AV 223 - Signals and Systems
Assignment on Convolution

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SC22B146

① Find the output $y[n]$ of the system $h[n]$ to the input $x[n]$. Use discrete-time convolution.

ⓐ $x[n] = \{1, 2, 3\}$ where $n = \{0, 1, 2\}$ and
$h[n] = \{1, 2, 3\}$ where $n = \{0, 1, 2\}$.

Soln: Start time $= 0+0 = 0$
Stop time $= 2+2 = 4$
$\therefore K = 0, 1, 2, 3, 4$

$$y[K] = \sum_{m=-\infty}^{\infty} x[m] \cdot h[K-m]$$
$$= \sum_{m=0}^{2} x[m] \cdot h[K-m]$$

$$y[0] = \sum_{m=0}^{2} x[m] \cdot h[-m]$$
$$= x(0) \cdot h(0) + x(1) \cdot h(-1) + x(2) \cdot h(-2)$$
$$= (1)(1) + (2)(0) + (3)(0)$$
$$= 1$$

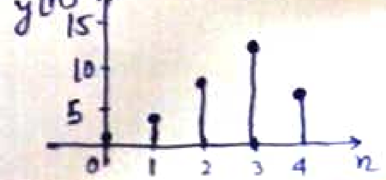
$$y[1] = x(0)\, h(1) + x(1)\, h(0) + x(2)\, h(-1)$$
$$= (1)(2) + (2)(1) + (3)(0)$$
$$= 2 + 2 = 4$$

$$y[2] = x(0) \cdot h(2) + x(1) \cdot h(1) + x(2) \cdot h(0)$$
$$= (1)(3) + (2)(2) + (3)(1)$$
$$= 3 + 4 + 3 = 10$$

$$y[3] = x(0) \cdot h(3) + x(2) \cdot h(2) + x(2) \cdot h(1)$$
$$= (1)(0) + (3)(3) + (3)(2)$$
$$= 9 + 6 = 15$$

$$y[4] = x(0) h(4) + x(2) h(3) + x(2) h(2)$$
$$= 0 + 0 + 3 \times 3$$
$$= 9$$

$\therefore y[n] = \{1, 4, 10, 15, 9\}$ for $n = \{0, 1, 2, 3, 4\}$.



(b) $x[n] = 10\delta[n+1] + 5\delta[n] - 5\delta[n-2] - 10\delta[n-3]$

$h[n] = -\delta[n-5] + \delta[n-7]$.

Soln: $y[n] = x[n] * h[n]$

$$= -10(\delta[n+1]*\delta[n-5]) - 5(\delta[n]*\delta[n-5]) + 5(\delta[n-2]*\delta[n-5])$$
$$+10(\delta[n-3]*\delta[n-5]) + 10(\delta[n+1]*\delta[n-7]) + 5(\delta[n]*\delta[n-7])$$
$$-5(\delta[n-2]*\delta[n-7]) - 10(\delta[n-3]*\delta[n-7]).$$

$$= -10\delta[n-4] - 5\delta[n-5] + 5\delta[n-7] + 10\delta[n-8] + 10\delta[n-6]$$
$$+5\delta[n-7] - 5\delta[n-9] - 10\delta[n-10]$$

$$= 5\left[-2\delta[n-4] - \delta[n-5] + 2\delta[n-6] + 2\delta[n-7] + 2\delta[n-8] - \delta[n-9] - 2\delta[n-10]\right]$$

(2) Perform discrete-time convolution on the following signals and systems. Also perform these convolutions numerically using MATLAB.

