सम्भाव्यता

PROBABILITY

31-07-2024

PROBABILITES

Random Event:

The possible outcomes are already known but which outcome we will get next is not known.

Classical Probability

↳ The no. of outcomes is finite and equally likely.

Probability: A function defined on a set.

Eg. $S = \{H, T\}$

$p(H) = p$

$p(T) = q$

$p + q = 1$

$\{H\} \cup \{T\} = S$

$\{H\} \cap \{T\} = \phi$

Power set of S : $\mathcal{P}(S) = \{\{H\}, \{T\}, S, \phi\}$

$P : \mathcal{P}(S) \to \mathbb{R}$

$P(\{H\}) = P$

$P(\{T\}) = q$

$P(S) = 1$ (from 1st and 2nd property)

$P(\phi) = 0$

Eg. $S = \{1, 2, 3, \dots, 6\}$

$A \subset S$

$P(A) = ?$

In this case, the probability function has to be the power set of S.

S : all possible outcomes / event points.

Suppose $p(i) = p_i$

such that $p_1 + p_2 + \dots + p_6 = 1$.

e.g. ① $p_i = 1/6 \quad \forall i$

② $p_1 = 1/2, \; p_2 = 1/9, \; p_3 = 1/8, \; p_4 = \dots$

$p_6 = 1 - (1/2 + 1/9 + 1/8 + \dots)$

Properties

① $p(S) = 1$ (No proof, by intuition)

② $A_1, A_2, \cdots \subset S$

$A_i \cap A_j = \phi$ (mutually disjoint)

Then, $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$

Eg. $S = \{1, 2, \ldots, 6\}$

$A = \{1, 2\}$

$= \{1\} \cup \{2\}$ (Either 1 or 2)

$p(A) = p(\{1\}) + p(\{2\})$

$= p_1 + p_2$

Eg. A coin is being tossed until head appears.

Known: $p(H) = p$, $P(T) = q$.

We know: $p + q = 1$

Experiment may end at the 1st toss: H

" " 2nd toss: TH

⋮

Possible outcomes, $S = \{H, TH, TTH, \ldots\}$

Exercise: In general,

$P\{\underbrace{TTT\ldots T}_{n}H\} = q^n p$

$|\wp(S)| = 2^{\aleph_0} = c$

the no. of elements if real.

01-08-2024

Countable Sets: A set is said to be countable, if the elements in the set can be numbered, like 1, 2, 3, ...

↳ All finite sets are countable.

$$A = \{A_1, \dots, A_n\}$$

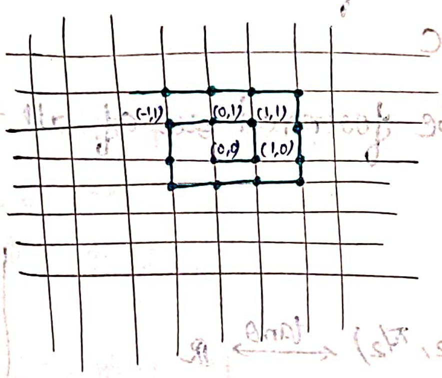
↳ Infinite Countable: Numbering keeps going on.

$\mathbb{N} = \{1, 2, \dots\}$ → Countable

$\mathbb{N} \cup \{0\} = \{\overset{1}{0}, \overset{2}{1}, \overset{3}{2}, \dots\}$ → Countable

$\mathbb{Z} = \{\dots, \underset{5}{-2}, \underset{3}{-1}, \underset{1}{0}, \underset{2}{1}, \underset{4}{2}, \dots\}$ → Countable

$\mathbb{Z} \times \mathbb{Z}$ → Countable



$\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ → Countable

$$\mathbb{Q} = \left\{ \frac{p}{q} \;\middle|\; \begin{array}{l} q \neq 0 \\ p, q \in \mathbb{Z} \\ \gcd(p,q) = 1 \end{array} \right\}$$ → Pair of integers: $p|q$ or (p,q)

↳ $\mathbb{Q}$ is countable

$\mathbb{R}$ is not countable.

$\mathbb{R} \setminus \mathbb{Q}$ → not countable. $[\mathbb{R} - \mathbb{Q}]$

→ A set is said to be infinite if it has a (proper) subset which is also infinite.

If A is infinite, then $\exists\, B \subsetneq A$ such that B is also infinite.

Eg. $\mathbb{N}$

$2\mathbb{N} \subsetneq \mathbb{N}$

$3\mathbb{N} \subsetneq \mathbb{N}$

⋮

→ $|\mathbb{N}| = \aleph_0$ (Aleph naught) → first infinite cardinal no.

↳ Cardinality of $\mathbb{N}$: no. of natural numbers.

$|A|$ = no. of elements in A

$A \rightsquigarrow |\mathcal{P}(A)| = 2^{|A|}$ → No. of elements in Power set of A

$|\mathcal{P}(\mathbb{N})| = 2^{\aleph_0}$ → infinite cardinal no. of 2nd kind.

→ $\aleph_0 < 2^{\aleph_0} = c$

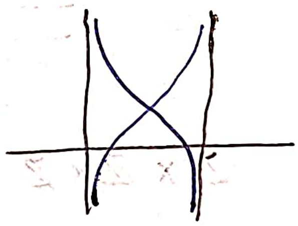
Between $\aleph_0$ & c, there is no cardinal no.

<u>Fact</u>: $\mathbb{R}$ has c-many no. of elements,
i.e., $|\mathbb{R}| = c$

→ The real no.s can be formed out of all the rational sequences.

$|(a,b)| = c$

$(a,b) \xleftrightarrow[\text{scaled}]{\text{can be}} (-\pi/2, \pi/2) \xleftrightarrow{\tan\theta} \mathbb{R}$



02-08-2024

S: collection of the outcomes of a random experiment, called the sample space.

The elements of S are called sample points or "event points".

$A \subset S$: A is called an event.

<u>Aim</u>: To find the probability of A.

Suppose ~~But~~ $\sigma(S)$ is a collection of subsets of S.
(i.e., $\sigma(S) \subset \mathcal{P}(S)$)

[We call $\sigma(S)$ to be a σ-algebra on S of whose we need to know probability.]

Probability on S or on $\sigma(S)$ is a function $P: \sigma(S) \to \mathbb{R}$, such that

① $P(A) \in [0,1] \quad \forall\, A \in \sigma(S)$

② $P(S) = 1$

③ If $A_1, A_2, \ldots \in \sigma(S)$
such that $A_i \cap A_j = \phi \quad (i \neq j)$

Then, $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$ [Countable additivity]

Cor: $S \in \sigma(S)$

Q) Is $\phi \in \sigma(S)$? ✓

Cor: $P(\phi) = 0$.

Proof: $A_1 = S, \quad A_i = \phi \;\forall\; i \geq 2$

$A_1 \cup A_2 \cup \cdots$
$= S \cup \phi \cup \phi \cup \ldots$
$= S$

$P(S) = P(S \cup \phi \cup \ldots)$
$= P(S) + P(\phi) + \ldots + P(\phi)$
$= P(S) + P(\phi)$

$$\left[\begin{array}{l} \because \phi = \phi \cup \phi \cup \phi \ldots \\ P(\phi) = P(\phi \cup \phi \cup \phi \ldots) \\ \quad\quad = P(\phi) + P(\phi) + \ldots \end{array}\right]$$

$\Rightarrow P(\phi) = 0.$

$\therefore P(\phi) = 0$ and $\phi \in \sigma(S)$.

Eg. Toss a coin until a head appears.

$S = \{H, TH, TTH, \ldots\}$

$P(H) = p, \quad P(T) = 1 - p = q$

$P(T^{n-1}H) = q^{n-1}p$

$\underline{P(S) =}$

$$S = \bigcup_{i=0}^{\infty} \{T^iH\} \qquad \left[\{T^iH\} \cap \{T^jH\} = \phi\right]$$

$$P(S) = \sum_{i=0}^{\infty} P(T^iH)$$

$$= \sum_{i=0}^{\infty} q^i p \qquad \text{[From property ③]}$$

$$= \frac{p}{1-q} = p/p = 1.$$

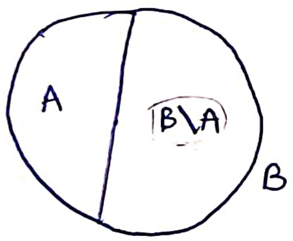
Corr.: Let $A, B \in \sigma(S)$

Suppose $A \subseteq B$ $\quad A \subsetneq B$

Then, $P(A) \leq P(B)$.

Proof: Assume that

$$P(A_1 \cup A_2 \cup \dots \cup A_n) = P(A_1) + P(A_2) + \dots + P(A_n), \quad A_i \cap A_j = \phi.$$



$B = B \setminus A \cup A$

$P(B) = P(B \setminus A) + P(A)$

$[\because \underline{B \setminus A \cap A = \phi}]$

$$P(B \setminus A) = P(B) - P(A) \geq 0$$

$$\Rightarrow P(B) \geq P(A).$$

To show:

$$P(A_1 \cup A_2 \cup \dots \cup A_n) = P(A_1) + P(A_2) + \dots + P(A_n), \quad A_i \cap A_j = \phi.$$

Given: $P(A_1 \cup A_2 \cup \dots \cup A_n \cup A_{n+1} \cup \dots)$

$$= P(A_1) + P(A_2) + \dots + P(A_n) + P(A_{n+1}) + \dots$$

Assign $A_{n+1} = A_{n+2} = \dots = \phi$

$P(A_1 \cup A_2 \cup \dots A_n \cup \phi \cup \phi \cup \dots)$

$$= P(A_1) + P(A_2) + \dots + P(A_n) + P(\phi) + \dots$$

$$\Rightarrow P(A_1 \cup A_2 \cup \dots \cup A_n) = P(A_1) + P(A_2) + \dots + P(A_n).$$

Cor.: If $A, B \in \sigma(S)$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

07-08-2024

Eg. $S = \{1, 2, \ldots, n\}$

Define a probability f^n

$$P : \mathcal{P}(S) \longrightarrow \mathbb{R}$$

$P(i) = a_i \quad \forall\, i = 1, 2, \ldots, n.$

$a_i \in [0,1]$ & $\sum_{i=1}^{n} a_i = 1 \overset{\text{ensure}}{\Longrightarrow} P(A) \in [0,1] \;\forall\, A \in \mathcal{P}(S)$

- $P(S) = 1.$

$$P\left(\bigcup_{i=1}^{n} \{i\}\right) = \sum_{i=1}^{n} P(i) = a_1 + a_2 + \ldots + a_n$$

Get a series that converges to 1.

