Tutorial -1

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① $f_n(x) = \frac{x}{x+n}$

$\epsilon = 0.1$

and, $|f_n(x)| < \epsilon$

$\Rightarrow \frac{x}{x+n} < \epsilon$

$\Rightarrow \frac{x+n}{x} > \frac{1}{\epsilon}$

$\Rightarrow 1 + \frac{n}{x} > \frac{1}{0.1} = 10$

$\Rightarrow \frac{n}{x} \geq 9$

$\Rightarrow n \geq 9x$

(i) $x = 10$

$\Rightarrow n \geq 90$

$\therefore N(x, \epsilon) = 90$

(ii) $x = 100$

$\Rightarrow n \geq 900$

$\therefore N(x, \epsilon) = 900$

② (i) $f_n(x) = x^n, \; 0 < x < 1, \; \epsilon = 0.1$

$|f_n(x)| < \epsilon$

$\Rightarrow |x^n| < \epsilon$

$\Rightarrow n \log x < \log \epsilon$

$\Rightarrow n > \frac{\log \epsilon}{\log x}$ (as $0 < x < 1 \Rightarrow -\infty < \log x < 0$)

$\Rightarrow n > \frac{-\log 10}{\log x}$ ($\because \epsilon = 0.1$)

(ii) $f_n(x) = \frac{nx}{1+n^2x^2}, \; -\infty < x < \infty, \; \epsilon = 0.1$

$|f_n(x)| < \epsilon$

$\Rightarrow \left|\frac{nx}{1+n^2x^2}\right| < \epsilon$

$\Rightarrow \frac{nx}{1+n^2x^2} < \frac{1}{nx} < \epsilon \Rightarrow n > \frac{1}{0.1x} \Rightarrow n > \frac{10}{x}$

As $1+n^2x^2 > n^2x^2$

$\Rightarrow \frac{1}{1+n^2x^2} < \frac{1}{n^2x^2}$

$\Rightarrow \frac{nx}{1+n^2x^2} < \frac{1}{nx}$

③ (i) $f_n(x) = x^n,\ 0 \le x \le 1$

$$f_n(x) = x^n \xrightarrow{p} f(x) = \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1 \end{cases}$$

$$\Rightarrow f_n(x) \not\xrightarrow{u} f(x)$$

(ii) $f_n = \dfrac{x}{1+nx},\ 0 < x < \infty$

$$f_n(x) \xrightarrow{p} f(x) = 0 \quad \forall x\ (0,\infty)$$

Now, $\delta_n = \underset{x \in (0,\infty)}{\text{Max}} \left| \dfrac{x}{1+nx} \right|$

Take $x = \frac{1}{n}$

$$\Rightarrow \delta_n = \underset{x \in (0,\infty)}{\text{Max}} \left| \frac{1/n}{2} \right| \to 0$$

$$\therefore f_n \xrightarrow{u} f = 0$$

④ (i) $f_n(x) = n^2 x e^{-nx},\ 0 \le x \le 1$

$$\xrightarrow{p} 0$$

Take $x_n = \frac{1}{n}$, then

$$\delta_n = \underset{x \in [0,1]}{\text{Max}} \left| \frac{n^2 x}{e^{nx}} \right|$$

$$= \underset{x \in [0,1]}{\text{Max}} \left| \frac{n}{e} \right| \to \infty \quad \text{as } n \to \infty$$

As $\delta_n \not\to 0$, $f_n \not\xrightarrow{u} f$

(ii) $f_n(x) = x^n (1-x)^n,\ 0 \le x \le 1$

$$\xrightarrow{p} 0$$

Now, $\delta_n = \underset{x \in [0,1]}{\text{Max}} \left| x^n (1-x)^n \right| = \left(\frac{1}{4}\right)^n$

