FOURIER SERIES HOMEWORK

AV223 - Signals and Systems

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Q. 1 How many hours did you spent on this homework?

Ans: Approximately 3 hours.

2 Convert the complex exponentials into sines and cosines (and no complex numbers) with Euler's formula:

$$\left.\begin{aligned} e^{-j\theta} &= \cos(\theta) - j\sin(\theta) \\ e^{+j\theta} &= \cos(\theta) + j\sin(\theta) \end{aligned}\right\} \Rightarrow \begin{aligned} &\tfrac{1}{2}(e^{j\theta} + e^{-j\theta}) = \cos(\theta) \quad \ldots (i) \\ &\tfrac{1}{2j}(e^{j\theta} - e^{-j\theta}) = \sin(\theta) \quad \ldots (ii) \end{aligned}$$

(a) $x_1(t) = e^{-j\pi t} + e^{+j\pi t}$

Soln: $x_1(t) = \cos(\pi t) - j\sin(\pi t) + \cos(\pi t) + j\sin(\pi t)$ [Put $\theta = \pi t$] $\Rightarrow x_1(t) = 2\cos(\pi t)$ [using (i)]

(b) $x_2(t) = \frac{2}{j}\left[e^{-j(12t-\pi/3)} - e^{+j(12-\pi/3)}\right]$

Soln: $x_2(t) = \cancel{\frac{2}{e^{j\pi/2}}\{\cos(12t-\pi/3)\}}$

$= \frac{2}{j}\left[-2j\sin(12t-\pi/3)\right]$

$= -4\sin(12-\pi/3)$ [using (ii)]

For $\cancel{\theta = \pi/2}$

$\cancel{e^{j\pi/2} = \cos(\pi/2) + j\sin(\pi/2) = j}$

(c) $x_3(t) = (1-j\sqrt{3})e^{-j12t} + (1+j\sqrt{3})e^{j12t}$

Soln: $x_3(t) = (e^{j12t} + e^{-j12t}) + j\sqrt{3}(e^{j12t} - e^{-j12t})$

$= 2\cos(12t) + j\sqrt{3}\,[2j\sin(12t)]$

$= 2\cos(12t) - 2\sqrt{3}\sin(12t)$

(d) $x_4(t) = (2+j)e^{+j\pi t} + (2-j)e^{-j\pi t} - (2-3j)e^{+j4\pi t} - (2+3j)e^{-j4\pi t}$

Soln: $x_4(t) = 2[e^{j\pi t} + e^{-j\pi t}] + j[e^{j\pi t} - e^{-j\pi t}] - 2[e^{j4\pi t} + e^{j4\pi t}] + 3j[e^{j4\pi t} - e^{-j4\pi t}]$

$= 2(2\cos(\pi t)) + j[2j\sin(\pi t)] - 2[2\cos(4\pi t)] + 3j[2j\sin(4\pi t)]$

$= 4\cos(\pi t) - 2\sin(\pi t) - 4\cos(4\pi t) - 6\sin(4\pi t)$

③ Convert cosines and sines into complex exponentials with Euler's formula:

$$\cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta}) \quad \text{(i)}, \qquad \sin(\theta) = \frac{1}{2j}(e^{+j\theta} - e^{-j\theta}) \quad \text{(ii)}$$

ⓐ $x_1(t) = \cos(2\pi t)$

ⓑ $x_2(t) = \cos(2\pi t + \pi/2)$

ⓒ $x_3(t) = \cos(3\pi t) + 5\sin(6\pi t) + 2\cos(6\pi t)$

Soln: ⓐ $x_1(t) = \frac{1}{2}\left[e^{j2\pi t} + e^{-j2\pi t}\right]$ [using (i)]

ⓑ $x_2(t) = \cos(2\pi t + \pi/2)$

$= -\sin(2\pi t)$

$= -\frac{1}{2j}\left[e^{j2\pi t} - e^{-j2\pi t}\right] = \frac{j}{2}\left[e^{j2\pi t} - e^{-j2\pi t}\right]$

