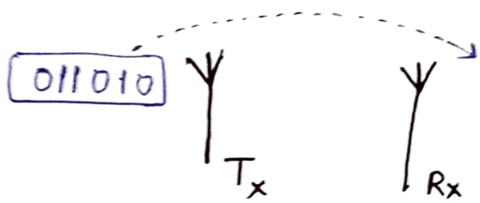
06-01-2025

AV 323 - Communication systems II

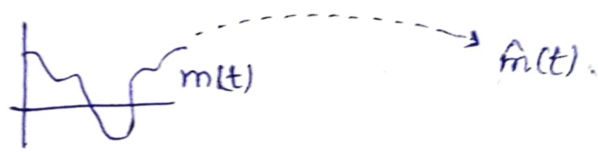
AV 342 - Lab - Software (MATLAB + Python)

This is Digital Communication Systems.

Point-to-point system.



Earlier



Pre-requisites:

① AV 314 – Comm. system I
- DSBSC
- FM
- Random Processes
  Gaussian Processes
  WSS, $S_x(f).|H(f)|^2 = S_y(f)$
- PLLs
- Complex BB

② DSP
- Filters
- Common Signal Processing
- Sampling

References:

① Lecture Notes (Write Your Own !!)

② Bernard Sklar – "Digital Comms"

3. Upamanyu Madhow –
   "Intro. to Comm. System"
4. Simon Haykin – "Communication Systems"
5. B.P. Lathi – "Modern Analog, Digital Comm. Systems"

<u>Course Evaluation</u> :

- Midterm (30) + Finals (50)
- Internals : 20
  - 10 Class Tests + (Surprise Tests)
  - 10 Assignments

What is Digital Communication?

Bits : Computer-to-computer communication

A sequence of bits at source

$\boxed{011010}$ Tx

Rx $\boxed{01\boxed{0}010}$

Rx ⇝ Makes bits-estimation
↳ Make sure that bits are received correctly!

The digital communication problem is to transmit those bits from source to sink such that the bit sequence is received correctly.

$B_0, B_1, B_2, \ldots, B_N$

$\hat{B}_0, \hat{B}_1, \hat{B}_2, \ldots, \hat{B}_N$

Bit error occurs when $B_i \neq \hat{B}_i$.

$$\boxed{BER = \frac{\sum_{i=0}^{M} \mathbb{I}_{\{\hat{B}_i \neq B_i\}}}{M+1}}$$

$\mathbb{I}$ → Indicator f^n ⇝ like box car f^n
↳ $\begin{cases} 1 & \text{when argument is zero} \\ 0 & \text{otherwise} \end{cases}$

Bit error rates $\leq \epsilon$ (for voice, $\epsilon = 10^{-2} - 10^{-3}$)

$\boxed{011010}$ ⇢ $01\boxed{0}010$ → $BER = \frac{1}{6}$

Simple Digital Comm. System :

Eg. Transmit a bit sequence from one computer to another.

Abstract bit : 0101100 (source)
↓
Stored as sequence in the memory
↓
Convert to time-signal (waveform)
[Can't send bits through the channel]

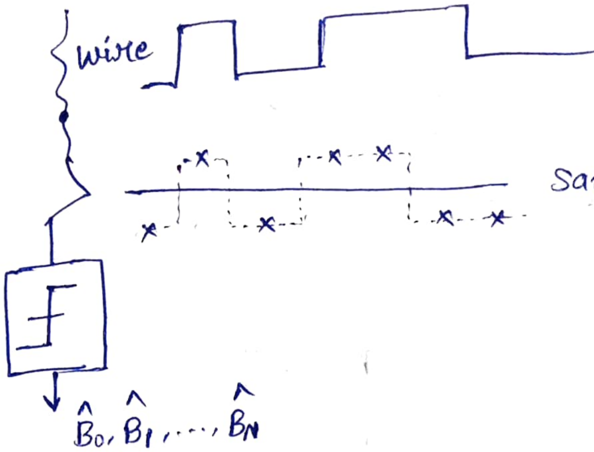
0101100 → Line Coder ↓

→ $m(t)$: Baseband analog signal (waveform)

Bit time (T_b)

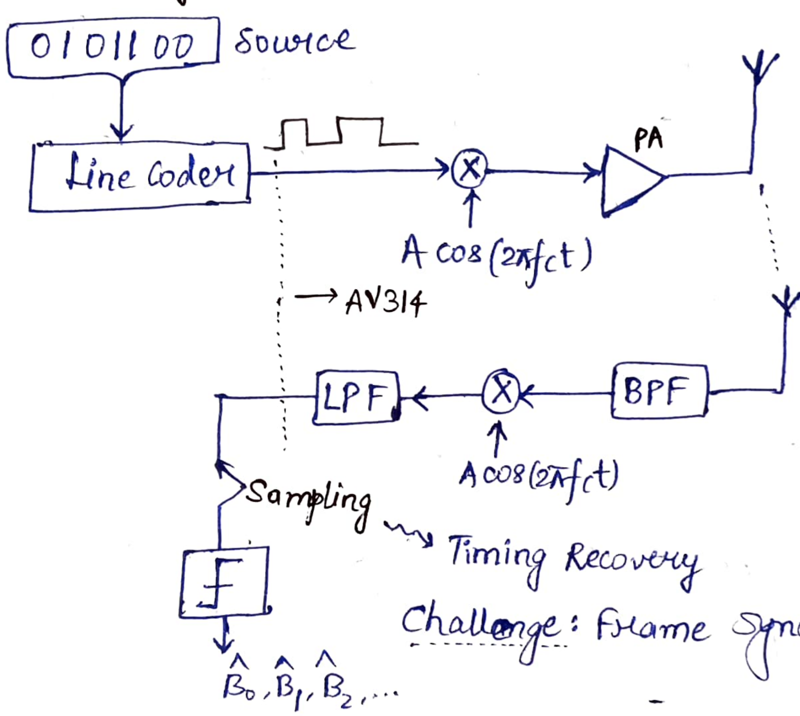
wire

$\boxed{\text{Bit rate} : R_b = 1/T_b}$



Sample at the mid-bit time

A passband system:



Timing Recovery

Challenge: Frame Synchronisation.

08-01-2025

Review

AC Problem: $d(\hat{m}(t), m(t)) < \epsilon$

Source $- m(t)$

$\downarrow$

Destination $- \hat{m}(t)$

$\Big\}$ AM, FM

DC Problem:

Source: sequence of bits

E.g. 0101100

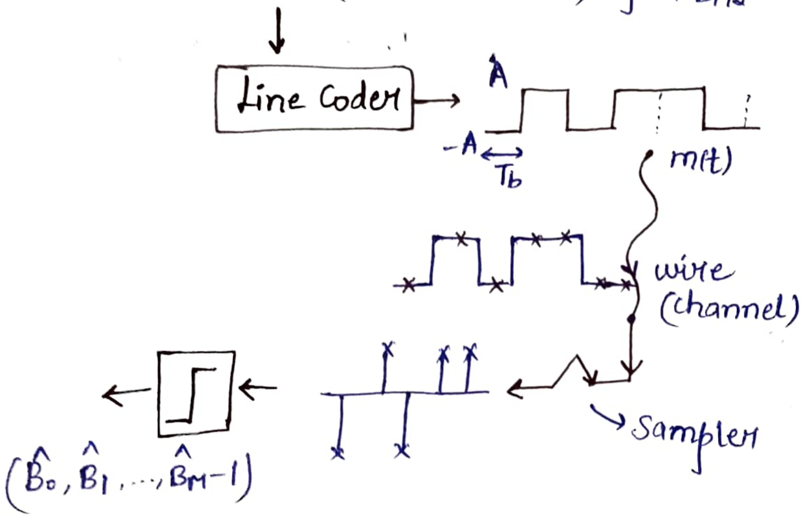
$\downarrow$

Dest.: 0001100

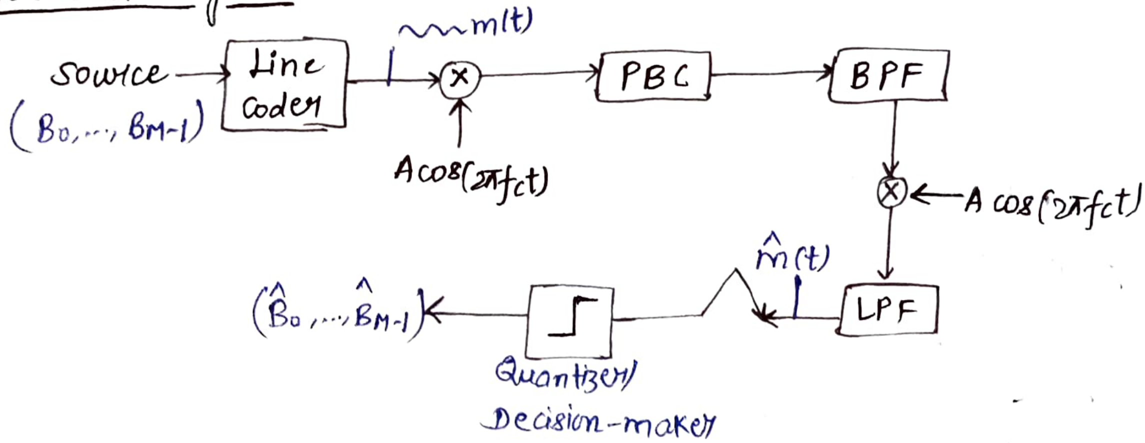
$\Big\}$ BER $\leq \epsilon$

Baseband System

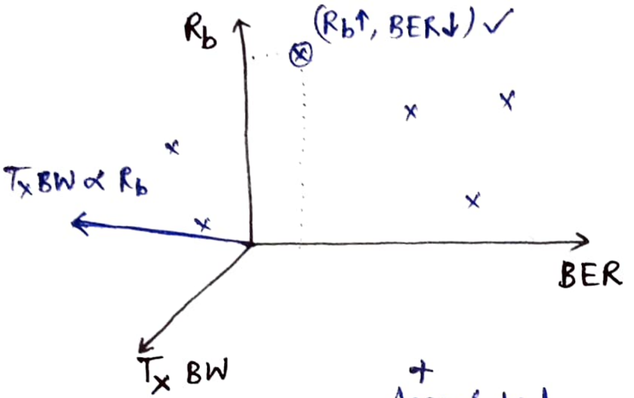
Source has $(B_0, B_1, \dots, B_{M-1})$ $\}$ M bits



Passband System



Tradeoffs



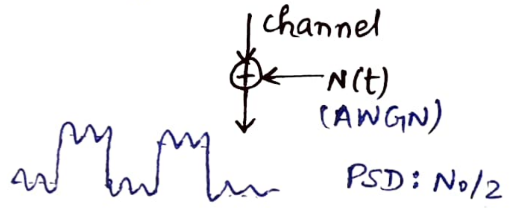
+ Associated metric
power, SNR

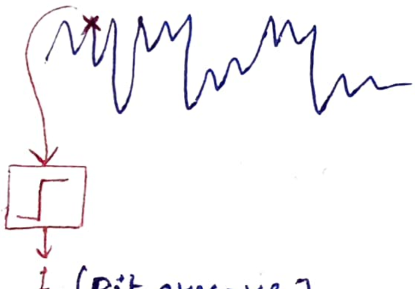
Multiple levels $\Rightarrow$ BER$\uparrow$

Challenges

① Noise

Eg. BB system





(Bit errors)

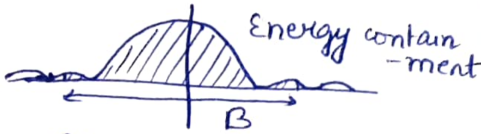
Mitigations:

① Filtering

② Error correction codes
- FEC
- ECC
- Parity

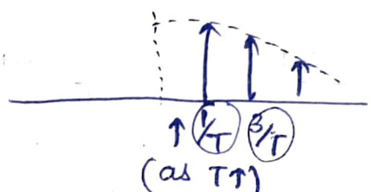
Bit rate, $R_b = 1/T_b$

$x(t) \xleftrightarrow{F} X(f)$



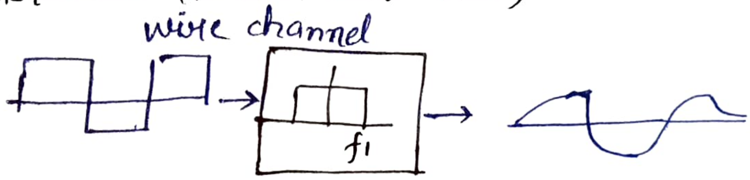
e.g

$\downarrow F$



[Bit rate (R_b) $\uparrow \Rightarrow$ BW $\uparrow$
$= 1/T_b \downarrow$]

② Distortion (Linear distortion)



Mitigation : Equalization
Pulse shaping

Reading Assignment:
Chapter 1 of Sklar

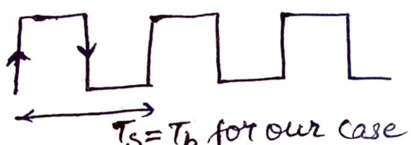
09-01-2025

③ Synchronisation

ⓐ Carrier Synchronisation : Phase of received signal should match with that of local oscillator

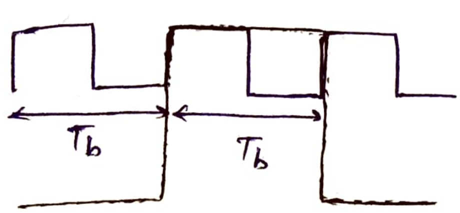
ⓑ Timing sync/recovery

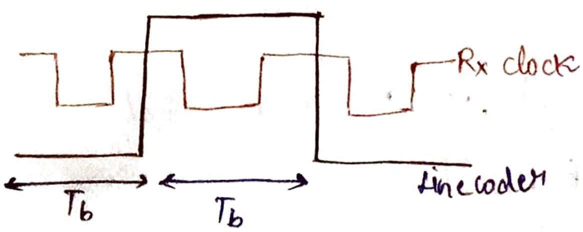
Sampler : Rx should have a clock ⇝ (Generated at the receiver)



- Line code at Tx is generated using a tx clock
- Sampling at Rx is done using a sampler clock.
  ↳ sync problem occurs due to mismatch b/w these clocks.

Tx clock and line code



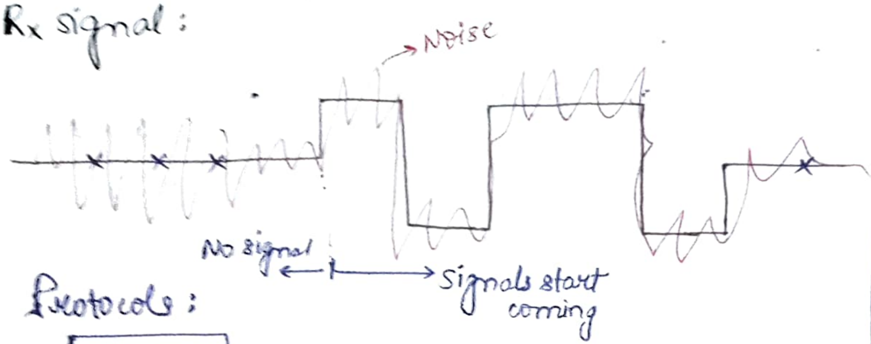


[Rx clock is not sync.]

© Frame sync / recovery

Sequence of bits

Rx signal :



Pulse transition signals the Rx that signal has started ~~too~~ coming

Protocols :

TCP	
IP	Packets
MAC	
PHY	Frames

Reading : Chapter 2 (Sklar)
↳ How we get bits from text/analog signals
↳ Starting with bits

Model this sequence

$(B_0 \dots B_{M-1})$ ⇝ DTRP (Discrete Time Random Process)

↳ Application generates these bits.

DTRP

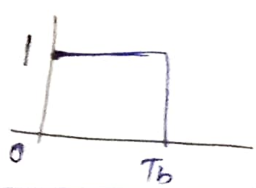
↳ IID Bernoulli Process : compression algorithm (ideally)

↳ Each bits is random and possibility of taking value '1' comes with probability (of 1/2).

$B_i \sim \text{Bernoulli}(1/2)$

Output of the line coder

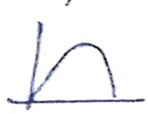
↳ in terms of $p(t)$

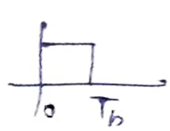


$$\boxed{\sum A_n\, p(t - nT_b)}$$ ⇝ Random signal

Amplitude, $A_n = \begin{cases} +A, & \text{if } B_n = 1 \\ -A, & \text{if } B_n = 0 \end{cases}$

Other possibilities:





[Sklar PCM codes]

13-01-2024

Plan: Study noise and distortion
Assume sync. is achieved.

Baseband Communication

Bits @ source : $(\underbrace{B_0}_{\text{Random Variable}}, B_1, \ldots, B_{M-1})$ → frame

DTRP model:

$\{0,1\} \Rightarrow$ Bernoulli R.V.

$P\{B_i = 1\} = 1/2$

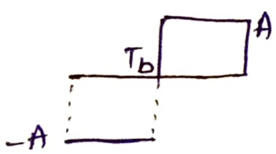
↳ IID R.P.

Sometimes, we will think about an ∞-length R.P.

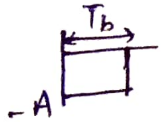
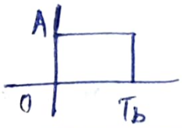
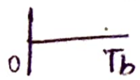
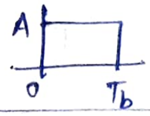
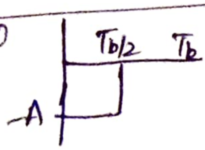
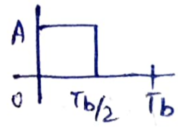
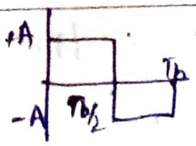
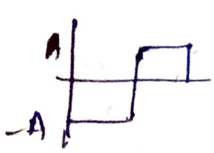
$B_0, B_1, \ldots, B_M$

$\downarrow \quad \downarrow$

$0 \quad 1$

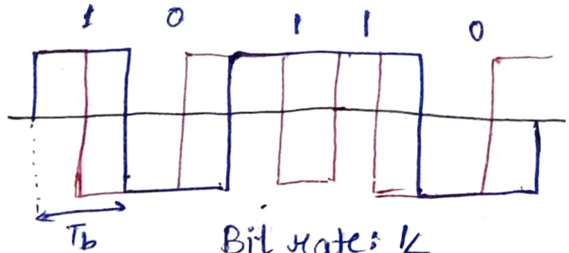


Examples of line codes
Bipolar NRZ
(Not return to zero)

0 	1 	Bipolar NRZ	Transition ↓ BW ↓
0 	1 	Unipolar NRZ	
0 	1 	Bipolar RZ	Transitions ↑ BW ↑ Synch ↑
1 	0 	Manchester Code	

↳ Pulse transitions in the bit is used for synchronization as they help recover the clock.

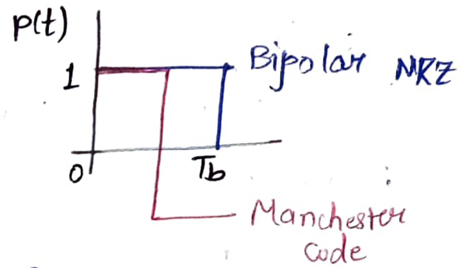
Bit → line coder →



Bit rate: $1/T_b$

$$X(t) = \sum A_i \cdot p(t - iT_b)$$

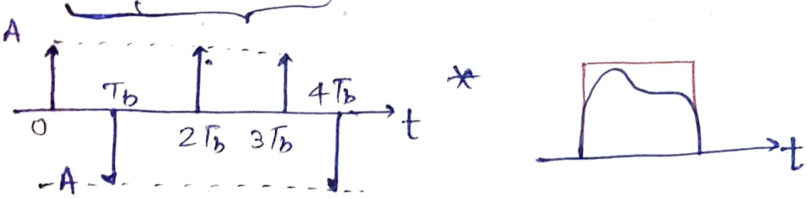
B_i	A_i
0	$-A$
1	A



Can $x(t)$ be represented as the o/p of a LTI filter?

?? → $\boxed{p(t)}$ → $x(t)$

$$x(t) = \underbrace{\sum A_i \delta(t - iT_b)}_{} * p(t)$$



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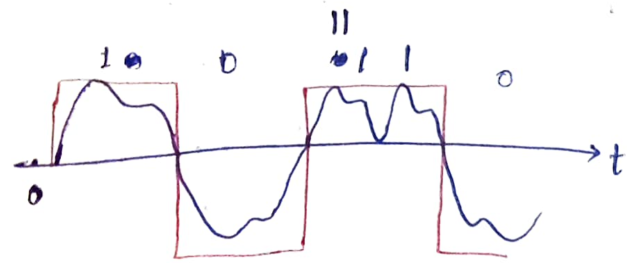


$A(t)$ → $\boxed{p(t)}$ → $X(t)$ → RP

$\sum A_i \delta(t - iT_b)$

$$\boxed{S_x(f) = |P(f)|^2 \, S_A(f)},$$

provided i/p is WSS.

Find the PSD of $x(t)$.

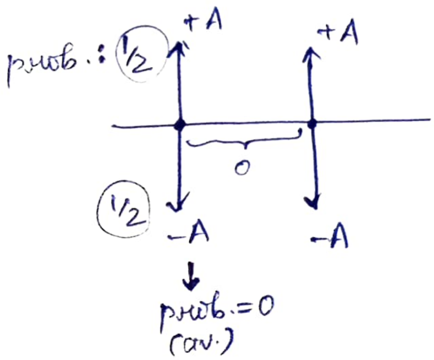
Tradeoffs:

Transmit BW

Recall, when is $A(t)$ WSS?

- $\mathbb{E}\,A(t) = \text{constant}$
- $\mathbb{E}\,A(t_1).A(t_2) = R_A(t_1 - t_2)$

Is $A(t)$ WSS?

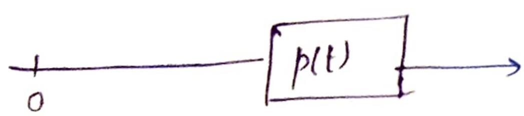


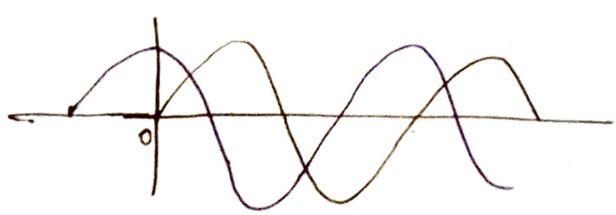
$\mathbb{E}\,A(t) = 0$ ✓

→ A or -A at integral value.

$$\underset{\substack{(2^{nd}\text{ moment})\\ [t_1 = t_2]}}{\mathbb{E}\,A^2(t)} = \begin{cases} A^2 & \text{at } t = nT_b \\ 0 & \text{, otherwise} \end{cases} \neq \text{constant}$$

$\therefore$ $A(t)$ is not WSS.





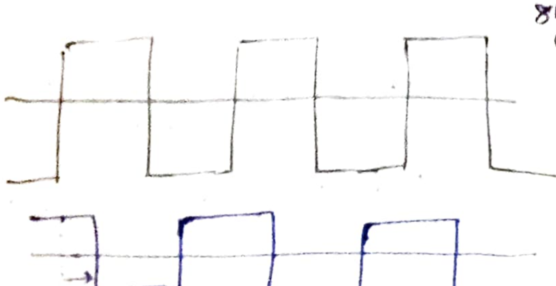
Let us consider

$\boxed{A(t-T)}$ → random delay $\in [0, T_b]$ → uniformly distributed in $[0, T_b]$

→ Reduce the significance of nT_b → Signal $A(t)$ can be $-A/A$ anywhere.

↔ Verify whether this is WSS.

Hint: Recall last ques. from AV314 exam.



[15-01-2025]

Look at $\sum A_i \cdot \delta(t - iT_b - T)$

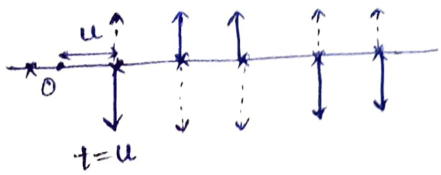
Random delay T is uniformly distributed in $[0, T_b)$.

Is $\sum A_i \,\delta(t - iT_b - T)$ or $A(t-T)$ WSS?

① $\mathbb{E}\left[\sum A_i \cdot \delta(t - iT_b - T)\right]$

$$\int_0^{T_b} \frac{1}{T_b}\, \underbrace{\mathbb{E}\left[\sum A_i \cdot \delta(t - iT_b - T)\right]}_{0} dt$$

Eg.



② $\mathbb{E}\left(A(t_1 - T)\, A(t_2 - T)\right)$

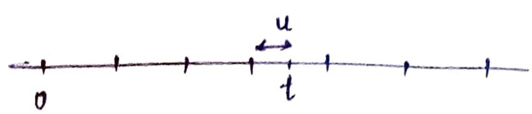
$$\int_0^{T_b} \frac{1}{T_b}\, \mathbb{E}\left(A(t_1 - u)\, A(t_2 - u)\right) du$$

$\mathbb{E}\left[A(t_1 - u)\, A(t_2 - u)\right]$

ⓐ Suppose $t_1 = t_2 = t$.

$\mathbb{E}\, A^2(t-u) = A^2 \delta(t - u - nT_b)$

$$\int_0^{T_b} \frac{1}{T_b} A^2 \delta(t - u - nT_b)\, du = \frac{A^2}{T_b}$$



ⓑ $t_1 \neq t_2$

$\mathbb{E}\left[A(t_1 - u)\, A(t_2 - u)\right]$

ⓘ $t_2 - t_1 > T_b$

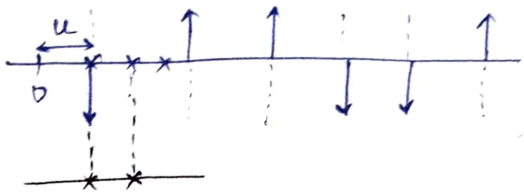
$\mathbb{E}\left[\left(\sum A_i \cdot \delta(t_1 - iT_b - u)\right)\left(\sum A_j \cdot \delta(t_2 - jT_b - u)\right)\right]$

$= \mathbb{E}\left(\sum A_i \cdot \delta(t_1 - iT_b - u)\right) \cdot \mathbb{E}\left(\sum A_j \cdot \delta(t_2 - iT_b - u)\right)$

$= 0$ $\quad [\cancel{\sum A_i}$

[A_i's are independent.]

(ii) $t_2 - t_1 < T_b$



$\left(\begin{array}{l} t_1 = t_2 \text{ should be equal.} \\ t_1 - u = nT_b \end{array} \right)$

$\delta(t_1 - t_2)\ \delta(t_1 - u - nT_b)$

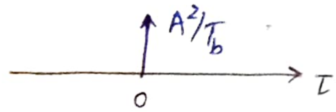
In general,

$$E(A(t_1 - u)\cdot A(t_2 - u)) = A^2\,\delta(t_1 - t_2)\cdot\delta(t_1 - u - nT_b)$$

$$\frac{1}{T_b}\int_0^{T_b} A^2\,\delta(t_1 - t_2)\ \delta(t_1 - u - nT_b)\,du$$

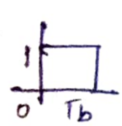
$$= \frac{A^2}{T_b}\ \underbrace{\delta(t_1 - t_2)}_{\tau}$$

$$R_A(\tau) = \frac{A^2}{T_b}\,\delta(\tau)$$

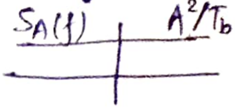


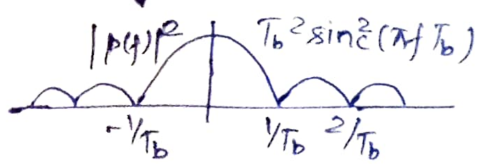
$\underbrace{A(t-T)}_{\text{is WSS}} \longrightarrow \boxed{p(t)} \longrightarrow x(t)$

with auto correlation of $R_A(\tau) = \frac{A^2}{T_b}\delta(\tau)$

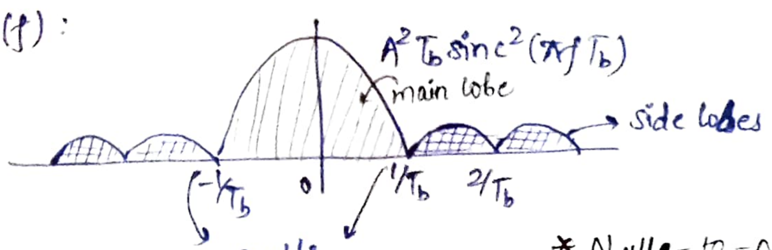


$\updownarrow$





$S_x(f)$:



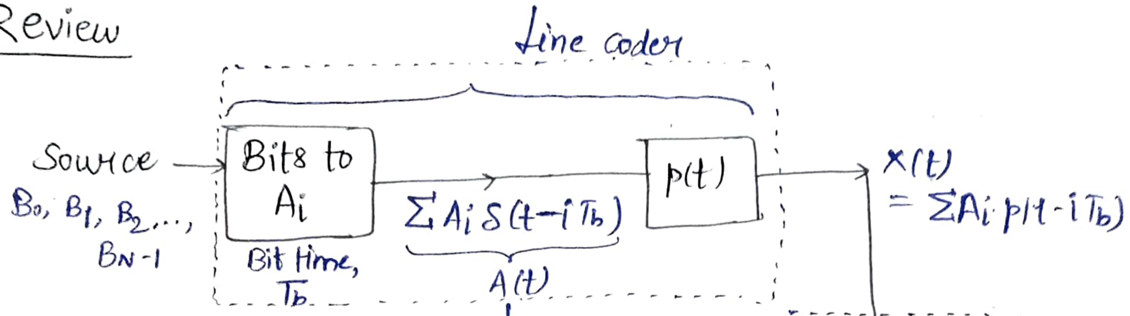
$\underbrace{\text{nulls}}$

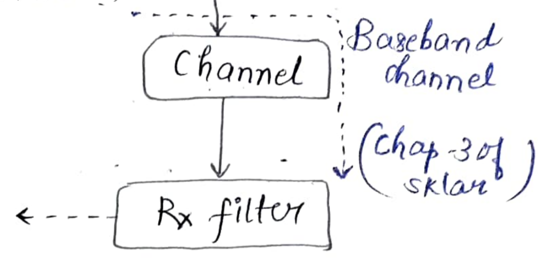
1st null-to-null BW $= 2/T_b$

* Nulls-to-Null BW

16-01-2024

Review





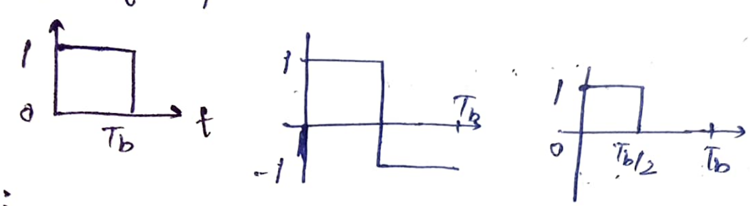
Signal into the channel:

— $S_x(f) = \frac{A^2}{T_b} |P(t)|^2$

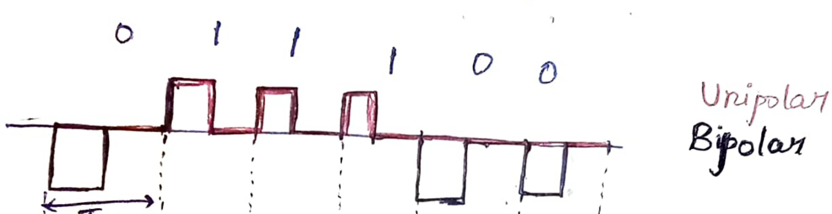
— Bit rate, $R_b = 1/T_b$

$BW \uparrow \Rightarrow$ Transitions $\uparrow$

Examples for p(t):

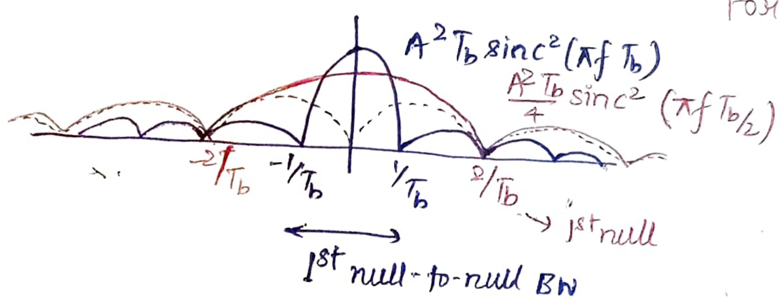


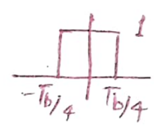
Eg:

