Indian Institute of Space Science and Technology

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MA211 - Linear Algebra

1. Find the Kernel and Image of the linear map $T : R^3 \to R^3$ defined as multiplication by the matrix A, that is $T(v) = A \times v$, where

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$$

2. Find the matrix representation of the linear map $T : R^3 \to R^2$ defined by $T((x, y, z)) = (z, x - y)$ with respect to the ordered bases $B_1 = \{1, 0, 1), (0, 2, 1), (0, 0, 1)\}$ for R^3 and $B_2 = \{(1, 1), (2, 3)\}$ for R^2.

3. The matrix representation of $T : R^2 \to R^3$ is $\begin{bmatrix} 1 & 1 \\ 2 & 3 \\ -1 & -1 \end{bmatrix}$ with respect to the bases $B_1 = \{(2, 3), (-1, 1)\}$ for R^2 and $B_2 = \{(0, 1, 0), (1, 1, 0), (0, 0, 1)\}$ for R^3. Find $T((4, 3))$

4. The matrix representation of $T : R^3 \to R^3$ is $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & 2 & 1 \end{bmatrix}$ with respect to the bases $B_1 = \{(2, 1, 0), (1, 0, 0), (0, 1, 1)\}$ for the domain and $B_2 = \{(0, 1, 0), (1, 1, 0), (0, 0, 1)\}$ for the codomain. Find T.

Answers

1. $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} : R^3 \to R^3.$

   $\text{Ker}(A) = \{v = (v_1, v_2, v_3) \in R^3 / A.v = 0\}$

$$= \{(v_1, v_2, v_3)/ \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} . \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0\}$$

$$= \{(v_1, v_2, v_3)/v_1 + v_2 + 2v_3 = 0, v_1 + 2v_2 + 3v_3 = 0, -v_2 - v_3 = 0\}$$

$$= \{(v_1, v_2, v_3)/v_3 = -v_2, v_1 = v_2\}$$

$$= \{(v_2, v_2, -v_2)/v_2 \in R\}.$$

   So $\text{Ker}(A) = \{(x, x, -x)/x \in R\}$.

   A basis for $\text{Ker}(A)$ is $\{(1, 1, -1)\}$ and dimension of $\text{Ker}(A)$ is 1.

$$\text{Image}(A) = \{(w_1, w_2, w_3) \in R^3 / \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} . \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \}$$

$$= \{w_1 = v_1 + v_2 + 2v_3, w_2 = v_1 + 2v_2 + 3v_3, w_3 = -v_2 - v_3 / v_1, v_2, v_3 \in R\}$$

$$= \{(w_1, w_2, w_3)/w_1 - w_2 = w_3\}$$

$$= \{(w_1, w_2, w_1 - w_2) \in R^3 / w_1, w_2 \in R\}.$$

A basis for Image(A) is $\{(1,0,1),(0,1,-1)\}$ and the dimension of Image(A) is 2.

2. $B_1 = \{v_1 = (1,0,1), v_2 = (0,2,1), v_3 = (0,0,1)\}$ and $B_2 = \{w_1 = (1,1), w_2 = (2,3)\}$.

$T((x,y,z)) = (z, x-y)$

So, $T(v_1) = (1,1) = 1.(1,1) + 0.(2,3) = 1.w_1 + 0.w_2$

$T(v_2) = (1,-2) = 7.(1,1) + -3.(2,3) = 7.w_1 + -3.w_2$

$T(v_3) = (1,0) = 3.(1,1) + -1.(2,3) = 3.w_1 + -1.w_2$

Therefore $[T] = \begin{bmatrix} 1 & 7 & 3 \\ 0 & -3 & -1 \end{bmatrix}$.

3. We have $v_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

And $w_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $w_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $w_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

We know $(T(v_1), T(v_2)) = (w_1, w_2, w_3)[T]$.

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 \\ 3 & 4 \\ -1 & -1 \end{bmatrix}.$$

Thus $T(v_1) = (2,3,-1)$ and $T(v_2) = (3,4,-1)$.

Now $(4,3) = \alpha_1 v_1 + \alpha_2 v_2 = \alpha_1(2,3) + \alpha_2(-1,1)$.

Solving $\alpha_1 = \frac{7}{5}$ and $\alpha_2 = \frac{-6}{5}$.

Then $T((4,3)) = T(\frac{7}{5}v_1 + \frac{-6}{5}v_2) = \frac{7}{5}T(v_1) + \frac{-6}{5}T(v_2) = \frac{7}{5}(2,3,-1) + \frac{-6}{5}(3,4,-1)$.

That is, $T((4,3)) = \frac{-1}{5}(4,3,1)$.

4. We have $v_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

And $w_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad w_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

We know $(T(v_1), T(v_2)) = (w_1, w_2, w_3)[T]$.

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & 0 \\ 3 & 4 & 1 \\ 1 & 2 & 1 \end{bmatrix}.$$

Consider $(x, y, z) \in R^3$.

$(x, y, z) = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$
$= \alpha_1(2,1,0) + \alpha_2(1,0,0) + \alpha_3(0,1,1)$.

Then, $2\alpha_1 + \alpha_2 = x, \;\; \alpha_1 + \alpha_3 = y, \;\; \alpha_3 = z$.

That is, $\alpha_1 = y - z, \;\; \alpha_2 = x - 2y + 2z, \; \alpha_3 = z$.

Therefore $T((x, y, z)) = T(\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3)$

$$= \alpha_1 T(v_1) + \alpha_2 T(v_2) + \alpha_3 T(v_3)$$

$$= (y - z)[(2,3,1)] + (x - 2y + 2z)[(3,4,2)] + z[(0,1,1)]$$

$$= (2(y - z) + 3(x - 2y + 2z) + 0.z, \;\; 3(y - z) + 4(x - 2y + 2z) + 1.z, \;\; (y - z) + 2(x - 2y + 2z) + 1.z)$$

$$= (3x - 4y + 4z, 4x - 5y + 6z, 2x - 3y + 4z).$$

Thus $T((x, y, z)) = (3x - 4y + 4z, 4x - 5y + 6z, 2x - 3y + 4z)$.