Indian Institute of Space Science and Technology

Thiruvananthapuram

MA 211 - Integral Transforms

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1. Using the definition of the Fourier transform verify whether the Heaviside unit step function H, defined by
$$H(x) = \begin{cases} 1, & \text{for} \quad x \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$
possess the Fourier transform.

2. Let $a > 0$. Do the following functions $f : \mathbb{R} \to \mathbb{R}$ possess the Fourier transform? Explain. If it is, then find it. Also find $\widehat{f}(0)$.
   (a) (One-sided exponential functions) $f(x) = e^{-ax}H(x)$ and $f(x) = e^{ax}H(-x)$.
   (b) (Rectangular function of height 1, width 2a and centre at 0) $\mathbf{rect}_a(x) = \begin{cases} 1, & \text{for} \quad |x| \leq a, \\ 0, & \text{otherwise.} \end{cases}$
   (c) (Hat function) $f(x) = \begin{cases} 1 - \dfrac{|x|}{a}, & \text{for} \quad |x| \leq a, \\ 0, & \text{otherwise.} \end{cases}$
   (d) (Gaussian function) $f(x) = e^{-ax^2}$.

3. Consider a rectangular function of height H, width W and centre at C. Using the Fourier transform of the rectangular function $\mathbf{rect}_a$, determine its Fourier transform.

4. Consider the "off-on-off" pulse $f(x)$, defined by
$$f(x) = \begin{cases} 0, & \text{for} \quad x < -2, \\ -1, & \text{for} \quad -2 \leq x < -1, \\ 1, & \text{for} \quad -1 \leq x \leq -1, \\ -1, & \text{for} \quad 1 < x \leq 2, \\ 0, & \text{for} \quad x > 2. \end{cases}$$
   (a) Use suitable "rectangular functions" g and h, such that $f(x) = a\,g(x) + b\,h(x)$, for some $a, b \in \mathbb{R}$.
   (b) Determine the Fourier transform of f using the Fourier transform of the rectangular functions g and h.

5. Let $a > 0$.
   (a) Find the Fourier transform of $\dfrac{1}{a^2 + x^2}$.
   (b) Find the Fourier transforms of $\dfrac{x}{a^2 + x^2}$.
   (c) Let $b \in \mathbb{R}$. Find the Fourier transforms of $\dfrac{\cos bx}{a^2 + x^2}$ and $\dfrac{\sin bx}{a^2 + x^2}$.
   (Hint: Use the frequency shift property)

6. Find $F^{-1}\left(\dfrac{e^{4iw}}{3+iw}\right)$ and $F^{-1}\left(we^{-w^2/8}\right)$.

7. Compute the convolution $(f*g)$ for the following pair of functions and also sketch the graph of it in each case.

   (a) $f(x) = \begin{cases} \frac{1}{2}, & \text{for} \quad 0 \le x \le \frac{1}{2}, \\ 0, & \text{otherwise,} \end{cases}$, $g(x) = \begin{cases} 1, & \text{for} \quad 0 \le x \le 1, \\ 0, & \text{otherwise,} \end{cases}$

   (b) $f(x) = \begin{cases} 1, & \text{for} \quad 0 \le x \le 1, \\ 0, & \text{otherwise,} \end{cases}$, $g(x) = \begin{cases} 1, & \text{for} \quad 0 \le x \le 1, \\ 0, & \text{otherwise,} \end{cases}$

   (c) $f(x) = \begin{cases} 2, & \text{for} \quad 0 \le x \le 2, \\ 0, & \text{otherwise,} \end{cases}$, $g(x) = \begin{cases} 1, & \text{for} \quad 0 \le x \le 1, \\ 0, & \text{otherwise,} \end{cases}$

   (d) $f(x) = \begin{cases} 2, & \text{for} \quad 0 \le x \le 3, \\ 0, & \text{otherwise,} \end{cases}$, $g(x) = \begin{cases} e^{-|x|}, & \text{for} \quad |x| \le 1, \\ 0, & \text{otherwise,} \end{cases}$

8. Assume that $f \in L_1(\mathbb{R})$ such that f is piecewise-smooth on every finite interval $[a,b]$ and is continuous on $\mathbb{R}$. Then, solve the following integral equation:

   (a) $\displaystyle\int_{-\infty}^{\infty} f(t-x)f(x)\,dx = e^{-t^2}.$

9. Use the Fourier Convolution Theorem, for the followings:

   (a) $F^{-1}\left(\dfrac{1}{3+4iw-w^2}\right)$

   (b) $F^{-1}\left(\dfrac{4}{(1+w^2)^2}\right)$

10. Let $a > 0$.

    (a) Find the Fourier cosine and sine transforms of e^{-ax}.
    (Hint: Use the Fourier transform of $e^{-a|x|}$.)

    (b) Find the Fourier cosine transform of $\dfrac{1}{a^2+x^2}$ and the Fourier sine transform of $\dfrac{x}{a^2+x^2}$.

11. Let $\mathbb{R}^+ = [0,\infty)$. Assume that $f \in L_1(\mathbb{R}^+)$ such that f is piecewise-smooth on every finite interval $[a,b](a<b)$ and is continuous on $\mathbb{R}^+$. Then, solve the following integral equation:

    (a) $\displaystyle\int_{0}^{\infty} f(x)\cos wx\,dx = e^{-w}.$

12. Use suitable transform (Fourier/Cosine/Sine transform) to solve the following differential equations:

    (a) $u' - 2u = H(x)e^{-2x}, \quad -\infty < x < \infty$, where $H(x)$ is the unit step function and $u(x) \to 0$ as $x \to \pm\infty$

    (b) $u'' - u = -f(x), \quad -\infty < x < \infty$, where $f \in L_1(\mathbb{R})$ and $u(x) \to 0$, $u'(x) \to 0$ as $x \to \pm\infty$

    (c) $u'' - u = e^{-2x}, \quad 0 < x < \infty$, where $u(x) = 0$, and $u(x) \to 0$, $u'(x) \to 0$ as $x \to \infty$