Vector Calculus Lecture Note

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Chapter 1

The Space of Vectors

1.1 Introduction

Though the need of the conception of vectors was realized after the discovery of physical quantities carrying multiple values together, e.g., velocity, force, e.t.c., significant development in the systematic study of vector is done in the recent era, in the mid of 19th century. The primary development in the theory of vectors is witnessed in the late 17th century, while studying the of structure of complex numbers. By the passage of time the theory got enriched, as the need grew, to understand the geometrical objects and physical phenomenons. The modern theory vectors which is a quite sophisticated in nature is developed between late 19th century and early 20th century.

In the early development, vectors were defined to be finite straight line segments with unique arrow heads, where the length of the straight line segments represented the "magnitude" of the vectors and the arrow determined the "direction" of the vector; but it failed to explain many properties of vectors which should be satisfied naturally. Later, a corrected model of vectors was formalized by defining vectors as pairs of points in the corresponding space; and subsequently more refined and formal definition represents vectors as points in the ambient space.

The modern definition of vectors is axiomatic in nature; it defines vector by enlisting all the properties which are wished to be imposed (manifest) upon the vectors. Although these properties altogether endows the set of vectors with two structures, one is algebraic structure and the other one is spatial structure, the set of vectors, in general, is defined only through algebraic properties; and afterwards the spatial structure is manifested on it.

1.2 Algebraic Structure of Vectors

Definition 1.2.1. A set of elements $V = \{\bar{v}_i | \; i \in \lambda\}$ is said to be a space of vectors (or a vector space) over the reals $\mathbb{R}$ (or complex numbers $\mathbb{C}$) if it has the following algebraic structure:

(1) (Existence of binary operation) There exists a function $+ : V \times V \longrightarrow V$ which is called the "addition" operation. Through this binary operation two elements $\bar{v}_1$ and $\bar{v}_2$ from V generate a third element which we denote by $\bar{v}_1 + \bar{v}_2$, in V.

(2) (Associativity of addition) Each $\bar{v}_1$, $\bar{v}_2$ and $\bar{v}_3$ in V should satisfy $(\bar{v}_1 + \bar{v}_2) + \bar{v}_3 = \bar{v}_1 + (\bar{v}_2 + \bar{v}_3)$.

(3) (Existence of additive identity) There exists an element $\bar{0}$ such that $\bar{v}+\bar{0} = \bar{0}+\bar{v} = \bar{v}$ for all $\bar{v}$ in V. Such element $\bar{0}$ is called an additive identity of V.

(4) (Existence of additive inverse) For each $\bar{v}$ in V there exists an element $\bar{v}'$ such that $\bar{v}+\bar{v}' = \bar{v}'+\bar{v} = \bar{0}$. Such element $\bar{v}'$ is called an additive inverse of $\bar{v}$ in V.

(5) (Commutativity of addition) Each $\bar{v}_1$ and $\bar{v}_2$ in V must satisfy $\bar{v}_1 + \bar{v}_2 = \bar{v}_2 + \bar{v}_1$.

(6) (Existence of outer multiplication) There exists a function $\cdot : \mathbb{R} \times V \longrightarrow V$ which is called the "outer multiplication" operation. For an element α form $\mathbb{R}$ and an element $\bar{v}$ from V, this operation generates a new element of V, which is denoted by $\alpha.\bar{v}$.

(7) (Distribution of addition) Any element α in $\mathbb{R}$ and $\bar{v}_1, \bar{v}_2$ in V should satisfy $\alpha.(\bar{v}_1 + \bar{v}_2) = \alpha.\bar{v}_1 + \alpha.\bar{v}_2$.

(8) (Distribution of multiplication) For α, β in $\mathbb{R}$ and for $\bar{v}$ in V we must have $(\alpha + \beta).\bar{v} = \alpha.\bar{v} + \beta.\bar{v}$.

(9) (Compatibility of multiplication) Any α, β in $\mathbb{R}$ any $\bar{v}$ in V must satisfy $(\alpha\beta).\bar{v} = \alpha(\beta.\bar{v}) = \beta.(\alpha\bar{v})$.

(10) (Identity of multiplication) For any $\bar{v}$ in V we should have $1.\bar{v} = \bar{v}$.

