UNIT 6 HIGHER ORDER PARTIAL DERIVATIVES

Structure

6.1 INTRODUCTION

In the last unit you studied partial derivatives of first order and differentiability of functions of several variables. You must have seen often, that partial derivatives of first order again define functions. For instance, if $f(x,y) = 3x^3+2xy^2+5y^2+6$, then $f_x(x,y) = 9x^2+2y^2$ and $f_y(x,y) = 4xy+10y$ are again real-valued functions of two variables with the domain $\mathbf{R}^2$. Thus we can talk of first order partial derivatives of these new functions. If we consider a function of two variables, there are two first order partial derivatives, which may give rise to four more partial derivatives, which might again turn out to be functions. If this chain continues, then we obtain higher order partial derivatives which constitute the subject matter of this unit. We shall be using these partial derivatives in the next block. In this unit you will study Euler's, Schwarz's and Young's theorems, which give some sets of conditions under which the mixed partial derivatives become equal.

Objectives

After studying this unit, you should be able to

- define and evaluate higher order partial derivatives,
- state and prove Euler's theorem,
- state Schwarz's and Young's theorems,
- decide about the commutativity of the operations of taking partial derivatives with respect to different variables.

6.2 HIGHER ORDER PARTIAL DERIVATIVES

In the introduction you have seen that the partial derivative f_x of the function $f(x, y) = 3x^3 + 2xy^2 + 5y^2 + 6$ is again a function of x and y. In general, let $D \subset \mathbf{R}^2$ and let $f : D \to \mathbf{R}$ have a first order partial derivative f_x at every point of D. Then we get a new function, say $g = f_x$, which is defined on D. This new function g may or may not possess first order partial derivatives. In case it does, then g_x and g_y are called the second order partial derivatives of f and are denoted by f_{xx} and f_{xy}, respectively. Similarly, if the function f has a first order partial derivative f_y at every point of D, then f_y defines a new function. And if this new function has first order partial derivatives, then we get two more second order partial derivatives, namely, f_{yx} and f_{yy}. Thus, if f(x, y) is a real-valued function defined in a neighbourhood of (a, b) having both the partial derivatives at all the points of the neighbourhood, then

$$f_{xx}(a, b) = \lim_{h\to 0} \frac{f_x(a+h, b) - f_x(a, b)}{h}$$

$$f_{xy}(a, b) = \lim_{k\to 0} \frac{f_x(a, b+k) - f_x(a, b)}{k}$$

$$f_{yx}(a, b) = \lim_{h \to 0} \frac{f_y(a+h, b) - f_y(a, b)}{h}$$

$$f_{yy}(a, b) = \lim_{k \to 0} \frac{f_y(a, b+k) - f_y(a, b)}{k}$$

provided each one of these limits exists.

We also denote the second order partial derivatives of f by

f_{xy}, f_{yx} are also called **mixed partial derivatives.**

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}\ ;\ f_{xy} = \frac{\partial^2 f}{\partial y\, \partial x}\ ;$$

$$f_{yx} = \frac{\partial^2 f}{\partial x\, \partial y}\ ;\ f_{yy} = \frac{\partial^2 f}{\partial y^2}\ .$$

If we want to indicate the particular point at which the second order partial derivatives are taken, then we write

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_{(a,\, b)},\ \frac{\partial^2 f(a, b)}{\partial x^2},\ f_{xx}(a, b),\ \left(\frac{\partial^2 f}{\partial x\, \partial y}\right)_{(a,\, b)}$$

$\dfrac{\partial^2 f(a, b)}{\partial x\, \partial y}$, $f_{xy}(a, b)$, and so on.

In a similar manner partial derivatives of order higher than two are defined. For example,

$$\frac{\partial^3 f}{\partial x\, \partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x\, \partial y}\right) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\right]$$

i.e., $\dfrac{\partial^3 f}{\partial x\, \partial x\, \partial y}$ stands for the partial derivative of $\dfrac{\partial^2 f}{\partial x\, \partial y}$ with respect to x and is written as $\dfrac{\partial^3 f}{\partial^2 x\, \partial y}$.

Similarly, we can extend the idea of partial derivatives of higher orders to functions of more than two variables. In general, if f is a function of n variables $x_1, x_2, \ldots, x_n$ defined on $D \subset \mathbf{R}^n$, then $\dfrac{\partial^2 f}{\partial x_i\, \partial x_j}$ denotes the second order partial derivative of f with respect to x_i and x_j, obtained by differentiating partially the partial derivative $\dfrac{\partial f}{\partial x_j}$ with respect to x_i. Further, $\dfrac{\partial^3 f}{\partial x_i\, \partial x_j\, \partial x_k}$ will denote the third order partial derivative of f with respect to the variables x_i, x_j and x_k, obtained by partial differentiation of $\dfrac{\partial^2 f}{\partial x_j\, \partial x_k}$ with respect to the variable x_i and so on.

In the following examples, we show how to calculate these higher order partial derivatives.

Example 1 : Let us find all the second order partial derivatives of the following functions:

i) $U(x, y) = x^3 + y^3 + 3axy$, a is constant,

ii) $U(x, y, z) = x^2 + yz + xz^3$.

Let's take these one by one.

i) Clearly, for $U(x, y) = x^3 + y^3 + 3axy$,

$\dfrac{\partial U}{\partial x} = 3x^2 + 3ay$ and $\dfrac{\partial U}{\partial y} = 3y^2 + 3ax$. Therefore,

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial}{\partial x}(3x^2 + 3ay) = 6x,$$

$$\frac{\partial^2 U}{\partial y\,\partial x} = \frac{\partial}{\partial y}(3x^2 + 3ay) = 3a = \frac{\partial^2 U}{\partial x\,\partial y} = \frac{\partial}{\partial x}(3y^2 + 3ax) \text{ and}$$

$$\frac{\partial^2 U}{\partial y^2} = \frac{\partial}{\partial y}(3y^2 + 3ax) = 6y.$$

ii) For $U(x, y, z) = x^2 + yz + xz^3$

$\frac{\partial U}{\partial x} = 2x + z^3$, $\frac{\partial U}{\partial y} = z$ and $\frac{\partial U}{\partial z} = y + 3xz^2$. Therefore,

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\right) = \frac{\partial}{\partial x}(2x + z^3) = 2,$$

$$\frac{\partial^2 U}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial U}{\partial x}\right) = \frac{\partial}{\partial y}(2x + z^3) = 0$$

$$\frac{\partial^2 U}{\partial z\,\partial x} = \frac{\partial}{\partial z}\left(\frac{\partial U}{\partial x}\right) = \frac{\partial}{\partial z}(2x + z^3) = 3z^2$$

$$\frac{\partial^2 U}{\partial x\,\partial y} = \frac{\partial}{\partial x}(z) = 0$$

$$\frac{\partial^2 U}{\partial y^2} = \frac{\partial}{\partial y}(z) = 0$$

$$\frac{\partial^2 U}{\partial z\,\partial y} = \frac{\partial}{\partial z}(z) = 1$$

$$\frac{\partial^2 U}{\partial x\,\partial z} = \frac{\partial}{\partial x}(y + 3xz^2) = 3z^2$$

$$\frac{\partial^2 U}{\partial y\,\partial z} = \frac{\partial}{\partial y}(y + 3xz^2) = 1$$

$$\frac{\partial^2 U}{\partial z^2} = \frac{\partial}{\partial z}(y + 3xz^2) = 6xz.$$

Example 2 : If $f(x, y) = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$, $x \neq 0$, $y \neq 0$,

we will prove that $\frac{\partial^2 f}{\partial x\,\partial y} = \frac{x^2 - y^2}{x^2 + y^2}$.

