MA221 - PDE

Assignment - I

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① Let $f : [0,\ell] \to \mathbb{R}$ be a continuous function such that $f(0) = f(\ell) = 0$ and $f'(x)$ is a piecewise-continuous in $(0,\ell)$.
Show that for given $k > 0$ and any $t > 0$ and $0 \le x \le \ell$, the series below is convergent.

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-(n\pi/\ell)^2 kt} \sin\left(\frac{n\pi x}{\ell}\right)$$

where, $$a_n = \frac{2}{\ell} \int_0^\ell f(x) \sin\left(\frac{n\pi x}{\ell}\right) dx.$$

Hence, show that $u(x,t)$ as defined above satisfies the initial boundary-value problem-

$$u_t = k u_{xx}, \quad t > 0, \quad 0 < x < \ell$$
$$u(0,t) = 0, \quad t \ge 0$$
$$u(\ell,t) = 0, \quad t \ge 0$$
$$u(x,0) = f(x), \quad 0 \le x \le \ell.$$

Sol^n: As $\sin\left(\frac{n\pi x}{\ell}\right) \le 1$

$$\Rightarrow f(x) \sin\left(\frac{n\pi x}{\ell}\right) \le f(x)$$

$$\Rightarrow \int_0^\ell f(x) \sin\left(\frac{n\pi x}{\ell}\right) dx \le \int_0^\ell f(x)\, dx$$

$$\Rightarrow \frac{2}{\ell} \int_0^\ell f(x) \sin\left(\frac{n\pi x}{\ell}\right) dx \le \frac{2}{\ell} \int_0^\ell f(x)\, dx$$

$$\Rightarrow a_n \le \frac{2}{\ell} \int_0^\ell f(x)\, dx$$

Also,

$$u(x,t) = a_n e^{-\left(\frac{n\pi}{\ell}\right)^2 kt} \sin\left(\frac{n\pi x}{\ell}\right) \le a_n e^{-\left(\frac{n\pi}{\ell}\right)^2 kt}$$
$$\le \left[\frac{2}{\ell} \int_0^\ell f(x)\, dx\right] \cdot e^{-\left(\frac{n\pi}{\ell}\right)^2 kt}$$

where $\frac{2}{\ell} \int_0^\ell f(x)\, dx = c \to$ constant.

Let $b_n = c \cdot e^{-\left(\frac{n\pi}{\ell}\right)^2 kt}$

Also,

$$\lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = \left| e^{-(2n+1)\frac{\pi^2}{\ell^2} kt} \right| < 1$$

$\Rightarrow \sum b_n$ is converging

As $u(x,t) = \sum_{n=1}^{\infty} a_n e^{-(\frac{n\pi}{l})^2 kt} \sin(\frac{n\pi x}{l}) \leq b_n$,

hence $u(x,t)$ is ~~con~~ also convergent.

Now,

$u_t = k u_{xx} = X(x)T(t)$

$\Rightarrow \frac{X''}{X} = \frac{1}{k}\frac{T'}{T} = c$

$\Rightarrow X'' - cX = 0,$

$T' - kcT = 0$

and, $u(0,t) = 0 \Rightarrow X(0) = 0$

$u(l,t) = 0 \Rightarrow X(l) = 0$

$\therefore X'' - cX = 0$ with $X(0)=0$, $X(l)=0$.

For $c \geq 0 \Rightarrow X = 0$

$c < 0 \Rightarrow$ let $c = -p^2$

$X'' - cX = 0 \Rightarrow X = A\cos px + B\sin px$

$X(l) = 0 \Rightarrow B\sin pl = 0$

$\Rightarrow p = \frac{n\pi}{l}$

$\therefore k_n = -(\frac{n\pi}{l})^2$

$X_n = B_n \sin \frac{n\pi}{l} x$

Also, $T' - kcT = 0$

$\Rightarrow T_n = C_n e^{-(\frac{n\pi}{l})^2 ct}$

Now,

$u_n = X_n T_n = a_n e^{-(\frac{n\pi}{l})^2 kt} \cdot \sin(\frac{n\pi}{l})x$

and, $u(x,t) = \sum u_n(x,t)$

$u(x,t) = \sum a_n e^{-(\frac{n\pi}{l})^2 kt} \cdot \sin(\frac{n\pi}{l}x)$

$u(x,0) = f(x)$

$\Rightarrow f(x) = \sum a_n \sin(\frac{n\pi}{l})x$

$\Rightarrow a_n = \frac{2}{l}\int_0^l f(x) \sin(\frac{n\pi}{l})x \, dx$ [given that $f(x)$ is continuous]