ⓒ $x_3(t) = \frac{1}{2}\left[e^{j3\pi t} + e^{-j3\pi t}\right] + \frac{5}{2j}\left[e^{j6\pi t} - e^{-j6\pi t}\right] + \left[e^{j6\pi t} + e^{j6\pi t}\right]$

$= \frac{1}{2}\left[e^{j3\pi t} + e^{-j3\pi t}\right] - \frac{5j}{2}\left[e^{j6\pi t} - e^{-j6\pi t}\right] + \left[e^{j6\pi t} + e^{j6\pi t}\right]$

④ Determine the Fourier Series coefficients for the following signals. For these signals, use the trigonometric (cosine and sine) form of the Fourier series, i.e.,

$$x(t) = a_0 + \sum_{K=1}^{\infty} a_K \cos(K\omega_0 t) + \sum_{K=1}^{\infty} b_K \sin(K\omega_0 t)$$

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt, \quad a_K = \frac{2}{T_0}\int_{T_0} x(t)\cos(K\omega_0 t)\,dt,$$

$$b_K = \frac{2}{T_0}\int_{T_0} x(t)\sin(K\omega_0 t)\,dt, \quad K \geq 1.$$

where ω_0 is the fundamental angular frequency of our periodic signal, $\omega_0 = 2\pi/T_0$.

ⓐ $x_1(t) = 3\sin(4\pi t) + 2$

Soln: $T_0 = 2\pi/4\pi = 1/2$

$\therefore a_0 = \frac{1}{1/2}\int_0^{1/2}(3\sin(4\pi t) + 2)\,dt$

$= 2\left[3\underbrace{\int_0^{1/2}\sin(4\pi t)\,dt}_{0} + 2\int_0^{1/2} dt\right]$

$= 4(1/2 - 0) = 2$

(b) $x_2(t) = 3\sin\left(\frac{1}{7}t\right) + 5\cos(21t) + 3\pi$

Soln:

$\omega_0 = HCF\left(\frac{1}{7}, 21\right) = HCF\left(\frac{1}{7}, \frac{147}{7}\right) = \frac{1}{7}$

$T_0 = 2\pi/\omega_0 = 14\pi$.

$a_0 = \frac{1}{14\pi}\int_0^{14\pi} x_2(t)\,dt = 3\pi$

$$a_k = \frac{2}{1/2}\int_0^{1/2}\left[3\sin(4\pi t) + 2\right].\cos(4\pi t)\,dt$$

$$= 4\cdot\frac{3}{2}\int_0^{1/2}\cancel{\sin(8\pi t)}^{\,0}\,dt + 4\cdot 2\int_0^{1/2}\cancel{\cos(4\pi t)}^{\,0}\,dt$$

$= 0$.

$$b_k\Big|_{k=1} = \frac{2}{1/2}\int_0^{1/2}\left[3\sin(4\pi t) + 2\right].\sin(4\pi t)\,dt$$

$$= 4\cdot 3\int_0^{1/2}\sin^2(4\pi t)\,dt + 8\int_0^{1/2}\cancel{\sin(4\pi t)}^{\,0}\,dt$$

$$= 12\int_0^{1/2}\frac{1}{2}\,dt + 12\int_0^{1/2}\cancel{\cos(8\pi t)}^{\,0}\,dt$$

$$= 6\left(1/2 - 0\right)$$

$\therefore b_1 = 3$.

$\therefore a_0 = 2,\ b_1 = 3,\ a_k = 0.$

(b) $x_2(t) = 3\sin\left(\frac{1}{7}t\right) + 5\cos(21t) + 3\pi$

Soln:

$\omega_0 = HCF(1/7, 21) = HCF(1/7, 147/7) = 1/7$.

$T_0 = 2\pi/\omega_0 = 14\pi$.