Here, $\frac{\partial f}{\partial y} = x^2 . \frac{1}{1 + y^2/x^2} . \frac{1}{x} - 2y \tan^{-1}\frac{x}{y} - y^2 . \frac{1}{1 + x^2/y^2}\left(-\frac{x}{y^2}\right)$

$$= \frac{x^3}{x^2 + y^2} + \frac{xy^2}{x^2 + y^2} - 2y \tan^{-1}\frac{x}{y}$$

$$= x - 2y \tan^{-1}\frac{x}{y}$$

And therefore,

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(x - 2y \tan^{-1}\frac{x}{y}\right)$$

$$= 1 - 2y . \frac{1}{1 + x^2/y^2} . \frac{1}{y}$$

$$= 1 - \frac{2y^2}{x^2 + y^2}$$

$$= \frac{x^2 - y^2}{x^2 + y^2}$$

In the next example we go a step further and calculate a third order partial derivative.

Example 3 : If $u(x, y, z) = e^{xyz}$, then we can show that

$$\frac{\partial^3 u}{\partial x\, \partial y\, \partial z} = (1 + 3xyz + x^2y^2z^2)\, e^{xyz}.$$

Now $u(x, y, z) = e^{xyz}$. Therefore,

$$\frac{\partial u}{\partial z} = xye^{xyz},$$

$$\frac{\partial^2 u}{\partial y\, \partial z} = x\, e^{xyz} + x^2yz\, e^{xyz}, \text{ and}$$

$$\frac{\partial^3 u}{\partial x\, \partial y\, \partial z} = e^{xyz} + xyze^{xyz} + 2xyz\, e^{xyz} + x^2y^2z^2\, e^{xyz}$$

$$= (1 + 3xyz + x^2y^2z^2)\, e^{xyz}.$$

We are sure you will be able to solve these exercises now.

E1) Find all the second order partial derivatives of the following functions.

a) $f(x, y) = \cos \dfrac{y}{x}$;

b) $f(x, y) = x^5 + y^4 \sin x^6$

c) $f(x, y, z) = \sin xy + \sin yz + \cos xz$

d) $f(x, y, z) = xyz^2 + xyz + x^3y$

E2) If $V(x, y, z) = (x^2+y^2+z^2)^{-1/2}$, show that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

E3) Verify that $\dfrac{\partial^2 f}{\partial x\, \partial y} = \dfrac{\partial^2 f}{\partial y\, \partial x}$ for each of the following functions.

a) $f(x, y) = x^3y + e^{xy^2}$

b) $f(x, y) = \tan(xy^3)$

E4) If $x^x y^y z^z = c$, show that at $x = y = z$, $\dfrac{\partial^2 z}{\partial x\, \partial y} = -(x \ln ex)^{-1}$.

(Hint : Take logarithms on both sides and differentiate.)

In Unit 5 you have seen that it is not always possible to find first order partial derivatives by direct differentiation (See Examples 5 and 6 of Unit 5). The same is true for higher order partial derivatives of some functions. This is illustrated by the following examples.

Example 4 : Consider the function

$$f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

We'll now evaluate the second order partial derivatives of f at (0, 0).

Since $f_{xx}(0, 0) = \lim\limits_{h \to 0} \dfrac{f_x(h, 0) - f_x(0, 0)}{h}$, we have to first evaluate $f_x(h, 0)$ and $f_x(0, 0)$.

$$f_x(0, 0) = \lim_{t \to 0} \frac{f(t, 0) - f(0, 0)}{t}$$

$$= \lim_{t \to 0} \frac{0 - 0}{t} = 0.$$

$$f_x(h, 0) = \lim_{t \to 0} \frac{f(h+t, 0) - f(h, 0)}{t}$$

$$= \lim_{t \to 0} \frac{0 - 0}{t} = 0.$$

Therefore,

$$f_{xx}(0, 0) = \lim_{h \to 0} \frac{0 - 0}{h} = 0.$$

Since $f_{xy}(0, 0) = \lim_{k \to 0} \dfrac{f_x(0, k) - f_x(0, 0)}{k}$, we must first evaluate $f_x(0, k)$.

Now, $f_x(0, k) = \lim_{t \to 0} \dfrac{f(t, k) - f(0, k)}{t}$

$$= \lim_{t \to 0} \frac{\dfrac{tk(t^2 - k^2)}{t^2 + k^2} - 0}{t}$$

$$= \lim_{t \to 0} \frac{k(t^2 - k^2)}{t^2 + k^2}$$

$$= -\frac{k^3}{k^2}$$

$$= -k.$$

Therefore, $f_{xy}(0, 0) = \lim_{k \to 0} \dfrac{-k - 0}{k} = -1.$

Since $f_{yx}(0, 0) = \lim_{h \to 0} \dfrac{f_y(h, 0) - f_y(0, 0)}{h}$, we first evaluate $f_y(h, 0)$ and $f_y(0, 0)$.

Now, $f_y(0, 0) = \lim_{s \to 0} \dfrac{f(0, s) - f(0, 0)}{s} = \lim_{s \to 0} \dfrac{0 - 0}{s} = 0.$

$$f_y(h, 0) = \lim_{s \to 0} \frac{f(h, s) - f(h, 0)}{s}$$

$$= \lim_{s \to 0} \frac{\dfrac{hs(h^2 - s^2)}{h^2 + s^2} - 0}{s}$$

$$= \lim_{s \to 0} \frac{h(h^2 - s^2)}{h^2 + s^2}$$

$$= \frac{h^3}{h^2}$$

$$= h.$$

Therefore, $f_{yx}(0, 0) = \lim_{h \to 0} \dfrac{h - 0}{h} = 1.$

Since, $f_{yy}(0, 0) = \lim_{k \to 0} \dfrac{f_y(0, k) - f_y(0, 0)}{k}$, we first evaluate $f_y(0, k)$.

Now, $f_y(0, k) = \lim_{s \to 0} \dfrac{f(0, k+s) - f(0, k)}{s} = \lim_{s \to 0} \dfrac{0 - 0}{s} = 0.$

Therefore, $f_{yy}(0, 0) = \lim_{k \to 0} \frac{0 - 0}{k} = 0.$

Thus, you can see that to evaluate the partial derivatives of this function, we had to resort to the definition of partial derivatives, and direct differentiation was not possible.

