**INDIAN INSTITUTE OF SPACE SCIENCE AND TECHNOLOGY**
**THIRUVANANTHAPURAM 695 547**

## End Semester Examination - May 2015

**B. Tech - II Semester**

**MA121 - Vector Calculus and Differential Equations**

**Date: 13/05/2015** **Time: 1.30 pm - 4.30 pm** **Max. Marks: 100**

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## SECTION A ( Attempt all 10 questions - 10x5= 50 marks.)

1. Is it true that if each $f_n$ and the point wise limit function $f(x) = \lim_{n\to\infty} f_n(x)$ are continuous, then $f_n$ converges to $f$ uniformly. Justify your answer with an example. [5]

2. Show that the series $f(x) = \displaystyle\sum_{n=0}^{\infty} e^{-nx}\cos nx$ converges uniformly on $0 < x < \infty$. Also check whether the following term by term derivative of the series can be justified: [5]
$f'(x) = -\displaystyle\sum_{n=0}^{\infty} ne^{-nx}(\cos nx + \sin nx)$.

3. Define directional derivative of a function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ at a point $P_0$ along a vector $\vec{v}$. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by $f(x, y) = \sqrt{x^2 + y^2}$. Let $\vec{v}$ be a **unit vector** in $\mathbb{R}^2$; does $D_{\vec{v}}(f)|_{(0,0)}$ exist? Is $f$ differentiable at $(0, 0)$? [3+2].

4. Define arc length function of a smooth curve $C : \vec{\gamma}(t),\ t \in [a, b]$ with initial point $\gamma(a)$. Let $C$ be a curve given by $C : x^2 + y^2 = 1,\ y = x \tan z$ with initial point $(1, 0, 0)$. Give a parametrization to the curve. Find the "arc length function" of the curve. [2+3]

5. Let $\vec{\gamma} : [a, b] \longrightarrow \mathbb{R}^3$ be a non-constant smooth curve. Show that $\vec{\gamma}\,'(t) \neq \vec{0}$ for all $t \in [a, b]$. Express $-\vec{\gamma}$ in terms of $\vec{\gamma}$. Is $-\vec{\gamma}$ a smooth curve? Justify your answer. [3+1+1]

6. Using Green's theorem find the area of the region bounded by $x^2 + y^2 = 4,\ y \geq 0$; $x - y - 2 = 0$; and $x + y + 2 = 0$ on the $XY$-plane. [5]

7. Does Picard theorem guarantee that there exist an interval on which the following initial value problems have a unique solution? If yes, find it and also justify your answer. [5]

   (a) $\dfrac{dy}{dx} = y^{\frac{1}{2}}, \quad y(0) = 0.$

   (b) $\dfrac{dy}{dx} = x^2|y|, \quad y(0) = \dfrac{1}{2}.$

8. Find the general solution of $y'' - f(x)y' + [f(x) - 1]y = 0$, where $f$ is a continuous function on $\mathbb{R}$. [5]

9. Using method of undetermined coefficients, find particular solution of the following differential equations. [5]

   (a) $y'' + y = \sin x$.

   (b) $y''' + 2y'' - y' = 3x^2 - 2x + 1$.

10. Find the first three terms of the Legendre series of the following functions. [5]

(a) $f(x) = \begin{cases} 0, & -1 \leq x < 0, \\ x, & 0 \leq x \leq 1. \end{cases}$

(b) $f(x) = e^x$.

