MA311 – 2024

MA 311 Mid Term Exam

Indian Institute of Space Science and Technology

Mid Term Examination - September 2024

B.Tech 5th Semester

MA 311 - Probability, Statistics and Numerical Methods

Date : 23rd Sept, 2024 **Time: 02.00 pm to 04.00 pm** **Maximum Marks: 30**

May attempt all questions

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1. Two coins are flipped. The first coin will land on heads with probability 0.6, the second with probability 0.7. Assume that the results of the flips are independent, and let X counts the total number of heads that result. Find the probability mass function of X. [4]

2. An airport has problems with birds. If the weather is sunny, the probability that there are birds on the runway is $1/2$; if it is cloudy, but dry, the probability is $1/3$; and if it is raining, then the probability is $1/4$. The probability of each type of the weather is $1/3$. Given that the birds are on the runaway, with explanation [2+2]

   (a) find the probability that the weather is sunny.

   (b) find the probability that the weather is cloudy (dry or rainy).

3. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a pdf of a continuous random variable X defined by $f(x) = ax^2 + bx + c$ for all $x \in (0,\ 1)$ and $f(x) = 0$ otherwise where $a, b, c \in \mathbb{R}$. Given that the 1st and 2nd moment of X are ~~2~~ 1/4 and ~~9~~ 1/10 respectively, find the values of a, b and c. [4]

4. With explanation, build a pmf of some discrete random variable such that the pmf takes countably infinitely many values and the pmf is not of any well know distribution. [4]

5. Let $X \sim p(\lambda)$. Show that $E(X^n) = \lambda E((X+1)^{n-1})$ for any integer $n \geq 0$. Hence find the mean and variance of X. [2+2]

6. [~~3+1~~ 2+2]

   (a) If $Z \sim N(0, 1)$, for $x > 0$, show that $P(|Z| < x) = 2P(Z < x) - 1$.

   (b) If $X \sim U(-1, 3)$, find $P(|X| > 2)$.

7. Computer the mgf of a $N(\mu, \sigma^2)$-variate. [4]

8. The mgf of a random variable X is given by $M_X(t) = e^{2e^t - 2}$ and of Y is given by $M_Y(t) = (\frac{2}{3}e^t + \frac{1}{3})^5$. If the events related to X are independent of the events related to Y and vice versa, find $P(X + Y = 2)$. [4]

9. [3+1]

   (a) Establish Markov's inequality: for a random variable X taking non-negative values, we have $P(X \geq a) \leq \dfrac{E(X)}{a}$ for any chosen $a > 0$.

   (b) Suppose that it is known that the number of items produced in a factory during a week is a random variable X with mean 50. Give a bound to the probability $P(|X| > 75)$.

*****END*****

Q.1 Let H_{C_i}, T_{C_i} represents the event getting head of ith. coin & tail of ith coin resp.

0.5

Given that $P(H_{C_1}) = 0.6$

$P(T_{C_1}) = 0.4$

$P(H_{C_2}) = 0.7$, $P(T_{C_2}) = 0.3$.

0.5

Let X events the no. of heads when C_1 & C_2 are flipped. Then X takes values 0, 1 and 2.

1

$P(X=0) = P(T_{C_1} \cap T_{C_2}) = P(T_{C_1}) \cdot P(T_{C_2})$ (Since the coins are flipped independently, the corr. events are indep)

$= 0.4 \times 0.3$

$= 0.12$

1

$P(X=1) = P((H_{C_1} \cap T_{C_2}) \cup (H_{C_2} \cap T_{C_1}))$

$= P(H_{C_1} \cap T_{C_2}) + P(H_{C_2} \cap T_{C_1})$

(Since the $H_{C_1} \cap T_{C_2}$ & $H_{C_1} \cap T_{C_1}$ are exclusive)

$= P(H_{C_1}) P(T_{C_2}) + P(H_{C_2}) \cdot P(T_{C_1})$

(Since the coins are flipped indep)

$= 0.6 \times 0.3 + 0.7 \times 0.4$

$= 0.18 + 0.28 = .46$.