In the next example we take up a function which is slightly more complicated.

Example 5 : Let us evaluate $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$, for the function f given by

$$f(x, y) = \begin{cases} (x^4 + y^4) \tan^{-1}(y^2/x^2), & x \neq 0 \\ \dfrac{\pi y^4}{2}, & x = 0 \end{cases}$$

We first note that

$$f_x(0, 0) = \lim_{t \to 0} \frac{f(t, 0) - f(0, 0)}{t} = \lim_{t \to 0} \frac{0 - 0}{t} = 0, \text{ and}$$

$$f_x(0, k) = \lim_{t \to 0} \frac{f(t, k) - f(0, k)}{t}$$

$$= \lim_{t \to 0} \frac{(t^4 + k^4) \tan^{-1}(k^2/t^2) - \pi k^4/2}{t}$$

We have applied L'Hopital's rule here, since $\frac{(t^4+k^4) \tan^{-1}(k^2/t^2) - \pi k^4/2}{t}$ is in the $\frac{0}{0}$ form as $t \to 0$.

By L'Hopital's rule, we have

$$f_x(0, k) = \lim_{t \to 0} \frac{4t^3 \tan^{-1} \dfrac{k^2}{t^2} + (k^4+t^4) \cdot \dfrac{1}{1+(k^4/t^4)} \left(-\dfrac{2k^2}{t^3}\right)}{1}$$

$$= \lim_{t \to 0} [4t^3 \tan^{-1}(k^2/t^2) - 2k^2 t]$$

$$= 0$$

Therefore, $f_{xy}(0, 0) = \lim_{k \to 0} \frac{0 - 0}{k} = 0$

$$f_y(0, 0) = \lim_{s \to 0} \frac{f(0, s) - f(0, 0)}{s}$$

$$= \lim_{s \to 0} \frac{(\pi s^4/2) - 0}{s}$$

$$= 0$$

Further, $f_y(h, 0) = \lim_{s \to 0} \frac{f(h, s) - f(h, 0)}{s}$

$$= \lim_{s \to 0} \frac{(h^4+s^4) \tan^{-1}(s^2/h^2) - 0}{s}$$

$$= \lim_{s \to 0} \frac{4s^3 \tan^{-1}(s^2/h^2) + (h^4+s^4). \left(\dfrac{1}{1 + s^4/h^4}\right)(2s/h^2)}{1}$$

$$= \lim_{s \to 0} [4s^3 \tan^{-1}(s^2/h^2) + 2sh^2]$$

$$= 0.$$

Consequently, $f_{yx}(0, 0) = \lim_{h \to 0} \frac{f_y(h, 0) - f_y(0, 0)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0.$

In Unit 5 you have seen some examples of functions whose partial derivatives f_x, f_y do not exist (see Example 6 of Unit 5).

Here we will give you an example of a function whose first order partial derivatives exist, but higher order ones do not exist. From this example you will also see that the existence of a partial derivative of a particular order does not imply the existence of other partial derivatives of the same order.

Example 6 : Let us examine whether the second order partial derivatives of f at (0, 0) exist or not, if $f : \mathbf{R}^2 \to \mathbf{R}$ is defined by

$$f(x, y) = \begin{cases} \dfrac{xy^2}{\sqrt{x^2 + y^2}}, & xy \neq 0 \\ 0, & xy = 0 \end{cases}$$

Now, $f_x(0, 0) = \lim\limits_{t \to 0} \dfrac{f(t, 0) - f(0, 0)}{t}$

$$= \lim_{t \to 0} \frac{0 - 0}{t} = 0.$$

Similarly, for $h \neq 0$, $f_x(h, 0) = \lim\limits_{t \to 0} \dfrac{f(h+t, 0) - f(h, 0)}{t}$

$$= \lim_{t \to 0} \frac{0 - 0}{t} = 0.$$

Therefore, $f_{xx}(0, 0) = \lim\limits_{h \to 0} \dfrac{f_x(h, 0) - f_x(0, 0)}{h} = 0$

Now to check the existence of f_{xy}, we will have to see whether

$$\lim_{k \to 0} \frac{f_x(0, k) - f_x(0, 0)}{k} \text{ exists or not.}$$

Therefore, let us find $f_x(0, k)$, for $k \neq 0$.

For $k \neq 0$, $f_x(0, k) = \lim\limits_{t \to 0} \dfrac{f(t, k) - f(0, k)}{t}$

$$= \lim_{t \to 0} \frac{1}{t} \left[\frac{tk^2}{\sqrt{t^2 + k^2}} \right]$$

$$= \lim_{t \to 0} \frac{k^2}{\sqrt{t^2 + k^2}}$$

$$= \frac{k^2}{\sqrt{k^2}}$$

$$= |k|.$$

Now, $\lim\limits_{k \to 0} \dfrac{f_x(0, k) - f_x(0, 0)}{k}$

$$= \lim_{k \to 0} \frac{|k|}{k},$$

which does not exist, showing that f_{xy} does not exist at (0, 0).

Now
$f_y(0, 0) = \lim\limits_{s \to 0} \dfrac{f(0, s) - f(0, 0)}{s} = \lim\limits_{s \to 0} \dfrac{0 - 0}{s} = 0,$

and for $h \neq 0$, $f_y(h, 0) = \lim\limits_{s \to 0} \dfrac{f(h, s) - f(h, 0)}{s}$

$$= \lim_{s \to 0} \frac{\dfrac{hs^2}{\sqrt{h^2 + s^2}} - 0}{s}$$

$$= \lim_{s \to 0} \frac{hs}{\sqrt{h^2 + s^2}} = 0$$

Therefore, $f_{yx}(0, 0) = \lim\limits_{h \to 0} \dfrac{f_y(h, 0) - f_y(0, 0)}{h} = \lim\limits_{h \to 0} \dfrac{0 - 0}{h} = 0$

Again, for $k \neq 0$, $f_y(0, k) = \lim\limits_{s \to 0} \dfrac{f(0, k+s) - f(0, k)}{s} = \lim\limits_{s \to 0} \dfrac{0 - 0}{s} = 0$

Therefore, $f_{yy}(0, 0) = \lim\limits_{k \to 0} \dfrac{f_y(0, k) - f_y(0, 0)}{k} = \lim\limits_{k \to 0} \dfrac{0 - 0}{k} = 0.$

Thus, f_{xx}, f_{yy} and f_{yx} exist at (0, 0) and are equal to 0, while $f_{xy}(0, 0)$ does not exist.

See if you can solve these exercises now.