Exercise 1.2.2. Show that in a space of vectors

a. Additive identity element is unique, i.e., there exists one and only one additive identity element in a space of vectors; and it is denoted by $\bar{0}$.

b. Each element $\bar{v}$ has unique additive inverse which is denoted by $-\bar{v}$.

c. For an element $\bar{v} \in V$, if we have $\bar{v} + \bar{v} = \bar{v}$, then $\bar{v} = \bar{0}$.

d. For any element $\bar{v}$, we must have $0.\bar{v} = \bar{0}$.

e. For any element $\bar{v}$, the element $-1.\bar{v}$ is the additive inverse of $\bar{v}$, i.e., $-1.\bar{v} = -\bar{v}$.

One may ask whether the above definition make sense. The following examples will give an affirmative answer to the question and hence establish the existence of the space of vectors.

Example 1.2.3.

(I) Let $V = \mathbb{R}^2 = \{(x, y) | \ x, y \in \mathbb{R}\}$. Define the addition operation "+" by $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$, i.e., coordinate wise addition. One can easily check that the operation + is associative and commutative in nature; $(0, 0)$ is the additive identity; and for any vector (x, y) in $\mathbb{R}$, $(-x, -y)$ is the additive inverse of it. Define the outer multiplication operation "." by $\alpha.(x, y) =$

$(\alpha x, \alpha y)$, i.e., coordinate wise multiplication and by definition it is obvious that "." satisfies the properties (6)-(10) of Definition 1.2.1. This shows that V is a space of vectors over $\mathbb{R}$.

(II) In a similar fashion $\mathbb{R}^n$, $n \geq 1$ forms space of vectors over $\mathbb{R}$ with respect to coordinate wise addition and multiplication.

(III) Suppose $(x_0, y_0) \in \mathbb{R}^2$. Let $L_{(x_0,y_0)} = \{(\lambda x_0, \lambda y_0) \mid \lambda \in \mathbb{R}\}$. One should be able to see that $L_{(x_0,y_0)}$ is a space of vectors over $\mathbb{R}$. Note that the space of vectors $L_{(x_0,y_0)}$ is a subset of the space of vectors $\mathbb{R}^2$.

(IV) Let $V = \mathcal{C}[0,1]$ be the set of all real valued continuous functions over $[0,1]$. For $f, g \in V$, define $f+g$ by $(f+g)(x) = f(x)+g(x)$ for all $x \in [0,1]$, i.e., point wise addition; and for $\alpha \in \mathbb{R}$ and $f \in V$ define $\alpha.f$ by $(\alpha.f)(x) = \alpha f(x)$ for all $x \in [0,1]$. It is easy to see that with respect these two operations V forms a space of vectors over $\mathbb{R}$. The additive identity of this space is the constant function zero, i.e., the function $\underline{0}(x) = 0$ for all $x \in [0,1]$; and for each element $f \in V$, the additive inverse of f is the function $-f$ given by $(-f)(x) = -f(x)$ for all $x \in [0,1]$.

(V) Let $V = \mathcal{M}_{m\times n}(\mathbb{R})$ be the set of all $m \times n$ matrices with real entries. Let "+" denote the usual matrix addition operation, i.e., for $A = (a_{ij})$ and $B = (b_{ij})$ in V, $A+B$ is defined by $A+B = (a_{ij}+b_{ij})$. Define an operation $. : \mathbb{R} \times V \longrightarrow V$ by $\alpha.(a_{ij}) = (\alpha a_{ij})$ where $\alpha \in \mathbb{R}$ and $A = (a_{ij}) \in V$. It is easy to check that with respect to these two operations V forms a space of vectors over $\mathbb{R}$.

Definition 1.2.4. *Let V be a space of vectors over $\mathbb{R}$ and $S := \{\bar{v}_1, \bar{v}_2, \cdots, \bar{v}_n\}$ be a collection of vectors from V. By a linear combination of the vectors in S we mean an element of the form $\alpha_1\bar{v}_1 + \alpha_2\bar{v}_2 + \cdots + \alpha_n\bar{v}_n$ where $\alpha_i \in \mathbb{R}$ are constants.*

Let $< S >:= \{\alpha_1\bar{v}_1 + \alpha_2\bar{v}_2 + \cdots + \alpha_n\bar{v}_n \mid \alpha_i \in \mathbb{R}\}$ be the collection of all possible linear combinations of vectors from S. It can be proved that $< S >$ forms a vector space "being in V". $< S >$ is called the linear span/hull of vectors in S. Sometimes $< S >$ is denoted by $L(S)$.