Hence, $u(x,t)$ satisfies the given initial value problem.

② Let $c:(x_0(t), y_0(t))$ be a curve in (xy)-plane with $(x_0')^2 + (y_0')^2 \neq 0$.
Consider the following problem: find u such that

$$\left.\begin{aligned} a(x,y,u)\, u_x + b(x,y,u)\, u_y &= c(x,y,u), \\ u(x_0(t), y_0(t)) &= u_0(t). \end{aligned}\right\} \quad ①$$

Now suppose $x_0(t)$, $y_0(t)$ and $u_0(t)$ are continuously differentiable function of t in a closed interval $0 \le t \le 1$ and a, b and c are functions of x, y and u with continuous first-order partial derivatives in some domain D in (x,y,u)-space containing the initial curve $\Gamma: (x_0(t), y_0(t), u_0(t))$, $0 \le t \le 1$, and also satisfy

$$y_0'(t)\, a(x_0(t), y_0(t), u_0(t)) - x_0'(t)\, b(x_0(t), y_0(t), u_0(t)) \neq 0.$$

Then show that there exists a unique solution of ① in the neighbourhood of c.

Soln: Let $a, b, c \in C^1$, then the general soln of PDE:

$f(\phi, \psi) = 0$, where f is an arbitrary function (smooth) of $\phi(x,y,z)$ and $\psi(x,y,z)$,

and, $\phi(x,y,z) = c_1$,
$\psi(x,y,z) = c_2$ are the two solutions of

$$\frac{dx}{a} = \frac{dy}{b} = \frac{dz}{c}.$$

Now, $\phi = c_1 \Rightarrow d\phi = \phi_x dx + \phi_y dy + \phi_z dz = 0$

$\psi = c_2 \Rightarrow d\psi = \psi_x dx + \psi_y dy + \psi_z dz = 0$

$\frac{dx}{a} = \frac{dy}{b} = \frac{dz}{c} = k \Rightarrow dx = ak,$
$dy = bk$
$dz = ck.$

Now,

$a\phi_x + b\phi_y + c\phi_z = 0$

$a\psi_x + b\psi_y + c\psi_z = 0.$

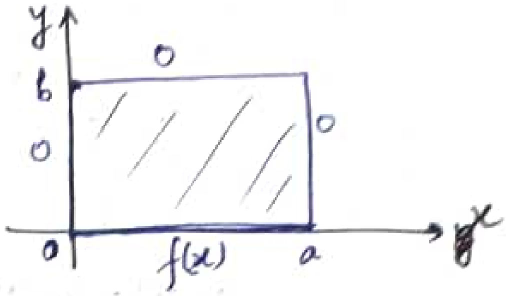
Given that $\frac{y_0'(t)}{x_0'(t)} \neq \frac{b(x_0(t), y_0(t), u_0(t))}{a(x_0(t), y_0(t), u_0(t))}$,

in $D \subseteq \mathbb{R}^3 \to$ open + connected curve in x, y, z plane,

then by Cauchy-theorem,

① has a unique solution in some neighbourhood of c.

3 An thin rectangular homogeneous thermally conducting plate lies in the xy-plane defined by $0 \le x \le a, 0 \le y \le b$. The edge $y=0$ is held at the temperature $Tx(x-a)$, where T is a constant, while the remaining edges are held at 0°. The other faces are insulated and no internal sources and sinks are present. Find the steady-state temperature inside the plate.



