31. Why *is* Einstein's theory called the theory of relativity? Would some other name characterize it better?

32. Is the classical concept of an incompressible fluid valid in relativity? Explain.

33. For a classical assembly of particles, the total angular momentum is the sum of the orbital and spin angular momenta. Can we regard the spin angular momentum as an example of a "proper" quantity in classical physics? (In the proper frame, the spin angular momentum equals the total angular momentum, the orbital part being zero.)

34. We have stressed the utility of relativity at high speeds. Relativity is also useful in cosmology, where great distances and long time intervals are involved. Show, from the form of the Lorentz transformation equations, why this is so.

Problems

1. (*a*) Assume, in Fig. 2-1, that S' is a train having a speed of 100 mi/hr and that it is 0.5 miles long (proper length). What is the elapsed time between the reception of the two wavefronts by O'? [Do this two ways: first by using the Lorentz transformation; second, by finding expressions for the time of receipt of the two signals by O' and subtracting. *Hint.* Remember that you are viewing the events from the ground (S) frame.] (*b*) What if the train were at rest on the tracks? What if the wavefronts traveled with infinite speed?

2. Show that Eqs. 2-6 for a_{44}, a_{11}, and a_{41} are the solutions to the equations preceding them.

3. Derive Eqs. 2-8 directly from Eqs. 2-7.

4. Suppose that an event occurs in S at $x = 100$ km, $y = 10$ km, $z = 1.0$ km at $t = 5.0 \times 10^{-6}$ seconds. Let S' move relative to S at $0.92c$ along the common x-x' axis, the origins coinciding at $t' = t = 0$. What are the coordinates x', y', z', and t' of this event in S'? Check the answer by using the inverse transformation to obtain the original data.

5. Two observers in the S frame, A and B, are separated by a distance of 60 m. Let S' move at a speed $\frac{3}{5}c$ relative to S, the origins of the two systems, O' and O, being coincident at $t' = t = 3 \times 10^{-7}$ sec $(90/c)$. The S' frame has two observers, one at A' and one at a point B' such that, according to clocks in the S frame, A' is opposite A at the same time that B' is opposite B (Fig. 2-1*a*). (*a*) What is the reading on the clock of B' when B' is opposite B? Do this twice: first, use the direct Lorentz transformation to find t'; second, use the inverse Lorentz transformation but again solve for t'. Do the answers agree? (*Careful:* x and x' are related as improper and proper lengths). (*b*) The S' system continues

moving until A' is opposite B. What is the reading on the clock of B when he is opposite A'? (c) What is the reading on the clock of A' when he is opposite B? Do this also in two ways: first, use the Lorentz transformations; second, use the concept of proper and improper time intervals. (*Note.* You may find it convenient to express time in units of $1/c$, i.e., 3×10^{-7} sec $= 90/c$ and so on.)

6. At what speed v will the Galilean and Lorentz expressions for x differ by 0.10 percent? By 1 percent? By 10 percent?

7. Prove the invariance of the electromagnetic wave equation in relativity by showing that the corresponding differential operator is an invariant. That is, show that

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}$$

when the space-time variables are related by the Lorentz transformations (see Problem 1-8).

8. Show that the proper time, given by Eq. 2-12 as $d\tau = dt\sqrt{1-\beta^2}$, is an invariant quantity with respect to a Lorentz transformation. [*Hint.* In $\beta^2 = v^2/c^2$, let $v^2 = (dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2$.]

9. Two events, one at position x_1, y_1, z_1 and another at a different position x_2, y_2, z_2 occur at the *same time* t according to observer S. (*a*) Do these events appear to be simultaneous to an observer in S' who moves relative to S at speed v? (*b*) If not, what is the time interval he measures between occurrences of these events? (c) How is this time interval affected as $v \to 0$? As the separation between events goes to zero?

10. A cart moves on a track with a constant velocity v (See Fig. 2-10). A and B are on the ends of the cart and observers C and D are stationed along the track. We define event AC as the occurrence of A passing C, and the others similarly. (*a*) Of the four events BD, BC, AD, AC, which are useful for measuring the rate of a clock carried by A for observers along the track? (*b*) Let Δt be the time interval between these two events for observers along the track. What time interval does the moving clock show? (c) Suppose that the events BC and AD are simultaneous in the track reference frame. Are they simultaneous in the cart's reference frame? If not, which is earlier?