Example 1.2.5. 1. *Let $\bar{v} \in \mathbb{R}^2$. Then $< \{\bar{v}\} >$ in $\mathbb{R}^2$ is a line passing though origin $(0,0)$ and $\bar{v}$.*

2. *Let $\bar{v}_1, \bar{v}_2 \in \mathbb{R}^3$, then $< \{\bar{v}_1, \bar{v}_2\} >$ is either a straight line passing though origin or a plane containing $(0,0), \bar{v}_1$ and $\bar{v}_2$ in $\mathbb{R}^3$.*

3. *Let $\bar{v}_1, \bar{v}_2 \in \mathbb{R}^2$, then $< \{\bar{v}_1, \bar{v}_2\} >$ is either a straight line passing though origin or the whole plane $\mathbb{R}^2$.*

4. *Let $e_1 = (1,0)$ and $e_2 = (0,1)$. It can be checked that $< \{e_1, e_2\} >= \mathbb{R}^2$.*

5. *Let $e_1 = (1,0,0), e_2 = (0,1,0)$ and $e_3 = (0,0,1)$. It can be checked that $< \{e_1, e_2, e_3\} >= \mathbb{R}^3$.*

Definition 1.2.6. *A subset S of a vector space V over $\mathbb{R}$ is called a basis, if S generates V, i.e., $< S >= V$; and any proper subset of S can not generate V*

Example 1.2.7. 1. *$\{e_1, e_2\}$ is a basis of $\mathbb{R}^2$.*

2. *$\{(1,1),\ (2,2)\}$ is not a basis of $\mathbb{R}^2$.*

3. $\{(1,1),\ (2,1)\}$ *is a basis of* $\mathbb{R}^2$. *(Prove it!)*

4. $\{(1,1),\ (2,0),\ (0,3)\}$ *is not a basis of* $\mathbb{R}^2$.

Exercise 1.2.8. *Show that any basis of* $\mathbb{R}^m$ *contains exactly* m *elements.*

We are now ready to introduce the axiomatic notion of length of a vector through a function called "norm" which will give a spatial structure on the space of vectors. The concept of length of vector is closely related to the thought of distance and hence endowing the space of vectors by this spatial structure will enable us to talk about continuity, differentiability, integrability of functions on the space of vectors.

1.3 The Spatial Structure of Vectors

Definition 1.3.1. Let V be a space of vectors over $\mathbb{R}$. A function $\|.\| : V \longrightarrow \mathbb{R}$ is called norm if it satisfies the following properties:

N-1: $\|\bar{v}\| \geq 0$ for all $\bar{v}$ in V; and $\|\bar{v}\| = 0$ if and only if $\bar{v} = \bar{0}$.

N-2: $\|\alpha\bar{v}\| = |\alpha|\|\bar{v}\|$ for all α in $\mathbb{R}$ and for all $\bar{v}$ in V.

N-3: $\|\bar{v_1} + \bar{v_2}\| \leq \|\bar{v_1}\| + \|\bar{v_2}\|$ for all v_1, v_2 in V.

For any vector $\bar{v}$ in V, $\|\bar{v}\|$ is called the norm or length of $\bar{v}$. A space of vectors endowed with a norm function is called a normed space. Geometrically norm of a vector $\bar{v}$ represents the length of line segment joining $\bar{0}$ and $\bar{v}$.

Example 1.3.2.

(I) Let $V = \mathbb{R}^m$ where n is a natural number. We already have seen that V is a vector space over $\mathbb{R}$. For any $\bar{v} = (v_1, v_2, \cdots, v_m)$ in V define $\|\bar{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots v_m^2}$. Then one can check that the function $\|.\| : V \longrightarrow \mathbb{R}$ is a norm on V. This norm is called the Euclidean norm on $\mathbb{R}^m$.

(II) Let $V = \mathcal{C}[0,1]$. Then V is a vector space over $\mathbb{R}$. Define a function $\|.\| : V \longrightarrow \mathbb{R}$ by $\|f\| = \sup \{|f(x)| \mid x \in [0,1]\}$. It can be seen that the function $\|.\|$ is a norm on V.

(III) For the same vector space as in the previous example, for any $f \in \mathcal{C}[0,1]$ define $\|f\| = \left(\int_0^1 f^2\right)^{1/2}$. One can show that the defined function $\|.\| : V \longrightarrow \mathbb{R}$ is a norm called the L^2-norm..

(IV) Let $V = \mathcal{M}_{m\times n}(\mathbb{R})$. It has already been established that V is a vector space over $\mathbb{R}$. Define $\|.\| : V \longrightarrow \mathbb{R}$ by $\|A\| = \sup \{\|A\bar{x}^T\| \mid \bar{x} \in \mathbb{R}^n,\ \|\bar{x}\| = 1\}$ where A is in V. The reader may take it as an exercise to show that the function $\|.\|$ is a norm on V.

