Assignment-2
AV223- Signals and Systems

SAURABH KUMAR
SC22B146

① Consider a system governed by the second-order differential eqn.

$$a\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + cy(t) = x(t),$$

where a, b, and c are non-negative real numbers.

ⓐ Show that this system is LTI.

Soln: Linearity: Input: $x_1(t)$, output $y_1(t)$

$$\Rightarrow a\frac{d^2y_1(t)}{dt^2} + b\frac{dy_1(t)}{dt} + cy_1(t) = x_1(t) \quad -①$$

Input: $x_2(t)$, output $y_2(t)$.

$$\Rightarrow a\frac{d^2y_2(t)}{dt} + b\frac{dy_2(t)}{dt} + cy_2(t) = x_2(t). \quad ⓘⓘ$$

$$K_1 ⓘ + K_2 ⓘⓘ \Rightarrow a\frac{d^2(K_1y_1 + K_2y_2)}{dt^2} + b\frac{d^2}{dt^2}(K_1y_1 + K_2y_2) + c(K_1y_1 + K_2y_2) = K_1x_1 + K_2x_2$$

∴ Hence, the system is additive and homogeneous.

⇒ System is linear.

Time Invariance: Replace $t \to t + a$.

$$a\frac{d^2y(t+a)}{dt^2} + b\frac{dy(t+a)}{dt} + cy(t+a) = x(t+a).$$

Delaying the input delayed the output.

⇒ ∴ System is time-invariant.

∴ The system is LTI.

ⓑ Consider a complex exponential input $x(t) = e^{j\omega t}$. Show that the resulting output is of the form

$$y(t) = H(\omega)e^{j\omega t}$$

for some complex no. $H(\omega)$.

Soln: $x(t) = e^{j\omega t}$

let $y(t) = Ke^{j\omega t}$

$\therefore a\frac{d^2(Ke^{j\omega t})}{dt^2} + b\frac{d(Ke^{j\omega t})}{dt} + cKe^{j\omega t} = e^{j\omega t}$

$\Rightarrow aK(j\omega)^2 e^{j\omega t} + bK(j\omega)e^{j\omega t} + cKe^{j\omega t} = e^{j\omega t}$

$\Rightarrow -aK\omega^2 + jbK\omega + cK = 1$

$\Rightarrow K = \frac{1}{(c-a\omega^2)+j(\omega b)} \times \frac{(c-a\omega^2)-j(\omega b)}{[(c-a\omega^2)-j\omega b]}$

$= (c-a\omega^2) - j(\omega b)$

$= H(\omega).$

$\therefore$ Output is of the form $H(\omega)e^{j\omega t}$.

(c) Consider now the sinusoidal input $x(t) = A\cos(\omega t+\theta)$ and express the resulting output as a sum of sinusoids with real coefficients.

Soln: $x(t) = A\cos(\omega t+\theta)$.

$\therefore a\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + cy(t) = A\cos(\omega t+\theta)$

Let $y(t) = K_1\cos(\omega t+\theta) + K_2\sin(\omega t+\theta)$

$\Rightarrow \dot{y}(t) = -K_1\omega\sin(\omega t+\theta) + K_2\omega\cos(\omega t+\theta)$

$\Rightarrow \ddot{y}(t) = -K_1\omega^2\cos(\omega t+\theta) - K_2\omega^2\sin(\omega t+\theta)$.

$\therefore A\cos(\omega t+\theta) = \cos(\omega t+\theta)\left[-aK_1\omega^2 + bK_2\omega + cK_1\right]$

$+ \sin(\omega t+\theta)\left[-aK_2\omega^2 - K_1 b\omega + K_2 c\right]$

Comparing coefficients,

$A = -aK_1\omega^2 + K_2 b\omega + cK_1$

$0 = -aK_2\omega^2 - K_1 b\omega + K_2 c$

$\Rightarrow K_1 = \frac{K_2(c-a\omega^2)}{b\omega}$

$\Rightarrow A = -a\frac{K_2}{b\omega}(c-a\omega^2)\cdot\omega^2 + K_2 b\omega + \frac{cK_2}{b\omega}(c-a\omega^2)$

