INDIAN INSTITUTE OF SPACE SCIENCE AND TECHNOLOGY
THIRUVANANTHAPURAM 695 547

End Semester Examination - May 2014

B. Tech - II Semester

MA121 - Vector Calculus and Differential Equations

Date: 28/04/2014 | **Time: 9.30 am - 12.30 pm** | **Max. Marks: 100**

SECTION A (Attempt all 10 questions - 10x5= 50 marks.)

1. Let $F_n(x) = nxe^{-nx^2}$. Show that each $F_n(x)$ and whose pointwise limit $F(x) = \lim_{n\to\infty} F_n(x)$ are continuous on $[0,1]$ but $\{F_n(x)\}$ does not converge uniformly on $[0,1]$.

2. (a) Let $\{F_n\}$ be a sequence defined by

$$F_n(x) = \begin{cases} n^2x, & 0 < x < \frac{1}{n} \\ 2n - n^2x, & \frac{1}{n} \le x < \frac{2}{n} \\ 0, & \frac{2}{n} \le x < 1. \end{cases}$$

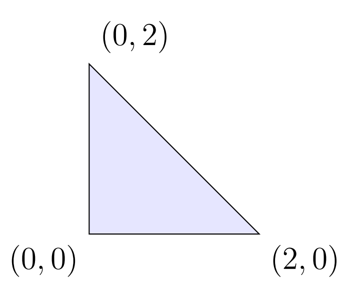
Then show that $\lim_{n\to\infty} \int_0^1 F_n(x)dx \neq \int_0^1 \lim_{n\to\infty} F_n(x)dx.$ **[4 Marks]**

(b) Let $\{f_n\}$ be a sequence of functions defined on $[a,b]$. Then under what conditions the following can be justified: $\frac{d}{dx}\Big(\lim_{n\to\infty} f_n(x)\Big) = \lim_{n\to\infty}\Big(\frac{d}{dx} f_n(x)\Big)$. **[1 Mark]**

3. Define directional derivative of a function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ at a point P_0 along a vector $\vec{v}$. Find the directional derivative of $f(x,y,z) = ze^{xy}$ at the point $P_0 = (1,1,1)$ along the vector $\vec{v} = (3,4,5)$. With explanation find the direction along which the rate of change of f is maximum. **[(1+2+2) Marks]**

4. Define arc length function of a smooth curve $C : \vec{\gamma}(t),\ t \in [a,b]$ with initial point $\gamma(a)$. Find the arc length function of the curve $C : y = \sqrt{\frac{3}{2}}x^2,\ z^2 = x^6,\ z \ge 0$ with initial point $(0,0,0)$. **[(2+3) Marks]**

5. Let $\vec{F}\,(x,y) = (ye^{x^2+y^2} + 2x^2ye^{x^2+y^2},\ xe^{x^2+y^2} + 2xy^2e^{x^2+y^2})$ be a vector field on xy-plane and C a curve joining straight line segments $(0,0) \longrightarrow (2,0) \longrightarrow (2,1) \longrightarrow (1,0)$ oriented accordingly. Evaluate $\int_C \vec{F}\,.d\vec{s}$. Is the integral independent of the path? Justify your answer. **[(3+2) Marks]**

6. Using Green's theorem find area of the triangle 

on xy-plane.

7. State the non-local existence theorem and with proper justification, find the largest interval in which the initial-value problem

$$\frac{dy}{dx} = (\tan x)y - \frac{\sin y}{16 - x^2}, \quad y(\pi) = y_0, \quad y_0 \in \mathbb{R},$$

has a unique solution.

8. (a) If the Wronskian of two functions f and g is $W(f,g)(x) = 3e^{4x}$, where $f(x) = e^{2x}$, then find $g(x)$.
(b) Determine whether the pair of functions $\{f(x) = x^5,\, g(x) = x^2|x|^3\}$ can be solutions of the differential equation $y'' + p(x)y' + q(x)y = 0$ with p and q continuous on $[-a, a]$, $a > 0$.

9. Find the singular point of the following differential equation and classify it. Also determine the interval of convergence of the corresponding series solution of

$$(x-\pi)y'' + \frac{1}{2\pi - 2\pi x + 2\pi^2}y' + \frac{1}{\frac{3}{2}(x-\pi)^2 + \frac{2}{3}}y = 0,$$

about that point.