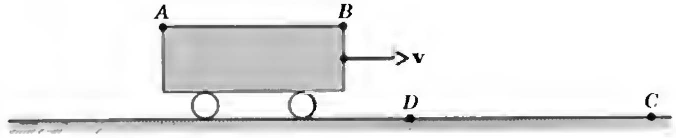

Fig 2-10. Problems 10 and 11

11. A cart moves on a track with constant velocity, as in Problem 10. Event AD is simultaneous with BC in the track frame. (*a*) The track observers set out to measure the length of the cart AB. They can do so either by using the events BD and AD and working through time measurements or by using the events BC and AB. In either case, the observers in the cart are not apt to regard these results as valid. Explain why for each case. (*b*) Suppose that the observers in the cart seek to measure the distance DC by making simultaneous marks on a long meter stick. Where (relative to A and B) would the observer, E, be situated such that AD is simultaneous with EC in the cart frame? Explain why in terms of synchronization. Can you see why there is a length contraction?

12. As seen from inertial system S an event occurs at point A on the x-axis and then 10^{-6} sec later an event occurs at point B further out on the x-axis. A and B are 600 m apart as seen from S. (*a*) Does there exist another inertial system S', moving with speed less than c parallel to the x-axis, such that the two events appear simultaneous as seen from S'? If so, what is the magnitude and direction of the velocity of S' with respect to S? What is the separation of events A and B according to S'? (*b*) Repeat part *a* for the case where A and B are only 100 m apart as seen from S.

13. What is the proper time interval between the occurrence of two events: (*a*) if in some inertial frame the events are separated by 10^9 m and occur 5 sec apart? (*b*) If . . . 7.5×10^8 m and occur 2.5 sec apart? (*c*) If . . . 5×10^8 m and occurs 1.5 sec apart?

14. In the usual set-up of frames S' and S having relative velocity v along x-x', the origins coinciding at $t = t' = 0$, we shall find that, at a later time, t, there is only one plane in S on which the clocks agree with those of S'. (*a*) Show that this plane is given by

$$x = \left(\frac{c^2}{v}\right)\left[1 - \left(1 - \frac{v^2}{c^2}\right)^{1/2}\right]t$$

and moves with velocity

$$u = \left(\frac{c^2}{v}\right)\left[1 - \left(1 - \frac{v^2}{c^2}\right)^{1/2}\right]$$

in S. (*b*) Show that this speed is less than v. (*c*) Suppose that an observer S'' moves with the velocity u relative to S. This means that clocks opposite him in S and S' will each measure the same improper time interval for an event in which an observer in the S'' frame carries the proper time. Using the result of (*b*) and the expression for time dilation, explain how this is possible.

15. (*a*) In Problem 14, let $v = \frac{3}{5}c$. Find u, the velocity of S'' relative to S. Is your answer consistent with Problem 14(*b*)? (*b*) Using this value of u in the velocity transformation equations, find the velocity of the frame S' relative to frame S''. Is your answer consistent with Problem 14(*c*)? (*c*) Prove that the result of (*b*) is a general result; that is, for any relative velocity v between frames S and S', an observer in this special frame S'' will see frame S moving with a velocity $-u$ and frame S' moving with a velocity $+u$. (*d*) Justify the result (*c*) logically using symmetry arguments and the fact that there is no preferred reference frame.

16. In our physical derivation of the length contraction (Section 2-4) we assumed that the time dilation was given. In a similar manner derive the time dilation for longitudinal light paths, assuming instead that the length contraction is given.

17. Show how the four results of the physical measurement processes of Section 2-4 can be combined to derive the Lorentz transformation equations of Section 2-2.

18. An airplane 40.0 m in length in its rest system is moving at a uniform velocity with respect to earth at a speed of 630 m/sec. (*a*) By what fraction of its rest length will it appear to be shortened to an observer on earth? (*b*) How long would it take by earth clocks for the airplane's clock to fall behind by one microsecond? (Assume that special relativity only applies).