Soln: Let $f(x) = Tx(x-a)$

2-D heat equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$$

At steady state, $\frac{\partial^2 y}{\partial t^2} = 0$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\Rightarrow \nabla^2 u = 0$$

$$\Rightarrow u_{xx} + u_{yy} = 0$$

Boundary conditions:

(1A) $u_{xx} + u_{yy} = 0$

(1B) $u(0,y) = u(a,y) = u(x,b) = 0$

(1C) $u(x,0) = Tx(x-a)$.

Let $u(x,y) = X(x) \cdot Y(y)$.

(1A) $\Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = K \Rightarrow X'' - KX = 0,$
$Y'' + KY = 0$

(1B) $\Rightarrow X(0) = 0,$
$X(a) = 0$

$\therefore X'' - KX = 0$ with $X(0) = 0, X(a) = 0$.

For $K=0 \Rightarrow X'' = 0 \Rightarrow X(x) = ax + b$

$$\begin{cases} X(0) = 0 \Rightarrow b = 0 \\ X(a) = 0 \Rightarrow a = 0 \end{cases}$$

$\Rightarrow X(x) = 0 \longrightarrow$ Not possible

For $K > 0 \Rightarrow$ Let $K = p^2,\ p > 0$

Now, $\lambda^2 - K = 0 \Rightarrow \lambda^2 - p^2 = 0$ [Auxiliary eqn]

$\Rightarrow \lambda = \pm p$

$\therefore X(x) = Ae^{px} + Be^{-px}$

$X(0)=0 \Rightarrow \quad A+B=0 \Rightarrow B=-A$

$X(a)=0 \Rightarrow \quad Ae^{pa} - Ae^{-pa}=0$

$\Rightarrow A(e^{pa}-e^{-pa})=0$

$\Rightarrow A=0, B=0 \rightarrow$ Not possible

For $K<0 \Rightarrow$ Let $k=-p^2$, $p>0$

$X''+p^2X=0$

$\Rightarrow \lambda^2+p^2=0 \Rightarrow \lambda=\pm p$

$\therefore X(x) = A\cos px + B\sin px$

$X(0)=0 \Rightarrow A+0=0$

$X(a)=0 \Rightarrow B\sin pa=0$

$\Rightarrow pa=n\pi$

$\Rightarrow p_n=\dfrac{n\pi}{a}$

$\therefore X_n = B_n \sin\left(\dfrac{n\pi}{a}x\right).$

For y: $\quad y'' - \left(\dfrac{n\pi}{a}\right)^2 y=0$

$\Rightarrow Y_n = C_n e^{\frac{n\pi}{a}y} \pm D_n e^{-\frac{n\pi}{a}y}$

Hence, $u_n(x,y) = \left[A_n e^{\frac{n\pi}{a}y} + B_n e^{-\frac{n\pi}{a}y}\right]\sin\dfrac{n\pi}{a}x.$

Now, $u(x,y) = \sum_{n=1}^{\infty} u_n(x,y)$

$u(x,b)=0 \Rightarrow \sum_{n=1}^{\infty} u_n(x,b)=0$

$\Rightarrow \sum_{n=1}^{\infty}\left[A_n e^{\frac{n\pi}{a}b} + B_n e^{-\frac{n\pi}{a}b}\right]\underbrace{\sin\left(\frac{n\pi}{a}x\right)}=0$

$\hookrightarrow$ cannot be zero as it will make $u_n(x,y)=0$.

$\Rightarrow A_n e^{\frac{n\pi}{a}b} + B_n e^{-\frac{n\pi}{a}b}=0$

$\Rightarrow B_n = -A_n \dfrac{e^{\frac{n\pi}{a}b}}{e^{-\frac{n\pi}{a}b}}$

Also, $u(x,0)=f(x)$.