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots$$

$$e^2 = 1 + 2 + \frac{2^2}{2!} + \ldots$$

$$1 = e^{-2} + 2e^{-2} + \frac{2^2}{2!} e^{-2} + \ldots$$

Define $P(i) = e^{-2} \dfrac{2^{i-1}}{(i-1)!} \quad \forall\, i = 1, 2, \ldots$

In general, choose $x = \lambda$.

$$e^\lambda = 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \ldots$$

$$1 = e^{-\lambda} + e^{-\lambda}\lambda + e^{-\lambda} \cdot \frac{\lambda^2}{2!} + \ldots$$

Define probability f^n on $S = \{1, 2, 3, \ldots\}$

$$P(i) = e^{-\lambda} \cdot \frac{\lambda^{(i-1)}}{(i-1)!} \quad \forall\, i = 1, 2, \ldots$$

It is guaranteed that $P(S) = 1$ & $P(A) \in [0,1]$.

Probability function on an uncountable set:

Eg. $S = [0,1]$

$S = \bigcup_{x \in [0,1]} \{x\}$

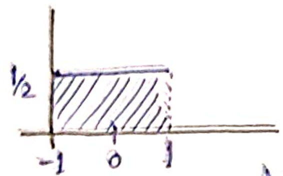
$P(x) = \lambda_x \in [0,1] \longrightarrow \times$

$P(S) = \sum_{x \in [0,1]} P(x) = \sum_{x \in [0,1]} \lambda_x \longrightarrow \times$

$P([0,x]) = \int_0^x dx = x$

$P([0,1]) = \int_0^1 dx = 1$

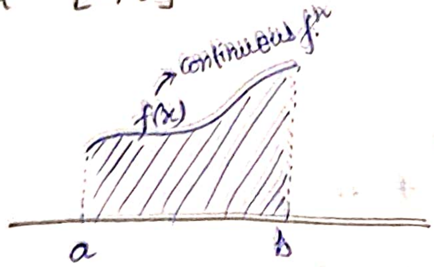
Eg. $S = [-1,1]$



$[a,b] \in [-1,1]$

$P([a,b]) = \int_a^b \frac{1}{2}\, dx$

Eg $S = [a,b]$



Take $f(x)$ such that

$f(x) \geq 0$

$\int_a^b f(x) = \alpha \geq 0$

$P([a,b]) = \int_a^b \frac{1}{\alpha} \cdot f(x) = 1$

$[x_1, x_2] \in [a,b]$

$P([x_1,x_2]) = \int_{x_1}^{x_2} 1/\alpha\, f(x)\, dx$.

08-08-2024

$P : \sigma([a,b]) \longrightarrow \mathbb{R}$

$S = \mathbb{R}$ or $S = \mathbb{R}^+$

$f(x) = e^{-x^2}$

$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

$P(\mathbb{R}) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-x^2} = 1$

$P([a,b]) = \int_a^b \frac{1}{\sqrt{\pi}} e^{-x^2}$

→ To cover whole $\mathbb{R}$, we only require countable many intervals : Generators.

$P : \underbrace{\sigma(\mathbb{R})} \longrightarrow \mathbb{R}$

$\mathcal{B}(\mathbb{R})$: The class generated by countable union & intersection of intervals.

$(a,b) = (-\infty, b) \setminus (-\infty, a)$: Ray generator

$= (-\infty, b) \cap (-\infty, a)^c$

$P(\underbrace{(-\infty,b) \cap (-\infty,a)^c}_{A}) = 1 - P(A^c)$

$= 1 - P((-\infty,b)^c \cup (-\infty,a))$

[Intersection can always be converted into union]

→ To get probability function over $\mathbb{R}$, we must know the integration of a function over an arbitrary interval.

$\xi = [a,b]$

or $\xi = \mathbb{R}$

$\int_{-\infty}^{\infty} f(x) dx = 1$

$P((-\infty, x]) = \int_{-\infty}^{\infty} f(t) dt$

$P([a,b]) = P((-\infty,b]) - P((-\infty,a))$

$\int_{(-\infty,x]} f(t)dt = \int_{(-\infty,x)} f(t)dt$

or, $\int_{(a,b)} f(t)dt = \int_{[a,b]} f(t)dt$

$[a,b] = (a,b) \cup \{a\} \cup \{b\}$

$\int_a^a f(t)dt = 0$

$\int_b^b f(t)dt = 0$

$\therefore P(\{a\}) = 0.$

Properties of Probability functions :

Corollary: $P(A^c) = 1 - P(A)$

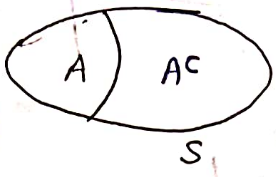
[Notation: $A \cap B = AB$ in Ross]

Proof: $1 = P(S) = P(A \cup A^c)$

$= P(A) + P(A^c)$

[$\because A$ & A^c are disjoint]

$\Rightarrow P(A^c) = 1 - P(A)$



Corollary: $A, B \in \sigma(S)$

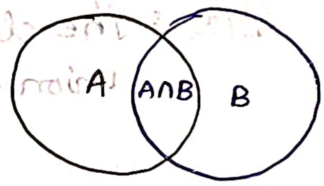
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof: $P(A \cup B) = P(A \cup (B \setminus A))$

$P(A \cup B) = P(A) + P(B \setminus A)$ ... ①

[$\because A \cap (B \setminus A) = \phi$]

[$B \setminus A = B \cap A^c$]



$B = (B \setminus A) \cup (A \cap B)$

$\Rightarrow P(B) = P(B \setminus A) + P(A \cap B)$ ... ②

[$\because (B \setminus A) \cap A = \phi$]

① & ② $\Rightarrow P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Corollary: $A, B, C \in \sigma(S)$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

Events

Exclusive events: Two events are said to be mutually exclusive if $A \cap B = \phi$.

Independent events: Two events are said to be independent if the occurance of one does not affect the other.

09-08-2024

\# $S = \{0, 1, \ldots, n\}$

$(p+q)^n = 1^n = 1$. [Binomial function] [Prob. of getting head]

$\Rightarrow {}^nC_0 p^0 q^n + {}^nC_1 p^1 q^{n-1} + {}^nC_2 p^2 q^{n-2} + \ldots + {}^nC_n p^n q^0 = 1$

$P(i) = {}^nC_i p^i q^{n-i} \quad \forall i = 0, \ldots, n$.

→ ~~Sets~~ A & B are two events.

- A & B are exclusive if $A \cap B = \phi$.
- A is called independent of B if happening of B does not affect the probability of A.

  i.e., if probability of happening of A = Probability of happening of A, given B has already occured. ...①

Eg. Toss a coin

$P(H) = p$

$P(T) = q$

"H|T" : Happening of head, provided T has occured.

$P(H|T) = 0 \neq p$

The same holds for $P(T|H) = 0 \neq q$.

⇒ H & T are dependent on each other.

⇒ H & T are not independent.

For ①,

Are we looking at $P(A \cap B)$?

e.g. $P(H \cap T) = 0$

Suppose $0 \neq P(A \cap B) \stackrel{?}{=} P(A|B)$

$$\underset{\text{(scaled)}}{\frac{P(B)}{P(B)}} = \underset{1}{\underset{\parallel}{P(B \cap B)}} \stackrel{?}{=} P(B|B).$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}. \quad [P(B) \neq 0]$$

Similarly,

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \text{ if } P(A) \neq 0.$$

$$\left.\begin{array}{l} \bullet\ P(A \cap B) = P(A) \cdot P(B|A), \quad P(A) \neq 0 \\ \bullet\ P(A \cap B) = P(B) \cdot P(A|B), \quad P(B) \neq 0. \end{array}\right\} (*)$$

→ If $P(A) \neq 0 \neq P(B)$,
then $P(A|B)$ and $P(B|A)$ exist.

→ If A is independent of B,
then $P(A) = P(A|B)$ ... ①

→ If B is independent of A,
then $P(B) = P(B|A)$ ... ⓘⓘ

If A and B are independent of each other, then ① and ⓘⓘ happen together. Then,

$$P(A \cap B) = P(A) \cdot P(B)$$
$$P(A \cap B) = P(B) \cdot P(A).$$

Defn: A and B are called independent if $P(A \cap B) = P(A) \cdot P(B)$.

Eg $P(H \cap T) = 0$

$P(H) \cdot P(T) = pq \neq 0$

$\Rightarrow P(H \cap T) \neq P(H) \cdot P(T)$

$\Rightarrow$ H and T are not independent.

→ Set A and B are independent.

$$P(A \cap B) = P(B).P(A)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) \neq 0.$$

$$= \frac{P(A).\cancel{P(B)}}{\cancel{P(B)}}$$

$$= P(A)$$

Similarly, $P(B|A) = P(B)$.

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$$P(A) = P(A|B), \quad P(B) \neq 0$$

$$= \frac{P(A \cap B)}{P(B)}, \quad P(B) \neq 0.$$

$$P(A \cap B) = P(B).P(A|B)$$

__Special case__: When $A \cap B = \phi$

Then, $P(A \cap B) = 0$

$$\Rightarrow 0 = P(A \cap B) = P(B).P(A|B)$$

Since $P(B) \neq 0$

$$\Rightarrow P(A|B) = 0$$

Is it possible/ ~~what happens when~~

$$(A \cap B \neq 0) \Rightarrow P(A \cap B) \neq 0$$

$$\& \cdot P(A|B) = 0.$$

$$\underset{\neq 0}{P(A \cap B)} = P(B) \cdot \underset{\neq 0}{P(A|B)}$$

__Special case__: $P(A) = 0$

$$A \cap B \subset A$$

$$\Rightarrow P(A \cap B) \leq P(A) = 0$$

$$\Rightarrow P(A \cap B) = 0$$

$$\Rightarrow P(A|B) = 0 \text{ as } P(A|B) = \frac{P(A \cap B)}{P(B)}$$

We have seen that there are examples when $A \cap B = \phi$, and A & B are not independent.

Is it possible to have
- A & B independent and $A \cap B = \phi$?
- A & B independent and $A \cap B \neq \phi$?
- $A \cap B \neq \phi$ and A & B dependent ?