$\therefore$ $\delta_n \to 0$ as $n \to \infty$

$$\therefore f_n \xrightarrow{u} f = 0$$

Max $|x - x^2|$

$\frac{d}{dx}(x - x^2) = 0 \Rightarrow 1 - 2x = 0$

$\Rightarrow x = \frac{1}{2}$

$\therefore$ Max $|x - x^2|$

$= \left|\frac{1}{2} - \frac{1}{4}\right| = \frac{1}{4}$

5) (i) $f_n(x) = \dfrac{n^2 x}{1+n^3x^2}$, $x \in (0,\infty)$

$1+n^3x^2 > n^3x^2$

$\Rightarrow \dfrac{n^2x}{1+n^3x^2} < \dfrac{n^2x}{n^3x^2} = \dfrac{1}{nx}$

$\xrightarrow{p} 0 \ \forall x \in (0,\infty)$

Now, $\delta_n = \max\limits_{x\in(0,\infty)} \left|\dfrac{n^2x}{1+n^3x^2}\right|$

As $\dfrac{n^2x}{1+n^3x^2} < \dfrac{1}{nx}$ & $\dfrac{1}{nx} \to 0$ as $n\to\infty$

$\therefore \delta_n \to 0$ as $n \to \infty$

$\Rightarrow f_n \xrightarrow{u} f = 0$.

(ii) $f_n(x) = \dfrac{n^2x^2}{1+n^3x^2}$, $x \in (0,\infty)$

$f_n(x) \xrightarrow{p} 0 \quad \forall x \in (0,\infty)$

Now $\delta_n = \max\limits_{x\in(0,\infty)} \left|\dfrac{n^2x^2}{1+n^3x^2}\right|$

Take $x = \frac{1}{n}$, then

$\delta_n = \max\limits_{x\in(0,\infty)} \left|\dfrac{1}{1+n}\right| \to 0$ as $n \to \infty$

$\therefore f_n \xrightarrow{u} f = 0$

6) $f_n(x) = \dfrac{nx^5}{1+nx^2}$, $x \in [-100, 100]$

$f_n(x) \xrightarrow{p} x^3 \ \forall x \in [-100,100]$

Now, $\delta_n = \max\limits_{x\in(0,\infty)} \left|\dfrac{nx^5}{1+nx^2} - x^3\right|$

$= \max\limits_{x\in(0,\infty)} \left|\dfrac{nx^5 - x^3 - nx^5}{1+nx^2}\right|$

$= \max\limits_{x\in(0,\infty)} \left|\dfrac{x^3}{1+nx^2}\right| \to 0$ as $n \to \infty$

$\therefore f_n \xrightarrow{u} f = 0$

7 Given that $\sum_{n=1}^{\infty} f_n(x) \xrightarrow{u} f(x)$ 8

$\Rightarrow$ ~~By weierstrass theorem,~~

~~$\exists M_n$ such that~~

~~$|f_n(x)| \leq M_n$~~

~~and $\sum M_n$ converges.~~

Let $S_n = f_1(x) + f_2(x) + \dots + f_{n-1}(x) + f_n(x)$

As S_n converges uniformly, by cauchy theorem, for $\epsilon > 0$, $\exists N(\epsilon)$ such that

$$|S_n(x) - S_m(x)| < \epsilon \quad \forall\ m, n \geq N(\epsilon)$$

Take $m = n-1$

$$\Rightarrow |S_n(x) - S_{n-1}(x)| < \epsilon$$

$$\Rightarrow |f_n - 0| < \epsilon \qquad (\because f_n = S_n(x) - S_{n-1}(x))$$

$$\Rightarrow f_n \rightarrow 0$$

(8) $\sum_{n=1}^{\infty} u_n(x), \quad 0<x<1$

$$u_n(x) = \frac{x}{((n-1)x+1)(nx+1)}$$

$$= \frac{1}{nx+1-x} - \frac{1}{nx+1}$$

$$S_n(x) = u_1(x) + u_2(x) + \dots + u_n(x)$$

$$= \left(1 - \cancel{\frac{1}{x+1}}\right) + \left(\cancel{\frac{1}{x+1}} - \cancel{\frac{1}{2x+1}}\right) + \dots + \left(\cancel{\frac{1}{nx+1-x}} - \frac{1}{nx+1}\right)$$

