INDIAN INSTITUTE OF SPACE SCIENCE AND TECHNOLOGY
THIRUVANANTHAPURAM 695 547

End Semester Examination - May 2016

B. Tech - II Semester

MA121 - Vector Calculus and Differential Equations

Date: 04/05/2016 **Time: 9.30 am - 12.30 pm** **Max. Marks: 50**

SECTION A (Attempt all 10 questions - 10x2.5= 25 marks.)

1. Check for uniform convergence of the sequence $\{f_n\}$ where

   (1) $f_n(x) = \tan^{-1} nx \ ; \ x \in [0,1]$

   (2) $f_n(x) = \dfrac{x}{e^{nx^2}} \ ; \ x \in [0,1]$

2. Show that the series $\dfrac{x}{x+1} + \dfrac{x}{(x+1)(2x+1)} + \dfrac{x}{(2x+1)(3x+1)} + \cdots \ ; \ x \geq 0$ is convergent in $[0,\infty]$ but the convergence is not uniform.

3. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a C^1-type function. Define $\phi(x,y) = \lim\limits_{h\to 0} \dfrac{f(hx,hy) - f(0,0)}{h}$ for all $(x,y) \in \mathbb{R}^2$ satisfying $x^2 + y^2 = 1$. Justify that the function ϕ exists, i.e., justify the limit exists. With explanation, find the condition for which $|\phi(x,y)|$ is maximum. [Hint: Directional Derivative]

4. Let $\gamma : [a,b] \longrightarrow \mathbb{R}^3$ be a C^1-type non-constant curve. Using properties of Arc-Length function of γ, show that $\|\gamma'(t)\| \neq 0$ for all $t \in [a,b]$.

5. Let D be an open set D in $\mathbb{R}^2$ and $\gamma : [a,b] \longrightarrow \mathbb{R}^2$ a C^1-type curve. If $f : D \longrightarrow \mathbb{R}$ be a C^1-type scalar field, show that $\displaystyle\int_\gamma \nabla f = f(\gamma(b)) - f(\gamma(a))$.

6. If $\vec{F}$ be a vector field such that $\displaystyle\int_C \vec{F} = 0$ for all loop C inside the domain of $\vec{F}$. Then show that $\displaystyle\int_{C_1} \vec{F} = \int_{C_2} \vec{F}$ for all paths C_1, C_2 from a point P_1 to a point P_2 inside the domain of $\vec{F}$.

7. Solve $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0$

8. Solve $\dfrac{d^2y}{dx^2} + 2y = x^3 + x^2 + e^{-2x} + \cos 3x$.

9. Suppose $\phi(x)$ is a solution of $(1-x^2)y'' - 2xy' + p(p+1)y = 0$. Let $\psi(t) = \phi(2t-1)$. Show that $\psi(t)$ satisfies the equation

$$(t - t^2)y'' + (1-2t)y' + p(p+1)y = 0$$

10. Prove that $P_n'(1) = \dfrac{1}{2}n(n+1)$, where $P_n(x)$ is the n^{th} degree Legendre polynomial.

SECTION B (Attempt any 5 questions - 5x5= 25 marks.)

11. (a) Let $f_n(x) = nx(1-x^2)^n \ ; \ 0 \leq x \leq 1$. Show that $\{f_n\}$ converges to a function f on $[0,1]$ and the convergent is not uniform by showing $\lim \displaystyle\int_0^1 f_n \neq \int_0^1 f$.

    (b) Let $f(x) = \displaystyle\sum_1^\infty \frac{\cos nx}{n^2}$. Prove that $\displaystyle\int_0^{\pi/2} f(x)dx = \sum_0^\infty \frac{(-1)^n}{(2n+1)^3}$