Having a norm function $\|.\|$ on a space of vectors we can define a metric, i.e., a distance function on V which measures the distance between any two elements in V such that the distance between $\bar{0}$ and any element $\bar{v}$ in V remains the desired one, $\|\bar{v}\|$.

Definition 1.3.3. Let V be a space of vectors over $\mathbb{R}$ and $\|.\| : V \longrightarrow \mathbb{R}$ a norm. Define a function $d : V \times V \longrightarrow \mathbb{R}$ by $d(\bar{x}, \bar{y}) = \|\bar{x} - \bar{y}\|$ for all $\bar{x}, \bar{y}$ in V. This function d is called the metric or the distance function on V induced by the norm $\|.\|$.

We see that this metric has some nice characteristics.

Proposition 1.3.4. *Suppose* V *be a space of vectors endowed with a norm* $\|.\|$ *and* d *the metric induced by the norm* $\|.\|$*. Then the metric* $d : V \times V \longrightarrow \mathbb{R}$ *has the following properties:*

M-1: (Non-negativity) $d(\bar{x}, \bar{y}) \geq 0$ *for all* $\bar{x}, \bar{y} \in \mathbb{R}^m$*; and* $d(\bar{x}, \bar{y}) = 0$ *if and only if* $\bar{x} = \bar{y}$*.*

M-2: (Symmetricity) $d(\bar{x}, \bar{y}) = d(\bar{y}, \bar{x})$ *for all* $\bar{x}, \bar{y} \in V$*.*

M-3: (Triangle Inequality) $d(\bar{x}, \bar{y}) + d(\bar{y}, \bar{z}) \geq d(\bar{x}, \bar{z})$ *for all* $\bar{x}, \bar{y}, \bar{z} \in V$*; and* $d(\bar{x}, \bar{y}) + d(\bar{y}, \bar{z}) = d(\bar{x}, \bar{z})$ *if and only if* $\bar{x}$*,* $\bar{y}$ *and* $\bar{z}$ *are collinear.*

N-1: $d(\bar{x}, \bar{y}) = d(\bar{0}, \bar{x} - \bar{y})$ *for all* $\bar{x}, \bar{y} \in V$*.*

N-2: $d(\alpha\bar{x}, \alpha\bar{y}) = |\alpha| d(\bar{x}, \bar{y})$ *for all* $\bar{x}, \bar{y} \in V$*.*

In this context it should be noted that if a metric is to be defined on a space, not necessarily space of vectors, it is defined axiomatically satisfying the properties M-1 to M-3 in Proposition 1.3.4.

Let us now compute some metrics corresponding to norms in the respective spaces.

Example 1.3.5.

(I) In Example 1.3.2, (I) if we compute the metric induced by the corresponding norm, then the metric is given by

$$d(\bar{x}, \bar{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_m - y_m)^2}$$

where $\bar{x} = (\bar{x}_1, x_2, \cdots, x_m), \bar{y} = (y_1, y_2, \cdots, y_m) \in \mathbb{R}^m$. This metric is on $\mathbb{R}^m$ is called the Euclidean metric on $\mathbb{R}^m$.

(II) In Example 1.3.2, (II) the induced metric will be given by

$$d(f, g) = \sup \{|(f - g)|(x) \mid x \in [0, 1]\} \text{ for all } f, g \in \mathcal{C}[0, 1].$$

This metric is called the supremum metric or the uniform metric on $\mathcal{C}[0, 1]$.

In high school Mathematics we have learned the measurement of angle between two vectors in $\mathbb{R}^2$ is done by dot product of vectors. For general space of vectors we introduce a notion called inner product of vectors which exactly catches the notion of angle between two vectors.

Definition 1.3.6. Let V be a space of vectors over $\mathbb{R}$. A function $\langle . \, , \, . \rangle$: $V \times V \longrightarrow \mathbb{R}$ is called an inner product if it satisfies the following properties:

IP-1 (Bilinearity): $\langle \alpha_1\bar{v}_1 + \alpha_2\bar{v}_2, \bar{v}_3 \rangle = \alpha_1\langle \bar{v}_1, \bar{v}_3 \rangle + \alpha_2\langle \bar{v}_2, \bar{v}_3 \rangle$ for all α_1, α_2 in $\mathbb{R}$ and $\bar{v}_1, \bar{v}_2, \bar{v}_3$ in V.

