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# LINEAR ALGEBRA

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# MATRICES AND SYSTEM OF LINEAR EQUATIONS

The system of linear equations

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

$a_{ij} \in \mathbb{R}$.

can be written in the matrix form as $Ax = b$

where $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & & a_{mn} \end{pmatrix}$

$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$.

We want to reduce the matrix form into a simple form so that solutions can be obtained.

For this we use row-reduced echelon form of a matrix. This method is called Gauss - Jordan elimination method.

For that we consider the $m \times (n+1)$ matrix $(A : b)$ called the augmented matrix.

## Elementary Row Operations

① A row is multiplied with a non-zero number.

$R(Ki \rightarrow i)$, $i$ is $i^{th}$ row.

② Interchanging rows

$R(i \rightarrow j, j \rightarrow i)$

③ Multiple of a row is added to another row.

$R(Ki + j \rightarrow j)$

**Note:** Elementary column operations can also be defined. But we are interested in row operations, since we wanted to solve system of linear equations.

**Defn:** Two $m \times n$ matrices are said to be row equivalent if one can be obtained from the other by means of a finite no. of elementary row operations.

**Defn:** An $m \times n$ matrix is said to be of row-echelon form if

(i) The first non-zero entry in a row is 1. (e.g., 0 0 1)

This entry is called the leading entry. (first non-zero entry)

(ii) The leading entry of $(i+1)^{th}$ row is to be right of the leading entry of the $i^{th}$ row.

Eg.

$$\begin{array}{l} 0\ 0\ 1\ 2\ 0\ 3 \quad 4^{th} = i^{th} \\ 0\ 0\ 0\ 0\ 1 \quad 5^{th} = (i+1)^{th} \end{array}$$

(iii) The rows with all entries zero must lie in the bottom.

→ An $m \times n$ matrix is said to be row-reduced echelon form if in addition to the above three conditions, it must satisfy:

(iv) Other entries in a column having the leading entry must be zero.

Eg.,
$$\begin{pmatrix} 0 & 1 & 0 & \cancel{2}\,0 & 5 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \rightarrow \text{Row-reduced echelon form}$$

Note: Using elementary row operations, we can find row-echelon form and row-reduced echelon form of ~~the~~ a matrix.
Row echelon form ~~form~~ is not unique.
(Row-reduced echelon form is <u>unique</u>.)

→ For $(S, \sim)$, ($S$: set, $\sim$: Relation)

$s_1, s_2, s_3 \in S$

(i) $s_1 \sim s_1$ : Reflexive
(ii) $s_1 \sim s_2 \Rightarrow s_2 \sim s_1$ : Symmetric
(iii) $s_1 \sim s_2,\ s_2 \sim s_3 \Rightarrow s_1 \sim s_3$ : Transitive

Q) Reduce $\begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & 4 \\ -2 & 8 & 2 \end{bmatrix}$ into row-reduced echelon form.

← choose any one

Soln: Look for the farther non-zero value (with greater absolute value, in case of tie) in each row (looking from right side), and interchange that row with the 1st.

$$\begin{bmatrix} 2 & -1 & 4 \\ 1 & 3 & 5 \\ -2 & 8 & 2 \end{bmatrix} \overset{R(i)\frac{1}{2}}{\sim} \begin{bmatrix} 1 & -\frac{1}{2} & 2 \\ 1 & 3 & 5 \\ -2 & 8 & 2 \end{bmatrix}$$

$$\underset{\substack{R(i)2+(iii) \\ \to (iii)}}{\overset{\substack{R(i)(-1)+(ii) \\ \to (ii)}}{\sim}} \begin{bmatrix} 1 & -\frac{1}{2} & 2 \\ 0 & \frac{7}{2} & 3 \\ 0 & 7 & 6 \end{bmatrix}$$

$$\overset{R(ii)\times\frac{2}{7}}{\sim} \begin{bmatrix} 1 & -\frac{1}{2} & 2 \\ 0 & 1 & \frac{6}{7} \\ 0 & 7 & 6 \end{bmatrix} \underset{\substack{R(ii)(-7)+(iii) \\ \to (iii)}}{\overset{\substack{R(ii)\times\frac{1}{2}+(i) \\ \to (i)}}{\sim}} \begin{bmatrix} 1 & 0 & \frac{17}{7} \\ 0 & 1 & \frac{6}{7} \\ 0 & 0 & 0 \end{bmatrix}$$

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Q) Solve by Gauss-Jordan Elimination method.

$$x_2 + 2x_3 = 3$$
$$2x_1 + 4x_2 + 6x_3 + 2x_4 = 4$$
$$x_1 + 2x_2 + 4x_3 + 2x_4 = 2$$
$$x_1 + 3x_2 + 6x_3 + x_4 = 5$$

Sol^n:

$$\overset{A}{\begin{bmatrix} 0 & 1 & 2 & 0 \\ 2 & 4 & 6 & 2 \\ 1 & 2 & 4 & 2 \\ 1 & 3 & 6 & 1 \end{bmatrix}} * \overset{X}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}} = \overset{B}{\begin{bmatrix} 3 \\ 4 \\ 2 \\ 5 \end{bmatrix}}$$

Augmented Matrix

$$\begin{bmatrix} 0 & 1 & 2 & 0 & 3 \\ 2 & 4 & 6 & 2 & 4 \\ 1 & 2 & 4 & 2 & 2 \\ 1 & 3 & 6 & 1 & 5 \end{bmatrix} \underset{\sim}{\xrightarrow{R\ (ii) \leftrightarrow (i)}} \begin{bmatrix} 2 & 4 & 6 & 2 & 4 \\ 0 & 1 & 2 & 0 & 3 \\ 1 & 2 & 4 & 2 & 2 \\ 1 & 3 & 6 & 1 & 5 \end{bmatrix} \underset{\sim}{(i)\tfrac{1}{2}}$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 2 \\ 0 & 1 & 2 & 0 & 3 \\ 1 & 2 & 4 & 2 & 2 \\ 1 & 3 & 6 & 1 & 5 \end{bmatrix} \underset{(iv) \to (iv) - (i)}{\overset{(iii) \to (iii) - (i)}{\sim}} \begin{bmatrix} 2 & 1 & 3 & 1 & 2 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 & 3 \end{bmatrix} \underset{(iv) \to (iv) - (ii)}{\overset{(i) \to (i) - 2(ii)}{\sim}}$$

$$\begin{bmatrix} 1 & 0 & -1 & 1 & -4 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \underset{(iv) \to (iv) - (iii)}{\overset{(i) \to (i) + (iii)}{\sim}} \begin{bmatrix} 1 & 0 & 0 & 2 & -4 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix} \overset{(ii) \to (ii) - 2(iii)}{\sim} \begin{bmatrix} 1 & 0 & 0 & 2 & -4 \\ 0 & 1 & 0 & -2 & 3 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

$(iii) \to (iii) + (iv)$
$(ii) \to (ii) + (-2)(iv)$
$(i) \to (i) + 2(iv)$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow \begin{aligned} x_1 &= -4 \\ x_2 &= 3 \\ x_3 &= 0 \\ x_4 &= 0. \end{aligned}$$

**Note:** (i) Any matrix can be reduced to Row-echelon or reduced row-echelon.

(ii) Row echelon form of a matrix need not be unique.
But reduced row-echelon form is unique.

(iii) If the square matrix $A_{m\times n}$ is invertible ($\det A \neq 0$) then the reduced row-echelon form of A is the identity matrix $m\times n$.

[Note that elementary row operation of A are multiplication with elementary matrices $E_i$ with A.. :

$$E_1 - E_0 A = I_{m\times n}$$

These elementary matrices are invertible

$\Rightarrow$ A is invertible.]

$$\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}2 & 1\\ 3 & 4\end{pmatrix} = \begin{pmatrix}2\alpha & \alpha\\ 3 & 4\end{pmatrix}$$

$$\begin{pmatrix}2 & 1\\ 3 & 4\end{pmatrix} \xrightarrow{i \to i\alpha} \begin{pmatrix}2\alpha & 1\alpha\\ 3 & 4\end{pmatrix}$$

(iv) The no. of rows will all entries zero is same in a row-echelon form of A.
The same number of zero rows occur in reduced row-echelon form ~~of A~~ also.

**Defn:** The number of non-zero rows in the row-echelon form (or reduced row-echelon form) of a matrix A is called the row rank of A.
(Similarly, we define column rank of A.)

**Theorem:**
For a matrix A, the column rank and row rank are same.

**Defn:** The ~~colu~~ common value of row rank and column rank of A is called the rank of A denoted by rank(A).

Q) find the rank of $\begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 2 & 3 & 4 \\ 4 & 7 & 7 & 9 \end{bmatrix}$.

Soln: The row echelon form is

$$A = \begin{bmatrix} 1 & 7/4 & 7/4 & 9/4 \\ 0 & 1 & 5 & 7 \\ 0 & 0 & 1 & 4/5 \end{bmatrix}$$

$\therefore$ rank $(A) = 3$.

Note: Suppose in a matrix, one row is sum of multiple of other row (some other rows).
Then, in the row-echelon (or reduced row-echelon) form, that row becomes zero row.
Such a row is called linearly depedent row. Otherwise we call linearly independent row.
Thus, rank(A) is the number of linearly independent rows (or number of linearly independant columns).

Defn: A system of linear equations is said to be consistent if it has a solution.

For a system, ~~thse~~ there are three possibilities:

(i) no solution exists.
(ii) a unique solution exists.
(iii) infinitely many solutions exist.

(iii) = 2×(ii) + 5×(v)

## Homogeneous Case:

Consider the system $Ax=0$, where $A$ is $m\times n$ matrix and $x=\begin{pmatrix}x_1\\x_2\\\vdots\\x_m\end{pmatrix}$ ($m$ equations in $n$-variables). | $A=(a_{ij})_{m\times n}$ $x=\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}_{n\times 1}$

This system is consistent always. ($\because x_1=x_2=\dots=x_n=0$ is a solution.)

Case-(i): $m=n$

$A$ is $n\times n$ square matrix.

$Ax=0.$

If $A$ is invertible ($\det A\neq 0$), then the system has trivial solution.

Non-trivial solution exists if $\det(A)=0$.

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If $\text{rank}(A)=n$, then only trivial solution exists for the system.

Non-trivial solution exists if $\det(A)=0$.

Case-(ii): $A$ is $m\times n$ matrix and $x=\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}$.

The system, $Ax=0$, has non-trivial solution, if and only if $\text{rank}(A)<n$.

Proof: Suppose $\text{rank}(A)=r<n$.

(If $A_E$ is the row-echelon form, then $Ax=0$ and $A_Ex=0$ has same zeroes.)

Since $\text{rank}(A)=r$, there are '$r$' equations in '$n$' variables.

Now $r<n$, then $(n-r)$ variables can be given arbitrary values and the values of $r$ variables can be obtained in terms of $(n-r)$ variables. | $x_1+2x_2+3x_3=1$, $x_2=\alpha$, $x_3=\beta$, $x_1=1-2\alpha-3\beta$

Therefore, there are infinitely many non-trivial solutions.

Conversely, suppose $Ax=0$ has non-trivial solutions.

To show that $\text{rank}(A)<n$.

# $A$ is a $m \times n$ matrix.

$\text{rank}(A) \le \min\{m, n\}$.

To show that $\text{rank}(A) \le n$.

Case ①: $m = n$, in this case $\text{rank}(A) \le n$ if $\text{rank}\, A = n$, then ~~ste~~ after reduction $A$ becomes identity matrix.

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

In this case $Ax = 0$ has trivial solution only.