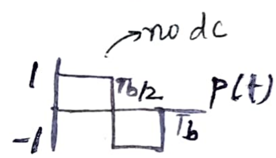


Eg:

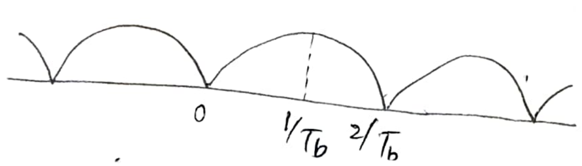
For unipolar p(t)





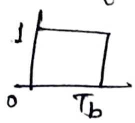


\# $sinc^2(\pi f \frac{T_b}{2})$



$$e^{-j2\pi f \frac{T_b}{4}} \cdot \frac{T_b}{2} sinc(\pi f T_b/2)$$
$$- e^{-j2\pi \frac{3T_b}{4}} \frac{T_b}{2} sinc(\pi f \frac{T_b}{2})$$
$$= e^{-j2\pi f \frac{T_b}{2}} \frac{T_b}{2} sinc(\pi f T_b/2)$$
$$(e^{j2\pi} - e^{-j2\pi f T_b/4})$$
$$= \left| e^{-j2\pi f \frac{T_b}{2}} \cdot 2j \sin(2\pi f \frac{T_b}{4}) \, sinc(\pi f T_b/2) \right|^2$$

Transmit power for p(t)

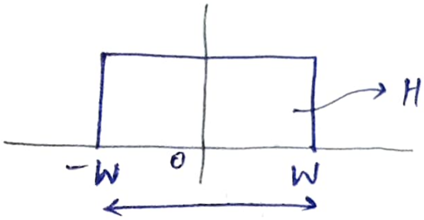
if p(t) = 

$$\underbrace{\begin{matrix} 0 \\ 1 \end{matrix}}_{B_i} \begin{matrix} \longrightarrow \\ \longrightarrow \end{matrix} \underbrace{\begin{matrix} -A \\ +A \end{matrix}}_{A_i}$$

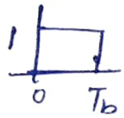
$$\frac{A^2}{T_b}\left(\int_{-\infty}^{\infty} |P(f)|^2\, df\right) \longrightarrow \frac{A^2}{T_b}\int_0^{T_b} |p(t)|^2\, dt = \frac{A^2}{T_b}\cdot T_b = A^2$$

Channel (BB)

↳ Filters



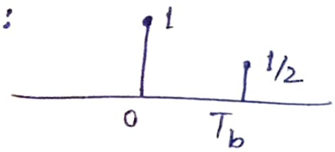
Assume $x(t)$ is Bipolar line coder with $p(t) =$



Bipolar NRZ

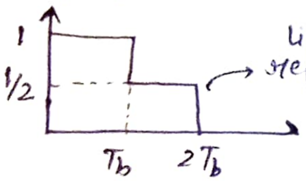
$$Y(t) = \sum A_i \boxed{(p*h)(t - iT_b)}$$

Eg. $h(t)$:



$$h(t) = \delta(t) + \tfrac{1}{2}\,\delta(t - T_b)$$

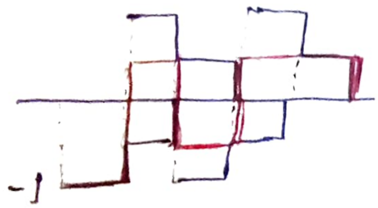
$(p*h)(t)$



like reflection with delay

For 0 1 0 1 → Bits @ Source

$Y(t)$:



Distortion
↓
Pulse waveforms extend to other pulses
↳ Intersymbol interference (ISI)

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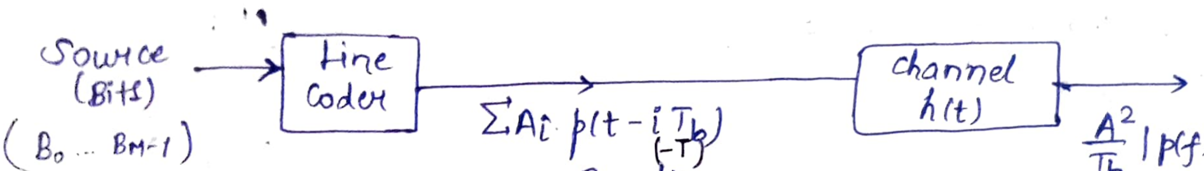
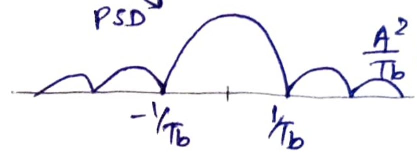


$\sum A_i \, p(t - iT_b)$ (-T)

$R_b = 1/T_b$

$\frac{A^2}{T_b}|P(f)|^2|H(f)|^2$

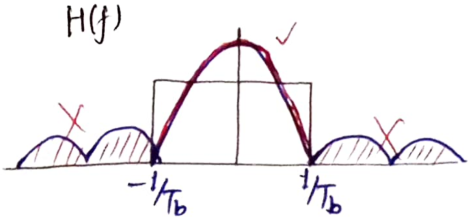
PSD



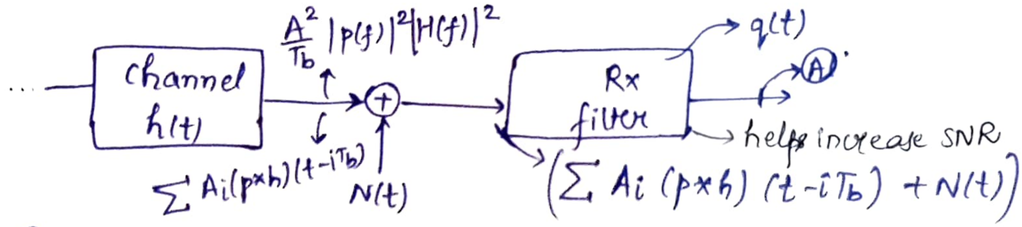
$\sum A_i (p*h)(t - iT_b)$

$(p*h)(t) \rightarrow$ New pulse shape

Eg.

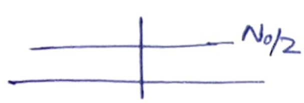


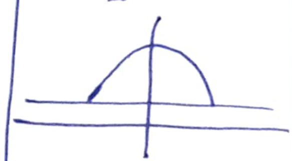
Generally, distortion and there is power cut off by the filter.



$\left(\sum A_i (p*h)(t - iT_b) + N(t)\right)$ ⇝ Noise is indep. of signal → PSDs add up

$N(t) \rightarrow$ C.T.R.P
- ↳ Mean is zero
- ↳ Gaussian random process
- ↳ White process
- ↳ AWGN

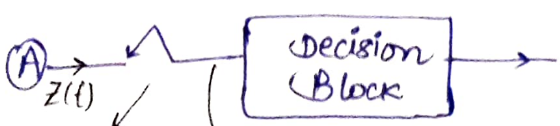




At A

$(p*h*q)(t) = g(t)$ → still Intersymbol interference

$\sum A_i \, g(t - iT_b) + N_1(t)$



Sample can be taken at mid or end of bit-time

$Z_k = Z\left(kT_b + \frac{T_b}{2}\right), \; k \in \mathbb{Z}$

or $Z((k+1)T_b)$

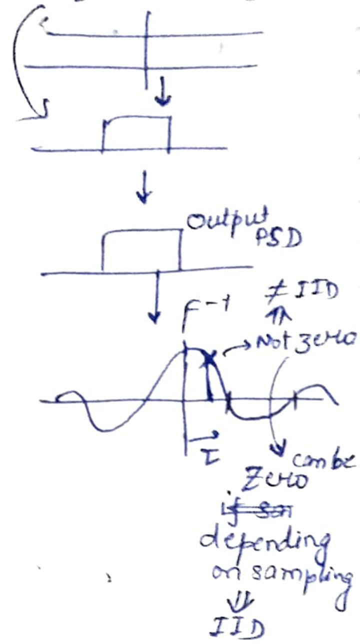
$Z_k = \sum A_i \, g((k-i)T_b + T_b/2) + N_1(kT_b + T_b/2)$

Not white; samples of coloured noise → Not IID (may be IID is some cases)

The sequence of Z_K's (Z_K)

Is Z_K IID?

N_1:



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Sampling $N_1(t)$

AWGN $N(t) \longrightarrow \boxed{q(t)} \longrightarrow N_1(t)$.

$N_1(t)$ is a CTGP.

Since $N(t)$ is WGN

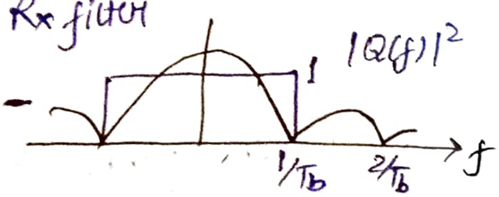
$\Rightarrow$ IID $\Rightarrow$ WSS

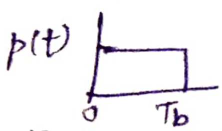
$\Rightarrow N_1(t)$ is WSS.

$S_{N_1}(f) = \frac{N_0}{2} |Q(f)|^2$

$\hookrightarrow$ Usually not white.

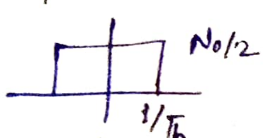
Eg: Rx filter





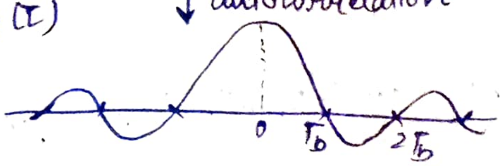
For this example,

$S_{N_1}(f)$



$\downarrow$ autocorrelation

$R_{N_1}(\tau)$



Autocorrelation of $N_1(t)$ has zeros at $\tau = nT_b$.

$(N_{1,k})$?

$\forall k$

$N_{1,k} \sim \mathcal{N}\left(0, \frac{N_0}{T_b}\right) \rightarrow \mathbb{E}(N_{1,k})^2$

$\mathbb{E}(N_{1,k_1} \times N_{1,k_2}) = 0$

$= R_{N_1}(\tau = k_2 T_b + T_b/2 - k_1 T_b - T_b/2)$

$= R_{N_1}(\tau = (k_2 - k_1) T_b)$

$\Rightarrow$ $N_{1,k}$ is IID

H/W : Suppose Rx filter BW (1-sided) is $2/T_b$.

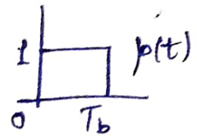
$$\sum_i A_i \, g((k-i)T_b + T_b/2)$$

$k - i = j$

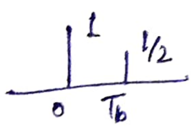
The above expression becomes

$$\sum_j A_{k-j} \, g(jT_b + T_b/2)$$

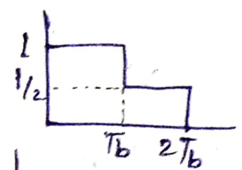
eg.

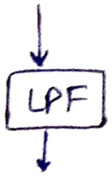


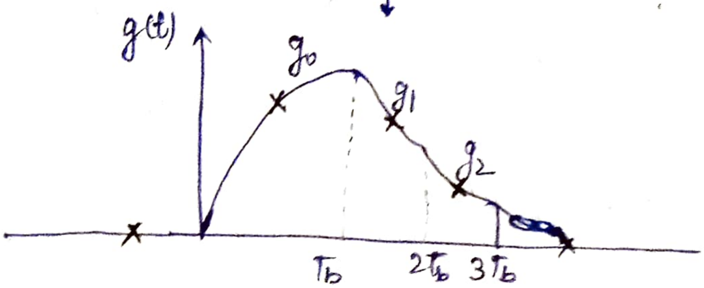
h(t)



$(p * h)(t) =$







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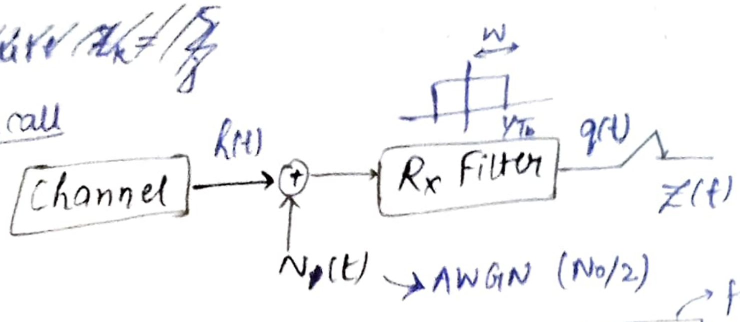
~~where $Z_k = \sum_j$~~

Recall



filtered N(t)

$$Z(t) = \sum A_i\, g(t - iT_b) + \boxed{N_1(t)}$$

↳ Sample every $Z_K = Z(KT_b + T_b/2)$

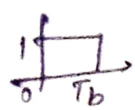
$$Z_K = \sum_i A_i\, g((K-i)T_b + T_b/2) + N_{1,K}$$

$N_{1,K}$:
- ↳ DTGP
- ↳ $\sim N(0, N_0 W)$
- ↳ IID in a special case.

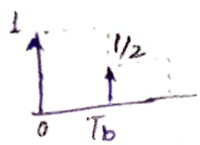
$N_1(t) \longrightarrow R_N(\tau)$

$$\sum_i A_i \cdot g((K-i)T_b + T_b/2)$$

eg, p(t)

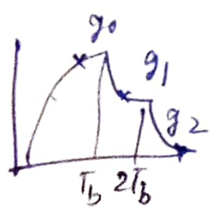


h(t)



→ Pulse spreading

g(t) could be



Write $Z_K = \boxed{\sum_j A_{K-j} \cdot \underbrace{g(jT_b + T_b/2)}_{g(j)}} + N_{1,K}$ → additional noise

$$Z_K = \sum_j A_{K-j} \cdot g_j + N_{1,K}$$

R.V. ↵

$$= \underbrace{A_K g_0}_{\text{ideally}} + \boxed{\sum_{j \neq K} A_{K-j} \cdot g_j} + N_{1,K}$$

→ $A_{K-1} \cdot g_1 + A_{K-2}\, g_2$

→ noise

→ Definition of ISI

Suppose in an "ideal case"

$h(t) = \delta(t)$, $g(t) = \delta(t)$ (& no noise)

g(t) would be p(t)

$$\boxed{Z_K = A_K \cdot g_0}$$

Put some assumption:

- Assume ISI is not there. → Increase channel BW or decrease bit rate → not good to fit the spectrum into the channel

$Z_K = A_K \cdot g_0 + N_{1,K}$ → 2 random quantities ($A_K g_0$ + Noise)

↳ Random Variable

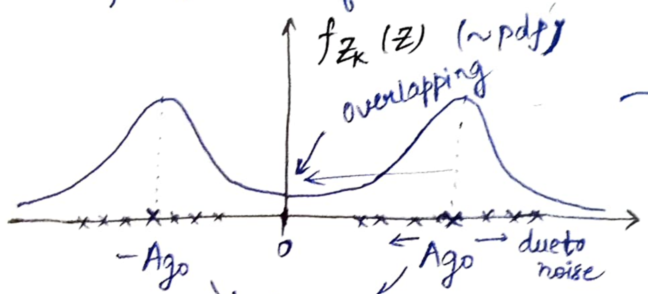
IID?

If $A_K = -A$, $Z_K \sim N(-Ag_0, N_0 W)$

$A_K = A$, $Z_K \sim N(Ag_0, N_0 W)$

↳ Mixture of Gaussian distribution

Histogram: frequencies of values vs. values.

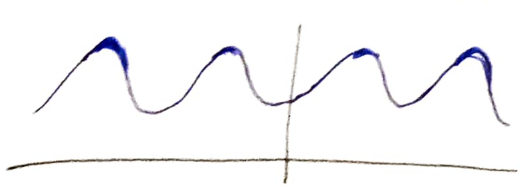


→ Noise margin to check bits.

$\boxed{\text{Decision}}$: $Z_K \gtrless_0^1 0$ → or (r) → some threshold value

Bit Error Rate: Condition on 1 or 0 being transmitted.

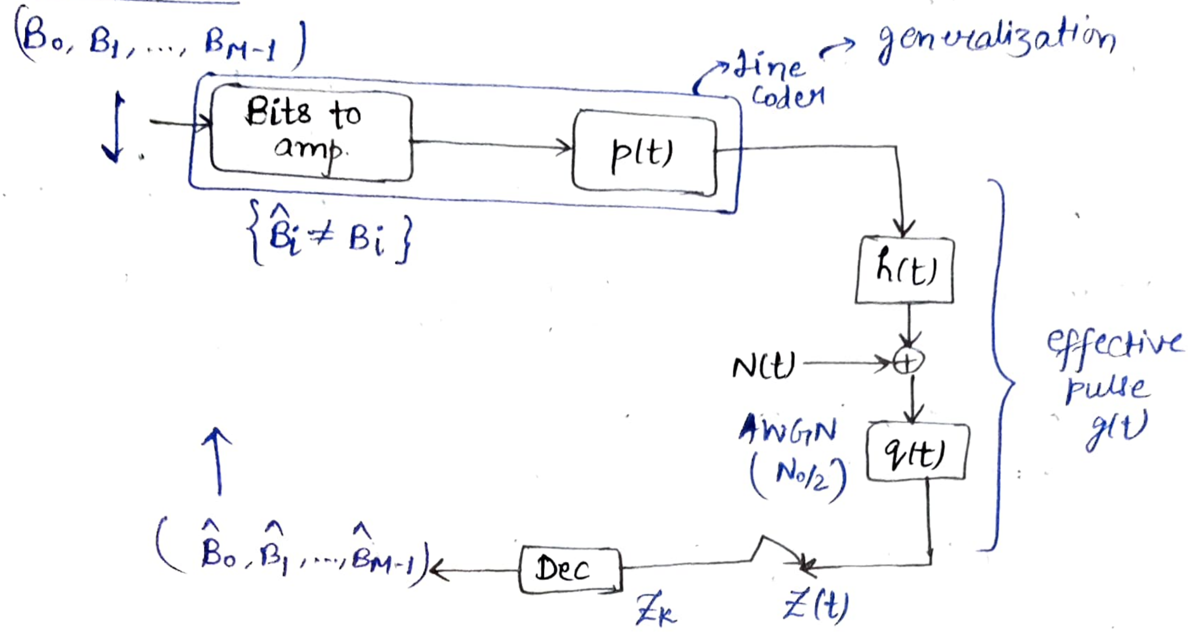
With ISI: $Z_K = A_K g_0 + A_{K-1} g_1 + N_{1,K}$ ⇒ Bit error rate ↑



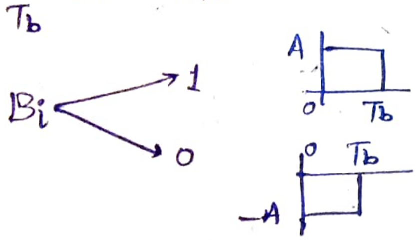
$\boxed{\text{\# Section 3.1 Sklar}}$

Review

27-01-2025



$\boxed{Z_K} \underset{0}{\overset{1}{\gtrless}} r$ (= 0 in the case that we have seen)



$B_i \to 1$: $S_1(t)$

$B_i \to 0$: $S_0(t)$

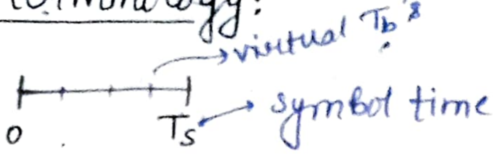
→ Bit rate is limited by the signal BW.

→ To transmit more bits (increase bit rate), send more bits in one T_b.

Bits ⟶ Symbols
↓
Group of bits is a symbol
↓
(power of 2)

0 0	$S_0(t)$
0 1	$S_1(t)$
1 0	$S_2(t)$
1 1	$S_3(t)$

Some terminology:



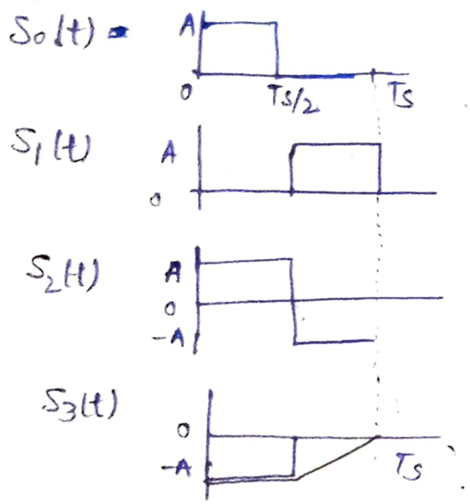
$0 \le t \le T_s$ for $\{S_0(t) \ldots S_3(t)\}$

$$\frac{T_s}{\text{no. of bits in a symbol}} = T_b$$

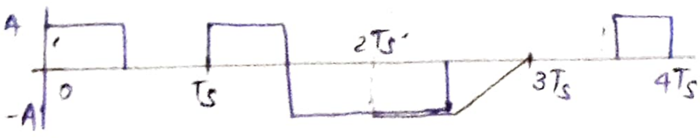
$$\frac{1}{T_s} = \text{Baud rate} \quad (\text{symbols/sec})$$

Baud rate × no. of bits per symbol = bit rate

Example:

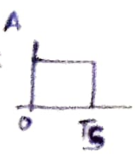


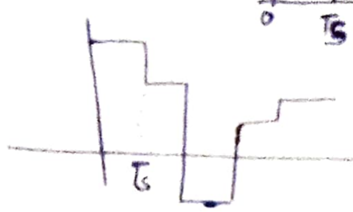
| 00 | 10 | 11 | 01 |



eg (HW) 3 bits is a symbol.

000	-1	A_i Amp.
001	1	
010	-3	
011	+3	
100	-5	
101	+5	
110	-7	
111	+7	

$S_i(t) = A_i \times$ 



$(B_0 \dots B_1) \longrightarrow$ Bits to symbols $\longrightarrow$ Symbol to signal mapper (Constellation mapper)

$\downarrow$

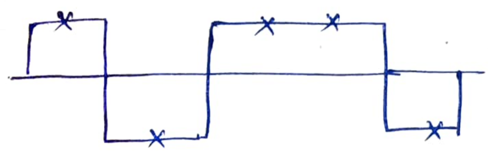
$c_0, c_1, \dots$ (seq. of symbols)

e.g. $c_0 \to B_0 B_1$, $c_1 \to B_2 B_3$

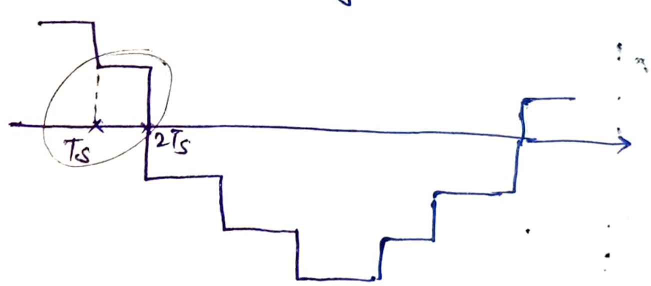
Symbol to signal mapper $\downarrow$ $\sum_i s_{c_i}(t - iT_s)$

$$\xrightarrow{\sum_i s_{c_i}(t-iT_s)} \boxed{h(t)} \xrightarrow{\sum_i s_{c_i}(t-iT_s)}$$

Earlier, for an ideal $h(t)$ and no noise.



Here, assuming that synchronization is there, at the receiver, we know where each signal corresponding to a symbol starts.



$\downarrow$

~~...~~ Correlation with s_i (Reverse Mapping)

$$\longrightarrow \boxed{h(t)} \xrightarrow{\sum_i s_{c_i}(t-iT_s)}$$

$\boxed{\int_0^{T_s} (\cdot)\, s_0(\tau)\, d\tau} \longrightarrow$

$\boxed{\int_0^{T_s} (\cdot)\, s_1(\tau)\, d(\tau)} \longrightarrow$

$\boxed{\int_0^{T_s} (\cdot)\, s_2(\tau)\, d\tau} \longrightarrow$

$\boxed{\int_0^{T_s} (\cdot)\, s_3(\tau)\, d\tau} \longrightarrow$

$s_{c_i}(\tau) \rightsquigarrow s_{c_0}(\tau)$
$s_{c_1}(\tau)$

$\hookrightarrow$ Correlation Type Receiver

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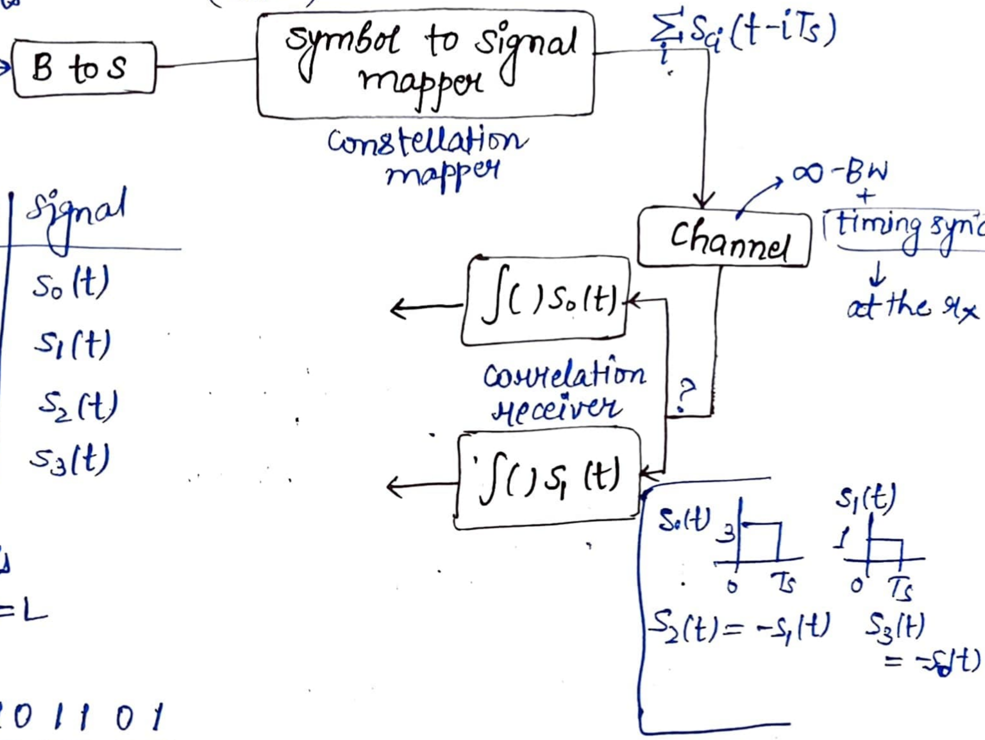
General Signalling in BBDC

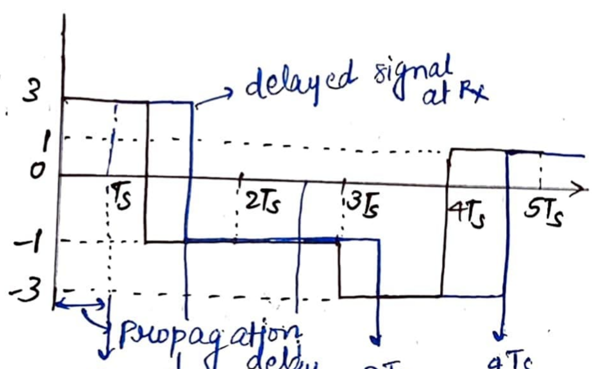
Bits to Symbols

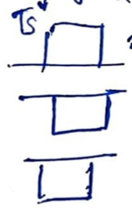
$(B_0, B_1, \dots)$ →



Bits	Symbols	Signal
0 0	0	$s_0(t)$
0 1	1	$s_1(t)$
1 0	2	$s_2(t)$
1 1	3	$s_3(t)$
g	Total 4 symbols $(2^g) = L$	

Eg. $\underbrace{0\,0}_{s_0(t)}\ \underbrace{1\,0}_{s_2(t)}\ \underbrace{1\,0}_{s_2(t)}\ \underbrace{1\,1}_{s_3(t)}\ \underbrace{0\,1}_{s_1(t)}$



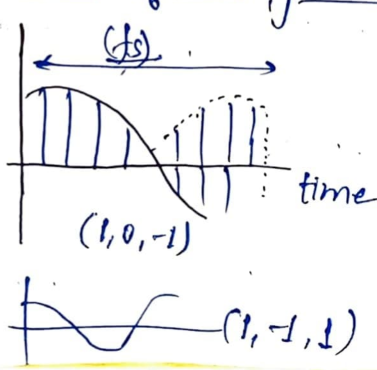


→ Break it up into segments one for every T_s.

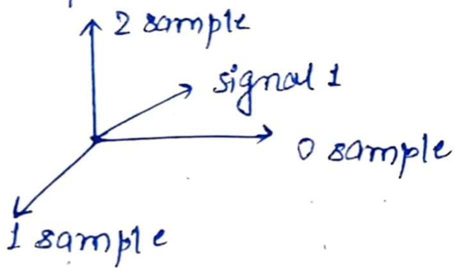
Std. Protocol: ITU 3GPP

Signals as Vectors

Fourier Series & Transforms



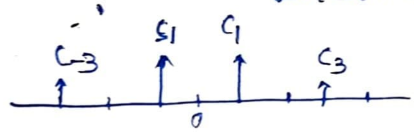
n samples $\Rightarrow \in \mathbb{R}^n$



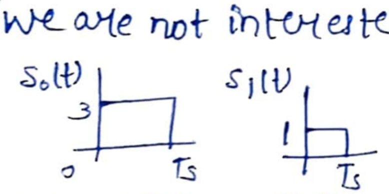
$$x(t) = \sum_{K=-\infty}^{\infty} c_K \left(e^{j2\pi K f_0 t}\right)$$

$e^{j2\pi K f_0 t}$ → orthogonal → Basis vectors → (Check correlation)

Linear combination

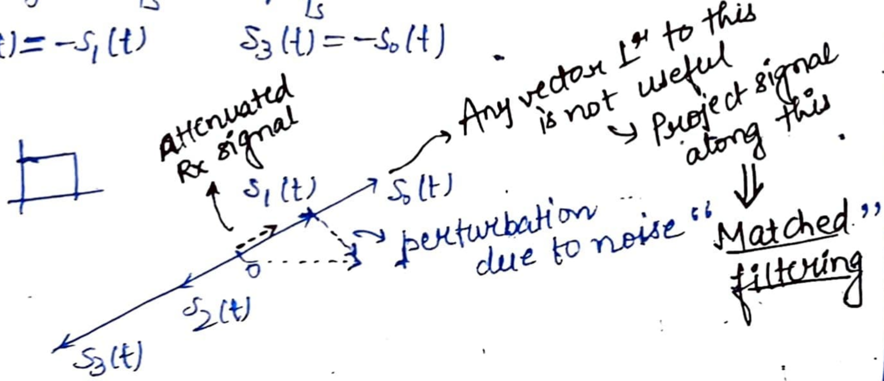


Find a "vector space" that contains $s_0(t) \ldots s_{L-1}(t)$. [L-symbols]
[we are not interested in other signals.]



$s_2(t) = -s_1(t)$ $\qquad$ $s_3(t) = -s_0(t)$

Eg.



[Find basis vectors such that signals lie in the span of these basis vectors.]

→ Basis vectors lie in the BW of the signal.

Any vector lying on this line (equally on all basis) → subspace → less no. of basis than $\mathbb{R}^2$

↳ n basis vectors

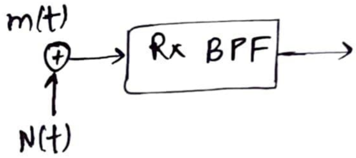
$L \to < n$

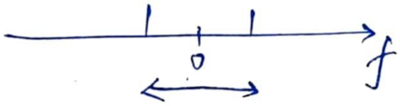
We want a "compact" representation of signals in a vector space. — How?
↳ Idea from "usual" vector spaces.
Suppose we are given

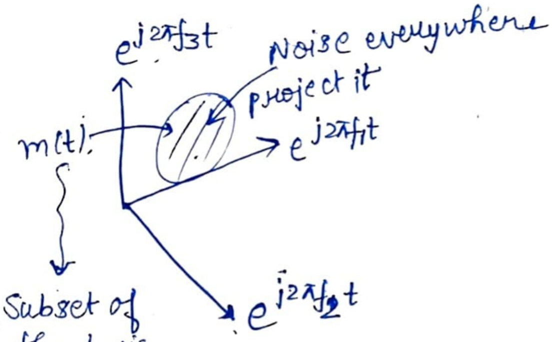
$$\left.\begin{aligned} \vec{x}_1 &= (2,0,0) \\ \vec{x}_2 &= (5,0,0) \\ \vec{x}_3 &= (2,1,1) \end{aligned}\right\}$$

Find a set of basis vectors $\vec{x}_1, \vec{x}_2, \vec{x}_3 \in$ span (basis).