$$= 1 - \frac{1}{nx+1}$$

$$= \frac{nx}{nx+1}$$

$$S_n(x) \xrightarrow{P} f(x) = 1 \quad \forall x \in (0,1)$$

Now, $\delta_n = \underset{x\in(0,1)}{\text{Max}} |S_n(x) - 1|$

$$= \underset{x\in(0,1)}{\text{Max}} \left|\frac{nx}{nx+1} - 1\right|$$

$$= \underset{x\in(0,1)}{\text{Max}} \left|\frac{1}{1+nx}\right|$$

Take $x = 1/n$, then

$$\delta_n = \underset{x\in(0,1)}{\text{Max}} \left|\frac{1}{2}\right| = \frac{1}{2}$$

As $\delta_n \not\to 0$, $\sum u_n = S_n \xrightarrow{u}\!\!\!\!\!/\ \ u(x)$

(9) $\sum_{n=1}^{\infty} u_n(x), \quad |x| < 1 \Rightarrow -1 < x < 1$

$$u_n(x) = x^{n-1} - x^n$$

$$S_n(x) = u_1(x) + u_2(x) + \dots + u_n(x)$$

$$= (1 - \cancel{x}) + (\cancel{x} - \cancel{x^2}) + (\cancel{x^2} - x^3) + \dots + (\cancel{x^{n-1}} - x^n)$$

$$= 1 - x^n$$

$$S_n(x) \xrightarrow{P} 1 \quad \forall |x| < 1$$

Now, $\delta_n = \underset{x\in(-1,1)}{\text{Max}} |1 - x^n - 1|$

$$= \underset{x\in(-1,1)}{\text{Max}} (x^n) \longrightarrow 0 \quad \text{as } n\to\infty$$

$\therefore f_n(x) \xrightarrow{u} f(x) = 1.$

(10) (i) $\sum_{n=1}^{\infty} (u_n(x) - u_{n-1}(x))$, $0 \le \alpha < x < \infty$

$$u_n(x) = \frac{x}{1+n^2x}$$

Let $S_n =$

$$\sum_{n=1}^{\infty} (u_n(x) - u_{n-1}(x)) = \sum_{n=1}^{\infty} \left(\frac{x}{1+n^2x} - \frac{x}{1+(n-1)^2x} \right)$$

$$= \left(\frac{\cancel{x}}{\cancel{1+x}} - x \right) + \left(\frac{\cancel{x}}{1+\cancel{4x}} - \frac{\cancel{x}}{1+\cancel{x}} \right) + \left(\frac{\cancel{x}}{1+\cancel{9x}} - \frac{x}{1+\cancel{4x}} \right) + \dots + \left(\frac{x}{1+n^2x} - \frac{\cancel{x}}{1+\cancel{(n-1)^2x}} \right)$$

$$= \frac{x}{1+n^2x} - x$$

$$= \frac{-n^2x^2}{1+n^2x}$$

$S_n(x) \xrightarrow{p} -x \quad \forall\, x \in [0, \infty)$

Now,

$$\delta_n = \max_{x \in [0,\infty)} \left| \frac{-n^2x^2}{1+n^2x} \right|$$

Take $x = \frac{1}{n}$, then

$$\delta_n = \max_{x \in [0,\infty)} \left| \frac{1}{1+n} \right| \to 0 \text{ as } n \to 0$$

