AV314 - Communication Systems

Assignment - 2 : Study Assignment

SC22B146
SAURABH KUMAR

① Properties of the Hilbert transform.

ⓐ We studied the Hilbert transform filter using its frequency domain characterization in class. Find out what is the impulse response of the Hilbert transform filter — here you have to derive the F.D. response as the Fourier transform of the impulse response or obtain the impulse response from the frequency domain characterization using Fourier transform.

ⓑ Suppose $g(t)$ is real signal and $\hat{g}(t)$ respresents the Hilbert transform of $g(t)$. Do the following tasks:

ⓘ Show that $|G(f)| = |\hat{G}(f)|$.

ⓘⓘ Show that $HT(HT(g(t))) = -g(t)$.

ⓘⓘⓘ Show that $\int_{-\infty}^{\infty} g(t)\,\hat{g}(t) = 0$.

ⓘⓥ Why is HT useful in the context of SSB?

Soln: ⓐ Frequency response of HT :

$$\hat{H}(f) = -j\,sgn(f)$$

$$\text{or, } \hat{H}(\omega) = -j\,sgn(\omega) = \begin{cases} -j, & f \geq 0 \text{ or } \omega \geq 0 \\ +j, & f < 0 \text{ or } \omega < 0. \end{cases}$$

Impulse response:

$$\hat{h}(t) = IFT(\hat{H}(\omega))$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{H}(\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{j}{2\pi}\int_{-\infty}^{0} e^{j\omega t}\, d\omega - \frac{j}{2\pi}\int_{0}^{\infty} e^{j\omega t}\, d\omega$$

$$= \frac{j}{2\pi}\lim_{a\to 0}\left[\int_{-\infty}^{0} e^{+a\omega} e^{j\omega t}\, d\omega - \int_{0}^{\infty} e^{-a\omega} e^{j\omega t}\, d\omega\right]$$

$$= \frac{j}{2\pi}\lim_{a\to 0}\left[\int_{-\infty}^{0} e^{(a+jt)\omega}\, d\omega - \int_{0}^{\infty} e^{-(a-jt)\omega}\, d\omega\right]$$

$$= \frac{j}{2\pi}\lim_{a\to 0}\left[\frac{e^{(a+jt)\omega}}{a+jt}\Bigg|_{-\infty}^{0} + \frac{e^{-(a-jt)\omega}}{a-jt}\Bigg|_{0}^{\infty}\right]$$

$$= \frac{j}{2\pi}\left[\frac{1}{jt} + \frac{1}{jt}\right] = \frac{j}{2\pi}\left[\frac{2}{jt}\right] = \boxed{\frac{1}{\pi t}}$$

(b) (i) To prove: $|G(f)| = |\hat{G}(f)|$

As $\hat{G}(f) = -j\,\text{sgn}(f)\, G(f)$

$\Rightarrow |\hat{G}(f)| = |-j|\,|\text{sgn}(f)|\,|G(f)|$

$\Rightarrow |\hat{G}(f)| = |G(f)|$

(ii) To prove: $HT(HT(g(t))) = -g(t)$.

$$F\{HT(HT(g(t)))\} = -j\,\text{sgn}(f).(-j\,\text{sgn}(f)).G(f)$$
$$= (-j)^2 (\text{sgn}(f))^2 G(f)$$
$$= -1 \times G(f)$$
$$= -G(f)$$

Take IFT both sides,

~~HT(HT)~~

$\Rightarrow HT(HT(g(t))) = -g(t)$.

