SAURABH KUMAR
SC22B146

Tutorial on Laplace Transforms

AV223 - Signals and systems

① Find the inverse laplace transform (ILT) of $X(s)$, if $x(t)$ is causal signal.

$$X(s) = \frac{2s^2+9s-47}{(s+1)(s^2+6s+25)}$$

Sol^n:
$$X(s) = \frac{2s^2+9s-47}{(s+1)(s^2+6s+25)}$$

$$= \frac{\frac{47}{10}s + \frac{41}{2}}{(s^2+6s+25)} + \frac{-27}{10(s+1)}$$

$$= \frac{47/10\, s + 41/2}{(s+3)^2+16} - \frac{27/10}{s+1}$$

$$= \frac{47/10\,(s+3)}{(s+3)^2+4^2} + \frac{41/2 - 47/10 \times 3}{(s+3)^2+4^2} - \frac{27/10}{s+1}$$

$$= \frac{47/10\,(s+3)}{(s+3)^2+4^2} + \frac{47/5\,(4) \times 1/4}{(s+3)^2+4^2} - \frac{27/10}{s+1}$$

ILT ⇒ $x(t) = \left[47/10\, e^{-3t}\cos 4t + 47/20\, e^{-3t}\sin 4t - 27/10\, e^{-t}\right] u(t)$

② Find ILT of $X(s)$: $X(s) = \dfrac{2}{(s+4)(s-1)}$

if the region of convergences (ROCs) are

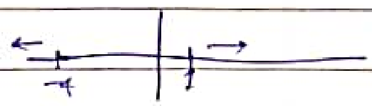
ⓐ $-4 < Re\{s\} < 1$

ⓑ $Re\{s\} > 1$

ⓒ $Re\{s\} < -4$.

Sol^n:
$$X(s) = \frac{2}{(s+4)(s-1)} = \frac{A}{s+4} + \frac{B}{s-1} \Rightarrow A = 2/-5 = -2/5, \quad B = 2/5$$

$$= \frac{-2/5}{s+4} + \frac{2/5}{s-1}$$



ⓐ ILT ⇒ $x(t) = \left[-2/5\, e^{-4t}u(t) - 2/5\, e^{t}u(-t)\right]$

ⓑ ILT ⇒ $x(t) = \left[-2/5\, e^{-4t} + 2/5\, e^{t}\right] u(t)$

ⓒ ILT ⇒ $x(t) = -\left[-2/5\, e^{-4t} + 2/5\, e^{t}\right] u(-t)$

③ Find the Laplace transform (LT) of $x(t) = e^{-2t}\sin(2t)\,u(t)$.

Sol^n: $x(t) = e^{-2t}\sin(2t)u(t)$

$$\sin(2t)u(t) \longleftrightarrow \frac{2}{s^2+2^2}$$

$$e^{-2t}\sin(2t)\,u(t) \longleftrightarrow \frac{2}{(s+2)^2+4}$$

$$\therefore\ X(s) = \frac{2}{(s+2)^2+4}$$

$$= \frac{2}{s^2+4s+8}$$

④ Find the transfer function and impulse response of the system given by

$$\frac{d^2y(t)}{dt^2} + 7\frac{dy(t)}{dt} + 6y(t) = 7x(t) + 11\frac{dx(t)}{dt}$$

Sol^n: Take LT with initial conditions as zero.

$$\Rightarrow s^2Y(s) + 7sY(s) + 6Y(s) = 7X(s) + 11sX(s)$$

$$\Rightarrow Y(s)[s^2+7s+6] = X(s)[7+11s]$$

$$\Rightarrow \frac{Y(s)}{X(s)} = G(s) = \frac{11s+7}{s^2+7s+6}$$

↑ ~~Impulse response~~ / transfer function

Now, $G(s) = \dfrac{11s+7}{s^2+7s+6}$

$$= \frac{11s+6}{(s+6)(s+1)} = \frac{A}{s+6} + \frac{B}{s+1} \quad \Rightarrow A = \frac{-66+6}{-5} = 12$$

$$= \frac{12}{s+6} + \frac{\cancel{B}-1}{s+1} \qquad B = \frac{-11+6}{5} = -1$$

$$ILT \Rightarrow g(t) = 12e^{-6t} - e^{-t}.$$

↑ Impulse response.

⑤ Find the LT of the following signals and specify ROC

ⓐ $x(t) = te^{-2|t|}$

ⓑ $x(t) = e^{-4t}u(t) + e^{-5t}\sin(5t)u(t)$.