10. Find the general solution of $xy'' + (1+2\lambda)y' + xy = 0, \quad x > 0$ in terms of Bessel's functions, using the substitution $y = \dfrac{u(x)}{x^\lambda}$, where λ is a positive real number.

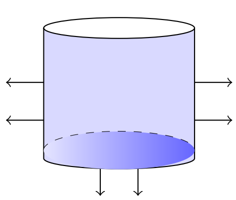
SECTION B (Attempt any 5 questions - 5x10= 50 marks.)

11. (a) Suppose that $\displaystyle\sum_{n=1}^{\infty} f_n(x)$ converges uniformly to a function S on an interval I. Suppose that each $f_n(x)$ is continuous on I. Then prove that S is also continuous on I. Check whether $\displaystyle\sum_{n=1}^{\infty} \frac{xe^{-nx}}{n^2}$, $x \in (0, +\infty)$ is a continuous function? **[(2+4) Marks]**

(b) Show that the series $\displaystyle\sum_{n=1}^{\infty} x^{n-1}$ converges uniformly on $[0, \frac{3}{4}]$ and hence evaluate the series $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n\, 4^n}$. **[4 Marks]**

12. (a) State Green's theorem for non-simply connected surfaces having two boundary components (i.e., only one hole). Verify the stated Green's theorem for the vector field $F(x,y) = (y^2, xy)$ over the annular region $S : 1 \le x^2 + y^2 \le 9$. **[(2+5) Marks]**

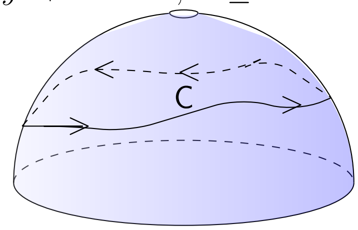
(b) Using Stoke's theorem find the surface integral $\displaystyle\int_S Curl(\vec{F}).d\vec{S}$



where $\vec{F}\,(x,y,z) = (-y, z, x)$ and S is a outward oriented surface given by $S = S_1 \sqcup S_2$ where $S_1 : x^2 + y^2 = 1,\ 0 \le z \le 1$ and $S_2 : x^2 + y^2 \le 1$.

[3 Marks]

13. Let S be the upper hemisphere $x^2+y^2+z^2=25,\ z\geq 0$ without the point $(0,0,5)$ and C a



positively oriented smooth loop on S.

Let $\vec{F}\,(x,y,z)=\left(\dfrac{-y}{x^2+y^2},\dfrac{x}{x^2+y^2},5\right)$ be a vector field. Find the domain of $\vec{F}$. Find the value of the integral $\displaystyle\int_C \vec{F}\,.d\vec{s}$. **[(1+9) Marks]**

14. (a) Using the Picard's theorem, verify whether the initial-value problem

$$\frac{dy}{dx}=1+y^{2/3},\ \ y(2014)=\frac{1}{2014},$$

has a unique solution around the point 2014.
If $y(2014)=0$, does the Picard's theorem guarantee the existence of a unique solution? Justify your answer. **[(2+2) Marks]**

(b) Find the general solution of $x^2\dfrac{d^2y}{dx^2}-6y=\ln x,\, x>0$, using the method of variation of parameters. **[6 Marks]**

15. (a) Find the indicial equation, the recurrence relation and the Frobenius series solution up to <u>first two terms</u> of $(x-x^2)y''+(1-x)y'-y=0$, about the point $x=0$ for $x>0$. Also write the proper form of another independent solution. **[(5+2) Marks]**

(b) Verify whether $y+(2x-ye^y)\dfrac{dy}{dx}=0$ is exact or not, and then solve it. **[3 Marks]**

16. (a) Using the definition of Bessel function of first kind of order $p\,(p>0)$, given by

$$J_p(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(n+p+1)}\left(\frac{x}{2}\right)^{2n+p}$$

establish the identity $\dfrac{d}{dx}[x^pJ_p(x)]=x^pJ_{p-1}(x)$. Also show that between any two positive roots of $J_{p+1}(x)=0$, there is a root of $J_p(x)=0$. **[(2+3) Marks]**

(b) Find the eigenvalues and eigenfunctions of the boundary value problem

$$\frac{d}{dx}(x\frac{dy}{dx})+\frac{\lambda}{x}y=0,\quad 1<x<l,\quad y(1)=0,\,y(l)=0,\quad \lambda\in\mathbb{R}.$$

[5 Marks]

END