(iii) To prove: $\int_{-\infty}^{\infty} g(t)\hat{g}(t)\,dt = 0$

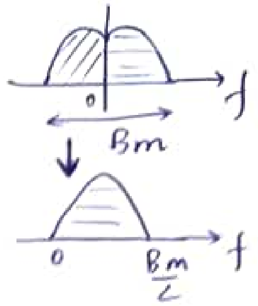
$$\int_{-\infty}^{\infty} g(t).\hat{g}(t)\,dt = \int_{-\infty}^{\infty} G(f).\hat{G}^*(f)\,df$$
$$= \int_{-\infty}^{\infty} G(f)\left[-j\,\text{sgn}(f).G(f)\right]^* df$$
$$= \int_{-\infty}^{\infty} G(f)G^*(f).\left[j\,\text{sgn}(f)\right] df$$
$$= j\int_{-\infty}^{\infty} |G(f)|^2\, \text{sgn}(f)\, df$$
$$= 0 \quad \left[\text{As } |G(f)|^2 \to \text{even} \Rightarrow |G(f)|^2\,\text{sgn}(f) \to \text{odd } f^n \text{ and } \int_{-\infty}^{\infty}(\text{odd } fn) = 0\right]$$

(iv) Hilbert transform of a signal produces a phase change of 90° for negative frequencies and −90° for positive frequencies. And in SSB, our aim is to retain only either the upper sideband (USB) or the lower sideband (LSB) of the signal. This is done by adding the HT resultant signal with the original signal.

$$m(t) \longrightarrow \boxed{\hat{H}(f)} \longrightarrow \hat{m}(t)$$

$m(t) \pm j\hat{m}(t) \xrightarrow{\mathcal{F}} M(f) \pm j\hat{M}(f)$

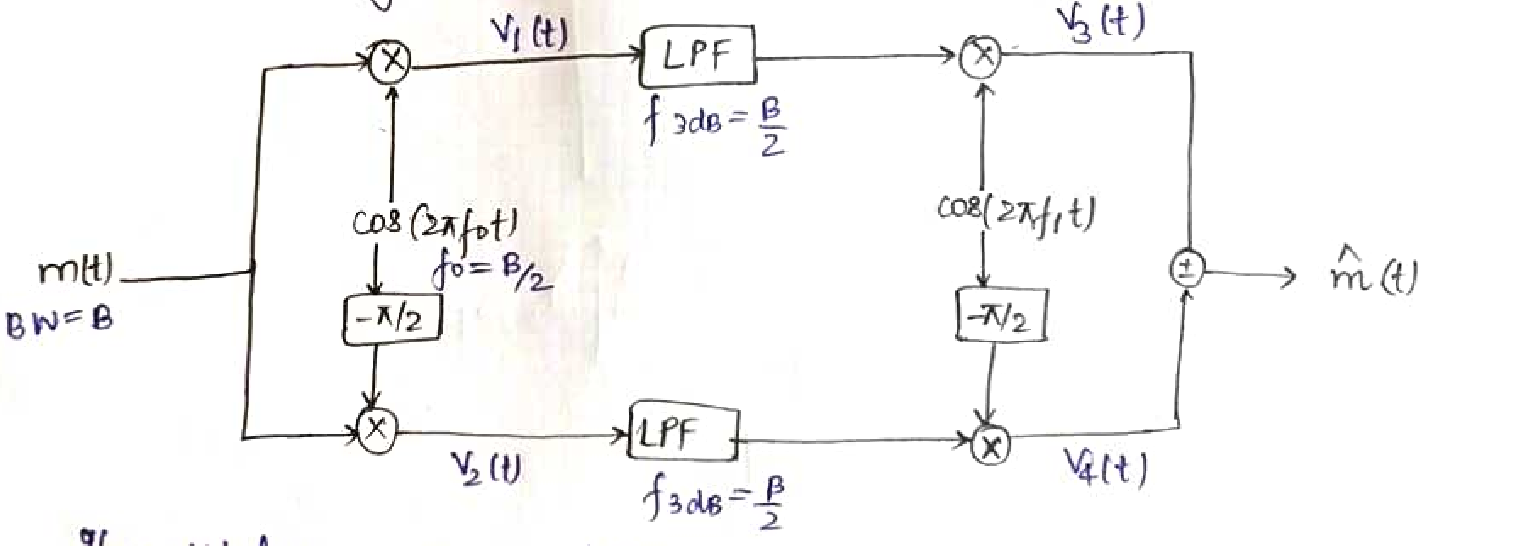
$$M(f) + j\hat{M}(f) = U(f) + L(f) + j[-jU(f) + jL(f)]$$
$$= U(f) + L(f) + U(f) - L(f)$$
$$= 2U(f) \longrightarrow \text{only USB}$$



② Write a short note on weaver's method for generating SSB signals. Include a derivation of the fourier transform of the generated SSB signal using weaver's method.

