COV - Tutorial

1. Solve the Euler-Lagrange equation for the functional

$$\int_{1/10}^{1} y'(1+x^2y')dx$$

subject to the end conditions $y(\frac{1}{10}) = 19, y(1) = 1$.

2. Derive Euler-Lagrange equation for the variational problem

$$Extremize\ I(y) = \int_{x_1}^{x_2} F(x,y,y')dx,\ \ y(x_1) = y_1\ and\ y(x_2) = y_2.$$

Deduce Beltrami identity from it.

*3. Find the curve on which the functional

$$\int_0^1 (y'^2 + 12xy)dx\ with\ y(0) = 0, y(1) = 1$$

has extremum value.

4. Find an extremal for the functional $I(y) = \displaystyle\int_0^{\pi/2} [y'^2 - y^2]dx$ which satisfies the boundary conditions $y(0) = 0$ and $y(\frac{\pi}{2}) = 1$.

5. Show that the Euler-Lagrange equation can also be written in the form

$$F_y - F_{y'x} - F_{y'y}y' - F_{y'y'}y'' = 0.$$

*6. It is required to determine the continuously differentiable function $y(x)$ which minimizes the integral $I(y) = \displaystyle\int_0^1 (1+y'^2)dx$, and satisfies the end conditions $y(0) = 0, y(1) = 1$.

   (a) Obtain the relevant Euler equation, and show that the stationary function is $y = x$.

   (b) With $y(x) = x$ and the special choice $\eta(x) = x(1-x)$ and with the notation $I(\epsilon) = \displaystyle\int_0^1 F(x, y+\epsilon\eta(x), y'+\epsilon\eta'(x))dx$, calculate $I(\epsilon)$ and verify directly that $\frac{dI(\epsilon)}{d\epsilon} = 0$ when $\epsilon = 0$.

7. Find the extremal of the following functionals

   (a) $I(y) = \displaystyle\int_{x_1}^{x_2} \left[y^2 - (y')^2 - 2y\cos hx\right] dx, \quad y(x_1) = y_1\ \&\ y(x_2) = y_2$

   (b) $I(y) = \displaystyle\int_{x_1}^{x_2} \frac{1+y^2}{y'^2}dx$

   *(c) $I(y) = \displaystyle\int_{x_1}^{x_2} \frac{\sqrt{1+y'^2}}{x}dx$

(d) $I(y) = \int_0^1 (xy + y^2 - 2y^2y')dx, \quad y(0) = 1, y(1) = 2$

(e) $I(y) = \int_{x_1}^{x_2} (y^2 + y'^2 - 2y\sin x)dx$

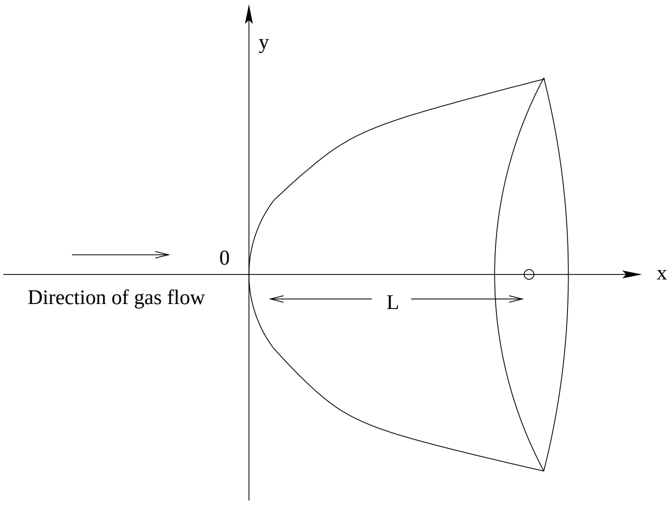
(f) $\int_0^{\pi/2} (y'^2 - y^2 + 2xy)dx, \quad y(0) = 0, y(\frac{\pi}{2}) = 0$

(g) $\int_{x_1}^{x_2} (y^2 + 2xyy')dx; \quad y(x_1) = y_1, y(x_2) = y_2$

*(h) $\int_0^{\pi} (4y\cos x - y^2 + y'^2)dx; \quad y(0) = 0, y(\pi) = 0$

*(i) $I(y) = \int_{x_0}^{x_1} (y^2 + y'^2 + 2ye^x)dx$

8. Determine the shape of solid of revolution moving in a flow of gas with least resistance.



(Hint : The total resistance experienced by the body is $I(y) = 4\pi\rho v^2 \int_0^L yy'^3 dx$ where ρ is the density, v is the velocity of gas relative to the solid).

9. Prove the following facts by using COV:
   (a) The shortest distance between two points in a plane is a **straight line.**
   (b) The curve passing through two points on xy plane which when rotated about $x - axis$ giving a minimum surface area is a **Catenary.**
   (c) The path on which a particle in absence of friction slides from one point to another in the shortest time under the action of gravity is a **Cycloid**(Brachistochrone Problem).

*10. Find the extremal of the functional

$$I(y) = \int_0^{\pi} (y'^2 - y^2)dx, \quad y(0) = 0, \ y(\pi) = 1$$

and subject to the constraint $\int_0^{\pi} y\,dx = 1.$

11. Find the extremal of the isoperimetric problem

$$\text{Extremize} \quad I(y) = \int_1^4 y'^2 dx, \quad y(1) = 3, y(4) = 24$$

subject to $\displaystyle\int_1^4 y\,dx = 36$.

*12. Determine $y(x)$ for which $\displaystyle\int_0^1 x^2 + y'^2 dx$ is stationary subject to $\displaystyle\int_0^1 y^2 dx = 2, \quad y(0) = 0, y(1) = 0$.