$\Rightarrow A = K_2\left[b\omega + \frac{c}{b\omega}(c-a\omega^2) - \frac{a}{b}(c-a\omega^2)\right]$

$$\Rightarrow K_2 = \frac{A}{b\omega + \frac{c}{b\omega}(c - a\omega^2) - \frac{a}{b}(c - a\omega^2)}$$

$$\Rightarrow K_1 = \frac{A\,(c - a\omega^2)}{b^2\omega^2 + c\,(c - a\omega^2) - a\omega\,(c - a\omega^2)}$$

$$\therefore\ y(t) = \frac{A\,(c - a\omega^2)}{b^2\omega^2 + c\,(c - a\omega^2) - a\omega(c - a\omega^2)} \cdot \sin(\omega t + \theta) + \frac{A}{b\omega + \frac{c}{b\omega}(c - a\omega^2) - \frac{a}{b}(c - a\omega^2)} \times \cos(\omega t + \theta)$$

(1.17) Consider a continuous-time system with input $x(t)$ and output $y(t)$ related by

$$y(t) = x(\sin(t)).$$

(a) Is this system causal?

(b) Is this system linear?

Soln: (a) Causality: For any time $t = \tau$.

~~Input~~ Output $y(\tau)$ requires input $x(\sin(\tau))$ at that time.

$\therefore$ system is causal.

(b) Linearity: Inputs: $x_1(t)$, $x_2(t)$; outputs: $y_1(t)$, $y_2(t)$.

$$y_1(t) = x_1 \sin(t)$$
$$y_2(t) = x_2 \sin(t)$$
$$\Rightarrow y_1(t) + y_2(t) = x_1 \sin(t) + x_2 \sin(t).$$

$\therefore$ It is additive.

Input: $\alpha x_1(t)$. Output: $Y(t)$.

$$Y(t) = \alpha x_1 \sin(t)$$
$$= \alpha y_1(t).$$

$\therefore$ system is homogeneous.

Hence, the system is linear.

(1.18) Consider a discrete-time system with input $x[n]$ and Output $y[n]$ related by

$$y[n] = \sum_{K=n-n_0}^{n+n_0} x[K],$$

where n_0 is a finite positive integer.

(a) Is this system linear?

(b) Is this system time-invariant?

(c) If $x[n]$ is known to be bounded by a finite integer B (i.e, $|x[n]| < B$ for all n), it can be shown that $y[n]$ is bounded by a finite no. C. We can conclude that the given system is stable. Express C in terms of B and n_0.

Soln: (a) Input: x_1, x_2, output: y_1, y_2.

$$y_1[n] = \sum x_1[K]$$
$$y_2[n] = \sum x_2[K]$$

Input: $x_1 + x_2$, output: y_3

$$y_3[n] = \sum (x_1+x_2)[K]$$
$$= \sum x_1[K] + \sum x_2[K]$$
$$= (y_1 + y_2)[n]$$

$\therefore$ System is additive.

Input: $a\,x_1$, output: y_4.

$$y_4[n] = \sum a x_1[K]$$
$$= a \sum x_1[K]$$
$$= a\,y_1$$

$\therefore$ System is homogeneous.

Hence, the system is linear.

(b) Replace $n \rightarrow n+c$.

$$y[n+c] = \sum_{K=(n+c)-n_0}^{(n+c)+n_0} x[K].$$

Delaying the input, delayed the output.

$\therefore$ System is time-invariant.

(c) $|x[n]| < B$, for all n.

$|y[n]| < c$, for finite c.

$|y[n]| = |x[n-n_0] + x[n-n_0+1] + \dots + x[n+n_0-1] + x[n+n_0]|$

$\Rightarrow |y[n]| \leq |x[n-n_0]| + |x[n-n_0+1]| + \dots + |x[n+n_0]|$.

$\Rightarrow |y[n]| \leq B(2n_0)$

$\Rightarrow |y[n]| \leq B(2n_0)$

$\Rightarrow \boxed{C \leq 2Bn_0}$.

(1.19) For each of the following i/p-o/p relationships, determine whether the corresponding system is linear, time-invariant or both.

(a) $y(t) = t^2 x(t-1)$.