$\therefore f_n(x) \xrightarrow{u} f.$

(ii) $\sum_{n=1}^{\infty} (u_n(x) - u_{n-1}(x))$, $0 \le \alpha < x < \infty$

$$u_n(x) = \frac{nx}{1+n^2x^2}.$$

Let $S_n = \sum_{n=1}^{\infty} (u_n(x) - u_{n-1}(x)) = \sum_{n=1}^{\infty} \left(\frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2} \right)$

$$= \left(\frac{\cancel{x}}{1+\cancel{x^2}} - 0 \right) + \left(\frac{2\cancel{x}}{1+\cancel{4x^2}} - \frac{\cancel{x}}{1+\cancel{x^2}} \right) + \left(\frac{3\cancel{x}}{1+\cancel{9x^2}} - \frac{2\cancel{x}}{1+4x^2} \right) + \dots + \left(\frac{nx}{1+n^2x^2} - \frac{(n-1)\cancel{x}}{1+(n-1)^2x^2} \right)$$

$$= \frac{nx}{1+n^2x^2}$$

$S_n \xrightarrow{p} 0.$

Now. $\delta_n = \underset{x \in [0,\infty)}{Max} \left| \frac{nx}{1+n^2x^2} \right|$

Take $x = 1/n$, then

$$\delta_n = \underset{x \in [0,\infty)}{Max} \left|\frac{1}{2}\right| = \frac{1}{2}$$

As $\delta_n \not\to 0$, $f_n \overset{u}{\not\to} f$

(11) $p > 1$

for $\sum_{n=1}^{\infty} \frac{\sin nx}{n^p}$, by weierstrass theorem, $\exists M_n$ such that

$$\left|\frac{\sin nx}{n^p}\right| \le M_n = \frac{1}{n^p} \text{ and } \Sigma M_n = \Sigma \frac{1}{n^p} \text{ converges.}$$

$\therefore \sum_{n=1}^{\infty} \frac{\sin nx}{n^p}$ converges uniformly.

Similarly, for $\sum_{n=1}^{\infty} \frac{\cos nx}{n^p}$,

$$\left|\frac{\cos nx}{n^p}\right| \le \frac{1}{n^p} \text{ and } \Sigma \frac{1}{n^p} \text{ converges.}$$

$\therefore \sum_{n=1}^{\infty} \frac{\cos nx}{n^p}$ converges uniformly.

(12) $S_n = \Sigma \frac{(-1)^{n-1}}{n+x^2}$, $x \in \mathbb{R}$

Fix $x = x_0$, $S_n = \Sigma \frac{(-1)^{n-1}}{n+x_0^2} \longrightarrow$ converges pointwise

$\therefore$ S_n converges for all $x \in \mathbb{R}$.

For absolute convergence,

$$\Sigma \left| \frac{(-1)^{n-1}}{n+x^2} \right| = \Sigma \frac{1}{n+x^2} \overset{p}{\not\to} \text{ doesn't converge pointwise.}$$

(13) To prove: $\sum_{n=1}^{\infty} \frac{(-1)^n (x^2+n)}{n^2} \xrightarrow{u} S$, in every bounded interval.

$x^2 + n \le x^2 + n^2$ | Let $(-1)^n \frac{x^2+n}{n^2} = u_n$

$\Rightarrow \frac{x^2+n}{n^2} \le \frac{x^2+n^2}{n^2} = x^2\left(\frac{1}{n^2}\right) + 1 = M_n$

As $|u_n| \le M_n$ and M_n is convergent for all x,

the series is uniformly convergent for all x.

For absolute pointwise convergence,

$$S_n = \sum_{n=1}^{\infty} \left| \frac{(-1)^n (x^2+n)}{n^2} \right| = \sum_{n=1}^{\infty} \frac{x^2+n}{n^2}$$

Fix $x = x_0$, $S_n = \sum \frac{x_0^2+n}{n^2} = \sum u_n$

Put $x=0$, $S_n = \sum \frac{1}{n}$, which is diverging.

For other x, fix $x = x_0$

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{x_0^2+n+1}{x_0^2+n} \cdot \frac{n^2}{(n+1)^2} \right| = \lim_{n\to\infty} \left| \left(1 + \frac{1}{x_0^2+n}\right) \frac{1}{(1+\frac{1}{n})^2} \right|$$
$$= 1$$

$\therefore$ The series doesn't converge absolutely for any real x.

