MA221 - Integral Transforms
Assignment

SAURABH KUMAR
SC22B146

Tutorial - 2

③ (a) Show that $f(x) = \cos x$ is of exponential order for any $\sigma \geq \sigma_c = 0$.

Soln: As $|\cos x| \leq 1 = 1 \times e^{0x} \quad \forall x$

$$\Rightarrow \cos x \leq |\cos x| \leq 1 e^{0x} \quad \forall x$$

$$\Rightarrow e^{-0x} \cos x \leq 1 \quad \forall x$$

Thus, $f(x) = \cos x$ is of exponential order with $\sigma \geq \sigma_c = 0$ and $M = 1$.

(b) without using definition, show that $f(x) = (x+1).\cos(x+1)$ possesses the laplace transform $\hat{f}(s)$ for $s > 0$. Also, compute $\hat{f}(s)$.

Soln: As $(x+1)$ and $\cos(x+1)$ are continuous functions, they are also piecewise-continuous.

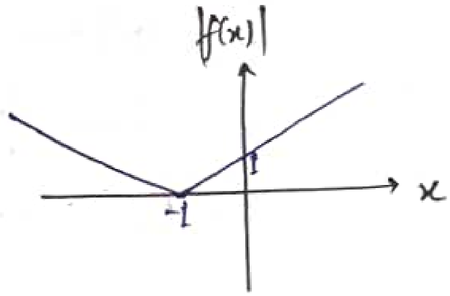
$\Rightarrow f(x)$ is also piecewise-continuous.

Also,

$$|\cos(x+1)| \leq 1$$

$$\Rightarrow |(x+1).\cos(x+1)| \leq |x+1|$$

$$\Rightarrow |f(x)| \leq |x+1| \leq 2.e^{\sigma x} \quad \forall x \geq 0$$



Here, $M = 2$, $\sigma \geq \sigma_c = 0$.

$\therefore f$ is of exponential order with $\sigma_c = 0$.

Now, as (i) f is P.C on $[0, \infty)$,

and (ii) f is of exponential order with $\sigma_c = 0$.

$\Rightarrow LT(f) = \hat{f}(s)$ exists for $s > \sigma_c = 0$.

Now, $f(x) = (x+1).\cos(x+1)$

$$\cos(x) \xleftrightarrow{LT} \frac{s}{s^2+1}$$

$$x\cos(x) \xleftrightarrow{LT} -\frac{d}{ds}\left(\frac{s}{s^2+1}\right) = -\frac{(s^2+1) - s(2s)}{(s^2+1)^2}$$

$$= \frac{s^2-1}{(s^2+1)^2}$$

$$(x+1)\cos(x+1) \xleftrightarrow{LT} e^{1.s}\,\frac{s^2-1}{(s^2+1)^2}$$

$$\therefore \hat{f}(s) = e^{s}.\,\frac{s^2-1}{(s^2+1)^2}.$$

Tutorial-3

(3) Suppose the laplace transform of a function $f:[0,\infty)\to\mathbb{R}$ is given by $\mathcal{L}_T(f(x)) = \hat{f}(s)$.

(a) Show that $\mathcal{L}_T\left(\int_0^x f(t)\,dt\right) = \frac{\hat{f}(s)}{s}$.

Soln: Let $g(x) = \int_0^x f(t)\,dt$

$$\Rightarrow \frac{d\,g(x)}{dx} = f(x)$$

$$\mathcal{L}T \Rightarrow \mathcal{L}T\left\{\frac{dg(x)}{dx}\right\} = \hat{f}(s)$$

$$\Rightarrow s\hat{g}(s) - g(0) = \hat{f}(s)$$

Now, $g(0) = \int_0^0 f(t)\,dt = 0$.

$$\Rightarrow s\hat{g}(s) = \hat{f}(s)$$

$$\Rightarrow \hat{g}(s) = \frac{\hat{f}(s)}{s}$$

$$\Rightarrow \boxed{\mathcal{L}_T\left(\int_0^x f(t)\,dt\right) = \frac{\hat{f}(s)}{s}}$$

(b) Using this, determine the inverse Laplace transform of $\frac{1}{s(s^2+1)}$.