Soln: ⓐ $x(t) = te^{-2|t|}$

$$\Rightarrow \begin{cases} te^{-2t}, & t>0 \\ te^{2t}, & t<0 \end{cases}$$

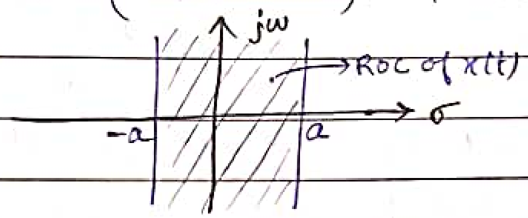
$$= te^{-2t}u(t) + te^{2t}u(-t)$$

$te^{-at}u(t) \leftrightarrow \frac{1}{(s+a)^2}$

ROC: $Re\{s\} > -a$

$$LT \Rightarrow X(s) = \frac{1}{(s+a)^2} + \frac{1}{(s-a)^2}$$

ROC: $(Re\{s\} > -a) \cap (Re\{s\} < a)$



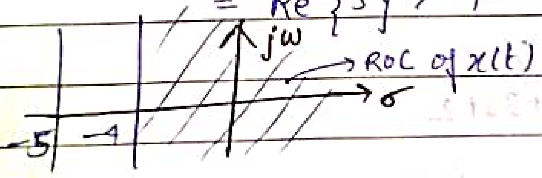
ⓑ $x(t) = e^{-4t}u(t) + e^{-5t}\sin(5t)u(t)$

$$LT \Rightarrow X(s) = \frac{1}{s+4} + \frac{5}{(s+5)^2+5^2}$$

$$= \frac{1}{s+4} + \frac{5}{s^2+10s+50}$$

ROC: $(Re\{s\} > -4) \cap (Re\{s\} > -5)$

$\equiv Re\{s\} > -4$



⑤ⓑ Find the unilateral LT of the signal $x(t) = e^{-2t}u(t+1)$ and find the ROC.

Soln: $x(t) = e^{-2t}u(t+1)$

$$X(s) = \int_{-\infty}^{\infty} e^{-2t}u(t+1)e^{-st}dt$$

$$= \int_{-1}^{\infty} e^{-2t}e^{-st}dt = \left[\frac{e^{-(2+s)t}}{-(s+2)}\right]_{-1}^{\infty} = \frac{0-e^{(2+s)}}{-(s+2)} = \boxed{\frac{e^{s+2}}{s+2}}$$

ROC: $Re\{s\}+2>0$

$\boxed{Re\{s\} > -2}$

(7) Consider a signal $y(t)$ expressed as $y(t) = x_1(t-2) * x_2(-t+3)$ where $*$ denotes convolution operation, $x_1(t) = e^{-2t}u(t)$ and $x_2(t) = e^{-3t}u(t)$. Determine $Y(s)$.

Sol^n:
$$y(t) = x_1(t-2) * x_2(-t+3)$$
$$= \left[e^{-2(t-2)}u(t-2)\right] * \left[e^{-3(-t+3)}u(-t+3)\right]$$

$$LT \Rightarrow Y(s) = LT\{e^{-2(t-2)}u(t-2)\} \times LT\{e^{-3(-t+3)}u(-t+3)\}$$
$$= \int_{2}^{\infty} e^{-2(t-2)}e^{-st}\,dt + \int_{-\infty}^{3} e^{-3(-t+3)}e^{-st}\,dt$$
$$= e^{4}\int_{2}^{\infty} e^{-(2+s)t}\,dt + e^{-9}\int_{-\infty}^{3} e^{(3-s)t}\,dt$$
$$= e^{4}\left[\frac{e^{-(2+s)t}}{-(2+s)}\right]_{2}^{\infty} + e^{-9}\left[\frac{e^{(3-s)t}}{3-s}\right]_{-\infty}^{3}$$
$$= e^{4}\,\frac{e^{-2(2+s)}}{(s+2)} + e^{-9}\,\frac{e^{-3(s-3)}}{(s-3)}$$
$$= \frac{e^{-2s}}{s+2} + \frac{e^{-3s}}{s-3}$$

(8) Find the initial and final values of the following transforms.