IP-2 (Symmetricity): $\langle \bar{v}_1, \bar{v}_2 \rangle = \langle \bar{v}_2, \bar{v}_1 \rangle$ for all $\bar{v}_1, \bar{v}_2$ in V.

IP-3 (Positive-definiteness): $\langle \bar{v}, \bar{v} \rangle \geq 0$ for all $\bar{v}$ in V; and $\langle \bar{v}, \bar{v} \rangle = 0$ if and only if $\bar{v} = \bar{0}$.

Example 1.3.7.

(I) Define $\langle . \ , \ . \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ by $\langle (x_1, y_1) \ , \ (x_2, y_2) \rangle = x_1x_2 + y_1y_2$ for all $(x_1, y_1), (x_2, y_2)$ in $\mathbb{R}^2$. This function is the well known dot product on $\mathbb{R}^2$. One can easily verify that the function $\langle . \ , \ . \rangle$ is an inner product on $\mathbb{R}^2$.

(II) In more generality, the function $\langle . \ , \ . \rangle : \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}$ defined by $\langle \bar{x} \ , \ \bar{y} \rangle = x_1y_1 + x_2y_2 + \cdots + x_my_m$ for all $\bar{x} = (x_1, x_2, \cdots, x_m)$, $\bar{y} = (y_1, y_2, \cdots, y_m)$ in $\mathbb{R}^m$ can be verified to be an inner product on $\mathbb{R}^m$, which called the Euclidean inner product.

(III) On the space of vectors $V = \mathcal{C}[0, 1]$, define $\langle . \ , \ . \rangle : V \times V \longrightarrow \mathbb{R}$ by $\langle f \ , \ g \rangle = \displaystyle\int_0^1 fg$ for all f, g in V. It is east to check that the function $\langle . \ , \ . \rangle$ is an inner product on V, and is called the L^2 inner product.

Note that the property IP-1 of Definition 1.3.6 implies that $\langle \alpha f \ , \ \alpha f \rangle = \alpha^2 \langle f \ , \ f \rangle$, i.e., $\sqrt{\langle \alpha f \ , \ \alpha f \rangle} = |\alpha| \sqrt{\langle f \ , \ f \rangle}$ for all f in V and for all α in $\mathbb{R}$. Thus looking at IP-1 and IP-3 of Definition 1.3.6 one may naturally ask whether inner product leads to giving a norm in the corresponding space. We shall see that the answer, in fact, is affirmative. To see this we shall observe some properties of inner product.

Proposition 1.3.8 (Cauchy-Schwarz inequality)**.** *Let* V *be a space of vectors and* $\langle . \ , \ . \rangle : V \times V \longrightarrow \mathbb{R}$ *an inner product on it. Then* $\langle f \ , \ g \rangle^2 \leq \langle f \ , \ f \rangle \langle g \ , \ g \rangle$ *for all* f, g *in* V.

Proof. Let f, g in V. Then $\langle \lambda f + g \ , \ \lambda f + g \rangle = \lambda^2 \langle f \ , \ f \rangle + 2\lambda \langle f \ , \ g \rangle + \langle g \ , \ g \rangle$. Since $\langle \lambda f + g \ , \ \lambda f + g \rangle \geq 0$ for all λ in $\mathbb{R}$, we have $\langle f \ , \ g \rangle^2 \leq \langle f \ , \ f \rangle \langle g \ , \ g \rangle$. $\square$

Corollary 1.3.9. *Let* V *be a space of vectors and* $\langle . \ , \ . \rangle : V \times V \longrightarrow \mathbb{R}$ *an inner product on it. Then* $\sqrt{\langle f + g \ , \ f + g \rangle} \leq \sqrt{\langle f \ , \ f \rangle} + \sqrt{\langle g \ , \ g \rangle}$.

Proof. Let f, g in V. Then $\langle f + g \ , \ f + g \rangle = \langle f \ , \ f \rangle + 2\langle f \ , \ g \rangle + \langle g \ , \ g \rangle$. Using Cauchy-Schwarz inequality we get $\langle f + g \ , \ f + g \rangle \leq \langle f \ , \ f \rangle + 2\sqrt{\langle f \ , \ f \rangle \langle g \ , \ g \rangle} + \langle g \ , \ g \rangle = (\sqrt{\langle f \ , \ f \rangle} + \sqrt{\langle g \ , \ g \rangle})^2$. This proves $\sqrt{\langle f + g \ , \ f + g \rangle} \leq \sqrt{\langle f \ , \ f \rangle} + \sqrt{\langle g \ , \ g \rangle}$. $\square$

Now we are ready to show that an inner product on a space of vectors defines a norm on the corresponding space.

