Indian Institute of Space Science and Technology

Vector Calculus   Spring 2023   Directional Derivatives   Tutorial-I

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Directional derivatives

1. Consider $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $f(x, y) := ||x| - |y|| - |x| - |y|$. Determine whether (i) f is continuous at $(0, 0)$, (ii) the partial derivatives $D_x f|_{(0,0)}$ and $D_y f|_{(0,0)}$ exist, and (iii) the directional derivative $D_{\vec{v}} f|_{(0,0)}$ exists. Is f differentiable at $(0, 0)$? Justify your answer.

2. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by $f(x, y) := 0$ if $xy = 0$, and $f(x, y) := 1$ otherwise. Show that f is not continuous at $(0, 0)$ although both the partial derivatives of f exist at $(0, 0)$.

3. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by $f(x, y) := x^2 + y^2$ if x and y are both rational, and $f(x, y) := 0$ otherwise. Determine the points of $\mathbb{R}^2$ at which (i) $D_x f$ exists, (ii) $D_y f$ exists.

4. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by one of the following functions. Check if $D_{\vec{v}} f|_{(0,0)}$ exists for any unit vector $\vec{v}$. Is f continuous at $(0, 0)$? Is f differentiable at $(0, 0)$?
(i) $f(x, y) = \sqrt{x^2 + y^2}$, (ii) $f(x, y) = |x| + |y|$

5. Consider $f : \mathbb{R}^2 \longrightarrow R$ defined by $f(0, 0) := 0$ and for $(x, y) \neq (0, 0)$, by one of the following. In each case, determine whether the directional derivative $D_{\vec{v}} f|_{(0,0)}$ exists for any unit vector $\vec{v}$ in $\mathbb{R}^2$. If it does, then check whether $D_{\vec{v}} f|_{(0,0)} = \langle \nabla f_{(0,0)}, \vec{v} \rangle$ for a unit vector $\vec{v}$ in $\mathbb{R}^2$. Finally, determine whether f is differentiable at $(0, 0)$.
(i) $\dfrac{x^2 y}{x^2 + y^2}$, (ii) $xy\dfrac{x^2 - y^2}{x^2 + y^2}$, (iii) $\dfrac{x^3}{x^2 + y^2}$, (iv) $\frac{xy^2}{x^4+y^2}$, (v) $\ln(x^2+y^2)$, (vi) $xy\ln(x^2+y^2)$, (vii) $\dfrac{xy}{x^2 + y^2}$.

6. Consider $f : \mathbb{R}^2 \longrightarrow R$ defined by $f(x, y) := (y/|y|)\sqrt{x^2 + y^2}$ if $y \neq 0$, and $f(x, y) := 0$ if $y = 0$. Show that f is continuous at $(0, 0)$, $D_{\vec{v}} f|_{(0,0)}$ exists for every unit vector $\vec{v}$ in R^2, but f is not differentiable at $(0, 0)$.

7. show that the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $f(x, y) = \dfrac{x^2 y^2}{x^4 + y^2}$ for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$ is differentiable at $(0, 0)$.

8. Starting from $(1, 1)$, in which direction should one travel in order to obtain the most rapid rate of decrease of the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $f(x, y) := (x + y - 2)^2 + (3x - y - 6)^2$?

9. About how much will the function $f(x, y) := \ln\sqrt{x^2 + y^2}$ change if the point (x, y) is moved from $(3, 4)$ a distance 0.1 unit straight toward $(3, 6)$?

10. Consider $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $f(x, y) := (x + y)/\sqrt{2}$ if $x = y$, and $f(x, y) := 0$ otherwise. Show that $D_x f|_{(0,0)} = D_x f|_{(0,0)} = 0$ and $D_{\vec{v}} f|_{(0,0)} = 1$, where $\vec{v} = (1/\sqrt{2}, 1/\sqrt{2})$. Deduce that f is not differentiable at $(0, 0)$.

11. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ a C^1-type function. Define $\phi(x, y) = \lim_{h\to 0} \dfrac{f(hx, hy) - f(0, 0)}{h}$ for all $(x, y) \in \mathbb{R}^2$ satisfying $x^2 + y^2 = 1$. Prove that the function ϕ exits, i.e., the given limit exists. Show that for any constant $\alpha \in \mathbb{R}$, the level curve $L := \{(x, y) \in \mathbb{R} \mid \phi(x, y) = \alpha\}$ represents a straight line. Find the normal vector at any point of this level curve.

12. Find the directional derivative, if exists, of the given function in the given point in the indicated direction
(i) $x^2y - y^2z - xyz$, at $(1, -1, 0)$ in the direction $(\hat{\imath} - \hat{\jmath} + 2\hat{k})$, (ii) $(x^2 + y^2 + z^2)^{\frac{3}{2}}$, at $(-1, 1, 2)$ in the direction $(\hat{\imath} - 2\hat{\jmath} + \hat{k})$, (iii) $e^x - yz$, at $(1, 1, 1)$ in the direction $(\hat{\imath} - \hat{\jmath} + \hat{k})$.

13. Let $h(x, y) = 2e^{-x^2} + e^{-3y^2}$ denote the height on a mountain at position (x, y). In what direction from $(1, 0)$ should one begin walking in order to climb the fastest?
(i) $x^2 + y^2 + z^2 = 9$ at $(0, \sqrt{3}, \sqrt{3})$, (ii) $x^3y^3 + y - z + 2 = 0$ at $(0, 0, 2)$, (iii) $z = 1/(x^2 + y^2)$ at $(1, 1, 1/2)$.

14. **(Theory)** Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a function. Show that f is differentiable at x_0 if and only if exists $\alpha \in \mathbb{R}$ such that $\lim_{h\longrightarrow 0} \dfrac{f(x_0 + h) - f(x_0) - \alpha h}{|h|} = 0$.

15. **(Theory)** Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a function. We know that f is differentiable at a point $(x_0, y_0) \in \mathbb{R}^2$ if there exist $\alpha, \beta \in \mathbb{R}$ such that $\lim_{(h,k)\longrightarrow(0,0)} \dfrac{f(x_0 + h, y_0 + k) - f(x_0, y_0) - \alpha h - \beta k}{\sqrt{h^2 + k^2}} = 0$. Show that if f is differentiable at (x_0, y_0), then α and β are the partial derivates of f at (x_0, y_0).

16. **(Theory)** Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is differentiable at a point (x_0, y_0). Show that f is continuous at (x_0, y_0).