Maths Assignment

MA121 – Vector Calculus

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Direction Derivatives and Differentiability

1. Let ~~~~ $\lim_{h\to0} \frac{f(x_0+h)-f(x_0)-\alpha h}{|h|} = 0$

If $h>0$, $\lim_{h\to0} \frac{f(x_0+h)-f(x_0)}{h} - \frac{\alpha h}{h} = 0$

$\Rightarrow \lim_{h\to0} \frac{f(x_0+h)-f(x_0)}{h} = \alpha$

$\therefore$ $\alpha = f'(x_0)$ is true iff α exists.

If $h<0$, $\lim_{h\to0} \frac{f(x_0+h)-f(x_0)}{-h} = -\alpha$

$\Rightarrow \lim_{h\to0} \frac{f(x_0+h)-f(x_0)}{h} = \alpha$

$\therefore$ We can say that $\lim_{h\to0} \frac{f(x_0+h)-f(x_0)}{h}$ exists iff α exits.

Thus, function is differentiable iff $\exists\, \alpha \in \mathbb{R}$.

2. Let $\theta(x) = \frac{g(x)-g(x_0)}{x-x_0}$

$\Rightarrow g(x) = g(x_0) + \theta(x)(x-x_0)$ – ①

If $g(x)$ is differentiable $\Rightarrow$ $g(x)$ is continuous

$\Rightarrow$ RHS of ① is combination of continuous fns. $\Rightarrow$ $\theta(x)$ is continuous

$\therefore \lim_{x\to x_0} \theta(x) = \lim_{x\to x_0} \frac{g(x)-g(x_0)}{x-x_0}$ must exist.

3. Given that $D_{\bar{v}}(f)|_{P_0}$ exists.

$D_{\bar{v}}(f)|_{P_0} = \lim_{t\to0} \frac{f(P_0+t\bar{v})-f(P_0)}{t}$

Let $\phi(t) = f(P_0+t\bar{v})$

and $g(t) = (P_0+t\bar{v})$

Then, $\frac{d}{dt}\phi(t)|_{t=0} = \lim_{t\to0} \frac{\phi(t)-\phi(0)}{t-0}$

$$\Rightarrow \frac{d}{dt} f(P_0+t\bar{v}) = \lim_{t\to 0} \frac{f(P_0+t\bar{v})-f(P_0)}{t}$$

$$\Rightarrow \frac{d}{dt} f(g(t))\Big|_{t=0} = D_v(f)\Big|_{P_0}$$

4. Given that f is differentiable.

$$\lim_{(h,k)\to(0,0)} \frac{f(x_0+h, y_0+k)-f(x_0,y_0) - \alpha h - \beta k}{\sqrt{h^2+k^2}} = 0$$

Along x-axis, $(y=0)$,

$$k=0 \Rightarrow \lim_{h\to 0} \frac{f(x_0+h, y_0)-f(x_0,y_0)-\alpha h}{|h|} = 0$$

$$\Rightarrow \lim_{h\to 0} \frac{f(x_0+h, y_0)-f(x_0,y_0)}{h} = \alpha$$

Now, $\frac{\partial f}{\partial x}\Big|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0+h, y_0)-f(x_0,y_0)}{h}$

$$\Rightarrow \alpha = \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)}$$

Similarly, $\beta = \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)}$

Hence, α and β are the partial derivatives of f at (x_0,y_0).

5. Given that $f: \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (x_0, y_0).

$\therefore f(x,y) = f(x_0,y_0) + (x-x_0) f_x\big|_{(x_0,y_0)} + (y-y_0) f_y\big|_{(x_0,y_0)}$,

where f_x and f_y are partial derivatives which are continuous.

$\Rightarrow$ RHS combination is continuous.

$\Rightarrow$ $f(x,y)$ is continuous.

6. Caratheodory theorem:

(1) For one variable case:

Let $G \subset \mathbb{R}$ be open, $a \in G$ and $f: G \to \mathbb{R}$ be a function.

Then f is differentiable at 'a' iff ~~and~~ there exists a function $f_1: G \to \mathbb{R}$ such that

ⓐ f_1 is continuous at 'a'.

ⓑ $f(x)-f(a) = f_1(x)(x-a) \quad \forall\, x \in G$

Here, $f_1(x) = \frac{d}{dx} f(x)\big|_a = f'(a)$.

(ii) <u>For two variable case:</u>

Let $G \subset \mathbb{R}^2$ be open, $(x_0, y_0) \in G$ and $f: G \to \mathbb{R}$ be a function. Then, f is differentiable at (x_0, y_0) iff there exists two functions $f_1, f_2: G \to \mathbb{R}$ such that

ⓐ f_1 and f_2 are continuous at (x_0, y_0).