0.5 Similarly,

$$P(X=2) = P(H_{C_1} \cap H_{C_2}) = P(H_{C_1}) \cdot P(H_{C_2})$$
$$= 0.6 \times 0.7$$
$$= 0.42.$$

0.5 So the prob. mass f^n of X is given by

$$f(x) = \begin{cases} 0.12 & \text{if } x=0 \\ 0.46 & \text{if } x=1 \\ 0.42 & \text{if } x=2 \\ 0 & \text{otherwise.} \end{cases}$$

Q2: S: Weather is sunny.
B: Birds are on the runway. | C: cloudy weather
CD: cloudy weather, but dry
R: Raining (ofcourse cloudy)

0.5 Given that $P(B|S) = 1/2$, $P(B|CD) = 1/3$
and $P(B|R) = 1/4$; $P(S) = P(CD) = P(R) = 1/3$.

It is given that the birds are on the runway.

(a) 0.5 Need to find $P(S|B) = \dfrac{P(S \cap B)}{P(B)}$

$$= \frac{P(S) \cdot P(B|S)}{P(B)}.$$

0.5 Now $P(B) = P((B\cap S) \cup (B \cap CD) \cup (B \cap R))$

$$= P(B\cap S) + P(B\cap CD) + P(B \cap R)$$
$$= P(S) \cdot P(B|S) + P(CD) \cdot P(B|CD) + P(R) \cdot P(B|R)$$
$$= (1/2 + 1/3 + 1/4) \cdot 1/3 = \frac{6+4+3}{12} \times \frac{1}{3} = \frac{13}{12} \times \frac{1}{3}$$

0.5 So, $P(S|B) = \dfrac{\frac{1}{3} \times \frac{1}{2}}{\frac{13}{12} \times \frac{1}{3}} = \dfrac{6}{13}$.

(b) Again $P(C|B) = P(CD \cup R | B)$

1 $= P(CD|B) + P(R|B)$ [CD & R are exclusive]

$= \dfrac{P(CD \cap B)}{P(B)} + \dfrac{P(R \cap B)}{P(B)}$

0.5 $= \dfrac{P(CD) \cdot P(B|CD) + P(R) \cdot P(B|R)}{P(B)}$

0.5 $= \frac{1}{3}\left(\frac{1}{3} + \frac{1}{4}\right) \Big/ \frac{13}{12} \times \frac{1}{3}$

$= 7/13$.

Alt: $P(C|B) = 1 - P(S|B)$

$= 1 - \frac{6}{13} = \frac{7}{13}$

as $P(\cdot|B)$ is a prob f^n

Q.3. given that the density f^n is

$f(x) = ax^2 + bx + c \quad \forall x \in [0,1]$.

s.t $\int_0^1 x f(x)\, dx = 1/4$ ...(1) and $\int_0^1 x^2 f(x)\, dx = 1/10$ ...(2)

Since f is a density, $\int_0^1 f(x)\, dx = 1$. — (3)

So, we have the following three eq^n:

3
$3a + 4b + 6c = 3$ — from (1)

$12a + 15b + 20c = 6$ — from (2)

$2a + 3b + 6c = 6$

1 One can find that $a = 3,\ b = -6,\ c = 3$.

So, the density f^n is $f(x) = 3x^2 - 6x + 3 \ \forall x \in [0,1]$

Q4: ① Let $x_{2n+1} = \frac{1}{2^{n+2}} \quad \forall n \geq 0$

$x_{2n} = \frac{1}{3^n} \quad \forall n \geq 1$

Then $\sum x_n = \sum_{n=1}^{\infty} \frac{1}{2^n} + \sum_{n=1}^{\infty} \frac{1}{3^n}$