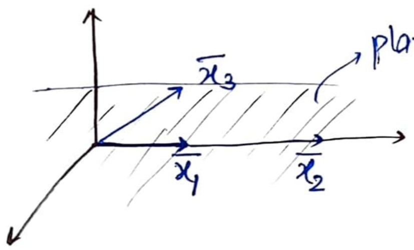
BB Ac







Subset of the basis vectors lying in the BW.



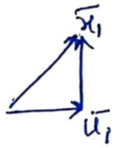
plane containing the vectors

↓

Find basis vector for the plane

Gram-Schmitt procedure → Algorithm to find a linearly independent (orthogonal) vectors from a set of basis vectors

$\vec{u}_1 = (1, 0, 0)$



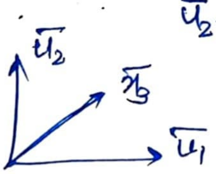
$\bar{x}_2 = 5\bar{u}_1$

$\bar{x}_1 - \bar{u}_1 < \bar{x}_1 . \bar{u}_1 > \; = 0$

$\bar{x}_2 - \bar{u}_1 < \bar{x}_2 . \bar{u}_1 > \; = 0$

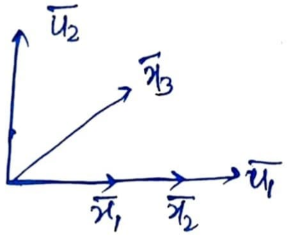
$\bar{x}_3 - \bar{u}_1 < \bar{x}_3 . \bar{u}_1 > \; = (0, 1, 1)$

$\bar{u}_2 = (0, 1/\sqrt{2}, 1/\sqrt{2})$



Book: Kreysig for vectors

$\therefore$



$$\bar{u}_1 = \frac{\bar{x}_1}{\|x_1\|}$$

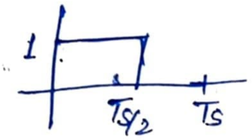
$$\boxed{\bar{u}_k = \frac{\bar{x}_k - \sum_{j=1}^{k-1} \bar{u}_j < x_k . \bar{u}_j >}{\left\| \bar{x}_k - \sum_{j=1}^{k-1} \bar{u}_j < \bar{x}_k . \bar{u}_j > \right\|}}$$

For signals:

$\|\cdot\|^2 \equiv$ Signal energy

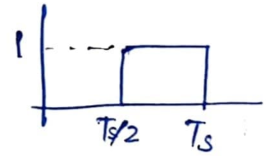
dot product $\equiv \int x(t)\, y(t)\, dt$
(correlation)

Eg. $S_0(t)$



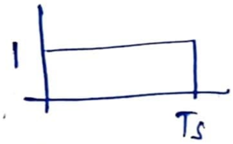
$\rightsquigarrow$ signal energy $= (1)^2 \cdot T_s/2 = T_s/2$

$S_1(t)$

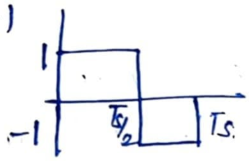


$\rightsquigarrow$ Orthogonal to $u_0(t)$

$S_2(t)$

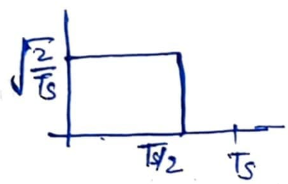


$S_3(t)$

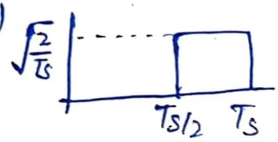


Unit vectors:

$\frac{\bar{S}_0}{\|\bar{S}_0\|} \equiv u_0(t)$



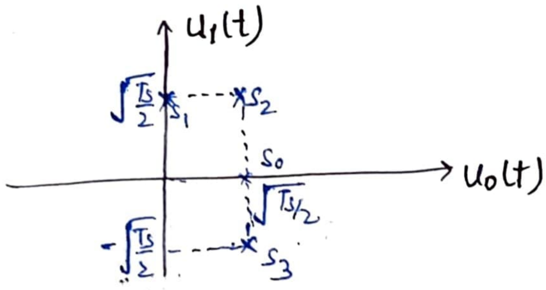
$u_1(t)$



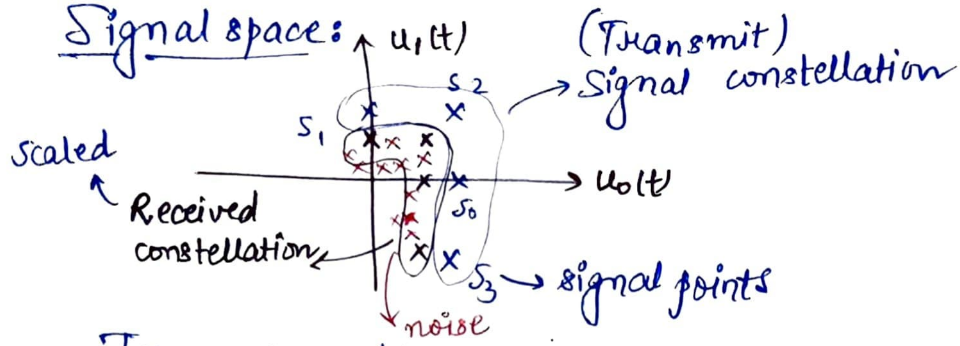
Since $S_2(t) = \sqrt{T_s/2}\, u_0(t) + \sqrt{T_s/2}\, u_1(t)$

$S_3(t) = \sqrt{T_s/2}\, u_0(t) - \sqrt{T_s/2}\, u_1(t)$

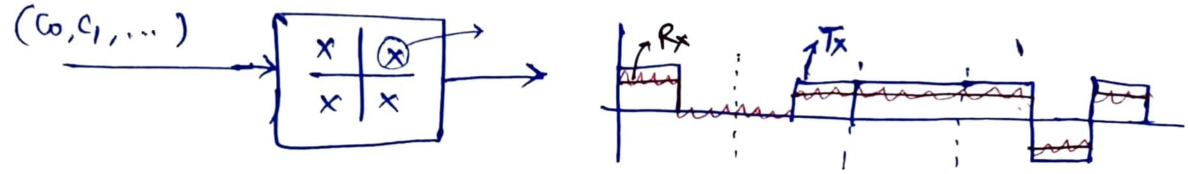
} No need of $\bar{u}_2$ & $\bar{u}_3$.



Signal space:



Transmitter side:



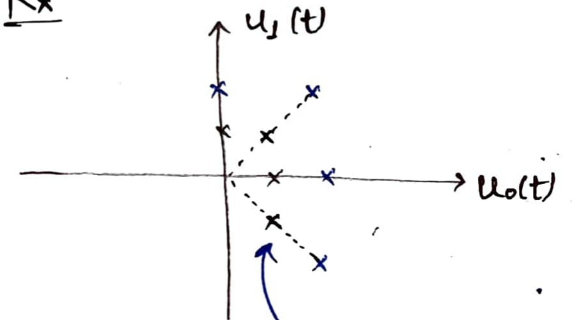
Channel → gain of $r<1$

Bits → symbols → constellation mapper → $\sum s_{c_i}(t_0 - iT_s)$

03-02-2025

Channel
↳ ∞ BW channel
↳ No noise
↳ Sync.

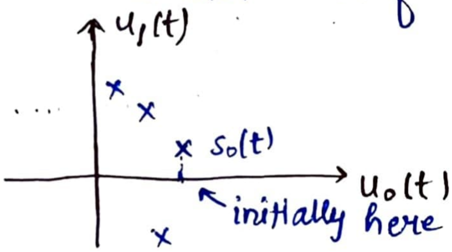
R_x



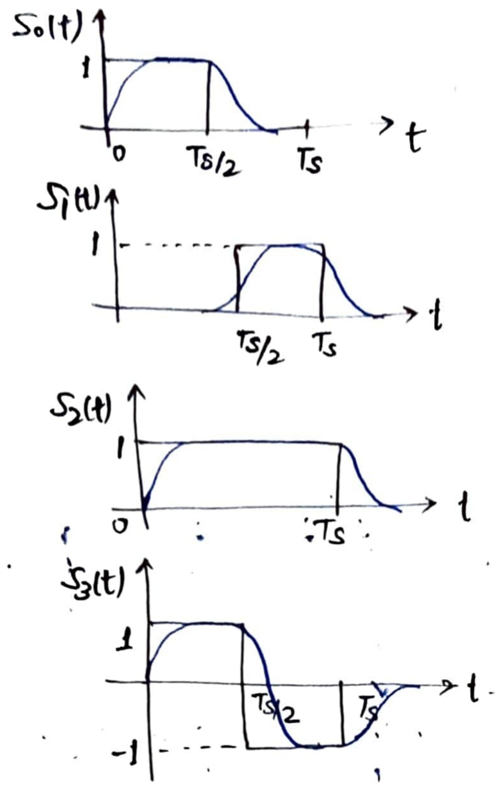
Ideally, the received signal space and signal constellation = T_x signal space.

→ Suppose we have a gain of $r<1$ for the channel.

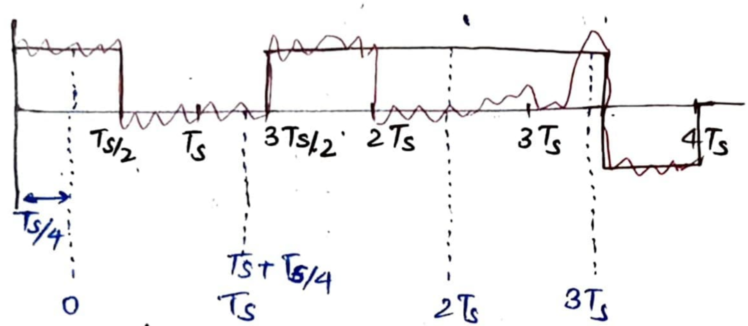
→ Suppose channel is a filter. (LPF)

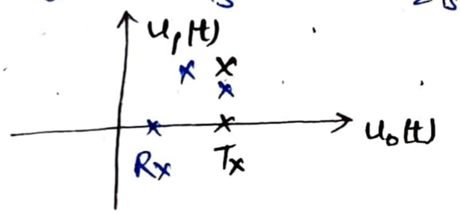


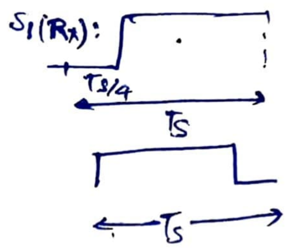
Initially, $s_0(t)$ have a component only along the $u_0(t)$ unit vector. Now, it also have a component along $u_1(t)$.



Synchronization:



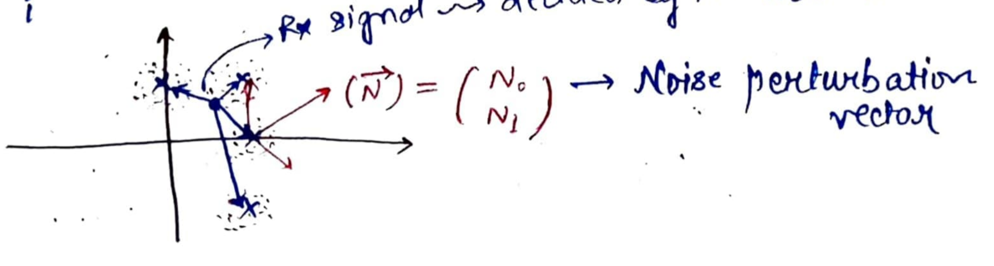




Header

Noise:

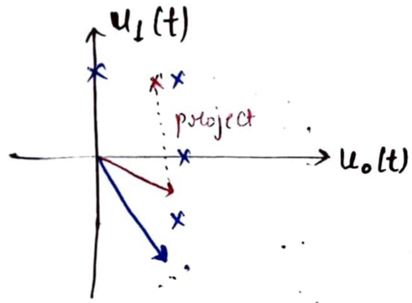
$$\sum_i S_{ci}(t - iT_s) + N(t)$$



$S_0(t) + N(t) \begin{cases} u_0(t) \\ u_1(t) \end{cases}$: Random signal around $u_0(t)$

$$\int_0^{T_s} N(t).u_0(t)\,dt = N_0$$

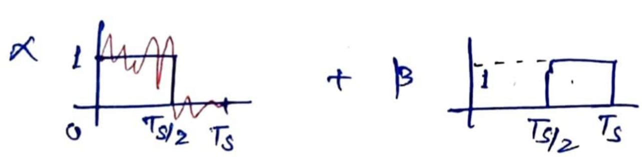
$$\int_0^{T_s} N(t).u_1(t)\,dt = N_1$$

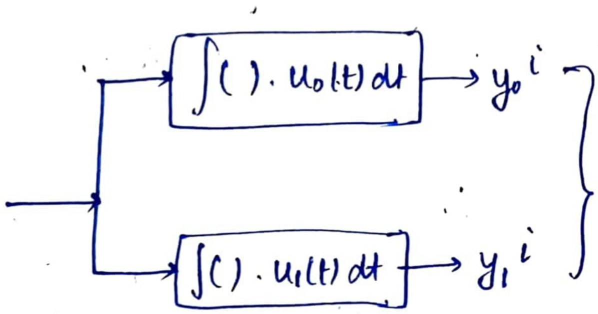


$N(t) = N_0 . u_0(t) + N_1 . u_1(t) + \underset{\sim}{N_r}$ → residue

↳ leftover which cannot be represented in u_0, u_1 space.

Vector in u_0, u_1 -space:



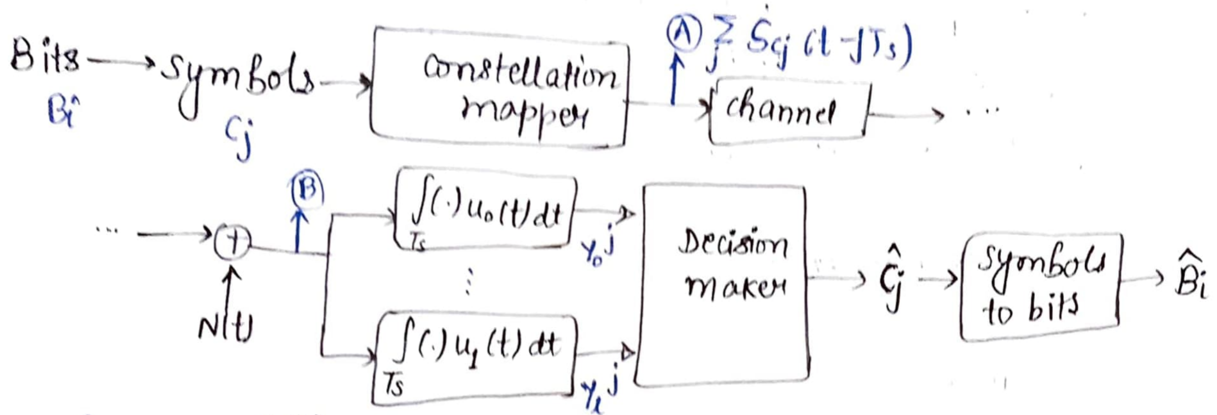


Rx signal → decision about symbol

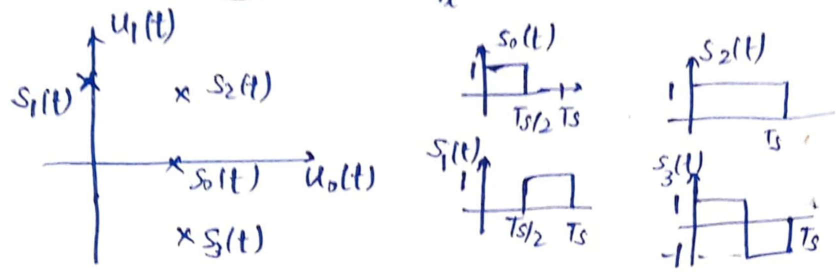
↓

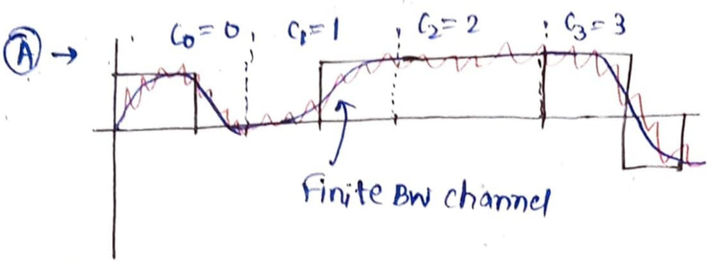
Minimum distance rule

05-02-2025

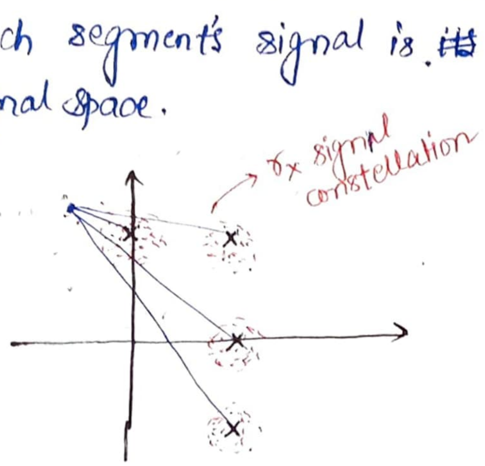


Eg.

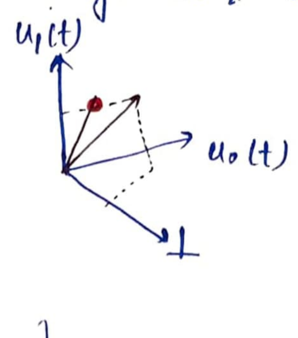




At the receiver, Ⓑ is divided into segments of duration T_s, each segment's signal is ~~its~~ then represented in the signal space.



The signal received in every segment ∉ the signal space.



"Good" decision rule :

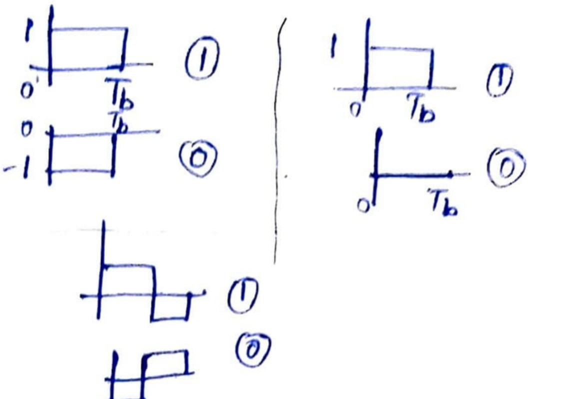
$$\underbrace{(Y_0^j, Y_1^j)}_{\bar{y}^j} \rightarrow \bar{S}_0, \bar{S}_1, \bar{S}_2, \bar{S}_3$$

$\| \bar{y}^j - \bar{S}_K \|^2$ ← this is min.

Find $K \in \{0 \ldots L-1\}$

"Min. distance" decoding rule

For usual line codes in a baseband system,

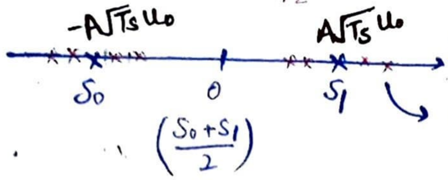


Error Probability Analysis

Error Probability $= P_r\{\text{error event}\}$

$= P_r\{s_0\ tx\}.P_r\{\text{error}|s_0\} + P_r\{s_1\ tx\}.P_r\{\text{error}|s_1\}$

$1/2$ $\quad$ $1/2$

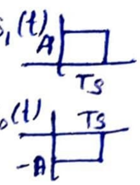


$u_0(t) \equiv 1/\sqrt{T_s}$ (0 to T_s)

$y^j \to S_1 + N^j$ if 1 tx

$y^j \to S_0 + N^j$ if 0 tx

eg. $S_1 : S_1(t)$; $S_0 : S_0(t)$



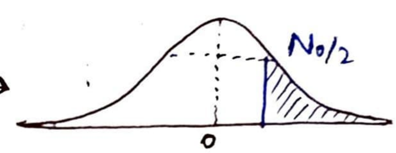
When do we make an error?

$$P_r\{\text{error}|s_0\} = P_r\{\overbrace{y^j > \frac{S_0+S_1}{2}}^{\text{error given } s_0} \mid S_0\ tx\}$$

$$P_r\{\text{error}|s_1\} = P_r\{\underbrace{y^j < \frac{S_0+S_1}{2}}_{\text{error given } s_1} \mid S_1\ tx\}$$

$$= P_r\{\underbrace{y^j}_{S_0+N^j} > \frac{S_0+S_1}{2} \mid S_0\ tx\}$$

$$= P_r\{N^j > \frac{S_1-S_0}{2} \mid S_0\ tx\} \longrightarrow$$



$$N^j = \int_0^{T_s} \underbrace{N(t)}_{\text{AWGN }(N_0/2)} . u_0(t)\,dt$$

$N(0, N_0/2)$

[Take $g(t) = u_0(-t)$]

$$= \int_0^{T_s} N(t)\, g(-t)\, dt$$

$$= \int_0^{T_s} N(\tau)\, g(0-\tau)\, d\tau$$

$$\int \frac{N_0}{2} . \underbrace{|G(f)|^2}_{\equiv \int u(t)\,dt = 1} = \frac{N_0}{2}$$

gaussian $\quad$ gaussian

$N(t) \longrightarrow$ LTI $\longrightarrow N^j$

06-02-2025

$$P_r(\text{error event}) = \cancel{P_r}\{s_0\}^{\,1/2} \cdot P_r\left\{y^j > \frac{s_0+s_1}{2} \,\Big|\, s_0\right\} + \cancel{P_r}\{s_1\}^{\,1/2} \cdot P_r\left\{y^j < \frac{s_0+s_1}{2} \,\Big|\, s_1\right\}$$

$$P_r\left\{s_0 + N^j > \frac{s_1+s_0}{2} \,\Big|\, s_0\right\}$$

$$= P_r\left\{N^j > \frac{s_1-s_0}{2} \,\Big|\, s_0\right\}$$

$$= P_r\left\{N^j > \frac{s_1-s_0}{2}\right\}, \quad N^j \sim N\left(0, \frac{N_0}{2}\right)$$

$$P_r\left\{\underbrace{\frac{N^j}{\sqrt{N_0/2}}}_{} > \frac{s_1-s_0}{2\sqrt{N_0/2}}\right\}$$

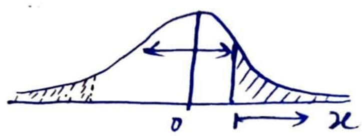
$$S \geq \frac{s_1-s_0}{\sqrt{2N_0}}$$

$$\| $$

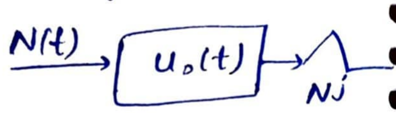
$$Q\left(\frac{s_1-s_0}{\sqrt{2N_0}}\right)$$

complementary cummulative distribution f^n

$$= \int_{\frac{s_1-s_0}{2}}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-u^2/N_0}\, du$$



$$N^j = \int_0^{T_s} N(t)\cdot u_0(t)\, dt$$



$$\boxed{Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-u^2/2}\, du}$$: definition of Q-function

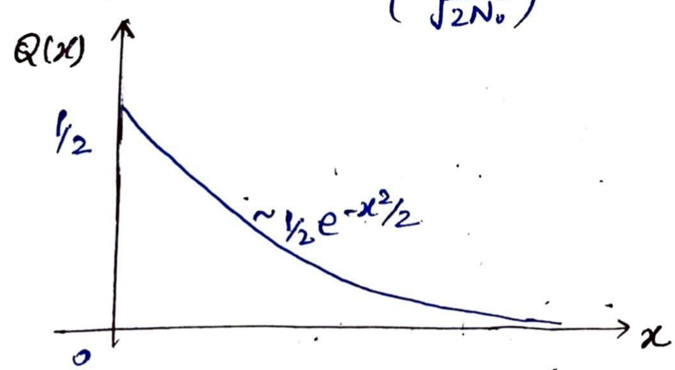
Similarly,

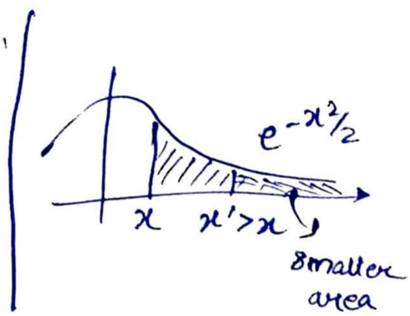
$$P_r\left\{y^j < \frac{s_1+s_0}{2} \,\Big|\, s_1\right\} = P_r\left\{y^j > \frac{s_1+s_0}{2} \,\Big|\, s_0\right\}$$

$$= Q\left(\frac{s_1-s_0}{\sqrt{2N_0}}\right)$$

Finally, we get

$$P_r(\text{error}) = Q\left(\frac{s_1-s_0}{\sqrt{2N_0}}\right).$$





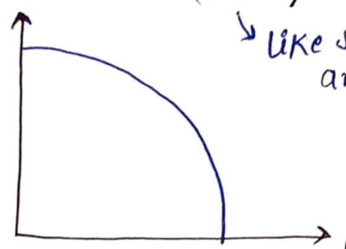
Eg. For the signals shown

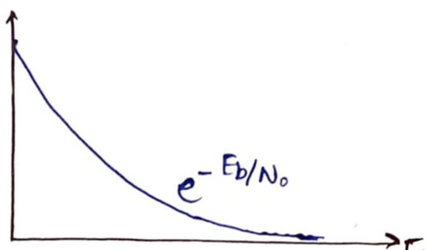
$$\left(\frac{2A\sqrt{T_S}}{\sqrt{2N_0}}\right) = \sqrt{2}\left(\frac{\sqrt{A^2 T_S}}{\sqrt{N_0}}\right)$$ → Energy per bit : $E_b = A^2 T_S$

$$= \left(\sqrt{2}\sqrt{\frac{E_b}{N_0}}\right)$$ → # see Sklar

$$\Pr\{\text{error}\} = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

↳ like SNR for analog comms.





Transmit energy (bit) ↑
⇒ A↑, T_S ↑

Setting 10-02-2025

Scalar value = $\underbrace{(S_1 \text{ or } S_0) + N^j}$

Noise corrupted version of S_1 or S_0

Decision Rule :
- minimum distance
  ↳ maximum likelihood decisions
  ↓
  Appendix B (Sklar)

Hypothesis testing problems :

$\left.\begin{matrix} H_0 \\ H_1 \end{matrix}\right\}$ Y (observation)
↳ Random variable

$\left.\begin{matrix} f_{Y|H_0} \\ f_{Y|H_1} \end{matrix}\right\}$ distribution given hypothesis

Apriori probability
→ $\Pr\{H_0\}$
$\Pr\{H_1\}$

→ Given the observation, come up with a rule that can tell the ~~given~~ which hypothesis is true.

Optimal decision rules

Decision rule $g : Y \to \{H_0, H_1\}$

Optimal is in the sense of minimum the chance of making errors.

Consider a randomized rule ind. of Y.

Choose H_0 w.p. $1/2$,

H_1 w.p. $1/2$.

(In our DC-problem,

s_0 w.p. $1/2$,

s_1 w.p. $1/2$)

s_0 chosen

s_0 tx $\quad \Pr\{H_0\}$

$$\underbrace{\tfrac{1}{2} \cdot \tfrac{1}{2}} + \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{1}{2}.$$

A possibly better rule:

Given an observation y,

$$\left.\begin{array}{l} f_{Y|H_1}(y) \\ f_{Y|H_0}(y) \end{array}\right\} \text{comp.}$$

Compare

$$\underbrace{f_{Y|H_1}(y)}_{\substack{\downarrow \\ \text{likelihood distribution}}} \underset{H_0}{\overset{H_1}{\gtrless}} f_{Y|H_0}(y)$$

This is the maximum likelihood rule.

For our example, what is

$$f_{Y|S_1}(y) = \mathcal{N}(s_1, N_0/2)$$

$$f_{Y|S_0}(y) = \mathcal{N}(s_0, N_0/2)$$

$$\frac{1}{\cancel{\sqrt{\pi N_0}}} e^{-(y-s_1)^2/N_0} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{1}{\cancel{\sqrt{\pi N_0}}} e^{-(y-s_0)^2/N_0}$$

$$\Rightarrow -\frac{(y-s_1)^2}{N_0} \underset{H_0}{\overset{H_1}{\gtrless}} -\frac{(y-s_0)^2}{\cancel{N_0}}$$

$$\Rightarrow (y-s_1)^2 \underset{H_1}{\overset{H_0}{\gtrless}} (y-s_0)^2$$

→ When apriori probabilities are uniform, ML rules have minimum error.

H_0 : disease is absent

H_1 : disease is present.

$Y \in \{0, 1\}$

$Pr\{Y=1 \mid H_1\} = 0.98$

$Pr\{Y=1 \mid H_0\} = 0.6$

$$Pr\{H_1|Y\} \underset{H_0}{\overset{H_1}{\gtrless}} Pr\{H_0|Y\}$$: Optimal rule → Maximum Aposteriori rule/ probability (MAP)

$\boxed{Pr\{H_1 \mid Y=y_0\}}$

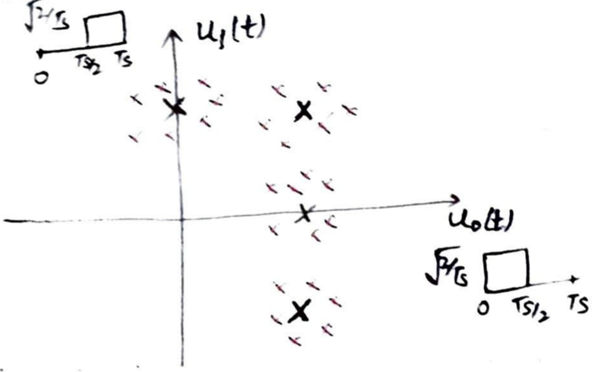
$$\frac{Pr\{H_1\}.Pr\{Y|H_1\}}{\cancel{Pr\{Y\}}} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{Pr\{H_0\}.Pr\{Y|H_0\}}{\cancel{Pr\{Y\}}}$$

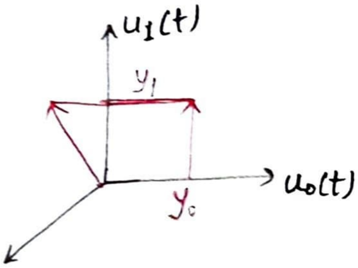
$Pr\{A|B\} = \frac{Pr\{AB\}}{Pr\{B\}} = \frac{Pr\{A\}.Pr\{B|A\}}{Pr\{B\}}$

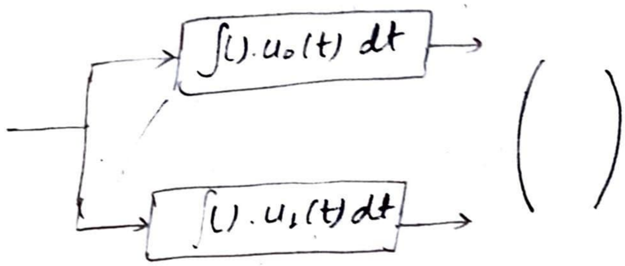
$$\underbrace{Pr\{H_0|Y=y_0\}}_{\text{Posterior Probability}} \propto Pr\{H_0\}.Pr\{Y=y_0|H_0\}$$

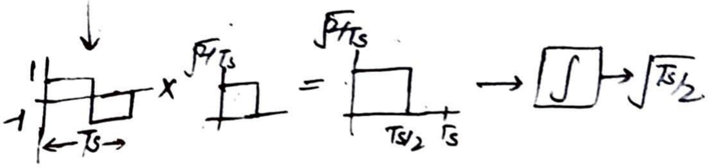
$$Pr\{H_1|Y=y_0\} \underset{H_0}{\overset{H_1}{\gtrless}} Pr\{H_0|Y=y_0\}$$ (MAP Rule)

Matched Filters (Theory + Lab)

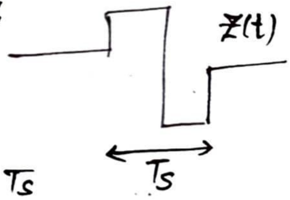








Signal:

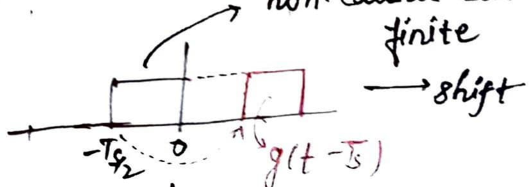


$$\int_0^{T_s} z(\tau)\cdot u_0(\tau)\,d\tau \quad \Big| \quad g(t)=u_0(-t)$$

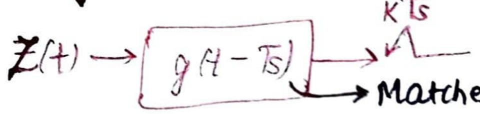
$$=\int_0^{T_s} z(\tau)\,g(0-\tau)\,d\tau$$

$$=z(t)*g(t)\Big|_{t=0}$$

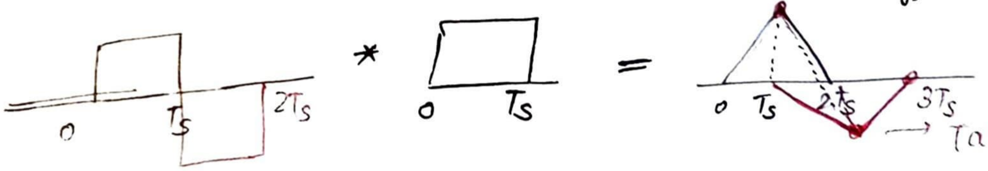
non-causal but finite



(sample at T_s)

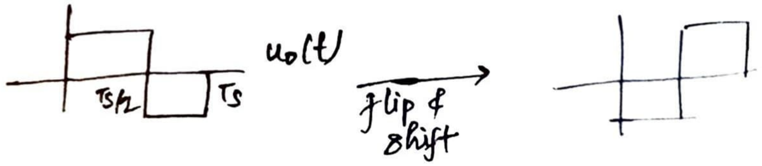


Same affect as the correlation



→ Take sample at the ends

Eg.