→ A is independent of B if $P(A) = P(A|B)$

→ A & B are independent

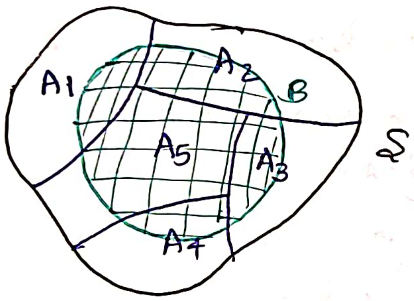
$$\Rightarrow P(A \cap B) = P(A) \cdot P(B)$$

① The theorem of total probability

② Bayes Theorem.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) \neq 0.$$

⇒ Gives rise to a new probability function — $P(\cdot|B)$
called conditional probability



called partition of S

$$S = \bigcup_{i=1}^{n} A_i, \quad A_i \cap A_j = \phi$$

$$B \subset S$$

$$B = S \cap B = \left(\bigcup_{i=1}^{n} A_i\right) \cap B$$

$$= \bigcup_{i=1}^{n} (A_i \cap B)$$

$$B = \bigcup_{i=1}^{n} (A_i \cap B) \rightarrow \text{is a partition of } B.$$

find $P(B) = \sum_{i=1}^{n} P(A_i \cap B)$ [Aim is to know $P(B)$]

$$P(A_i \cap B) = P(B).\, P(A_i \mid B)$$
$$P(A_i \cap B) = P(A_i).\, P(B \mid A_i)$$
$\Big\} \times \Rightarrow$ Not helpful

Hence,

$$\boxed{P(B) = \sum_{i=1}^{n} P(A_i).\, P(B \mid A_i)}$$

called the formula for total probability.

Bayes Theorem (on conditional Probability)

$$P(A_i \mid B) = \frac{P(A_i \cap B)}{P(B)}$$

$$\boxed{P(A_i \mid B) = \frac{P(A_i).\, P(B \mid A_i)}{\sum_{i=1}^{n} P(A_i).\, \cancel{P(A_i \mid B)}\, P(B \mid A_i)}}$$

called Bayes theorem.

$$P(B) = \underbrace{P(A_1).\, P(B \mid A_1)}_{P(A_1 \cap B)} + \underbrace{P(A_2).\, P(B \mid A_2)}_{P(A_2 \cap B)} + \underbrace{P(A_3).\, P(B \mid A_3)}_{P(A_3 \cap B)}$$

$$\frac{P(A_1 \cap B)}{P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B)}$$: Classical case

Eg. A lab blood test is 95% effective in detecting a certain disease when infact it is present. However, the test also yields a false positive result for 1% of the healthy patient tested. If 0.5% of the population actually has the disease, what is probability that a person has a disease given the test result is positive?

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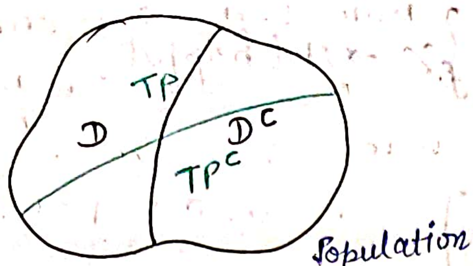


$S = D \cup D^c$

$S = TP \cup TP^c$

D: Has health issue

D^c: Has no health issue

TP: Test result is positive

TP^c: " " negative

$P(TP|D) = 0.95$

$P(TP|D^c) = 0.01$

$P(D) = 0.005$

$\Rightarrow P(D^c) = 0.995$

$TP = (TP \cap D) \cup (TP \cap D^c)$

$$P(TP) = P(TP \cap D) + P(TP \cap D^c)$$

$$= P(D) \cdot P(TP|D) + P(D^c) \cdot P(TP|D^c)$$

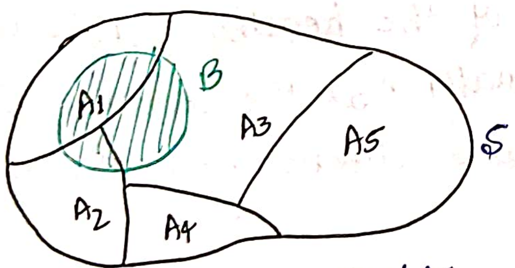
$$= (0.005 \times .95) + (.995 \times 0.01) = 0.0147$$

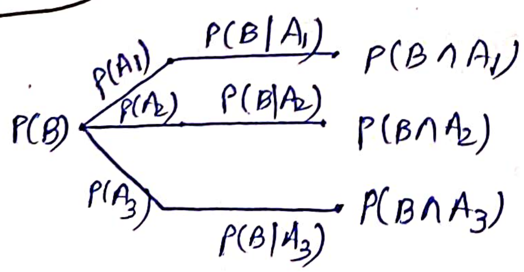
$$P(D|TP) = \frac{P(D \cap TP)}{P(TP)}$$

$$= \frac{P(D) \cdot P(TP|D)}{P(TP)}$$

$$= \frac{0.05 \times 0.95}{0.0147} = 3.231$$

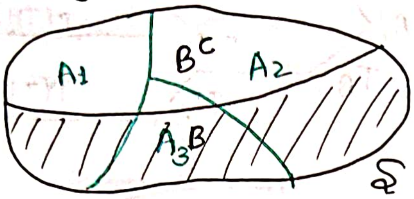
→





Eg. (Walpole) In a certain assembly plant, 3 machines B_1, B_2 & B_3, respectively make 30%, 45% and 25% of the product. It is known from the 1st experience 2%, 3% and 2% of the product by the machine gives defective product. Suppose a finished product is chosen randomly, what is the probability that the product is defective?

Soln:



B : The product is defective

To calculate $P(B)$

A_i : Product is coming from B_i

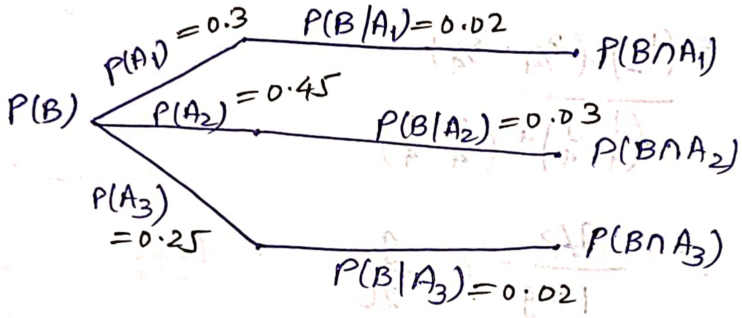
$$B = (B \cap A_1) \cup (B \cap A_2) \cup (B \cap A_3)$$

$$P(B) = \sum_{i=1}^{\infty} P(B \cap A_i) = \sum_{i=1}^{\infty} P(A_i) \cdot P(B|A_i)$$

$P(A_1) = 0.3$

$P(A_2) = 0.45$

$P(A_3) = 0.25$



$$P(B) = (0.3 \times 0.02) + (0.45 \times 0.03) + (0.25 \times 0.02) = 0.0245$$

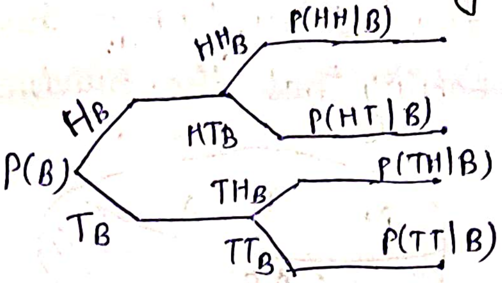
If a product is chosen randomly and was found to be defective, what is the probability it was made by B_3?

Soln:
$$P(A_3|B) = \frac{P(A_3 \cap B)}{P(B)}$$

$$= \frac{P(A_3) \cdot P(B|A_3)}{P(B)}$$

$$= \frac{0.25 \times 0.02}{0.0245} = 0.204$$

Eg. When ~~point A~~ coin A is flipped, it comes up with a head with probability one-fourth, while for coin B it is 3/4. Suppose one of the coins is randomly chosen and flipped twice. If both the flips give head, what is the probability coin B was flipped?



Soln: $P(T_B) = 1/4$

$P(H_B) = 3/4$

$P(H_A) = 1/4$

$P(T_A) = 3/4$

$$P(B|HH) = \frac{P(B \cap HH)}{P(HH)}$$

$$= \frac{P(B) \cdot P(HH|B)}{P(HH|A) + P(HH|B)}$$

$$= \frac{(1/2) \cdot (3/4 \cdot 3/4)}{\left(\frac{1}{4} \cdot \frac{1}{4}\right) + \left(\frac{3}{4} \cdot \frac{3}{4}\right)}$$

$$\frac{9/2}{1+9} = \frac{9}{20}$$

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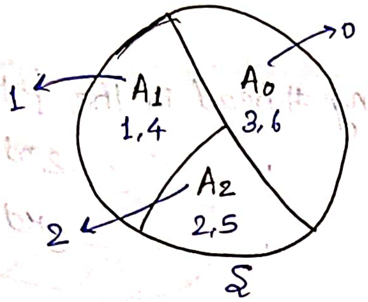
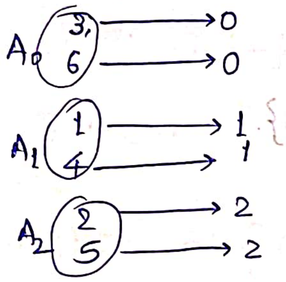
Random Variable

Eg. Remainder when the no. on the dice is divided by 3.



X: random variable
↳ function

$S : X \longrightarrow \mathbb{R}$



$P(A_1) = ?$
$P(A_2) = ?$
$P(A_3) = ?$

$A_0 = X^{-1}(\{0\})$

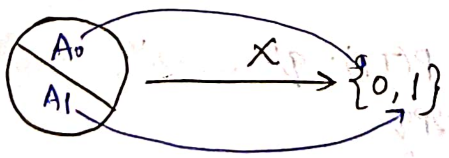
$P(A_0) = P(X^{-1}(\{0\}))$

$= P(X=0)$ [Probability that the rem.=0 when divided by 3]

Defn: S is a sample space.

Any function $X: S \to \mathbb{R}$ is called a random variable.

$X(s) = 1 \ \forall \ s \in S$ } X ↓ not helpful



[Like a coin]

$X=0 \equiv A_0$
$X=1 \equiv A_1$
$r \neq 0,1,\ X=r \equiv \phi$

↳ a function, not a variable.
↳ called variable b/c it takes a value.

Eg. Tossing a coin 3 times.

$S = \{TTT, TTH, THT, \ldots\}$

$|S| = 2^3$

$P(TTH) = P(T) \cdot P(T) \cdot P(H)$

X_1 : a random variable counting # head in the 1^{st} toss

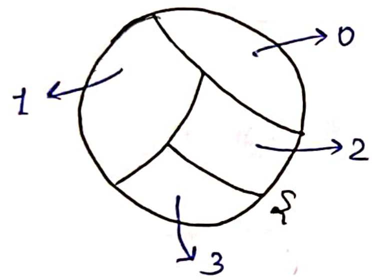
X_2 : " " 2^{nd} toss

X_3 : " " 3^{rd} toss

X_i shall take value in $\{0,1\}$

$X = X_1 + X_2 + X_3$ ... ~~countes~~ counts

X takes values in the set $\{0,1,2,3\}$.