(a) $x[n] = 5\delta[n] + 10\delta[n-1] + 15\delta[n-2] + 20\delta[n-3]$

$h[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 4\delta[n-3]$

Soln: $y[n] = x[n] * h[n]$

$$= 5(\delta[n]*\delta[n]) + 10(\delta[n-1]*\delta[n]) + 15(\delta[n-2]*\delta[n]) + 20(\delta[n-3]*\delta[n])$$
$$+ 10(\delta[n]*\delta[n-1]) + 15(\delta[n]*\delta[n-2]) + 20(\delta[n]*\delta[n-3])$$
$$+ 20(\delta[n-1]*\delta[n-1]) + 30(\delta[n-1]*\delta[n-2]) + 40(\delta[n-1]*\delta[n-3])$$
$$+ 30(\delta[n-2]*\delta[n-1]) + 45(\delta[n-2]*\delta[n-2]) + 60(\delta[n-2]*\delta[n-3])$$
$$+ 40(\delta[n-3]*\delta[n-1] + 60(\delta[n-3]*\delta[n-2]) + 80(\delta[n-3]*\delta[n-3])$$

$$= 5\delta[n] + 10\delta[n-1] + 15\delta[n-2] + 20\delta[n-3] + 10\delta[n-1] + 15\delta[n-2]$$
$$+ 20\delta[n-3] + 20\delta[n-2] + 30\delta[n-3] + 40\delta[n-4] + 30\delta[n-3]$$
$$+ 45\delta[n-4] + 60\delta[n-5] + 40\delta[n-4] + 60\delta[n-5] + 80\delta[n-6]$$

$$= 5\delta[n] + 20\delta[n-1] + 50\delta[n-2] + 70\delta[n-3] + 125\delta[n-4]$$
$$+ 120\delta[n-5] + 80\delta[n-6].$$

③

ⓑ $x[n] = -\delta[n+5] - 3\delta[n+2] - 4\delta[n-1]$

$h[n] = 2\delta[n-100] + 4\delta[n-102]$.

Soln: $y[n] = x[n] * h[n]$

$$= -2(\delta[n+5] * \delta[n-100]) - 4(\delta[n+5] * [n-102]) - 6(\delta[n+2] * \delta[n-100])$$
$$-12(\delta[n+2] * \delta[n-102]) - 8(\delta[n-1] * \delta[n-100]) - 16(\delta[n-1] * \delta[n-102])$$
$$= -2\delta[n-95] - 4\delta[n-97] - 6\delta[n-98] - 12\delta[n-100] - 8\delta[n-101]$$
$$-16\delta[n-103]$$

③ Compute and plot the convolution for each of the pairs of signals.

ⓐ $v(t) = 10e^{-10t}u(t)$, $x(t) = u(t)$.

Soln:
$$y_1(t) = 10e^{-10t}u(t) * u(t)$$
$$= 10\int_{-\infty}^{\infty} e^{-10\tau}u(\tau)\cdot u(t-\tau)\,d\tau = 10\int_{-\infty}^{\infty} u(\tau)\, e^{-10(t-\tau)} u(t-\tau)\,d\tau$$
$$= 10\int_0^{t} e^{-10(t-\tau)}\,d\tau = 10e^{-10t}\int_0^t e^{10\tau}\,d\tau$$
$$= 10e^{-10t}\left[\frac{e^{10\tau}}{10}\right]_0^t$$
$$= \frac{10}{11}e^{-10t}(e^{11t} - 1)$$
$$= \frac{10}{11}(e^{t} - e^{-10t})$$

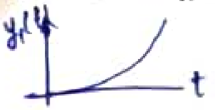
ⓑ $v(t) = 2e^{-2t}u(t)$, $x(t) = u(t)$

Soln:
$$y_2(t) = 2e^{-2t}u(t) * u(t)$$
$$= 2\int_0^t e^{-2(t-\tau)}\,d\tau = 2e^{-2t}\int_0^t e^{2\tau}\,d\tau$$
$$= 2e^{-2t}\left[\frac{e^{3\tau}}{3}\right]_0^t$$
$$= \frac{2}{3}e^{-2t}(e^{3t} - 1)$$
$$= \frac{2}{3}(e^{t} - e^{-2t})$$

ⓒ Compare parts ⓐ and ⓑ. Which is the ~~faster~~ response?

Soln: $y_1(t) = \frac{10}{11}(e^{t} - 10^{-10t})$

$y_2(t) = \frac{2}{3}(e^{t} - e^{-2t})$



$\therefore$ $y_1(t)$, i.e., ⓐ has the faster response.

ⓓ $v(t) = 2e^{-2t}u(t)$, $x(t) = u(t+1)$.