E5) Show that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$ for the function $f : \mathbf{R}^2 \to \mathbf{R}$ defined by

$$f(x, y) = \begin{cases} \dfrac{xy^5}{x^2+y^4}, & \text{if } (x, y) \neq (0, 0) \\ 0 \quad , & \text{if } (x, y) = (0, 0) \end{cases}$$

E6) Examine the following functions for equality of f_{xy} and f_{yx} at (0, 0).

a) $$f(x, y) = \begin{cases} \dfrac{x^2y^2}{\sqrt{x^4+y^4}}, & \text{if } (x, y) \neq (0, 0) \\ 0 \quad , & \text{if } (x, y) = (0, 0) \end{cases}$$

b) $$f(x, y) = \begin{cases} \dfrac{xy^3}{\sqrt{x^2+y^4}}, & \text{if } (x, y) \neq (0, 0) \\ 0 \quad , & \text{if } (x, y) = (0, 0) \end{cases}$$

E7) Show that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$ for the function f defined by

$$f(x, y) = \begin{cases} xy, & \text{if } |y| \leq |x| \\ -xy, & \text{if } |y| > |x|. \end{cases}$$

The study of the above examples and exercises must have convinced you that we have to be careful about the order of variables with respect to which higher order derivatives are taken. For instance, from Example 4 it is clear that f_{xy} need not be equal to f_{yx}. Example 6 goes a step further, where f_{xy} exists at (0, 0), while f_{yx} does not, showing that the question of their equality does not arise at all. If you look at the definitions of f_{xy} and f_{yx} at a point (a, b) more carefully, you would see why the expectation of the equality $f_{xy}(a, b) = f_{yx}(a, b)$ is farfetched. By definition

$$\begin{aligned} f_{xy}(a, b) &= \lim_{k \to 0} \frac{f_x(a, b+k) - f_x(a, b)}{k} \\ &= \lim_{k \to 0} \left[\frac{1}{k} \left\{ \lim_{h \to 0} \frac{f(a+h, b+k) - f(a, b+k)}{h} - \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h} \right\} \right] \\ &= \lim_{k \to 0} \left[\lim_{h \to 0} \left\{ \frac{f(a+h, b+k) - f(a, b+k) - f(a+h, b) - f(a, b)}{hk} \right\} \right], \end{aligned}$$

Similarly,

$$f_{yx}(a, b) = \lim_{h \to 0} \left[\lim_{k \to 0} \left\{ \frac{f(a+h, b+k) - f(a+h, b) - f(a, b+k) - f(a, b)}{hk} \right\} \right]$$

and we have already seen in Unit 4 that repeated limits are not equal, in general.

In the next section we will study the conditions under which these mixed partial derivatives become equal.

6.3 EQUALITY OF MIXED PARTIAL DERIVATIVES

We shall now give a set of sufficient conditions which would ensure that the order of the variables with respect to which higher order partial derivatives are taken is immaterial. In other words, if a function f satisfies these conditions, then its mixed partial derivatives will be equal.

Theorem 1 : Let f(x, y) be a real-valued function such that the two second order partial derivatives f_{xy} and f_{yx} are continuous at a point (a, b). Then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Proof : The continuity of f_{xy} and f_{yx} at (a, b) implies that f_x, f_y, f_{xy} and f_{yx} exist in a neighbourhood, say D of (a, b).
Consider the expression

$$\psi(h, k) = f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b),$$

which is defined for all those real numbers h, k for which $(a+h, b+k) \in D$.

Let I_h denote the closed interval [a, a+h] or [a+h, a] according as $h > 0$ or $h < 0$. Let G(x) be a real-valued function defined on the closed interval I_h by

$$G(x) = f(x, b+k) - f(x, b)$$

so that $G(a+h) - G(a) = \psi(h, k)$. Since, for all x in I_h, the points (x, b+k) and (x, b) belong to D, it follows that $f_x(x, b+k)$ and $f_x(x, b)$ exist for all $x \in I_h$.
Now we can write

$$G'(x) = f_x(x, b+k) - f_x(x, b).$$

Therefore, the function G(x) is differentiable on the closed interval I_h. Thus G(x) satisfies the requirements of Lagrange's mean value theorem, and we get

$$\begin{aligned}\psi(h, k) &= G(a+h) - G(a) = h\,G'(a+\theta h)\\ &= h\,[f_x(a+\theta h, b+k) - f_x(a+\theta h, b)], \quad \ldots\ldots\ldots\ldots(1)\end{aligned}$$

where $0 < \theta < 1$.

Now we define a function $F : I_k \to \mathbf{R}$ by

$$F(t) = f_x(a+\theta h, t),$$

where I_k is the closed interval [b, b+k] or [b+k, b] according as $k > 0$ or $k < 0$. Since f_{xy} exists on D, it follows that the function F is differentiable on I_k. Therefore, by Lagrange's mean value theorem, we get

$F(b+k) - F(b) = kF'(b+\theta' k)$ for some θ', $0 < \theta' < 1$. This means that

$$f_x(a+\theta h, b+k) - f_x(a+\theta h, b) = k\,f_{xy}(a+\theta h, b+\theta' k)$$

Using Equation (1), we get

$$\psi(h, k) = hk\,f_{xy}(a+\theta h, b+\theta' k)$$

and consequently

$$\begin{aligned}\lim_{\substack{h\to 0\\ k\to 0}} \frac{\psi(h, k)}{hk} &= \lim_{\substack{h\to 0\\ k\to 0}} f_{xy}(a+\theta h, b+\theta' k)\\ &= f_{xy}(a, b)\end{aligned}$$

as f_{xy} is given to be continuous at (a, b).

Starting with the function

$$H(y) = f(a+h, y) - f(a, y)$$

for $y \in I_k$ and proceeding exactly as above we can prove that

$$\lim_{\substack{h\to 0\\ k\to 0}} \psi(h, k) = f_{yx}(a, b)$$

and conclude $f_{xy}(a, b) = f_{yx}(a, b)$.

(L. Euler (1707-1783))

This result was proved by L. Euler around 1734, when he was working on some problems in hydrodynamics. Later the German mathematician Hermann Amandus Schwarz (1843-1921) proved another theorem about the equality of mixed partial derivatives. ThePEL- conditions in Schwarz's theorem are less restrictive than those in Euler's theorem (Theorem 1). We give only the statement of Schwarz's theorem here.

Theorem 2 : (Schwarz's Theorem) : Let f(x, y) be a real-valued function defined in a neighbourhood of (a, b) such that

i) f_y exists on a certain neighbourhood of (a, b).

ii) f_{xy} is continuous at (a, b).

Then f_{yx} exists at (a, b) and $f_{yx}(a, b) = f_{xy}(a, b)$.

We now give an example to illustrate this.

Example 7 : Let us evaluate f_{xy} at a point (x, y) for the function f defined by $f(x, y) = x^4 + x^2y^2 + y^6$. Then we'll use Schwarz's theorem to evaluate f_{yx} at the point (x, y).

By direct differentiation you can see that

$f_x(x, y) = 4x^3 + 2xy^2$. Therefore, $f_{xy}(x, y) = 4xy$.