$a_0 = \frac{1}{14\pi}\int_0^{14\pi} x_2(t)\,dt = 3\pi$

$$a_k\Big|_{k=1} = \frac{2}{14\pi}\int_0^{14\pi} 3\cancel{\sin(1/7\,t)\cos(1/7\,t)}^{\,0}\,dt + \frac{1}{7\pi}\int_0^{14\pi} 5\cos\left(\frac{1}{7}t\right).\cos(21t)\,dt + \frac{1}{7\pi}\int_0^{14\pi} 3\pi\cos\left(\frac{1}{7}t\right)dt$$

$$= \frac{5}{7\pi}\int_0^{14\pi}\left[\cos\left(\frac{148}{7}t\right) + \cos\left(\frac{146}{7}t\right)\right]dt + \frac{3}{7}\left[-\frac{\sin(1/7\,t)}{1/7}\right]_0^{14\pi}$$

$$= \frac{5}{7\pi}\left[\cancel{\frac{\sin\left(\frac{148}{7}t\right)}{\frac{148}{7}} + \frac{\sin\left(\frac{146}{7}t\right)}{\frac{146}{7}}}^{\,0}\right]_0^{14\pi} - \frac{3}{7}\times 7\left(\cancel{\sin 2\pi - \sin 0}^{\,0}\right)$$

$\Rightarrow a_1 = 0$

$$b_K\Big|_{K=1} = \frac{2}{14\pi}\int_0^{14\pi}\left[3\sin^2(1/7\, t) + \cancel{5\cos(2/t)\cdot\sin(\tfrac{1}{7}t)} + \cancel{3\pi\sin(\tfrac{1}{7}t)}\right] dt$$ ④

$$= \frac{3}{7\pi}\int_0^{14\pi}\frac{1}{2}\,dt - \frac{3}{7\pi}\int_0^{14\pi}\cancel{\frac{\cos(2/7\, t)}{2}}\,dt$$

$$= \frac{3}{7\pi}\times\frac{1}{2}(14\pi)$$

$$\Rightarrow b_1 = 3.$$

$\therefore\ a_0 = 3\pi,\ b_1 = 3,\ a_1 = 0,\ a_{147} = 5.$

© $x_3(t) = 3\sin(\pi t) + 123\cos(\pi t) + 5\cos(3\pi t) + 5\cos(4\pi t) + 6\cos(7\pi t).$

Soln: $\omega_0 = HCF(\pi, \pi, 3\pi, 4\pi, 7\pi) = \pi \Rightarrow T_0 = 2$

$$a_0 = \int_0^2 x_3(t)\,dt = 0$$

$$a_K = \frac{2}{2}\int_0^2 x(t)\cdot\cos(2Kt)\,dt$$

$\Rightarrow a_1 = 123$

$a_3 = 5$

$a_4 = 5$

$a_7 = 6.$

and, $b_K = \frac{2}{2}\int_0^2 x(t)\ \cancel{\cos}\sin(2Kt)\,dt.$

$\Rightarrow b_1 = 3.$

ⓓ $x_4(t) = 4\cos(3\pi t + \pi/3)$

Soln: $\omega_0 = 3\pi \Rightarrow T_0 = \frac{2}{3}$

$$x_4(t) = 4\cos(3\pi t)\cos(\pi/3) - 4\sin(3\pi t)\sin(\pi/3)$$

$$= 2\cos(3\pi t) - 2\sqrt{3}\sin(3\pi t)$$

$$a_0 = \int_0^{2/3} x_4(t)\,dt = 0$$

$$a_K = \frac{1}{2/3}\int_0^{2/3} x_4(t)\cos(\tfrac{2}{3}Kt)\,dt$$

$\Rightarrow a_1 = 2$

and, $b_K = 3/2\int_0^{2/3} x_4(t)\sin(2/3\,Kt)\,dt$

$\Rightarrow b_1 = -2\sqrt{3}.$

(5) Answer the following problems using the complex exponential form of the Fourier Series.

