Indian Institute of Space Science and Technology

Complex Analysis

TUTORIAL - V

1. In each case, write the principal part of the function at its isolated singular point and determine whether that point is a pole, a removable singular point, or an essential singular point:

   (a) $z \exp\left(\frac{1}{z}\right)$; (b) $\frac{z^2}{1+z}$; (c) $\frac{\sin z}{z}$; (d) $\frac{\cos z}{z}$; (e) $\frac{1}{(2-z)^3}$.

2. Show that the singular point of each of the following functions is a pole. Determine the order m of that pole and the corresponding residue.

   (a) $\frac{1-\cosh z}{z^3}$; (b) $\frac{1-\exp(2z)}{z^4}$; (c) $\frac{\exp(2z)}{(z-1)^2}$.

3. Show that

   (a) $\operatorname*{Res}_{z=-1} \frac{z^{1/4}}{z+1} = \frac{1+i}{\sqrt{2}} \quad (|z|>0,\ 0<\arg z<2\pi)$;

   (b) $\operatorname*{Res}_{z=i} \frac{\log z}{(z^2+1)^2} = \frac{\pi+2i}{8}$;

   (c) $\operatorname*{Res}_{z=i} \frac{z^{1/2}}{(z^2+1)^2} = \frac{1-i}{8\sqrt{2}} \quad (|z|>0,\ 0<\arg z<2\pi)$.

4. Find the value of the integral

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}dz,$$

   taken counterclockwise around the circle

   (a) $|z-2|=2$;

   (b) $|z|=4$.

5. Find the value of the integral

$$\int_C \frac{dz}{z^3(z+4)},$$

   taken counterclockwise around the circle

   (a) $|z|=2$;

   (b) $|z+2|=3$.

6. Show that

   (a) $\operatorname*{Res}_{z=\pi i} \frac{z-\sinh z}{z^2\sinh z} = \frac{i}{\pi}$;

   (b) $\operatorname*{Res}_{z=\pi i} \frac{\exp(zt)}{\sinh z} = -2\cos \pi t$.

7. Use residues to evaluate the improper integrals.

   (a) $\displaystyle\int_0^\infty \frac{dx}{(x^2+1)^2}$.

   (b) $\displaystyle\int_0^\infty \frac{dx}{x^4+1}$.

   (c) $\displaystyle\int_0^\infty \frac{x^2\,dx}{(x^2+1)(x^2+4)}$.

   (d) $\displaystyle\int_{-\infty}^\infty \frac{dx}{x^2+2x+2}$.

8. Use residues to evaluate the improper integrals.

   (a) $\displaystyle\int_0^\infty \frac{\cos ax}{x^2+1}dx\ (a>0)$.

   (b) $\displaystyle\int_0^\infty \frac{\cos ax}{(x^2+b^2)^2}dx\ (a>0, b>0)$.

   (c) $\displaystyle\int_0^\infty \frac{x\sin 2x}{x^2+3}dx$.

   (d) $\displaystyle\int_{-\infty}^\infty \frac{x\sin ax}{x^4+4}dx\ (a>0)$.

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