Indian Institute of Space Science and Technology

Thiruvananthapuram

MA 211 - Integral Transforms

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1. Determine the inverse Laplace transform of the following functions.
   (a) $\dfrac{s}{s^2-2s+2}$ (b) $\dfrac{e^{-2s}}{s^2-s}$.

2. Use "Convolution" theorem to determine the inverse Laplace transform of the following functions.
   (a) $\dfrac{1}{s^2(s+2)^2}$, (b) $\dfrac{1}{\sqrt{s}(s-1)}$, (use $errf(t) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^t e^{-x^2}\,dx$)

3. Suppose the Laplace transform of a function $f:[0,\infty)\to\mathbb{R}$ is given by $L_T(f(x)) = \widehat{f}(s)$.

   (a) Show that $L_T\Big(\displaystyle\int_0^x f(t)\,dt\Big) = \dfrac{\widehat{f}(s)}{s}$. [Hint: Let $g(x) = \int_0^x f(t)\,dt$ such that $\frac{dg}{dx} = f(x)$]

   (b) Using this determine the inverse Laplace transform of $\dfrac{1}{s(s^2+1)}$.

4. Let $a>0$. Consider the function $f:[0,\infty)\to\mathbb{R}$, defined by $f(x) = \begin{cases} x^2, & \text{for } 0\le x<a, \\ 0, & \text{for } x\ge a. \end{cases}$
   Express f in terms of the Heaviside unit step functions and obtain the Laplace transform of it.

5. Consider the function $f:\mathbb{R}\to\mathbb{R}$, defined by $f(x) = \begin{cases} 0, & \text{for } x<1, \\ -1, & \text{for } 1\le x<3, \\ 1, & \text{for } 3\le x<5, \\ -1, & \text{for } 5\le x<6, \\ 0, & \text{for } x\ge 6. \end{cases}$
   Express f in terms of the Heaviside unit step functions and obtain the Laplace transform of it.

6. Let $a>0$. Determine the Laplace transform of the following functions.
   (a) $f(x) = \begin{cases} x/a, & \text{for } 0\le x<a, \\ (2a-x)/a, & \text{for } a\le x\le 2a, \end{cases}$ such that $f(x+2a) = f(x)$.

   (b) $f(x) = \begin{cases} \frac{1-e^{3x}}{x}, & \text{for } x>0 \\ 1, & \text{for } x=0. \end{cases}$

7. Use Laplace transform to solve the following initial-value-problems.

   (a) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - 3y = xe^x$, $x>0$ with $y(0)=2$, $\frac{dy}{dx}(0)=1$.

   (b) $x\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = -1$, $x>0$ with $y(0)=0$ and $\frac{dy}{dx}(0)=B$,
   where B is a constant.

8. Use Laplace transform to solve the following initial-value-problems.

(a) $\dfrac{dy}{dx} - y = f(x), \quad x > 0$ with $y(0) = 0$, where $f(x) = \begin{cases} 0, & \text{for} \quad 0 \le x < 3, \\ (\sin x)^2, & \text{for} \quad x \ge 3. \end{cases}$

(b) $\dfrac{d^2y}{dx^2} + y = f(x), \quad x > 0$ with $y(0) = 1, \quad \frac{dy}{dx}(0) = 0$, where

i. $f(x) = \begin{cases} \sin x, & \text{for} \quad 0 \le x < \pi, \\ 0, & \text{for} \quad \pi \le x < 2\pi, \end{cases}$ such that $f(x + 2\pi) = f(x)$.

ii. $f(x) = (\sqrt{x} + 1)^3$.

9. Use "Laplace Convolution theorem" to solve

(a) $y(x) = x + \displaystyle\int_0^x e^{(x-t)} y(t)\, dt, \quad x \ge 0.$

10. A system consists of two unit masses lying in a straight line on a smooth surface and conducted together to two fixed points by three springs. When a sinusoidal force is applied to the system, the displacements $y_1(x)$ and $y_2(x)$ of the respective masses from their equilibrium positions satisfy the equations

$$\left\{ \frac{d^2y_1}{dx^2} = y_2 - 2y_1 + \sin 2x, \quad \frac{d^2y_2}{dx^2} = -2y_2 + y_1. \right.$$

Given that the system is initially at rest in the equilibrium position ($y_1 = y_2 = 0$). Use Laplace transform to solve the equations for $y_1(x)$ and $y_2(x)$.