(a) Show that the real part of C_k is even.

Soln: $f(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$

$$C_k = \frac{1}{T_0}\int_{T_0} f(t)\, e^{-jk\omega_0 t}\, dt$$

$$g(t) = \text{Real}(C_k) = \frac{1}{2}\left[\frac{1}{T_0}\int_{T_0} f(t)\, e^{-jk\omega_0 t}\, dt + \frac{1}{T_0}\int_{T_0} f(t)\, e^{+jk\omega_0 t}\, dt\right]$$

$$g(-t) = \frac{1}{2}\left[\frac{1}{T_0}\int_{T_0} f(t)\, e^{+jk\omega_0 t}\, dt + \frac{1}{T_0}\int_{T_0} f(t)\, e^{-jk\omega_0 t}\, dt\right] \quad (\because f(-t)=f(t))$$

$$= g(t)$$

$\therefore$ $g(t)$ or Real (C_k) is even.

(b) Show that the imaginary part of C_k is odd.

Soln: $$\text{Im}(C_k) = \frac{1}{2j}\left[\frac{1}{T_0}\int_{T_0} f(t)\, e^{-jk\omega_0 t}\, dt - \frac{1}{T_0}\int_{T_0} f(t)\, e^{+jk\omega_0 t}\, dt\right]$$

$$= h(t).$$

$$h(-t) = \frac{1}{2j}\left[\frac{1}{T_0}\int_{T_0} f(t)\, e^{+jk\omega_0 t}\, dt - \frac{1}{T_0}\int_{T_0} f(t)\, e^{-jk\omega_0 t}\, dt\right]$$

$$= -h(t).$$

$\therefore$ $h(t)$ or Im (C_k) is odd.

(c) Show that the magnitude of C_k (i.e., $|C_k|$) is ~~odd~~ even.

Soln: $$C_k = \frac{1}{T_0}\int_{T_0} f(t)\left[\cos(k\omega_0 t) - j\sin(k\omega_0 t)\right] dt$$

$$\Rightarrow |C_k| = \frac{1}{T_0}\left|\int_{T_0} f(t)\left[\cos(k\omega_0 t) - j\sin(k\omega_0 t)\right] dt\right|$$

OR, $$|C_k| = \sqrt{[\text{Re}(C_k)]^2 + [\text{Im}(C_k)]^2}$$

$$= \sqrt{a^2 + b^2}, \quad \text{Re}\{C_k\} = \text{Re}\{C_{-k}\} = a$$
$$\text{Im}\{C_k\} = \text{Im}\{C_{-k}\} = b$$

$\therefore$ $|C_k|$ is even.

⑥ Determine the Fourier Series coefficients, c_k, for the following signals. For these signals, use the complex exponential form of the Fourier series, i.e.,

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, \quad c_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}\,dt.$$

where ω_0 is the fundamental angular frequency of our periodic signal, $\omega_0 = 2\pi/T_0$.

ⓐ $x_1(t) = 3\sin(4\pi t) + 2$.

Sol^n: $\omega_0 = 4\pi \Rightarrow T_0 = \frac{2\pi}{4\pi} = \frac{1}{2}$.

$$\therefore c_k = \frac{1}{1/2}\int_0^{1/2}[3\sin(4\pi t)+2]\,e^{-j4\pi k t}\,dt$$

$$= 2\int_0^{1/2}[3\sin(4\pi t)+2][\cos(4\pi t k) - j\sin(4\pi t k)]\,dt$$

$$= 2\cdot 3\int_0^{1/2}\sin(4\pi t k)\cos(4\pi t k)\,dt \to 0 + 2\cdot 2\int_0^{1/2}\cos(4\pi t k)\,dt \to 0$$

$$- 2\cdot 3j\int_0^{1/2}\sin^2(4\pi t k)\,dt - 2j\int_0^{1/2}\sin(4\pi t k)\,dt \to 0$$