→ MF response

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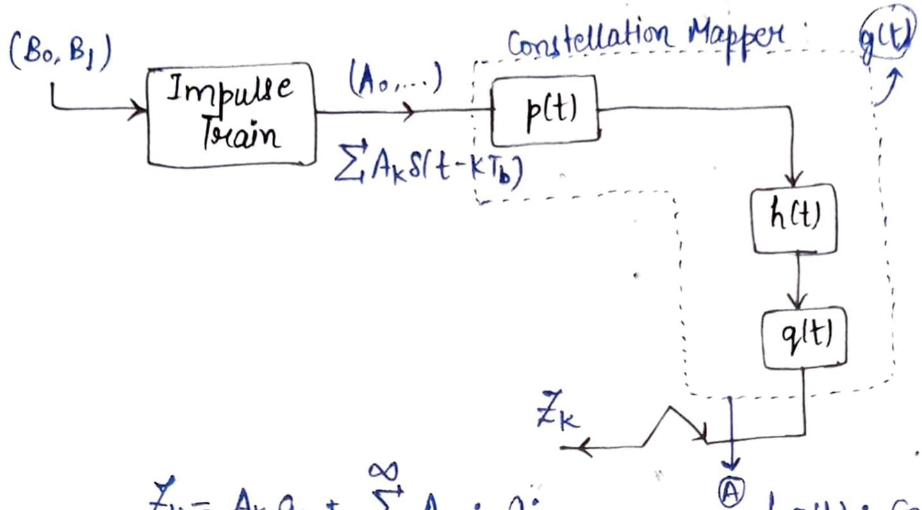
Intersymbol Interference



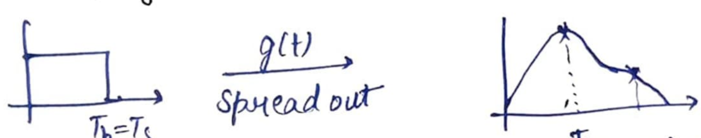
$$Z_K = \underbrace{A_K g_0}_{\text{sampled at the end of matched filtering}} + \underbrace{\sum_{\substack{i=1 \\ (i \neq 0)}}^{\infty} A_{K-i}\, g_i}_{\text{ISI}}$$

sampled at the end of matched filtering

ISI → Remove this completely

$q(t)$: Correlator or Matched function

Pulse Shaping : To remove ISI



T_b → spread out to other T_b intervals also. $(2T_b, 3T_b, \dots)$

But for Z_K

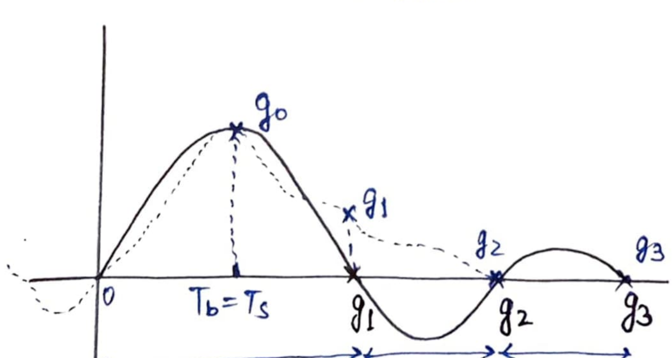
↳ if we sample $g(t)$ at $2T_b, 3T_b, \dots$

such that $g_i = 0$ → can remove g_i.

↓

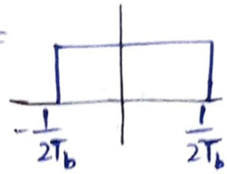
Requires signal which is spread but still zero at nT_b.

↳ sinc.



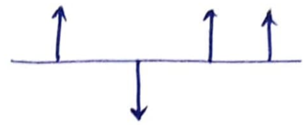
: $g(t) = \text{sinc}\left(\frac{t - T_b}{T_b}\right)$

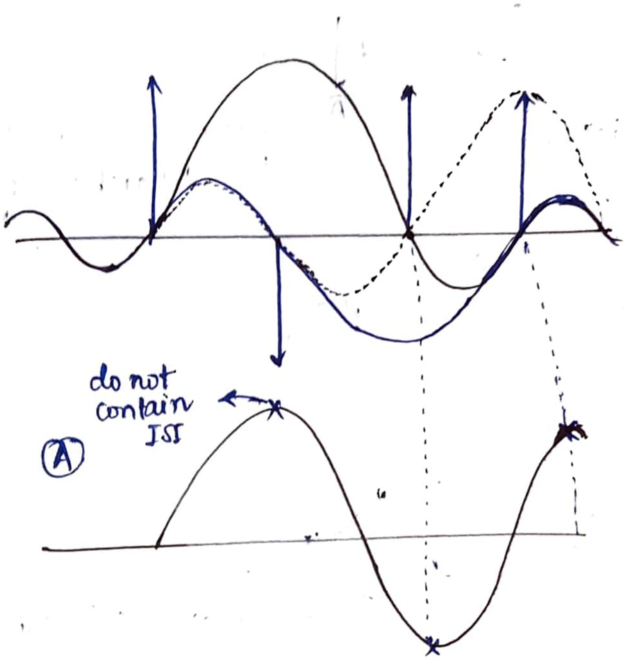
$G(f) =$



$= P(f)\,\boxed{H(f)}\,Q(f)$

Eg. 1011 →





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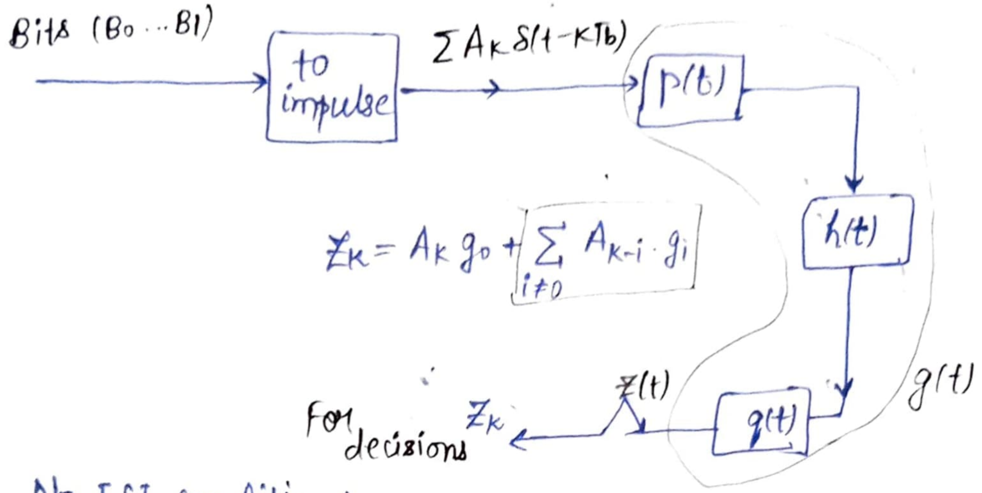
Pulse shaping (for ISI)



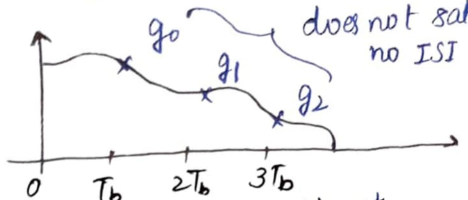
$Z_K = A_K g_0 + \boxed{\sum_{i \neq 0} A_{K-i} \cdot g_i}$

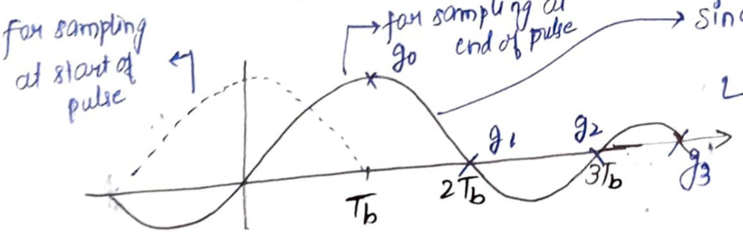
No ISI condition :

$g_0 \neq 0$

$g_i = 0, \; i \neq 0$

$$Z_K = A_K g_0 + \sum_{i \neq 0} A_{K-i} \cdot g_i$$





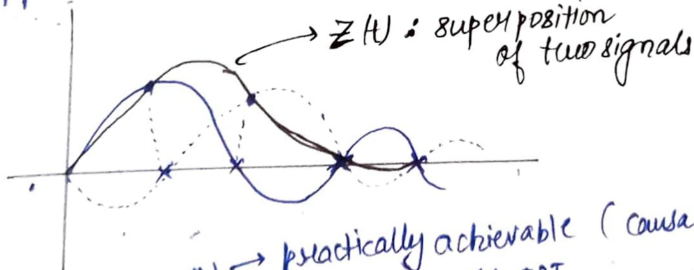
$\text{sinc}\left(\frac{t - T_b}{T_b}\right)$: choose $p(t)$, $q(t)$ for given $h(t)$, accordingly

↳ (filter) satisfies no ISI condition

↳ but not achievable ~~(non-causal)~~

↳ infinitely spread & non-causal

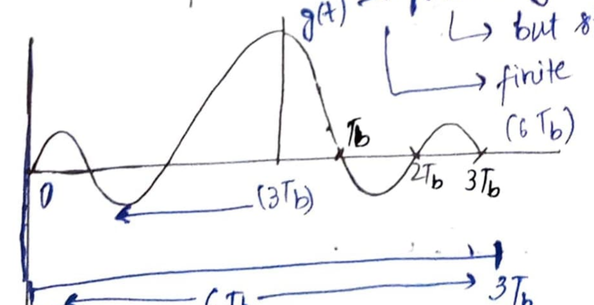
Suppose we send a sequence, say 1, 1.

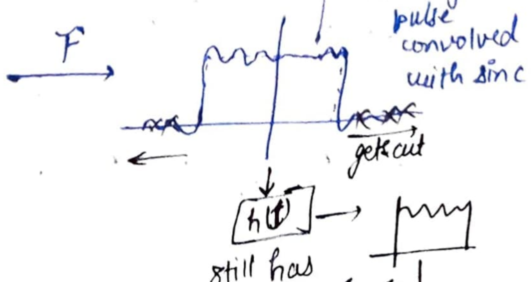


$g(t)$ → practically achievable (causal) → cannot get as it is band-limited.

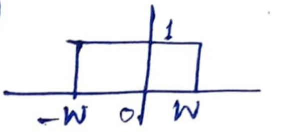
↳ but still, ISI

↳ finite ($6T_b$)

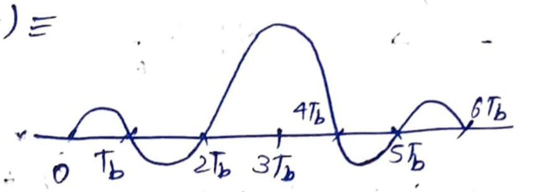




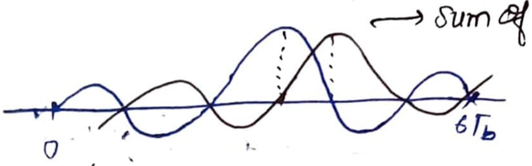
Eg. Suppose $h(t) \equiv H(f)$



$p(t) \equiv$



1, 1



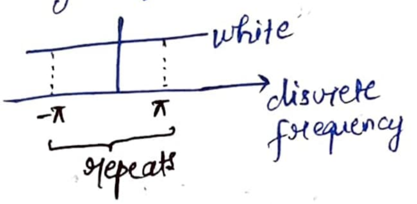
→ sum of these two signals go into the channel

$Z(t)$: signal that we get

$(g_0, g_1) \rightarrow$ Pulse shape $g(t)$ that satisfies no ISI.

$\overset{\Vert}{1} \quad \overset{\Vert}{0}$ $\downarrow$ dtft

$[g_0 = 1, \; g_i = 0 \; \forall \; i \neq 0]$



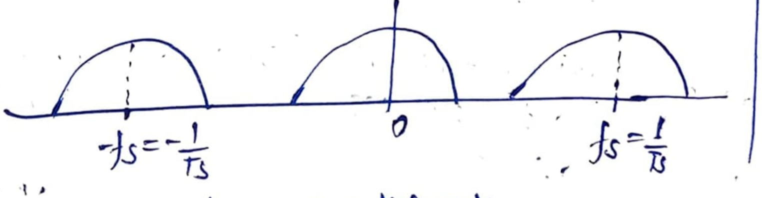
From sampling theory,

$x(t)$

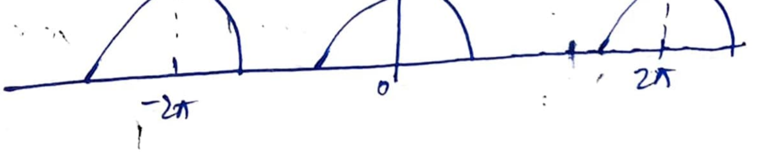
$\downarrow$

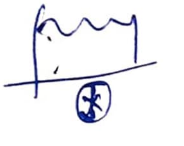
$\sum_i x[nT_s] \cdot \delta(t - nT_s)$

$\downarrow$ CTFT

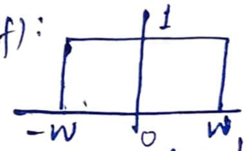


$x_n \rightarrow$ DTFT : $\updownarrow$ same but different scaling



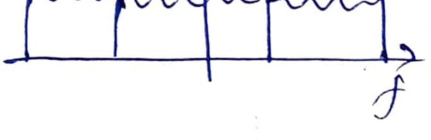


$H(f)$:



$\left(= \frac{1}{2T_b}, \text{choose} \right)$

$\downarrow$



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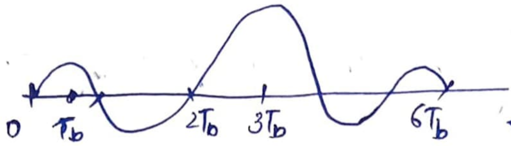
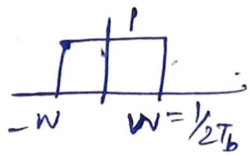
Suppose $g(t)$ is

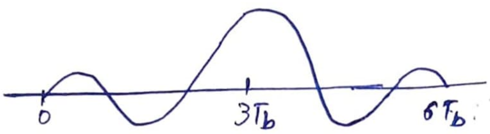


→ Band-limited : not achievable
↪ But finite → No ISI
(truncated sink)

Eg: If $h(t) \equiv H(f)$ is



and $p(t)$ is

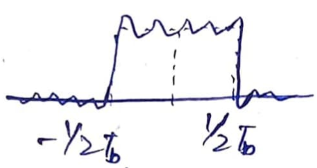


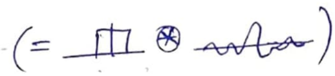
and $q(t) = \delta(t)$,

then we get a $g(t)$ — almost like

but with some ISI.

$p(t) \xrightarrow{\mathcal{F}}$

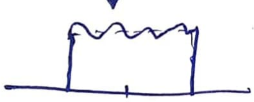




↓

$H(f)$

↓

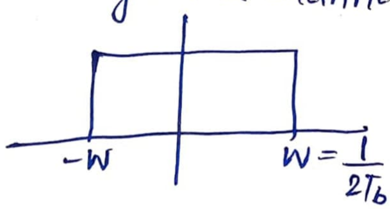


↓

this $\tilde{g}(t)$ that we actually get is not $g(t)$ but $\approx g(t)$.

But we ~~should~~ showed that $\tilde{g}(t)$ has some ISI.

If we are given a channel



for min BW ← (others use more BW)

The max^m rate through which we can signal through the channel (theoretically)

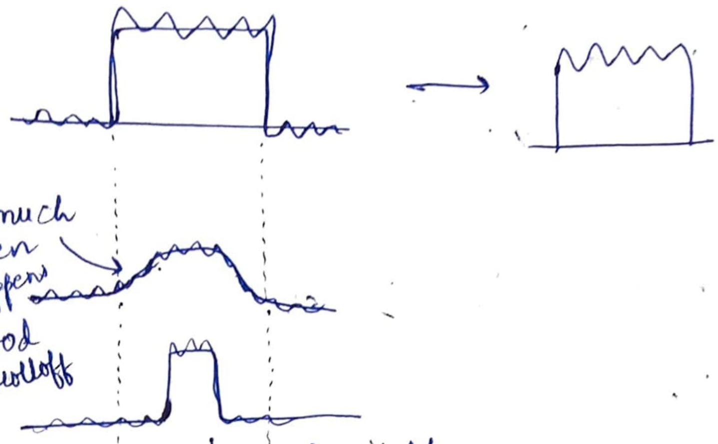
$1/T_b = R_b = 2W$

: $\boxed{\text{when } g(t) \text{ is } \operatorname{sinc}(t/T_b)}$

↳ optimal Nyquist pulse

or, min.^{m} B/W to signal at rate R_b is $R_b/2$ (one sided).

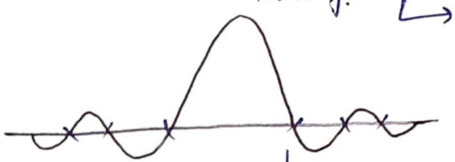
H/W: compare with B/W for rectangular pulse for the same rate.



not much loss when cutoff happens
⇒ Good rolloff

↳ not suitable
↳ ISI (large main lobe of the sinc)

ideal $g(t) \times \underbrace{rc(t)}_{\text{some } f^n}$ ⟶ Raised cosine pulse shape
↳ To get good rolloff.



↳ Zero at integral T_b's

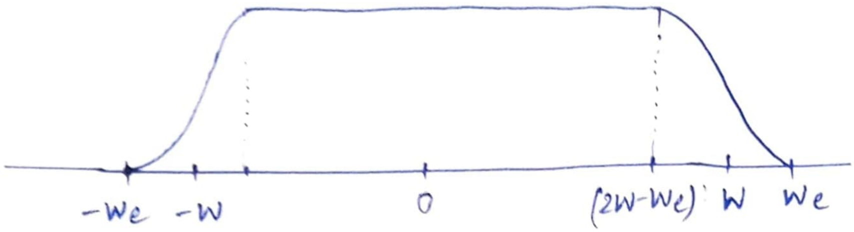
Effective pulse shape:

parameter to tune (choose)

$$\text{raised cosine } g(t) = \underbrace{2W \,\text{sinc}(2Wt)}_{=\sin\left(\frac{t}{T_b}\right) \text{ for } W=\frac{1}{2T_b}} \times \underbrace{\frac{\cos(2\pi(W_e - W)t)}{1-(4(W_e-W)t)^2}}_{rc(t)}$$

$$G(f) = \begin{cases} 1, & |f| < 2W - W_e \\ \cos^2\left(\frac{\pi}{4}\,\frac{|f| + W_e - 2W}{W_e - W}\right), & 2W - W_e < |f| < W_e \\ 0, & |f| > W_e \end{cases}$$

G(f) :



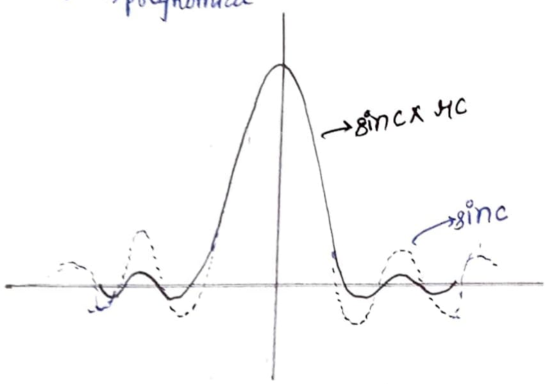
~~2W sinc~~

$\underbrace{g(t)}_{\text{sinc}\left(\frac{t}{T_b}\right)} \times \underbrace{rc(t)}_{\sim \frac{1}{t^2}}$:

$1 \rightarrow$ bounded $\rightarrow \cos^2$

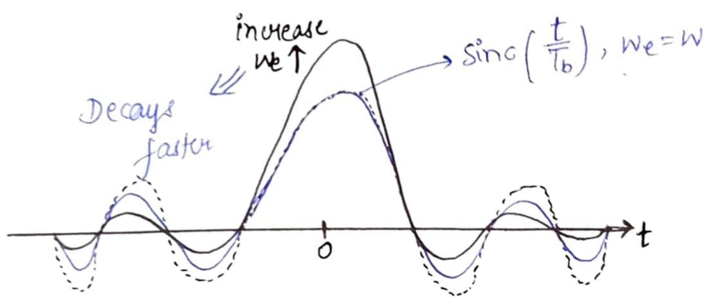
$t^2 \rightarrow$ polynomial

if $W = \frac{1}{2T_b}$



$\boxed{17\text{-}02\text{-}2025}$

g(t) :



→ Truncate

⇓

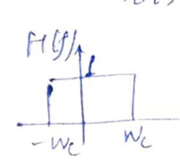
→ Spread above We (ISI)

↓

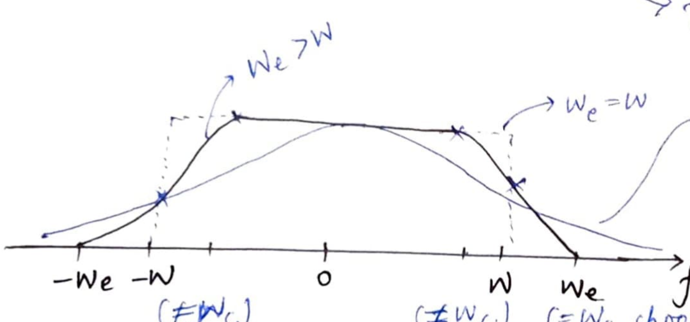
but ripple will be smaller above We

Bandlimited channel

h(t)

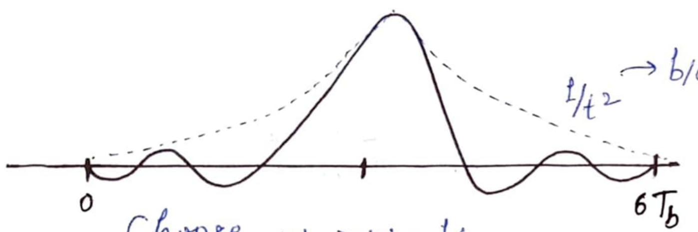


G(f) :



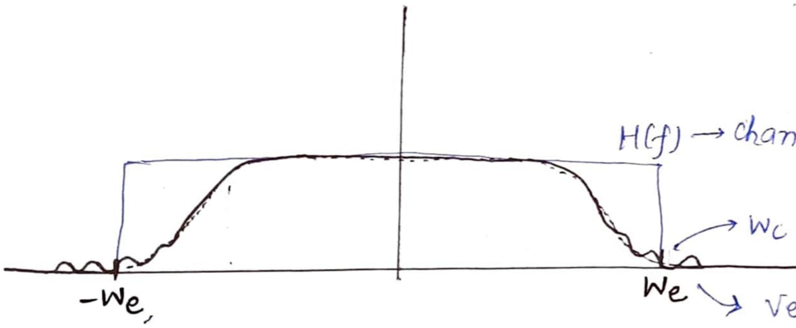
Truncated version of this g(t) is what we aim to achieve.

A more practical scenario:



Choose $W_e > W = 1/2T_b$

$p(t) \xleftrightarrow{F} P(f)$.

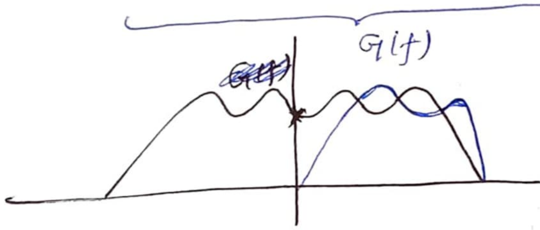


<u>What should we do if we have noise?</u>

$$s(t) \longrightarrow \boxed{p(t)} \longrightarrow \boxed{h(t)} \xrightarrow{y(t)} \boxed{q(t)} \longrightarrow g(t)$$

$\underbrace{\qquad\qquad\qquad\qquad}_{G(f)}$

Matched filter: $y(T_y - t)$



$Y(f) \longleftrightarrow y(t)$

$Y^*(f) \longleftrightarrow y(-t)$

$\Rightarrow G(f) = |Y(f)|^2$. (some phase)

$\sqrt{G(f)} = Y(f)$..

LAB Session

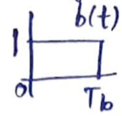
Baseband Modelling:

$y = \text{channel}(x, fs, fc, g)$

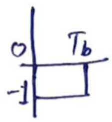
$h = \text{firpm}(n, [0, \frac{fc}{fs/2},], [g, g, 0, 0])$

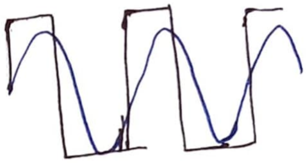
$y = x * h$

Effect of ISI:

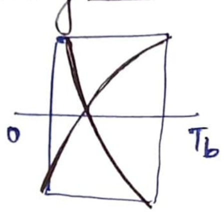


$T_b = 0.1s$





Eye diagram



Ideal:

Recall

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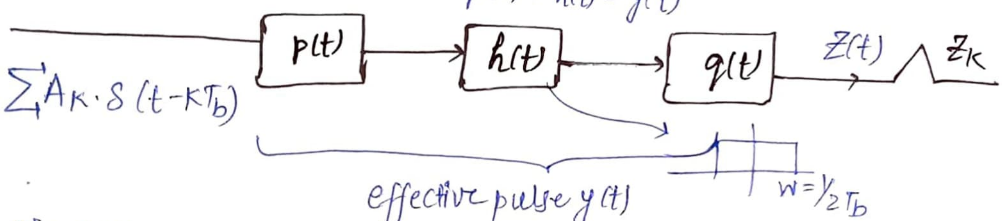
$p(t) * h(t) = y(t)$



No ISI $\Rightarrow g_0 \neq 0,\ g_i = 0\ \forall i$, g_i : from sampling $g(t)$.

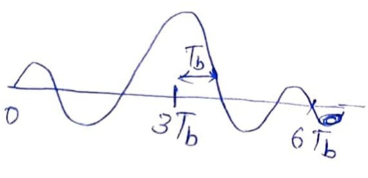
Theoretically, sinc raised cosine are band limited $g(t)$ which satisfy in ISI conditions.

Practically, truncated pulse shapes — $g(t)$ can be used.

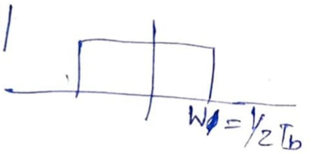
Truncation $\Rightarrow$ Some ISI

Truncated sinc

$p(t) \equiv$



$h(t) \longrightarrow |H(f)|$



$g(t) = \delta(t)$ (all pass)

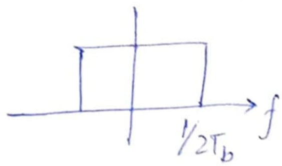
$g(t) \approx p(t)$

} this scheme does not consider noise.

→ No ISI + Matched filter

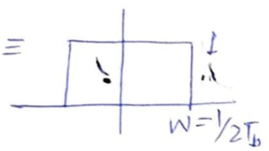
where $g(t)$ is $\text{sinc}\left(\frac{t-T_b}{T_b}\right)$

$G(f) \equiv$



In this case,

$H(f) \equiv$



$G(f) = \boxed{P(f) \cdot H(f)} \cdot Q(f)$

$Q(f)$ being a matched filter?

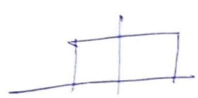
$q(t) = y(T_b - t)$

$Q(f) = e^{-j2\pi f T_b} \cdot Y(f)$

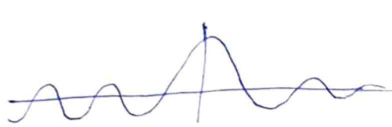
$Q(f) = e^{-j2\pi f T_b} \, P^*(f) \cdot H^*(f)$.

$G(f) = P(f) \cdot P^*(f) \cdot e^{-j2\pi f T_b} \, |H(f)|^2$.

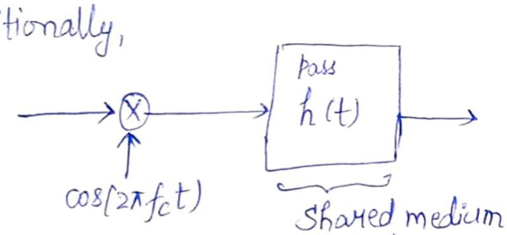
$P(f) \equiv$



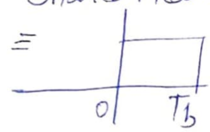
$p(t) \equiv$

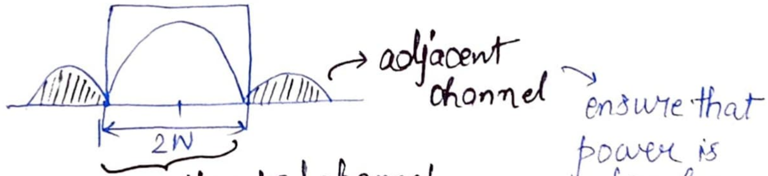


Additionally,



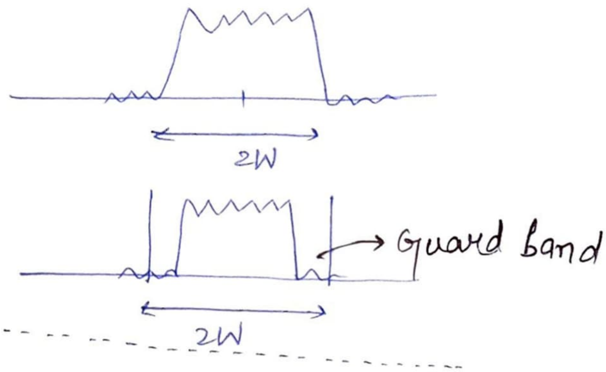
Suppose $p(t) \equiv$





Suppose $p(t)$

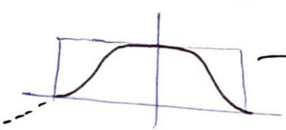
For truncated sinc,



→ Any other ~~no~~ (nyquist) pulse which satisfy no ISI (not just sinc) can be used.

Redo for $g(t)$ = raised cosine

for $H(f)$ =



→ raised cosine

$G(f)$

\# Sklar

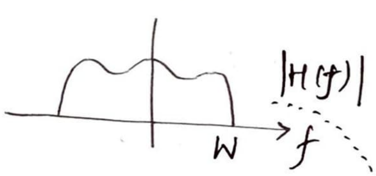
$P(f) = \sqrt{G(f)}$ → Square Root Raised cosine (SRRC) signal.