Soln: As $\mathcal{L}_T(\sin x) = \frac{1}{s^2+1}$,

$$\Rightarrow \mathcal{L}_T\left(\int_0^x \sin t\,dt\right) = \frac{1}{s}\cdot\frac{1}{s^2+1} \qquad \text{[Using the property]}$$

$$\Rightarrow \mathcal{L}_T\left[-\cos t\Big|_0^x\right] = \frac{1}{s}\cdot\frac{1}{s^2+1}$$

$$\Rightarrow \mathcal{L}_T\left[1-\cos x\right] = \frac{1}{s}\cdot\frac{1}{s^2+1}$$

Take ILT $\Rightarrow$ $\mathcal{L}T^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s^2+1}\right\} = (1-\cos x)$.

⑦ Use Laplace transforms to solve the following initial-value-problems.

ⓑ $x\frac{d^2y}{dx^2} - \frac{dy}{dx} = -1$, $x>0$ with $y(0)=0$ and $\frac{dy}{dx}(0) = B$, where B is a constant.

Sol^n: Using property of LT of derivatives,

$$LT\left\{x\frac{d^2y}{dx^2}\right\} - LT\left\{\frac{dy}{dx}\right\} = LT\{-1\}, \text{ for } x>0$$

$$\Rightarrow -\frac{d}{ds}\left\{s^2 Y(s) - s\cancelto{0}{y(0)} - \cancelto{B}{y'(0)}\right\} - \left[sY(s) - \cancelto{0}{y(0)}\right] = -\frac{1}{s}$$

$$\Rightarrow -s^2Y'(s) - 2sY(s) - sY(s) = -\frac{1}{s}$$

$$\Rightarrow Y'(s) + \frac{2}{s}Y(s) + \frac{1}{s}Y(s) = \frac{1}{s^3}$$

$$\Rightarrow Y'(s) + \frac{3}{s}Y(s) = \frac{1}{s^3} \longrightarrow \text{ODE(linear)}$$

Here, $I.F. = e^{\int \frac{3}{s}ds} = e^{3\ln s} = s^3$

$$\therefore Y(s).s^3 = \int \frac{1}{s^3}.s^3\,ds = s + c, \ c: \text{constant}.$$

$$\Rightarrow Y(s) = \frac{s+c}{s^3} = \frac{1}{s^2} + \frac{c}{s^3}.$$

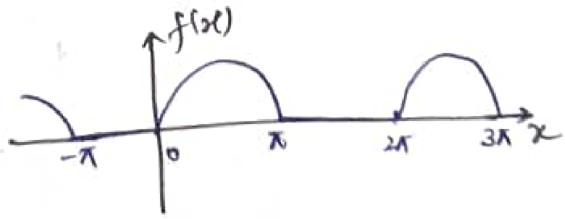
Taking ILT on both sides,

$$y(x) = x + cx^2, \text{ where } c \text{ is a constant}.$$

⑧ Use Laplace transform to solve the following initial-value-problems.

ⓑ $\frac{d^2y}{dx^2} + y = f(x)$, $x>0$ with $y(0)=1$, $\frac{dy}{dx}(0) = 0$, where

ⓘ $f(x) = \begin{cases} \sin x, & \text{for } 0 \le x < \pi \\ 0, & \text{for } \pi \le x < 2\pi, \end{cases}$ such that $f(x+2\pi) = f(x)$.



Sol^n: $f(x)$ is periodic with period $T = 2\pi$.

Now, $\int_0^{T=2\pi} e^{-sx} f(x)\,dx$

$$= \int_0^{\pi} \sin x . e^{-sx}dx + \cancelto{0}{\int_{\pi}^{2\pi} 0.e^{-sx}dx} \quad -①$$

Take $I = \int_0^{\pi} \sin x \cdot e^{-sx}\, dx$

$$= \sin x \frac{e^{-sx}}{-s} + \frac{1}{s}\int \cos x\, e^{-sx}\, dx \Big|_0^{\pi}$$

$$= -\frac{\sin x}{s} \cdot e^{-sx} + \frac{1}{s}\left[+ \frac{\cos x\, e^{-sx}}{-s} + \frac{-1}{s}\underbrace{\int e^{-sx} \sin x\, dx}_{I}\right]\Big|_0^{\pi}$$

$$\Rightarrow I\left(1+\frac{1}{s^2}\right) = -\frac{e^{-sx}}{s} \cdot \sin x - \frac{e^{-sx}}{s^2}\cos x \Big|_0^{\pi}$$

$$= +\frac{e^{-s\pi}}{s^2} - \left[-\frac{1}{s^2}\right]$$

$$= \frac{e^{-s\pi}}{s^2} + \frac{1}{s^2} = \frac{e^{-s\pi}+1}{s^2}$$

$$\Rightarrow I(s^2+1) = 1 + e^{-s\pi} \Rightarrow I = \frac{1+e^{-s\pi}}{s^2+1} \quad -(2)$$