But given $Ax = 0$ has non-trivial solution.

$\therefore \text{rank}(A) < n$.

(ii) $m < n$, then $\text{rank}(A) \le m < n$.

So, $\text{rank}(A) < n$.

(iii) $m > n$, then $\text{rank}(A) \le n$.

If $\text{rank}(A) = n$, then $A$ is invertible and there is only trivial solution for $Ax = 0$. ($\because Ax = 0$ has non-trivial solution)

$\therefore \text{rank}(A) < n$

Q.① Solve the system of eqns.

$$x_1 + 2x_2 + x_3 + 7x_4 = 0$$

$$3x_1 + 6x_2 + 4x_3 + 24x_4 + 3x_5 = 0$$

$$x_1 + 4x_2 + 4x_3 + 12x_4 + 3x_5 = 0.$$

Soln: $A = \begin{bmatrix} 1 & 2 & 1 & 7 & 0 \\ 3 & 6 & 4 & 24 & 3 \\ 1 & 4 & 4 & 12 & 3 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 8 & 3 \\ 0 & 1 & 0 & -2 & -3 \\ 0 & 0 & 1 & 3 & 3 \end{bmatrix}$$

rank(A) = 3 < 5 = n.

Put arbitrary values for 2 variables.

Put $x_4 = \alpha$, $x_5 = \beta$, $\alpha, \beta \in \mathbb{R}$.

$$A_E x = 0 \Rightarrow \begin{aligned} x_1 + 8x_4 + 3x_5 &= 0 \\ x_2 - 2x_4 - 3x_5 &= 0 \\ x_3 + 3x_4 + 3x_5 &= 0 \end{aligned}$$

$\therefore$ $x_4 = \alpha$, $x_5 = \beta$

$x_1 = -8\alpha - 3\beta$

$x_2 = 2\alpha + 3\beta$

$x_3 = -3\alpha - 3\beta$

Q.2 Let $A = \begin{bmatrix} 1 & 4 & 1 & 2 \\ 1 & 3 & 0 & 1 \\ 2 & 1 & 1 & 1 \\ 4 & 9 & 3 & 5 \\ 5 & 5 & 2 & 3 \end{bmatrix}$ be the coefficient matrix of a system $Ax = 0$, $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$

Find solutions.

Soln: $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, rank(A) = 3.

$x_1 = 0$

$x_2 + \frac{1}{3}x_4 = 0$

$x_3 + \frac{2}{3}x_4 = 0$.

Put arbitrary value for one variable.

Put $x_4 = \alpha$.

Solutions are:

$x_1 = 0$

$x_2 = -\frac{1}{3}\alpha$

$x_3 = -\frac{2}{3}\alpha$

$x_4 = \alpha$.

## Non-Homogeneous case:

$Ax=b$ where $A=(A_{ij})_{m\times n}$, $x=\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $b=\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$

with atleast one of $b_i$ is non-zero.

Denote $\text{rank}(A)=r$,

$\text{rank}[A:b]=r_a$

Case-(i): If $r=r_a<n$, the system has infinitely many solutions.

$$\begin{pmatrix} a_{11} & \cdots & a_{1m} & b_1 \\ a_{21} & \cdots & a_{2m} & b_2 \\ \cdots & \cdots & \cdots & \vdots \\ a_{m1} & \cdots & a_{mm} & b_m \end{pmatrix} = [A:b]$$

In this case, there are only $r$ equations in $n$ variables $(r<n)$.
Then $(n-r)$ variables can be given arbitrary values.
Therefore, we have infinitely many solutions.

Case-(ii): If $r=r_a=n$, then the system has unique solutions.

$$\left(\begin{array}{c|c} \boxed{I_{m\times n}} & \begin{matrix} b_1' \\ \vdots \\ b_n' \neq b_n \end{matrix} \\ \cdots & \end{array}\right)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Case (iii): $r<r_a$, then system has no solution.

$[A:b]_{m\times(n+1)}$

$$\begin{pmatrix} \boxed{\begin{matrix} a & a & \cdots \\ \cdots & A & \cdots \\ - & - & \cdots \end{matrix}} & \begin{matrix} \cdots \\ \cdots \\ \cdots \end{matrix} & \begin{matrix} \vdots \\ \vdots \\ \vdots \end{matrix} \\ 0 \quad 0 \quad \cdots & 0 & 1 \end{pmatrix} \qquad 0x_1+0.x_2+\dots+0.x_n=1$$

In this case, the reduced form of $[A:b]$ has the last row.
That is, $0.x_1+\dots+0.x_2=1$, which is not possible.

This implies that the system has no solution.

Result: The system $Ax=b$ has solution

iff $\text{rank}(A) = \text{rank}[A:b]$

Q) Find the values of $c$ and $k$ for which the system

$$x+y+z=3$$
$$x+2y+cz=4$$
$$2x+3y+2cz=k.$$

has (i) no solution (ii) unique solution (iii) infinitely many sol^ns.

Soln:

$$[A:b] = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & c & 4 \\ 2 & 3 & 2c & k \end{bmatrix} \underset{(i)(-2)+(iii)}{\overset{(i)(-1)+(ii)}{\sim}} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & (c-1) & 1 \\ 0 & 1 & 2(c-1) & (k-6) \end{bmatrix}$$

$$\underset{(ii)(-1)+(iii)}{\overset{(ii)(-1)+(i)}{\sim}} \begin{bmatrix} 1 & 0 & 2-c & 2 \\ 0 & 1 & c-1 & 1 \\ 0 & 0 & c-1 & k-7 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 0 & 2-c \\ 0 & 1 & c-1 \\ 0 & 0 & c-1 \end{bmatrix}$$

Case (i): If $c \neq 1$, and $k$ be any value,

then $\text{rank}(A)=3$, $\text{rank}[A:b]=3$.

$r = r_a = 3 = n (=3)$

$\Rightarrow$ System has unique soln.

Case (ii): If $c=1$, $k \neq 7$,

then $\text{rank}(A)=2$, $\text{rank}[A:b]=3$

$r \neq r_a$.

$\Rightarrow$ The system has no solution.

(iii) If $c=1$ and $K=7$.

$r = r_a = 2 < n (=3)$

$\Rightarrow$ The system has infinitely many sol^ns.

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# Eigenvalues and Eigenvectors

Let $A$ be an $m\times n$ matrix (real or complex entries).
A number $\lambda$ is said to be an eigenvalue of $A$ if
$Ax=\lambda x$, for some non-zero vector $x=\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}$.

$Ax-\lambda I x=0$ where $I$ is the identity matrix $I_{n\times n}$.

$(A-\lambda I)x=0$ ($n$ eqns in $n$ variables)

Then, this has non-trivial solution if

$$\det(A-\lambda I)=0.$$

[Note that $\det(A-\lambda I)$ is a polynomial in $\lambda$]

# $\lambda\underbrace{\begin{pmatrix}1&0\\0&1\end{pmatrix}}_{I_{2\times 2}}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}\lambda x_1\\ \lambda x_2\end{pmatrix}$

# $A-\lambda I$

$\begin{pmatrix}2&1\\3&4\end{pmatrix}-\lambda\begin{pmatrix}1&0\\0&1\end{pmatrix}$

$=\begin{pmatrix}2-\lambda&1\\3&4-\lambda\end{pmatrix}$ ↳ polynomial in $\lambda$

Say,

$$\det(A-\lambda I)=(-1)^n\left(\lambda^n-c_1\lambda^{n-1}+c_2\lambda^{n-2}+\dots+(-1)^n c_n\right)$$

where $c_1,c_2,\dots,c_n$ can be written in terms of the entries of $A$.

$(A-\lambda I)x=0$ has non-trivial solution

if $\lambda^n-c_1\lambda^{n-1}+c_2\lambda^{n-2}+\dots+(-1)^n c_n=0$

This equation is called the characteristic equation of $A$.

The polynomial $P_n(\lambda)=\lambda^n-c_1\lambda^{n-1}+\dots+(-1)^n c_n$ is called the characteristic polynomial.

Characteristic equation has $n$ roots,

say $\lambda_1,\lambda_2,\dots,\lambda_n$.

So, an $m\times n$ matrix has '$n$' eigenvalues.

Sum of the eigenvalues:

$$\sum_{i=1}^{n}\lambda_i=c_1=\text{Trace of } A \quad (a_{11}+a_{22}+\dots+a_{nn})$$

$$\begin{pmatrix}a_{11}&\dots\dots&a_{1n}\\a_{21}&a_{22}\dots&a_{2n}\\ \dots&\dots&\dots\\a_{n1}&a_{n2}\dots&a_{nn}\end{pmatrix}$$

Product of the eigenvalues:

$$\lambda_1\lambda_2\dots\lambda_n=\det(A).$$

$$\det(A-\lambda I) = (-1)^n P_n(\lambda)$$
$$= (-1)^n (\lambda-\lambda_1)(\lambda-\lambda_2)\dots(\lambda-\lambda_n)$$
$$P_n(0) = \det(A) = \lambda_1 \lambda_2 \dots \lambda_n$$

→ $A$ : $n\times n$ matrix.

$\lambda_1, \lambda_2, \dots, \lambda_n$ : eigenvalues or characteristic values.

$Ax = \lambda x$, $\quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ → non-zero vector

Q) Find the eigenvalues of $A = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$.

Also, find an eigenvector for each eigenvalue.

Soln: $|A-\lambda I| = 0$, i.e., $\begin{vmatrix} 2-\lambda & 1 \\ 4 & -1-\lambda \end{vmatrix} = 0$

$$\Rightarrow -2-2\lambda+\lambda+\lambda^2-4=0$$
$$\Rightarrow \lambda^2-\lambda-6=0$$
$$\Rightarrow (\lambda-3)(\lambda+2)=0$$
$$\Rightarrow \lambda = 3, -2.$$

$\underline{\lambda = 3}$

$(A-3I)x = 0$

$$\begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$A \sim \begin{bmatrix} 1 & -1 \\ 4 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

eq becomes

$$x_1 - x_2 = 0$$

Take $x_2 = 1$, then $x_1 = 1$.

Eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ corresponding to $\lambda = 3$.

[Any multiple is also an eigenvector.]

**Note:** For an eigenvalue $\lambda$, the associated eigenvector is not unique.
**Note:** The set of all eigen values of $A$ is called the spectrum of $A$.

The largest eigenvalue in absolute value is called the spectral radius of $A$, denoted by $\rho(A)$.

$$spec(A) = \{3, -2\}.$$

## Cayley Hamilton Theorem

Any square matrix will satisfy its characteristic equation.

Let $A$ be an $n \times n$ matrix.
Then, the characteristic eq$^n$ is

$$P_n(\lambda) = \lambda^n - c_1 \lambda^{n-1} + \ldots + (-1)^{n-1} c_{n-1} \lambda + (-1)^n c_n = 0.$$

The C-H theorem says

$$A^n - c_1 A^{n-1} + \ldots + (-1)^{n-1} c_{n-1} A + (-1)^n c_n = 0.$$

### Applications:

① Let the ch. eq$^n$ of $A$ be $\lambda^n - c_1 \lambda^{n-1} + \ldots + (-1)^n c_n I = 0$.