$P(X=0)$: the probability that no head appears $\} \to$ no need $= {}^3C_0 q^3$

$P(X=1) = {}^3C_1 q^2 p$

$P(X=2) = {}^3C_2 p^2 q$

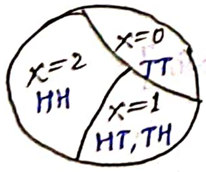
$P(X=3) = {}^3C_3 p^3$

$P(H) = p$

$P(T) = q$

$p + q = 1$

$\{TT, HT, TH, HH\}$



$P(X=0) = P(TT) = q^2$

$P(X=1) = 2pq$

$P(X=2) = p^2$

$1 = p^2 + 2pq + q^2 = (p+q)^2$

$1 = (p+q)^3 = {}^3C_0 q^3 p^0 + {}^3C_1 q^2 p^1 + {}^3C_2 q p^2 + {}^3C_3 q^0 p^3$

X takes values in $\mathbb{Z}$ or a set equivalent to $\mathbb{Z}$. [Infinite set]

$$\cdots < r_{-2} < r_{-1} < r_0 < r_1 < r_2 < \cdots$$

Assume that X takes countable values, r_i, $i \in \mathbb{Z}$ and they are ordered accordingly:

<u>Questions:</u> $P(X=r_i)=?$, $i \in \mathbb{Z}$

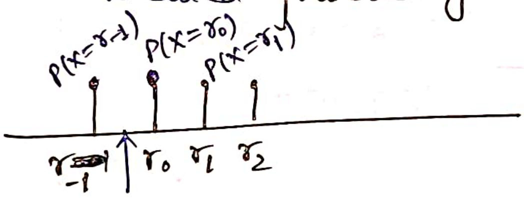
Suppose $P(X=r_i)=p_i$

$$\Rightarrow \sum_{i \in \mathbb{Z}} p_i = 1$$

This function $f: \{r_i \mid i \in \mathbb{Z}\} \longrightarrow [0,1]$

$$f(r_i) = P(r=r_i) = p \quad , \quad p \text{ in } f.$$

is called probability mass function



$f(x)=0 \;\forall\, x \neq r_i$

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X takes values in $\{r_i \mid i \in \mathbb{Z}\}$.

If $r \neq r_i \;\forall\, i$,

then $P(X=r)=0$.

($\because$ the event $X=r$ is ϕ)

<u>Define a f^n:</u>

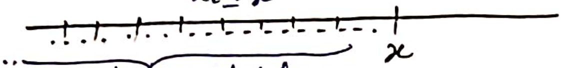
$$f_X = P(X=x) = \begin{cases} p_i, & \text{if } x=x_i \\ 0, & \text{if } x \neq x_i \;\forall\, i \end{cases}$$

This function is called probability mass function of X, or in short p.m.f. of X.

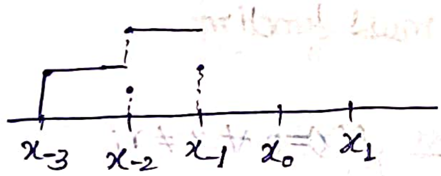
Let $r_i < r_{i+1}$

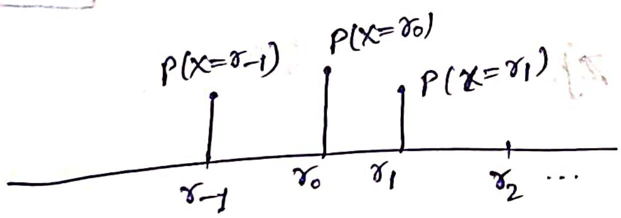
<u>Define a function:</u>

$$F_X(x) = P(X \le x)$$
$$= \sum_{\substack{y \in \mathbb{R} \\ y \le x}} P(X = y)$$ is called cumulative distributive function of X. (c.d.f.)
$$= \sum_{x_i \le x} P(X = x_i)$$



$$F_X(x) = \sum_{y \le x} f_X(y)$$
$$= \sum_{x_i \le x} f_X(x_i)$$





$r_i < r_{i+1}$

$$F_X(r_{i+1}) - F(r_i) = P(X = r_{i+1})$$
$$= f_X(r_{i+1})$$

Eg. Let X be a random variable that counts the # heads when a coin is tossed n-times.

Known: $P(H) = p$.
$P(T) = q$.

Find the distribution of the probability of getting heads.

Soln: Shall find the p.m.f. of X.

The possible values of X are $0, 1, \ldots, n$.

$P(X=i) = {}^nC_i p^i q^{n-i}$.

So, the p.m.f. of X is given by

$$f_X(x) = \begin{cases} {}^nC_i p^i q^{n-i}, & \text{if } x=i, \quad i=0,\ldots,n \\ 0, & \text{otherwise.} \end{cases}$$

$$F(x) = \begin{cases} 0 & , \text{ if } x<0 \\ \left.\begin{array}{ll} {}^nC_0 p^0 q^n & , \text{ if } x=0 \\ {}^nC_0 p^0 q^n & , \text{ if } 0<x<1 \end{array}\right\} x<1 \\ \left.\begin{array}{ll} {}^nC_0 p^0 q^n + {}^nC_1 p q^{n-1} & , \text{ if } x=1 \\ {}^nC_0 p^0 q^n + {}^nC_1 p q^{n-1} & , \text{ if } x<2 \end{array}\right\} x<2 \\ \left.\begin{array}{ll} {}^nC_0 p^0 q^n + {}^nC_1 p q^{n-1} + {}^nC_2 p^2 q^{n-2} & , \text{ if } x=2 \end{array}\right\} x<3 \\ \vdots \end{cases}$$

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Properties

X: a random variable taking values $\{x_i \mid i \in \mathbb{Z}\}$.

$f_X : \mathbb{R} \longrightarrow \mathbb{R}$ is p.m.f. of X. $[f_X(x) = P(X=x)]$

Then, f_X has the following properties:

① $0 \leq f_X(x) \leq 1$

② $1 = \sum_{x \in \mathbb{R}} f_X(x)$

③ $P(X \leq x) = \sum_{y \leq x} f_X(y)$
$(= F_X(x))$

Q1. Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a given function. How can we say that g is a p.m.f. of some r.v.?

Ans: g will be a p.m.f. of some r.v. (say y) if

① $0 \leq g(x) \leq 1$

② $\sum_{x \in \mathbb{R}} g(x) = 1$

③ $P(y \leq z) = \sum_{y \leq z} g(y)$ } Automatically; need not be shown; to be ensured.

Eg. Let $g(x) = \frac{1}{2} \cdot \frac{2^x}{x!}$ $\forall x = 0, 1, 2, \ldots$

Can g be a p.m.f. of some r.v.?

Soln:

$$\sum_{x=0}^{\infty} \frac{1}{2} \cdot \frac{2^x}{x!} = \frac{1}{2} \sum_{x=0}^{\infty} \frac{2^x}{x!} = \frac{1}{2} e^2 \neq 1$$

$$\left[e^{\lambda} = 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \ldots \right]$$

$\Rightarrow$ 2nd property does not hold.

$\Rightarrow g$ is not a p.m.f. of some r.v.

1^{st} property: If $\frac{1}{2} \cdot \frac{2^x}{x!} \leq 1 \quad \forall x = 0, 1, 2, \ldots$

or, If $\frac{2^{x-1}}{x!} \leq 1 \quad \forall x = 0, 1, 2, \ldots$

or, If $2^{x-1} \leq x!$ ✓

$$\underbrace{2 \cdot 2 \cdot 2 \cdot \ldots 2}_{n} \leq n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1 \quad ✓$$

Eg: Let a coin is being tossed until head appears. Suppose X is the no. of tosser, which ensures the end of the experiment. Given $P(H) = p$, $P(T) = q$, $(p+q=1)$, find the distribution of X.

Soln: X takes values in $\mathbb{N}$.

Infact all values of $\mathbb{N}$ are taken by X.

So, we need to find

$$P(X=i), \quad i \in \mathbb{N}$$
$$= q^{i-1} p \quad \forall i \in \mathbb{N}$$

Random variable (X) and their distribution	Name of the r.v or name of the distribution
① $f_X(x) = {}^nC_x p^x q^{n-x}$, $x = 0, 1, \ldots, n$	Binomial distribution or X is a binomial variable $X \sim b(n, p)$ ↑(sim)
② $f_X(x) = q^x p$ $\forall x = 1, 2, \ldots$	Geometric distribution or X is a geometric variable with parameter p. $X \sim g(p)$

Eg. ~~Let~~ $g(x) = e^{-2}\frac{2^x}{x!}$ $\forall$ $x = 0, 1, 2, \dots$

↳ Examples: Accidents [2 : avg. no. of accidents happened]
Printing mistake (due to less ink, etc.)

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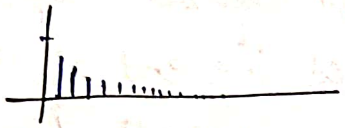
③ $f_X(x) = e^{-\lambda}\frac{\lambda^x}{x!}$, $\lambda \geq 0$
$\forall$ $x = 0, 1, \dots$



Poisson distribution with parameter λ.
$X \sim P(\lambda)$

Eg. Random variable : $X \to$ binomial
$X \sim b(n, p)$ $[X = 0, 1, \dots, n]$
$Y_1 = \alpha X + \beta$ (discrete)
$Y_2 = X^2$ (discrete)
Find dist. of Y_1 and Y_2.

Soln: Dist. of Y_2: Find p.m.f of Y_2.

$$F_{Y_2}(z) = P(Y_2 = z) = \begin{cases} 0, & z \neq i^2 \\ ?, & z = i^2 \end{cases}$$

When $z = i^2$, $i = 0, 1, \dots, n$
Then find $P(Y_2 = i^2)$
$= P(X^2 = i^2)$
"$X^2 = i^2$" is an event, to find its equivalent event in terms of X.
"$X^2 = i^2$" $\equiv$ "$X = i$" [X takes non-neg. values]
So, $P(X^2 = i^2) = P(X = i)$
$= {}^nC_i p^i q^{n-i}$

So, $$F_{Y_2}(z) = \begin{cases} 0, & \text{if } z \neq i^2 \ \forall\ i = 0, 1, \dots, n \\ {}^nC_i p^i q^{n-i} & \text{if } z = i^2 \ \forall\ i = 0, 1, \dots, n \end{cases}$$

↳ Not binomial r.v. [It is jumping with i^2]

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Eg. Let X be a r.v. such that X takes uncountably many values, possibly $\mathbb{R}$. Assume X takes all the values in $\mathbb{R}$.

x_λ is countable whole $\mathbb{R}$

" $X \in \mathbb{R}$ " $= \Omega$

$\overset{\equiv}{\text{"}} X = x \;\forall\, x \in \mathbb{R}\text{"} = \Omega$

So,

$P(X \in \mathbb{R}) = 1$

$\quad\;\; \parallel$

$\cancel{f_{\mathbb{R}}} \int_{\mathbb{R}} f_X(x) \underset{\text{has to be}}{=} 1$. } not proven; Belief

So, we define X, a r.v. to be continuous if $\exists$ a f^n $f_X : \mathbb{R} \to \mathbb{R}$ such that

$P(X \in A) = \int_A f_X(x)\,dx$, $A \in \mathcal{B}(\mathbb{R})$

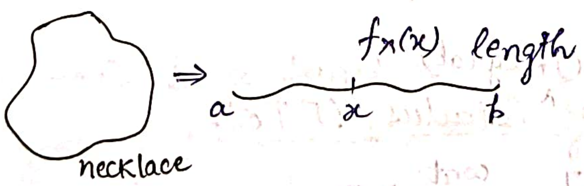
e.g., if $A = [a, b]$

$P(X \in [a,b]) = \int_a^b f_X(x)\,dx$

& $P(X \in \mathbb{R}) = \int_{-\infty}^{\infty} f_X(x)\,dx$.