(a) $X(s) = \dfrac{s+5}{s^2+3s+2}$

Sol^n:
$$x(0^+) = \lim_{s\to\infty} sX(s)$$
$$= \lim_{s\to\infty} \frac{s(s+5)}{s^2+3s+2}$$
$$= \lim_{s\to\infty} \frac{1+5/s}{1+3/s+2/s^2} = 1$$

$$x(\infty) = \lim_{s\to 0} sX(s) = \lim_{s\to 0} \frac{s(s+5)}{s^2+3s+2} = 0.$$

(6) $X(S) = \dfrac{s^2+5s+7}{s^2+3s+2}$

Soln: $x(0^+) = \lim_{s\to\infty} sX(s)$

$= \lim_{s\to\infty} \dfrac{s(s^2+5s+7)}{s^2+3s+2}$

$= \lim_{s\to\infty} \dfrac{1+5/s+7/s^2}{1/s+3/s^2+2/s^3} = \infty$

(8) $x(\infty) = \lim_{s\to 0} s\,X(s) = 0.$

(9) Consider an LTI system with input $x(t) = e^{-t}u(t)$ and impulse response $h(t) = e^{-2t}u(t)$. Determine the output signal $y(t)$.

Soln: $y(t) = x(t) * h(t)$

$Y(S) = X(S) \cdot H(S)$

$x(t) = e^{-t}u(t) \longleftrightarrow X(S) = \dfrac{1}{S+1}$

$h(t) = e^{-2t}u(t) \longleftrightarrow X(S) = \dfrac{1}{S+2}$.

$\therefore Y(S) = \dfrac{1}{(S+1)} \cdot \dfrac{1}{(S+2)} = \dfrac{A}{(S+1)} + \dfrac{B}{(S+2)} \Rightarrow A = 1,\ B = -1$

$= \dfrac{1}{S+1} - \dfrac{1}{S+2}$

ILT $\Rightarrow e^{-t}u(t) - e^{-2t}u(t) = y(t)$.

(10) An LTI system is defined as:

$$\frac{d^3y(t)}{dt^3} + 6\frac{d^2y(t)}{dt^2} + 11\frac{dy(t)}{dt} + 6y(t) = x(t),$$

with $x(t) = e^{-4t}u(t)$. Find $y(t)$ if $y(0^-) = 1$, $\dot{y}(0^-) = -1$ and $\ddot{y}(0^-) = 1$.

Soln:

$$\frac{d^3x(t)}{dt^3} \longleftrightarrow s^3X(s) - s^2x(0^-) - s^1\dot{x}(0^-) - \ddot{x}(0^-)$$

$$\frac{d^2x(t)}{dt^2} \longleftrightarrow s^2X(s) - s\,x(0^-) - \dot{x}(0^-)$$

$$\frac{dx(t)}{dt} \longleftrightarrow sX(s) - x(0^-).$$

By LT

$$\Rightarrow \left[s^3Y(s) - s^2\overset{1}{\cancel{y(0^-)}} - s\overset{-1}{\cancel{\dot{y}(0^-)}} - \overset{1}{\cancel{\ddot{y}(0^-)}}\right] + 6\left[s^2Y(s) - s\,\overset{1}{\cancel{y(0^-)}} - \overset{-1}{\cancel{\dot{y}(0^-)}}\right]$$

$$+ 11\left[sY(s) - \overset{1}{\cancel{y(0^-)}}\right] + 6Y(s) = \frac{1}{s+4}$$

$$\Rightarrow Y(s)\left[s^3 + 6s^2 + 11s + 6\right] - s^2 + \cancel{s} - \cancel{1} \overset{-5s}{-\cancel{6s}} + \overset{-6}{\cancel{6}} - \cancel{11} = \frac{1}{s+4}$$

$$\Rightarrow Y(s)\left[s^3 + 6s^2 + 11s + 6\right] = \frac{1}{s+4} + s^2 + 5s + 6 = \frac{s^3 + 5s^2 + 6s + 4s^2 + 20s + 24}{s+4}$$

$$\Rightarrow Y(s) = \frac{s^3 + 9s^2 + 26s + 25}{(s^3 + 6s^2 + 11s + 6)(s+4)}$$

$$= \frac{1}{s+4} + \frac{3s^2 + 15s + 19}{(s^3 + 6s^2 + 11s + 6)(s+4)}$$

$$= \frac{1}{s+4} + \frac{A}{(s+1)} + \frac{B}{s+2} + \frac{C}{s+3} + \frac{D}{s+4}$$

$$= \frac{1}{s+4} + \frac{7/6}{s+1} + \frac{-1/2}{s+2} + \frac{1/2}{s+3} + \frac{-7/6}{s+4}$$

$$= \frac{7/6}{s+1} + \frac{-1/2}{s+2} + \frac{1/2}{s+3} + \frac{-1/6}{s+4}.$$

$$\text{ILT} \Rightarrow y(t) = \left[\tfrac{7}{6}e^{-t} - \tfrac{1}{2}e^{-2t} + \tfrac{1}{2}e^{-3t} - \tfrac{1}{6}e^{-4t}\right]u(t).$$