Indian Institute of Space Science and Technology
Thiruvananthapuram-695 547

B.Tech Summer Examination - July 2012

MA121 - Linear Algebra and Differential Equations (2010 & 2009 Batch)

Date : 3^{rd}July, 2012 **Time: 9.30 am to 12.30 pm** **Max. Marks: 100**

SECTION A (Attempt all 10 questions - 10x5= 50 marks.)

1. Does the sequence $f_n(x) = \dfrac{n^2x^2}{1+n^3x^2}$, $x \in (0, \infty)$ converges uniformly?

2. Show that the series $\sum u_n(x) = \sum \dfrac{\sin nx}{n^3}$ converges pointwise on [0,1] to a limit $'u'$. Examine the validity of the relation $u'(x) = \displaystyle\sum_{n=1}^{\infty} u_n'(x)$.

3. Find all eigenvalues and the corresponding eigenvectors of $A = \begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix}$.

4. Find λ so that $B = \{(1,\lambda,1),(2,1,2),(2\lambda,3,2)\}$ is a basis for $\mathbb{R}^3$.

5. Check whether $W = \Big\{$ polynomial $p(t)$ of degree ≤ 3 such that $p(0) = 0$, $p(1) = 0, p(2) = 0 \Big\}$ is a subspace of P_3 and if so, find a basis.

6. Find $Ker(T)$, $range(T)$ and a basis for both kernel and range spaces of the linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$ by $T(x,y,z) = (x, y+z)$.

7. Verify whether the pair of functions $\{f(x) = x^5,\ g(x) = |x|^5\}$ are linearly independent on $\mathbb{R}$ and also compute their Wronskian. Determine whether they can be solutions of the differential equation $y'' + p(x)y' + q(x)y = 0$ with p and q continuous on $[-a, a]$, $a > 0$.

8. Solve $e^x dx + \big(e^x \cot y + 2y\, cosec\, y\big)dy = 0$, by finding appropriate integrating factor if necessary.

9. Find the path of steepest ascent up the mountain $z = 5 - \sqrt{(x^2-4)^3} + \sqrt{(y^2-9)^3}$ starting from the point $P = (2,3,5)$, using **orthogonal trajectory**.

10. Find the possible general solutions of the differential equation
$$\lambda^2 \frac{d^2y}{dt^2} + \frac{\lambda^2}{t}\frac{dy}{dt} + \left(\lambda^4 - \frac{p^2}{t^2}\right)y = 0, \quad t > 0$$
in terms of Bessel's functions, using the substitution $w = \lambda t$, where λ and p are positive integers.

[P.T.O.]

SECTION B (Attempt any 5 questions - 5x10= 50 marks.)

11. Show that the sequence $f_n(x) = nxe^{-nx^2}$ converges pointwise on [0,1]. Examine whether the relation $\lim_{n\to\infty} \int_0^1 \lim_{n\to\infty} f_n(x)dx$ holds and also whether the convergence of $\{f_n\}$ is uniform.

12. (a) Find the matrix of $T : P_3 \to P_2$ given by $T(a_0 + a_1x + a_2x^2 + a_3x^3) = a_0 + a_2x^2$ with respect to the ordered basis $B_1 = \{1, 1+x, 1+x^2, 1+x^3\}$ for P_3 and $B_2 = \{2, x, x^2\}$ for P_2.

    (b) Find the change of basis matrix P from the ordered basis $B_1 = \{(1,0),(0,1)\}$ to $B_2 = \{(1,3),(2,5)\}$ of $\mathbb{R}^2$.

13. (a) Let $[T] = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -3 & -4 \end{bmatrix}$ be the matrix representation of the linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$ with respect to the ordered bases $\{(1,-1,1),(2,3,-1),(1,1,-1)\}$ of $\mathbb{R}^3$ and $\{(1,1),(2,3)\}$ of $\mathbb{R}^2$. Find T.

    (b) Let $B_1 = \{1, x, x^2\}$ and $B_2 = \{2, 3+x, x+x^2\}$ be two bases of P_2.
    Let $[T]_{B_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ be a linear operator on P_2. Find $[T]_{B_2}$.

14. (a) Does the **Picard's theorem** guarantee the existence of a unique solution of the initial value problem $\dfrac{dy}{dx} = 2\sqrt{y},\ y(2012) = 0$, on some interval about $x = 2012$? Justify your answer.

    (b) Solve $y^2(y - px) = x^4p^2,\ p = \dfrac{dy}{dx}$, by reducing to Clairaut's form using the substitution $X = \dfrac{1}{x}, Y = \dfrac{1}{y}$.

15. (a) Compute the indicial equation, its roots and hence find the first three terms of each of two linearly independent series solutions of $x^2y'' + (x^2 - 2x)y' + 2y = 0$, about the point $x = 0$ for $x > 0$.

    (b) Find the real-valued general solution of $\mathbf{L}(y) \equiv y^{vi} - y^{iv} - 2y'' = 0$. Also find the correct linear combination to obtain particular integral of $\mathbf{L}(y) = x\sinh(\sqrt{2}\,x)$ by the method of undetermined coefficients. (Hint: $\sinh x = (e^x - e^{-x})/2$.)

16. (a) Using the identity $\dfrac{d}{dx}[x^pJ_p(x)] = x^pJ_{p-1}(x)$, satisfied by the Bessel function $J_p(x)$ of first kind of order p and Rolle's theorem, show that between any two positive roots of $J_{p+1}(x) = 0$, there is a root of $J_p(x) = 0$.

    (b) Using the generating function $\left(1 - 2xt + t^2\right)^{-1/2}$ for the n-th Legendre polynomial $P_n(x)$, show that

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x).$$

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