Do part ⓓ in two ways — use convolution directly; also, use the time-delay property and the solution from ⓑ.

Soln: $y(t) = v(t) * x(t)$

$$= 2e^{-2t}u(t) * u(t+1)$$

$$= 2\int_{-\infty}^{\infty} u(\tau+1)\, e^{-2(t-\tau)}\, u(t-\tau)\, d\tau \qquad \Big|\ \begin{matrix}\tau > -1 \\ t-\tau > 0 \Rightarrow \tau < t\end{matrix}$$

$$= 2\int_{-1}^{t} e^{-2(t-\tau)} d\tau = 2e^{-2t}\int_{-1}^{t} e^{2\tau} d\tau$$

$$= 2e^{-2t}\left[\frac{e^{+3\tau}}{+3}\right]_{-1}^{t}$$

$$= 2e^{-2t}\left(\frac{e^{3t} - e^{-3}}{+3}\right) = \frac{2}{3}\left(e^{t} - e^{-2t-3}\right)$$

$$= \cancel{e^{-2(t+1)}u(t+1)}.$$

OR, by using shifting property,

as $2e^{-2t}u(t) * u(t) = \cancel{\frac{2}{3}e^{-2t}u(t)}\ \frac{2}{3}(e^{t} - e^{-2t})$

$\Rightarrow 2e^{-2t}u(t) * u(t+1) = \cancel{e^{-2(t+1)}u(t+1)}\ \frac{2}{3}(e^{t} - e^{-2(t-1)-1})$

④ Find the output signal, $y(t)$, given

$h(t) = 10e^{-10t}u(t)$, $x(t) = e^{-t}u(t)$.

Soln: $y(t) = h(t) * x(t)$

$$= 10\int_{-\infty}^{\infty} e^{-\tau}u(\tau)\cdot e^{-10(t-\tau)}u(t-\tau)\, d\tau$$

$$= 10\int_{0}^{t} e^{-10t}\cdot e^{+9\tau}\, d\tau \qquad \Big|\ \left.\begin{matrix}\tau > 0 \\ t-\tau > 0 \Rightarrow \tau < t\end{matrix}\right\} \Rightarrow t > 0$$

$$= 10\, e^{-10t}\left[\frac{e^{10\tau}}{10}\right]_{0}^{t}$$

$$= 10e^{-10t}\left(\frac{10t - 1}{10}\right)$$

$$= e^{-9t} - e^{-10t}$$

⑤ Find the output signal, $y(t)$, given
$h(t) = 10e^{-10t}u(t)$, $x(t) = u(t) - u(t-1)$.
Also perform this convolution numerically using MATLAB and the "conv" function.

Soln: $y(t) = x(t) * h(t)$

$$= 10\int_{-\infty}^{\infty} u(\tau)\, e^{-10(t-\tau)}\, u(t-\tau)\, d\tau - 10\int_{-\infty}^{\infty} u(\tau-1)\, e^{-10(t-\tau)}\, u(t-\tau)\, d\tau$$

$$= 10\int_{0}^{t} e^{-10t} \cdot e^{10\tau}\, d\tau - 10\, e^{-10t}\int_{1}^{t} e^{10\tau}\, d\tau$$

$\tau > 1$
$\tau < t$

$$= 10e^{-10t}\left[\frac{e^{11\tau}}{11}\right]_0^t - 10e^{-10t}\left[\frac{e^{11\tau}}{11}\right]_1^t$$

$$= 10e^{-10t}\left(\frac{e^{11t}-1}{11}\right) - 10e^{-10t}\left(\frac{e^{11t}-e^{11}}{11}\right)$$

$$= \frac{10}{11}\left(e^{t} - e^{-10t}\right) - \frac{10}{11}\left(e^{t} - e^{-10t+11}\right)$$

$$= \frac{10}{11}\left(e^{11-10t} - e^{-10t}\right)$$

$$= \frac{10}{11} e^{-10t}\left(e^{11} - 1\right)$$

⑥ Find the output signal, $y(t)$, given
$h(t) = e^{-t}u(t)$, $x(t) = \sin(t).u(t)$.