19. The rest radius of the earth may be taken as 6400 km and its orbital speed about the sun as 30 km/sec. By how much would the earth's diameter appear to be shortened to an observer on the sun, due to the earth's orbital motion?

20. Consider a universe in which the speed of light $c = 100$ mi/hr. A Lincoln Continental traveling at a speed v relative to a fixed radar speed trap overtakes a Volkswagon traveling at the speed limit of 50 mi/hr $= c/2$. The Lincoln's speed is such that its length is measured by the fixed observer to be the same as that of the Volkswagon. By how much is the Lincoln exceeding the speed limit? The proper length of the Lincoln is twice that of the Volkswagon.

21. A 100-Mev electron, for which $\beta = 0.999975$, moves along the axis of an evacuated tube which has a length l' of 3.00 m, as measured by a laboratory observer S' with respect to whom the tube is at rest. An observer S moving with the electron would see the tube moving past at a speed v. What length would observer S measure for this tube?

22. The length of a spaceship is measured to be exactly half its proper length. (*a*) What is the speed of the spaceship relative to the observer's frame? (*b*) What is the dilation of the spaceship's unit time?

23. The radius of our galaxy is 3×10^{20} m, or about 3×10^{4} light-years. (*a*) Can a person, in principle, travel from the center to the edge of our galaxy in a normal lifetime? Explain, using either time-dilation or length-contraction arguments. (*b*) What constant velocity would he need to make the trip in 30 years (proper time)?

24. Two spaceships, each of proper length 100 m, pass near one another heading in opposite directions. If an astronaut at the front of one ship measures a time interval of 2.50×10^{-6} sec for the second ship to pass him, then (*a*) what is the relative velocity of the spaceships? (*b*) What time interval is measured on the first ship for the front of the second ship to pass from the front to the back of the first ship?

25. Suppose that a pole vaulter, holding a 16 ft long pole parallel to his direction of motion, runs through an 8 ft long shed which is open at each end. Is it possible to close sliding doors at each end of the shed such that the pole is entirely in the shed before it strikes the exit door? Discuss the situation from the point of view of the pole-vaulter and an observer on the shed roof [see Ref. 16].

26. A rod of rest length 1.0 m is moving longitudinally on a smooth table with a velocity $0.8c$ relative to the table. A circular hole of rest diameter 1.0 m lies in its path. (*a*) What is the diameter of the hole as seen by the rod? (*b*) What is the length of the rod as seen by the hole? (*c*) Does the rod fall into the hole (gravity acting) or not? Explain (see Refs. 17 and 18).

27. (*a*) If the average (proper) lifetime of a μ-meson is 2.3×10^{-6} sec, what average distance would it travel in vacuum before dying as measured in reference frames in which its velocity is $0.00c$, $0.60c$, $0.90c$, and $0.99c$? (*b*) Compare each of these distances with the distance the meson sees itself traveling through.

28. A π^+ meson is created in a high-energy collision of a primary cosmic-ray particle in the earth's atmosphere 200 km above sea level. It descends vertically at a speed of $0.99c$ and disintegrates, in its proper frame, 2.5×10^{-8} sec after its creation. At what altitude above sea level is it observed from earth to disintegrate?

29. The mean lifetime of μ-mesons stopped in a lead block in the laboratory is measured to be 2.3×10^{-6} sec. The mean lifetime of high-speed μ-mesons in a burst of cosmic rays observed from the earth is measured to be 1.6×10^{-5} sec. Find the speed of these cosmic-ray μ-mesons.

30. Laboratory experiments on μ-mesons at rest show that they have a (proper) average lifetime of about 2.3×10^{-6} sec. Such μ-mesons are produced high in the earth's atmosphere by cosmic-ray reactions and travel at a speed $0.99c$ relative to the earth a distance of from 4000 to 13000 m after formation before decaying. (*a*) Show that the average distance a μ-meson can travel before decaying is much less than even the shorter distance of 4000 m, if its lifetime in flight is only 2.3×10^{-6} sec. (*b*) Explain the consistency of the observations on length traveled and lifetime by computing the lifetime of a μ-meson in flight as measured by a ground observer. (*c*) Explain the consistency by computing the length traveled as seen by an observer at rest on the meson in its flight through the atmosphere.