By C-H theorem,

$$A^n - c_1 A^{n-1} + c_2 A^{n-2} + \ldots + (-1)^{n-1} c_{n-1} A + (-1)^n c_n = 0$$

Suppose $A$ is invertible and multiply the above eq$^n$ with $A^{-1}$.

$$A^{-1}\left(A^n - c_1 A^{n-1} + \ldots + (-1)^{n-1} c_{n-1} A + I(-1)^n c_n\right) = 0$$

$$\Rightarrow A^{n-1} - c_1 A^{n-2} + \ldots + (-1)^{n-1} c_{n-1} I + (-1)^n c_n A^{-1} = 0$$

$$\therefore A^{-1} = \frac{(-1)^{n+1}}{c_n}\left[A^{n-1} - c_1 A^{n-2} + \ldots + (-1)^{n-1} c_{n-1} I\right].$$

② We have

$$A^n - c_1 A^{n-1} + \dots + (-1)^n c_n I = 0$$

$$\Rightarrow A^n = c_1 A^{n-1} - c_2 A^{n-2} + \dots + (-1)^n c_n \quad \dots ①$$

$$A^{n+1} = c_1 A^n - c_2 A^{n-1} + \dots + (-1)^n c_n A$$

By replacing $A^n$ using eqⁿ ①,

$$\therefore A^{n+1} = \alpha_1 + \alpha_2 A + \dots + \alpha_{n-1} A^{n-1}$$

Note that $A^l$ (where $l \geq n$) can be written in terms of powers of $A$ upto the order $(n-1)$.

Q ① Product of two eigenvalues of the matrix $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ is 16. Find the other eigenvalue.

② Verify the Cayley-Hamilton theorem and find $A^4$ and $A^{-1}$ for $A = \begin{bmatrix} 5 & 2 & 1 \\ 1 & 1 & 7 \\ 3 & 0 & 11 \end{bmatrix}$.

Soln: ② $|A - \lambda I| = \begin{vmatrix} 5-\lambda & 2 & 1 \\ 1 & 1-\lambda & 7 \\ 3 & 0 & 11-\lambda \end{vmatrix} = 0$

$$\Rightarrow (5-\lambda)(11 - \lambda - 11\lambda + \lambda^2) - 2(11 - \lambda - 21) + (-3 + 3\lambda) = 0$$

$$\Rightarrow 55 - 5\lambda - 55\lambda + 5\lambda^2 - 11\lambda + \lambda^2 + 11\lambda^2 - \lambda^3 - 22 + 2\lambda + 42 - 3 + 3\lambda = 0$$

$$\Rightarrow 72 - 66\lambda + 17\lambda^2 - \lambda^3 = 0$$

$$\Rightarrow \lambda^3 - 17\lambda^2 + 66\lambda - 72 = 0$$

and, $A^3 - 17A^2 + 66A - 72 = \begin{bmatrix} 252 & 72 & 444 \\ 393 & 57 & 987 \\ 618 & 102 & 1454 \end{bmatrix} + \begin{bmatrix} -510 & -204 & -510 \\ -459 & -51 & -1445 \\ -816 & -102 & -2108 \end{bmatrix} +$

$$\begin{bmatrix} 330 & 132 & 66 \\ 66 & 66 & 462 \\ 198 & 0 & 726 \end{bmatrix} + \begin{bmatrix} -72 & 0 & 0 \\ 0 & -72 & 0 \\ 0 & 0 & -72 \end{bmatrix} = 0$$

$\therefore$ Cayley-Hamilton theorem is satisfied.

① $|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$

$$\Rightarrow (6-\lambda)\left[(3-\lambda)^2-1\right] + 2(-6+2\lambda+2) + 2(2-6+2\lambda) = 0$$

$$\Rightarrow (6-\lambda)\left[8+\lambda^2-6\lambda\right] + 2(2\lambda-4) + 2(2\lambda-4) = 0$$

$$\Rightarrow 48 + 6\lambda^2 - 36\lambda - \cancel{8\lambda} - \lambda^3 + 6\lambda^2 + \cancel{4\lambda} - 8 + \cancel{4\lambda} - 8 = 0$$

$$\Rightarrow 32 + 12\lambda^2 - 36\lambda - \lambda^3 = 0$$

$$\Rightarrow \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

$$\cancel{\Rightarrow (\lambda-2)^2(\lambda-8) = 0}$$

$$\cancel{\Rightarrow \lambda = 2, 2, 8}$$

Product of eigenvalues $= \lambda_1\lambda_2\lambda_3 = 32$

$$\Rightarrow (16)\lambda_3 = 32$$

$$\Rightarrow \lambda_3 = 2$$

$$\therefore (\lambda-2)(\lambda^2-10\lambda+16) = 0$$

$$\Rightarrow (\lambda-2)(\lambda-2)(\lambda-8) = 0$$

$$\Rightarrow \lambda = 2, 2, 8$$

20-10-2023

# VECTOR SPACES

Consider the equation $x_1 + 2x_3 = 0$.

Let $S$ be the set of all solutions of this equation

$$S = \{(x_1, x_2) \mid x_1 + 2x_2 = 0\}$$

$S$ is non-empty.

$+ : S \times S \to S$

$((x_1, x_2), (y_1, y_2)) \mapsto (x_1 + y_1, x_2 + y_2)$

$S$ is closed under addition. Binary Operation $+$

(i) $(x_1, x_2) + (y_1, y_2) = (y_1, y_2) + (x_1, x_2)$

Commutative (abelion)

(ii) There exists a unique element (0, 0) in $S$, such that

$$(x_1, x_2) + (0, 0) = (x_1, x_2)$$

This unique element (0,0) is called the (additive) identity.

(iii) For $(x_1, x_2)$, there exists (additive) inverse $(-x_1, -x_2)$ such that

$$(x_1, x_2) + (-x_1, -x_2) = (0, 0).$$

(iv) $((x_1, x_2) + (y_1, y_2)) + (z_1, z_2) = (x_1, x_2) + ((y_1, y_2) + (z_1, z_2))$.

Associative

Define scalar multiplication.

- $\mathbb{R} \times S \to S$

  $\alpha \cdot (x_1, x_2) \mapsto (\alpha x_1, \alpha x_2)$

This operation satisfies $[\alpha \in \mathbb{R}, (x_1, x_2), (y_1, y_2) \in S]$

1. $\alpha \cdot ((x_1, x_2) + (y_1, y_2)) = \alpha \cdot (x_1, x_2) + \alpha \cdot (y_1, y_2)$

② $(\alpha+\beta)\cdot(x_1,x_2) = \alpha\cdot(x_1,x_2) + \beta\cdot(x_1,x_2)$

③ $(\alpha\beta)\cdot(x_1,x_2) = \alpha\cdot(\beta\cdot(x_1,x_2))$

④ $1\cdot(x_1,x_2) = (x_1,x_2)$

→ $(S,+,\cdot)$ satisfying the above 8 properties is called a vector space over $\mathbb{R}$.

Note: ① $(\mathbb{Z},+)$

② $(\mathbb{R},+)$

③ $(\mathbb{R}\setminus\{0\},\times)$ ⇝ multiplicative identity, multiplicative inverse

$\begin{bmatrix} \times : \mathbb{R}\times\mathbb{R}\to\mathbb{R} \\ \alpha \cdot \beta \mapsto \alpha\beta \end{bmatrix}$

④ $(\mathbb{Q},+)$

⑤ $(\mathbb{Q}\setminus\{0\},\times)$

$\times : \mathbb{Q}\times\mathbb{Q}\to\mathbb{Q}$

⑥ $(\mathbb{Q}^c,+) \to \times$ $[\mathbb{Q}^c$ : irrational$)$

⑦ $(M_{2\times3}(\mathbb{R}),+)$

→ A non-empty set $G$ with a binary operation

$* : G\times G \to G$

satisfying

(i) Commutitivity (ii) Existence of identity

(iii) Existence of Inverse (iv) Associativity,

is called a commutative group.

↳ Eg. Above egs ① to ⑦.

So,

A vector space over $\mathbb{R}$ is a non-empty set $V$ satisfying

(i) $(V,+)$, where $+ : V\times V \to V$, is a commutative group.

(ii) There is a scalar multiplication.

$\therefore \mathbb{R} \times V \to V$

$\alpha, v \mapsto \alpha \cdot v$

satisfying

ⓐ $\alpha \cdot (v_1 + v_2) = \alpha \cdot v_1 + \alpha \cdot v_2$

ⓑ $(\alpha + \beta) \cdot v = \alpha \cdot v + \beta \cdot v$

ⓒ $(\alpha \beta) \cdot v = \alpha \cdot (\beta \cdot v)$

ⓓ $1 \cdot v = v$

Note: $V$ is a real vector space.

Members of $V$ are called vectors.

Members of $\mathbb{R}$ are called scalars.

The identity in $(V, +)$ is called zero vector.

$S = \{3, 5, 7\}$

$+ : S \times S \to S$

Scalar set $\mathbb{R}$:

$(\mathbb{R}, +, \times)$ is a Field. $\xrightarrow{\text{conditions}}$
- $(\mathbb{R}, +) \to$ Commutative group
- $(\mathbb{R} \setminus \{0\}, \times) \to$ Commutative group.
- $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$
  ↳ Distributive law

$+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$\cdot : \mathbb{R} \setminus \{0\} \times \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$.

→ Complex set:
- $(\mathbb{C}, +)$
- $(\mathbb{C} \setminus \{(0,0)\}, \times)$
- Distributive law

$\Rightarrow (\mathbb{C}, +, \times)$ is a Field.

→
- $(\mathbb{Q}, +)$
- $(\mathbb{Q} \setminus \{0\}, \times)$
- Distributive law

$\Rightarrow (\mathbb{Q}, +, \times)$ is a Field.

30-10-2023

→ $(V, +, \cdot)$ is a vector space.

$(V,+)$ is a commutative group.

$+ : V \times V \to V$

$\therefore R \times V \to V$

$(\alpha, v) \mapsto \alpha v$

① $\alpha(v_1 + v_2) = \alpha v_1 + \alpha v_2$

② $(\alpha+\beta) v = \alpha v + \beta v$

③ $(\alpha\beta)(v) = \alpha.(\beta.v)$

④ $1.v = v$

↳ Eg: $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{Q}$ Fields.

Eg finite field

$\mathbb{Z}_3 = \{0,1,2\}$

$+_3$ : addition modulo 3 $\left[a +_3 b = \text{remainder of } \frac{a+b}{3}\right]$

| $+_3$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 1 | 2 |
| 1 | 1 | 2 | 0 |
| 2 | 2 | 0 | 1 |

$(\mathbb{Z}_3, +_3)$ is a commutative group.

$\times_3$ : multiplication modulo 3 $\left[a \times_3 b = \text{remainder of } \frac{ab}{3}\right]$

| $\times_3$ | 1 | 2 |
|---|---|---|
| 1 | 1 | 2 |
| 2 | 2 | 1 |

$(\mathbb{Z}_3 \setminus \{0\}, \times_3)$ : commutative group.

$2 \times_3 (2) = 1$

$1 \times_3 (1) = 1$

$a \times_3 (b +_3 c) = (a \times_3 b) +_3 (a \times_3 c)$

$\therefore (\mathbb{Z}_3, +_3, \times_3)$ is a field.

Eg. $(\mathbb{Z}_4, +_4, \times_4)$ is not a field.

| $+_4$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | | | | |
| 1 | | | | |
| 2 | | | | |
| 3 | | | | |

✓

| $\times_4$ | 1 | 2 | 3 |
|---|---|---|---|
| 1 | | | |
| 2 | | 0 | |
| 3 | | | |

×

(As $2 \times_4 2 = 0$ for 2 non-zero nos)

Eg. $(\mathbb{Z}_5, +_5, \times_5)$ is a field

$(\mathbb{Z}_6, +_6, \times_6)$ is not a field.

<u>Note:</u> $(\mathbb{Z}_p, +_p, \times_p)$ is a field where $p$ is a <u>prime</u>.