Soln: I/p: x_1, x_2, O/p: y_1, y_2

$y_1(t) = t^2 x_1(t-1)$

$y_2(t) = t^2 x_2(t-1)$

I/p: $x_1 + x_2$, O/p: y_3

$y_3(t) = t^2 (x_1 + x_2)(t-1)$

$= t^2 x_1(t-1) + t^2 x_2(t-1)$

$= y_1 + y_2$

and I/p: αx_1, O/p: y_4.

$y_4(t) = \alpha t^2 x_1(t-1)$

$= \alpha y_1(t)$.

$\therefore$ System is linear

TI: $t \to t + c$

$y(t+c) = (t+c)^2 x(t+c-1)$

$= t^2 (t+c-1) + c^2 x(t+c-1) + 2tc\, x(t+c-1)$

O/p didn't change proportionally for delayed i/p.

$\therefore$ System is not time-invariant.

(b) $y[n] = x^2[n-2]$

Soln: $y_1 = x_1^2[n-2]$

$y_2 = x_2^2[n-2]$

$y_3 = (x_1+x_2)^2[n-2]$

$= x_1^2[n-2] + x_2^2[n-2] + 2x_1x_2[n-2]$

$\neq y_1 + y_2$

$\therefore$ System is not linear.

and $n \to n+c$

$y[n+c] = x^2[n+c-2]$

$\therefore$ System is TI.

(c) $y[n] = x[n+1] - x[n-1]$

Soln: $y_1 = x_1[n+1] - x_1[n-1]$

$y_2 = x_2[n+1] - x_2[n-1]$

$y_3 = (x_1+x_2)[n+1] - (x_1+x_2)[n-1]$

$= y_1 + y_2$

and, $y_4 = \alpha x[n+1] - \alpha x[n-1]$

$= \alpha y_1$

$\therefore$ Linear.

And, $n \to n+c$.

$y[n+c] = x[n+c+1] - x[n+c-1]$

$\therefore$ TI.

(d) $y[t] = \text{odd}\{x(t)\}$.

Soln: $y_1 = \text{odd}\{x_1(t)\}$

$y_2 = \text{odd}\{x_2(t)\}$

$y_3 = \text{odd}\{(x_1+x_2)(t)\}$

$= y_1 + y_2$

and, $y_4 = \text{odd}\{\alpha x_1(t)\}$

$= \alpha\, \text{odd}\{x_1(t)\}$

$= \alpha y_1(t)$.

$\therefore$ Linear.

And, $y[t+c] = \text{odd}\{x(t+c)\}$

$\therefore$ TI.

(P.20) A continuous-time linear system S with i/p $x(t)$ and o/p $y(t)$ yields the following i/p - o/p pairs:

$$x(t) = e^{j2t} \xrightarrow{S} y(t) = e^{j3t}$$
$$x(t) = e^{-j2t} \xrightarrow{S} y(t) = e^{-j3t}.$$

(a) If $x_1(t) = \cos(2t)$, determine the corresponding o/p $y_1(t)$ for system S.

(b) If $x_2(t) = \cos(2(t-\frac{1}{2}))$, determine the corresponding o/p $y_2(t)$ for system S.

Soln: (a) $x_1 = \cos(2t) = \dfrac{e^{j2t} + e^{-j2t}}{2}$

Let $x_2(t) = \dfrac{e^{j2t}}{2}$, $x_3 = \dfrac{e^{-j2t}}{2}$.

$\Rightarrow y_2(t) = \dfrac{e^{j3t}}{2}$, $y_3 = \dfrac{e^{-j3t}}{2}$

and, $y_1(t) = y_2(t) + y_3(t)$

$\Rightarrow y_1(t) = \dfrac{e^{j3t} + e^{-j3t}}{2} = \cos(3t).$

(b) $x_2(t) = \cos(2(t - \frac{1}{2})) = \cos(2t - 1) = \dfrac{e^{j(2t-1)} + e^{-j(2t-1)}}{2}$

$= \dfrac{e^{j2t} \cdot e^{-j} + e^{-j2t} \cdot e^{j}}{2}$

$\therefore y_2(t) = e^{j3t}\left(\dfrac{e^{-j}}{2}\right) + e^{-j3t}\left(\dfrac{e^{j}}{2}\right)$

$= \dfrac{e^{j(3t-1)} + e^{-j(3t-1)}}{2}$

$= \cos(3t-1).$

(4) Determine if the following system properties are valid.