Corollary 1.3.10. *Let* V *be a space if vectors and* $\langle . \ , \ . \rangle : V \times V \longrightarrow \mathbb{R}$ *be an inner product on it. Then the function* $\|.\| : V \longrightarrow \mathbb{R}$ *defined by* $\|f\| = \sqrt{\langle f \ , \ f \rangle}$ *for all* f *in* V *is a norm on* V.

Proof. By the definition we see that $\|f\| \geq 0$ for all f in V, and $\|\alpha f\| = |\alpha| \|f\|$ for all f in V. By corollary 1.3.9 we see that $\|f + g\| \leq \|f\| + \|g\|$ for all f, g in V. This completes the proof. $\square$

Example 1.3.11.

(I) Let $\mathbb{R}^m$ be endowed with Euclidean inner product. Then it can be seen easily that the corresponding norm is the Euclidean norm on $\mathbb{R}^m$.

(II) The L^2 inner product in $\mathcal{C}[0,1]$ induces the L^2 norm on the corresponding space.

Example 1.3.12.

(I) Let $\bar{x}, \bar{y} \in \mathbb{R}^3$ be such that the line segments $\overline{\bar{0}, \bar{x}}$ and $\overline{\bar{0}, \bar{y}}$ are perpendicular to each other. Then one can check that the Euclidean inner product of $\bar{x}$ and $\bar{y}$ is zero.

(II) Let $\bar{x}, \bar{y} \in \mathbb{R}^3$ be such that the line segments $\overline{\bar{0}, \bar{x}}$ and $\overline{\bar{0}, \bar{y}}$ are parallel to each other. Then one can check that the absolute value of the Euclidean inner product of $\bar{x}$ and $\bar{y}$ is product of the norm of $\bar{x}$ and $\bar{y}$.

In view of the above examples, for a general space of vectors we define the following:

Definition 1.3.13. Let V be a space of vectors with an inner product $\langle . \ , \ . \rangle$. Two vectors $\bar{v}_1$ and $\bar{v}_2$ in V are called

(I) Orthogonal, if $\langle \bar{v}_1 \ , \ \bar{v}_2 \rangle = 0$.

(II) Parallel, if $|\langle \bar{v}_1 \ , \ \bar{v}_2 \rangle| = \|\bar{v}_1\| \|\bar{v}_2\|$.

In this chapter so far we have discussed on general space of vectors. But in the rest of topics we shall mainly restrict our discussion in the space of vectors $\mathbb{R}^m$, specially $\mathbb{R}^2$ and $\mathbb{R}^3$, with Euclidean inner product. In the next section we shall continue to explore the spatial structure of $\mathbb{R}^m$.

1.4 Topology of the Euclidean Space $\mathbb{R}^m$

Roughly speaking, topology is a spatial structure to study continuous functions which preserves "nearness" of points. Thus topology, actually, meant to define the nearness between elements in a set. In real analysis at first we have measured this "nearness" in terms of distance, e.g., if x is a point in $\mathbb{R}$ then the points which are near to x by distance less than δ are given by the set $(x-\delta, x+\delta) = \{y \in \mathbb{R} \mid |x-y| < \delta\}$; and later without specifying the distance we defined the "nearness" by open sets (or open neighborhoods) , e.g., the points near to $x \in \mathbb{R}$ are given by points in open sets containing x; smaller the open sets nearer the points to x. We shall buildup this concept of "nearness" for the space of vectors $\mathbb{R}^m$, and later through which we shall study continuous functions on $\mathbb{R}^m$.

Definition 1.4.1. An open m-ball in $\mathbb{R}^m$ centered at $\bar{x}_0$ with radius $r > 0$ is denoted by $B(\bar{x}_0, r)$ and is defined by

$$B(\bar{x}_0, r) = \{ \bar{x} \in \mathbb{R}^m \mid d(\bar{x}_0, \bar{x}) < r \}$$

A closed n-ball in $\mathbb{R}^m$ centered at $\bar{x}_0$ with radius $r > 0$ is denoted by $\overline{B}(\bar{x}_0, r)$ and is defined by

$$\overline{B}(\bar{x}_0, r) = \{ \bar{x} \in \mathbb{R}^m \mid d(\bar{x}_0, \bar{x}) \leq r \}$$

If $r = 1$, the n-balls are called unit n-balls and are, generally, denoted by $B(\bar{x}_0)$ and $\overline{B}(\bar{x}_0)$ for open and closed respectively.