<u>vector space $(V, +, \cdot)$ over $\mathbb{R}$:</u>

Eg ① $V = \mathbb{R}$ is a vector space over $\mathbb{R}$.

$+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ : usual addition

$\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ : usual multiplication.

② $V = \mathbb{R}^2$ is a vector space over $\mathbb{R}$.

$+ : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $\quad [(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)]$

Zero vector is $(0,0)$.

$\cdot : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ $\quad [\alpha (x_1, x_2) \longmapsto (\alpha x_1, \alpha x_2)]$

[$\mathbb{R}^n$ is a vector space over $\mathbb{R}$]

③ $V = \mathbb{C}$ over $\mathbb{R}$.

$+ : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$

$\cdot : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$

$[\alpha \cdot (x + iy) \longmapsto \alpha x + i \alpha y]$

$\therefore$ $\mathbb{C}$ is a vector space over $\mathbb{R}$.

④ $V = M_{2\times 3}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \middle| \ a,b,c,d,e,f \in \mathbb{R} \right\}$

$+ : M_{2\times3}(\mathbb{R}) \times M_{2\times3}(\mathbb{R}) \longrightarrow M_{2\times3}(\mathbb{R})$

$\cdot : \mathbb{R} \times M_{2\times3}(\mathbb{R}) \longrightarrow M_{2\times3}(\mathbb{R})$

$$\left[ \alpha \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \longmapsto \begin{pmatrix} \alpha a & \alpha b & \alpha c \\ \alpha d & \alpha e & \alpha f \end{pmatrix} \right]$$

$M_{2\times3}(\mathbb{R})$ is a vector space over $\mathbb{R}$.

$M_{m\times n}(\mathbb{R})$ is a vector space over $\mathbb{R}$, $m,n$ are fixed positive integers.

\# $V = P_3 = \{$Real polynomials of degree $= 3\}$
↳ is not a v.s. over $\mathbb{R}$
(as addition of (degree 3) polynomials may give polynomials of degree $< 3$)

⑤ $V = P_3 = \{$real polynomials of degree $\le 3\}$

$(P_3, +)$ : commutative group

$\cdot : \mathbb{R} \times P_3 \longrightarrow P_3$

$$\left[ \alpha \cdot (a_0 + a_1 x + a_2 x^2 + a_3 x^3) \longmapsto \alpha a_0 + \alpha a_1 x + \alpha a_2 x^2 + \alpha a_3 x^3 \right]$$

$\therefore$ $P_3$ is a vector ~~the~~ space over $\mathbb{R}$.

$P_n = \{$real polynomials of degree $\le n\}$ is a vector space over $\mathbb{R}$.

⑥ $V = \mathbb{Q}$ over $\mathbb{R}$ is not a vector space.
↳ ~~Scalarly~~ scaling will fail.

$\exists$ $V = \mathbb{Q}$ over $\mathbb{Q}$ is a vector space.

⑦ $V = \mathbb{R}$ over $\mathbb{Q}$ is a vector space.

$+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$\cdot : \mathbb{Q} \times \mathbb{R} \to \mathbb{R}$

# Subspace

Let $V$ be a vector space over $\mathbb{R}$.

A non-empty subset $W$ of $V$ is said to be a subspace of $V$ if $W$ itself is a vector space under the same vector addition and scaling defined for $V$.

<u>Note</u>: $(V, +, \cdot)$ is a vector space over $\mathbb{R}$.

$W \subset V$ : non-empty subset.

$+ : W \times W \to W$

$\cdot : \mathbb{R} \times W \to W$

$w_1, w_2 \in W$, then $w_1 + w_2 \in W$

$\alpha \in \mathbb{R}$, $w \in W$, then $\alpha w \in W$

Only to check whether $W$ is closed under vector addition and scaling.

<u>Eg</u> ① $V = \mathbb{R}$ is vector space over $\mathbb{R}$.

$W = \{0\}$ is a subspace (Trivial subspace).

$W = \mathbb{R}$ is also a subspace.

(For any vector space, these $W = \{0\}$ and $W = V$ are subspaces.)

② $V = \mathbb{R}^2$ is vector space over $\mathbb{R}$.

$W_1 = \{(0,0)\} \to$ subspace of $\mathbb{R}^2$.

$W_2 = \{$Any line passing through origin is a subspace of $\mathbb{R}^2\}$

$W = \mathbb{R}^2 \to$ subspace of $\mathbb{R}^2$.

③ $V = \mathbb{R}^3$ over $\mathbb{R}$.

$W_1 = \{(0,0,0)\}$

$W_2 = \{$Any line passing through origin$\}$

$W_3 = \{$Any plane passing through origin$\}$ is a subspace.

$W = \mathbb{R}^3$

03-11-2023

**Note:**

Let $V$ be a vector space over $\mathbb{R}$.

Take $v \in V$ and consider

$$W = \{\alpha . v \mid \alpha \in \mathbb{R}\}$$

$w_1 = \alpha_1 v, \ w_2 = \alpha_2 v$

$w_1 + w_2 = \alpha_1 v + \alpha_2 v = (\alpha_1 + \alpha_2) v = \beta v; \ \beta = \alpha_1 + \alpha_2$

$\Rightarrow w_1 + w_2 \in W$

$\beta . w_1 = \beta . (\alpha_1 v)$

$= \beta \alpha_1 v$

$= \beta_1 v \ ; \ \beta_1 = \beta \alpha_1 \in \mathbb{R}$

$\in W$

$\therefore W$ is a subspace of $V$

If $v = 0$ zero vector in $V$,

then $W = \{0\}$.

(Trivial subspace).

Take $v_1, v_2 \in V$, consider

$W = \{\alpha_1 v_1 + \alpha_2 v_2 \mid \alpha_1, \alpha_2 \in \mathbb{R}\} \subset V$

$v_1 \in W, \ v_2 \in W$

$w_1 = \alpha_1 v_1 + \alpha_2 v_2$

$w_2 = \beta_1 v_1 + \beta_2 v_2$

$w_1 + w_2 = (\alpha_1 + \beta_1) v_1 + (\alpha_2 + \beta_2) v_2$

$\Rightarrow w_1 + w_2 \in W$

$\alpha . w_1 = \alpha (\alpha_1 v_1 + \alpha_2 v_2) = \alpha \alpha_1 v_1 + \alpha \alpha_2 v_2 \in W$

$\Rightarrow W$ is a subspace of $V$.

If both $v_1, v_2$ are zero vectors,

then subspace $W = \{0\}$. (Trivial subspace)

If $v_2$ is a multiple of $v_1$, then
$$W = \{\alpha \cdot v_1 \mid \alpha \in \mathbb{R}\}.$$

If $v_1, v_2$ are multiples of each other, then $W$ is generated by one vector.

Take $v_1, v_2, v_3 \in (V, +, \cdot)$.

Consider $W = \{\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}\}$

non-empty $\subset V$ $\quad [\because \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \in V]$

$W$ is closed under vector addition and scaling.

$\therefore$ $W$ is a subspace of $V$.

(i) If $v_1 = v_2 = v_3 =$ zero vector,
then $W = \{0\}$.

(ii) If $v_3$ is the sum of scaling of $v_1$ and $v_2$, and $v_1$ and $v_2$ are not multiples of each other,
$$[v_3 = \beta_1 v_1 + \beta_2 v_2]$$
then $W = \{\alpha_1 v_1 + \alpha_2 v_2 \mid \alpha_1, \alpha_2 \in \mathbb{R}\}$.

(iii) If $v_2$ is a multiple of $v_1$
and $v_3$ is a multiple of $v_1$,
then $W = \{\alpha \cdot v_1 \mid \alpha \in \mathbb{R}\}$.

In general, for $v_1, v_2, \dots, v_n \in V$,
(finite number of vectors)
$$W = \{\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n \mid (\alpha_1, \alpha_2, \dots, \alpha_n) \in V\}$$
is a subspace of $V$.

**Defn:** Let $V$ be a vector space over $\mathbb{R}$ and $\{v_1, v_2, \dots, v_n\}$ be a finite set of vectors in $V$.

A linear combination of $\{v_1, \dots, v_n\}$ in an expression of the form $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$ for some

$$\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}$$

Note that $\alpha_1 v_1 + \dots + \alpha_n v_n \in V$.

Eg. ① $V = \mathbb{R}$ v.s. over $\mathbb{R}$.

ⓐ $\{v_1 = 0.75\}$

linear combination

$$4 \times v_1 = 4 \times 0.75 = 3.$$

ⓑ $\{v_1 = 2, \; v_2 = 0.5\}$

$\alpha_1 2 + \alpha_2 . (0.5)$

② $V = P_3 =$ real polynomial of degree $\leq 3$.

$1, 1+x, 2+x^2, x+x^3.$

$$\alpha_1 . 1 + \alpha_2 (1+x) + \alpha_3 (2+x^2) + \alpha_4 (x+x^3).$$

$$= (\alpha_1 + \alpha_2 + 2\alpha_3) + (\alpha_2 + \alpha_4) x + \alpha_3 x^2 + \alpha_4 x^3.$$

**Defn:** A finite set of vectors $\{v_1, v_2, \dots, v_n\}$ in a vector space $V$ over $\mathbb{R}$ is said to be linearly dependent

if $\sum_{i=1}^{n} \alpha_i v_i = 0$, then $\exists$ at least one $\alpha_i$ is non-zero,

i.e., linearly dependent means

$\sum_{i=1}^{n} \alpha_i v_i = 0 \Rightarrow$ all the scalars $\alpha_i$ cannot be zero.

$\{v_1, v_2, \dots, v_n\}$ is said to be linearly independent

if $\sum_{i=1}^{n} \alpha_i v_i = 0 \Rightarrow$ all $\alpha_i$'s are zero.

**Note:** ① If $\{v_1, \ldots, v_n\}$ are linearly dependent, then one of the vectors can be written as a linear combination of other vectors.

$$v_3 = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_4 v_4 + \cdots + \alpha_n v_n$$

$$\Rightarrow \alpha_1 v_1 + \cdots + \alpha_n v_n - 1 \cdot v_3 = 0$$

atleast one $\alpha$ is non-zero.

② If one of the vector in $\{v_1, \ldots, v_n\}$ is a zero vector, then $\{v_1, \ldots, v_n\}$ is linearly dependent.

Eg., (i) In $V = \mathbb{R}$,
linearly in dependent set of vectors are
$\{\alpha\}$, where $0 \neq \alpha \in \mathbb{R}$.

(ii) In $V = \mathbb{R}^2$ over $\mathbb{R}$,
$(e_1, e_2) \rightarrow$ linearly independent.

Any set of two non-zero vectors which are not collinear form a linearly independent set in $\mathbb{R}^2$.

(iii) $V = P_4$ = real poly. of degree $\leq 4$.

$\{1, x, x^2, x^3, x^4\}$ is linearly independent.

$$\alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 + \alpha_5 x^4 = 0$$
$$= 0 + 0x + 0 \cdot x^2 + 0 \cdot x^3 + 0 \cdot x^4$$

$$\Rightarrow \alpha_1 = 0 = \alpha_2 = \alpha_3 = \alpha_4 = 0.$$

e.g. $\{3,\ 3+x,\ 3+x+x^2,\ 3+x+x^3\}$.

Check whether this set is LI in $P_4$.

Consider

$$\alpha_1 3 + \alpha_2 (3+x) + \alpha_3 (3+x+x^2) + \alpha_4 (3+x+x^3) = 0$$
$$= 0 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 + 0 \cdot x^4$$

$$\Rightarrow 3(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) + (\alpha_2 + \alpha_3 + \alpha_4)x + \alpha_3 x^2 + \alpha_4 x^3$$
$$= 0 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 + 0 \cdot x^4$$

$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$.