(a) $y(t) = x(-t)$. Causal?

(b) $y(t) = (t+5)\,x(t)$ Memoryless?

(c) $y(t) = x(5)$ Memoryless?

(d) $y(t) = 2x(t)$. Stable (BIBO)?

Soln: (a) For $t = -1$,

$y(-1)$ requires i/p at $t = 1$.

$\therefore$ System is <u>non-causal</u>.

(b) $y(t) = (t+5)\,x(t)$.

For all t, the output depends only on the present i/p.

$\therefore$ System is <u>memoryless</u>.

(c) $y(t) = x(5)$.

Output depends on the input at $t=5$.

$\therefore$ System is not <u>memoryless</u>.

(d) $y(t) = 2x(t)$.

For $|x(t)| < B$,

$|y(t)| < 2B+1$.

As $|y(t)| < C$, for some C, the system is <u>stable</u>.

(1.34) In this problem, we explore several of the properties of even and odd signals.

(a) Show that if $x[n]$ is an odd signal, then

$$\sum_{n=-\infty}^{+\infty} x[n] = 0.$$

Soln: For odd function,

$$x[n] = -x[-n]$$

$$\Rightarrow x[n] + x[-n] = 0$$

$$\therefore \sum_{-\infty}^{\infty} x[n] = \sum_{-\infty}^{0} x[n] + \sum_{1}^{\infty} x[-n]$$

$$= \cancel{x[-\infty]} + \dots + \cancel{x[-1]} + \cancel{x[0]} + \cancel{x[1]} + \dots + \cancel{x[\infty]}$$

$$\Rightarrow \sum_{-\infty}^{\infty} x[n] = 0$$

(b) Show that if $x_1[n]$ is an odd signal and $x_2([n])$ is an even signal, then $x_1[n]\,x_2[n]$ is an odd signal.

Soln: $x_1[n] = x_1[-n]$

and, $x_2[n] = -x_2[-n]$.

$$\therefore x_1[n]\,x_2[n] = -x_1[-n]\,x_2[-n]$$

$$= -(x_2[-n]\,x_2[-n])$$

$\therefore x_1[n]\,x_2[n]$ is an odd signal.

(c) Let $x[n]$ be an arbitrary signal with even and odd parts denoted by $x_e[n] = \mathcal{E}v\{x[n]\}$,

and $x_o[n] = \mathcal{O}d\{x[n]\}$.

Show that $\sum_{n=-\infty}^{+\infty} x^2[n] = \sum_{n=-\infty}^{+\infty} x_e^2[n] + \sum_{n=-\infty}^{\infty} x_o^2[n]$.

Soln: As $x_e[n]\, x_o[n] =$ odd

$$\Rightarrow 2\sum x_e[n] x_o[n] = 0$$

$$\therefore \sum_{-\infty}^{\infty} x^2[n] = \sum_{-\infty}^{\infty} (x_e[n] + x_o[n])^2$$

$$= \sum_{-\infty}^{\infty} (x_e^2[n] + x_o^2[n] + 2x_e[n]\, x_o[n])$$

$$= \sum_{-\infty}^{\infty} (x_e^2[n] + x_o^2[n])$$

$\therefore$ Hence proved.

(5) Determine if the following system properties are valid.

(a) $y(t) = x(t) + a$ Linear?

(b) $y(t) = t\, x(2t)$ Linear?

(c) $y(t) = \int_0^T x(t-T)\, dT$ Time Invariant?

(d) $y(t) = x(2t)$ Time Invariant?

Soln: I/P: x_1, x_2 ; O/P: y_1, y_2.

$$y_1(t) = x_1(t) + a$$

$$y_2(t) = x_2(t) + a$$

I/P: $x_1 + x_2$; O/P: y_3.

$$y_3(t) = (x_1 + x_2)(t) + a$$

$$= x_1(t) + x_2(t) + a$$

$$\neq y_1 + y_2$$

$\therefore$ System is not linear.

(b) $y(t) = t\, x(2t)$.

$$y_1(t) = t\, x_1(2t)$$

$$y_2(t) = t\, x_2(2t)$$