$$= -6j\int_0^{1/2}\frac{1}{2}\,dt + 6j\int_0^{1/2}\cos(8\pi t k)\,dt \to 0$$

$$= -3j\left(\tfrac{1}{2} - 0\right)$$

$$\therefore c_1 = -\frac{3}{2}j, \quad c_0 = 2, \quad c_{-1} = 3/2j$$

ⓑ $x_2(t) = 3\sin\left(\frac{1}{7}t\right) + 5\cos(21t) + 3\pi$

Sol^n: $\omega_0 = \frac{1}{7} \Rightarrow T_0 = \frac{2\pi}{1/7} = 14\pi$.

$$\therefore c_1 = \frac{1}{14\pi}\Bigg[\int_0^{14\pi} 3\sin\left(\tfrac{t}{7}\right)\cos\left(\tfrac{t}{7}\right)dt \to 0 + \int_0^{14\pi} -j3\sin\left(\tfrac{t}{7}\right)\sin\left(\tfrac{t}{7}\right)dt$$

$$+ \int_0^{14\pi} 5\cos(21t)\cos(t/7)\,dt \to 0 + \int_0^{14\pi} -j3\cos(21t)\sin(t/7)\,dt \to 0$$

$$+ 3\pi\int_0^{14\pi}\cos(t/7) - j\sin(t/7)\,dt \to 0\Bigg]$$

$$= \frac{-j3}{14\pi}\int_0^{14\pi}\sin^2(t/7)\,dt = \frac{-j3}{14\pi}\int_0^{14\pi}\left[\frac{1}{2} - \frac{\cos(2t/7)}{2}\right]dt$$

$$= \frac{-j3}{14\pi}\cdot\frac{1}{2}(14\pi) = -\frac{3}{2}j.$$

$\therefore c_0 = 3\pi, \ c_1 = 3/2j, \ c_{-1} = -3/2j$

$c_{147} = 5/2, \ c_{-147} = 5/2$.

(c) $x_3(t) = 3\sin(\pi t) + 123\cos(\pi t) + 5\cos(3\pi t) + 5\cos(4\pi t) + 6\cos(7\pi t)$

$\omega_0 = \text{HCF}(\pi, 3\pi, 4\pi, 7\pi) = \pi.$

$$x_3(t) = \frac{3}{2j}\left[e^{j\pi t} - e^{-j\pi t}\right] + \frac{123}{2}\left[e^{j\pi t} + e^{-j\pi t}\right]$$
$$+ \frac{5}{2}\left[e^{j3\pi t} + e^{-j3\pi t}\right] + \frac{5}{2}\left[e^{j4\pi t} + e^{-j4\pi t}\right] + \frac{6}{2}\left[e^{j7\pi t} + e^{-j7\pi t}\right]$$
$$= \frac{3}{\cancel{2j}}\left[\cancel{2j}\sin(\pi t)\right] + \frac{123}{\cancel{2}}\left[\cancel{2}\cos(\pi t)\right] + \frac{5}{\cancel{2}}\left[\cancel{2}\cos(3\pi t)\right]$$
$$+ \frac{5}{\cancel{2}}\left[\cancel{2}\cos(4\pi t)\right] + 3\left[2\cos(7\pi t)\right]$$
$$= 3\sin\pi t + 123\cos(\pi t) + 5\cos(3\pi t) + 5\cos(4\pi t) + 6\cos(7\pi t).$$

$\therefore C_0 = 0,\ C_1 = \frac{3}{2j} + \frac{123}{2},\ C_3 = \frac{5}{2},\ C_4 = \frac{5}{2},\ C_7 = \frac{6}{2}$

$C_{-1} = \frac{-3}{2j} + \frac{123}{2},\ C_{-3} = \frac{5}{2},\ C_{-4} = \frac{5}{2},\ C_{-7} = \frac{6}{2} = 3.$

