Indian Institute of Space Science and Technology

Thiruvananthapuram-695547

End Semester Examination-May 2012

B.Tech 2nd Semester

MA121 - Vector Calculus and Differential Equations

Date : 14^{th}May, 2012 **Time: 9.30 am to 12.30 pm** **Max. Marks: 100**

SECTION A (Answer all 10 questions - 10x5= 50 marks.)

1. Verify whether the following sequences converge uniformly.

   (a) $f_n(x) = x^n + 2x^{n+1} + 3\cos(\pi x)x^{n+2}, x \in [0,1]$.

   (b) $f_n(x) = x^{n+3} + 3x^{n+2} - 2x^{n+1} + 100x^n, x \in [0,1]$.

2. Show that the sequence $f_n(x) = nxe^{-nx^2}$ converges pointwise on $[0,1]$. Examine whether the relation $\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 \lim_{n\to\infty} f_n(x)dx$ holds. Also verify whether the convergence is uniform.

3. Let a surface S be given by $\vec{r}(u,v) = (2\cos u, 2\sin u, v)$ where $u \in [0, 2\pi]$ and $v \in [-1,1]$. Represent the surface S pictorially. Using surface integral find the surface area of S.

4. Let D be a elliptical region on xy-plane given by $D = \left\{(x,y) \middle| \frac{x^2}{9} + \frac{y^2}{4} \leq 1\right\}$. Using **Green's theorem** find the area of the region D.

5. Using **Stoke's theorem** find the surface integral $\int_S Curl(\vec{F}).d\vec{S}$ where $\vec{F} = (x, 0, 3y^2)$ and S is the upper hemisphere "$x^2 + y^2 + z^2 = 1,\ z \geq 0$".

6. Verify **Gauss divergence theorem** for the vector field $\vec{F} = (0,0,z)$ over the volume $V : x^2 + y^2 + z^2 \leq 1$.

7. Find the path of steepest ascent up the mountain $z = \alpha^2 - \sqrt{(x^2 - \beta^2)^3} + \sqrt{(y^2 - \gamma^2)^3}$ starting from the point $P = (\beta, \gamma, \alpha)$, using **orthogonal trajectory**. Here β, γ, α are all non-zero constants.

8. Verify whether the pair of functions $\{f(x) = x^3, g(x) = |x|^3\}$ are linearly independent on $\mathbb{R}$ and also compute their Wronskian. Determine whether they can be solutions of the differential equation $y'' + p(x)y' + q(x)y = 0$ with p and q continuous on $[-a, a]$, $a > 0$.

9. Find the power series expansions of $\frac{1}{1 - x + \pi}$ and $\frac{1}{9(x-\pi)^2 + 4}$ about the point $x = \pi$. Also determine the interval of convergence of the corresponding series solution of

$$(x-\pi)y'' + \frac{1}{2\pi - 2\pi x + 2\pi^2}y' + \frac{1}{\frac{3}{2}(x-\pi)^2 + \frac{2}{3}}y = 0,$$

   about that point. (Hint:Use the power series expansion: $1 + x + x^2 + \ldots$ of $\frac{1}{1-x}$ about $x = 0$.)

10. Find the general solution of $xy'' + (1 + 2\lambda)y' + xy = 0, \quad x > 0$ in terms of Bessel's functions, using the substitution $y = \frac{u(x)}{x^\lambda}$. Here $\lambda \in \mathbb{R}$ is not an integer.

SECTION B (Answer any 5 questions - 5x10= 50 marks.)

1. Is the function $f(x) = \sum_{1}^{\infty} \frac{[nx]}{n^3}$ for $0 \leq x \leq 1$, when $[x]$ = greatest integer$< x$, integrable over $[0, 1]$? Justify your answer.

2. (a) Let C be a curve in xyz-space satisfying $y^2 = 4x$, $z = \frac{4}{3}\sqrt{xy}$, $y \geq 0$. Parameterize the curve C. Find the "arc length function" of C with initial point $(0, 0, 0)$ and hence calculate the length of the curve from $(0, 0, 0)$ to $(4, 4, 16/3)$.
   (b) Find the directional derivative of $xy^2 + yx^2 + e^z$ at the point $(1, 1, 1)$ along the unit vector $\vec{v} = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$.

3. (a) State the Fundamental Theorem of Line integral.
   (b) Let $\vec{F} = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}, e^z\right)$. Calculate $Curl(\vec{F})$. Find the domain of $\vec{F}$. Is the domain of $\vec{F}$ simply connected? Is $\vec{F}$ a conservative vector field? Evaluate $\int_C \vec{F}.d\vec{r}$ and explain whether the integral is path dependent where $C := \{(cos\ t,\ sin\ t, t) |\ t \in [0, \pi]\}$.

4. (a) Does the **Picard's theorem** guarantee the existence of a unique solution of the following initial value problems
      i. $\frac{dy}{dx} = 1 + y^{2/3}$, $y(2012) = 0$, on some interval about $x = 2012$?
      ii. $\frac{dy}{dx} = xy^2$, $y(0) = 0$, on the entire interval $|x| \leq 2$?

   Justify your answers.
   (b) Solve $x^3p^2 + x^2yp + a^3 = 0$, $p = \frac{dy}{dx}$, by reducing to Clairaut's form using the substitution $Y = y, X = \frac{1}{x}$. Here a is a non-zero constant. What does the singular solution of the given differential equation represent pictorially?

5. (a) Compute the indicial equation, its roots and hence find the first three terms of each of two linearly independent series solutions of
$$x^2y'' + (x^2 - 2x)y' + 2y = 0,$$
   about the point $x = 0$ for $x > 0$.
   (b) Find the general solution of $y''+6y'+9y = e^{-3x}/x$, using method of variation of parameters.

6. (a) Using the identity $\frac{d}{dx}[x^pJ_p(x)] = x^pJ_{p-1}(x)$, satisfied by the Bessel function $J_p(x)$ of first kind of order p and Rolle's theorem, show that between any two positive roots of $J_{p+1}(x) = 0$, there is a root of $J_p(x) = 0$.
   Also using Rodrigues' formula $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n$, for the Legendre polynomial $P_n(x)$ of degree n, find first three terms of the Legendre series of $f(x) = |x|$, $-1 \leq x \leq 1$.
   (b) Find the eigenvalues and eigenfunctions of the boundary value problem
$$(xy')' + (\lambda/x)y = 0, \quad 1 < x < l, \quad y(1) = 0, y(l) = 0, \quad \lambda \in \mathbb{R}.$$