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**SECTION B ( Attempt any 5 questions - 5x10= 50 marks.)**

11. Let $f_n(x) = n^c x(1-x^2)^n$ for $x$ real and $n \geq 1$.

(a) Prove that $\{f_n\}$ converges pointwise on $[0,1]$ for every real $c$. [2]

(b) Find the values of $c$ for which the convergence is uniform on $[0,1]$. [4]

(c) Also determine the values of $c$ for which the following is true [2]

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 \lim_{n\to\infty} f_n(x)dx.$$

(d) Suppose $\{g_n\}$ is a sequence of continuous functions satisfying the integral identity given in question (c). Does it imply the uniform convergence of $\{g_n\}$? [2]

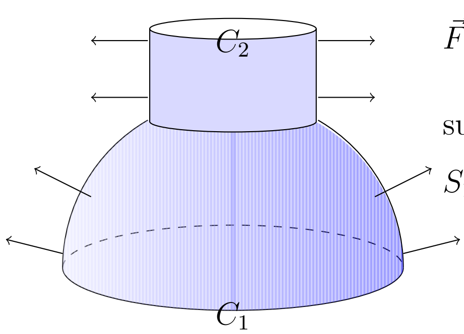
12. (a) Check whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$ converges uniformly on $\mathbb{R}$. [3]

(b) Prove the following with appropriate justification: $\displaystyle\int_0^1 \sum_{n=1}^{\infty} \frac{x}{(n+x^2)^2} dx = \frac{1}{2}$. [7]

13. (a) Let $\vec{F}\,(x,y,z) = (ye^z + z\cos x,\ xe^z + 1,\ xye^z + \sin x + 1)$ be a vector field and $C$ a curve defined by $C = C_1 * C_2 * C_3$ where $C_1$ is the straight line segment $(0,0,0) \longrightarrow (\pi, 0, -\pi)$; $C_2 : \gamma(t) = (\pi+t,\ (\pi+t)\sin(\pi+t),\ (\pi+t)\cos(\pi+t))$ where $t \in [0,\pi]$; and $C_3$ is the straight line segment $(2\pi, 0, 2\pi) \longrightarrow (1,0,0)$. Evaluate $\displaystyle\int_C \vec{F}$, if exists. Is the integral path independent? Justify your answer. [3+2]

(b) Suppose that at time $t = 0$ a particle is ejected from the surface $x^2 + y^2 - z^2 = -1$ at the point $(1, 1, \sqrt{3})$ along the normal to the surface, which is directed toward the $xy$ plane with a speed of 10 units per second. When and where does it cross the $xy$ plane? [5]

14. (a) Verify multiply connected version of Stoke's theorem for the vector field




$\vec{F}(x,y,z) = \left( \dfrac{x}{x^2+y^2}, \dfrac{y}{x^2+y^2}, e^z \right)$ over the surface $S = S_1 \sqcup S_2$ where

$S_1 : x^2 + y^2 + z^2 = 5,\ 0 \leq z \leq 2$ and

$S_2 : x^2 + y^2 = 1,\ 2 \leq z \leq 4$ with outward orientation (see corresponding figure). [5]

(b) Let $\vec{F}$ be a smooth vector field with domain $\mathbb{R}^2 - \{(0,0)\}$ having Curl zero. Let $C_1$ and $C_2$ be any two positively oriented loops on $XY$-plane around $(0,0)$ such that $C_1$ and $C_2$ do not intersect each other (e.g. concentric circles). Using **Green's theorem on simply-connected domains** show that the line integrals of $\vec{F}$ along $C_1$ and along $C_2$ are the same. [5]

15. Find the general solution of the following differential equation

$$x^2y'' + (x^2 - 3x)y' + 3y = 0.$$

[10]

16. Consider the following second order differential equation

$$y'' + P(x)y' + Q(x)y = R(x), x \in I = (-1, 1). \quad (1)$$

Then prove or disprove the following for arbitrary real constants $c_1$ and $c_2$:

(a) If $R(x) = 0 \quad \forall x \in I$, $y_1$ and $y_2$ are any two solutions of (1), then $c_1y_1 + c_2y_2$ is the general solution of (1). [3]

(b) If $R(x) \neq 0$ for some $x \in I$, $y_1$ and $y_2$ are any two solutions (1), then $c_1y_1 + c_2y_2$ is the general solution of (1). [3]

(c) Let $R(x) = 0 \quad \forall x \in I$, then there exist $P$ and $Q$ such that $x^2$ and $x|x|$ are solutions of (1). Justify your answer. [4]

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