Example 1.4.2.

(A) In $\mathbb{R}$, the open interval $(0,1)$ is the open 1-ball with center $\frac{1}{2}$ and radius $\frac{1}{2}$; i.e., $(0,1) = B(\frac{1}{2}, \frac{1}{2})$. In general, for $a, b \in \mathbb{R}$ where $a \neq b$ we have $(a,b) = B(\frac{a+b}{2}, \frac{a-b}{2})$ and $[a,b] = \overline{B}(\frac{a+b}{2}, \frac{a-b}{2})$.

The unit open 1-ball, i.e, the unit open ball in $\mathbb{R}$, is given by $B(0) = (-1,1)$; and similarly $\overline{B}(0) = [-1,1]$.

(B) The unit closed 2-ball is the circular disk $\{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1 \}$ where $\bar{0} = (0,0) \in \mathbb{R}^2$, the and unit open 2-ball $B(\bar{0})$ is the "boundary-less" circular disk $\{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1 \}$ where $\bar{0} = (0,0) \in \mathbb{R}^2$.

(C) The closed 3-ball with center (x_0, y_0, z_0) and radius r is the solid sphere $\{ (x,y,z) \in \mathbb{R}^3 \mid (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 \leq r^2 \}$. The open 3-ball $B((x_0, y_0, z_0), r)$ is the "boundary-less" solid sphere $\{ (x,y,z) \in \mathbb{R}^3 \mid (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 < r^2 \}$.

Definition 1.4.3. A set $U \subseteq \mathbb{R}^m$ is called open if for each point $\bar{x} \in U$ there exists an open ball $B(\bar{x}, r)$ such that $B(\bar{x}, r) \subseteq U$. A set $C \subseteq \mathbb{R}^m$ is called closed if $\mathbb{R} \backslash C$ is open.

Example 1.4.4.

(A) Any open ball $B(\bar{x}, r)$ in $\mathbb{R}^m$ is open, as for each $\bar{y} \in B(\bar{x}, r)$, we have $B(\bar{y}, r_0) \subseteq B(\bar{x}, r)$ where $r_0 = min\,\{d(\bar{x}, \bar{y}),\ r - d(\bar{x}, \bar{y})\}$.

(B) Any closed interval $[a,b]$ is not open, for the points a and b in $[a,b]$ there exists no open 1-ball, i.e., open intervals, centered at a and centered at b, which lies in $[a,b]$. Note that for $a = b$, we have $[a,b] = \{a\}$.

(C) The closed interval $[a, b]$ is a closed set. To see this we need to observe that $\mathbb{R}\backslash[a, b] = (-\infty, a) \cup (b, \infty)$ is open. Suppose $x \in \mathbb{R}\backslash[a, b]$. Let $r_0 = \min\{d(x, a),\ d(x, b)\}$. Then it is easy to show that $B(x, r_0) \subseteq \mathbb{R}\backslash[a, b]$. Since $x \in \mathbb{R}\backslash[a, b]$ was arbitrary, we get that $\mathbb{R}\backslash[a, b]$ is open.

Proposition 1.4.5.

1. *Finite intersection of open sets is open.*
2. *Arbitrary union of open sets is open.*

Proof. Exercise □

Proposition 1.4.6.

1. *Finite union of closed sets is closed.*
2. *Arbitrary intersection of closed sets is closed.*

Proof. The proof is very similar to that of Proposition 1.4.5. □

Corollary 1.4.7. *Any finite set in* $\mathbb{R}^m$ *is closed.*

Proof. Follows from Proposition 1.4.6. □

Example 1.4.8. Arbitrary intersection of open sets may not be open. Set $U_n := B(\bar{0}, 1/n) \subset \mathbb{R}^m$. Then each U_m is an open n-ball and hence an open set. But $\bigcap_{n\geq 1} U_n = \{\bar{0}\}$, which is closed.

