Indian Institute of Space Science and Technology

Complex Analysis

TUTORIAL - III

1. Use the Cauchy-Riemann equations show that the function $f(z) = \exp \bar{z}$ is not analytic anywhere.

2. Show in two ways that the function $\exp(z^2)$ is entire. What is its derivative?

3. Let the function $f(z) = u(x, y) + i\, v(x, y)$ be analytic in some domain D. State why the functions

$$U(x, y) = e^{u(x,y)} \cos v(x, y), \quad V(x, y) = e^{u(x,y)} \sin v(x, y)$$

   are harmonic in D and why $V(x, y)$ is, in fact, a harmonic conjugate of $U(x, y)$.

4. Verify that when $n = 0, \pm 1, \pm 2, \cdots,$

   (a) $\log e = 1 + 2n\, \pi\, i;$

   (b) $\log i = \left(2n + \frac{1}{2}\right) \pi\, i;$

   (c) $\log(-1 + \sqrt{3}i) = \ln 2 + 2\left(n + \frac{1}{3}\right) \pi\, i.$

5. Show that

   (a) $\mathrm{Log}(1 + i)^2 = 2\, \mathrm{Log}(1 + i)$;

   (b) $\mathrm{Log}(-1 + i)^2 \neq 2\, \mathrm{Log}(-1 + i).$

6. Given that the branch $\log z = \ln r + i\theta (r > 0,\ \alpha < \theta < \alpha + 2\pi)$ of the logarithmic function is analytic at each point z in the stated domain, obtain its derivative by differentiating each side of the identity $\exp(\log z) = z$ and using the chain rule.

7. Suppose that the point $z = x + i\, y$ lies in the horizontal strip $\alpha < y < \alpha + 2\pi$. Show that when the branch $\log z = \ln r + i\theta (r > 0, \alpha < \theta < \alpha + 2\,\pi)$ of the logarithmic function is used, $\log(e^z) = z$.

8. Use the Cauchy-Riemann equations show that neither $\sin \bar{z}$ nor $\cos \bar{z}$ is an analytic function of z anywhere.

9. Show that

   (a) $\overline{\cos(iz)} = \cos(i\bar{z})$ for all z;

   (b) $\overline{\sin(iz)} = \sin(i\bar{z})$ if and only if $z = n\,\pi\, i\, (n = 0, \pm 1, \pm 2, \cdots).$

10. Show that

    (a) $\sinh(z + \pi i) = -\sinh z;$

    (b) $\cosh(z + \pi i) = -\cosh z;$

    (c) $\tanh(z + \pi i) = \tanh z.$

11. Why is the function $\sinh(e^z)$ entire? Write its real part as a function of x and y, and state why that function must be harmonic everywhere.

12. $f(z) = z - 1$ and C is the arc from $z = 0$ to $z = 2$ consisting of

    (a) the semicircle $z = 1 + e^{i\theta}\, (\pi \leq \theta \leq 2\pi);$

(b) the segment $0 \leq x \leq 2$ of the real axis.

13. $f(z)$ is the branch

$$z^{-1+i} = \exp[(-1+i)\log z] \quad (|z| > 0,\ 0 < \arg z < 2\pi)$$

of the indicated power function, and C is the positively oriented unit circle $|z| = 1$.

END