31. (*a*) Derive Eq. 2-18 in the same way in which Eq. 2-19 was derived. (*b*) Derive Eq. 2-19 directly, rather than by taking the inverse of u_y'.

32. In Fig. 2-11, A and B are the points of intersection of the x-axes (stationary rod) and an inclined rod (moving rod) at two different times. The inclined rod is moving in the $+y$-direction (without turning) with a speed v. (*a*) Show that the point of intersection of the rods has a speed $u = v \cot \theta$ to the left. (*b*) Let $\theta = 60°$ and $v = \frac{1}{2}c$. Show that u then exceeds c and explain why no contradiction with relativity exists.

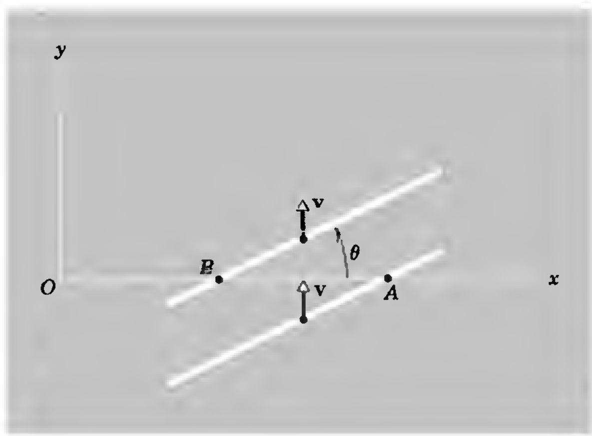

Fig 2-11. Problem 32

33. One cosmic-ray particle approaches the earth along its axis with a velocity 0.8c toward the North Pole and another with a velocity 0.6c toward the South Pole. What is the relative speed of approach of one particle with respect to the other? (*Hint.* It is useful to consider the earth and one of the particles as the two inertial systems.)

34. Suppose that a particle moves parallel to the x-x' axis, that $v = 25{,}000$ mi/hr and $u_x' = 25{,}000$ mi/hr. What percent error is made in using the Galilean rather than the Lorentz equation to calculate u_x? The speed of light is 6.7×10^8 mi/hr.

35. Consider three inertial frames of reference S, S', and S''. Let S' move with velocity v with respect to S, and let S'' move with velocity v' with respect to S'. All velocities are colinear. (*a*) Write the transformation equations relating x, y, z, t with x', y', z', t' and also those relating x', y',

z', t' with x'', y'', z'', t''. Combine these equations to get the relations between x, y, z, t and x'', y'', z'', t''. (*b*) Show that these relations are equivalent to a direct transformation from S to S'' in which the relative velocity v'' of S'' with respect to S is given by the relativistic addition theorem

$$v'' = \frac{v + v'}{1 + vv'/c^2}.$$

(*c*) Explain how the above analysis proves that two successive Lorentz transformations are equivalent to one direct transformation.

36. Suppose that a particle moves relative to the primed system with the velocity u' in the x'-y' plane so that its trajectory makes an angle θ' with the x'-axis. (*a*) Show that its equations of motion in S' are given by

$$x' = u't' \cos\theta' \qquad y' = u't' \sin\theta' \qquad z' = 0.$$

(*b*) In the S-frame, the corresponding velocity u and angle θ will be given by the equations

$$x = ut\cos\theta \qquad y = ut\sin\theta \qquad z = 0.$$

Justify this statement. (*c*) Show, using the Lorentz transformation equations, that the magnitude and direction of the velocity in S is given by

$$u^2 = \frac{u'^2 + v^2 + 2u'v\cos\theta' - (u'^2v^2/c^2)\sin^2\theta'}{[1 + (u'v/c^2)\cos\theta']^2}$$

and

$$\tan\theta = \frac{u'\sin\theta'\sqrt{1-\beta^2}}{u'\cos\theta' + v}$$

(*d*) How is this result related to the relativistic equation for the aberration of light, Eq. 2-27*a*? (*Hint.* What is u' in the case of light?) (*e*) Show that the expression for u^2 in part (*c*) is identical to that obtained by using Eqs. 2-18 and 2-19 with $u^2 = u_x^2 + u_y^2$.