Since 4xy is a polynomial, f_{xy} is a continuous function.

Further, $f_y(x, y) = 2x^2y + 6y^5$ exists. Hence f satisfies the conditions of Schwarz's theorem and so $f_{yx}(x, y) = f_{xy}(x, y) = 4xy$.

Are you ready for an exercise now?

E8) Evaluate f_{xy} at a point (x, y) for each of the following functions.

a) $f(x, y) = x^2 + xy + y^2$

b) $f(x, y) = e^x \cos y - e^y \sin x$

c) $f(x, y) = \dfrac{x^2-y^2}{x^2+y^2}$, $x \neq 0, y \neq 0$

Verify that each of these functions satisfies the requirements of Schwarz's theorem and hence evaluate $f_{yx}(x, y)$.

In Euler's Theorem we assume that both the mixed partial derivatives are continuous, whereas in Schwarz's theorem we assume that only one of them, say f_{xy} is continuous, and that f_y exists. But even though the conditions of Schwarz's theorem are less strict, these are still not necessary for the equality of mixed partial derivatives. In other words, we can have functions whose mixed partial derivatives at some point are equal, but which do not satisfy the requirements of Schwarz's theorem. We give one such function in the following example.

Example 8 : Consider the function f defined by

$$f(x, y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & x = 0 = y \end{cases}$$

We will show that $f_{xy}(0, 0) = f_{yx}(0, 0)$, even though f does not fulfil the requirements of Schwarz's theorem.

$$\text{Now, } f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h}$$

$$= \lim_{h \to 0} \frac{0 - 0}{h}$$

$$= 0.$$

Also, for $y \neq 0$,

$$f_x(0, y) = \lim_{h \to 0} \frac{f(h, y) - f(0, y)}{h}$$

$$= \lim_{h \to 0} \frac{h^2y^2}{h^2+y^2} \cdot \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{hy^2}{h^2+y^2}$$

$$= 0.$$

Therefore, $f_{xy}(0, 0) = \lim_{k \to 0} \frac{f_x(0, k) - f_x(0, 0)}{k}$

$$= 0.$$

Similarly, you can check that

$f_y(0, 0) = 0$ and for $x \neq 0$, we have

$$f_y(x, 0) = \lim_{k \to 0} \frac{f(x, k) - f(x, 0)}{k}$$

$$= \lim_{k \to 0} \frac{x^2k^2}{x^2+k^2} \cdot \frac{1}{k}$$

$$= 0.$$

From this we get

$$f_{yx}(0, 0) = \lim_{h \to 0} \frac{f_y(x, 0) - f_y(0, 0)}{h}$$

$$= 0.$$

Hence, we have shown that

$f_{xy}(0, 0) = f_{yx}(0, 0)$.

We'll now show that the conditions of Schwarz's theorem are **not** satisfied. Now, for $x \neq 0$, $y \neq 0$, we can find the partial derivatives of f at (x, y) by differentiating directly. Thus,

$$f_x(x, y) = \frac{\partial}{\partial x}\left[\frac{x^2y^2}{x^2+y^2}\right]$$

$$= \frac{2(x^2+y^2)\,xy^2 - 2x^3y^2}{(x^2+y^2)^2}$$

$$= \frac{2xy^4}{(x^2+y^2)^2}.$$

Further, $f_{xy}(x, y) = \frac{\partial}{\partial y}\left[\frac{2xy^4}{(x^2+y^2)^2}\right]$

$$= \frac{8x(x^2+y^2)^2y^3 - 8xy^5(x^2+y^2)}{(x^2+y^2)^4}$$

$$= \frac{8xy^3\,(x^2+y^2)\,[x^2+y^2-y^2]}{(x^2+y^2)^4}$$

$$= \frac{8x^3y^3}{(x^2+y^2)^3}$$

Now, $\lim_{(x, y) \to (0, 0)} \frac{8x^3y^3}{(x^2+y^2)^3}$ does not exist. Put $y = mx$ in $\frac{8x^3y^3}{(x^2+y^2)^3}$ and take the limit as $x \to 0$. You will find that the limit is different for different values of m. This means that $\lim_{(x, y) \to (0, 0)} f_{xy}(x, y)$ does not exist, which implies that f_{xy} is not continuous at (0, 0).

There is another criterion which tells us when f_{xy} equals f_{yx} at a particular point. We state this also without proof.

Theorem 3 (Young's Theorem) : Let f(x, y) be a real-valued function defined in a neighbourhood of a point (a, b) such that both the first order partial derivatives f_x and f_y are differentiable at (a, b). Then $f_{xy}(a, b) = f_{yx}(a, b)$.

As in the case of Schwarz's theorem the conditions stated in Young's theorem are less strict than in Theorem 1. However, these are not necessary for the equality of mixed partial derivatives. Our next example illustrates this fact.

Example 9 : Consider the function f in Example 8.

$$f(x, y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0 & , x = y = 0. \end{cases}$$

We have seen that $f_x(0, 0) = 0$ and $f_{xy}(0, 0) = 0$.

You can easily check that $f_x(h, 0) = 0$. Now we'll prove that f_x is not differentiable at (0, 0). For this, let us start with the assumption that f_x is differentiable at (0, 0). Then there exist functions $\phi(h, k)$ and $\psi(h, k)$, such that

$$f_x(h, k) - f_x(0, 0) = h\, f_{xx}(0, 0) + k\, f_{xy}(0, 0) + h\phi(h, k) + k\psi(h, k) \qquad \ldots(2)$$

and $\phi(h, k) \to 0$ as $(h, k) \to (0, 0)$,

$\psi(h, k) \to 0$ as $(h, k) \to (0, 0)$.

Now let us calculate $f_{xx}(0, 0)$.

$$f_{xx}(0, 0) = \lim_{h \to 0} \frac{f_x(h, 0) - f_x(0, 0)}{h}$$

$$= 0.$$

Therefore, (2) becomes

$$f_x(h, k) = h\phi(h, k) + k\psi(h, k).$$

or, $$\frac{2hk^4}{(h^2+k^2)^2} = h\,\phi(h, k) + k\,\psi(h, k).$$

Now if we put $h = r\cos\theta$ and $k = r\sin\theta$ we get

$$2\cos\theta\sin^4\theta = \cos\theta\,\phi(r\cos\theta, r\sin\theta) + \sin\theta\,\psi(r\cos\theta, r\sin\theta) \qquad \ldots(3)$$

Now, if $r \to 0$, $r\cos\theta \to 0$ and $r\sin\theta \to 0$.
This means as $r \to 0$, $h \to 0$ and $k \to 0$, and therefore,

$\phi(r\cos\theta, r\sin\theta) \to 0$ and $\psi(r\cos\theta, r\sin\theta) \to 0$.

Thus, taking the limit of (3) as $r \to 0$, we get

$2\cos\theta\sin^4\theta = 0$, for all θ.