$\therefore \{3, 3+x, \dots\}$ is LI.

Q) check whether

$\{(1,2,1), (1,0,1), (1,1,1)\}$ is LI or not in $\mathbb{R}^3$.

04-11-2023

Soln: $\{(1,2,1), (1,0,1), (1,1,1)\}$ in $\mathbb{R}^3$.

$\alpha_1(1,2,1) + \alpha_2(1,0,1) + \alpha_3(1,1,1) = (0,0,0)$.

$(\alpha_1+\alpha_2+\alpha_3), (2\alpha_1+\alpha_3), (\alpha_1+\alpha_2+\alpha_3) = (0,0,0)$.

we can find $\alpha_1, \alpha_2, \alpha_3$ such that all $\alpha$'s are not zero,

and $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$

$\Rightarrow \{v_1, v_2, v_3\} \to$ linearily dependent.

$v_1, v_2, v_3 \in V$.

$\alpha_1 + \alpha_2 + \alpha_3 = 0$

$2\alpha_1 + \alpha_3 = 0$

$\alpha_1 + \alpha_2 + \alpha_3 = 0$

$W = \{\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}\}$.

Defn: Span:

Let $S = \{v_1, v_2, \dots, v_n\}$ be a finite set of vectors is a vector space $V$ over $\mathbb{R}$.

Define span of $S$ as

$$\text{span}(S) = \{a_1 v_1 + a_2 v_2 + \dots + a_n v_n \mid a_1, a_2, \dots, a_n \in \mathbb{R}\}.$$

(Set of all linear combinations.)

~~14-11-2023~~

~~Eg. {(1,2,1), (1,0,1), (1,1,1)}, ℝ³~~

**Defn:** Basis : (Independent vector space)

Let $V$ be a vector space over $\mathbb{R}$.

A basis for $V$ is a set of $B$ vectors in $V$ satisfying:

(i) vectors in $B$ are linearly independent.

(ii) $\text{Span}(B) = V$.

condition (ii) means any vector $v \in V$ can be written as linear combination of vectors from basis $B$.

condition (i) means any vector $v \in V$ can be uniquely written as linear combination of vectors in $B$.

$$v = \alpha_1 v_1 + \dots + \alpha_n v_n$$

$$= \beta_1 v + \dots + \beta_n v_n$$

$$(\alpha_1 - \beta_1) v_n + \dots + (\alpha_n - \beta_n) v_n$$

$$\Rightarrow \alpha_1 - \beta_1 = 0, \ \alpha_2 - \beta_2 = 0, \ \dots, \ \alpha_n - \beta_n = 0.$$

## Finite Dimensional

A vector space $V$ over $\mathbb{R}$ is said to be finite dimensional if its basis has only finite number of vectors.

→ For a finite dimensional vector space, the no. of vectors in different bases will be the same.

→ This unique number is called the dimension of the vector space, $\dim_{\mathbb{R}} V$.

Eg ① $V = \mathbb{R}$ over $\mathbb{R}$.

Any singleton set $\{\lambda\}$ where $0 \neq \lambda \in \mathbb{R}$ is a basis.

$$\text{span}(\{0.05\}) = \{\alpha.(0.05) \mid \alpha \in \mathbb{R}\}$$
$$= \mathbb{R}$$

$$\therefore \boxed{\dim_{\mathbb{R}} \mathbb{R} = 1}.$$

② $V = \mathbb{R}^2$ v.s. over $\mathbb{R}$.

$$(v_1, v_2) = v_1(1,0) + v_2(0,1)$$

$B_1 = \{(1,0), (0,1)\}$.

span $B_1 = \mathbb{R}^2 \rightarrow$ linear independent also

$B_1$ is a Basis for $\mathbb{R}^2$ over $\mathbb{R}$.

$B_2 = \{(1,2), (3,4)\}$.

Any two non-zero vectors which span $\mathbb{R}^2$ is a basis for $\mathbb{R}^2$ over $\mathbb{R}$.

$$\therefore \boxed{\dim_{\mathbb{R}} \mathbb{R}^2 = 2}$$

③ $V = P_3$ over $\mathbb{R}$

$B_1 = \{1, 1+x, 1+x+x^2, 1+x+x^3)$

$$\therefore \boxed{\dim_{\mathbb{R}} P_3 = 4}.$$

④ $\dim_{\mathbb{R}} \mathbb{C} = 2$

$\dim_{\mathbb{C}} \mathbb{C} = 1$

Note: Let F be a field.

Then, $V = F$ is a vector space over F.

Eg ① $V = \mathbb{R}$ v.s. over $\mathbb{Q}$.

All irrational numbers should be in basis to have $\mathbb{R}$ over $\mathbb{Q}$ and one rational.

INFINITE DIMENSIONAL

[For generating irrational no., we cannot generate it by taking element from $\mathbb{Q}$ & scaling $\mathbb{R}$ (does not contain irrational)]

② Let $X$ be any non-empty set.

$$V = \{ f : X \to \mathbb{R} \mid f \text{ is a function} \}$$

$$f(x) = 2 \quad \forall\, x \in X$$
$$0(x) = 0 \quad \forall\, x \in X$$

$V$ is non-empty.

($\because$ constant functions are there)

→ $+ : V \times V \to V$

$(f, g) \longrightarrow f+g$

$f+g$ is defined by $f+g : V \to \mathbb{R}$

$$\underbrace{(f+g)(x)}_{\text{vector addition}} = \underbrace{f(x) + g(x)}_{\text{Addition in reals}} \quad \forall\, x \in X.$$

$\therefore\ \mathbb{R} \times V \to V$

$(\alpha, f) \longmapsto \alpha f$

where $\alpha f : X \to \mathbb{R}$, $\alpha \in \mathbb{R}$.

$$(\alpha f)(x) = \underbrace{\alpha . f(x)}_{\text{multiplication of reals}}$$

- Additive identity is zero function.

  $0(x) = 0 \quad \forall\, x \in \mathbb{R}$.

- Inverse is $-f$.

- $(\alpha + \beta) f = \alpha f + \beta f$

  $\alpha . (f+g) = \alpha f + \alpha g$

  $(\alpha\beta) f = \alpha . (\beta f)$

  $1 . f = f$

→ $V$ is a v.s. over $\mathbb{R}$.

Infinite dimensional :

$$\dim_{\mathbb{R}} V = \infty.$$

All polynomial, sin, cos and other functions would be there in the basis.

→ $W = \{f : X \to \mathbb{R} \mid f \text{ is continuous}\}$ is a subspace of $V$.

$$X = (0,1)$$

$$V = \{f : (0,1) \to \mathbb{R} \mid f \text{ funct.}\}$$

$$C((0,1)) = \{f : (0,1) \to \mathbb{R} \mid f \text{ is continuous}\}$$

$\dim_{\mathbb{R}} C((0,1))$ is infinite.

→ $V$ = v.s. over $\mathbb{R}$

$B = \{v_1, \ldots, v_n\}$ basis for $V$.

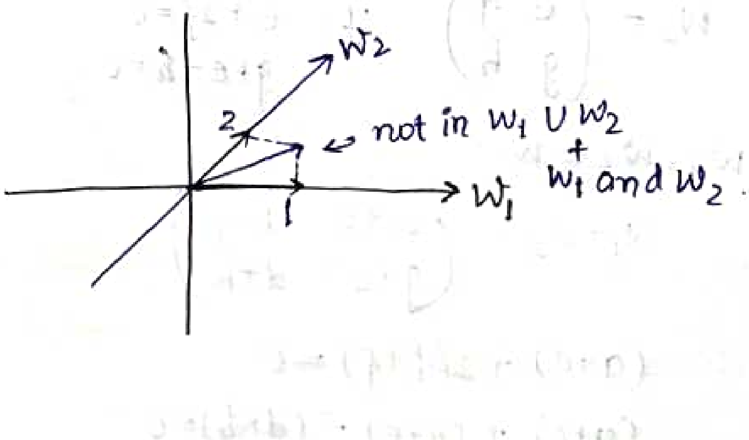
$$\dim_{\mathbb{R}} V = n.$$

<u>Note</u>: If a set of $n$ vectors span the v.s. $V$ of dimension $\dim_{\mathbb{R}} V = n$.

<u>Note</u>: Let $W_1, W_2$ be subspaces of $V$.

(i) $W_1 \cup W_2$ → is not a subspace.

(ii) $W_1 \cap W_2$ → is a subspace.



(iii) $W_1 + W_2 = \{ \omega_1 + \omega_2 \mid \omega_1 \in W_1, \omega_2 \in W_2 \}$.

$v_1 = \omega_1 + \omega_2 \in W_1 + W_2$

$v_2 = \mu_1 + \mu_2 \in W_1 + W_2$

$$v_1 + v_2 = (\omega_1 + \omega_2) + (\mu_1 + \mu_2)$$

$$= \underbrace{(\omega_1 + \mu_1)}_{W_1} + \underbrace{(\omega_2 + \mu_2)}_{W_2} \in W_1 + W_2$$

$\alpha v_1 = \alpha \omega_1 + \alpha \omega_2 \in W_1 + W_2$

$\therefore$ $W_1 + W_2$ is a subspace of $V$.

Eg. $V = \mathbb{R}^2$ over $\mathbb{R}$

$W_1 = \text{span}\{(0,0,1)\} \rightarrow$ Z-axis

$W_2 = \text{span}\{(2,0,0)\} \rightarrow$ x-axis.

$W_1 + W_2 = XZ$ plane is a subspace of $\mathbb{R}^3$.

Q) Check whether $W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{array}{l} a+2b=0 \\ c+a-d=0 \\ a,b,c,d \in \mathbb{R} \end{array} \right\}$ is a subspace of $M_{2\times2}(\mathbb{R})$ or not. If so, find a basis for $W$.

Soln: $W$ is non-empty such that

$W_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ni \begin{array}{l} a+2b=0 \\ c+a-d=0 \end{array}$

$W_2 = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \ni \begin{array}{l} e+2f=0 \\ g+e-h=0. \end{array}$

$w_1, w_2 \in W$.

$$\omega_1 + \omega_2 = \begin{pmatrix} a+e & b+f \\ g+c & d+h \end{pmatrix}$$

$(a+e) + 2(b+f) = 0$

$(g+c) + (a+e) - (d+h) = 0$

$\Rightarrow w_1 + w_2 \in W$.

$$\alpha \cdot w_1 = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}$$

$\alpha a + 2(\alpha b) = 0$

$\alpha c + \alpha a - \alpha d = 0$

$\alpha w_1 \in W$.

$\Rightarrow W$ is a subspace of $M_{2\times 2}(\mathbb{R})$.

$\therefore \dim_{\mathbb{R}} M_{2\times 2}(\mathbb{R}) = 4$; $B = \left( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$

for $\dim_{\mathbb{R}} M_{2\times 2}(\mathbb{R}) = 4$.

$a + 2b = 0 \Rightarrow b = -\frac{1}{2}a$

$c + a - d = 0 \Rightarrow d = c + a$

$$W = \left\{ \begin{pmatrix} a & -\frac{1}{2}a \\ c & c+a \end{pmatrix} \middle| \, a, c \in \mathbb{R} \right\}$$

$a = 1,\ c = 0 \mid a = 0,\ c = 1$

$$B = \left\{ \begin{pmatrix} 1 & -1/2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\}$$

$\searrow$ linearly independent.

and Span $(B) = W$.

$$a \begin{pmatrix} 1 & -1/2 \\ 0 & 1 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

$\therefore \dim_{\mathbb{R}} W = 2$

Q) $V = P_3$ over $\mathbb{R}$.