37. (*a*) Show that, with $u'^2 = u_x'^2 + u_y'^2$ and $u^2 = u_x^2 + u_y^2$, we can write

$$c^2 - u^2 = \frac{c^2(c^2 - u'^2)(c^2 - v^2)}{(c^2 + u_x'v)^2}.$$

(*b*) From this result show that if $u' < c$ and $v < c$, then u must be less than c. That is, the relativistic addition of two velocities, each less than c, is itself a velocity less than c. (*c*) From this result, show that if $u' = c$ or $v = c$, then u must equal c. That is, the relativistic addition of any velocity to the velocity of light merely gives again the velocity of light [Ref. 19].

38. Consider a radioactive nucleus moving with uniform velocity $0.05c$ relative to the laboratory. (*a*) The nucleus decays by emitting an electron with a speed $0.8c$ along the direction of motion (the common x-x'

axis). Find the velocity (magnitude and direction) of the electron in the lab frame, S. (*b*) The nucleus decays by emitting an electron with speed $0.8c$ along the positive y'-axis. Find the velocity (magnitude and direction) of the electron in the lab frame. (*c*) The nucleus decays by emitting an electron with a speed $0.8c$ along the positive y-axis (i.e., perpendicular to the original motion of the nucleus in the lab frame). Find the speed of the electron in the lab frame and the direction of emission in the original rest frame of the nucleus, S'.

39. Suppose that event A causes event B in frame S, the effect being propagated with a speed *greater than c*. Show, using the velocity addition theorem, that there exists an inertial frame S', which moves relative to S with a velocity less than c, in which the order of these events would be reversed. Hence, if concepts of cause and effect are to be preserved, it is impossible to send signals with a speed greater than that of light.

40. A stick at rest in S has a length L and is inclined at an angle θ to the x-axis (see Fig. 2-12). Find its length L' and angle of inclination θ' to the x'-axis as measured by an observer in S' moving at a speed v relative to S along the x-x' axes.

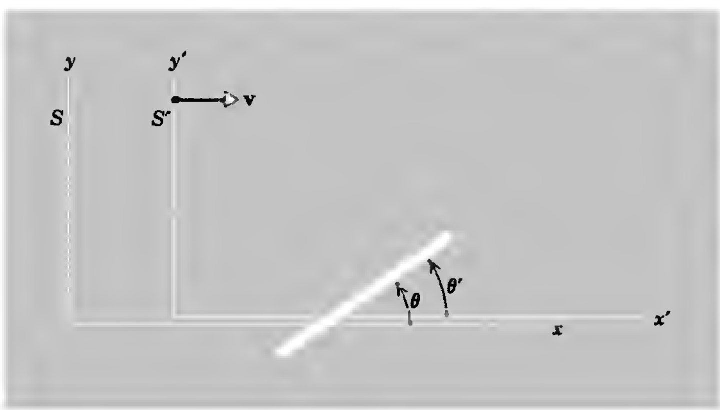

Fig 2-12. Problem 40

41. An object moves with speed u at an angle θ to the x-axis in system S. A second system S' moves with speed v relative to S along x. What speed u' and angle θ' will the object appear to have to an observer in S'?

42. Derive the relativistic acceleration transformation

$$a_x' = \frac{a_x\left(1 - \dfrac{v^2}{c^2}\right)^{3/2}}{\left(1 - \dfrac{u_x v}{c^2}\right)^3},$$

in which $a_x = du_x/dt$ and $a_x' = du_x'/dt'$. [*Hint.*

$$du_x/dt' = (du_x/dt)(dt/dt').]$$

43. Imagine a source of light emitting radiation uniformly in all directions in rest-frame S'. Find the distribution of radiation in the laboratory frame S in which the source moves at a speed $\frac{4}{5}c$. (*Hint.* Find the corresponding angle θ for $\theta' = 0, 30, 60, 90, 120, 150$, and $180°$. A polar graph plot of the data would be helpful.) Can you guess why this phenomena is often referred to as the "headlight effect"?

