Indian Institute of Space Science and Technology

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MA 221 - Integral Transforms, PDE and Calculus of Variations

Tutorial I - PDE

1. (a) Show that the family of right circular cones whose axes coincide with the z-axis

$$x^2 + y^2 = (z - c)^2 \tan^2 \alpha$$

satisfies the first-order, partial differential equation

$$yp - xq = 0.$$

(b) Show that the surfaces of revolution, $z = f(x^2 + y^2)$ with the z-axis as the axis of symmetry, where f is an arbitrary function, satisfy the partial differential equation

$$yp - xq = 0.$$

(c) Show that the two-parameter family of curves $u - ax - by - ab = 0$ satisfies the nonlinear equation

$$xp + yq + pq = u.$$

2. Find the partial differential equation arising from each of the following surfaces:

(a) $z = x + y + f(xy)$, (b) $z = f(x - y)$
(c) $z = xy + f(x^2 + y^2)$, (d) $2z = (\alpha x + y)^2 + \beta$

3. Find the general solution of each of the following equations:

(a) $u_x + yu_y = 0$, (b) $(y + u)u_x + yu_y = x - y$,
(c) $y^2 u_x - xyu_y = x(u - 2y)$, (d) $yu_y - xu_x = 1$,
(e) $y^2 up + u^2 xq = -xy^2$

4. Find the solution of the following Cauchy problems:

(a) $yu_x + xu_y = 0$, with $u(0, y) = \exp(-y^2)$,
(b) $xu_x + yu_y = 2xy$, with $u = 2$ on $y = x^2$,
(c) $u_x + xu_y = 0$, with $u(0, y) = \sin y$,
(d) $yu_x + xu_y = xy, x \geq 0, y \geq 0$, with $u(0, y) = \exp(-y^2)$ for $y > 0$, and $u(x, 0) = \exp(-x^2)$ for $x > 0$,

5. Find the solution of the Cauchy problem

$$2xyu_x + (x^2 + y^2)u_y = 0, \text{ with } u = \exp\left(\frac{x}{x - y}\right) \text{ on } x + y = 1.$$

6. Solve the equation

$$u_x + xu_y = y$$

with the Cauchy data

$$(a)\ u(0, y) = y^2, \qquad (b)\ u(1, y) = 2y.$$

7. Show that $u_1 = e^x$ and $u_2 = e^{-y}$ are solutions of the nonlinear equation

$$(u_x + u_y)^2 - u^2 = 0$$

but that their sum $(e^x + e^{-y})$ is not a solution of the equation.

8. Solve the Cauchy problem $(y + u)u_x + yu_y = (x - y)$, with $u = 1 + x$ on $y = 1$.

9. Show that the solution of the equation

$$yu_x - xu_y = 0$$

containing the curve $x^2 + y^2 = a^2$, $u = y$, does not exist.

10. Find the solution surface of the equation

$$(u^2 - y^2)u_x + xyu_y + xu = 0, \text{ with } u = y = x,\ x > 0.$$

a

11. Solve the following equations:

(a) $(y + u)u_x + (x + u)u_y = x + y$,

(b) $xu(u^2 + xy)u_x - yu(u^2 + xy)u_y = x^4$,

(c) $yu_x + xu_y = xy(x^2 - y^2)$,

12. Obtain the family of curves which represent the general solution of the partial differential equation

$$(2x - 4y + 3u)u_x + (x - 2y - 3u)u_y = -3(x - 2y).$$

13. Find the solution of the equation

$$yu_x - 2xyuy = 2xu$$

with the condition $u(0, y) = y^3$.

14. Obtain the general solution of the equation

$$(x + y + 5z)p + 4zq + (x + y + z) = 0 \quad (p = z_x, q = z_y),$$

and find the particular solution which passes through the circle

$$z = 0, \quad x^2 + y^2 = a^2.$$

15. Obtain the general solution of the equation

$$z^2 - 2yz - y^2)p + x(y + z)q = x(y - z) \qquad (p = z_x, q = z_y).$$

Find the integral surfaces of this equation passing through (a) the x-axis, (b) the y-axis, and (c) the z-axis.

16. Solve the Cauchy problem

$$(x + y)u_x + (x - y)u_y = 1, \quad u(1, y) = \frac{1}{\sqrt{2}}$$

17. Show that the PDEs
$$xp = yq \text{ and } z(xp + yq) = 2xy$$
are compatible and hence find its solution.

18. Show that the equations
$$p^2 + q^2 = 1 \text{ and } (p^2 + q^2)x = pz$$
are compatible and hence find its solution.

19. Find a complete integral of the equation
$$p^2 + q^2)x = pz$$
where $p = \partial z/\partial x,\ q = \partial z/\partial y$.

20. Find a complete integral of the equations
    (a) $px^5 - 4q^3x^2 + 6x^2z - 2 = 0$
    (b) $2(z + xp + yq) = yp^2$.

21. Find a complete integral of the equation
$$p + q = pq$$

22. Find a complete integral of the following equations:
    (a) $zpq = p + q$
    (b) $p^2q^2 + x^2y^2 = x^2q^2(x^2 + y^2)$

23. Find a complete integral of the PDE
$$z = px + qy - \sin(pq)$$

24. Find a complete integral of the following PDEs:
    (a) $xp^3q^2 + yp^2q^3 + (p^3 + q^3) - zp^2q^2 = 0$
    (b) $pqz = p^2(xq + p^2) + q^2(yp + q^2)$.

25. Find a complete integral of the equation
$$(p^2 + q^2)x = pz$$
where $p = \dfrac{\partial z}{\partial x}, q = \dfrac{\partial z}{\partial y}$

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