$W = \{ p(x) \in P_3 \mid p(1) = 0,\ p(2) = 0 \}$.

Is $W$ a subspace of $P_3$? If so, what is ~~the~~ its dimensions? ~~given basis.~~

Soln: $p_1(x) \in W$, $p_2(x) \in W$

$\Rightarrow p_1(x) + p_2(x) \in W$

$p_1(1) + p_2(1) = 0$

$p_2(2) + p_2(2) = 0$

$p(x) = a + bx + cx^2 + dx^3$.

$p(1) = 0 \Rightarrow a + b + c + d = 0$

$p(2) = 0 \Rightarrow a + 2b + 4c + 8d = 0$

Let $\left.\begin{array}{l} c = \alpha \\ d = \beta \end{array}\right\} \rightarrow$ find $a, b$.

So, dimension is 2.

$a + b = -\alpha - \beta$

$a + 2b = -4\alpha - 8\beta$

$b = -3\alpha - 7\beta$

$a = 2\alpha + 6\beta$.

$W = \{(2\alpha + 6\beta) + (-3\alpha - 7\beta)x + \alpha x^2 + \beta x^3 \mid \alpha, \beta \in \mathbb{R}\}$.

$\underline{\beta = 0, \ \alpha = 1}$

$2 - 3x + x^2 = 0$

$\underline{\alpha = 0, \ \beta = 1}$

$6 - 7x + x^3 = 0$.

# Inner Product Space

Defn: Let $V$ be a vector space over $\mathbb{R}$.

An inner product $\langle , \rangle$ is a map,

$$\langle , \rangle : V \times V \to \mathbb{R} \text{ which}$$

associates scalar with each pair of vectors and satisfies:

(i) $\langle v, v \rangle \geq 0$ and is zero iff $v = 0$.

(ii) $\langle v_1, v_2 \rangle = \langle v_2, v_1 \rangle$

(iii) $\langle v_1 + v_2, v_3 \rangle = \langle v_1, v_3 \rangle + \langle v_2, v_3 \rangle$

(iv) $\langle \alpha v_1, v_2 \rangle = \alpha \langle v_1, v_2 \rangle$,

$\alpha \in \mathbb{R}$

$v_1, v_2, v_3 \in V$

A vector space with an inner product is called an inner product space.

Eg. ① $V = \mathbb{R}^2$ over $\mathbb{R}$

$x = (x_1, x_2)$, $y = (y_1, y_2)$ $\in \mathbb{R}^2$

$\langle x, y \rangle = x_1 y_1 + x_2 y_2$ $(= x \cdot y)$

② $V = \mathbb{R}^n$ over $\mathbb{R}$

$x = (x_1, x_2, \ldots, x_n)$

$y = (y_1, y_2, \ldots, y_n)$

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n = \sum_{i=1}^{n} x_i y_i$$

- $(\mathbb{R}^n, \langle , \rangle)$ is an inner product space.

③ $V = C((0,1))$ vector space over $\mathbb{R}$.

$$\langle f, g \rangle = \int_0^1 f(x) g(x)\, dx$$

Defn: Let $(V, \langle , \rangle)$ be an inner product space vectors. $v_1, v_2 \in V$ are said to be orthogonal if $\langle v_1, v_2 \rangle = 0$.

A set of vectors $\{v_1, v_2, v_3, \dots, v_n\}$ is said to be an orthogonal set if $\langle v_i, v_j \rangle = 0$ for all $v_i, v_j \in V$ with $i \neq j$.

Eg. In $V = \mathbb{R}^2$,

$\{(1,0), (0,1)\}$
$\{(1,1), (-1,1)\}$ $\Big\}$ orthogonal set

In $V = \mathbb{R}^n$,

$\{e_1, e_2, e_3, \dots, e_n\} \leftarrow$ orthogonal.

Defn: Let $(V, \langle , \rangle)$ be an inner product space and $v \in V$.

Define length of $v$

$$\|v\| = \sqrt{\langle v, v \rangle}$$

Eg. In $V = \mathbb{R}^2$,

$x = (x_1, x_2)$

$$\|x\| = \sqrt{\langle x_1, x_2 \rangle} = \sqrt{\langle (x_1, x_2), (x_1, x_2) \rangle}$$
$$= \sqrt{x_1^2 + x_2^2}$$

In $V = \mathbb{R}^n$,

$x = (x_1, x_2, \dots, x_n)$

$$\|x\| = \sqrt{x_1^2 + \dots + x_n^2}$$

Defn: A set of vectors $\{v_1, v_2, \dots, v_n\}$ is $(V, <,>)$ is said to be an orthonormal set if

(i) the set is orthogonal set (any two vectors are $\perp$).

(ii) each vector is of unit length,

i.e., $\|v_i\| = 1$ for $i = 1, 2, \dots, n$.

Eg. (i) $\{(1,0), (0,1)\}$ in $\mathbb{R}^2$

(ii) $\left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)\right\}$.

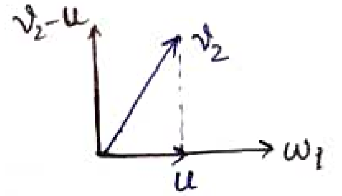
# Gram Schmidt Process

Given an ordered basis of an inner product space we can find an orthonormal basis from that.

Proof:

Let $\{v_1, v_2, \dots, v_n\}$ be the given ordered basis.

Take $w_1 = \dfrac{v_1}{\|v_1\|}$




$u$ is the ~~comonent~~ component of $v_2$ along $w_1$.

$\therefore \; u = \alpha w_1, \; \alpha \in \mathbb{R}$

$v_2 - u$ is $\perp^r$ to $w_1$.

$\Rightarrow \; \langle v_2 - \alpha w_1, w_1 \rangle = 0$

$\Rightarrow \; \langle v_2, w_1 \rangle - \alpha \langle w_1, w_1 \rangle = 0$

$\Rightarrow \; \alpha = \langle v_2, w_1 \rangle$

Take $w_2 = \dfrac{v_2 - \langle v_2, w_1 \rangle w_1}{\| v_2 - \langle v_2, w_1 \rangle w_1 \|}$

$\|w_2\| = 1$ and $w_1 \perp^r w_2$.

$\text{span}(w_1) = \text{span}(v_1)$

$\text{span}(\{w_1, w_2\}) = \text{span}(\{v_1, v_2\})$

Take $w_3 = \dfrac{v_3 - \langle v_3, v_1\rangle w_1 - \langle v_3, w_2\rangle w_2}{\| v_3 - \langle v_3, w_1\rangle w_1 - \langle v_3, w_2\rangle w_2\|}$

Then, $\{w_1, w_2, w_3\}$ is an orthonormal set and

$$\text{span}(\{w_1, w_2, w_3\}) = \text{span}(\{v_1, v_2, v_3\}).$$

Continuing like this, we get an orthonormal set $\{w_1, w_2, \dots, w_n\}$ such that (over span $(v_1, v_2, \dots, v_n)$)

$$\text{span}(\{w_1, \dots, w_n\}) = \text{span}(\{v_1, v_2, \dots, v_n\}).$$

→ Note that $\{w_1, \dots, w_n\}$ is orthonormal basis of $V$.

Q Find an orthonormal basis from the ordered basis $\{\underset{v_1}{(1,0,3)}, \underset{v_2}{(2,2,0)}, \underset{v_3}{(3,1,2)}\}$ of $\mathbb{R}^3$.

06-11-2023

# LINEAR MAPS

## (Linear Transformations)

Let $V, W$ be vector spaces over $\mathbb{R}$. A map $T: V \to W$ is said to be an linear map if

(i) $T(v_1 + v_2) = T(v_1) + T(v_2)$ → vector addition in $W$

(ii) $T(\alpha . v) = \alpha . T(v)$, $\alpha \in \mathbb{R}$

$v_1, v_2, v_3 \in V$.

vector addition in $V$

**Note:**

① $T$ (zero vector in $V$) = Zero vector in $W$

zero vector in $V$ ↘ $0_V$; zero vector in $W$ ↘ $0_W$

$$T(0_V) = T(0 . v) = 0 . T(v) = 0_W.$$

② $T$ is completely determined by its action on a basis of $V$. For let $\{v_1, \ldots, v_n\}$ be a basis for $V$.

$T(v_1), T(v_2), \ldots, T(v_n)$ are known.

$\Rightarrow T$ is completely known.

For that we take $v \in V$.

Then, $v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$ for $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$.

Now,

$$T(v) = T(\alpha_1 v_1 + \ldots + \alpha_n v_n)$$
$$= \alpha_1 T(v_1) + \alpha_2 T(v_2) + \ldots + \alpha_n T(v_n).$$

This output is known since $T(v_1) + \ldots + T(v_n)$ and $\alpha_1, \ldots, \alpha_n$ are known.

e.g. ① $V = \mathbb{R}$, $W = \mathbb{R}$ v.s. over $\mathbb{R}$.

$T : \mathbb{R} \to \mathbb{R}$

ⓐ $T(x) = 2.5x$ is linear

ⓑ $T(x) = 2 \ \forall \ x \in \mathbb{R}$

$$\left.\begin{aligned} &T(x+y) = 2 \\ &T(x) = 2 \\ &T(y) = 2 \\ &T(x+y) \neq T(x) + T(y) \end{aligned}\right\} \times \to \text{not linear}$$

ⓒ $T(x) = \alpha x$, $\alpha \in \mathbb{R}$ linear

ⓓ $T : \mathbb{R}^2 \to \mathbb{R}$ linear

$T((x_1, x_2)) = x_1 + x_2$.

ⓔ $T : \mathbb{R}^2 \to \mathbb{R}^2$ linear

$T((x_1, x_2)) = (x_1 + x_2, \ x_1 - x_2)$

ⓕ $T : \mathbb{R}^2 \to \mathbb{R}$ not linear

$T((x_1, x_2)) = x_1 x_2$

ⓖ $T : \mathbb{R}^3 \to \mathbb{R}^4$

$T((x_1, x_2, x_3)) = (x_1, x_1 + x_2, x_1 + x_2 + x_3, x_3)$ is linear.

ⓗ $T : P_2 \to P_3$ linear map

$T(a_0 + a_1 x + a_2 x^2) = a_2 x + a_0 x^3$

$T((a_0 + a_1 x + a_2 x^2) + (b_0 + b_1 x + b_2 x^2))$

$= T((a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2)$

$= (a_2 + b_2)x + (a_0 + b_0)x^3$

$= T(a_0 + a_1 x + a_2 x^2) + T(b_0 + b_1 x + b_2 x^2)$

$a_2 x + a_0 x^3 + b_2 x + b_0 x^3$

eg. $V = C((0,1)) = \{f : (0,1) \to \mathbb{R} \mid f \text{ is continuous}\}$

$T : V \to V$

Is it well defined?

$T(f(x)) = \frac{d}{dx}((f(x)))$

If so, is it linear?

eg. $\det : M_{2\times2}(\mathbb{R}) \to \mathbb{R}$. Is it well defined? If so, is it linear?

**Defn:**

Let $T : V \to W$ be a linear map.

Define Kernal of T as

$$\ker(T) = \{v \in V \mid T(v) = \underbrace{0}_{\text{Zero vector in } W}\}$$

$\ker(T)$ is a non-empty subset $(\because 0_V \in \ker(T))$.