Soln: $y(t) = x(t) * h(t)$

$$= \int_{-\infty}^{\infty} \sin(\tau)\, u(\tau)\, e^{-(t-\tau)}\, u(t-\tau)\, d\tau$$

$$= e^{-t}\int_{0}^{t} \sin(\tau)\, e^{\tau}\, d\tau$$

$$I = \int \sin\tau\, e^{\tau}\, d\tau$$

$$= \sin\tau\, e^{\tau} - \int \cos\tau\, e^{\tau}\, d\tau$$

$$= \sin\tau\, e^{\tau} - \Big[\cos\tau\, e^{\tau} + \underbrace{\int \sin\tau\, e^{\tau}\, d\tau}_{I}\Big]$$

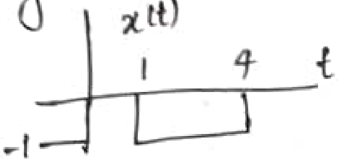
$$\Rightarrow I = \frac{\sin\tau\, e^{\tau} - \cos\tau\, e^{\tau}}{2}$$

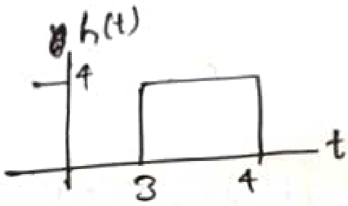
$$\therefore\ y(t) = e^{-t}\left[\frac{\sin\tau\, e^{\tau} - \cos\tau\, e^{\tau}}{2}\right]_0^t$$

$$= \frac{e^{-t}}{2}\left[(\sin t\, e^{t} - \cos t\, e^{t}) - (0-1)\right]$$

$$= \frac{\sin t - \cos t + e^{-t}}{2}$$

⑦ Find the output, $y(t)$, given the input, $x(t)$, and the impulse response, $h(t)$, using convolution.





Soln:

$$h(t) = \begin{cases} 4, & t<0 \\ 4, & 3<t<4 \\ 0, & \text{elsewhere} \end{cases} \qquad x(t) = \begin{cases} -1, & t<0 \\ -1, & 1<t<4 \\ 0, & \text{elsewhere} \end{cases}$$

$$h(t) = 4u(-t) + 4u(t-3) - 4u(t-4)$$

$$x(t) = -u(-t) - u(t-1) + u(t-4)$$

$$y(t) = x(t) * h(t)$$

$$= -4\int_{-\infty}^{\infty} u(-\tau)\,u(-t+\tau)\,d\tau - 4\int_{-\infty}^{\infty} u(\tau-1)\,u(-t+\tau)\,d\tau + 4\int_{-\infty}^{\infty} u(\tau-4)\,u(-t+\tau)\,d\tau$$

$$-4\int_{-\infty}^{\infty} u(\tau-3)\,u(-t+\tau)\,d\tau - 4\int_{-\infty}^{\infty} u(\tau-3)\,u(t-\tau-1)\,d\tau + 4\int_{-\infty}^{\infty} u(\tau-3)\,u(t-\tau+4)\,d\tau$$

$$+4\int_{-\infty}^{\infty} u(\tau-4)\,u(-t+\tau)\,d\tau + 4\int_{-\infty}^{\infty} u(\tau-4)\,u(t-\tau-1)\,d\tau - 4\int_{-\infty}^{\infty} u(\tau-4)\,u(t-\tau-4)\,d\tau$$

$$= -4\int_0^0 d\tau - 4\int_1^t d\tau + 4\int_4^t d\tau - 4\int_3^t d\tau - 4\int_3^{t+1} d\tau + 4\int_3^{t+4} d\tau$$

$$+4\int_4^t d\tau + 4\int_4^{t+1} d\tau - 4\int_4^{t+4} d\tau$$

$$= -4(t-1) + 4(t-4) - 4(t-3) - 4(t+1-3) + 4(t+4-3)$$

$$+4(t-4) + 4(t+1-4) - 4(t+4-4)$$

$$= +4 - 16 + 12 + 8 + 4 - 16 - 12$$

$$= -16$$