(14) $$S_n(x) = \sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$$

For uniform convergence, using weierstrans theorem,

$|S_n(x)| \le M_n \quad \forall x$

<u>Calculating M_n:</u>

Let $u(x) = \frac{x}{n(1+nx^2)}$

$$u'(x) = 0 \Rightarrow \frac{n(1+nx^2) - x(2n^2x)}{n^2(1+nx^2)^2} = 0 \Rightarrow n + n^2x^2 = 2n^2x^2$$
$$\Rightarrow n = n^2x^2$$
$$\Rightarrow x = \frac{1}{\sqrt{n}}$$

$$\therefore u(x)\Big|_{max} = \frac{1/\sqrt{n}}{n(1+n\cdot\frac{1}{n})} = \frac{1}{2n\sqrt{n}} = \frac{1}{2n^{3/2}} = M_n.$$

and $\frac{1}{2n^{3/2}}$ is convergent.

$\therefore S_n(x)$ is uniformly convergent.

For absolute pointwise convergence,

$$\lim_{n\to 0} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to 0} \left| \frac{x}{(n+1)(1+(n+1)x^2)} \cdot \frac{n(1+nx^2)}{x} \right|$$
$$= \lim_{n\to 0} \left| \frac{n+n^2x^2}{(n+1)+(n+1)^2x^2} \right| < 1$$

$\therefore S_n$ is pointwise convergent for all x.

$$S_n = \sum_{n=1}^{\infty} 3^n \sin\frac{1}{4^n x} = \sum_{n=1}^{\infty} u_n$$

For absolute convergent,

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}\sin\frac{1}{4^{n+1}x}}{3^n \sin\frac{1}{4^n x}}\right| = 3\lim_{n\to\infty}\left|\frac{\left(\dfrac{\sin\frac{1}{4^{n+1}x}}{\frac{1}{4^{n+1}x}}\right)\left(\frac{1}{4^{n+1}x}\right)}{\left(\dfrac{\sin\frac{1}{4^n x}}{\frac{1}{4^n x}}\right)\left(\frac{1}{4^n x}\right)}\right| = \frac{3}{4} < 1$$

$\therefore S_n$ is absolutely convergent.

For uniform convergence,

$|u_n(x)| \le M_n$

Calculating M_n:

$$3^n \sin\frac{1}{4^n x} \le 3^n\left|\frac{1}{4^n x}\right| = \left(\frac{3}{4}\right)^n \cdot \left|\frac{1}{x}\right|$$

As $x > a$,

$$\left(\frac{3}{4}\right)^n \left|\frac{1}{x}\right| < \left(\frac{3}{4}\right)^n \cdot \frac{1}{a} = M_n$$

and $M_n = \left(\frac{3}{4}\right)^n\left(\frac{1}{a}\right)$ converges.

$\therefore S_n$ is uniformly convergent on (a, ∞).

(16) (i) $\sum_{n=1}^{\infty} x^n \rightarrow$ power series; $a_n = 1$

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} 1 = 1$$

The series converges for all $x \in (-1, 1)$.

(ii) $\sum_{n=1}^{\infty} n!\, x^n$; $a_n = n!$

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n!}{(n+1)!}\right| = \lim_{n\to\infty}\left|\frac{1}{n+1}\right| = 0$$

The series converges for $x=0$ only.

(17) (i) $\sum_{n=1}^{\infty} \frac{2n!}{2^{2n}(n!)^2} x^n$

Here $a_n = \frac{2n!}{2^{2n}(n!)^2}$, $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$

$$R = \lim_{n\to\infty} \left| \frac{2n!}{2^{2n}(n!)^2} \times \frac{2^{2(n+1)}((n+1)!)^2}{2(n+1))!} \right|$$

$$= \lim_{n\to\infty} \frac{4(n+1)^2}{(2n+1)(2n+2)} = \lim_{n\to\infty} \frac{4(n^2+2n+1)}{4n^2+6n+2} = 1$$

$\therefore$ Series converges for $x \in (-1, 1)$.