44. A, on earth, signals with a flashlight every six minutes. B is on a space station that is stationary with respect to the earth. C is on a rocket traveling from A to B with a constant velocity of $0.6c$ relative to A (see Fig. 2-13). (*a*) At what intervals does B receive the signals from A? (*b*) At what intervals does C receive signals from A? (*c*) If C flashes a light using intervals equal to those he received from A, at what intervals does B receive C's flashes?

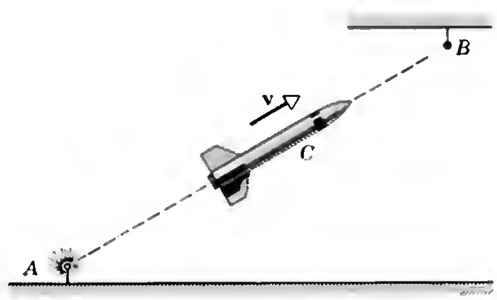

Fig 2-13. Problem 44

45. A radar transmitter (T) is fixed to a system S_2 which is moving to the right with speed v relative to system S_1 (see Fig. 2-14). A timer in S_2, having a period τ_0 (measured in S_2) causes transmitter T to emit radar pulses, which travel at the speed of light, and are received by R, a receiver fixed to S_1. (*a*) What would be the period (τ) of the timer relative to observers A and B, spaced a distance $v\tau$ apart? (*b*) Show that the receiver R would observe the time interval between pulses arriving from S_2 not as τ or as τ_0 but as $\tau' = \tau_0\sqrt{(c+v)/(c-v)}$. (*c*) Explain why the observer at R measures a different period for the transmitter than do observers A and B who are in his own reference frame. (*Hint.* Compare the events measured by R to the events measured by A and B. What is meant by the proper time in each case?)

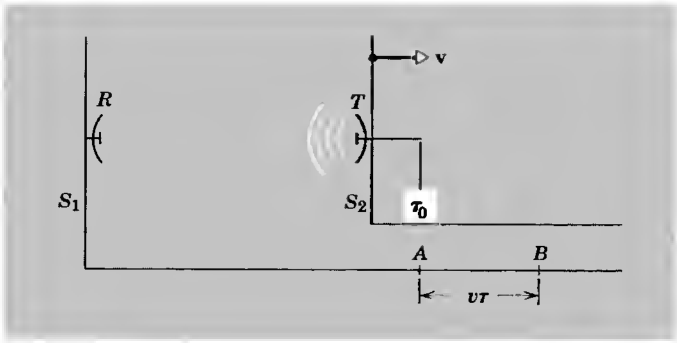

Fig 2-14. Problem 45

46. In the case of wave propagation in a medium, the Doppler shifts for the case of source moving through medium and observer moving through medium are different, whereas for light in vacuo the two situations are equivalent. Show that if we take the geometric mean of the two former results, we get exactly the relativistic Doppler shift (see Section 40-5, Ref. 10).

47. A rocketship is receding from the earth at a speed of $0.2c$. A light in the rocketship appears blue to passengers on the ship. What color would it appear to be to an observer on the earth?

48. Give the wavelength shifts in the relativistic longitudinal Doppler effect for the sodium D_1 line (5896 Å) for source and observer approaching at relative velocities of $0.1c$, $0.4c$, $0.8c$. Is the classical (first-order) result a good approximation?

49. Give the wavelength shift in the relativistic Doppler effect for the 6563 Å H_α line emitted by a star receding from the earth with a relative velocity $10^{-3}c$, $10^{-2}c$, and $10^{-1}c$. Is the classical (first-order) result a good approximation?

50. Give the wavelength shift, if any, in the Doppler effect for the sodium D_2 line (5890 Å) emitted from a source moving in a circle with constant speed $0.1c$ measured by an observer fixed at the center of the circle.

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