(d) $x_4(t) = 4\cos(3\pi t + \pi/3)$

$\omega_0 = 3\pi$

$$\therefore x_4(t) = \frac{4}{2}\left[e^{j3\pi t}e^{j\pi/3} + e^{-j3\pi t}e^{-j\pi/3}\right]$$
$$= 2\left[\left(\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)e^{j3\pi t} + \left(\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)e^{-j3\pi t}\right]$$
$$= (1 + j\sqrt{3})e^{j3\pi t} + (1 - j\sqrt{3})e^{-j3\pi t}$$
$$= 2\cos(3\pi t) + j\sqrt{3}\left[2j\sin(3\pi t)\right]$$
$$= 2\cos(3\pi t) - 2\sqrt{3}\sin(3\pi t).$$

$\therefore C_0 = 0,\ C_1 = \frac{2}{3}(1 + j\sqrt{3}),\ C_{-1} = \frac{2}{3}(1 - j\sqrt{3}).$

⑦ Given the following F.S. coefficients c_k (for the complex exp. form of F.S.), determine the corresponding periodic signal, $x(t)$. Write the result as a real-valued function if possible (i.e., not as complex exponentials).

ⓐ $c_0 = 1,\ c_1 = -2j,\ c_{-1} = 2j$ and all other $c_k = 0$. Assume $\omega_0 = 2\pi$.

Soln: $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$

$\cancel{= [c_0 + c_2 + c_{-1}] e^{j2\pi t}}$

$\cancel{= [1 - 2j + 2j] e^{j2\pi t}}$

$\cancel{= e^{j2\pi t}}$

$\cancel{= \cos(2\pi t) + j\sin(2\pi t)}$

$$= c_0 + c_2 e^{j\omega_0 t} + c_{-1} e^{-j\omega_0 t}$$

$$= 1 + (-2j)e^{j2\pi t} + 2j e^{-j2\pi t}$$

$$= 1 + 2j\left[e^{-j2\pi t} - e^{+j2\pi t}\right]$$

$$= 1 + 2j\left[-2j \sin(2\pi t)\right]$$

$$= 1 + 4\sin(2\pi t)$$

ⓑ $c_0 = 0,\ c_1 = j - 9,\ c_{-1} = -j - 9$, and all other $c_k = 0$. Assume $\omega_0 = 3$.

Soln: $x(t) = \cancel{[0 + j - 9 - j - 9] e^{j3t}}$

$\cancel{= -18 e^{j3t}}$

$$0 + (j-9)e^{j3t} - (j+9)e^{-j3t}$$

$$= j(e^{j3t} - e^{-j3t}) - 9(e^{j3t} + e^{-j3t})$$

$$= j(2j\sin(3t)) - 9(2\cos(3t)) = -2\sin 3t - 18\cos 3t$$

ⓒ $c_0 = 10,\ c_2 = -2j+1,\ c_{-2} = 2j+1,\ c_7 = 1,\ c_{-7} = 1$, and all other $c_k = 0$. Assume $\omega_0 = 1/5$.

Soln: $x(t) = \cancel{[10 - 2j + 1 + 2j + 1 + 1 + 1] e^{j\frac{1}{5}t}}$

$\cancel{= 14 e^{jt/5}}$

$$= 10e^{0} + (-2j+1)e^{j\frac{2}{5}t} + (2j+1)e^{-j\frac{2}{5}t} + e^{j7/5 t} + e^{-j7/5 t}$$

$$= 10 - 2j\left[2j\sin(2/5\, t)\right] + 2\cos(2/5\, t) + 2\cos(7/5\, t)$$

$$= 10 + 4\sin(2/5\, t) + 2\cos(2/5\, t) + 2\cos(7/5\, t)$$

ⓓ $c_2 = e^{j\pi/2},\ c_{-2} = e^{-j\pi/2},\ c_3 = \sqrt{2}e^{-j\pi/4},\ c_{-3} = \sqrt{2}e^{j\pi/4}$, and all other $c_k = 0$. Assume $\omega_0 = 2\pi$.