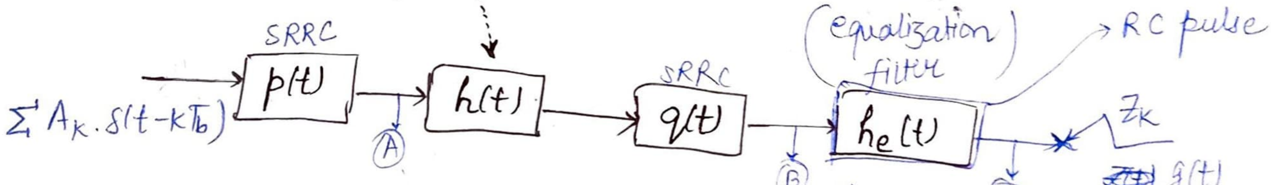
↓

$p(t)$ → SRRC pulse

Equalisation (for ISI)

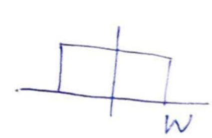
\# Sklar – 3.4





$p(t), q(t)$ = SRRC under assumption that

$h(t) \approx H(f)$ ideal



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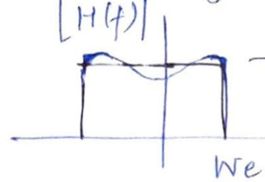
Now, the channel

$$h(t) \xrightarrow{F} H(f)$$



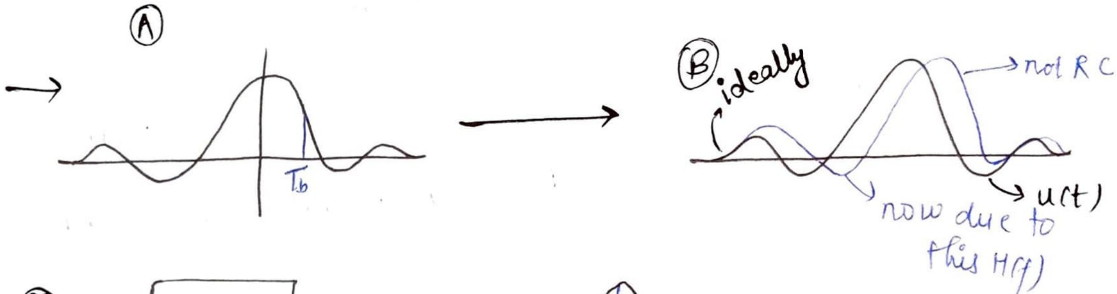
→ freq-selective

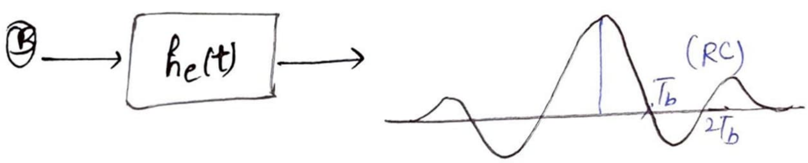
We still want no ISI

$$H(f) = |H(f)|\, e^{-j\theta(f)}$$

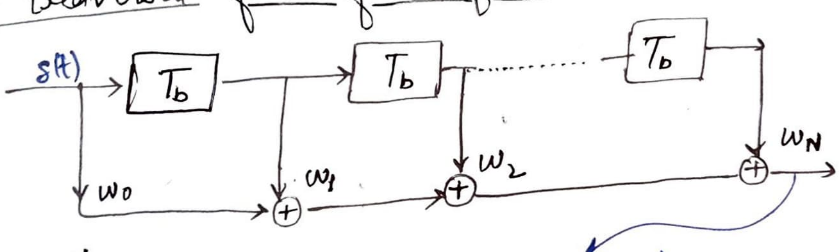
$$H_e(f) = \frac{1}{|H(f)|} \cdot e^{-j\alpha f + j\theta(f)}$$

Sklar → Max likelihood sequence det.
↳ Viterbi equaliser





Linear traversal filter form for $h_e(t)$:



$$\sum_{n=0}^{N} w_n \delta(t - nT_b) \rightarrow \text{equally spaced equalizer}$$

Sampled output : $$g(t) = \sum_{n=0}^{N} w_n \cdot u(t - nT_b)$$

$$\left.\begin{aligned} g(T_b) &= g_0 = 1 \\ g(2T_b) &= g_1 = 0 \\ &\vdots \\ g_i &= 0 \end{aligned}\right\} \text{No ISI condition}$$

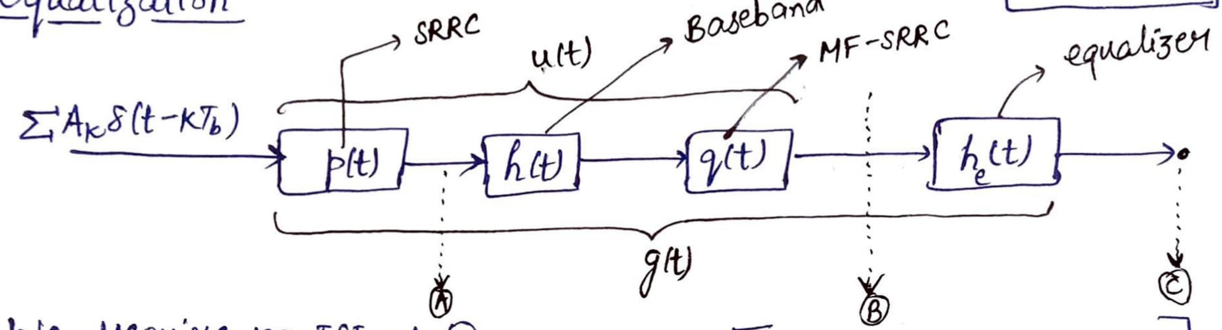
$$\text{no ISI} \Rightarrow \sum_{n=0}^{N} w_n\, u(\underbrace{t}_{iT_b} - nT_b)$$

$$g_0 = 1 \Rightarrow \sum_{n=0}^{N} w_n\, u(T_b - nT_b) = 1$$

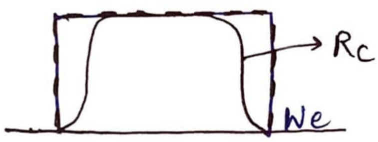
$$g_1 = 0 \Rightarrow \sum_{n=0}^{N} w_n\, u(2T_b - nT_b) = 0.$$

Equalization

24-02-2025



We require no ISI at C.
Given a bandlimited and freq. selective $H(f)$, say



$$\left[\begin{aligned} u(t) &= p(t) * h(t) * q(t) \\ g(t) &= u(t) * h_e(t) \end{aligned}\right]$$

= not a RC @ c, there will be ISI

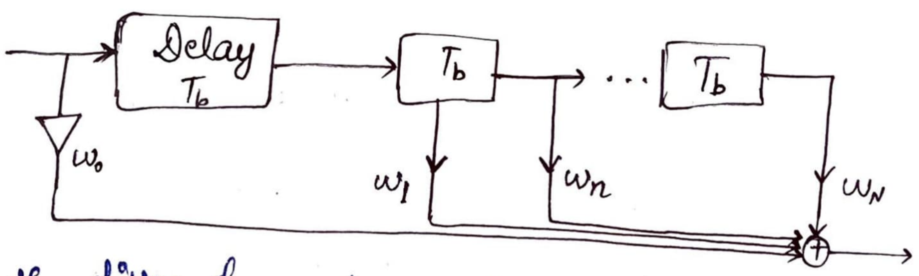
Equalizer design:

$$H_e(f) = \frac{1}{H(f)} \quad \text{within } [-W_c, W_c].$$

A particular arch. for $h_e(t)$:

Linear transversal form:

$$h_e(t) = \sum_{n=0}^{N} w_n \cdot \delta(t - nT_b)$$

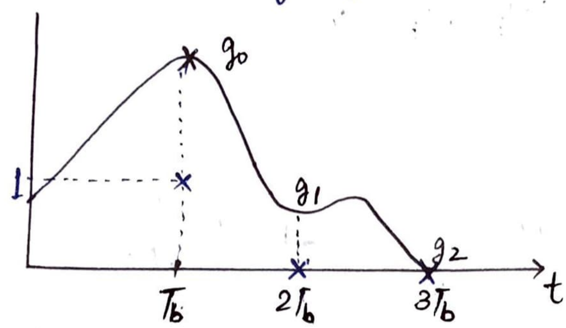


For the filter shown above, what is $g(t)$?

$$g(t) = \sum_{n=0}^{N} w_n \cdot u(t - nT_b)$$

Our requirement :

No ISI conditions for $g(t)$.



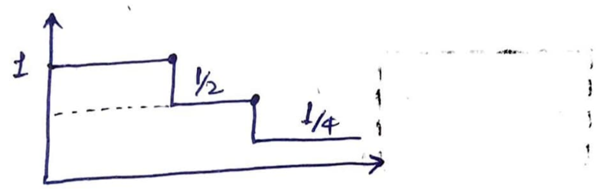
We want the equalizer :

$g_0 = 1$

$g_1 = 0$, $g_i = 0 \;\forall\; i \neq 0$.

$g_2 = 0$.

$u(t)$: $\longrightarrow$ assume



$$g(t) = \sum_{n=0}^{N} w_n \cdot u(t - nT_b) \longrightarrow u(kT_b - nT_b) = u((k-n)T_b) = u_{k-n}$$

Suppose $N=2$.

$$g_0 = g(T_b) = \sum_{n=0}^{2} w_n \cdot u((1-n)T_b)$$

$$= w_0 \cdot 1 + w_1 \cdot 0 + w_2 \cdot 0$$

$$g_1 = g(2T_b) = w_0 \cdot \tfrac{1}{2} + w_1 \cdot 1 + w_2 \cdot 0$$

$$g_2 = g(3T_b) = w_0 \cdot \tfrac{1}{4} + w_1 \cdot \tfrac{1}{2} + w_2 \cdot 1$$

$$g_3 = g(4T_b) = w_0 \cdot 0 + w_1 \cdot \tfrac{1}{4} + w_2 \cdot \tfrac{1}{2}$$

$$g_4 = g(5T_b) = w_0 \cdot 0 + w_1 \cdot 0 + w_2 \cdot \tfrac{1}{4}$$

constraints by our requirements

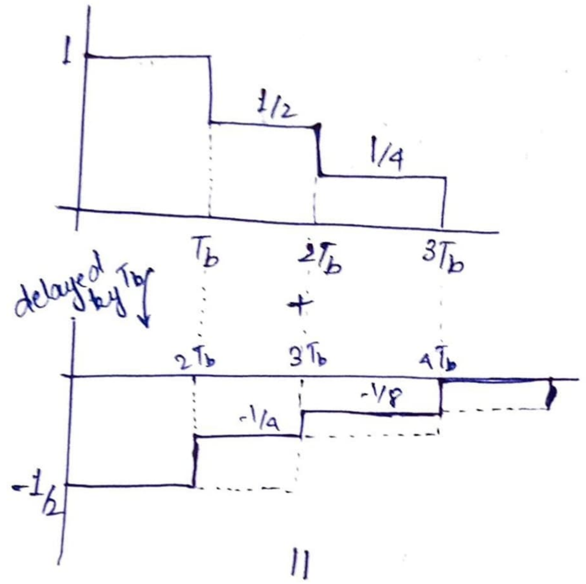
constraints by the system ($p(t), q(t)$).

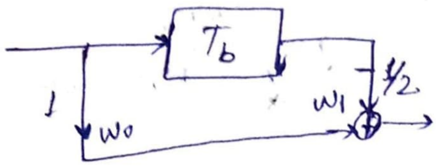
Zero forcing equalizer:

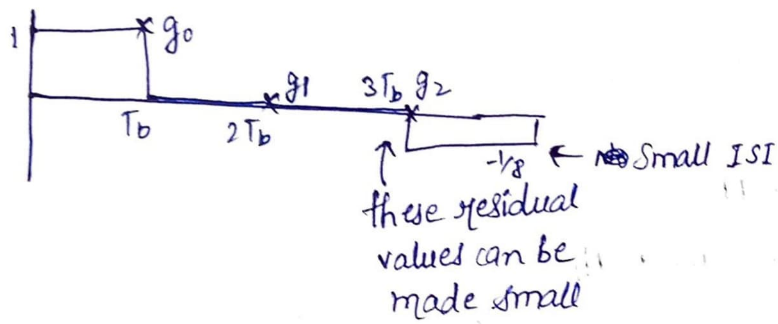
$$1 = w_0 \cdot 1$$

$$0 = w_0 \cdot \tfrac{1}{2} + w_1 \Rightarrow w_1 = -\tfrac{1}{2}$$

$$0 = w_0 \cdot \tfrac{1}{4} + w_1 \cdot \tfrac{1}{2} + w_2 \Rightarrow w_2 = 0$$







Equalizer:
- symbol-spaced equalizer
- fractional-spaced equalizer

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$$\longrightarrow \boxed{p(t)} \longrightarrow \boxed{h(t)} \longrightarrow \boxed{q(t)} \longrightarrow \boxed{h_e(t)} \longrightarrow g(t)$$

$$h_e(t) = \sum_{n=0}^{N} w_n \delta(t - nT_b)$$ → linear traversal structure with delays, sums

$$g(t) = \sum_{n=0}^{N} w_n u(t - nT_b)$$

No ISI conditions:

$$g_0 = g(T_b) = 1 \Rightarrow \sum_{n=0}^{N} w_n u(T_b - nT_b) = 1$$

$$g_i = g((i+1)T_b) = 0 \Rightarrow \sum_{n=0}^{N} w_n \cdot u((i+1)T_b - nT_b) = 0$$

Zero-forcing eq^ns : Choose N_1 equations, solve for w_ns.

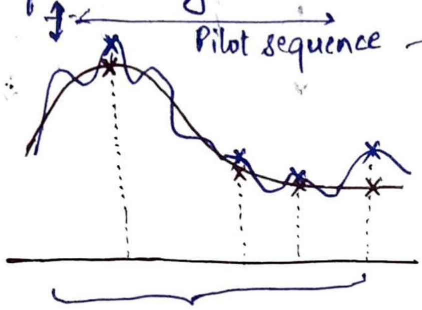
Suppose channel $h(t)$ is static. (Ideal case)

Find out $h(t)$ and precompute w_ns.

But this cannot be done.

So, practically, we need to compute w_ns in an "online" manner.

Pilot sequence → used for computing the term.



Zero-forcing compute w_ns → $\boxed{h_c(t)}$

Minimum Mean-Square Error Equaliser (MMSE)

Take (N+1) equations,

$$g_i = \sum_{n=0}^{N} w_n \cdot u_{i+1-n} \quad ; \quad u_{i+1-n} = u((i+1)T_b - nT_b)$$

$$g_0 = w_0 u_1 + N_0$$
$$g_1 = w_1 u_2 + w_1 u_1 + N_1$$
$$g_2 = w_0 u_3 + w_1 u_2 + w_1 u_1 + N_2$$

[$N_0, N_1, N_2, \ldots$ are noises]

$$\boxed{\vec{g} = U\vec{w} + \vec{N}} \quad , \quad \vec{w} = \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_N \end{pmatrix}$$

(m×1) (m×N+1) (N+1 × 1)

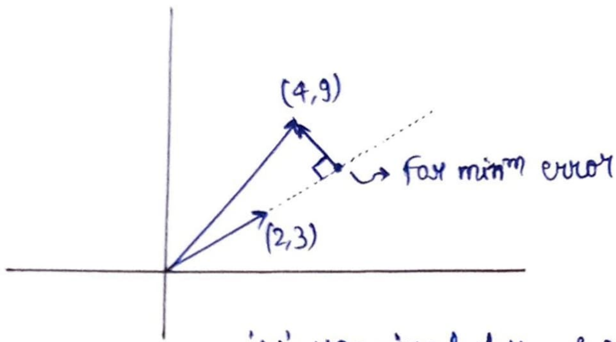
$$\min_{\vec{w}} \|\vec{g} - U\vec{w}\|^2$$ → Min^m MSE equaliser tries to achieve (like Max^m likelihood)

Eg. $\begin{pmatrix} 2 \\ 3 \end{pmatrix} w = \begin{pmatrix} 4 \\ 9 \end{pmatrix}$, with $\vec{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\vec{g} = \begin{pmatrix} 4 \\ 9 \end{pmatrix}$ [Assuming scalar w]

$2w = 4$

$3w = 9$

→ No sol^n



'ω' required for min^m error :

Projection of $\vec{g}$: $\left(\dfrac{u^T g}{\|u\|}\right) \cdot \dfrac{\vec{u}}{\|u\|}$

$$= \frac{u^T g}{\underbrace{\|u\|^2}_{u^T u}} \cdot u$$

$$= \boxed{\boxed{(u^T u)^{-1}\, u^T}\, g} \cdot u \qquad \text{[Adaptive equalizer]}$$

pseudo-inverse of $u = (u^T u)^{-1} \cdot u^T$
($\because$ u is not a square matrix)

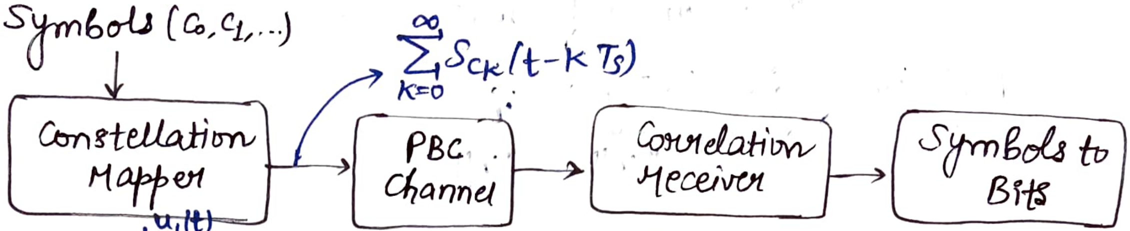
Passband Systems

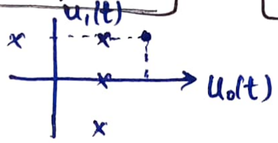
ch. 4 of Sklar

Recall

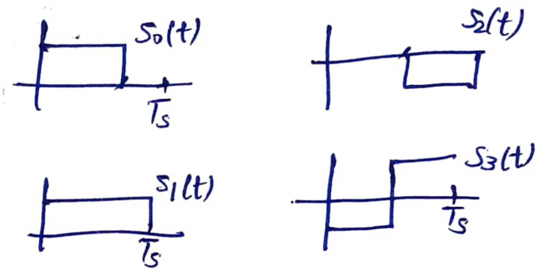
$(s_0(t), s_1(t), s_2(t), s_3(t))$ → Gram Schmidth → $\boxed{u_0(t), u_1(t)}$ → Passband signals

Bits $(B_0, B_1, \dots)$
↓
Symbols $(c_0, c_1, \dots)$
↓



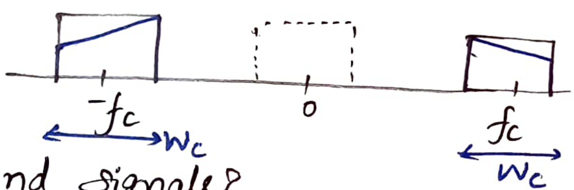


Bits	Symbols	Signals
0 0	0	$s_0(t)$
0 1	1	$s_1(t)$
1 0	2	$s_2(t)$
1 1	3	$s_3(t)$



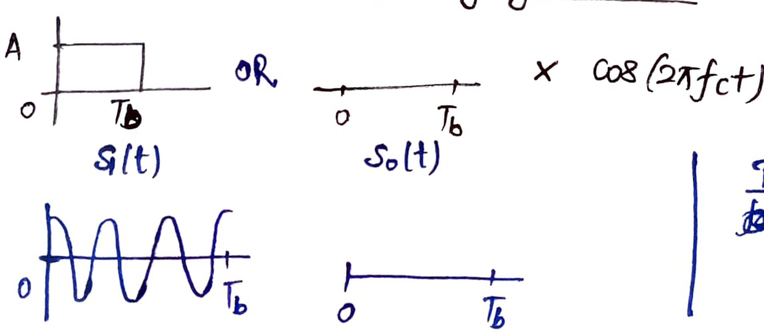
T_s ↳ symbol duration

Channel

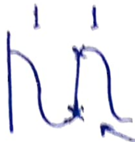


Passband signals?
↳ Obtained by AM/FM baseband signals

Binary Amplitude Shift Keying (BASK)

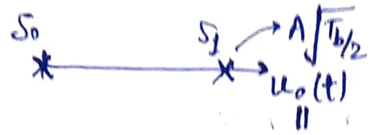


T_b is integer multiple of $1/f_c$
⇓
continuous



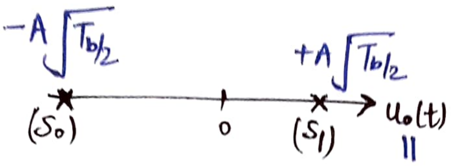
phase discontinuity

↳ if T_b is not $\frac{n}{f_c}$.



$$= \sqrt{2/T_b}\cos(2\pi f_c t)$$

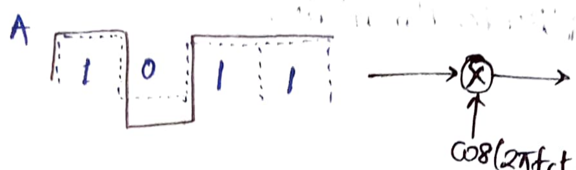
Binary Phase Shift Keying (BPSK)

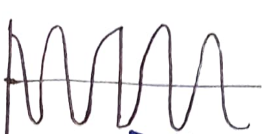


$$= \sqrt{2/T_b}\cos(2\pi f_c t)$$

$1 \longrightarrow A\cos(2\pi f_c t)$

$0 \longrightarrow A\cos(2\pi f_c t + \pi)$





discontinuity

Binary Frequency shift Keying (BFSK)

$1 \longrightarrow A\cos(2\pi(f_c+\Delta f)t)$

$0 \longrightarrow A\cos(2\pi(f_c-\Delta f)t)$

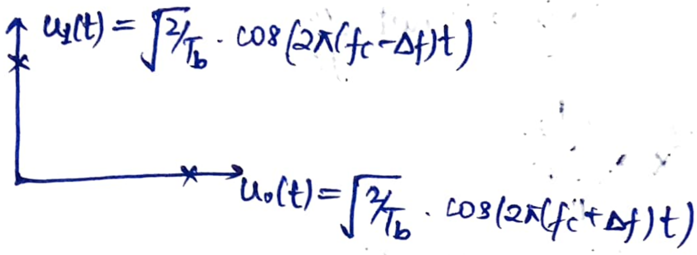
↳ Requires 2D constellation

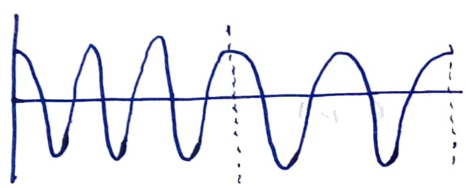
$\left.\begin{matrix} f_c+\Delta f \\ f_c-\Delta f \end{matrix}\right\}$ orthogonal $\Rightarrow A^2\int_0^{T_b}\cos(2\pi(f_c+\Delta f)t)\cdot\cos(2\pi(f_c-\Delta f)t)\,dt = 0$

$$\Rightarrow \int_0^{T_b} \cos(2\pi\, 2\Delta f\, t)\, dt = 0 \,, \quad \int_0^{T_b} \cos(2\pi\, 2f_c t)\, dt = 0$$

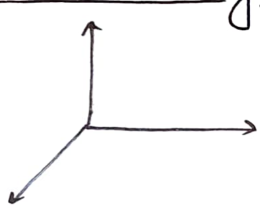
$$\Rightarrow \frac{n}{2\Delta f} = T_b \,, \quad \frac{n}{2f_c} = T_b$$

$$\Rightarrow \Delta f = \frac{n}{2T_b}$$



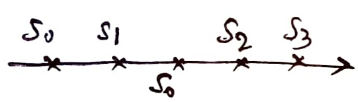


M-FSK or M-ary FSK



$\log_2 M$ symbols for M bits

M-ary PAM [Pulse Amplitude Modulation]



20-08-2025

Recall

Bits → symbols → $\boxed{\text{Constellation mapper}}$ → $\boxed{\text{PBC}}$

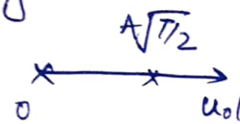
Bits	Symbols	Signals (Passband)
0 0	0	$s_0(t)$
0 1	1	$s_1(t)$
1 0	2	$s_2(t)$
1 1	3	$s_3(t)$

Symbols 0–3: M

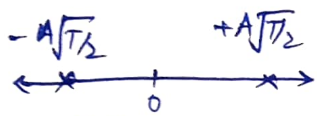
Signals → $\boxed{\text{Signal space}}$

Basis : $u_0(t)$, $u_1(t)$

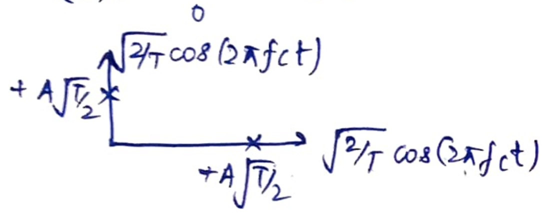
Binary Schemes:



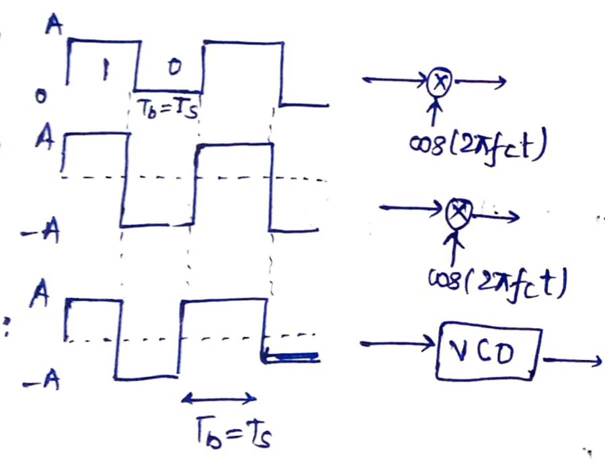
$u_0(t) = \sqrt{2/T}\cos(2\pi f_c t)$ $\boxed{\text{BASK}}$



$\boxed{\text{BPSK}}$

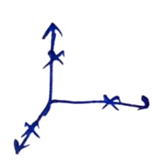


$\boxed{\text{BFSK}}$

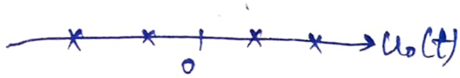


Mary FSK :

$f_0, f_1, \ldots, f_{M-1}$

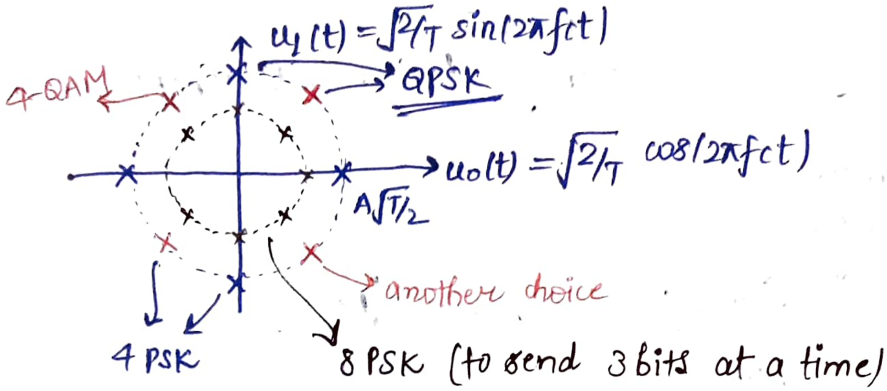


M-ary PAM :



M-ary PSK :

0	$A\cos(2\pi f_c t)$
1	$A\cos(2\pi f_c t + \pi/2)$
2	$A\cos(2\pi f_c t + \pi)$
3	$A\cos(2\pi f_c t + 3\pi/2)$



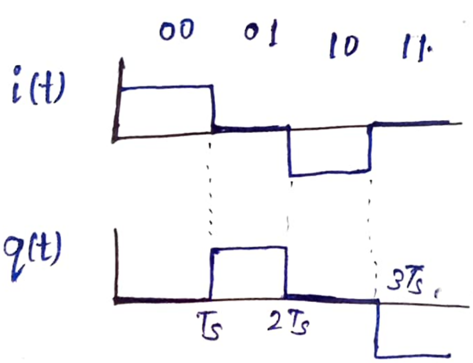
APK (Amplitude-Phase Keying)

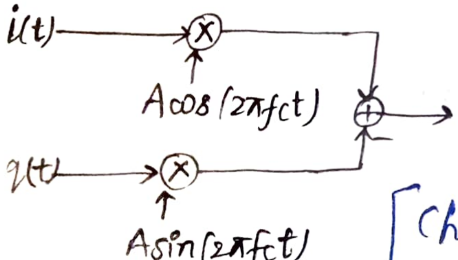
↳ Combine PSK and PAM

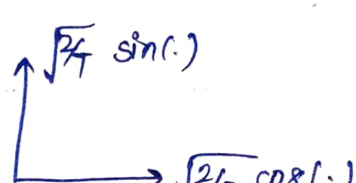
X X X → 16 APK → 4 bits at a time
↓ ↓ ↓
4 4 8

$A\cos(2\pi f_c t + \phi_i)$

$$= A\cos(2\pi f_c t)\underbrace{\cos(\phi_i)}_{i(t)} - A\sin(2\pi f_c t)\underbrace{\sin(\phi_i)}_{q(t)}$$



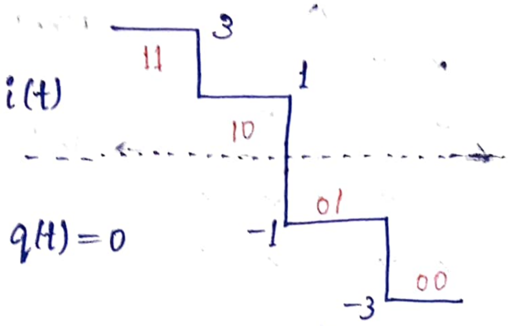




[Choose appropriate $i(t)$, $q(t)$

$$p(t) = i_p(t)\cos(2\pi f_c t) - q_p(t)\sin(2\pi f_c t).$$

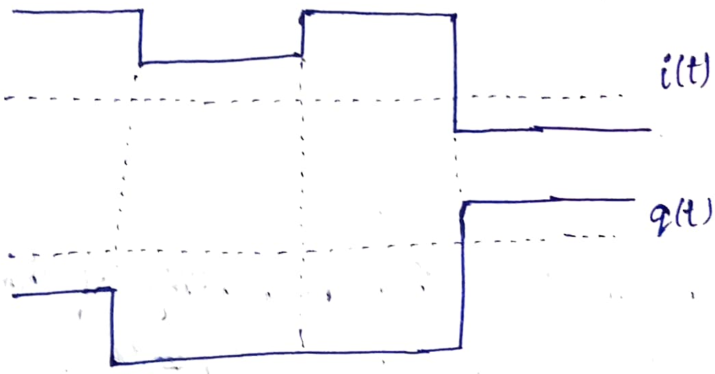
M-ary PAM :



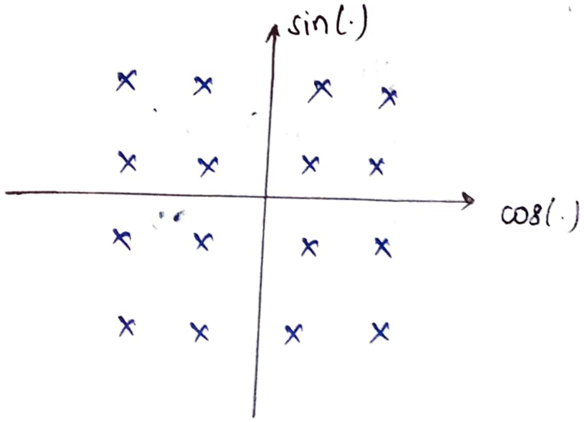
$\boxed{11}$ (01) $\boxed{10}$ (00) $\boxed{11}$ (00) $\boxed{01}$ (10)

3 1 3 -1

i(t) ↗ ↑ q(t)



i(t) → 4 possibilities
q(t) → 4 possibilities
} 16 possibilities
↳ 4 bits per symbol



→ M-ary QAM
(16-ary QAM)

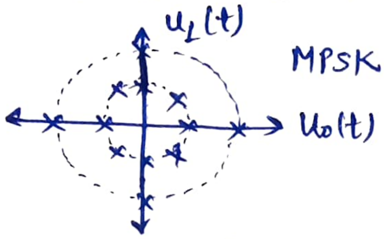
4-QAM
(Quadrature QAM)
= 4-PSK
(QPSK)
↓
(4× or +×)

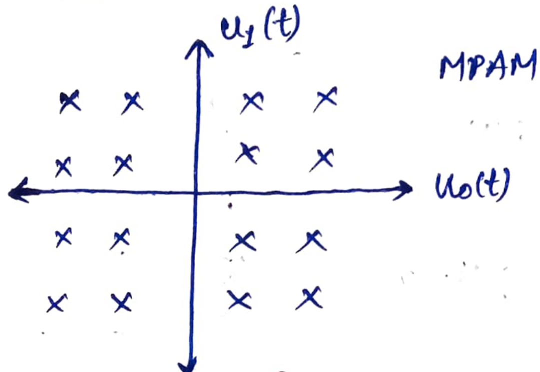
24-03-2025

Recall

Passband Digital Communication:

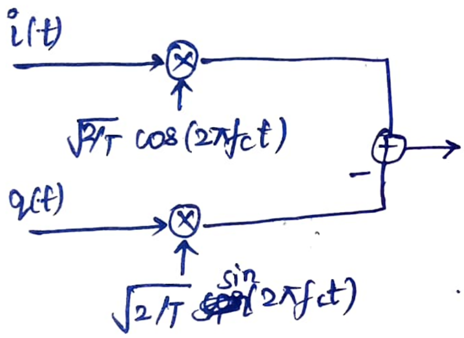
Bits $(B_0, B_1, \dots)$ $\longrightarrow$ Symbols $(c_0, c_1, \dots)$ $\longrightarrow$ $\boxed{\text{Constellation Mapper}}$ $\longrightarrow$

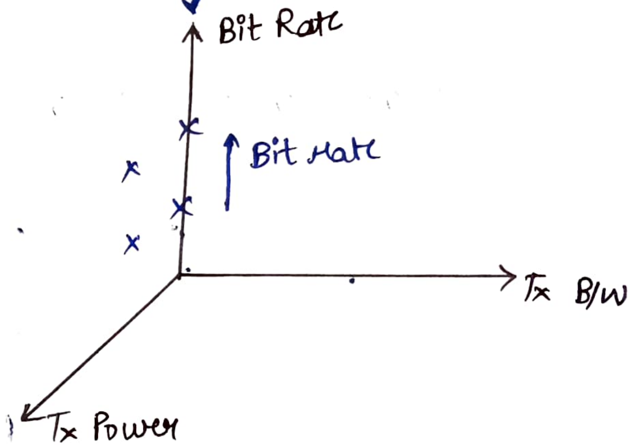




$u_0(t) = \sqrt{2/T}\cos(2\pi f_c t)$

$u_1(t) = \sqrt{2/T}\sin(2\pi f_c t)$



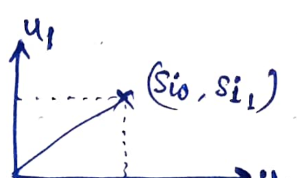


→ If bits are Bernoulli (½), then probability of choosing symbol is $1/M$, otherwise find this (for $p \neq 1/2$).

$$s_1(t) = \frac{1}{T_s}\int (s_i(t))^2\, dt \longrightarrow P_i$$

$$\text{avg}(P_i) = \sum_{i=1}^{M} \frac{1}{M} P_i = \frac{1}{M}\sum_{i=1}^{M} P_i$$

$$P_i = s_{i_0}^2 + s_{i_1}^2$$



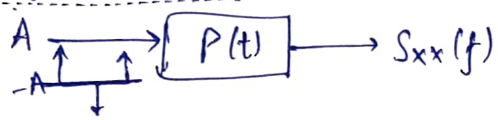
Eg.