ⓑ $f(x,y) - f(x_0,y_0) = (x-x_0) f_1(x,y) + (y-y_0) f_2(x,y) \quad \forall (x,y) \in G$

Further, $f_1(x_0,y_0) = f_x(x_0,y_0)$ and $f_2(x_0,y_0) = f_y(x_0,y_0)$.

⑦ By caratheodory theorem for two variable case, we know that

$$f(x,y) = f(x_0,y_0) + (x-x_0) f_x\big|_{(x_0,y_0)} + (y-y_0) f_y\big|_{(x_0,y_0)}$$

if f is differentiable at (x_0, y_0).

Tangent plane: $z - z_0 = f_x(x-x_0) + f_y(y-y_0)$

$$\Rightarrow \Delta z = \Delta x \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)} + \Delta y \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)}$$

The approximations of f in the neighbourhood of (x_0,y_0) have higher occurance when $\Delta z \to 0$.

⑧ For a single varible case, the function can be approximated by a tangent line:

$$f(x) = f(a) + (x-a) f'(a)$$

$$\Rightarrow \Delta y = f'(a) \Delta x.$$

Arc Length Function

① Given that c is a parametric c^1-type curve.

We know that $c^{-1}(t) = c(a+b-t)$, $c : [a,b] \to \mathbb{R}^2$

Let $s = a+b-t$, $s \in [a,b]$

$\therefore c^{-1}(t) = c(s)$

$\Rightarrow c^{-1}(t)$ must also be c^1-type.

② Given that $c : \gamma(t)$, $t \in [a,b]$ is a c^1-type curve.

$c^1 : \gamma'$

$\gamma^{-1}(t) = \gamma(a+b-t)$

$$\Rightarrow \int_a^b \|\gamma^{-1}{}'(t)\| dt = \int_a^b \|\gamma'(a+b-t)\| dt \qquad (\because \|\gamma^{-1}{}'\| = \|\gamma'\|)$$

As $\ell(c) = \int_a^b \|\gamma'(t)\| dt$

$$\ell(c^{-1}) = \int_a^b \|\gamma'(a+b-t)\| dt$$

Let $m = a+b-t$

$\Rightarrow dm = -dt$

$$\therefore \ell(c^{-1}) = -\int_b^a \|\gamma'(m)\| dm$$

$$= \int_a^b \|\gamma'(m)\| dm$$

$$= \int_a^b \|\gamma'(t)\| dt$$

$\therefore \ell(c^{-1}) = \ell(c)$

3 Given that $\gamma : [a,b] \to \mathbb{R}^3$ is a c^1-type curve.

(a) As $s(x) = \int_a^x \|\gamma'(t)\| dt$

For $x_2 > x_1$, $\int_a^{x_1} \|\gamma'(t)\| dt < \int_a^{x_2} \|\gamma'(t)\| dt$

$\left[\text{As } \int_a^{x_1} f(t)\,dt < \int_a^{x_2} f(t)\,dt \text{ holds true when } f(t) > 0 \right]$

$\Rightarrow s(x_2) > s(x_1)$

$\therefore$ s is a non-decreasing function.

(b) $\gamma(t)$ is c^1-type $\Rightarrow \gamma'(t)$ is continuous.

$\Rightarrow \|\gamma'(t)\|$ is continuous.

As integral of a continuous function is also continuous.

$\therefore$ $s(x) = \int_a^x \|\gamma'(t)\| dt$ is a continuous function.

(c) As s is differentiable, $\|\gamma'\|$ is continuous & $\frac{d}{dx} s(x) = \underset{\text{continuous.}}{\underset{\downarrow}{\|\gamma'(t)\|}}$

$\Rightarrow$ s is a c^1-type function.

$s'(x) = \|\gamma'(x)\|$.

(d) As $\|\gamma'(t)\| \neq 0$ and $\|\gamma'(t)\| > 0$

$\Rightarrow s = \int_a^x \|\gamma'(t)\| dt \geq 0$ is zero only when $x = a$

$\therefore$ $s(x) > 0$ for all $x \in [a,b]$.

(e) As $s(x) = \int_a^x \|\gamma'(t)\| dt \geq 0 \quad \forall x \in [a,b]$

At $x = a$, $s(a) = \int_a^a \|\gamma'(t)\| dt = 0$

Since s is non-decreasing and smooth, $s(x) = 0$ iff $x = a$.

For $\forall x \in [a,b]$, $s(x) > 0$.

Thus, $s(x)$ is strictly increasing function.