① $= \frac{1}{4} \cdot \frac{1}{1-1/2} + \frac{1}{3} \cdot \frac{1}{1-1/3}$

$= 1/2 + \frac{1}{3} \cdot \frac{3}{2}$

$= 1.$

① So define the p.m.f as $f(n) = \begin{cases} \frac{1}{2^{n+2}} & \forall n \geq 0 \\ \frac{1}{3^n} & \forall n \geq 1 \\ 0 & \text{otherwise.} \end{cases}$

① clearly $f(n) \geq 0$ & $\sum_{n=1}^{\infty} f(n) = 1.$

Q.5: $X \sim p(\lambda)$; $f_X(x) = \frac{e^{-\lambda} \lambda^x}{x!} \quad \forall x = 0, 1, \ldots$

0.5 So, $E(X^n) = \sum_{x=0}^{\infty} x^n e^{-\lambda} \frac{\lambda^x}{x!} = \sum_{x=1} x^n e^{-\lambda} \frac{\lambda^x}{x!}$

$= \sum_{x=1}^{\infty} x^{n-1} e^{-\lambda} \frac{\lambda^{x-1} \cdot \lambda}{(x-1)!}$

$E(X^n) = \lambda \sum_{x=1}^{\infty} x^{n-1} e^{-\lambda} \frac{\lambda^{x-1}}{(x-1)!} \quad \text{---} (1)$

0.5 Let $y = x-1$. then for $x=1$, we have $y=0$ So, we get from (1) the following.

0.5 $$E(X^n) = \lambda \sum_{y=0}^{\infty} (y+1)^{n-1} e^{-\lambda} \frac{\lambda^y}{y!}$$

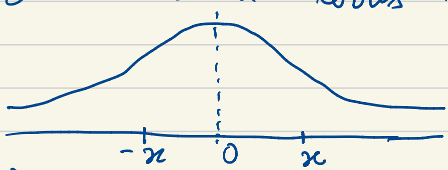
0.5 Since $e^{-\lambda} \frac{\lambda^y}{y!}$ $\forall y = 0, 1 \ldots$ is p.m.f $P(\lambda)$,
we get
$$E(X^n) = \lambda E\left((X+1)^{n-1}\right) \quad \text{where } X \sim P(\lambda).$$

1 So mean of X is $E(X) = \lambda E\left((X+1)^0\right) = \lambda E(1)$
$= \lambda.$

1 & $$\begin{aligned} \operatorname{var}(X) &= E(X^2) - (E(X))^2 \\ &= \lambda E(X+1) - \lambda^2 \\ &= \lambda(E(X) + E(1)) - \lambda^2 \\ &= \lambda(\lambda+1) - \lambda^2 = \lambda. \end{aligned}$$

So mean is λ & variance is λ.

Q6: $Z \sim N(0,1)$. We know that $E(Z) = 0$ & $\operatorname{Var}(Z) = 1$.
0.5 further the density of Z is symmetric around $z = 0$. & looks like the following.



Let $x > 0$, then $P(|Z| < x)$
$$= P(-x < Z < x)$$
$$= P(-x < Z < 0) + P(0 < Z < x)$$

0.5 $= 2P(0 < Z < x)$ since the density is symmetric around 0.

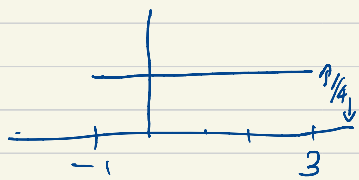
0.5 $= 2\left(\left[P(-\infty < Z < 0) + P(0 < Z < x)\right] - P(-\infty < Z < 0)\right)$

0.5 $= 2\left(P(-\infty < Z < x) - P(-\infty < Z < 0)\right)$

$= 2\left(P(Z < x) - 1/2\right)$ [Since the density is sym. around 0]

$= 2P(Z < x) - 1.$

Q6.(b) Let $X \sim U(-1, 3)$.



1 then $f_X(x) = \frac{1}{4} \quad \forall\, x \in (-1, 3)$.