But this is impossible. Hence f_x is not differentiable. Thus this function f does not satisfy the requirements of Young's theorem, even though we have $f_{xy}(0, 0) = f_{yx}(0, 0)$.
For most of the functions that we come across, all the partial derivatives are continuous, and therefore the value of the mixed partial derivatives does not change when there is a change in the order of variables with respect to which the partial derivatives are taken. Let us look at a few more examples.
Example 10 : Consider the function $f : \mathbf{R}^3 \to \mathbf{R}$ defined by

$$f(x, y, z) = \begin{cases} \dfrac{x^3y - xy^3 + yz^2}{x^2+y^2+z^2} & (x, y, z) \neq (0, 0, 0) \\ 0 & (x, y, z) = (0, 0, 0) \end{cases}$$

We'll show that $f_{xy}(0, 0, 0) \neq f_{yx}(0, 0, 0)$, whereas $f_{xz}(0, 0, 0) = f_{zx}(0, 0, 0)$.

Let us first calculate $f_{xy}(0, 0, 0)$. For this we need to evaluate $f_x(0, 0, 0)$ and $f_x(0, k, 0)$. Now

$$f_x(0, 0, 0) = \lim_{p \to 0} \frac{f(p, 0, 0) - f(0, 0, 0)}{p} = \lim_{p \to 0} \frac{0-0}{p} = 0 \text{ and}$$

$$f_x(0, k, 0) = \lim_{p \to 0} \frac{f(p, k, 0) - f(0, k, 0)}{p}$$

$$= \lim_{p \to 0} \frac{\frac{p^3k - pk^3}{p^2+k^2} - 0}{p}$$

$$= \lim_{p \to 0} \frac{p^2k - k^3}{p^2+k^2} = -k.$$

Therefore, $$f_{xy}(0, 0, 0) = \lim_{k \to 0} \frac{f_x(0, k, 0) - f_x(0, 0, 0)}{k}$$

$$= \lim_{k \to 0} \frac{-k-0}{k}$$

$$= -1.$$

Next, we'll evaluate $f_{yx}(0, 0, 0)$. For this we need $f_y(h, 0, 0)$ and $f_y(0, 0, 0)$.

Now, $$f_y(0, 0, 0) = \lim_{q \to 0} \frac{f(0, q, 0) - f(0, 0, 0)}{q} = 0, \text{ and}$$

$$f_y(h, 0, 0) = \lim_{q \to 0} \frac{f(h, q, 0) - f(h, 0, 0)}{q}$$

$$= \lim_{q \to 0} \frac{\frac{h^3q - hq^3}{h^2+q^2} - 0}{q}$$

$$= \lim_{q \to 0} \frac{h^3 - hq^2}{h^2+q^2}$$

$$= h.$$

Therefore,

$$f_{yx}(0, 0, 0) = \lim_{h \to 0} \frac{f_y(h, 0, 0) - f_y(0, 0, 0)}{h}$$

$$= \lim_{h \to 0} \frac{h-0}{h} = 1$$

So $f_{xy}(0, 0, 0) \neq f_{yx}(0, 0, 0)$

Now, $$f_z(0, 0, 0) = \lim_{r \to 0} \frac{f(0, 0, r) - f(0, 0, 0)}{r} = 0, \text{ and}$$

$$f_z(h, 0, 0) = \lim_{r \to 0} \frac{f(h, 0, r) - f(h, 0, 0)}{r} = 0$$

Therefore, $$f_{zx}(0, 0, 0) = \lim_{h \to 0} \frac{f_z(h, 0, 0) - f_z(0, 0, 0)}{h} = 0$$

Also $$f_x(0, 0, r) = \lim_{p \to 0} \frac{f(p, 0, r) - f(0, 0, r)}{p} = 0$$

This means, $$f_{xz}(0, 0, 0) = \lim_{r \to 0} \frac{f_x(0, 0, r) - f_x(0, 0, 0)}{r} = 0$$

Hence $f_{xz}(0, 0, 0) = f_{zx}(0, 0, 0)$.

Here is another example to show that the conditions in Theorem 1 are not necessary for the equality of mixed partial derivatives.

Example 11 : Let us show that $f_{xy}(0, 0, 0) = f_{yx}(0, 0, 0)$, but neither f_{xy} nor f_{yx} is continuous at (0, 0, 0) for the function $f : \mathbf{R}^3 \to \mathbf{R}$ defined by

$$f(x, y, z) = \begin{cases} \frac{1}{x} + \frac{1}{y} + \frac{1}{z}, & x \neq 0, y \neq 0, z \neq 0. \\ 0 & , \text{otherwise} \end{cases}$$

Now, $$f_x(0, 0, 0) = \lim_{h \to 0} \frac{f(h, 0, 0) - f(0, 0, 0)}{h} = 0, \text{ and}$$

$$f_x(0, k, 0) = \lim_{h \to 0} \frac{f(h, k, 0) - f(0, k, 0)}{h} = 0$$

$\lim\limits_{h\to 0}\frac{-1}{y^2h}$ does not exist because $\lim\limits_{h\to 0^+}\frac{-1}{y^2h} = -\infty$ and $\lim\limits_{h\to 0^-}\frac{-1}{y^2h} = \infty$

Therefore, $f_{xy}(0, 0, 0) = 0$.

Similarly we can show that $f_{yx}(0, 0, 0) = 0$. However, for $y \neq 0$, $z \neq 0$,

$\lim\limits_{h\to 0}\frac{f_y(h, y, z) - f_y(0, y, z)}{h} = \lim\limits_{h\to 0}\frac{-1/y^2}{h}$ does not exist, and we conclude that $f_{yx}(0, y, z)$ does not exist. Since in any neighbourhood of (0, 0, 0) there exist points (0, y, z) with $y \neq 0$, $z \neq 0$, it follows that f_{yx} is not defined in any neighbourhood of (0, 0, 0), and hence f_{yx} cannot be continuous at (0, 0, 0). In a similar way, we can show that f_{xy} is not continuous at (0, 0, 0).

You can try this exercise now.

E9) Let $f : \mathbf{R}^3 \to \mathbf{R}$ be defined by

$$f(x, y, z) = \begin{cases} \frac{x}{y} + \frac{y}{z}, & y \neq 0, z \neq 0 \\ 0 & , \text{otherwise} \end{cases}$$

Show that at the origin f_{xy}, f_{yx}, f_{xz} and f_{zx} all exist, but neither f_{zy} nor f_{yz} exists.

Now let us briefly recall what we have covered in this unit.

