Indian Institute of Space Science and Technology

Complex Analysis

TUTORIAL - IV

1. Let C_0 denote the circle $|z - z_0| = R$, taken counterclockwise. Use the parametric representation $z = z_0 + Re^{i\theta}(-\pi \le \theta \le \pi)$ for C_0 to derive the following integration formulas:

   (a) $\displaystyle\int_{C_0} \frac{dz}{z - z_0} = 2\pi i;$

   (b) $\displaystyle\int_{C_0} (z - z_0)^{n-1} dz = 0 \quad (n = \pm 1, \pm 2, \cdots).$

2. Apply the Cauchy-Goursat theorem to show that
$$\int_C f(z)\, dz = 0$$
when the contour C is the circle $|z| = 1$, in either direction, and when

   (a) $f(z) = \dfrac{z^2}{z-3}$; (b) $f(z) = ze^{-z}$; (c) $f(z) = \dfrac{1}{z^2 + 2z + 2}$;
   (d) $f(z) = \operatorname{sech} z$; (e) $f(z) = \tan z$; (f) $f(z) = \log(z + 2)$.

3. Let C denote the positively oriented boundary of the half disk $0 \le r \le 1$, $0 \le \theta \le \pi$, and let $f(z)$ be a continuous function defined on that half disk by writing $f(0) = 0$ and using the branch
$$f(z) = \sqrt{r}e^{i\theta/2} \quad \left(r > 0, \frac{-\pi}{2} < \theta < \frac{3\pi}{2}\right)$$
of the multiple-valued function $z^{1/2}$. Show that
$$\int_C f(z)dz = 0$$
by evaluating separately the integrals of $f(z)$ over the semicircle and the two radii which make up C. Why does the Cauchy-Goursat theorem not apply here?

4. Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate each of these integrals:

   (a) $\displaystyle\int_C \frac{e^{-z}\, dz}{z - (\pi i/2)}$;

   (b) $\displaystyle\int_C \frac{\cos z}{z(z^2 + 8)} dz$;

   (c) $\displaystyle\int_C \frac{z\, dz}{2z + 1}$;

   (d) $\displaystyle\int_C \frac{\cosh z}{z^4} dz$;

   (e) $\displaystyle\int_C \frac{\tan(z/2)}{(z - x_0)^2} dz \, (-2 < x_0 < 2).$

5. Let C be the circle $|z| = 3$, described in the positive sense. Show that if
$$g(w) = \int_C \frac{2z^2 - z - 2}{z - w} dz \quad (|w| \neq 3),$$
then $g(2) = 8\pi i$. What is the value of $g(w)$ when $|w| > 3$?

6. Show that if f is analytic within and on a simple closed contour C and z_0 is not on C, then

$$\int_C \frac{f'(z)\,dz}{z-z_0} = \int_C \frac{f(z)\,dz}{(z-z_0)^2}.$$

7. Let f denote a function that is *continuous* on a simple closed contour C. Prove that the function

$$g(z) = \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{s-z}$$

is *analytic* at each point z interior to C and that

$$g'(z) = \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z)^2}$$

at such a point.

8. Obtain the Maclaurin series representation

$$z\cosh(z^2) = \sum_{n=0}^{\infty} \frac{z^{4n+1}}{(2n)!} \qquad (|z| < \infty).$$

9. Obtain the Taylor series

$$e^z = e\sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} \qquad (|z-1| < \infty)$$

for the function $f(z) = e^z$ by

(a) using $f^{(n)}(1)$ $(n = 0, 1, 2, \cdots)$;

(b) writing $e^z = e^{z-1}e$.

10. Show that when $0 < |z| < 4$,

$$\frac{1}{4z - z^2} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}.$$

11. Give two Laurent series expansions in powers of z for the function

$$f(z) = \frac{1}{z^2(1-z)},$$

and specify the regions in which those expansions are valid.

12. Show that when $0 < |z-1| < 2$,

$$\frac{z}{(z-1)(z-3)} = -3\sum_{n=0}^{\infty} \frac{(z-1)^n}{2^{n+2}} - \frac{1}{2(z-1)}.$$

13. Write the two Laurent series in powers of z that represent the function

$$f(z) = \frac{1}{z(1+z^2)}$$

in certain domains, and specify those domains.

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