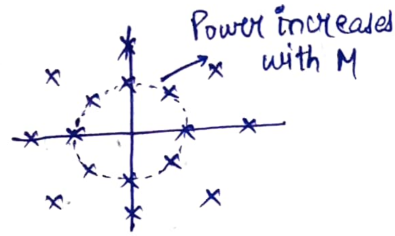
Eg.

$$E_b = \frac{\frac{1}{M}\sum_{i=1}^{M} \|s_i\|^2}{\log_2 M}$$

Tx BW:

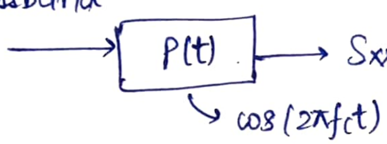
Recall in baseband:



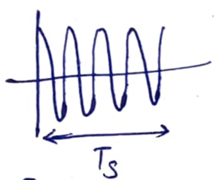


PSD for BPSK

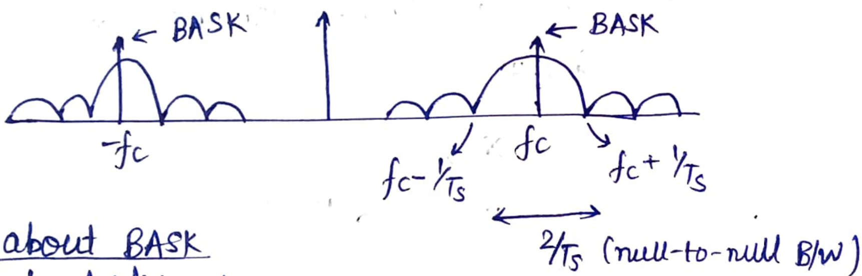
Passband



$$S_{xx}(f) = \frac{A^2}{T_s}|P_i|^2$$

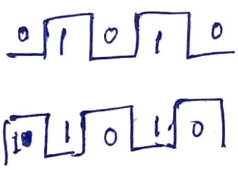


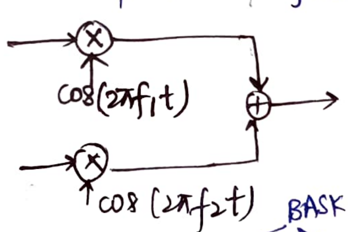
$$P(t) = I[0, T_s] \cdot \cos(2\pi f_c t)$$

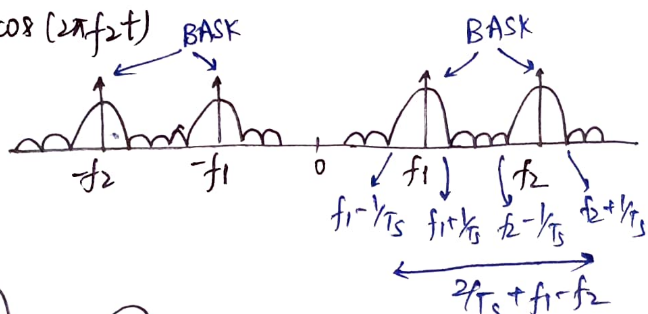


What about BASK

↳ Additional carrier component at f_c and $-f_c$.

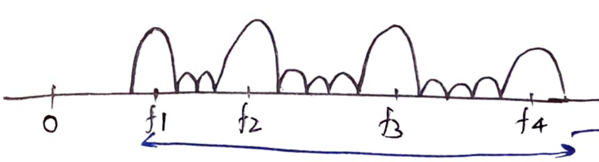






M-FSK

4-FSK

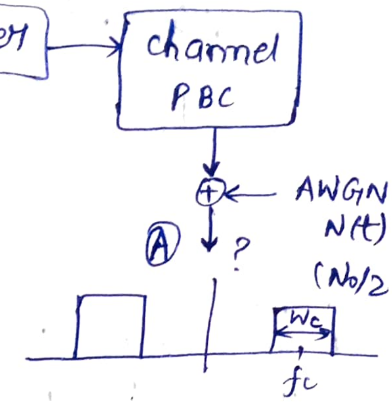


$\rightarrow f_4 - f_1 + 2/T_s$

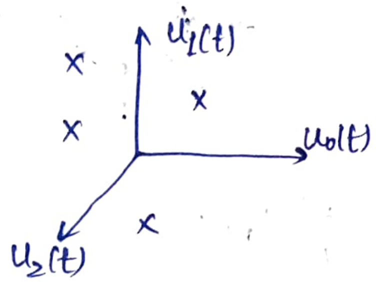
26-03-2025

Passband Digital Communication – Receiver

Bits — Symbols → [Constellation mapper] → [channel PBC]



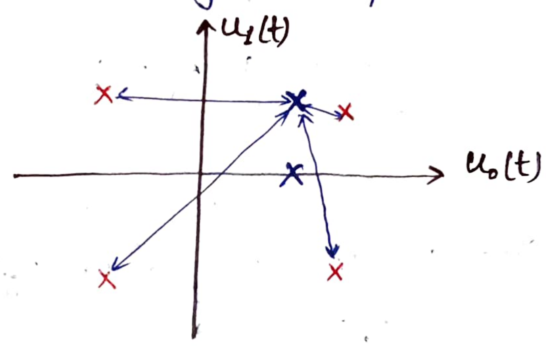
Bits	Symbols	Signals
00	0	$S_0(t)$
01	1	$S_1(t)$
10	2	$S_2(t)$
11	3	$S_3(t)$



Receivers – Coherent / Non-coherent Receivers

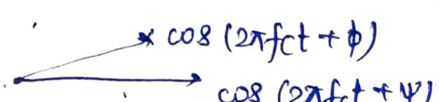
Assuming that timing sync is there. . . .

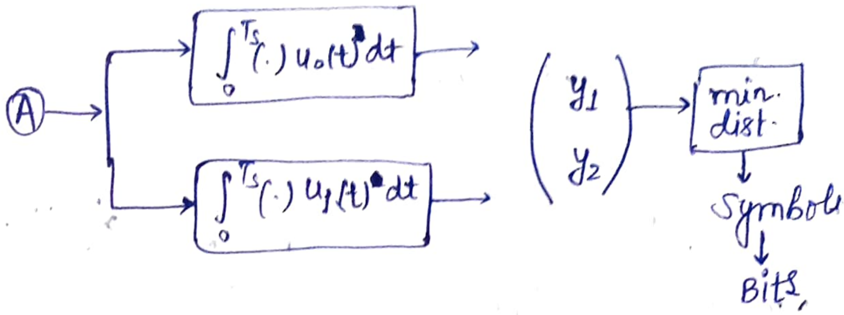
Coherent Receivers: we have a LO at the r_x which can generate a carrier signal in phase with the $r_x$ed carrier signal.



$\cos(2\pi f_c t)$ → → $\cos(2\pi f_c t + \phi)$

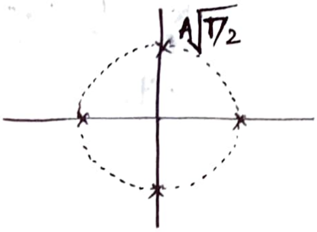
$\cos(2\pi f_c t + \psi)$



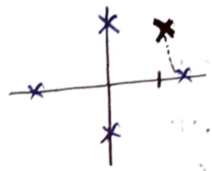


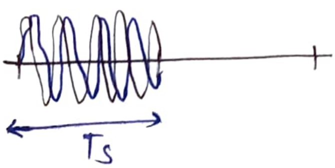
A
$\int_0^{T_s} (\cdot)\, u_0(t)^* dt$
$\int_0^{T_s} (\cdot)\, u_1(t)^* dt$
y_1
y_2
min. dist.
Symbols
Bits

Eg.

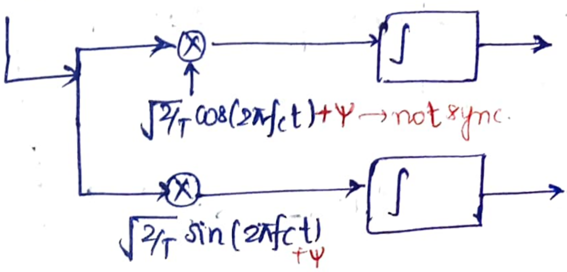


$A\sqrt{T/2}$

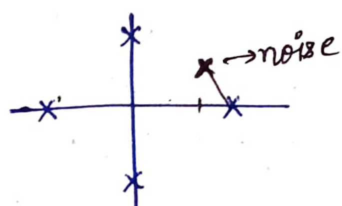




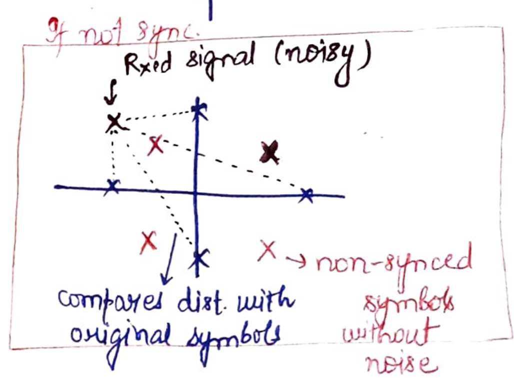
T_s



$\sqrt{2/T}\cos(2\pi f_c t) + \psi \rightarrow$ not sync.
$\sqrt{2/T}\sin(2\pi f_c t) + \psi$



noise

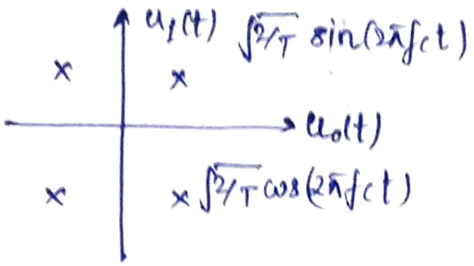


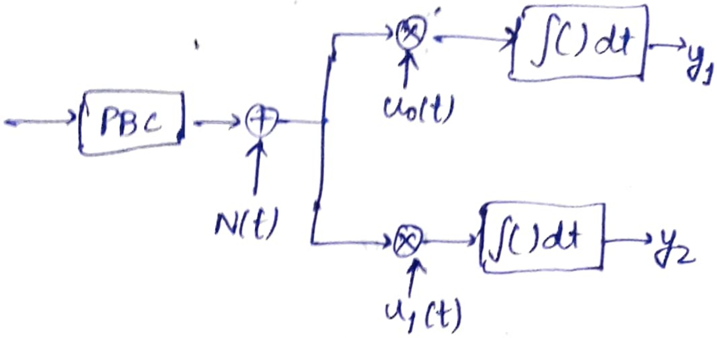
If not sync.
Rx'd signal (noisy)
compares dist. with original symbols
X → non-synced symbols without noise

27-03-2025

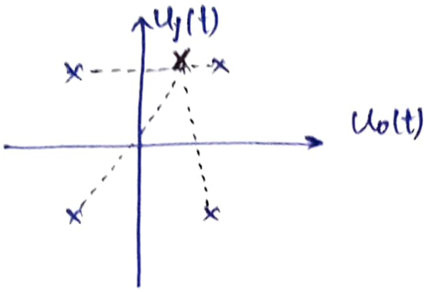
Passband Digital Communication Receivers

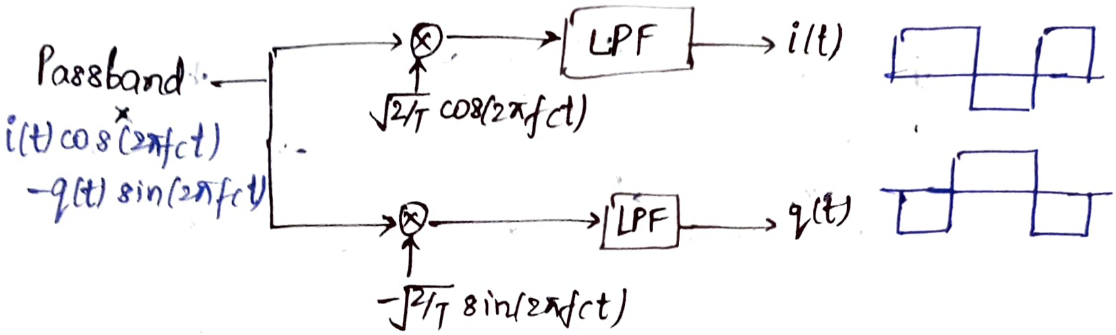
Coherent receiver



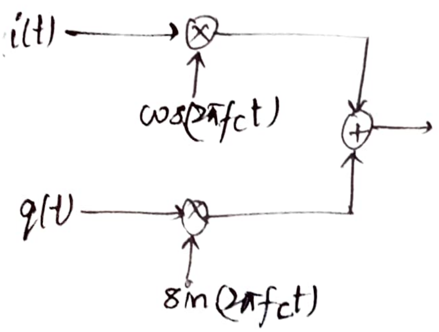


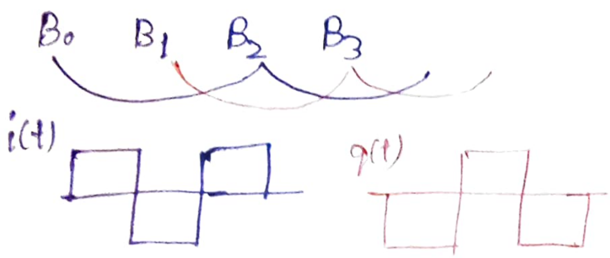
Min^m distance rule:





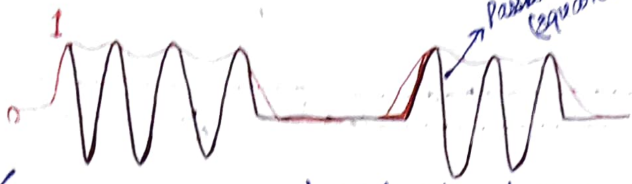
Eg. Recall QPSK



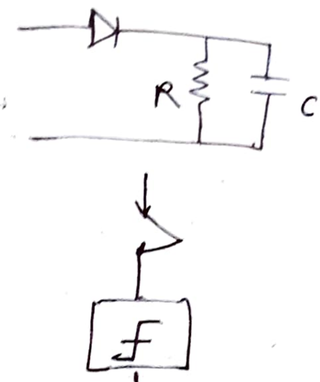


Non-Coherent Receivers

BASK / OOK:

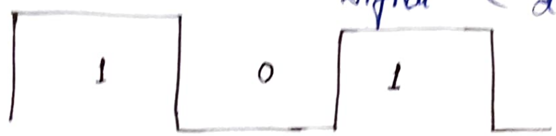


→



[## Sklar Differential PSK]

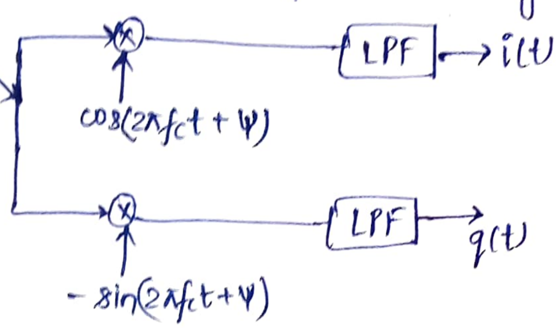
↓ get baseband signal (use envelope detector)



↓ Make decisions

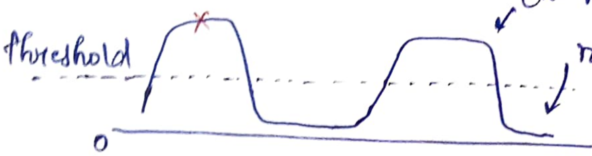
Baseband signal

Another receiver: (to get baseband signal)



→ Energy detector

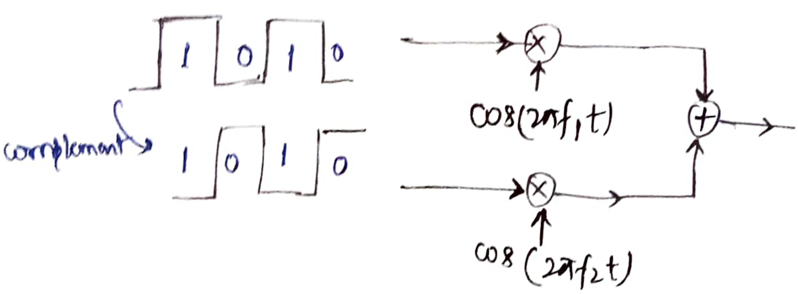
$i^2(t) + q^2(t)$

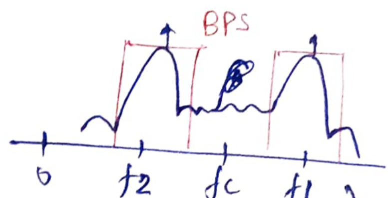


[~ complex Baseband energy]

→ sampling synchronization still needed

BFSK:





BASK Rx   BASK Rx

Make decisions

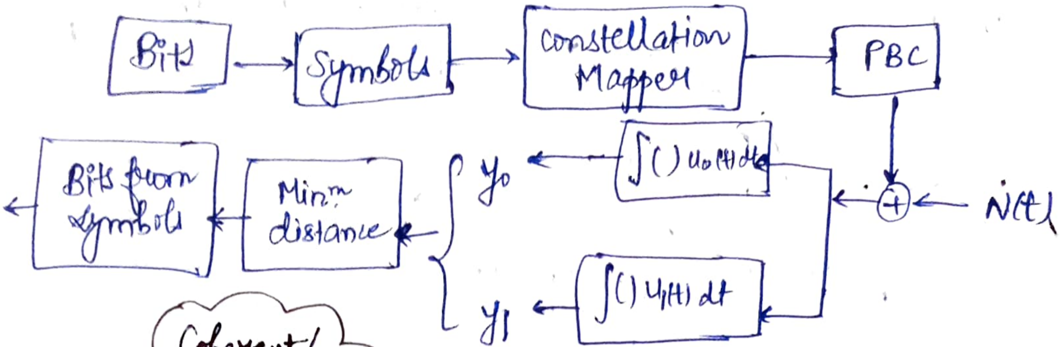
[We can make joint decisions here to check for error]

(Check if bits are complement)

(Take the one far away from threshold in case of error)

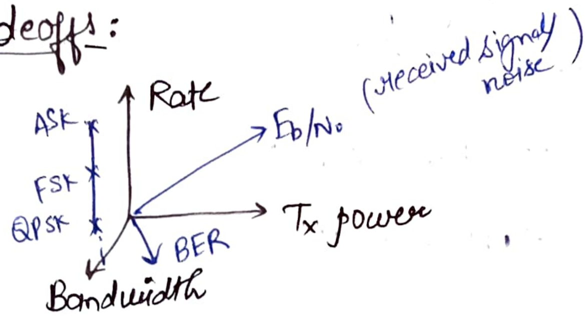
02-04-2025

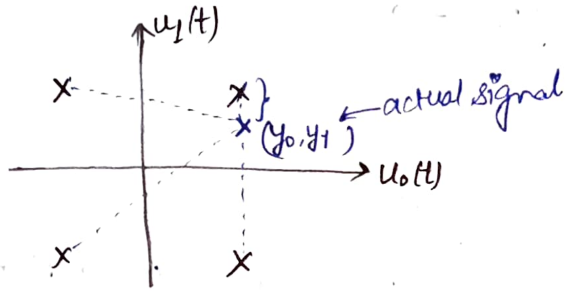
Recall

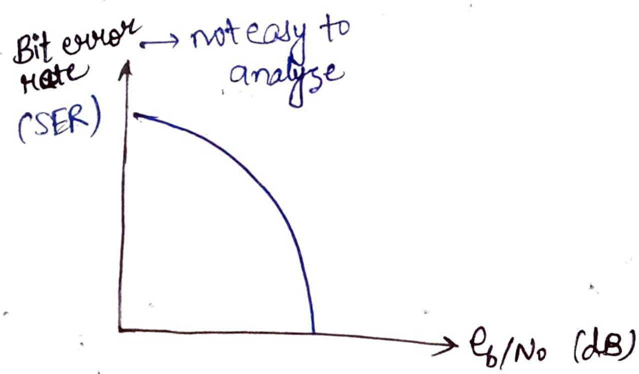


Coherent/ Non Coherent

Tradeoffs:





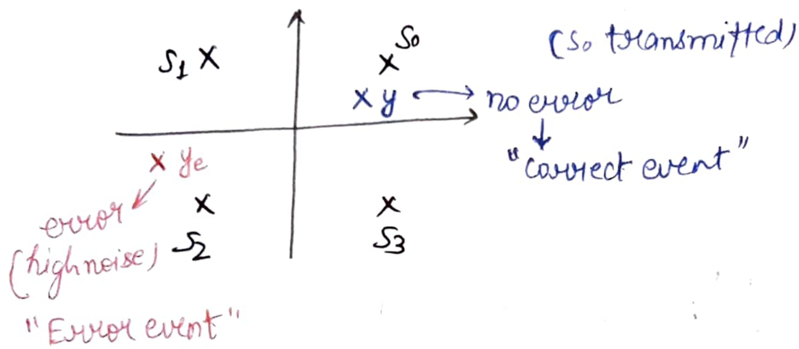


#Read Upamonyu Madhow for this part

SER: Symbol error rate → Easier to analyze

$$= \frac{\sum_{i=1}^{N_s} \mathbb{I}\{\hat{a}_i \neq a_i\}}{N_s}$$

4-PSK example:



Error events:

$\{md(y) \neq s_i | s_i\}$ → [Error made by Tx-Rx after applying min^m dist. rule, given s_i was transmitted]

$Pr\{md(y) \neq s_i | s_i\}$ → Probability that m.d. rule applying does not give the symbol that was txed. [y: random]

Avg. probability of symbol error

⇓

SER

$$= \underbrace{\sum Pr\{s_i\}}_{=\left(\frac{1}{N}\right)} \cdot Pr\{md(y) \neq s_i | s_i\}$$

, N: no. of symbols/ order of modulation scheme

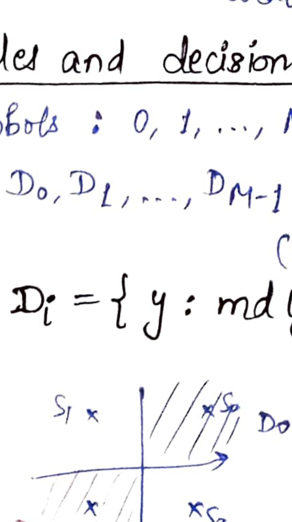
Decision rules and decision regions

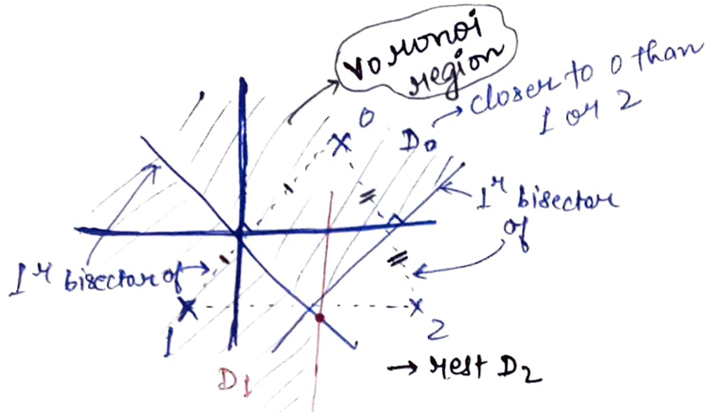
Signals/symbols : 0, 1, ..., M−1

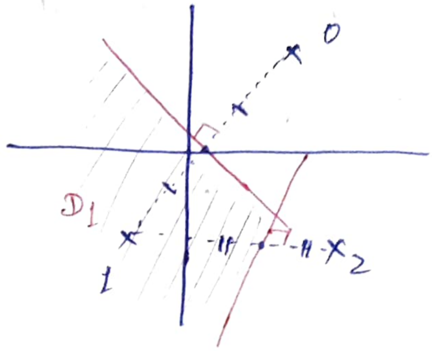
Define $D_0, D_1, \ldots, D_{M-1} \subseteq$ signal space

(subset of signal space)

$D_i = \{y : md(y) = s_i\}$: set of points that Rx thinks have come from transmitting s_i





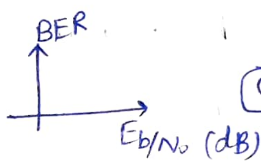


$$P_r\{md(y) \neq s_i \mid s_i\} = P_r\{y \notin D_i \mid s_i\}$$

→ Decision region gives decision rules.
↳ Both are equivalent.

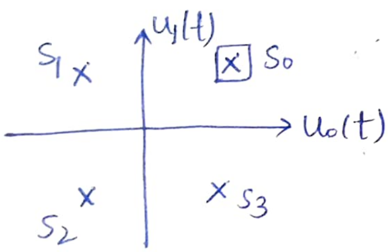
Make decision rule to
minimize value of $\sum_i P(s_i) \cdot P(y \notin D_i \mid s_i)$

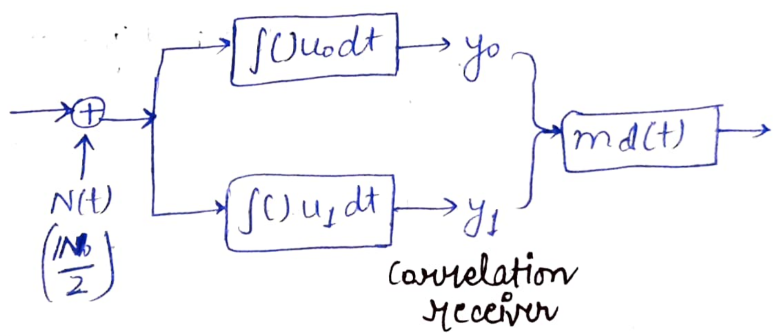
Find out $(D_0, D_1, \ldots, D_{M-1})$
such that $\sum_i P(s_i) \cdot P(y \notin D_i \mid s_i)$
is minimized.