6.4 SUMMARY

In this unit, we have

1) Introduced partial derivatives of order more than one.
2) Evaluated these higher order partial derivatives for various functions.
3) Studied examples of functions which show that, in general, the two partial derivatives of order more than one obtained by changing the order of variables are not equal in value, even if both exist.
4) Applied the following three sets of sufficient conditions which ensure the equality of $f_{xy}(a, b)$ and $f_{yx}(a, b)$.
   - Euler's theorem says that if f_{xy} and f_{yx} are both continuous at a point (a, b), then

     $f_{xy}(a, b) = f_{yx}(a, b)$.
   - Schwarz's theorem tells us that if f_{xy} is continuous at (a, b), and if f_y exists at (a, b), then $f_{yx}(a, b) = f_{xy}(a, b)$.
   - Young's theorem says that if f_x and f_y are differentiable at (a, b) then $f_{xy} = f_{yx}(a, b)$.
5) Seen, through some examples, that the conditions stated in the above three theorems are only sufficient and not necessary.

6.5 SOLUTIONS AND ANSWERS

E1) a) $f(x, y) = \cos\frac{y}{x}$. Then

$$f_x = -\sin\left(\frac{y}{x}\right)\cdot\left(-\frac{y}{x^2}\right) = \frac{y}{x^2}\sin\frac{y}{x}$$

$$f_y = -\sin\left(\frac{y}{x}\right)\cdot\left(\frac{1}{x}\right) = -\frac{1}{x}\sin\frac{y}{x}$$

$$f_{xx} = \frac{y}{x^2}\cos\left(\frac{y}{x}\right)\cdot\left(-\frac{y}{x^2}\right) + \left(-\frac{2y}{x^3}\right)\sin\frac{y}{x}$$

$$= -\frac{y^2}{x^4}\cos\frac{y}{x} - \frac{2y}{x^3}\sin\frac{y}{x}$$

$$f_{yx} = \frac{1}{x^2}\sin\frac{y}{x} + \frac{y}{x^2}\left(\cos\frac{y}{x}\right)\left(\frac{1}{x}\right)$$

$$= \frac{1}{x^2}\sin\frac{y}{x} + \frac{y}{x^3}\cos\frac{y}{x}$$

$$f_{xy} = \frac{1}{x^2}\sin\frac{y}{x} - \frac{1}{x}\left(\cos\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)$$

$$= \frac{1}{x^2}\sin\frac{y}{x} + \frac{y}{x^3}\cos\frac{y}{x}$$

$$f_{yy} = -\frac{1}{x^2}\cos\frac{y}{x}$$

b) $f(x, y) = x^5 + y^4 \sin(x^6)$

$\therefore f_x = 5x^4 + 6x^5y^4\cos(x^6)$

$f_y = 4y^3 \sin x^6$

$f_{xx} = 20x^3 + 30x^4y^4\cos(x^6) - 36x^{10}y^4\sin(x^6)$

$f_{yx} = 24x^5y^3\cos(x^6) = f_{xy}$

$f_{yy} = 12y^2\sin(x^6)$

c) $f(x, y, z) = \sin xy + \sin yz + \cos xz$

$\therefore f_x = y\cos xy - z\sin xz$

$f_y = x\cos xy + z\cos yz$

$f_z = y\cos yz - x\sin xz$

$f_{xx} = -y^2\sin xy - z^2\cos xz$

$f_{yx} = \cos xy - xy\sin xy = f_{xy}$

$f_{zx} = -\sin xz - xz\cos xz = f_{xz}$

$f_{yy} = -x^2\sin xy - z^2\sin yz$

$f_{zy} = \cos yz - yz\sin yz = f_{yz}$

$f_{zz} = -y^2\sin yz - x^2\cos xz$

d) $f(x, y, z) = xyz^2 + xyz + x^3y$

$f_x = yz^2 + yz + 3x^2y$

$f_y = xz^2 + xz + x^3$

$f_z = 2xyz + xy$

$f_{xx} = 6xy$

$f_{yx} = z^2 + z + 3x^2 = f_{xy}$

$f_{zx} = 2yz + y = f_{xz}$

$f_{zz} = 2xy$

$f_{yy} = 0$

$f_{zy} = 2xz + x = f_{yz}$

E2) $v(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$

$$\therefore \frac{\partial v}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x) = -\frac{x}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial^2 v}{\partial x^2} = -\frac{1}{(x^2+y^2+z^2)^{3/2}} + \frac{3}{2}\frac{2x}{(x^2+y^2+z^2)^{5/2}} \cdot x$$

$$= -\frac{1}{(x^2+y^2+z^2)^{3/2}} + \frac{3x^2}{(x^2+y^2+z^2)^{5/2}}$$

$$= \frac{-(x^2+y^2+z^2)+3x^2}{(x^2+y^2+z^2)^{5/2}}$$

$$= \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}$$

Similarly,

$$\frac{\partial^2 v}{\partial y^2} = \frac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{5/2}} ; \frac{\partial^2 v}{\partial z^2} = \frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{3/2}}$$

So, $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$

E3) a) $f(x, y) = x^3y + e^{xy^2}$

$$\frac{\partial f}{\partial x} = 3x^2y + y^2 e^{xy^2}$$

$$\frac{\partial^2 f}{\partial y\, \partial x} = 3x^2 + 2y\, e^{xy^2} + 2xy^3\, e^{xy^2}$$

$$\frac{\partial f}{\partial y} = x^3 + 2xy\, e^{xy^2}$$

$$\frac{\partial^2 f}{\partial x\, \partial y} = 3x^2 + 2y\, e^{xy^2} + 2xy^3\, e^{xy^2}$$

So, $\frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial^2 f}{\partial y\, \partial x}$

b) $f(x, y) = \tan(xy^3)$

$$\therefore \frac{\partial f}{\partial x} = y^3 \sec^2(xy^3)$$

$$\frac{\partial^2 f}{\partial y\, \partial x} = 3y^2 \sec^2(xy^3) + 6xy^5 \sec^2(xy^3)\tan(xy^3)$$

$$\frac{\partial f}{\partial y} = 3xy^2 \sec^2(xy^3)$$

$$\frac{\partial^2 f}{\partial x\, \partial y} = 3y^2 \sec^2(xy^3) + 6xy^5 \sec^2(xy^3)\tan(xy^3)$$

So, $\frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial^2 f}{\partial x\, \partial y}$

E4) $x^x y^y z^z = c$

Taking logarithms on both sides, we get

$x \ln x + y \ln y + z \ln z = \ln c$.