(ii) $\sum_{n=1}^{\infty} \frac{n^n}{n!} x^n$

Here, $a_n = \frac{n^n}{n!}$

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left| \frac{n^n}{\cancel{n!}} \frac{\cancel{(n+1)!}}{(n+1)^{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n^n}{(n+1)^n} \right| = 1$$

$\therefore$ Series converges for $x \in (-1, 1)$.

(18) (i) $f_n(x) = x^n \xrightarrow{p} f(x) = \begin{cases} 0, & \text{if } x \in [0,1) \\ 1, & \text{if } x = 1 \end{cases}$

$x \in [0,1]$

$\therefore f_n(x) \not\xrightarrow{u} f(x)$.

(ii) $f_n(x) = e^{-nx^2} \xrightarrow{p} f(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x \in [0,1) \end{cases}$

$x \in [0,1]$

$\therefore f_n(x) \not\xrightarrow{u} f(x)$.

$$f_n(x) = (1-x)x^n \xrightarrow{\quad p \quad} f(x) = \begin{cases} 0, & \text{if } x=0 \\ 0, & \text{if } x=1 \\ 0, & \text{if } x\in(0,1) \end{cases} = 0 \quad \forall x \in [0,1]$$

For $x=0$,

$$f_n(x) = 0 \xrightarrow{p} 0$$

For $x=1$,

$$f_n(x) = 0 \xrightarrow{p} 0$$

For $x \in (0,1)$, fix $x = x_0$

$$f_n(x) = (1-x_0)x_0^n \longrightarrow 0 \text{ as } n\to\infty.$$

$\therefore f_n(x)$ converges pointwise to zero on $[0,1]$.

Now,

$$f_n'(x) = (-1)x^n + (1-x)nx^{n-1}$$
$$= -x^n + nx^{n-1} - nx^n$$
$$= nx^{n-1} - (1+n)x^n$$

$$\lim_{n\to\infty} \int f_n'(x)\,dx = \lim_{n\to\infty} \int (nx^{n-1} - (1+n)x^n)\,dx$$
$$= \lim_{n\to\infty} (x^n - x^{n+1})$$
$$= \lim_{n\to\infty} (1-x)x^n = \lim_{n\to\infty} f_n(x) = f(x).$$

(20) $u_n(x) = \frac{1-x}{n} x^n \xrightarrow{p} 0$

For uniform convergence,

$$\delta_n = \max_{x\in[0,1]} |u_n(x) - u(x)|$$
$$= \max_{x\in[0,1]} \left| \frac{(1-x)x^n}{n} \right| \to 0 \text{ as } n\to\infty$$

$\therefore S_n(x) = \sum u_n(x)$ is uniformly convergent and hence can be integrated term-by-term.

Now, $$\int_0^1 \left(\sum_{n=1}^{\infty} \frac{1-x}{n} x^n\right) dx = \sum_{n=1}^{\infty} \int_0^1 \left(\frac{x^n - x^{n+1}}{n}\right) dx \quad \left(\text{as } \sum \frac{1-x}{n}x^n \text{ is uniformly convergent}\right)$$
$$= \sum_{n=1}^{\infty} \left(\frac{x^{n+1}}{n(n+1)} - \frac{x^{n+2}}{n(n+2)}\right)\Bigg|_0^1$$
$$= \sum_{n=1}^{\infty} \left(\frac{1}{n(n+1)} - \frac{1}{n(n+2)}\right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n(n+1)} - \sum_{n=1}^{\infty}\left(\frac{1}{n(n+2)}\right)$$

$$= \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) - \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+2}\right)$$

$$= 1 - \frac{1}{2}\left(1 - \cancel{\frac{1}{3}} + \frac{1}{2} - \cancel{\frac{1}{4}} - \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} \cdots\right)$$

$$= 1 - \frac{1}{2}\left(1 + \frac{1}{2}\right)$$

$$= 1 - \frac{3}{4}$$

$$= \frac{1}{4}.$$