Indian Institute of Space Science and Technology, Thiruvananthapuram

MA111-CALCULUS

TUTORIAL I

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1. Show that each of the following sequences $\{a_n\}$ converges to a limit α (say). For given $\epsilon > 0$, find an $N(\epsilon) \in \mathbb{N}$ as required in the definition of limit.

(i) $a_n = \dfrac{1}{an+b}$, for $a, b > 0$.

(ii) $a_n = \dfrac{\sqrt{n}}{n^2+1}$

(iii) $a_n = \dfrac{n}{n^2-n+1}$

(iv) $a_n = \sqrt{n^2+1} - n$

(v) $a_n = \dfrac{n+1}{2n+3}$

(vi) $a_n = \dfrac{2^n}{5^{n+1}}$

(vii) $a_n = \dfrac{3+5n^2}{n+n^2}$

(viii) $a_n = \dfrac{cos^2 n}{3^n}$

(ix) $a_n = \dfrac{1}{\ln(1+n)}$

2. Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences. Let $s_n := \min\{a_n, b_n\}$ and $t_n := \max\{a_n, b_n\}$. Discuss whether the sequences $\{s_n\}$ and $\{t_n\}$ are convergent or divergent?

3. Let $a_n := (1-\frac{1}{2})(1-\frac{1}{3})...(1-\frac{1}{n+1})$. Discuss the monotonicity, boundedness, and convergence of $\{a_n\}$.

4. Let $a_n := \frac{1}{1^2+1} + \frac{1}{2^2+2} + ... + \frac{1}{n^2+n}$. Show that the sequence $\{a_n\}$ is monotonically increasing and bounded. Justify its convergence.

5. Let $a_n := \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{2n}$. Show that the sequence $\{a_n\}$ is convergent to a limit at most one.

6. Show that each of the following sequences $\{a_n\}$ are null sequences, i.e., $a_n \to 0$ as $n \to \infty$.

(i) $a_n = \sqrt{n+1} - \sqrt{n}$

(ii) $a_n = n^2\, b^n$ for $0 < b < 1$.

(iii) $a_n = \dfrac{n!}{n^n}$

(iv) $a_n = \dfrac{n^3}{n!}$

(v) $a_n = \dfrac{3^n}{n!}$

(vi) $a_n = \dfrac{a^n}{n!}$ for fixed $a \in \mathbb{R}$

(vii) $a_n = \dfrac{1}{(n+1)^2} + \dfrac{1}{(n+2)^2} + ... + \dfrac{1}{4n^2}$

(viii) $a_n = \dfrac{1.3.5...(2n-1)}{2.4.6...2n}$

7. Let $a_n := \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + ... + \frac{1}{\sqrt{n^2+n}}$. Show that the sequence $\{a_n\}$ is convergent.

8. Let $a > 0$ and $a_n := a^{\frac{1}{n}}$. Show that the sequence $\{a_n\}$ is convergent.

9. Let $0 < a < b$. Show that the sequence $\{(a^n + b^n)^{\frac{1}{n}}\} \to b$.

10. Let $\{a_n\}$ be a bounded sequence. Assume that $a_{n+1} \geq a_n - 3^{-n}$. Show that $\{a_n\}$ is convergent.

11. Determine the limits of the following sequences:

(i) $a_n = \sqrt{4n^2+n} - \sqrt{2n}$

(ii) $a_n = (n+1)^{1/\ln(n+1)}$

(iii) $a_n = (n!)^{1/n^2}$

(iv) $a_n = \sqrt{(n+a)(n+b)} - n$ for $a > 0, b > 0$.

12. Let $a_n = (1+\frac{1}{n})^n$. Show that the sequence $\{a_n\}$ is monotonic and bounded with limit at most 3.

• Evaluate the followings:

13. $\lim_{n\to+\infty} (1^2+2^2+3^2+...+n^2)^{1/n}$.

14. $\lim_{n\to+\infty} \frac{1}{\sqrt{n}}\Big(\frac{1}{\sqrt{1}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{5}} + ... + \frac{1}{\sqrt{2n-1}+\sqrt{2n+1}}\Big)$.

15. $\lim_{n\to+\infty} \Big(\frac{1}{n^2+1} + \frac{2}{n^2+2} + ... + \frac{n}{n^2+n}\Big)$.

16. $\lim_{n\to+\infty} \frac{n!}{2^{n^2}}$.

17. Study the convergence or divergence of the following sequences:
(a). $a_n = \Big(1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2}\Big)$.
(b). $a_n = \Big(1 + \frac{1}{2^2} + \frac{1}{3^3} + ... + \frac{1}{n^n}\Big)$.

18. Justify the convergence or divergence of the sequence defined by

$$a_1 = 0, \quad a_2 = \frac{1}{2}, \quad a_{n+1} = \frac{1+a_n+a_{n-1}^3}{3}, \quad n > 1$$

Find its limit if it converges.

19. Discuss the convergence or divergence of the sequence defined by

$$a_1 = \sqrt{2}, \quad a_n = \sqrt{2+\sqrt{a_{n-1}}}, \quad n > 1$$

Justify its limit.

20. Let $a > 0$, let a_1 be any positive real number. Justify the convergence or divergence of the sequence $\{a_n\}$ defined by

$$a_{n+1} = \frac{1}{2}\Big(a_n + \frac{a}{a_n}\Big), \quad n \geq 1$$

Find its limit if it exists.

21. Show that the sequence $\{a_n\}$ defined by

$$0 < a_1 < a_2, \quad \frac{2}{a_{n+2}} = \frac{1}{a_{n+1}} + \frac{1}{a_n}, \quad n \geq 1$$

converges to the limit $\frac{3a_1a_2}{2a_1+a_2}$.

22. Given $a > 0$, define the following sequence $\{a_n\}$ by putting

$$a_1 < 0, \qquad a_{n+1} = \frac{a}{a_n} - 1, \qquad \forall\, n \in \mathbb{N}$$

Show that $\{a_n\}$ is convergent sequence.

23. Determine the n-th term of the following sequence

$$\left\{2, \frac{2}{1+2}, \frac{2}{1+\dfrac{2}{1+2}}, \ldots\ldots\right\}$$

Show that it converges to 1.

24. Let $a_1 = 5$. The sequence $\{a_n\}$ defined by

$$a_{n+1} = 3^{\frac{1}{4}}(a_n)^{\frac{3}{4}}, \quad n \geq 1$$

Discuss the convergence of $\{a_n\}$.

25. Let $\{a_n\}$ be a sequence of real numbers such that $a_1 = 3$, and for $n \geq 1$

$$a_{n+1} = \frac{a_n^2 - 2a_n + 4}{2}$$

Discuss whether $\{a_n\}$ is convergent or divergent.

26. Let $\{a_n\}$ be a sequence of real numbers such that $a_1 = 2$, and for $n \geq 1$

$$a_{n+1} = \frac{2a_n + 1}{a_n + 1}$$

Discuss whether $\{a_n\}$ is convergent or divergent. Find its limit if it converges.

27. $\displaystyle\lim_{n\to+\infty} \frac{1 + \frac{1}{2} + \ldots + \frac{1}{n}}{(\pi^n + e^n)^{\frac{1}{n}} \log_e n}$ equals __________.