Take $v_1, v_2 \in \ker(T)$,

i.e., $v_1, v_2 \in V \ni T(v_1) = 0,$
$T(v_2) = 0.$

$v_1 + v_2 \in V$

$T(v_1 + v_2) = T(v_1) + T(v_2) = 0.$

$\Rightarrow v_1 + v_2 \in \ker(T).$

Take $v_1 \in V \ni T(v_1) = 0$ $(v_1 \in \ker(T))$,

then $\alpha v_1 \in V$ and $T(\alpha v_1) = \alpha T(v_1)$
$= \alpha . 0$
$= 0.$

$\Rightarrow \alpha v_1 \in \ker(T)$, $\alpha \in \mathbb{R}$.

$\therefore \ker(T)$ is a subspace of $V$.

→ Define Image of T as

$$\text{Image}(T) = \{ w \in W \mid w = T(v), \text{ for some } v \in V \}.$$

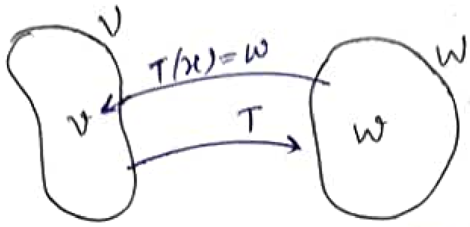



Image (T) is a non-empty subset of W. ($0_W \in \text{Image}(T)$).

Take $w_1, w_2 \in \text{Img}(T)$.

Then, there are $v_1, v_2 \in V$ such that

$$T(v_1) = w_1$$
$$T(v_2) = w_2.$$

$$w_1 + w_2 \in W.$$

$$T(v_1 + v_2) = T(v_1) + T(v_2)$$
$$= w_1 + w_2.$$
$$\Rightarrow w_1 + w_2 \in \text{Img}(T).$$

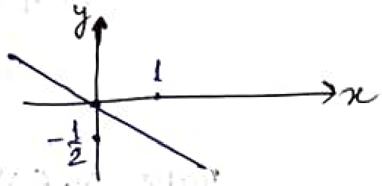
$$\alpha . w_1 \in \text{Img}(T)$$

Thus, Image (T) is a subspace of W.

eg. ① Find the ker (T) and Img (T) and also the dimension of them for $T: \mathbb{R}^2 \to \mathbb{R}$, $T(x_1, x_2) = x_1 + 2x_2$.

Soln: $\text{Ker}(T) = \{ (x_1, x_2) \in \mathbb{R}^2 \mid T((x_1, x_2)) = 0 \}.$

$$= \{ (x_1, x_2) \in \mathbb{R}^2 \mid x_1 + 2x_2 = 0 \}$$
$$= \{ (x_1, -\tfrac{1}{2}x_1) \mid x_1 \in \mathbb{R} \}$$




Basis of $\{ (1, -\frac{1}{2}) \}$

$\therefore \dim_{\mathbb{R}} (\ker (T)) = 1.$

$$\text{Image}(T) = \{ w \in \mathbb{R} \mid w = T((v_1, v_2)), (v_1, v_2) \in \mathbb{R}^2 \}.$$
$$= \{ w \in \mathbb{R} \mid w = v_1 + 2v_2, \ v_1, v_2 \in \mathbb{R} \}.$$
$$= \mathbb{R}$$

$\therefore \dim_{\mathbb{R}} (\text{Image}(T)) = 1.$

Q) Find the Ker(T) and Image(T) and dimension of them for

$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$T(x_1, x_2) = (x_1 - x_2,\ x_1 - x_2).$$

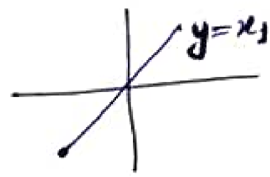
Sol$^n$: $\mathrm{Ker}(T) = \{(x_1, x_2) \in \mathbb{R}^2 \mid T((x_1, x_2)) = (0,0)\}$

$= \{(x_1, x_2) \in \mathbb{R}^2 \mid (x_1 - x_2,\ x_1 - x_2) = (0,0)\}$

i.e., $x_1 - x_2 = 0$

i.e., $x_1 = x_2$.

$= \{(x_1, x_1) \in \mathbb{R}^2 \mid x_1 \in \mathbb{R}\}$



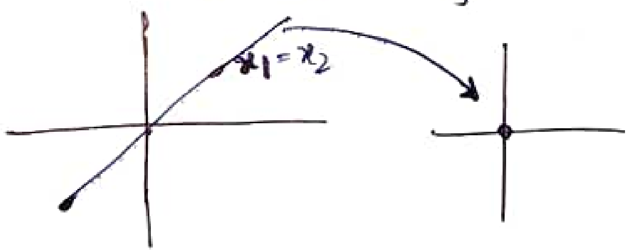

$\therefore \dim_{\mathbb{R}} \mathrm{Ker}(T) = 1.$

$\mathrm{Image}(T) = \{w \in \mathbb{R}^2 \mid w = T((v_1, v_2)),\ (v_1, v_2) \in \mathbb{R}^2\}.$

$= \{w = (w_1, w_2) \in \mathbb{R}^2 \mid (w_1, w_2) = (v_1 - v_2,\ v_1 - v_2),\ v_1, v_2 \in \mathbb{R}\}$

$= \{(v_1 - v_2,\ v_1 - v_2) \mid v_1, v_2 \in \mathbb{R}\}$

$= \{(x, x) \mid x \in \mathbb{R}\}.$




$T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$T((x_1, x_2)) = (x_1 - x_2,\ x_1 - x_2)$

$\therefore \dim_{\mathbb{R}} \mathrm{Img}(T) = 1$

$\therefore$ $\boxed{\text{dim ker + dim Img = dim domain}}$

- Ker (T) is subspace of V.
- Image (T) is subspace of W.

9-11-2023

→ $T: V \to W$, $V, W$ are v.s. over $\mathbb{R}$.

$Ker(T) \subset V$ subspace

$Image(T) \subset W$ subspace

Q1 find $\ker T$ and $Image(T)$, for

$T: P_2 \to P_3$ defined by

$$T(a_0 + a_1 x + a_2 x^2) = a_2 x + a_0 x^3.$$

Soln: $\ker T = \{ a_0 + a_1 x + a_2 x^2 \in P_2 \mid T(a_0 + a_1 + a_2 x^2) = 0 + 0.x + 0.x^2 + 0.x^3 \}$

$= \{ a_0 + a_1 x + a_2 x^2 \mid a_2 x + a_0 x^3 = 0.x + 0.x^3 \}$

$= \{ a_0 + a_1 x + a_2 x^2 \mid a_0 = 0, a_2 = 0, a_0, a_1, a_2 \in \mathbb{R} \}$

$= \{ a_1 x \mid a_1 \in \mathbb{R} \}$

$\therefore$ Bases: $\{x\}$

$\dim_{\mathbb{R}} \ker T = 1$

$Image(T) = \{ w \in P_3 \mid w = T(x) \text{ for } x \in P_2 \}$

$= \{ p(x) \in P_3 \mid p(x) = T(a_0 + a_1 x + a_2 x^2) = a_2 x + a_0 x^3 \}$

$= \{ a_2 x + a_0 x^3 \mid a_2, a_0 \in \mathbb{R} \}$

$\therefore$ Bases: $\{x, x^3\}$

$\dim_{\mathbb{R}} Image(T) = 2$.

Eg. $T: \mathbb{R}^2 \to \mathbb{R}^3$, $T((x_1, x_2)) = (x_1 + x_2, x_1, x_2)$.

Find $\ker T$, $Image(T)$.

Soln: $Ker(T) = \{ (x_1, x_2) \mid T(x_1, x_2) = (0,0,0) \}$

$= \{ (x_1, x_2) \mid (x_1 + x_2, x_1, x_2) = (0,0,0) \}$

$= \{(0,0)\}$

$Image\, T = \{ w \in \mathbb{R}^3 \mid w = T(v), v \in \mathbb{R}^2 \}$

$= \{ w \in \mathbb{R}^3 \mid w = T(v_1, v_2) = (v_1 + v_2, v_1, v_2) \text{ where } v_1, v_2 \in \mathbb{R} \}$

$x_2=1,\ x_1=0$

$v_1=1,\ x_2=0$

$= \{(v_1+v_2, v_1, v_2) \mid v_1, v_2 \in \mathbb{R}\} - \mathbb{R}^3$

Basis $\{(1,1,0), (1,0,1)\}$

$a\{a_1 \neq 0$

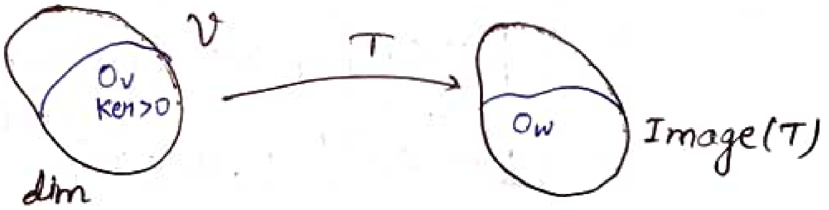
$\dim_{\mathbb{R}} \ker T = 0$

$\dim_{\mathbb{R}} \text{Image}(T) = 2$

## Rank- Nullity Theorem

Let $T: V \to W$ be a linear map between finite dimensional vector space over $\mathbb{R}$. Then

$$\dim_{\mathbb{R}} V = \text{rank}(T) + \text{nullity}(T).$$




Note: Dimension of Image of $T$ is called rank of $T$.

$$\text{rank}(T) = \dim_{\mathbb{R}} \text{Image}(T)$$

Dimension of Ker $T$ is called nullity.

$$\text{nullity}(T) = \dim_{\mathbb{R}} \ker(T).$$

# Matrix Representation of a Linear Map

Let $T: V \to W$ be linear where $V$ and $W$ be finite dimensional vector spaces.

$$\dim_{\mathbb{R}} V = n, \quad \dim_{\mathbb{R}} W = m.$$

Let $B_V = \{v_1, \dots, v_n\}$ and $B_W = \{w_1, \dots, w_m\}$ ~~are~~ be ordered bases for $V$ and $W$, respectively.

Then,

$$T(v_1), T(v_2), \dots, T(v_n) \in W$$

So,

$$T(v_1) = a_1 w_1 + a_2 w_2 + \dots + a_m w_m$$

$$T(v_2) = b_1 w_1 + b_2 w_2 + \dots + b_m w_m$$

$$\vdots \qquad \vdots$$

$$T(v_n) = \alpha_1 w_1 + \alpha_2 w_2 + \dots + \alpha_m w_m.$$

The matrix representation of $T$ is

$$[T] = \begin{pmatrix} a_1 & b_1 & \dots & \alpha_1 \\ a_2 & b_2 & & \alpha_2 \\ \vdots & \vdots & & \vdots \\ a_m & b_m & & \alpha_m \end{pmatrix}_{m \times n}$$

$$(T(v_1), T(v_2), \dots, T(v_n))_{1 \times n}$$

$$= (w_1, w_2, \dots, w_m)_{1 \times m} \begin{pmatrix} a_1 & b_1 & \dots & \alpha_1 \\ \vdots & \vdots & & \vdots \\ a_m & b_m & & \alpha_m \end{pmatrix}_{n \times m}$$

$$T(B_V) = B_W [T].$$

<u>Note</u>: Matrix representation depends on ordered bases.

Eg. $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $T((x_1, x_2)) = (x_1, x_1 + x_2)$.

Find $[T]$ with respect to the ordered bases.