1 So, $P(|Z| > 2) = P(Z < -2) + P(Z > 2)$

$= 0 + \int_2^\infty f_X(x)\,dx$

$= \int_2^3 1/4\,dx = 1/4.$

So $P(|Z| > 2) = 1/4$

Q7: Let $X \sim N(\mu, \sigma^2)$. then $f_X(x) = \frac{1}{\sqrt{2}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad \forall\, x \in \mathbb{R}.$

1 $M_X(t) = E(e^{tX})$

$= \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sqrt{2}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\,dx$

1

Let $\frac{1}{\sqrt{2}} \frac{x-\mu}{\sigma} = u$. $\Rightarrow x-\mu = \sqrt{2}\sigma u$

as $x \in (-\infty, \infty)$

$u \in (-\infty, \infty)$.

So, $M_X(t) = \int_{-\infty}^{\infty} e^{t(\sqrt{2}\sigma u + \mu)} \cdot e^{-u^2} \cdot du$

$$= e^{t\mu} \int_{-\infty}^{\infty} e^{-(-\sqrt{2} t\sigma u + u^2)} du$$

$$= e^{t\mu} \int_{-\infty}^{\infty} e^{-(u^2 - 2\cdot\frac{\sigma}{\sqrt{2}} t\cdot u + \frac{1}{2}\sigma^2 t^2 - \frac{1}{2}\sigma^2 t^2)} du$$

1

$$= e^{\mu t} \cdot e^{-\frac{1}{2}\sigma^2 t^2} \int_{-\infty}^{\infty} e^{-(u - \frac{1}{\sqrt{2}}\sigma t)^2} du$$

1

$$= e^{\mu t + \frac{1}{2}\sigma^2 t^2} \cdot 1.$$

$$= e^{(\mu t + \frac{1}{2}\sigma^2 t^2)}.$$

So, $M_X(t) = e^{-(\mu t + \frac{1}{2}\sigma^2 t^2)}$.

Q8. Let X & Y are r.v.s s.t

$$M_X(t) = e^{2e^t - 2}$$

1

& $M_Y(t) = \left(\frac{2}{3} e^t + \frac{1}{3}\right)^5$

Then $X \sim P(2)$ & $Y \sim b(5, 2/3)$.

1 Now, $P(X+Y=2) = P(X=0, Y=2) + P(X=1, Y=1) + P(X=2, Y=0)$

1 Since the events of X & Y are indep we have -

$P(X+Y=2) = P(X=0)\cdot P(Y=2) + P(X=1)\cdot P(Y=1) + P(X=2, Y=0)$

1
$$= e^{-2}\frac{2^0}{1}\cdot {}^5C_2\left(\frac{2}{3}\right)^2\cdot\left(\frac{1}{3}\right)^3 + e^{-2}\frac{2^1}{1}\cdot {}^5C_1\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^4 + e^{-2}\frac{2^2}{2!}\,{}^5C_0\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^5$$

$$= e^{-2}\left({}^5C_2\left(\frac{2}{3}\right)^2\cdot\left(\frac{1}{3}\right)^3 + 2\,{}^5C_1\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^4 + 2\left(\frac{1}{3}\right)^5\right).$$

9.(a) Given that X is a non-neg r.v. Let $a > 0$

So 1
$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^{\infty} x f_X(x) + \int_{-\infty}^{0} x f_X(x)$$

$$= \int_0^{\infty} x f_X(x)\,dx$$

1
$$= \int_0^a x f_X(x)\,dx + \int_a^{\infty} x f_X(x)\,dx$$

$$\geq \int_a^{\infty} x f_X(x)\,dx \quad \text{since } x \geq 0$$

$$\geq \int_a^{\infty} a f_X(x)\,dx \quad \text{since } x \geq a$$

$$= a P(X \geq a).$$

1 So, $P(X \geq a) \leq E(X)/a$.

9. (b) X is given to be non-neg. r.v. s.t $E(X) = 50$

So, $P(X \geq 75) \leq \dfrac{E(X)}{75}$ by Markov's

$= \dfrac{\cancel{50}\,2}{\cancel{75}\,3}$

So $P(X \geq 75) \leq 2/3$.

1