& $f_X(x) \geq 0 \;\forall\, x \in \mathbb{R}$

$f_X(x)$ is called the density function of X or probability density function of X (p.d.f. of X).



2 : $f_X(x)$ is not a probability f^n or does not represent probability of an event.

But in discrete case, we had the p.m.f. as probability of single-ton event,

$f_X(x) = P(X = x)$ [Discrete]

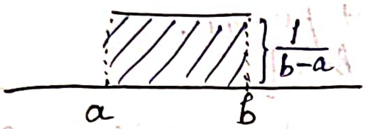
However, the c.d.f of X is

$$F_X(x) = P(X \leq x)$$
$$= \int_{-\infty}^{x} f_X(x)\,dx$$

is a probability function.

Let X be a r.v. such that

$$f_X(x) = \frac{1}{b-a} \quad \forall\, x \in [a,b]$$
(p.d.f of x) $= 0$, otherwise



is a r.v. with its density and the corresponding c.d.f. is

$$F_X(x) = P(X \leq x) \quad \forall\, x \in \mathbb{R}$$
$$= \int_{-\infty}^{x} f_X(t)\,dt$$
$$= \int_{-\infty}^{x} \frac{1}{b-a}\,dt = \frac{1}{b-a}\int_{-\infty}^{x} dt$$
$$= \begin{cases} 0, & \text{if } x \leq a \\ \dfrac{x-a}{b-a}, & \text{if } a < x \leq b \\ 1, & \text{if } x \geq b \end{cases}$$

<u>Fundamental Theory of (Integral) Calculus (F.T.C):</u>

$$f : [a,b] \xrightarrow{\text{cont.}} \mathbb{R}$$

Then, the f^n $F : [a,b] \longrightarrow \mathbb{R}$ given by

$$F(x) = \int_{a}^{x} f(t)\,dt$$

is differentiable and

$$F'(x) = f(x) \quad \forall\, x \in [a,b].$$

From F.T.I.C, we see that if $f_X(x)$ is piecewise continuous.

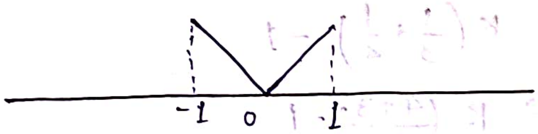
Then, $\frac{d}{dx} F_X(x) = f_X(x) \quad \forall\, x \in \mathbb{R}$

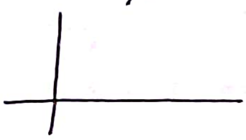
where, F_X : c.d.f X

f_X : p.d.f. X.

Eg. Let X be a r.v. with density

$$f_X(x) = |x|, \quad x \in [-1,1]$$
$$= 0, \quad \text{otherwise}.$$



Density of X	Name of X
① $f_X(x) = \frac{1}{b-a}$	X is called Uniform variant with parameters a, b. $X \sim U(a,b)$.
② $f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$	X is called normal variant with parameters μ, σ. $X \sim N(\mu, \sigma^2)$
③ $f_X(x) = \frac{1}{\lambda} e^{-\lambda x}, \quad x \geq 0$ 	$X \sim e(\lambda)$ $\exp(\lambda)$

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Eg. Let $g : \mathbb{R} \to \mathbb{R}$ be a function.

Is g a ~~good~~ p.d.f. of a r.v.?

Ans: g is a p.d.f. of some r.v. if

① $g(x) \geq 0 \quad \forall\, x \in \mathbb{R}$

② $\int_{\mathbb{R}} g(x)\,dx = 1$,

and if y is the corresponding r.v., then $P(Y \leq y) = \int_{-\infty}^{y} g(x)\,dx$.

Eg. Let $g(x) = \begin{cases} 0 & , \text{ if } x \notin [0,1] \\ (x^2+x^3).K & \forall\ x \in [0,1] \end{cases}$

Find the value of K so that g is a p.d.f. of some r.v..

Soln: $g(x)$ has to be positive $\Rightarrow K \geq 0$

and, $\displaystyle\int_{-\infty}^{\infty} g(x)\,dx = 1 \Rightarrow \int_0^1 (x^2+x^3)\,K\,dx = 1$

$$\Rightarrow K\left(\frac{x^3}{3}+\frac{x^4}{4}\right)\Bigg|_0^1 = 1$$

$$\Rightarrow K\left(\frac{1}{3}+\frac{1}{4}\right) = 1$$

$$\Rightarrow K\left(\frac{4+3}{12}\right) = 1$$

$$\Rightarrow K = \frac{12}{7}.$$

Eg. $g(x) = \begin{cases} 0 & \text{if } x \notin [-1,1] \\ ax^2+b & \text{if } x \in [-1,1] \end{cases}$

Find values of a and b so that g is a p.d.f. of some r.v.

Soln: $g(x) \geq 0 \Rightarrow ax^2+b \geq 0$

$\left[-1 \leq x < 1 \Rightarrow 0 \leq x^2 < 1 \Rightarrow 0 \leq ax^2 \leq a \Rightarrow b \leq ax^2+b \leq a+b\right]$

$\Rightarrow b \geq 0,\ a \geq 0$

and, $\displaystyle\int_{-\infty}^{\infty} g(x)\,dx = 1 \Rightarrow \int_{-1}^{1} (ax^2+b)\,dx = 1$

$$\Rightarrow \left(\frac{ax^3}{3}+bx\right)\Bigg|_{-1}^{1} = 1$$

$$\Rightarrow \left(\frac{a}{3}+b\right)-\left(-\frac{a}{3}-b\right) = 1$$

$$\Rightarrow \frac{a}{3}+b = 1/2$$

$$\Rightarrow a = 3/2 - 3b$$

$$\Rightarrow a \leq 3/2$$

$b \geq 0$
$3b \geq 0$
$-3b \leq 0$
$3/2 - 3b \leq 3/2$

$\therefore\ b \geq 0,\ a \in [0, 3/2]$.

Expectation of a Random Variable

Eg. (Average value)

H $\longrightarrow$ 2 pt. : value of X

T $\longrightarrow$ 1 pt. : value of X

In 100 tosses, one gets 45 heads and rest tails.

The avg. value of $X = \dfrac{(45\times2)+(100-45)\times1}{100}$

$= 2\times\frac{45}{100} + 1\times\frac{55}{100}$

= sum of (values of X & ~~probability~~ product of the freq.)

for $P(H)=p$, $P(T)=q$.

$p\times2 + 1\times q$ = avg. value of X.

Eg. Rolling dice

$P(i) = p_i$

$\sum_{i=1}^{6} p_i = 1$

Let X be a r.v. which counts the face value of the dice after rolling.

X takes values $1, 2, \ldots, 6$.

Average value of $X = 1\times p_1 + 2\times p_2 + \cdots + n\times p_6$

called expectation of X and is denoted by

$$E(X) = \sum_{i=1}^{6} i.p_i = \sum_{i=1}^{6} i.\,P(X=i).$$

(30-08-2024)

Let X be a r.v.

Then, the mean of X or expectation of X is given by

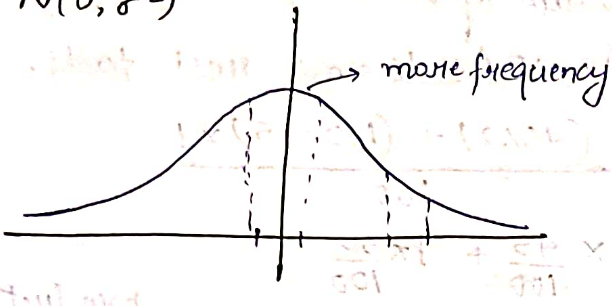
$$E(X) = \sum_{x_i} x_i.\,P(X=x_i), \quad x_i : \text{values taken by } X.$$

$$= \sum_{x_i} x_i.\,f_X(x_i)$$

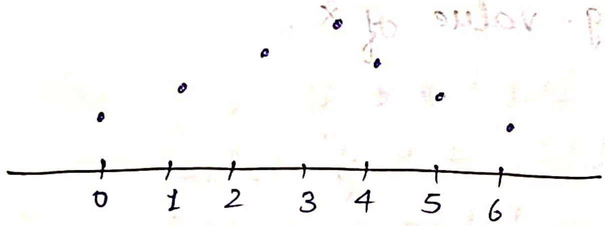
If X is continuous, then

$$E(X) = \int_{-\infty}^{\infty} x f_X(x) \quad , \quad f_X(x) : \text{density of } X.$$

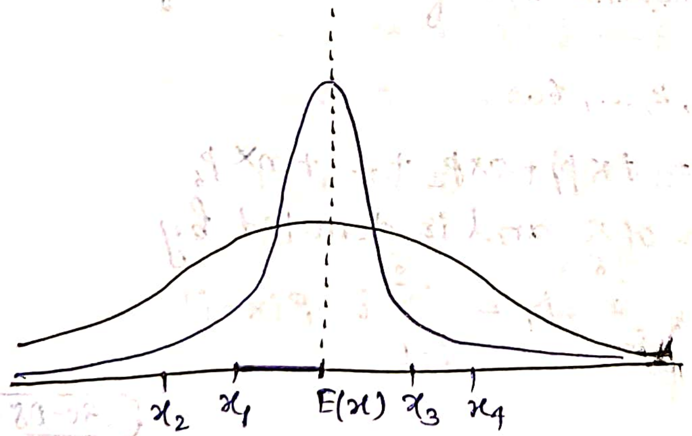
$X \sim N(0, \sigma^2)$



$X \sim b(n, p)$, $p = 1/2$
$n = 6$



X is r.v.