ⓐ $B_1 = \{ \overset{v_1}{(1,0)}, \overset{v_2}{(0,1)} \}$, $B_2 = \{(1,0),(0,1)\}$

ⓑ $B_1 = \{(1,2),(3,4)\}$, $B_2 = \{(1,1),(0,1)\}$

Soln: ⓐ $T(v_1) = T((1,0)) = (1,1) = 1\cdot(1,0) + 1\cdot(0,1)$

$T(v_2) = T((0,1)) = (0,1) = 0\cdot(1,0) + 1\cdot(0,1)$.

$$\therefore [T] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

ⓑ $T(v_1) = T((1,2)) = (1,3) = \alpha\cdot(1,1) + \alpha_2(0,1)$
$= 1\cdot(1,1) + 2(0,1)$

$\alpha = 1$
$\alpha_1 + \alpha_2 = 3$
$\Rightarrow \alpha_2 = 2$

$T(v_2) = T((3,4)) = (3,7) = \alpha_1\cdot(1,1) + \alpha_2\cdot(0,1)$
$= 3(1,1) + 4\cdot(0,1)$

$$\therefore [T] = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}.$$

Eg. Find $[T]$ for $T : \underset{\text{dim}=4}{P_3} \to \underset{\text{dim}=3}{P_2}$ by $T(a_0 + a_1x + a_2x^2 + a_3x^3) = a_3x + a_2x^2$; with respect to the bases $B_1 = \{\overset{v_1}{1}, \overset{v_2}{x}, \overset{v_3}{x^2}, \overset{v_4}{x^3}\}$ for $P_3$ and $B_2 = \{\overset{w_1}{2}, \overset{w_2}{x}, \overset{w_3}{2+x^2}\}$ for $P_2$.

Soln: $T(v_1) = T(1) = 0$

$T(v_2) = T(x) = 0$

$T(v_3) = T(x^2) = x^2$

$T(v_4) = T(x^3) = x$.

$\therefore T(v_1) = 0 = 0\cdot 2 + 0\cdot x + 0\cdot(2+x^2)$

$T(v_2) = 0 = 0\cdot w_1 + 0\cdot w_2 + 0\cdot w_3$

$T(v_3) = x^2 = -1\cdot 2 + 0\cdot x + 1\cdot(2+x^2)$

$T(v_4) = x = 0\cdot 2 + 1\cdot x + 0\cdot(2+x^2)$

$$[T] = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}_{3\times 4}.$$

Note: $A \in M_{2\times 2}(\mathbb{R})$, then we can define a linear map

$T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x) = A\cdot x$ (matrix multiplication).

Qn: Given $T: \mathbb{R}^n \to \mathbb{R}^m$ linear and $[T]$ its matrix ~~multiplication~~ representation with respect to ordered bases $B_1, B_2$, for $\mathbb{R}^n$ and $\mathbb{R}^m$, resp.

$$T(v) \overset{?}{=} [T].v, \quad v \in \mathbb{R}^n$$

Ans: Not in general, but if we choose $B_1, B_2$ as standard bases then it is true.

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Eg. $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(x_1, x_2) = (x_1, x_1 + x_2)$.

① $[T] = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ w.r.t. $B_1 = \{(1,0), (0,1)\} = B_2$

② $[T] = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ w.r.t. $B_1 = \{(1,2), (3,4)\}, B_2 = \{(1,0), (0,1)\}$

$v = (3,7)$.

$T(v) = T(3,7) = (3,10)$.

① $[T].v = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 7 \end{pmatrix}$

$= (3, 10)$

$T(v) = [T].v$

② $[T].v = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 3 \\ 7 \end{pmatrix}$

$= (24, 34)$

$T(v) \neq [T].v$

Q Let $[T] = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -3 & -4 \end{bmatrix}$ be the matrix representation of $T: \mathbb{R}^3 \to \mathbb{R}^2$ with respect to the ordered bases

$B_1 = \{(1,-1,1), (2,3,-1), (1,1,-1)\}$

$B_2 = \{(1,1), (2,3)\}$.

Find $T(1,2,3)$.

Soln: $T: V \to W$ linear

$B_V = \{v_1, \dots, v_n\}$, $B_W = \{w_1, \dots, w_m\}$

$$(T(v_1), T(v_2), \dots, T(v_n)) = (w_1, w_2, \dots, w_m) \cdot [T]$$

---

$$(T(v_1), T(v_2), T(v_3)) = \begin{array}{cc} w_1 & w_2 \end{array} \atop \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 2 & -3 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & -4 & -7 \\ 7 & -7 & -11 \end{pmatrix}$$
$$\begin{array}{ccc} T(v_1) & T(v_2) & T(v_3) \end{array}$$

$\therefore T(v_1) = (5,7)$

$T(v_2) = (-4,-7)$

$T(v_3) = (-7,-11)$

$\alpha_1 + 2\alpha_2 + \alpha_3 = 1$

$-\alpha_1 + 3\alpha_2 + \alpha_3 = 2$

$\alpha_1 - \alpha_2 - \alpha_3 = 3$

$\Rightarrow \alpha_1 = \frac{3}{4}, \alpha_2 = \frac{5}{2}, \alpha_3 = -\frac{19}{4}$

$$(1,2,3) = \alpha_1(1,-1,1) + \alpha_2(2,3,-1) + \alpha_3(1,1,-1)$$
$$= \frac{3}{4}v_1 + \frac{5}{2}v_2 - \frac{19}{4}v_3$$

$$T(1,2,3) = T\left(\frac{3}{4}v_1 + \frac{5}{2}v_2 - \frac{19}{4}v_3\right) = \frac{3}{4}T(v_1) + \frac{5}{2}T(v_2) - \frac{19}{4}T(v_3).$$
$$= \frac{3}{4}(5,7) + \frac{5}{2}(-4,-7) - \frac{19}{4}(-7,-11)$$
$$= (27, 40).$$

Q1 Let $T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 2 & 2 \end{bmatrix}$ be the matrix representation of

$T : \mathbb{R}^3 \to \mathbb{R}^2$ with respect to

$B_1 = \{ \overset{v_1}{(1,0,1)}, \overset{v_2}{(1,1,0)}, \overset{v_3}{(0,1,1)} \}$

$B_2 = \{ \overset{w_1}{(1,0,0)}, \overset{w_2}{(0,1,0)}, \overset{w_3}{(0,0,1)} \}$

Find T.

Soln: $$(T(v_1), T(v_2), T(v_3)) = \begin{array}{c} \begin{array}{ccc} w_1 & w_2 & w_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} [T] \\ \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 2 & 2 \end{pmatrix} \end{array}$$

$$= \begin{array}{c} \begin{array}{ccc} T(v_1) & T(v_2) & T(v_3) \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 2 & 2 \end{pmatrix} \end{array} ;$$

$T(v_1) = (0,1,2)$

$T(v_2) = (0,2,2)$

$T(v_3) = (0,1,2)$.

Take any vector $(x_1, x_2, x_3) \in \mathbb{R}^3$.

$$(x_1, x_2, x_3) = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$$
$$= \alpha_1 (1,0,1) + \alpha_2 (1,1,0) + \alpha_3 (0,1,1)$$
$$= (\alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_3)$$

$$\Rightarrow \left. \begin{array}{l} \alpha_1 + \alpha_2 = x_1 \\ \alpha_2 + \alpha_3 = x_2 \\ \alpha_1 + \alpha_3 = x_3 \end{array} \right\} \Rightarrow \alpha_2 - \alpha_3 = x_1 - x_3$$

$\Rightarrow \alpha_2 = \frac{1}{2}(x_2 + x_1 - x_3)$;

$\alpha_1 = x_1 - \alpha_2$
$= \frac{1}{2}(x_1 - x_2 + x_3)$;

$\alpha_3 = x_2 - \alpha_2$
$= \frac{1}{2}(x_2 - x_1 + x_3)$; i.e

$T((x_1,x_2,x_3)) = T(\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3)$

$= \frac{1}{2}(x_1 - x_2 + x_3)\,T(v_1) + \frac{1}{2}(x_2 + x_1 - x_3)\,T(v_2)$

$+ \frac{1}{2}(x_2 - x_1 + x_3)\,T(v_3)$

$= \frac{1}{2}\underbrace{(x_1 - x_2 + x_3)}_{\alpha_1}(0,1,2) + \frac{1}{2}\underbrace{(x_2 + x_1 - x_3)}_{\alpha_2}(0,2,2)$

$+ \frac{1}{2}\underbrace{(x_2 - x_1 + x_3)}_{\alpha_3}(0,1,2)$

$= (0,\ x_1 + x_2,\ x_1 + x_2 + x_3).$

$\therefore\ T((x_1,x_2,x_3)) = (0,\ x_1+x_2,\ x_1+x_2+x_3).$

Q) Let $[T] = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ be the matrix representation for

$T: \mathbb{R}^3 \to \mathbb{R}^2$ with respect to the bases

$B_1 = \{\overset{v_1}{(1,1,0)},\ \overset{v_2}{(1,0,1)},\ \overset{v_3}{(0,0,1)}\}$

and $B_2 = \{\overset{w_1}{(1,0)},\ \overset{w_2}{(1,2)}\}$.

Find the linear map $T$.

Soln: $(T(v_1), T(v_2), T(v_3)) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}_{2\times2} \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}_{2\times3}$

$= \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 2 \end{pmatrix}$

$T(v_1) = (3,2)$

$T(v_2) = (2,2)$

$T(v_3) = (1,2)$.

Take $(x_1,x_2,x_3) \in \mathbb{R}^3$.

$(x_1,x_2,x_3) = \alpha_1(1,1,0) + \alpha_2(1,0,1) + \alpha_3(0,0,1)$

$(\alpha_1 + \alpha_2) = x_1$

$\alpha_1 = x_2$ $\Rightarrow$ $\alpha_2 = x_1 - x_2$

$\alpha_2+\alpha_3 = x_3 \Rightarrow \alpha_3 = x_3 - x_1 + x_2$

$$T(x_1, x_2, x_3) = T(\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3)$$
$$= \alpha_1 T(v_1) + \alpha_2 T(v_2) + \alpha_3 T(v_3)$$
$$= x_2 (3,2) + (x_1 - x_2)(2,2) + (x_3 - x_1 + x_2)(1,2)$$
$$= (3x_2 + 2x_1 - 2x_2 + x_3 - x_1 + x_2 ,\ 2x_2 - 2x_2 + 2x_3 - 2x_1 + 2x_2)$$
$$= (x_1 + 2x_2 + x_3,\ -2x_1,\ 2x_2 + 2x_3).$$

→ $V$ is a vector space over $\mathbb{R}$ of dimension '$n$'.

Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for $V$.

Take any $v \in V$.

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n \;; \quad \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$$

uniquely written

$$v \in V \longmapsto (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$$

$$\therefore \; V \longrightarrow \mathbb{R}^n$$

$$v \longmapsto (\alpha_1, \alpha_2, \ldots, \alpha_n).$$

→ $L(V, W)$ = set of all linear maps from $V$ to $W$.

$\dim_{\mathbb{R}} V = n$

$\dim_{\mathbb{R}} W = m$.

Fix $L(\mathbb{R}^n, \mathbb{R}^m) \longrightarrow M_{m \times n}(\mathbb{R})$

$B_1$ $B_2$ bases $\quad T \longmapsto [T]_{m \times n}$

$\quad T_1 \longleftarrow A$

$$T_1 + T_2 : \mathbb{R}^n \to \mathbb{R}^m$$

$$(T_1 + T_2)(x) = T_1(x) + T_2(x).$$

$$T_1, T_2 : \mathbb{R}^n \to \mathbb{R}^m$$

$$\alpha T_1 : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$(\alpha T_1)(x) = \alpha T_1(x) \in \mathbb{R}^m.$$