$\Rightarrow \sum_{n=1}^{\infty}\left[A_n e^{\frac{n\pi}{a}x} + B_n e^{-\frac{n\pi}{a}y}\right]\sin\left(\frac{n\pi}{a}\right)x = f(x).$

$\Rightarrow \sum_{n=1}^{\infty}\left[A_n+B_n\right]\sin\left(\frac{n\pi}{a}x\right) = T.x(x-a)$

$$\Rightarrow \sum_{n=1}^{\infty}\left[A_n - \frac{A_n e^{\frac{n\pi}{a}b}}{e^{-\frac{n\pi}{a}b}}\right]\sin\frac{n\pi x}{a} = Tx(x-a)$$

$$\Rightarrow Tx(x-a) = \sum_{n=1}^{\infty} 2e^{\frac{n\pi}{a}b}A_n\left[\frac{e^{-\frac{n\pi b}{a}} - e^{\frac{n\pi b}{a}}}{2}\right]\sin\left(\frac{n\pi}{a}x\right)$$

$$\Rightarrow \sum_{n=1}^{\infty}\Big[\underbrace{-\sinh\left(\frac{n\pi}{a}b\right)\cdot 2e^{\frac{n\pi}{a}b}A_n}_{\text{Fourier sine coefficient}}\Big]\sin\left(\frac{n\pi}{a}x\right) = Tx(x-a)$$

$$\Rightarrow -2\sinh\left(\frac{n\pi}{a}b\right)e^{\frac{n\pi b}{a}}A_n = \frac{2}{a}\int_0^a Tx(x-a)\sin\left(\frac{n\pi}{a}x\right)dx$$

$$\Rightarrow -\sinh\left(\frac{n\pi}{a}b\right)e^{\frac{n\pi b}{a}}A_n = \frac{T}{a}\int_0^a x(x-a)\sin\left[\frac{n\pi}{a}x\right]dx$$

$$= \frac{T}{a}\int_0^a x^2\sin\left(\frac{n\pi}{a}x\right) - ax\sin\left[\frac{n\pi}{a}x\right]dx$$

Let $I = \int_0^a x^2\sin\left(\frac{n\pi}{a}x\right)dx = -\frac{x^2\cos(n\pi/a\, x)}{n\pi/a} + \int\frac{2x\cos n\pi/a\, x}{n\pi/a}dx$

$$= \frac{-x^2\cos n\pi/a\, x}{n\pi/a} + \frac{2}{\frac{n\pi}{a}}\left[\frac{x\sin\frac{n\pi}{a}x}{\frac{n\pi}{a}} - \int\frac{\sin\frac{n\pi}{a}x}{\frac{n\pi}{a}}dx\right]$$

$$= \frac{-x^2\cos\frac{n\pi}{a}x}{\frac{n\pi}{a}} + \frac{2}{\left(\frac{n\pi}{a}\right)^2}\left[x\sin\left(\frac{n\pi}{a}\right)x - \frac{1}{\left(\frac{n\pi}{a}\right)^2}\cos\left(\frac{n\pi}{a}x\right)\right]$$

$$= -\frac{a^2\cos n\pi}{n\pi/a} + \frac{2}{(n\pi/a)^2} + \frac{1}{(n\pi/a)^2}\cos n\pi - \frac{2}{(n\pi/a)^4}$$

$$= -\frac{2}{(n\pi/a)^4}\left[1 + \frac{a^2\cos n\pi}{n\pi/a}\right]$$

$\therefore$ Integral: $\int_0^a \left[x^2\sin\left(\frac{n\pi}{a}x\right) - ax\sin\left(\frac{n\pi}{a}\right)x\right]dx = \frac{-a^4}{12}\sin\left(\frac{n\pi}{a}\right)$

$$\therefore -\sinh\left(\frac{n\pi b}{a}\right)e^{\frac{n\pi b}{a}}A_n = \frac{T}{a}\left[\frac{-a^4}{12}\sin\frac{n\pi}{a}\right]$$

$$\Rightarrow A_n = \frac{\frac{Ta^3}{12}\sin n\pi/a}{\sinh\left(\frac{n\pi b}{a}\right)e^{n\pi b/a}}$$

Hence, $u(x,y) = \sum_{n=1}^{\infty} 2A_n e^{-\frac{n\pi b}{a}}\sinh\left[\frac{n\pi}{a}(y-b)\right]\sin\frac{n\pi}{a}x.$