Mean (avg.) deviation from the mean($E(X)$) $= \sum_{x_i} |E(x) - x_i| f_x(x_i)$

Variance of X, $Var(X) = \sum_{x_i} (E(x) - x_i)^2 f_X(x)$

↓ kill square

(x^2) : differentiable → better function

$\sqrt{Var(X)}$ is called standard deviation of X, $sd(X)$.

for cont. X, $Var(X) = \int_{-\infty}^{\infty} (E(X) - x)^2 f_X(x)\, dx$

$E(x)$ is denoted by μ_X.
$Var(X)$ is denoted by σ_X^2.
$sd(X)$ is denoted by σ_X.

$$Var(X) = \int_{\mathbb{R}} (E(x)-x)^2 f_X(x)\,dx$$

$$E(X) = \int_{\mathbb{R}} x f_X(x)\,dx$$

It can be checked that

$$E\left((E(X)-X)^2\right) = Var(X).$$

Eg. Let X be a r.v., then $g(X)$ is also a r.v.
Then, $g(X)$ has a distribution.
Let $g(X)$ is continuous r.v., and hence has p.d.f. $f_{g(X)}$.

$$X \rightsquigarrow f_X$$

$$g(X) \rightsquigarrow f_{g(X)}$$

$$E(g(X)) = \int_{\mathbb{R}} g(t)\cdot f_{g(X)}(t)\,dt$$

takes special cases

$[\mu_X = E(X)]$

$$g(X) = (\mu_X - X)^2$$

$$E(g(X)) = Var(X) = \int_{\mathbb{R}} (\mu_X - x)^2 f_X(x)\,dx$$

$$= \int_{\mathbb{R}} (\mu_X - x)^2 f_{g(x)}(x)\,dx.$$

Eg. $X \sim b(n, p)$

Find $E(X)$, $Var(X)$

$$E(X) = \sum_{i=0}^{n} i \cdot {}^nC_i p^i q^{n-i}$$

$$(x+q)^n = \sum {}^nC_i \cdot x^i q^{n-i}$$

differentiate $\Rightarrow n(x+q)^{n-1} = \sum i \, {}^nC_i x^{i-1} q^{n-i}$

$$\Rightarrow n(x+q)^{n-1} \cdot x = \sum i ({}^nC_i) x^i \cdot q^{n-i}$$

Plug in $x = p = 1-q$

$$\Rightarrow np = \sum_{i=0}^{n} i \cdot {}^nC_i p^i q^{n-i} = \mu_x.$$

$$Var(X) = \sum_{x_i} (E(x) - x_i)^2 f_X(x_i)$$

$$= \sum_{i=0}^{n} (\mu_x - i)^2 \cdot {}^nC_i \cdot p^i \cdot q^{n-i}$$

$$= \sum_{i=1}^{n} (\mu_x^2 - 2\mu_x i + i^2) \cdot {}^nC_i \cdot p^i q^{n-i}$$

$$= \sum_{i=1}^{n} \mu_x^2 \, {}^nC_i p^i q^{n-i} - 2\mu_x \sum i \cdot {}^nC_i p^i q^{n-i} + \sum i^2 \, {}^nC_i p^i q^{n-i}$$

$$= \mu_x^2 - 2\mu_x \cdot \mu_x + (n^2p^2 + npq)$$

$$= npq$$

$n(x+q)^{n-1} \cdot x = \sum i \cdot {}^nC_i x^i q^{n-i}$

$\Rightarrow n(n-1)(x+q)^{n-2} \cdot x + n(x+q)^{n-1} = \sum i^2 \, {}^nC_i x^{i-1} q^{n-i}$

$\Rightarrow n(n-1)(x+q)^{n-2} \cdot x^2 + n(x+q)^{n-1} x = \sum i^2 \cdot {}^nC_i \cdot x^i \cdot q^{n-i}$

Put $x = p$

$\Rightarrow n(n-1)p^2 + np = \sum i^2 \, {}^nC_i p^i q^{n-i}$

$= n^2p^2 + npq$

(04-09-2024)

Q.1 Let x be a r.v. and $g(x)=Y$ is also a r.v.

$$E(Y)=\int_{-\infty}^{\infty} y f_Y(y)\,dy$$
$$\parallel$$
$$E(g(x))=\int_{-\infty}^{\infty} g(x) f_X(x)\,dx$$

} why are they same?

(Functions of r.v. and their distribution)

Let $g(x)$ and $h(x)$

$\alpha g(x) + \beta h(x)$ such that α and β are continuous ($\alpha, \beta \in \mathbb{R}$).

$$E(\alpha g(x)+\beta h(x)) = \alpha E(g(x)) + \beta E(h(x))$$

... Linearity property of expectation.

So, E as an operation is linear.

$$Var(\alpha g(x)+\beta h(x)) = ??$$

$$Var(\alpha g(x)) = \alpha^2 Var(g(x))$$

- $Var(\alpha X) = E((E(\alpha X)-\alpha X)^2)$
$$= E((\alpha E(X)) - \alpha X)^2)$$
$$= E(\alpha^2(E(X)-X)^2)$$
$$= \alpha^2 E((E(X)-X)^2)$$
$$= \alpha^2 Var(X)$$

- $Var(X) = E((E(X)-X)^2)$
$$= E(E(X)^2 - 2E(X)X + X^2)$$
$$= E(E(X)^2) - E(2E(X)X) + E(X^2)$$
$$= E(X)^2 E(1) - 2E(X).E(X) + E(X^2)$$
$$= E(X)^2 - 2E(X)^2 + E(X^2)$$
$$= E(X^2) - E(X)^2$$

So, $Var(X) = E(X^2) - (E(X))^2$

→ $E(X^n)$ is called the nth moment of X (around the origin).

→ $E((X-a)^n)$ is called the nth moment of X around a point a.

→ Var(X) is the 2nd moment around the mean of X.

- $E(X^n) = \sum_{x} x^n f_X(x)$
- $E(X^n) = \int_{\mathbb{R}} x^n f_X(x)\,dx$.

Eg. Find nth moment of poisson distribution.

Soln: $X \sim P(\lambda)$

$$f_X(x) = e^{-\lambda}\frac{\lambda^x}{x!}, \quad x = 0, 1, 2, \ldots$$

$$E(X^n) = \sum_{x=0}^{\infty} x^n e^{-\lambda}\frac{\lambda^x}{x!}$$

$$= e^{-\lambda}\sum_{x=0}^{\infty} x^n \cdot \frac{\lambda^x}{x!}$$

$$= e^{-\lambda}\left[0 + \sum_{x=1}^{\infty} x^{n-1}\frac{\lambda^x}{(x-1)!}\right]$$

$$= e^{-\lambda}\sum_{x=1}^{\infty} x^{n-1}\frac{\lambda^{x-1}}{(x-1)!}\cdot\lambda$$

$$= \lambda e^{-\lambda}\sum_{x=1}^{\infty} x^{n-1}\frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda\sum_{x=1}^{\infty} x^{n-1}\left(e^{-\lambda}\frac{\lambda^{x-1}}{(x-1)!}\right)$$

$x-1 = k$

$$= \lambda\sum_{k=0}^{\infty}(k+1)^{n-1}\left(e^{-\lambda}\frac{\lambda^k}{k!}\right)$$

$$= \lambda\left(E\left((k+1)^{n-1}\right)\right)$$

Eg Find $E(X^n)$ for all the distributions.

$Var(\alpha X+\beta)$
$= \alpha^2 Var(X) + 0$
$= \alpha^2 Var(X)$

$\left[\begin{array}{c} Var(\beta) \\ = 0 \end{array} \right]$

$Var(\alpha X+\beta)$
$= E\left(\left(E(\alpha X+\beta) - (\alpha X+\beta)\right)^2\right)$
$= E\left(\left((\alpha E(X)+\beta) - (\alpha X+\beta)\right)^2\right)$
$= E\left((\alpha E(X) - \alpha X)^2\right)$
$= \alpha^2 E\left((E(X)-X)^2\right)$
$= \alpha^2 Var(X).$

Functions of R.V. and their distribution

Eg. $X \sim b(n,p)$

$f_X(i) = {}^nC_i \cdot p^i \cdot q^{n-i}, \quad i = 0, 1, \ldots, n$

$f_{X^2}(i) = {}^nC_i \cdot p^i \, q^{n-i} \quad \forall \, i = 0, 1^2, 2^2, 3^2, \ldots, n^2$

Functions of R.V. and their dist. (for Discrete): $\boxed{05\text{-}09\text{-}2024}$

Eg. Let X be a R.V. with known p.m.f. / p.d.f. and $g(x)$ is a function of r.v. X.

Find p.m.f. / p.d.f. / distribution of $g(x)$.

Eg. $X \sim b(n,p)$

Find distribution of X^2.

$f_X(x) = {}^nC_x \, p^x q^{n-x} \quad \forall \, x = 0, 1, \ldots, n$

$f_{X^2}(x) = {}^nC_y \, p^y q^{n-y} \quad \forall \, x = 0, 1^2, 2^2, \ldots, n^2.$

But X^2 does not follow binomial distribution.

Eg. X is r.v. such that X takes value in $\{-1, 0, 1\}$
such that $f_X(x) = \frac{1}{3}$ $\forall$ $x = -1, 0, 1$.
Find the p.m.f. of X^2.

$Y = X^2$ takes values $\{0, 1\}$.

Aim: To find f_Y.

$$f_Y(y) = P(Y=y) \;\; \forall \; y \in \mathbb{R}$$
$$= P(X^2 = y)$$

$$P(X^2=y) = \begin{cases} 0, & \text{if } y \neq 0, 1 \\ 1/3, & y=0 \\ 2/3, & y=1 \end{cases}$$

$P(X^2=0)$ since "$X^2=0$" $\equiv$ "$X=0$".

$\Rightarrow P(X^2=0) = \boxed{P(X=0) = 1/3}$

$P(X^2=1)$, "$X^2=1$" $\equiv$ "$X=1$" $\cup$ "$X=-1$"

$P($"$X=1$" $\cup$ "$X=-1$"$) = P(X=1) + P(X=-1)$

$\Rightarrow \boxed{P(X=1) = 2/3}$

→ Let X be a R.V. and g is a one-to-one function on the values of X. X is discrete.

$f_{g(X)}(y) = f_X(x)$ such that
$y = g(x)$ $\forall x$, i.e., $\forall y$.

$f_X(x) = P(X=x)$
$f_{g(X)}(y) = P(g(X)=y)$

The value is same if "$X=x$" $\equiv$ "$g(X)=y$".
This happens when g is one-to-one values of X.

Functions of R.V. and their distribution (for Continuous)

X is continuous.

Find $f_{g(x)}(y)$ when you know ~~that~~ $f_X(x)$; $y = g(x)$.

Let g is not one-to-one on the values of x.

