max}^2}{L} = T_{max}$

$\Rightarrow v_{max} = \sqrt{\frac{L}{m}T_{max}}$

(b) For particle to perform the circular motion,



T_Q at highest point $= 0$

$m\frac{v_Q^2}{L} = mg \Rightarrow v_Q^2 = \sqrt{gL}$

Using energy conservation b/w P and Q,

$K_P + U_P = K_Q + U_Q$

$\Rightarrow \frac{1}{2}mv_P^2 = \frac{1}{2}mv_Q^2 + mg(2L)$

$\Rightarrow v_P^2 = 2\left[\frac{gL}{2} + 2gL\right] = gL + 4gL$

$\Rightarrow v_P = \sqrt{5gL}$

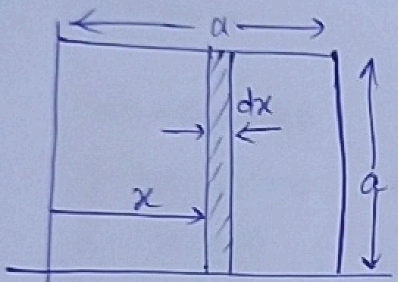
At the lowest point (P),

$$T = mg + m\frac{v_P^2}{L}$$

$$= mg + m\frac{(5gL)}{L} = 6mg \Rightarrow g = \frac{T}{6m}$$

$$\therefore V_{max} = \sqrt{5gL} = \sqrt{\frac{5Tl}{6m}}$$

(10)



$$\sigma = \alpha(1+x^2)$$

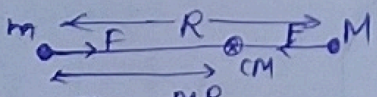
$$dm = \sigma\, a\, dx = a\alpha(1+x^2)x\, dx$$

$$\therefore x_{CM} = \frac{\int dm.x}{\int dm} = \frac{a\alpha\int_0^a (1+x^2)x\, dx}{a\alpha\int_0^a (1+x^2)dx}$$

$$= \frac{\int_0^a (x+x^3)dx}{\int_0^a (1+x^2)dx} = \frac{\frac{a^2}{2}+\frac{a^4}{4}}{a+\frac{a^3}{3}} = \frac{\frac{2a^2+a^4}{4}}{\frac{3a+a^3}{3}} = \frac{3}{4}\left(\frac{2a^2+a^4}{3a+a^3}\right) = \frac{3}{4}\frac{(2a+a^3)}{(3+a^2)}$$

$$\therefore \vec{r}_{CM} = \frac{3}{4}\left(\frac{2a+a^3}{3+a^2}\right)\hat{i} + \frac{a}{2}\hat{j}$$

(6)



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