03-04-2025

Recall

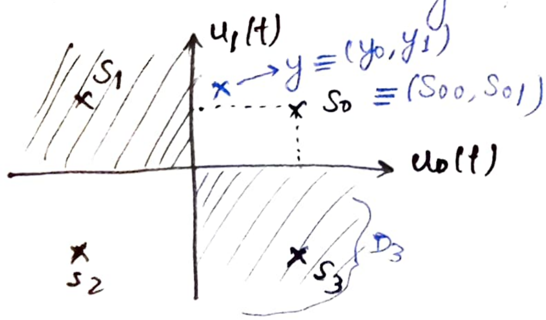




Probability of symbol error

$$= \sum_i P_r\{s_i\} \cdot P_r\{md(y) \neq s_i \mid s_i\}$$

Decision rule = decision regions



Probability of symbol error

$$= \sum_i P_r(s_i) \cdot P_r\{y \notin D_i \mid s_i\}$$

$$D_i = \{y : \text{decision rule}(y) = s_i\}$$

e.g. $md(\cdot)$

$$\left.\begin{array}{l} D_i \cap D_j = \text{null set} \\ \bigcup_i D_i = \text{signal space} \end{array}\right\} \text{partition}$$

<u>Minimum probability of symbol error</u> – How ?

$$P_s = \sum_i P_r\{s_i\} \cdot P_r\{y \notin D_i \mid s_i\}$$

$$P_r\{y \notin D_i \mid s_i\} = \int_{D_i^c} f_{Y|S}(y)\,dy$$

complement

$$\boxed{\begin{array}{l} y_0 = s_{00} + N_0 \rightarrow N(0, N/2) \\ y_1 = s_{01} + N_1 \rightarrow N(0, N/2) \end{array}}$$

$N_0 \perp N_1 \rightarrow$ independent random variables

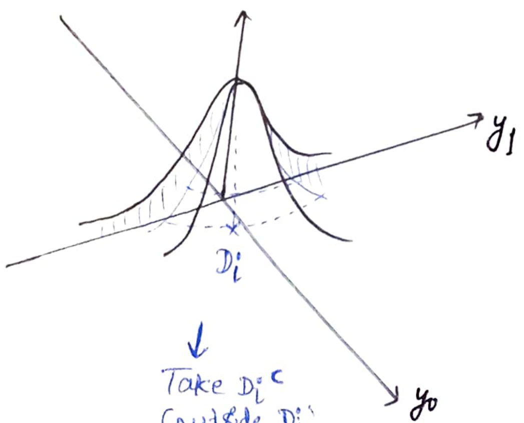
$$f_{Y|s_i}(y = (y_0, y_1) \mid s_i)$$

$Y = (y_0, y_1)$ is the same as

$y_0 - s_{i0}$, which is N_0

$y_1 - s_{10}$, which is N_1.

$$f_{Y|S} = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(y_0 - s_{i0})^2}{N_0}} \cdot \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(y_1 - s_{i1})^2}{N_0}}$$

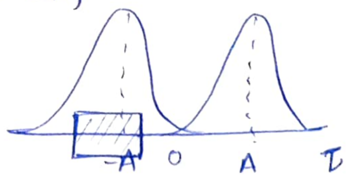


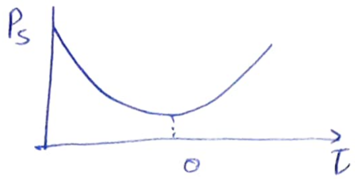
$\downarrow$

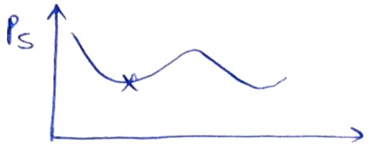
Take D_i^c

(outside D_i)

In 2D,







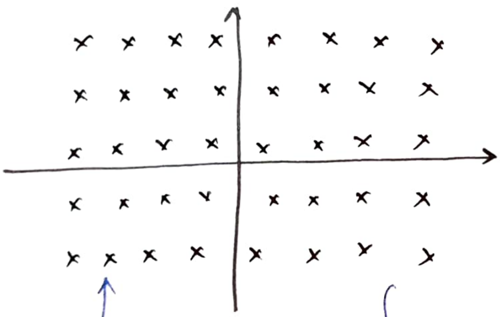
(D_i's)

To minimize the error,

D_i (decision rule) $\longrightarrow$ only thing that can be changed.

$$P_s = \sum_i P_r\{s_i\} \cdot P_r\{y \notin D_i \mid s_i\}$$
$$= 1 - \sum_i P_r\{s_i\} \cdot P_r\{y \in D_i \mid s_i\}$$

$$\min_{D_i} P_s = \max_{D_i} \sum_i P_r\{s_i\} \cdot P_r\{y \in D_i \mid s_i\}.$$



All possible points (discrete)

$D_i \rightarrow$ sets of such points

$$\sum_i P_r\{s_i\} \cdot \sum_{y \in D_i} f_{Y|s_i}(y)$$

$$\sum_i P_r\{s_i\} \cdot \sum_y \left[\mathbb{I}\{y \in D_i\} \cdot f_{Y|s_i}(y_i) \right]$$

$$= \sum_y \left[\sum_i P_r\{s_i\} \mathbb{I}\{y \in D_i\} \cdot f_{Y|s_i}(y) \right]$$

For each y,

$$\sum_i \Pr\{s_i\} \cdot \mathbb{I}\{y \in D_i\} \cdot f_{Y|s_i}(y)$$

Put y in D_i such that

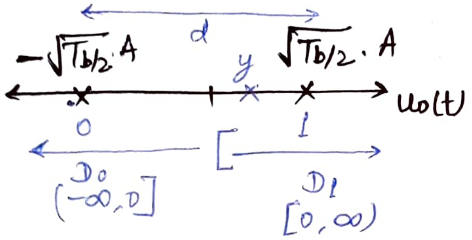
$$\max \underbrace{\Pr\{s_i\}}_{1/M} \cdot f_{Y|s_i}(y)$$

= MAP rule

Characterize P_s for MAP = MAP Rule for differential signal constellation:

BPSK: $Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left(\sqrt{\frac{d^2}{2N_0}}\right)$

$$P_s = \sum \underbrace{\boxed{\Pr\{s_i\}}}_{1/2} \cdot \Pr\{y \notin D_i \mid s_i\}$$

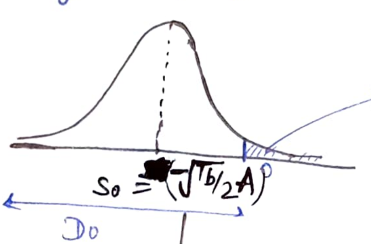


$$\Pr\{y \mid s_0\} \gtrless \Pr\{y \mid s_1\}$$

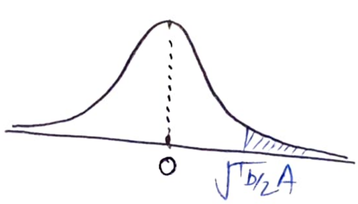
$$\mathcal{N}(y; s_0, N_0/2) \underset{s_1}{\overset{s_0}{\gtrless}} \mathcal{N}(y; s_1, N_0/2)$$

$$e^{-\frac{(y+\sqrt{T_b/2})^2}{N_0}} \gtrless e^{-\frac{(y-\sqrt{T_b/2})^2}{N_0}}$$

$$\Pr\{y \notin D_0 \mid s_0\}$$



$$\frac{1}{\sqrt{\pi N_0}} \int_0^\infty e^{-\frac{(y+\sqrt{T_b/2}A)}{N_0}} \, dy$$



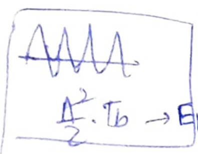
$$\underbrace{\boxed{\frac{1}{\sqrt{\pi N_0}}}}^{\frac{1}{\sqrt{2\pi}}} \int_{\boxed{\sqrt{T_b/2}A}}^{\infty} e^{-\boxed{u^2/N_0}} \, du$$

$\sqrt{T_b/N_0}\, A$

$v = \frac{u}{\sqrt{N_0/2}}$

$v^2 = \frac{u^2}{N_0/2}$

$dv = \frac{du}{\sqrt{N_0/2}}$



$$\frac{1}{\sqrt{2\pi}} \int_{\sqrt{T_b/N_0}A}^{\infty} e^{-v^2/2} \, dv = Q\left(\sqrt{\frac{T_b A^2}{N_0}}\right)$$

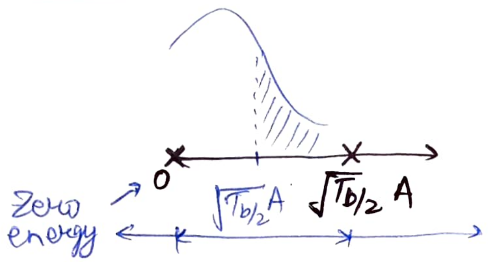
$\therefore$ $\boxed{\Pr\{y \notin D_0 \mid s_0\} = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)}$

Distance squared is

$$\left(2\sqrt{\frac{T_b}{2}}\,A\right)^2 = \underbrace{\left(2\cdot T_b\cdot A^2\right)}_{d^2}$$

$$\therefore\ Q\left(\sqrt{\frac{T_b A^2}{N_0}}\right) = Q\left(\sqrt{\frac{d^2}{2N_0}}\right)$$

BASK



A B

Translation

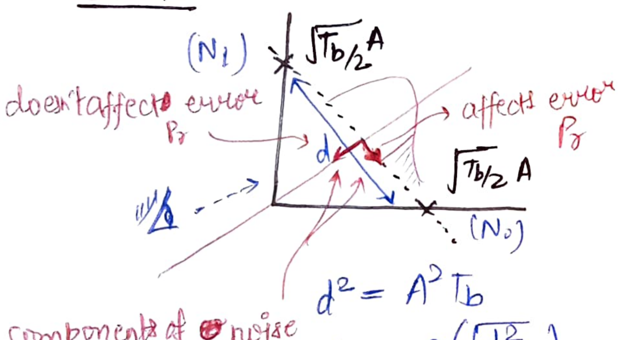
Rotation

→ Less T_r power than BPSK.

→ Less error P_r than BPSK

$$P_s = Q\left(\sqrt{\frac{T_b A^2}{4N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right), \quad E_b = \frac{T_b A^2}{4}$$

BFSK

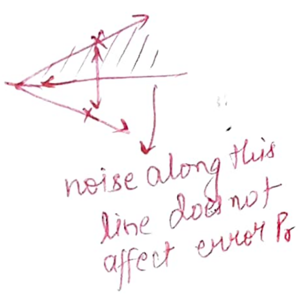


→ Same T_r power ~~than~~ as BPSK

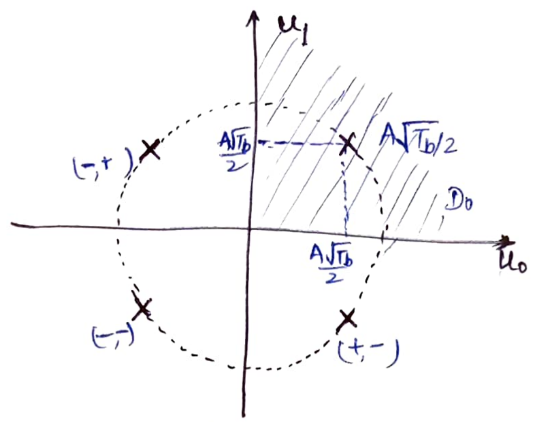
→ More error probability than BPSK.

$$d^2 = A^2 T_b$$

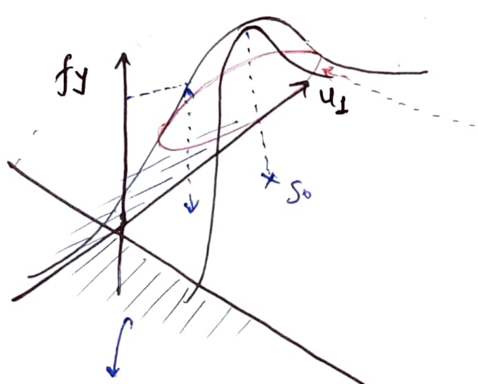
$$P_s = Q\left(\sqrt{\frac{d^2}{2N_0}}\right) = Q\left(\sqrt{\frac{A^2 T_b}{2N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right), \quad E_b = \frac{A^2 T_b}{2}$$



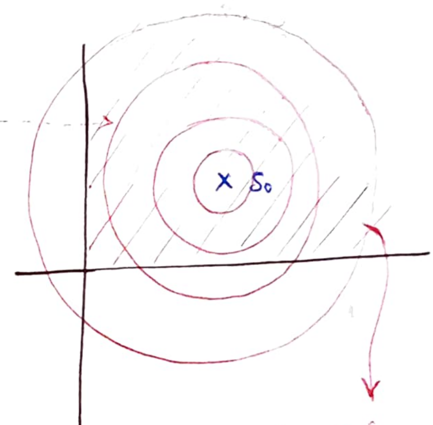
QPSK - Symbol Error Probability



$$P_s = \sum_{i=0}^{3} P_r\{s_i\} \cdot P_r\{y \notin D_i \mid s_i\}$$



$P_r\{y \notin D_0 \mid s_0\}$



$1 - P_r\{y \in D_0 \mid s_0\}$

$P_r\{y \in D_0 \mid s_0\}$

$$= \int_{D_0} \underbrace{f_{Y|s_0}(y)\, dy}$$

$$y = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} s_{00} \\ s_{01} \end{pmatrix} + \begin{pmatrix} N_0 \\ N_1 \end{pmatrix}$$

$$y_0 \perp y_1$$

$$N_0(s_{00}, N_0/2) \qquad N(s_{01}, N_0/2) = \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{(y_1 - s_{01})^2}{N_0}\right]$$

$f_{Y|s_0}(y) \rightarrow (y_0, y_1)$

$$f_{Y|S_0}(y) = f_{Y_0Y_1|S_0}(y_0,y_1)$$
$$= f_{Y_0|S_0}(y_0)\cdot f_{Y_1|S_0}(y_1)$$
$$= \frac{1}{\sqrt{\pi N_0}}\cdot e^{-\frac{(y_0-s_{00})^2}{N_0}}\cdot \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(y_1-s_{01})^2}{N_0}}$$

$$Pr\{y\in D_0|s_0\} = \int_{D_0} f_{Y|S_0}(y)\,dy$$
$$= \int_0^\infty\int_0^\infty \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(y_0-s_{00})^2}{N_0}}\cdot\frac{1}{\sqrt{\pi N_0}} e^{-\frac{(y_1-s_{01})^2}{N_0}}\,dy_0\,dy_1$$
$$= \boxed{\int_0^\infty \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(y_0-s_{00})^2}{N_0}}\,dy_0}\cdot\boxed{\int_0^\infty \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(y_1-s_{01})^2}{N_0}}\,dy_1}$$

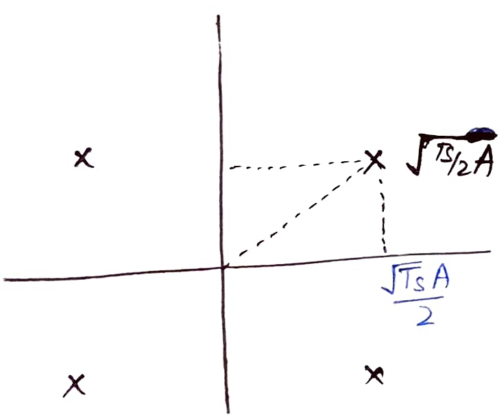
$$\left(\frac{-s_{00}}{\sqrt{N_0/2}}\right) \qquad \frac{1}{\sqrt{2\pi}} \qquad -v^2/2$$

$$v = \frac{y_0-s_{00}}{\sqrt{N_0/2}}$$

$$\frac{1}{\sqrt{2\pi}}\int_{\frac{-s_{00}}{\sqrt{N_0/2}}}^{\infty} e^{-v^2/2}\,dv$$

$$1-Q\left(\frac{s_{00}}{\sqrt{N_0/2}}\right)$$

$$\therefore Pr\{y\in D_0|s_0\} = \left(1-Q\left(\frac{s_{00}}{\sqrt{N_0/2}}\right)\right)\left(1-Q\left(\frac{s_{01}}{\sqrt{N_0/2}}\right)\right)$$



$$\frac{s_{00}}{\sqrt{N_0/2}} = \sqrt{\frac{T_sA^2}{2N_0}} = \sqrt{\frac{E_s}{N_0}} = \sqrt{\frac{2E_b}{N_0}}$$

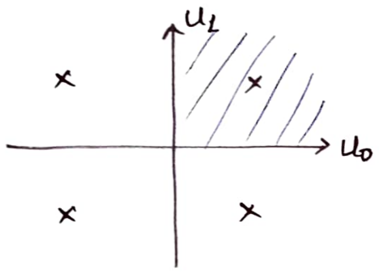
$$P_r\{y \in D_0 | s_0\} = \left(1 - Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right)\left(1 - Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right)$$

$$P_r\{y \notin D_0 | s_0\} = P_r\{y \notin D_i | s_i\}$$
$$= 1 - P_r\{y \in D_0 | s_0\}$$
$$= 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - \left(Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right)^2$$
$$= P_{s,\,QPSK}$$

04-04-2025

Probability of Symbol Error (P_s)

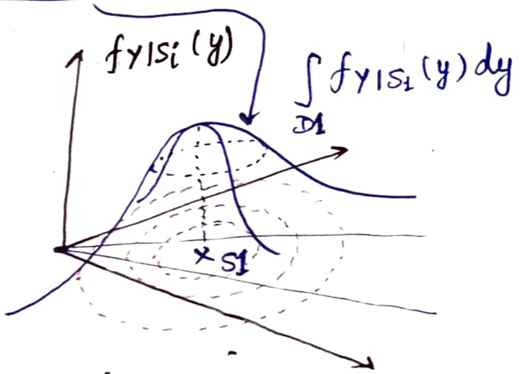
Recall



$$P_s = \sum_i P_r(s_i) \cdot P_r\{y \notin D_i | s_i\}$$

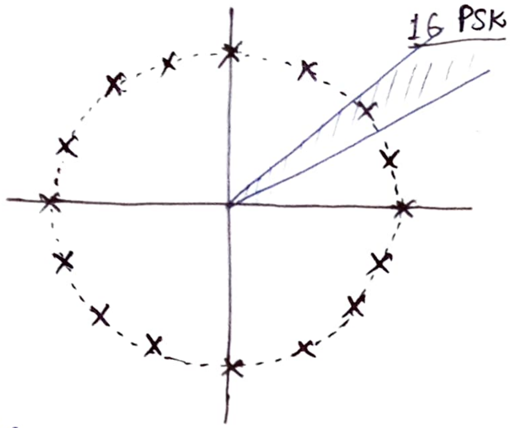
D_i under ML (MAP with $P_r\{s_i\} = 1/M$)

$$P_r\{y \in D_i | s_i\} = 1 - P_r\{y \notin D_i | s_i\}$$

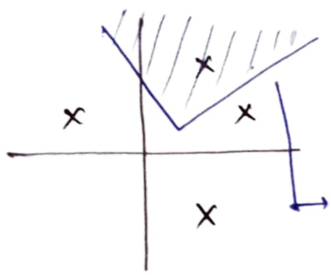


QPSK : $P_s = 1 - \left(1 - Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right)\left(1 - Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right)$

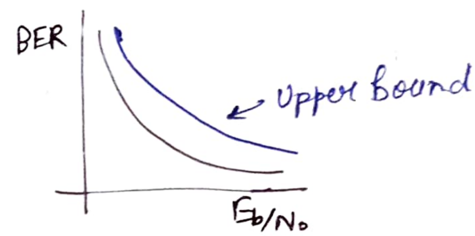
A general constellation or even M-PSK



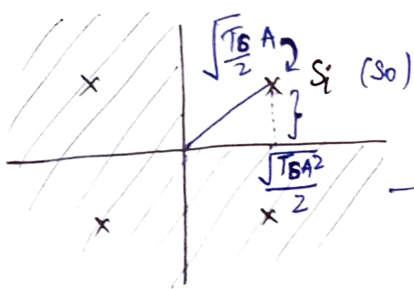
General:



→ Find upper and lower bound



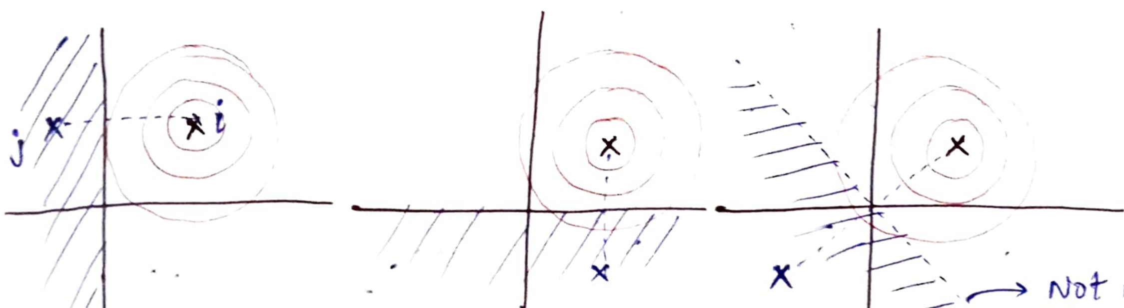
QPSK:



→ Find its upper bound

$$P_r\{y \notin D_i \mid S_i\} \le \sum_{j\neq i} Q\left(\sqrt{\frac{d_{ij}^2}{2N_o}}\right)$$

→ Upper bound OR Union bound



→ Not needed

[All these are P_e of two point symbol points] → Sum of all these = Upper bound

$$= Q\left(\sqrt{\frac{d_{ij}^2}{2N_o}}\right)$$

Union Bound :

$$P(A \cup B) \leq P(A) + P(B)$$

$$\{y \notin D_0 \mid s_0\}$$

$$= \{\underbrace{\{y \to s_1\}}_{y \text{ evaluates to } s_1} \cup \{y \to s_2\} \cup \{y \to s_3\} \mid s_0\}$$

Intelligent union bound :

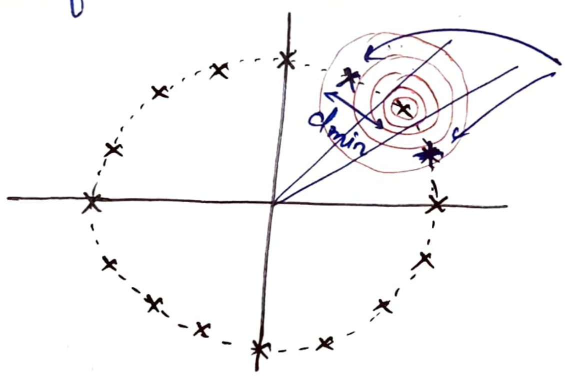
$$Q\left(\sqrt{\frac{d_{01}^2}{2N_0}}\right) + Q\left(\sqrt{\frac{d_{03}^2}{2N_0}}\right) \quad [\text{Exclude } 3^{rd} \text{ area}]$$

$$= 2Q\left(\sqrt{\frac{A^2 T_B}{2N_0}}\right) = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) > P_{s,QPSK}$$

upper bound (intelligent)

Nearest Neighbour Approximation

↳ Useful when noise level is not high (noise variance is small).



Nearest neighbours

↳ consider only them for upper bound

$$\underbrace{N_{min}}_{\text{No. of nearest neighbours}} \cdot Q\left(\sqrt{\frac{d_{min}^2}{2N_0}}\right)$$

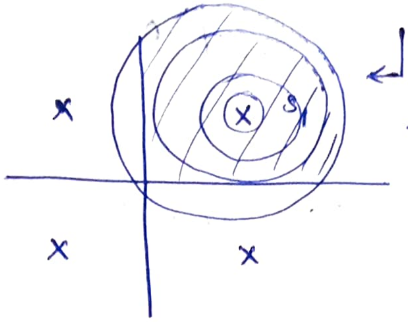
07-04-2025

Recall

Prob. of Symbol Error Calculation

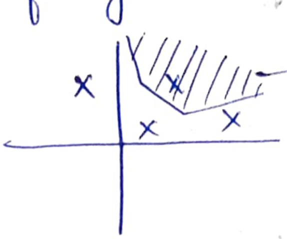
$$\sum \Pr\{s_i\} \cdot \underbrace{\Pr\{y \notin D_i \mid s_i\}}$$

or $1 - \Pr\{y \in D_i \mid s_i\}$



computed by this integral

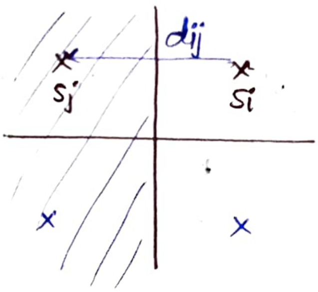
Not easy for general constellations



this integral is difficult

Main idea: Look at "pairwise" errors

$s_i \xrightarrow[\text{in error}]{\text{gets decoded}} s_j$



Easier to calculate

$$Q\left(\sqrt{\frac{d_{ij}^2}{2N_0}}\right)$$

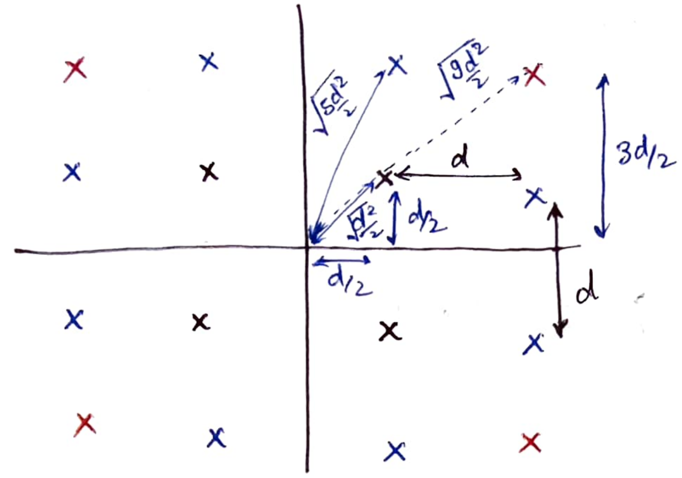
→ Use this idea for UB, IUB, nearest neighbours approx (NN).

NN Approx for QPSK:

$$2Q\left(\sqrt{\frac{d_{ij}^2}{2N_0}}\right) = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

NN → approximate $\Pr\{y \notin D_i \mid s_i\}$

16-QAM



NN approx:

$\times$ points $= 2Q\left(\sqrt{\frac{d^2}{2N_0}}\right)$

$\times$ points $= 3Q\left(\sqrt{\frac{d^2}{2N_0}}\right)$

$\times$ points $= 4Q\left(\sqrt{\frac{d^2}{2N_0}}\right)$

$$P_S = \frac{4}{16}\times 2Q\left(\sqrt{\frac{d^2}{2N_0}}\right) + \frac{8}{16}\times 2Q\left(\sqrt{\frac{d^2}{2N_0}}\right) + \frac{4}{16}\times 4Q\left(\sqrt{\frac{d^2}{2N_0}}\right)$$

$$= 3Q\left(\sqrt{\frac{d^2}{2N_0}}\right)$$

$$= 3Q\left(\sqrt{\frac{4E_b}{5N_0}}\right)$$

So, $E_S = \frac{5d^2}{2}$

$\frac{8E_b}{5} = d^2$

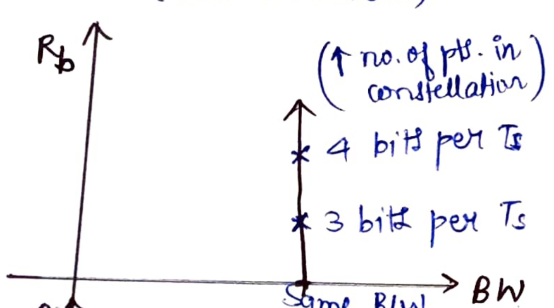
$$E_S = \frac{4}{16}\,\frac{9d^2}{2} + \frac{8}{16}\,\frac{5d^2}{2} + \frac{4}{16}\,\frac{d^2}{2}$$

$$= \frac{80d^2}{32} = \frac{5}{2}d^2$$

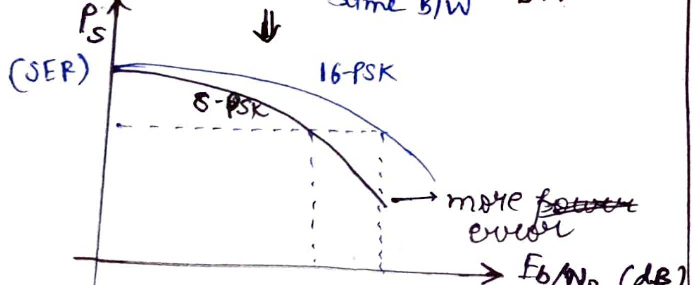
Schemes used for Bandlimited Scenarios

Same baud rate $(1/T_S)$ $\rightarrow$ same T_S $\rightarrow$ same B/W

Increase M $\rightarrow$ increase bit rate, $R_b = \frac{\log_2 M}{T_S}$
(modulation order)

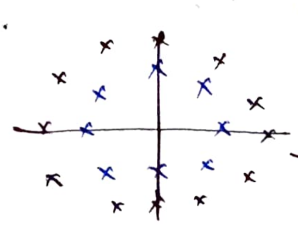


⇓



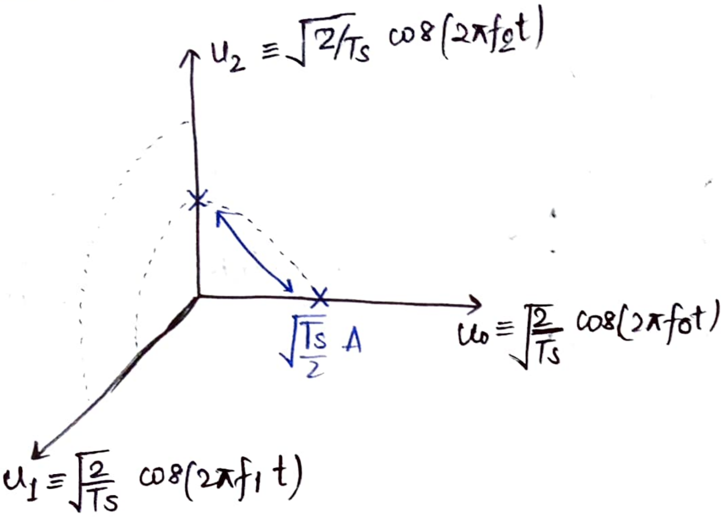
Adding pts. in b/w:

Same power (E_b), Same BW, More BER



Power ↑ same BER

M-FSK



Symbol pts:

$\left.\begin{matrix}(1,0,0,\dots) \\ (0,1,0,\dots) \\ (0,0,1,\dots)\end{matrix}\right\}$ pairwise dist. $=2$

$$\frac{A^2 T_s}{2} = E_s$$

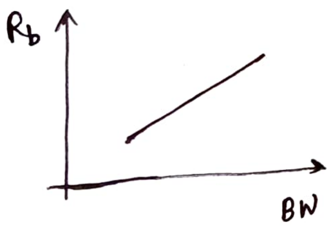
$$A^2 T_s = 2E_s = 2(\log_2 M)E_b$$

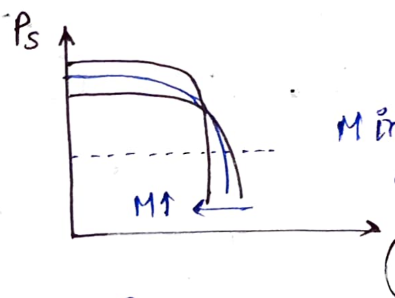
NN-approximation for M-FSK:

$$P_s = \sum_i P_r\{s_i\}.\underbrace{P_r\{y \notin D_i \mid s_i\}}_{\approx (M-1)\,Q\left(\sqrt{\frac{T_s A^2}{2N_0}}\right)}$$

$$= M.\frac{1}{M}.(M-1)\,Q\left(\sqrt{\frac{T_s A^2}{2N_0}}\right)$$

$$=(M-1)\,Q\left(\sqrt{\frac{(\log_2 M)E_b}{N_0}}\right)$$





M increases for same A^2T_s

Power-limited regime $\longrightarrow$ Increase bit rate by keeping power same
$\Downarrow$
BW$\uparrow$

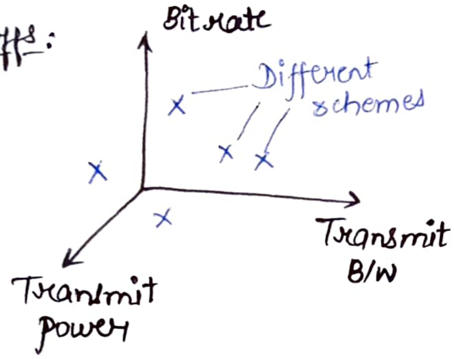
09-04-2025

Recall

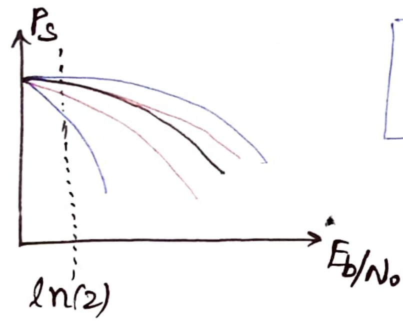
Modulation Schemes (and corresponding demodulation)

M-PSK, M-QAM, M-FSK

Tradeoffs:

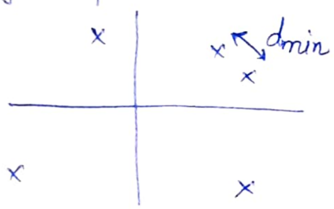


Claude Shannon's Capacity Results



$$\boxed{Q\left(\sqrt{\frac{d^2}{2N_0}}\right)}$$

d_{min} is very important.
Min. distance b/w signal points.



Information theory (Coding theorem)

$0 \longleftrightarrow 1$

Hamming distance

0 0 1 1
1 1 0 0

$0 \rightarrow 000$

$1 \rightarrow 111$

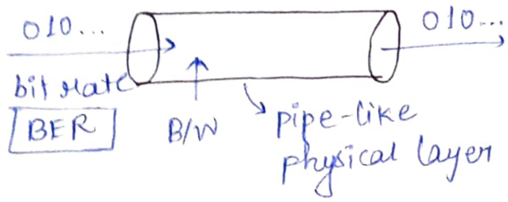
P_s: probability of symbol error $\rightarrow$ SER

$$SER = \frac{\sum_{i=1}^{N} \mathbb{I}_{\{\hat{c}_i \neq c_i\}}}{N} \quad \text{as } N \rightarrow \infty$$

$\downarrow$

find in practice

P_s to Probability of bit error (P_b) and BER :



Layers in Computer Networks:

Application
Transport
Network
MAC
PHY

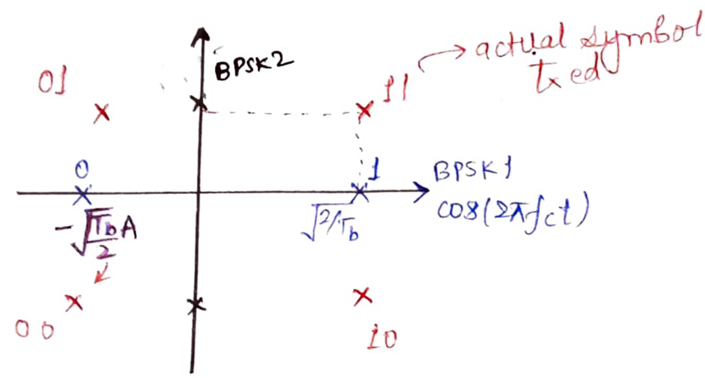
Cases: Get P_b from P_s.

QPSK

BPSK 1 ⟶ $\cos(2\pi f_c t)$

BPSK 2 ⟶ $\sin(2\pi f_c t)$

$$Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

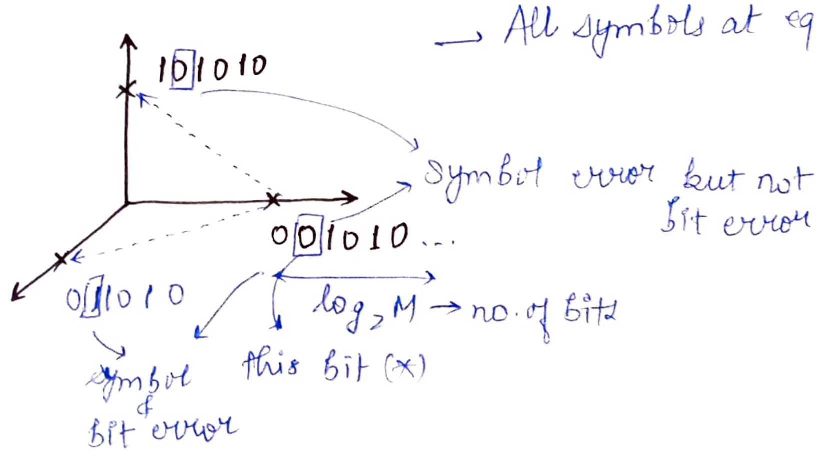


11
01
00
10
} gray-coded
↓
differs by 1 bit
↓
Possible error in only neighbours
↓
overall, less error.