Soln: $x(t) = \cancel{\left[e^{j\pi/2} + e^{-j\pi/2} + \sqrt{2}e^{-j\pi/4} + \sqrt{2}e^{j\pi/4}\right] e^{j2\pi t}}$

$\cancel{= \left[2\cos(\pi/2) + 2\sqrt{2}\cos(\pi/4)\right] e^{j2\pi t}}$

$\cancel{= 2\sqrt{2}\cdot\frac{1}{\sqrt{2}} e^{j2\pi t}}$

$\cancel{= 2e^{j2\pi t}}$

$$= e^{j\pi/2} e^{j4\pi t} + e^{-j\pi/2} e^{-j4\pi t} + \sqrt{2}e^{-j\pi/4} e^{j6\pi t} + \sqrt{2}e^{j\pi/4} e^{-j6\pi t}$$

$$= e^{j(4\pi t + \pi/2)} + e^{-j(4\pi t + \pi/2)} + \sqrt{2}e^{j(6\pi t - \pi/4)} + \sqrt{2}e^{-j(6\pi t - \pi/4)}$$

$$= 2\cos(4\pi t + \pi/2) + 2\sqrt{2}\cos(6\pi t - \pi/4)$$

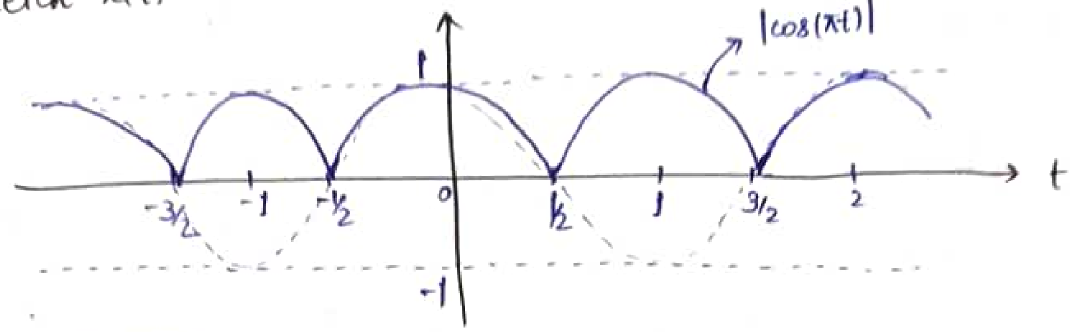
$$= -2\sin(4\pi t) + 2\sqrt{2}\cos(6\pi t - \pi/4).$$

⑧ Consider the following signal.

$$x(t) = |\cos(\pi t)|$$

ⓐ Sketch $x(t)$.

Soln:



$\omega_0 = 2\pi$

$T_0 = \frac{2\pi}{2\pi} = 1$

ⓑ Compute F.S. Coefficients c_k (for the complex form of F.S.) for the signal $x(t)$.

Soln:

$$c_k = \frac{1}{1}\int_0^1 |\cos(\pi t)|\, e^{-j2\pi k t}\, dt$$

$$= \int_0^{1/2} \cos(\pi t)\, e^{-jk2\pi t}\, dt - \int_{1/2}^{1} \cos(\pi t)\, e^{-j2\pi k t}\, dt$$

$$= \int_0^{1/2} \left(\frac{e^{j\pi t} + e^{-j\pi t}}{2}\right) e^{-j2\pi k t}\, dt - \int_{1/2}^{1} \left(\frac{e^{j\pi t} + e^{-j\pi t}}{2}\right) e^{-jk2\pi t}$$

$$= \frac{1}{2}\int_0^{1/2} \left(e^{(j\pi - j2\pi k)t} + e^{-(j\pi + j2\pi k)t}\right) dt - \frac{1}{2}\int_{1/2}^{1} \left(e^{(j\pi - j2\pi k)t} + e^{-(j\pi + jk2\pi)t}\right) dt$$