13. Find the extremal of $I = \displaystyle\int_0^\pi y'^2 dx$ subject to $\displaystyle\int_0^\pi y^2 dx = 1$ and satisfying $y(0) = y(\pi) = 0$.

*14. Given $F(x, y, y') = (y')^2 + xy$. Compute ΔF and δF for $x = x_0$, $y = x^2$ and $\delta y = \epsilon x^n$.

15. Find the extremals of the isopermetric problem

$$I(y) = \int_{x_0}^{x_1} y'^2 dx$$

given that $\displaystyle\int_{x_0}^{x_1} y dx = \text{constant}$.

16. Prove the following facts by using COV:

*(a) The geodesics on a sphere of radius a are its great circles.

(b) The sphere is the solid figure of revolution which, for a given surface area has maximum volume.

17. If y is an extremizing function for

$$I(y) = \int_{x_1}^{x_2} F(x, y, y'), y(x_1) = y_1, \mathit{and}\ y(x_2) = y_2$$

then show that of $\delta I = 0$ for the function y.

*18. Find $y(x)$ for which

$$\delta \left\{ \int_{x_0}^{x_1} (\frac{y'^2}{x^3}) dx \right\} = 0$$

and $y(x_1) = y_1$ and $y(x_2) = y_2$.

19. Write down the Euler-Lagrange equation for the following extremization problems

(i) Extremize $I(u, v) = \displaystyle\int_D\int F(x, y, u, v, u_x, u_y, v_x, v_y) dx\ dy$ where x, y are independent variables and u, v are dependent variables. D is a domain in xy plane and u and v are prescribed on the boundary of D.

(ii) Extremize $I(y) = \displaystyle\int_{x_0}^{x_1} F(x, y, y^{(1)}, y^{(2)}, ..., y^{(m)} dx$

$$y(x_0) = y_0, \ y(x_1) = y_1$$

:

$$y'(x_0) = y'_0, \ y'(x_1) = y'_1$$

....

$$y^{(m-1)}(x_0) = y_0^{(m-1)}, \ y^{(m-1)}(x_1) = y_1^{(m-1)}$$

(iii) Max or Min $I(y) = \int_{x_1}^{x_2} F(x, y, y')dx$ where y is prescribed at the end points $y(x_1) = y_1$, $y(x_2) = y_2$, and y is also to satisfy the integral constraint condition $J(y) = \int_{x_1}^{x_2} G(x, y, y')dx = k$, where k is a prescribed constant.

*20. Show that the extremals of the problem
Extremize $I(y) = \int_{x_1}^{x_2} [p(x)y'^2 - q(x)y^2]dx$
where $y(x_1)$ and $y(x_2)$ are prescribed and y satisfies a constraint $\int_{x_1}^{x_2} r(x)y^2(x)dx = 1$, are solutions of the differential equation $\frac{d}{dx}(p\frac{dy}{dx}) + (q + \lambda r)y = 0$
where λ is a constant.

*21. Reduce the BVP

$$\frac{d}{dx}(x\frac{dy}{dx}) + y = x, \ y(0) = 0, \ y(1) = 1$$

into a variational problem and use Rayleigh-Ritz method to obtain an approximate solution in the form

$$y(x) \approx x + x(1 - x)(c_1 + c_2x)$$

22. (Principle of least Action) A particle under the influence of a gravitational field moves on a path along which the kinetic energy is minimal. Using calculus of variation prove that the trajectory is parabolic.
(Hint: Minimize $I = \int \frac{1}{2}mv^2dt = \int \frac{1}{2}mvds = \int \sqrt{u^2 - 2gy}\sqrt{1 + y'^2}dx$)
where u is the initial speed.

23. Show that the curve which extremizes the functional $I(y) = \int_0^{\pi/4} (y''^2 - y^2 + x^2)dx$ under the conditions $y(0) = 0, y'(0) = 1, y(\pi/4) = y'(\pi/4) = \frac{1}{\sqrt{2}}$ is $y = \sin x$.

24. Find a function $y(x)$ such that $\int_0^{\pi} y^2dx = 1$ which makes $\int_0^{\pi} (y'')^2dx$ a minimum if $y(0) = 0 = y(\pi), \quad y''(0) = 0 = y''(\pi)$.

*25. Find the extremals of the following functional

$$I(y) = \int_{x_1}^{x_2} 2xy + (y''')^2dx$$

26. Find the extremals of the functional

$$I(u, v) = \int_{x_0}^{x_1} 2uv - 2u^2 + u'^2 - v'^2dx$$

where u and v are prescribed at the end points.

27. Find a function $y(x)$ such that $\int_0^{\pi} y^2 dx = 1$ which makes $\int_0^{\pi} y''^2 dx$ a minimum if $y(0) = 0 = y(\pi), \quad y''(0) = 0 = y''(\pi)$

*28. Show that the functional $\int_0^{\pi/2} 2xy \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 dt$ such that $x(0) = 0,\ x(\pi/2) = -1,\ y(0) = 0,\ y(\pi/2) = 1$ is stationary for $x = -\sin t,\ y = \sin t$.

*29. Explain Rayleigh - Ritz method to find an approximate solution of the variational problem

$$\text{Extremize}\, I(y) = \int_{t_0}^{t_1} F(x, y, y')dx$$

with prescribed end conditions $y(x_1) = y_1 \quad \& \quad y(x_2) = y_2$.

30. Solve the BVP $y'' + y + x = 0, \quad y(0) = y(1) = 0$ by Rayleigh - Ritz method.

31. Use Rayleigh - Ritz method to find an approximate solution of the problem
$y'' - y + 4xe^x = 0, \quad y'(0) - y(0) = 1, \quad y'(1) + y(1) = -e$.