Other cases :

M-FSK

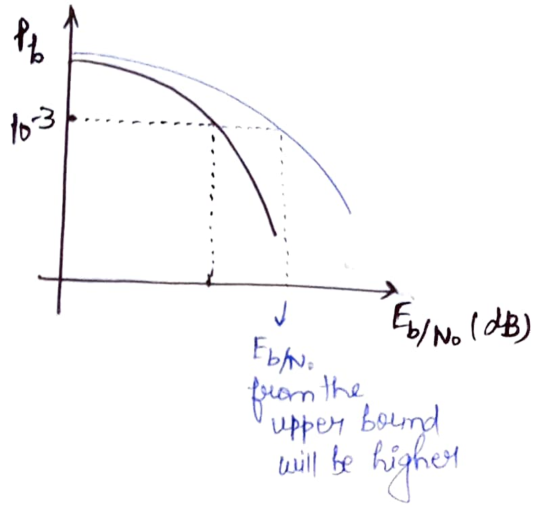
→ All symbols at equal distances.



$$1 - P_b = \underbrace{\frac{P_s}{M-1}}_{\text{probability of other symbol}} \times \underbrace{2^{(\log_2 M - 1)}}_{\text{probability of other symbols having the same particular bit (x)}}$$

Bound

$$\underbrace{P_b \leq P_s}_{\text{As Every bit error result in symbol error}} \leq \underbrace{(\log_2 M) \cdot P_b}_{\text{Error probability of any one bit}} \quad \text{(for any scheme)}$$



21-04-2025

Link Budget Analysis

0101... → [Bit pipe] → 0101, ... Point-to-point

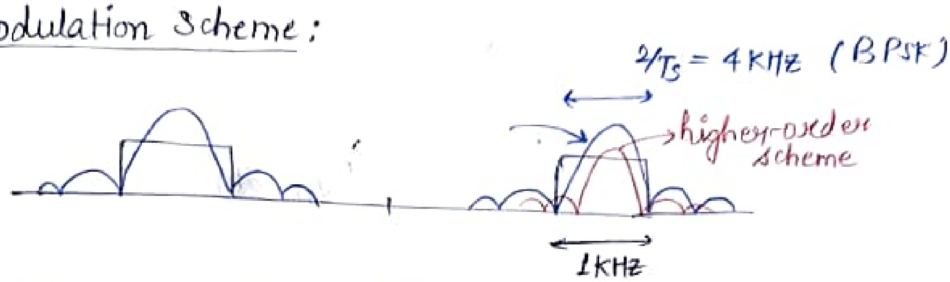
A given [bit-rate], [BER requirement]

application dependent

Other constraints, say [B/W], [power].

Candidate Modulation Scheme:

Eg. 1 KHz



Need 2 kbps.

Suppose we use BPSK. → Not possible

Higher M(order) scheme:

$2/T_s = 1$ KHz $\Rightarrow T_s = 2\times10^{-3}$ s

$1/T_s = 1/2$ ksps

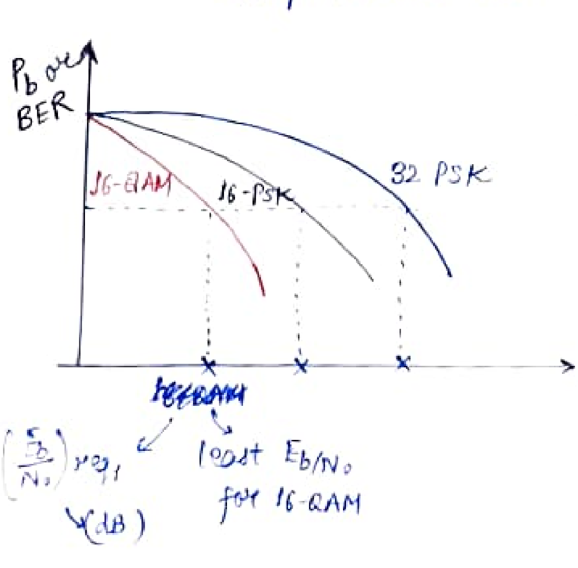
4 bits per symbol

→ $\boxed{\text{16 QAM}}$ / $\boxed{\text{16-PSK}}$

Say 32-PSK.

5 bits per symbol

2 kbps, baud rate = $2/5$ ksps.



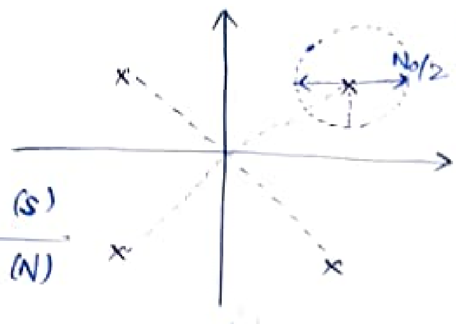
Requirements: A given BER and E_b/N_0

[N_0 → constant for all schemes]

$$\underbrace{\left(\frac{E_b}{N_0}\right)_{req_1}}_{(dB)} + \underbrace{\text{link margin}}_{\text{safety factor}} = \underbrace{\left(\frac{E_b}{N_0}\right)_{req}}_{dB}$$

E_b/N_0 :
$\underbrace{}$
SNR

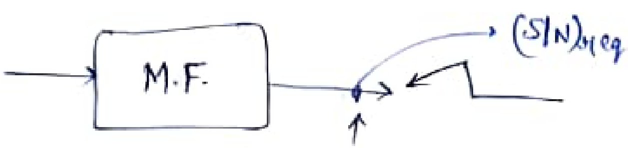
$= \frac{\text{signal power (S)}}{\text{noise power (N)}}$



Rx constellation

$E_s/\log_2 M = E_b$: bit energy

$N_0/2 \rightarrow$ noise variance
↳ determines the width/spread of noise



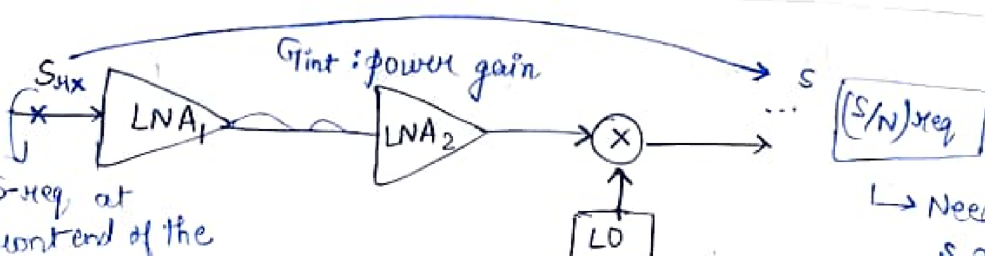
$\frac{E_b}{N_0}$: $S = E_b/T_s$

$N = \frac{N_0}{2} \times \underbrace{2W}_{\text{one-sided}}$

$= N_0 W$

$SNR = \frac{S}{N} = \frac{E_b/T_b}{N_0 W} = \frac{E_b \cdot R_b}{N_0 \cdot W}$ ... conversion formula (not in dB)

$\Rightarrow \left(\frac{S}{N}\right)_{req.} = \left(\frac{E_b}{N_0}\right)_{req.} \left(\frac{R_b}{W}\right)$: $\left(\frac{S}{N}\right)$ requirement. ⟶ Requires R_b, W.

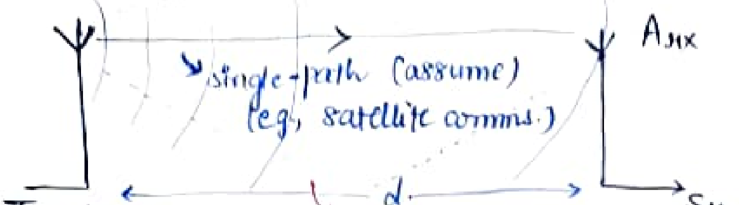


S-req at front end of the system

↳ Needs separated S and N calculation

↳ Not much control over N.

S_{rx} : power at the o/p of receiver antenna



Tx antenna
↳ omni-directional antenna
↳ transmits in all dir^n

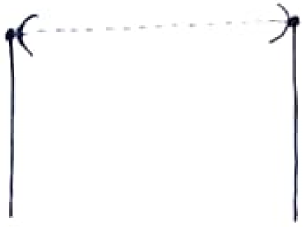
multi-path
↳ reflection
(eg, terrestial comms.)

Transmit power: P_{tx}

At Rx, power density $= \dfrac{P_{tx}}{4\pi d^2}$

$$S_{rx} = \frac{P_{tx} \cdot A_{rx}}{4\pi d^2}$$ → effective area

Directive antenna:



↳ Given a gain towards a particular dir^n. : G_{tx}

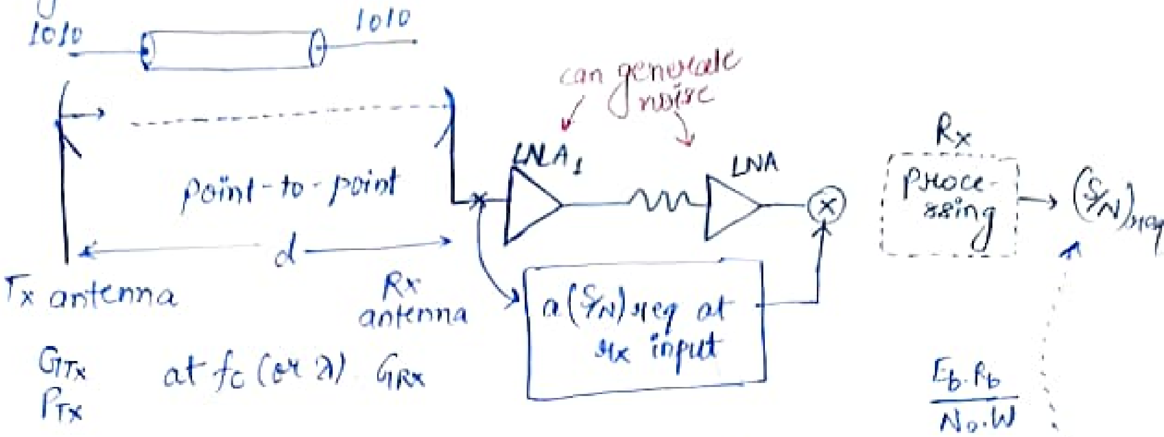
$$S_{rx} = \frac{P_{tx} \cdot G_{tx} \cdot A_{rx}}{4\pi d^2}, \quad A_{rx} = G_{rx} \cdot \frac{\lambda^2}{4\pi}$$

$$= \frac{P_{tx} \cdot G_{tx} \cdot G_{rx} \cdot \lambda^2}{(4\pi d)^2}, \quad P_{tx} \cdot G_{tx} : \text{EIRP}$$

(equivalent isotropic radiated power)

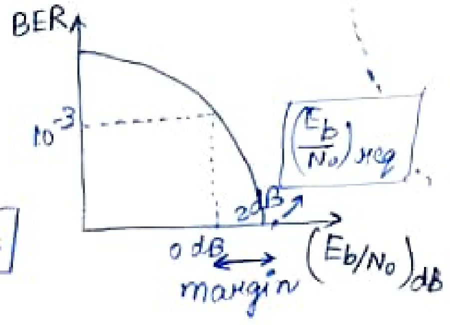
$$= \frac{\text{EIRP} \cdot G_{rx} \cdot \lambda^2}{(4\pi d)^2}$$

Link Budget continued ...



$$P_{Rx} = \frac{P_{Tx}\, G_{Tx}\, G_{Rx}\, \lambda^2}{(4\pi d)^2}$$

or, $$P_{Rx} = \frac{EIRP \cdot G_{Rx} \cdot \lambda^2}{(4\pi d)^2}$$ [Absolute scale]



$$P_{Rx}(dB) = EIRP(dB) + G_{Rx}(dB) - \boxed{20 \log\left(\frac{4\pi d}{\lambda}\right)}$$

path loss

[in dB scale, took 10 log, as power terms]

[= given power − path loss + gain]

- cable loss (dB)
- rain / atmospheric / ionospheric (dB)
- pointing loss (dB)
- polarization loss (dB)

} due to misalignment of antennas

Look into Sklar book

Eg. EIRP of 1 dBm

$G_{Rx}(dB) = 0$

$d = 36000$ Km

$f_c = 2$ GHz

dBm : power w.r.t 1 mW.

$$EIRP\,(dBm) = 10 \log_{10}\left(\frac{EIRP}{1mW}\right)$$

$$\Rightarrow EIRP = 10^{x/10} \cdot 1mW$$

$$\text{path loss} = 20 \log_{10}\left(\frac{4\pi \times \overset{d}{\boxed{36 \times 10^6}} \times \overset{f_c}{\boxed{2 \times 10^9}}}{\underset{c}{\boxed{3 \times 10^8}}}\right) = 190\, dB$$

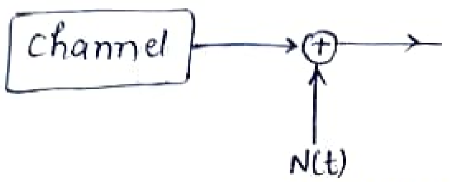
$190 = 10 \log_{10}(?)$

$\Rightarrow$ Reduction by 10^{190}.

If txed power $= 1\,mW \Rightarrow$ Rxed power $= -190$ dBm.

Think about noise (N) now

Recall, our model:



AWGN ($N_0/2$)

↳ source: any components in the ~~path~~ receiver, unwanted signal from channel.

Undesired signals : Noise + Interference

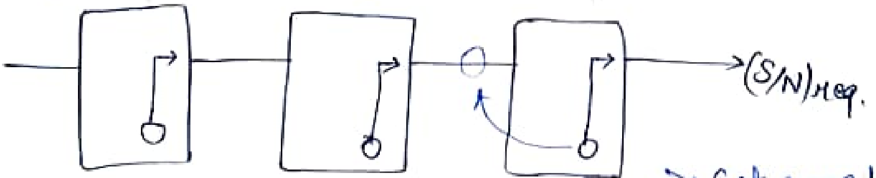
Noise ↓ totally random

Interference ↓ deterministic

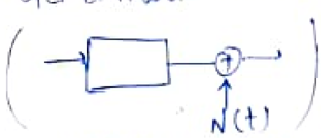
→ Any components ~~at~~ the receiver can generate noise.

↳ we will look into thermal noise : Generated by components not at absolute temperature.

Think of them as having a noise source inside them.

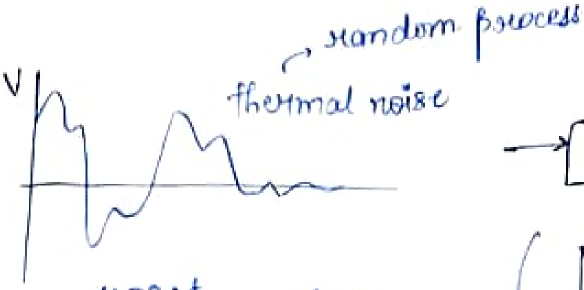


→ Get a model like

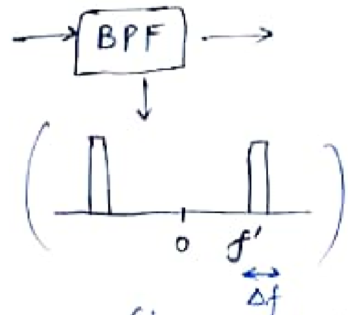


(pull the noise to outside the block)

Resistor at non-absolute temp. :



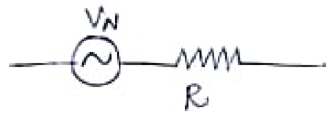
↳ constant for all f'



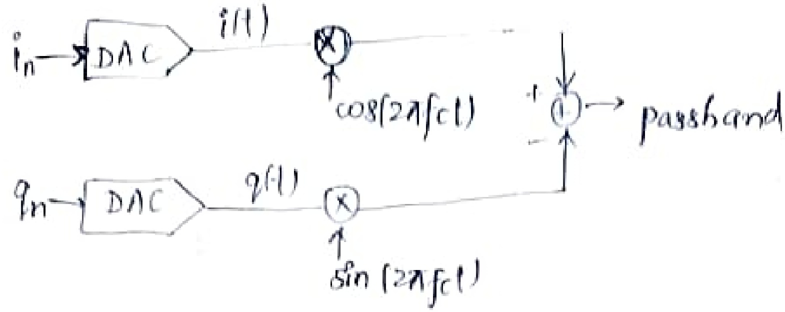
(look into spectrum at freq. f')

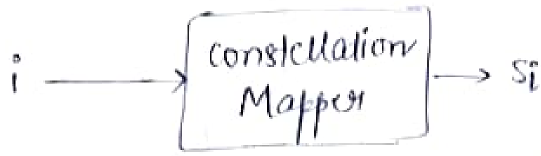
⇒ Zero mean
↳ Gaussian distributed

<u>Model</u> :



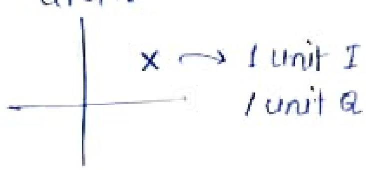
Synchronization issues

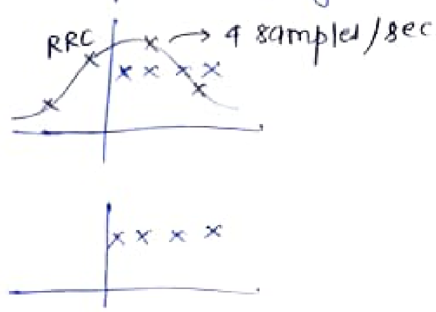




↳ Constellation Modulator block (in GNU Radio)
↳ also does pulse shaping (RRC)

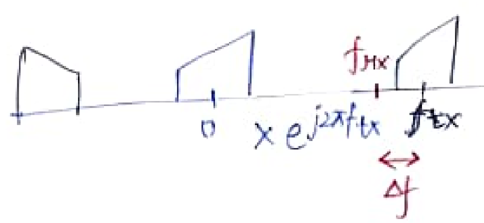
QPSK:





Complex baseband:

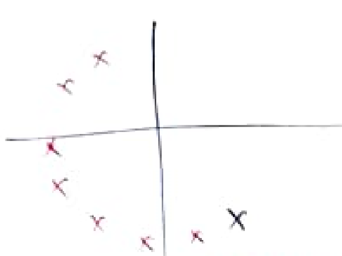
$1, 1 \rightarrow \underset{\text{(complex)}}{1+j} \equiv \underset{\text{(polar)}}{\sqrt{2}\, e^{j\pi/4}}$



$$\mathrm{Re}\left\{ \sqrt{2}\, e^{j\pi/4} \cdot e^{j2\pi f_{tx} t} \right\}$$

$$\downarrow \quad f_{tx} = \Delta f + f_{rx}$$

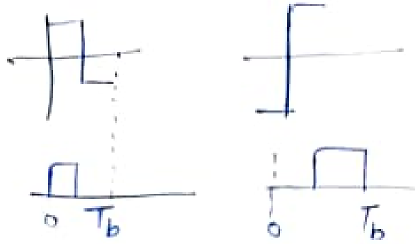
$$\mathrm{Re}\left\{ \boxed{\sqrt{2}\, e^{j\pi/4} \cdot e^{j2\pi \Delta f t}}\, e^{j2\pi f_{rx} t} \right\}$$



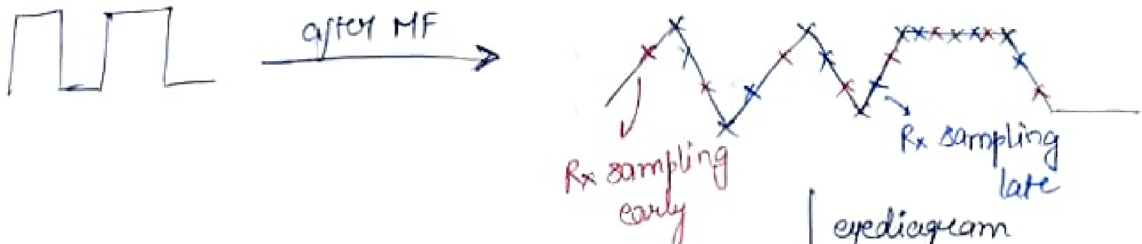
Rotates → constellation works circular

Symbol Timing Synchronization

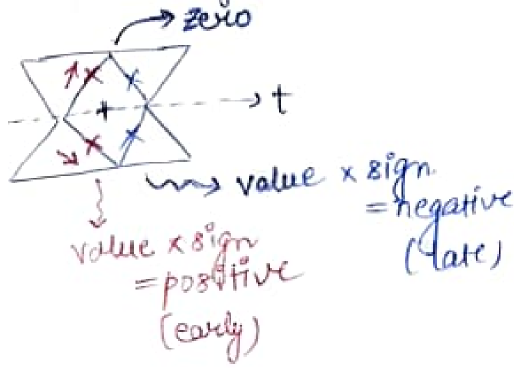
Transmitted bitstream ← recover



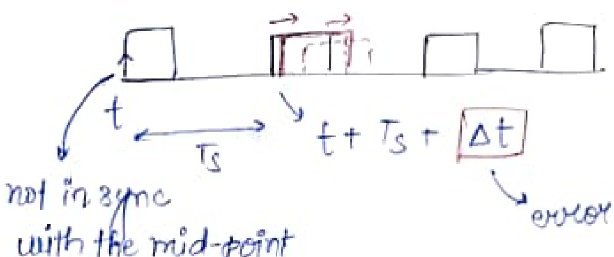
Early-late gating / $x.\dot{x}$:



↓ eyediagram



Consider clock

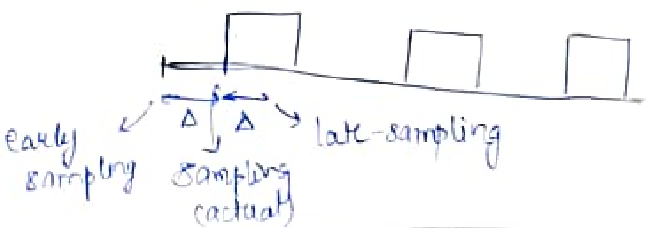


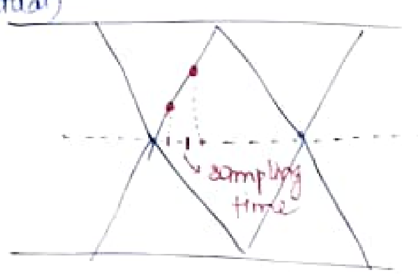
error correction added $= \alpha\,(x.\dot{x})$

↳ Feedback loop

$\Delta t \rightarrow$ ultimately goes to zero

Early-late gating: Take 3 samples

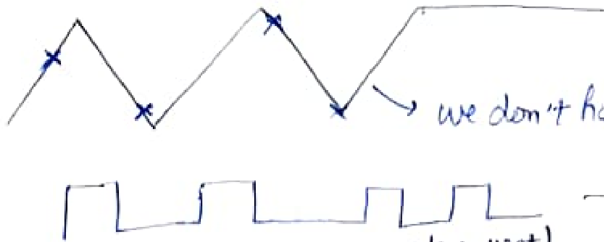




Value at late sample
>
Value at early sample

OR

late > early (again)

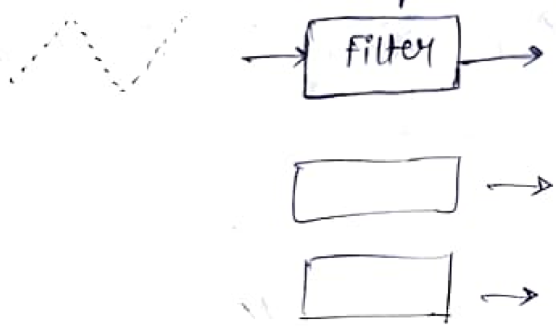


→ we don't have this actual time signal
→ we have samples

→ clock adjusted

Δt (offset)

Instead use Filter.
(a no. of filters, 32 here)



we can get as many no. of samples as we want by feeding Δt.

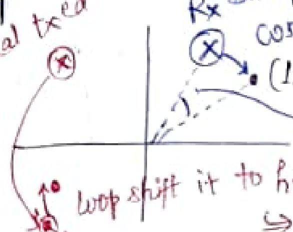
Costas loop:

Rx sample

costas loop tries to rotate sample to the nearest ideal sample point

actual txed

$(1,1) \equiv \sqrt{2}\, e^{j\pi/4}$



loop shift it to here (ideal pt.)
⇒ Error

rotated due to Δf or noise

θ → VCO →

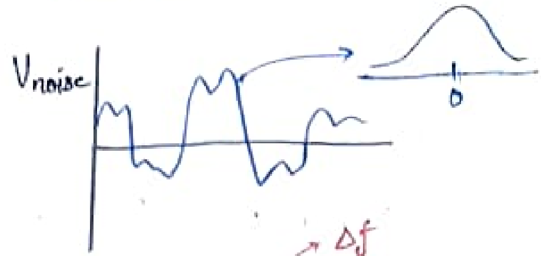
multiplies sample ⊗ by $e^{-j\theta}$

21-01-2025

Noise - in link Budget Analysis

Recall

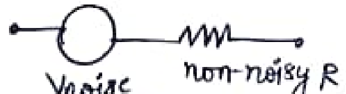
at temp. T K.



$V^2_{rms} = 4KTR\,B$ in any bandwidth B at any freq.

$B \rightarrow \Delta f$

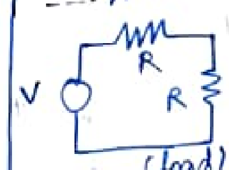
Model as :



↓ max. noise power from this source

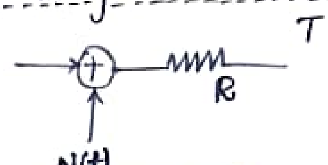
$$\boxed{V^2_{worst,\,rms} = KTB}$$ at any freq.

Max. power transfer:

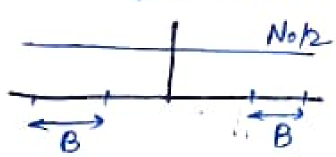


$\left(\frac{V}{2R}\right)^2 \cdot R = \frac{V^2}{4R}$

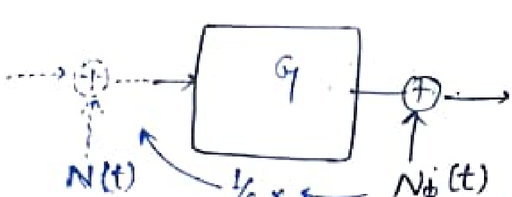
Receiver with just a resistor :



$$\boxed{N_0 = KT}$$



For an actual receiver :



We can still think of the receiver having just a resistor (noise being produced by a resistor) put at a different temperature (equivalent temp.).

$N(t)$ → input-referred Te ✓

$N_o(t)$ → may not be AWGN if the receiver is a filter ⇒ $N_o(t)$ will be filtered version of AWGN in this case.

Output-referred Te

Equivalent temp. of the receiver : Te .

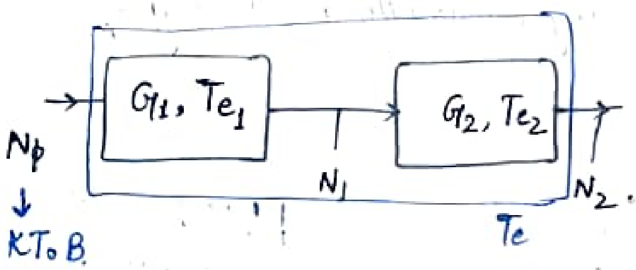
For receivers,

G_{rx}/T_e is a metric.

$$P_{rx} = EIRP \cdot G_{rx} \cdot \frac{\lambda^2}{(4\pi d)^2}$$

$$SNR = \frac{P_{rx}}{\left(\frac{N_0}{2}\right) \cdot 2B} = \frac{P_{rx}}{N_0 B} = \frac{EIRP}{K \boxed{T_e} B} \boxed{G_{rx}} \frac{\lambda^2}{(4\pi d)^2},$$

K: Boltzmann's constant



$$N_1 = G_1 (KT_0 B + KT_{e_1} B)$$

$$N_2 = G_2 \big(\underbrace{G_1 (KT_0 B + KT_{e_1} B)}_{N_1} + KT_{e_2} B \big)$$

i/p-referred noise

For overall receiver,

$$N_2 = G_1 G_2 (KT_0 B + KT_e B)$$

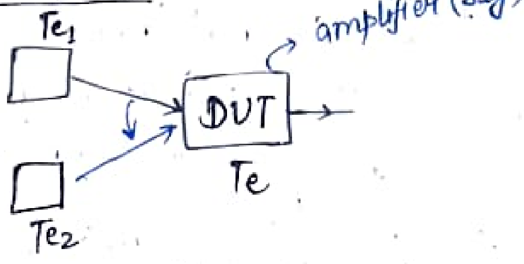
$$\Rightarrow \cancel{G_1 G_2 T_0} + G_1 G_2 T_{e_1} + G_2 T_{e_2} = \cancel{G_1 G_2 T_0} + G_1 G_2 T_e$$

$$\Rightarrow T_e = \boxed{T_{e_1}} + \frac{T_{e_2}}{G_1} + \frac{T_{e_3}}{G_1 G_2} + \frac{T_{e_4}}{G_1 G_2 G_3}$$

reduce its effect by taking high G_1

Te characterization

Y-factor method:



$$Y = \frac{K(T_e + T_{e_1})B}{K(T_e + T_{e_2})B}$$

$$\Rightarrow Y = \frac{(T_e + T_{e_1})}{(T_e + T_{e_2})}$$

$$\Rightarrow T_e(Y-1) = T_{e_1} - T_{e_2}Y$$

$$\Rightarrow T_e = \frac{T_{e_1} - T_{e_2}Y}{Y-1}$$

Noise Figure

$$SNR_i \longrightarrow \boxed{G} \longrightarrow SNR_o$$

$$F = \frac{SNR_i}{SNR_o}$$

$$SNR_i = \frac{S}{KT_{room}B}$$

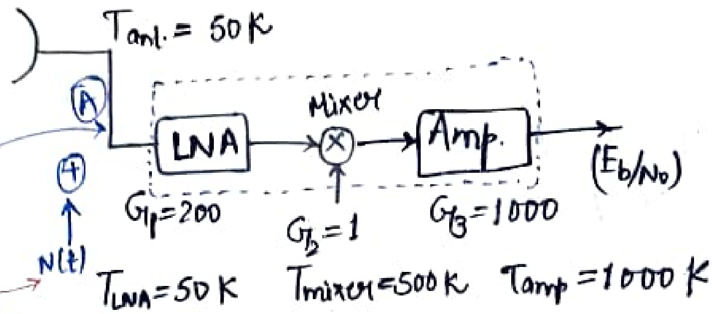
$$SNR_o = \frac{GS}{G(KT_{room}B + KT_eB)}$$

$$\therefore F = \frac{T_{room} + T_e}{T_{room}}$$

$$\Rightarrow \boxed{F = 1 + T_e/T_{room}}$$

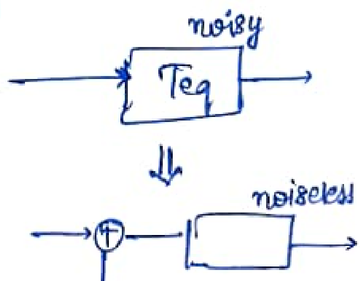
Put T_e to get equivalent F.

Link Budget Example:



Look at (S/N) or (C/N_0)

carrier signal



$N(t)$ - PSD $= N_0/2$

where $N_0 = K\,T_{eq}$.

$$T_{eq} = T_{LNA} + \frac{T_{mixer}}{G_1} + \frac{T_{amp}}{G_1 G_2}$$

$$\underbrace{50}_{\text{antenna}} + 50 + 2.5 + \frac{1000}{200}$$

$$= 107.5\,K.$$

$\therefore N_0 = K \cdot 107.5K. = 1.38 \times 10^{-23} \times 107.5 \equiv -208.28\,dB.$

EIRP $= 46.5$ dBW

$G_r = 45$ dB

Compute G/T ratio.
(in ~~normal~~ absolute scale)

Path loss: $\times$ ——— 40000 km ——— $\times$ $f = 2$ GHz

$$\left(\frac{c}{4\pi d f}\right)^2 = -206\,dB$$

$$\text{Rxed power} = \underbrace{46.5\,(\text{dB})}_{\text{directed towards Rx}} - \underbrace{206\,(\text{dB})}_{\text{(path loss)}} + 45\,(\text{dB})$$

$$= -114.5\,(\text{dBW})$$

$$C/N_0\ \text{dB} = -114.5 - (-208.28)$$

Suppose a bandwidth of W.

$$\frac{C}{WN_0} = S/N \text{ at } (A) \xrightarrow[\times\, G_1, G_2 G_3]{\times\, G_1 G_2 G_3} S/N \text{ (same) at the o/p of amplifier.}$$

$$\frac{C}{WN_0} = \frac{S}{N} = \boxed{\frac{E_b R_b}{N_0 W}}$$

$$\Rightarrow \boxed{\frac{E_b}{N_0}} = \frac{C}{N_0} \times \frac{1}{R_b}$$