$$= \frac{1}{2}\left[\frac{e^{(j\pi - j2\pi k)t}}{j\pi - j2\pi k} + \frac{e^{-(j\pi + j2\pi k)t}}{-(j\pi + j2\pi k)}\right]_0^{1/2} - \frac{1}{2}\left[\frac{e^{(j\pi - j2\pi k)t}}{j\pi - j2\pi k} + \frac{e^{-(j\pi + j2k\pi)t}}{-(j\pi + j2\pi k)}\right]_{1/2}^{1}$$

$$= \frac{1}{2}\left[\frac{e^{j\pi/2 - j\pi k}}{j\pi - j2\pi k} + \frac{e^{-j\pi/2 - j\pi k}}{-(j\pi + j2\pi k)}\right] - \frac{1}{2}\left[\frac{1}{j\pi - j2\pi k} + \frac{1}{-(j\pi + j2\pi k)}\right]$$

$$- \frac{1}{2}\left[\frac{e^{j\pi - j2\pi k}}{j\pi - j2\pi k} + \frac{e^{-(j\pi + j2k\pi)}}{-(j\pi + j2\pi k)} - \frac{e^{j\pi/2 - j\pi k}}{j\pi - j2k\pi} - \frac{e^{j\pi/2 - j\pi k}}{-(j\pi + j2\pi k)}\right]$$

$$= \frac{\overset{j}{e^{j\pi/2}} e^{-j\pi k}}{j\pi - j2\pi k} + \frac{\overset{-j}{e^{-j\pi/2}} e^{-j\pi k}}{-(j\pi + j2\pi k)} - \frac{1}{2}\left[\frac{\overset{-1}{e^{j\pi}} e^{-j2\pi k} + 1}{j\pi - j2\pi k} + \frac{\overset{-1}{e^{-j\pi}} e^{-j2k\pi} + 1}{-(j\pi + j2\pi k)}\right]$$

$$= \frac{\cancel{j} e^{-j\pi k}}{\cancel{j}\pi - \cancel{j}2\pi k} + \frac{\cancel{j} e^{-j\pi k}}{\cancel{j}\pi + \cancel{j}2\pi k} + \frac{1}{2}\,\frac{e^{-j2\pi k} + 1}{j\pi - j2\pi k} - \frac{1}{2}\,\frac{e^{-j2k\pi} + 1}{j\pi + j2\pi k}$$

$$= e^{-jk\pi}\left[\frac{2}{\pi - 4\pi k^2}\right] + \frac{1}{2j}\left[\frac{-4k}{\pi - 4k\pi}\right] + \frac{e^{-j2k\pi}}{2j}\left[\frac{-4k}{\pi - 4\pi k^2}\right]$$

$$\Rightarrow C_k = j\frac{2k}{\pi(1-4k^2)} + \frac{e^{-jk\pi}}{\pi}\left[\frac{2}{1-4k^2}\right] + \frac{je^{-j2k\pi}}{\pi}\frac{2k}{(1-4k^2)}$$

$$= \frac{2}{\pi(1-4k^2)}\left[jk + e^{-jk\pi} - jke^{-j2k\pi}\right] = \frac{2}{\pi\cdot(1-4k^2)}(-1)^k$$

(c) Sketch C_k for $-5 \leq k \leq 5$.

Soln: $C_{-5} = \frac{2}{99\pi}$, $C_{-4} = -\frac{2}{63\pi}$, $C_{-3} = \frac{2}{35\pi}$, $C_{-2} = \frac{-2}{15\pi}$, $C_{-1} = \frac{2}{3\pi}$

$C_0 = \frac{2}{\pi}$, $C_1 = \frac{2}{3\pi}$, $C_2 = -\frac{2}{15\pi}$, $C_3 = \frac{2}{35\pi}$, $C_4 = \frac{-2}{63\pi}$, $C_5 = \frac{2}{99\pi}$