Eg. x takes values in $[-1,1]$

$g(x) = x^2$

x^2 takes values in $[0,1]$.

To find $f_{g(x)}(y)$ given $f_X(x)$, $y = g(x)$

$$f_{g(x)}(y) = P(g(x) = y) \quad \times$$

Plan: Find $F_{g(x)}(y)$ and differentiate it to get $f_{g(x)}(y)$.

$$F_{g(x)}(y) = P(g(x) \leq y).$$

Finding probability for all $g(x)$ is tough. Here, we only focus on special case, $g(x) = x^2$.

Special case: $g(x) = x^2$.

Assume that X takes values in $[-1,1]$.

$g(x) = x^2$ in $[0,1]$

$P(x^2 \leq y)$, $y[0,1]$

Convert "$x^2 \leq y$" in terms of x.

"$x^2 \leq y$" $\equiv$ "$x \leq |\sqrt{y}|$"

$$F_{X^2}(y) = F(x^2 \leq y)$$

$$= P(-\sqrt{y} \leq x \leq \sqrt{y})$$

$\because$ "$x^2 \leq y$" $\equiv$ "$-\sqrt{y} \leq x \leq \sqrt{y}$"

$$F_{X^2}(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$= P(X \leq \sqrt{y}) - P(X \leq -\sqrt{y})$$

Differentiating $F_{X^2}(y)$ to get f_{X^2}.

$$~~f_{X^2}(y)~~ \quad f_{X^2}(y) = \frac{d}{dy} F_{X^2}(y)$$

$$= \frac{d}{dy}\left(F_X(\sqrt{y}) - F_X(-\sqrt{y})\right)$$

$$= F_X'(\sqrt{y})\frac{1}{2\sqrt{y}} + F_X'(-\sqrt{y})\left(\frac{1}{2\sqrt{y}}\right)$$

$$= \boxed{\frac{1}{2\sqrt{y}}\left(\underbrace{F_X'(\sqrt{y})}_{f_X(\sqrt{y})} + \underbrace{F_X'(-\sqrt{y})}_{f_X(-\sqrt{y})}\right)}$$

Eg. [Chapter-5, Pg. 232]

$$f_X(x) = \begin{cases} 0, & \text{if } x \notin [-1,1] \\ c(1-x^2), & \text{if } x \in [-1,1]. \end{cases}$$

Find c such that $f_X(x)$ is p.d.f.. If corresponding r.v. is X, find density of X.

(dd-mm-2024)

Eg. X is a r.v. such that the p.d.f. of X is given by
$$f_X(x) = \begin{cases} a + bx^2 & \forall\, x \in [0,1] \\ 0, & \text{otherwise.} \end{cases}$$

Given that $E(X) = 3/5$. Find the value of a and b, and find the distribution of X^2.

Soln: Since f_X is p.d.f.,
$$\int_{\mathbb{R}} f_X(x)\,dx = 1$$
$$\Rightarrow \int_0^1 (a+bx^2)\,dx = 1$$
$$\Rightarrow ax\Big|_0^1 + \frac{bx^3}{3}\Big|_0^1 = 1$$
$$\Rightarrow a + \frac{b}{3} = 1 \Rightarrow 3a+b=3.$$

$E(X) = 3/5$
$$\Rightarrow \int_{\mathbb{R}} x f_X(x)\,dx = 3/5$$
$$\Rightarrow \int_0^1 x(a+bx^2)\,dx = 3/5$$
$$\Rightarrow \frac{ax^2}{2} + \frac{bx^4}{4}\Big|_0^1 = 3/5$$
$$\Rightarrow \frac{a}{2} + \frac{b}{4} = \frac{3}{5}$$
$$\Rightarrow 2a + b = 12/5$$
$$\Rightarrow a = 3 - 12/5 = \frac{15-12}{5} = 3/5$$
$$b = 3 - 3a = 3 - 9/5 = \frac{15-9}{5} = \frac{6}{5}$$

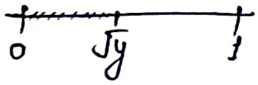
So, $f_X(x) = a + bx^2 \;\forall\, x \in [0,1]$.
where $a = 3/5$, $b = 6/5$.

To find the dist. of X^2,

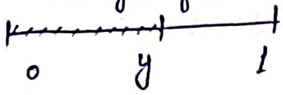
$F_{X^2}(y) = P(X^2 \le y)$

"$X^2 \le y$" $\equiv$ "$X \le \sqrt{y}$"

Range of X



Range of X^2



$F_{X^2}(y) = P(X \le \sqrt{y})$
$= F_X(\sqrt{y})$

Diff. w.r.t. y,

$f_{X^2}(y) = \frac{d}{dy}(F_{X^2}(y))$
$= F_X'(\sqrt{y}) \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{y}}$
$= \frac{1}{2} f_X(\sqrt{y}) \cdot \frac{1}{\sqrt{y}}$

So, $f_{X^2}(y) = \frac{1}{2} \cdot \frac{1}{\sqrt{y}}(a+by) \;\forall\; y \in [0,1]$

Eg. $f_X(x) = cx^2 \;\forall\; x \in [-1,2]$

Find value of c such that f_X is a density and find the p.d.f. of X^2.

Soln: $\int_{\mathbb{R}} f_X(x)\,dx = 1 \Rightarrow \int_{-1}^{2} cx^2\,dx = 1$

$\Rightarrow c\frac{x^3}{3}\Big|_{-1}^{2} = 1$

$= c\,\frac{9}{3} = 1$

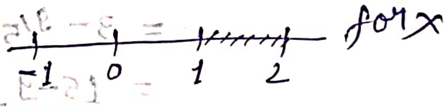
$\Rightarrow c = 1/3.$

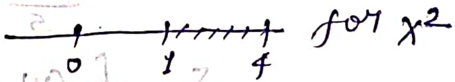
To find p.d.f. of X,

$F_{X^2}(y) = P(X^2 \le y)$

X was in $[-1,2]$

X^2 will be in $[0,4]$.





Case: $y \in [0,1]$

"$X^2 \le y$" $\equiv$ "$-\sqrt{y} \le X \le \sqrt{y}$"

Case: $y \in [1,4]$

"$X^2 \le y$" $\equiv$ "$X \le \sqrt{y}$"

When $y \in [0,1]$,

then $F_{X^2}(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$

$\forall\, y \in [0,1]$

When $y \in [1,4]$,

then $F_{X^2}(y) = P(X^2 \le y)$ [Don't write $= F_X(\sqrt{y})$ directly]

$$
\begin{aligned}
&= \cancel{P(X \le \sqrt{y})} \\
&= \cancel{P(X \in (-\infty, 1)) + P} \\
&= P(0 \le X^2 \le y) \\
&= P(0 \le X^2 \le 1) + P(1 \le X^2 \le y) \\
&= P(-1 \le X \le 1) + P(1 \le X^2 \le y) \\
&= (F_X(1) - F_X(-1)) + P(1 \le X \le \sqrt{y}) \\
&= \cancel{F_X(1)} - \cancel{F_X(-1)}^{\,0} + F_X(\sqrt{y}) - \cancel{F_X(1)} \\
&= F_X(\sqrt{y})
\end{aligned}
$$

11-09-2024

[6th Ed: A 1st course in Prob. (pg. 152)]

Eg. $x \sim b(n, p)$

$E(X^k) = ?$

$E(X^k) = npE((Y+1)^{k-1})$

$Y = b(n-1, p)$

$$E(X^k) = \sum_{i=0}^{n} i^k \, {}^nC_i \, p^i q^{n-i}$$

$= \vdots$

$$= np \sum_{j=0}^{n-1} (j+1)^{k-1} \, {}^{n-1}C_j \, p^j q^{(n-1)-j}$$, j cannot be X $\Rightarrow$ It is Y

$= npE((Y+1)^{k-1})$, $Y \sim b(n-1, p)$

$E(X^0) = E(1) = 1$

$E(X) = npE((y+1)^{1-1})$

$= np$

$E(X^2) = npE((Y+1))$, $Y \sim b(n-1, p)$

$= np[E(n) + E(1)$

$= np[(n-1)p + 1]$

$X \sim p(\lambda)$

$$E(X^k) = \lambda \sum_{x=1}^{\infty} x^{k-1} e^{-\lambda} \frac{\lambda^{x-1}}{(x-1)!}$$

$[y = x-1 \Rightarrow y = 0 \text{ for } x = 1]$

$$= \lambda \sum_{y=0}^{\infty} (y+1)^{k-1} \cdot e^{-\lambda} \frac{\lambda^y}{y!}$$

$= \lambda E((x+1)^{k-1})$, $x \sim p(\lambda)$ [since distribution didn't change, r.v. won't change.]

Aim: Calculating moments $E(X^k)$

$$a_k = E(X^k), \quad k=0,1,\ldots$$

A function that can generate all ~~set~~ such a_k is given by

$$M(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k$$

$$\left(\frac{d}{dt}\right)^n M(t) \Big|_{t=0} = a_n \quad \forall n.$$

Assume all sorts of convergence required.

Defn: $S_n = \sum_{k=0}^{n} \frac{a_k}{k!} t^k$

$$\lim_{n\to\infty} S_n = \sum_{k=0}^{\infty} \frac{a_k}{k!} \cdot t^k$$

What is S_n?

$$S_n = \sum_{k=0}^{n} \frac{E(X^k)}{k!} t^k$$

$$\Rightarrow S_n = E\left(\sum_{k=0}^{n} \frac{x^k}{k!} t^k\right), \text{ as E is a linear function.}$$

What happens when $n \to \infty$ to $\sum_{k=0}^{n} \frac{x^k}{k!} t^k$.

Assume that $\lim_{n\to\infty} \sum_{k=0}^{n} \frac{x^k}{k!} t^k$

$$= \sum_{k=0}^{\infty} \frac{x^k}{k!} t^k = e^{xt}$$

If we further assume that

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} E\left(\sum_{k=0}^{n} \frac{x^k}{k!} t^k\right)$$

$$= E\left(\lim_{n\to\infty} \sum_{k=0}^{n} \frac{x^k}{k!} t^k\right)$$

$$= E\left(e^{tx}\right)$$

So, $M(t) = E(e^{tX})$.