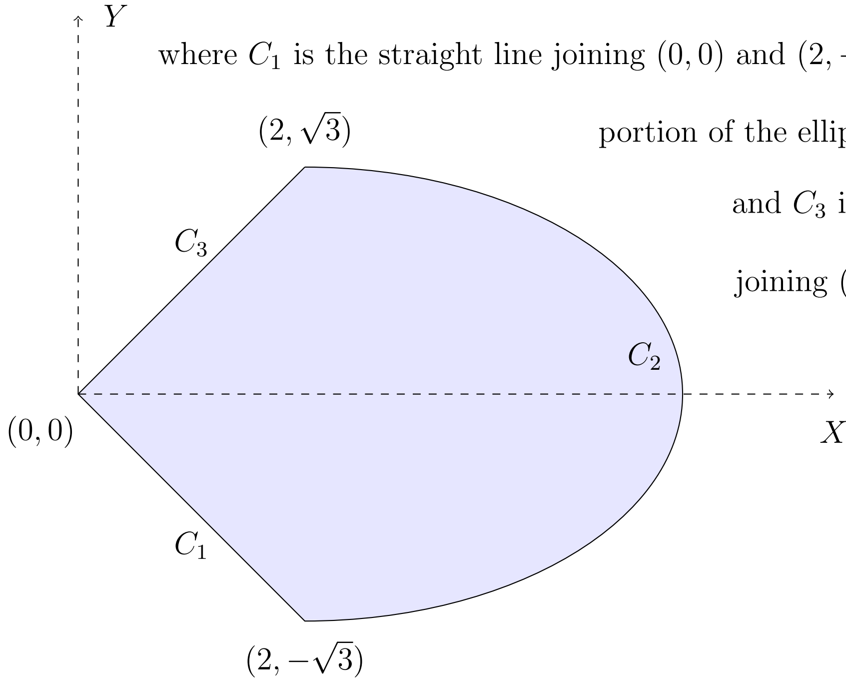
12. (a) Show that $f_n(x) = \dfrac{\ln(1+nx^2)}{2n}$; $x \in [1,2]$ converges uniformly by using the sequence $\{f_n'(x)\}$ of derivatives of $f_n(x)$.

(b) Is $\displaystyle\sum_{1}^{\infty} \frac{x}{n(1+nx^2)}$; $x \in (0,\infty)$ a continuous function.

13. (a) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be given by $f(x,y) = \sqrt{|xy|}$. Find all unit vectors $\vec{v}$ such that $D_{\vec{v}}(f)|_{(0,0)}$ exists. Is f differentiable at $(0,0)$? Justify your answer. Argue whether f is continuous at $(0,0)$. [1.5+0.5+0.5]

(b) Let $\gamma(t) = (t - \sin t,\ 1 - \cos t)$ for all $t \in [0, 2\pi]$ be a curve. Find the Arc-Length function of the curve γ and hence find the length of the curve. [2.5]

14. (a) State Green's Theorem. Find the area of the shaded region using Green's theorem.



where C_1 is the straight line joining $(0,0)$ and $(2,-\sqrt{3})$, C_2 is the portion of the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$, and C_3 is the straight line joining $(2,\sqrt{3})$ and $(0,0)$. [2.5]

(b) Let $\vec{F}(x,y) = (x \exp^{\frac{x^2+y^2}{2}} + \cos(x+y),\ y \exp^{\frac{x^2+y^2}{2}} + \cos(x+y))$ form all $(x,y) \in \mathbb{R}^2$. Suppose $C = C_1 * C_2$ be a curve in $\mathbb{R}^2$ given by $C_1 : \dfrac{x^2}{4} + \dfrac{y^2}{9} = 1, y \geq 0$ oriented positively and C_2 is the straight line segment from $(-2,0)$ to $(-4,2)$. Find $\displaystyle\int_C \vec{F}$. Is the vector field $\vec{F}$ conservative? Justify your answer. [1.5+1]

15. Find two linearly independent power series solution of $(1-x^2)y'' - 2xy' + 2y = 0$.

16. Consider the ODE

$$y'' + a_1(x)y' + a_2(x)y = 0$$

Show that the above ODE has a solution of the form

$$p(x) = \exp\left[\int_0^x p(t)\ dt\right],$$

if and only if the function p satisfies the nonlinear ODE

$$y' = -y^2 - a_1(x)y - a_2(x)$$

END