Soln: Weaver's method for SSB signal generation consists of ~~B~~ four modulators, two carrier signal generator, two low-pass filters and two $90°$ phase-shift networks.

First, the input signal is mixed with a low-frequency carriers. Then, the output of two mixer outputs are modulated with the RF carrier passing through low-pass filters. It is followed by a summing / difference circuit which generates the desired sideband signal.



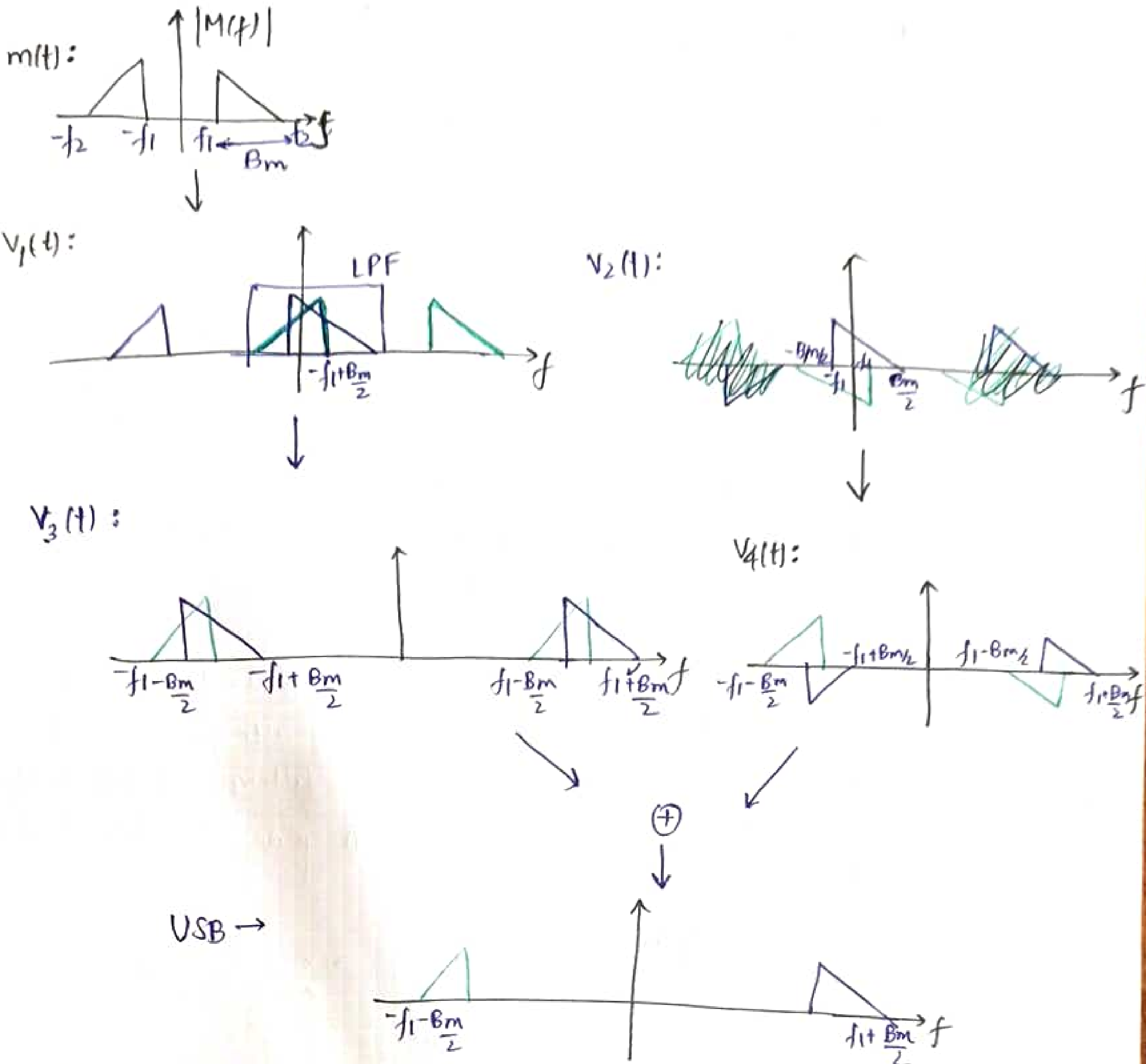
If $m(t)$ has maximum frequency component B, then