Differentiating with respect to y, treating z as a function of x and y, we get

$$\ln y + y \cdot \frac{1}{y} + \left[\ln z + z\frac{1}{z}\right]\frac{\partial z}{\partial y} = 0$$

$$\therefore \frac{\partial z}{\partial y} = -\frac{\ln y + 1}{\ln z + 1}$$

$$\therefore \frac{\partial^2 z}{\partial x\, \partial y} = \frac{\ln y + 1}{(\ln z + 1)^2} \cdot \frac{1}{z}\frac{\partial z}{\partial x} = -\frac{\ln ey \ln ex}{z\,(\ln ez)^3}$$

Putting $x = y = z$, we get

$$\frac{\partial^2 z}{\partial x\, \partial y} = -\frac{1}{x \ln ex} = -(x \ln ex)^{-1}$$

E5) $f_x(0, 0) = \lim\limits_{t \to 0} \dfrac{f(t, 0) - f(0, 0)}{t} = 0$. Similarly, $f_y(0, 0) = 0$,

and for $k \neq 0$, $f_x(0, k) = \lim\limits_{h \to 0} \dfrac{f(h, k) - f(0, k)}{k} = 1$

So, $f_{yx}(0, 0) = \lim\limits_{k \to 0} \dfrac{f_x(0, k) - f_x(0, 0)}{k} = 1.$

Also, for $h \neq 0$, $f_y(h, 0) = \lim\limits_{k \to 0} \dfrac{f(h, k) - f(h, 0)}{k}$

$$= \lim_{k \to 0} \frac{hk^4}{h^2 + k^4} = 0$$

So, $f_{xy}(0, 0) = \lim\limits_{h \to 0} \dfrac{f_y(h, 0) - f_y(0, 0)}{h} = 0.$

Hence, $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.

E6) a) $f_x(0, 0) = 0$.

For $k \neq 0$, $f_x(0, k) = \lim\limits_{h \to 0} \dfrac{f(h, k) - f(0, k)}{h} = \lim\limits_{h \to 0} \dfrac{hk^2}{\sqrt{h^4 + k^4}} = 0.$

So, $f_{yx}(0, 0) = \lim\limits_{k \to 0} \dfrac{f_x(0, k) - f_x(0, 0)}{k} = 0$

Similarly, $f_{xy}(0, 0) = 0$.

Hence, $f_{xy}(0, 0) = f_{yx}(0, 0)$

b) $f_x(0, 0) = 0 = f_y(0, 0)$

For $k \neq 0$, $f_x(0, k) = \lim\limits_{h \to 0} \dfrac{f(h, k) - f(0, k)}{h} = \lim\limits_{h \to 0} \dfrac{k^3}{\sqrt{h^2 + k^4}} = k.$

So, $f_{yx}(0, 0) = \lim\limits_{k \to 0} \dfrac{f_x(0, k) - f_x(0, 0)}{k} = 1$

For, $h \neq 0$, $f_y(h, 0) = \lim\limits_{k \to 0} \dfrac{f(h, k) - f(h, 0)}{k} = \lim\limits_{k \to 0} \dfrac{hk^2}{\sqrt{h^2 + k^4}} = 0$

So, $f_{xy}(0, 0) = \lim\limits_{h \to 0} \dfrac{f_y(h, 0) - f_y(0, 0)}{h} = 0$

Consequently, $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.

E7) $f_x(0, 0) = 0 = f_y(0, 0)$

For, $k \neq 0$, $f_x(0, k) = \lim\limits_{h \to 0} \dfrac{f(h, k) - f(0, k)}{h}$

$= \lim\limits_{h \to 0} \dfrac{-hk - 0}{h}$ (since k is fixed we can suppose that $|h| < |k|$)

$$= -k.$$

For, $h \neq 0$, $f_y(h, 0) = \lim_{k \to 0} \frac{f(h, k) - f(h, 0)}{k}$

$= \lim_{k \to 0} \frac{hk - 0}{k}$ (since h is fixed we can suppose that $|k| < |h|$)

$= h.$

Therefore, $f_{yx}(0, 0) = \lim_{k \to 0} \frac{f_x(0, k) - f_x(0, 0)}{k} = \lim_{k \to 0} \frac{-k - 0}{k} = -1$

and $f_{xy}(0, 0) = \lim_{h \to 0} \frac{f_y(h, 0) - f_y(0, 0)}{h} = \lim_{h \to 0} \frac{h - 0}{h} = 1$

Hence, $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.

E8) a) $f(x, y) = x^2 + xy + y^2$. Then

$f_y(x, y) = x + 2y$

$f_{xy}(x, y) = 1$

Clearly f_y exists everywhere and f_{xy} is continuous being a constant function. This shows that f satisfies the requirements of Schwarz's theorem. Hence, by Schwarz's theorem, f_{yx} exists and $f_{yx} = f_{xy} = 1$.

b) $f(x, y) = e^x \cos y - e^y \sin x$

$\therefore f_y(x, y) = -e^x \sin y - e^y \sin x$

and $f_{xy}(x, y) = -e^x \sin y - e^y \cos x$

It is easy to see that f_y exists and f_{xy} is continuous. So, in view of Schwarz's theorem, f_{yx} exists and

$f_{yx}(x, y) = f_{xy}(x, y) = -e^x \sin y - e^y \cos x$.

c) $f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$, $x \neq 0$, $y \neq 0$.

$$f_y(x, y) = \frac{-4x^2 y}{(x^2 + y^2)^2}$$

$$f_{xy}(x, y) = \frac{8xy(x^2 - y^2)}{(x^2 + y^2)^3}$$

Since f_y exists and f_{xy} is continuous at all points (x, y), where $x \neq 0$, $y \neq 0$, by Schwarz's theorem, we have

$$f_{yx}(x, y) = f_{xy}(x, y) = \frac{8xy(x^2 - y^2)}{(x^2 + y^2)^3}$$

E9) Now $f_x(0, 0, 0) = f_y(0, 0, 0) = f_z(0, 0, 0) = 0$.

For $k \neq 0$, $f_x(0, k, 0) = \lim_{h \to 0} \frac{f(h, k, 0) - f(0, k, 0)}{h} = 0$

Therefore, $f_{yx}(0, 0, 0) = \lim_{k \to 0} \frac{f_x(0, k, 0) - f_x(0, 0, 0)}{k} = 0.$

Similarly, $f_{xy}(0, 0, 0) = f_{xz}(0, 0, 0) = f_{zx}(0, 0, 0) = 0$.

For, $r \neq 0$, $f_y(0, 0, r) = \lim_{k \to 0} \frac{f(0, k, r) - f(0, 0, r)}{k}$

$= \lim_{k \to 0} \frac{k/r}{k}$

$= \lim_{k \to 0} \frac{1}{r} = \frac{1}{r}$

So, $f_{zy}(0, 0, 0) = \lim\limits_{r \to 0} \dfrac{f_y(0, 0, r) - f_y(0, 0, 0)}{r} = \lim\limits_{r \to 0} \dfrac{1}{r^2}$

But $\lim\limits_{r \to 0} \dfrac{1}{r^2}$ does not exist.

Hence, $f_{zy}(0, 0, 0)$ does not exist.

Since $f_z(0, k, 0) = \lim\limits_{r \to 0} \dfrac{f(0, k, r) - f(0, k, 0)}{r} = k \lim\limits_{r \to 0} \dfrac{1}{r^2}$,

therefore, $f_z(0, k, 0)$ does not exist as $\lim\limits_{r \to 0} \dfrac{1}{r^2}$ does not exist.

Hence $f_{yz}(0, 0, 0) = \lim\limits_{k \to 0} \dfrac{f_z(0, k, 0) - f_z(0, 0, 0)}{k}$ does not exist.