Defn: Let X be a random variable, then the function $M_X(t) = E(e^{tX})$ is called moment generating function of X, or simply m.g.f. of X and it generates moments of X in the following

$$\left(\frac{d}{dt}\right)^n M_X(t) \Big|_{t=0} = E(X^n).$$

Eg. Let $X \sim b(n, p)$

m.g.f. of X is

$$M_X(t) = E(e^{tX})$$

$$= \sum e^{ti} \, {}^nC_i \, p^i \, q^{n-i}$$

$$M_X(t) = \sum_{i=0}^{n} {}^nC_i \, (pe^t)^i \, q^{n-i}$$

$$= (pe^t + q)^n$$

\# $E(1) = 1$

$$E(X) = \frac{d}{dt} M_X(t) \Big|_{t=0}$$

$$= np\, e^t (pe^t + q)^{n-1} \Big|_{t=0}$$

$$= np(p+q)$$

$$= np.$$

\# Pg. 374 & 375
↳ M.V & m.g.f. of all dist. (ch. 7)

Continuous Probability Distribution

	p.d.f, $f(x)$	m.g.f., $(M(t))$	Mean	Variance
Uniform over (a,b)	$f(x)=\begin{cases}\frac{1}{b-a}, & a<x<b\\ 0, & \text{otherwise}\end{cases}$	$\frac{e^{tb}-e^{ta}}{t(b-a)}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential with parameter, $\lambda>0$	$f(x)=\begin{cases}\lambda e^{-\lambda x}, & x\geq 0\\ 0, & x<0\end{cases}$	$\frac{\lambda}{\lambda-t}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gamma with parameters (s,λ), $\lambda>0$	$f(x)=\begin{cases}\frac{\lambda e^{-\lambda x}(\lambda x)^{s-1}}{\Gamma(s)}, & x\geq 0\\ 0, & x<0\end{cases}$	$\left(\frac{\lambda}{\lambda-t}\right)^s$	$\frac{s}{\lambda}$	$\frac{s}{\lambda^2}$
Normal with parameters μ,σ^2	$f(x)=\frac{1}{\sqrt{2\pi}\,\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $-\infty<x<\infty$	$\exp\left(\mu t+\frac{\sigma^2 t^2}{2}\right)$	μ	σ^2

Discrete Probability Distribution

	p.m.f, $f(x)$	m.g.f., $M(t)$	Mean	Variance
Binomial with parameter n,p; $0\leq p\leq 1$	${}^nC_x\, p^x(1-p)^{n-x}$, $x=0,1,\dots,n$	$(pe^t+1-p)^n$	np	$np(1-p)$
Poisson with parameter $\lambda>0$	$e^{-\lambda}\frac{\lambda^x}{x!}$, $x=0,1,2,\dots$	$\exp[\lambda(e^t-1)]$	λ	λ
Geometric with parameter $0\leq p\leq 1$	$p(1-p)^{x-1}$, $x=0,1,2,\dots$	$\frac{pe^t}{1-(1-p)e^t}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Negative binomial with parameters r,p; $0\leq p\leq 1$	${}^{n-1}C_{r-1}\, p^r(1-p)^{n-r}$, $n=r,r+1,\dots$	$\left[\frac{pe^t}{1-(1-p)e^t}\right]^r$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$

24-10-2024

$P(A \cap B) = P(A).P(B)$, A & B are independent.

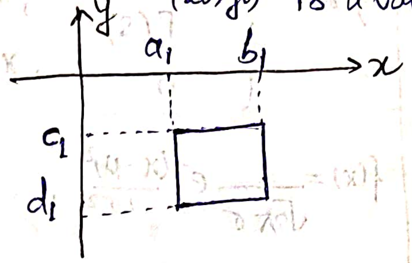
X and Y are r.v. such that all events related to X are independent of all events related to Y and vice-versa, then X & Y are said to be independent.

$$F_X(x) = P(X \le x)$$

X, Y are r.v.s.

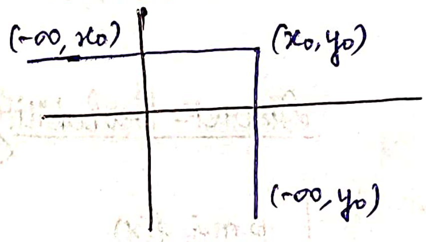
→ Building blocks for computing probability lying inside some rect. $P((x,y) \in \square)$

• (x_0, y_0) is a value of (X, Y).



$P((x,y) \in (-\infty, x_0) \times (-\infty, y_0))$

↑
Building blocks



So, corresponding to $F_X(x) = P(x \in (-\infty, x))$ shall have

$$F_{XY}(x,y) = P((x,y) \in ((-\infty, x) \times (-\infty, y)))$$

is called a joint distribution of (X,Y).

$$F_{XY}(x,y) = P(X \le x, Y \le y)$$

Suppose X and Y are independent

Then, $F_{X,Y}(x,y) = P(X \le x, Y \le y)$
$= P(X \le x).P(Y \le y)$
$= F_X(x).F_Y(y)$

$F_X(x) = P(X \le x)$
$= \int_{-\infty}^{x} f_X(x).dx$

If X and Y are continuous, then $\exists\ f_{X,Y} : \mathbb{R}^2 \to \mathbb{R}^2$ called joint density of (X,Y).

$$F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{XY}(x,y)\,dx\,dy$$

and in general, if $G \subset \mathbb{R}^2$,

$$P((x,y)\in G) = \iint_{(x,y)\in G} f_{XY}(x,y)\,dx\,dy$$

If X and Y are independent, then $F_{XY} = F_X F_Y$

$\boxed{f_{XY} = f_X f_Y}$ → condn for independence of X & Y

Suppose

$X_1 \sim N(\mu_1, \sigma_1^2)$

$X_2 \sim N(\mu_2, \sigma_2^2)$

$X_1 + X_2$

$M_{X_1}(t)$

$M_{X_2}(t)$

$$M_{X_1}(t) + M_{X_2}(t) = E\left(e^{(X_1+X_2)t}\right)$$

$$= \int e^{(x_1+x_2)t} f_{X_1,X_2}(x_1,x_2)\,dx_1\,dx_2$$

if X_1 & X_2 are indep: $\displaystyle\iint_{\mathbb{R}} e^{tX_1} e^{tX_2} f_{X_1}(x_1) f_{X_2}(x_2)\,dx_1\,dx_2$

$$= \int e^{tX_1} f_{X_1}(x_1)\,dx_1 \int e^{tx_2} f_{X_2}(x_2)\,dx_2$$

$$= E(e^{tX_1}) \cdot E(t X_2)$$

$$= M_{X_1}(t) \cdot M_{X_2}(t)$$

In general, $M_{X_1+X_2}(t) = M_{X_1}(t) \cdot M_{X_2}(t)$ [if X and Y are indep.]

$M_{X_1}(t) = e^{(\mu_1 t + \frac{1}{2}\sigma_1^2 t^2)}$

$M_{X_2}(t) = e^{(\mu_2 t + \frac{1}{2}\sigma_2^2 t^2)}$

$M_{X_1+X_2}(t) = e^{(\mu_1+\mu_2)t + \frac{1}{2}t^2(\sigma_1^2+\sigma_2^2)}$

$\boxed{X_1 + X_2 \sim N(\mu_1+\mu_2,\ \sigma_1^2+\sigma_2^2)}$

25-10-2024

→ $X \sim b(n,p)$

$Y \sim b(n,p)$

$X+Y \sim b(n_1+n_2, p)$

$M_X(t) = (pe^t+q)^{n_1}$, $q=1-p$

$M_Y(t) = (pe^t+q)^{n_2}$

$M_{X+Y} = (pe^t+q)^{n_1+n_2}$

$\boxed{X+Y \sim b(n_1+n_2, p)}$.

→ $X_i \sim N(\mu_i, \sigma_i^2)$

$i=1$ and these are independent.

Eg. a_1, a_2, a_3 are ~~cond~~ continuous, then find the distribution of $a_1X_1 + a_2X_2 + a_3$.

Let X be a r.v. and a be a constant.

$$M_{aX}(t) = E(e^{taX}) = M_X(at)$$

$$M_{X+a}(t) = e^{at}M_X(t)$$

$X_i \sim N(\mu_i, \sigma_i^2)$

$a_iX_i \sim N(a_i\mu_i, a_i^2\sigma_i^2)$

$X_i + a_3 \sim N(\mu_i + a_3, \sigma_i^2)$

$a_iX_i \sim N(a_i\mu_i, a_i^2\sigma_i^2)$, a_iX_i are indep. $\forall i$.

$\Rightarrow$ $a_1X_1 + a_2X_2 \sim N(a_1\mu_1 + a_2\mu_2, a_1^2\sigma_1^2 + a_2\sigma_2^2)$,

$a_1X_1 + a_2X_2 + a_3 \sim N(a_1\mu_1 + a_2\mu_2 + a_3, a_1^2\sigma_1^2 + a_2^2\sigma_2^2)$

X_i, $i=1,2,\dots,n$ are iid (independently identically distributed)

e.g., $X_i \sim N(\mu, \sigma^2)$

$$\bar{X} = \sum_{i=1}^{n} X_i/n$$

$$E(\bar{X}) = E\left(\frac{\sum_{i=1}^{n} X_i}{n}\right)$$

$$= \frac{1}{n} E\left(\sum_{i=1}^{n} X_i\right)$$

$$= \frac{1}{n} \sum_{i=1}^{n} E(X_i) = \frac{1}{n} \cdot n\mu = \mu$$ ⇝ constant wrt. n

↳ Comes with error

$$\mathrm{Var}(\bar{X}) = \frac{1}{n^2} n\sigma^2$$

$$= \frac{\sigma^2}{n}$$ ⇝ varies with n

$n \to \infty$, $\mathrm{var}(\bar{X}) \to 0$. $\Big\} \Rightarrow \bar{X}_i \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

If X_i are not independent, then also this holds.

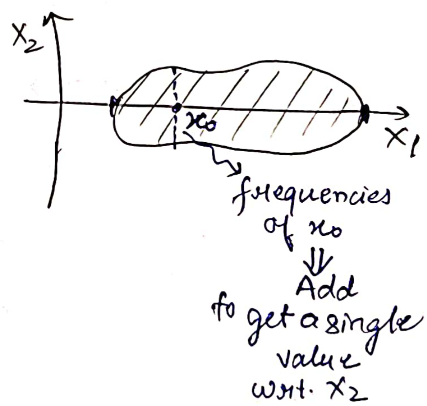
For $i = 1, 2$,

$$E(X_1 + X_2) = \iint_{\mathbb{R}^2} (x_1 + x_2) \cdot f_{X_1 X_2}(x_1, x_2)\, dx_1\, dx_2$$

$$= \iint_{\mathbb{R}^2} x_1 f_{X_1 X_2}(x_1, x_2)\, dx_1\, dx_2 + \iint_{\mathbb{R}^2} x_2 f_{X_1 X_2}(x_1, x_2)\, dx_1\, dx_2$$

$$= \int x_1 \underbrace{f_{X_1}(x_1)}_{\text{marginal distribution}}\, dx_1 + \int x_2 f_{X_2}(x_2)\, dx_2$$

[x_2 got integrated]



$$= E(X_1) + E(X_2)$$