cut-off freq. of $LPF = B/2$

carrier freq., $f_0 = B/2$

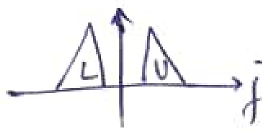
$$f_1 = f_c + B/2 \quad (\text{For USB})$$
$$= f_c - B/2 \quad (\text{For LSB})$$

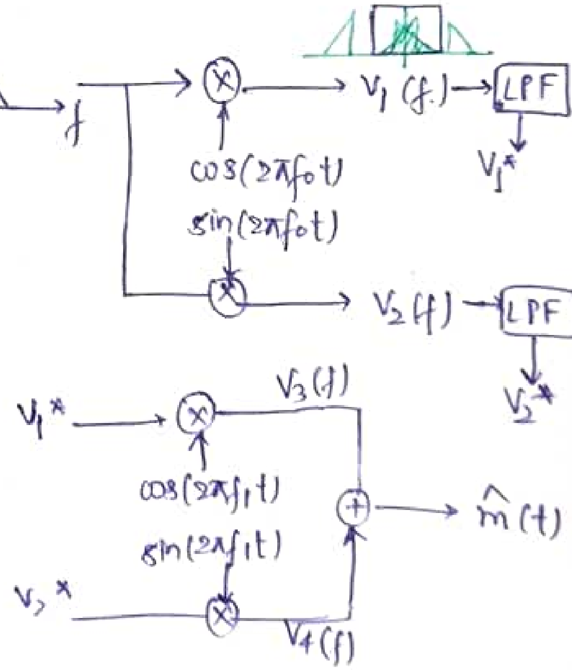
Eg.



Mathematically

$m(t) \longrightarrow M(f) = U(f) + L(f)$





$$V_1(f) = \frac{1}{2}\left[M(f-f_0) + M(f+f_0)\right]$$

$$V_2(f) = \frac{1}{2j}\left[M(f-f_0) - M(f+f_0)\right]$$

After LPF,

$$V_1^* = \frac{1}{2}\left[L(f-f_0) + U(f+f_0)\right]$$

$$V_2^* = \frac{1}{2j}\left[L(f-f_0) - U(f+f_0)\right]$$

After 2nd mixer,

$$V_3(f) = \frac{1}{4}\left[U(f+f_0-f_1) + L(f-f_0-f_1) + U(f+f_0+f_1) + L(f-f_0+f_1)\right]$$

$$V_4(f) = \frac{1}{4}\left[U(f+f_0-f_1) - L(f-f_0-f_1) - U(f+f_0+f_1) + L(f-f_0+f_1)\right]$$

After summation: $\hat{M}(f) = \frac{1}{2}\left[U(f+f_0-f_1) + L(f-f_0+f_1)\right] \leftarrow$ USB

or $\hat{M}(f) = \frac{1}{2}\left[L(f-f_0-f_1) + U(f+f_0+f_1)\right] \leftarrow$ LSB