Now, $LT\{f(x)\} = \frac{1}{1-e^{-2s\pi}} \int_0^{2\pi} e^{-sx} \cdot f(x)\, dx$

$$= \frac{1+e^{-s\pi}}{(1-e^{-2s\pi})(s^2+1)} \quad -(3)$$

Now, $\frac{d^2y}{dx^2} + y = f(x)$.

Take LT on both sides,

$$s^2 Y(s) - s\cancel{y(0)}^{1} - \cancel{y'(0)}^{0} + Y(s) = \frac{1+e^{-s\pi}}{(s^2+1)(1-e^{-2s\pi})}$$

$$\Rightarrow Y(s)\left[s^2+1\right] = s + \frac{1+e^{-s\pi}}{(s^2+1)(1-e^{-2s\pi})}$$

$$\Rightarrow Y(s) = \frac{s}{s^2+1} + \frac{1+e^{-s\pi}}{(s^2+1)^2(1-e^{-2s\pi})}$$

$$= \frac{s}{s^2+1} + \frac{1}{(s^2+1)^2(1-e^{-\pi s})} \quad \left[\because (1-e^{-2\pi s}) = (1+e^{-\pi s})(1-e^{-\pi s})\right]$$

$$= \frac{s}{s^2+1} + \sum_{n=0}^{\infty} e^{-n\pi s} \cdot \frac{1}{(s^2+1)^2} \quad -(4) \quad \left[\because \sum_{n=0}^{\infty} e^{-n\pi s} = \frac{1}{1-e^{-\pi s}}\right]$$

Now, to find $L_T^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$,

as $L_T^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$.

$$\Rightarrow L_T(\sin t\, u(t) * \sin(t)\, u(t)) = \frac{1}{(s^2+1)^2} \quad [\text{Using convolution}]$$

Let $g(t) = \int_{-\infty}^{\infty} \sin(t-\tau)\, u(t-\tau) \sin(\tau)\, u(\tau)\, d\tau$ $\quad [= \sin t\, u(t) * \sin(t) u(t)]$

$$= \int_0^t \sin(t-\tau) . \sin\tau\, d\tau$$

$$= \frac{1}{2}\left[\int_0^t \cos(2\tau - t) - \int_0^t \cos t\, d\tau\right]$$

$$= \frac{1}{2}\left[\frac{\sin(2\tau-t)}{2}\right]_0^t - \frac{1}{2}\cos t\, [\tau]_0^t$$

$$\Rightarrow g(t) = \frac{\sin t}{2} - \frac{t\cos t}{2}$$

$$\Rightarrow g(x) = \frac{\sin x}{2} - \frac{x\cos x}{2}$$

Now, using $L_T\{g(x-a)\} = e^{-as}\hat{g}(s)$,

as $L_T\{g(x)\} = L_T\left\{\frac{\sin x}{2} - \frac{x\cos x}{2}\right\} = \frac{1}{(s^2+1)^2}$

$$\Rightarrow L_T\{g(x-n\pi)\} = L_T\left\{\frac{\sin(x-n\pi)}{2} - \frac{(x-n\pi)\cos(x-n\pi)}{2}\right\}$$

$$= \frac{e^{-n\pi s}}{(s^2+1)^2}$$

Now, taking inverse LT both sides in eqn ④,

$$Y(s) = \frac{s}{s^2+1} + \sum_{n=0}^{\infty} e^{-n\pi s} . \frac{1}{(s^2+1)^2}$$

$$\text{ILT} \Rightarrow \boxed{y(x) = \cos x + \sum_{n=0}^{\infty} \frac{\sin(x-n\pi) - (x-n\pi)\cos(x-n\pi)}{2}}.$$