Definition 1.4.9. *Any function* $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ *is called a scalar field, and any function* $g : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ *is called a vector field. The vector field* g *can be represented by* n*-many scalar fields, in the following fashion:*
$g = (g_1, g_2, \cdots, g_n)$ *where* $g_i : \mathbb{R}^m \longrightarrow \mathbb{R}$ *are scalar fields.*

Example 1.4.10. *1.* $f(x, y) = x^2 + y^2$ *for all* $(x, y) \in \mathbb{R}^2$ *is a scalar field.*

2. $g(x, y) = (x^2, y^2, 1)$ *for all* $(x, y) \in \mathbb{R}^2$ *is a vector field.*

3. $h(x, y, z) = (x^2, y + z)$ *for all* $(x, y, z) \in \mathbb{R}^3$ *is a vector field.*

Definition 1.4.11. *Let* G *be an open set in* $\mathbb{R}^m$*, and* $f : G \longrightarrow \mathbb{R}$ *a scalar field.* f *is called continuous at* $\bar{a}$ *if for any* $\epsilon > 0$ *there exists* $\delta > 0$ *such that whenever* $\bar{x} \in B(\bar{a}, \delta)$ *(, i.e.,* $||\bar{x} - \bar{a}|| < \delta$*, i.e.,* $d(\bar{x}, \bar{a}) < \delta$*) we have* $f(\bar{x}) \in B(f(\bar{a}), \epsilon)$ *(i.e.,* $|f(\bar{x}) - f(\bar{a})| < \epsilon$*, i.e.,* $d(f(\bar{x}), f(\bar{a})) < \epsilon$*). Equivalently,* f *is continuus at* $\bar{a}$*, if for every sequence* $\{\bar{x}_n\}$ *in* G *converging to* $\bar{a}$*, the corresponding sequence* $f(\bar{x}_n)$ *converges to* $f(\bar{a})$*.*

A vector field $F : G \longrightarrow \mathbb{R}^n$ *is called continuous at* $\bar{a}$*, if each of the scalar component functions/fields of* F *are continuous.*

In general continuous functions are called $\mathcal{C}^0$*-type functions.*

Definition 1.4.12. *Let* $G \subset \mathbb{R}^m$ *be open and* $f : G \longrightarrow \mathbb{R}$ *a scalar field.* f *is called a* $\mathcal{C}^n$*-type function at* $\bar{a} \in G$ *if all the* n*th partial derivatives of* f *at* $\bar{a}$ *exists, and are continuous at* $\bar{a}$*.* f *is called smooth if* f *is* $\mathcal{C}^1$*-type and the partial derivatives of* f *do not vanish simultaneously at the corresponding point/points.*

If the partial derivatives of f *of all order exists, and are continuous,* f *is called to be of* $\mathcal{C}^\infty$*-type.*

Similar definitions for vector fields, as it is defined in Definition 1.4.11, exist.

Proposition 1.4.13. *(I) Let* $G \subset \mathbb{R}^m$ *be open and* $f_1, f_2, \cdots, f_s : G \longrightarrow \mathbb{R}$ *be* $\mathcal{C}^n$*-type at* $\bar{a}$ *(*$n \geq 0$*), then* $\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_s f_s : G \longrightarrow \mathbb{R}$ *and* $f_1 f_2 \cdots f_s : G \longrightarrow \mathbb{R}$ *are* $\mathcal{C}^n$*-type at* $\bar{a}$ *where* α_i *are constants in* $\mathbb{R}$*.*

(II) If $f_2 \neq 0$ *in* G*, then* f_1/f_2 *is* $\mathcal{C}^n$*-type at* $\bar{a}$*.*

(III) Let $V \subset \mathbb{R}^m$ *be open. Suppose* $f : G \longrightarrow \mathbb{R}^t$ *and* $g : V \longrightarrow \mathbb{R}^s$ *are* $\mathcal{C}^n$*-type (*$n \geq 0$*), then* $g \circ f : G \longrightarrow \mathbb{R}^s$*, if exists, is* $\mathcal{C}^n$*-type (*$G \xrightarrow{f} \mathbb{R}^t \supset V \xrightarrow{g} \mathbb{R}^s$*).*

Proof. Exercise! Please note that $g \circ f$ will have domain G iff $f(G) \subset V$. □

By repeated applications of Proposition 1.4.13 it is obvious that

Example 1.4.14.
1. $f(x, y, z) = x + yz + z^3$ *for all* $(x, y, z) \in \mathbb{R}^3$ *is continuous throughout.*
2. $f(x, y) = 2 + x\exp(y) + \sin(x + y)$ *for all* $(x, y) \in \mathbb{R}^2$ *is continuous on* $\mathbb{R}^2$*.*
3. $f(x, y) = x/\sqrt{x^2 + y^2}$ *for all* $(x, y) \in \mathbb{R}^2 \backslash (0, 0)$ *is continuous on* $\mathbb{R}^2 \backslash (0, 0)$*.*
4. $f(x) = |x|$ *is* $\mathcal{C}^0$*-type on* $\mathbb{R}$*, but not of* $\mathcal{C}^1$*-type on* $\mathbb{R}$*.*
5. $f(x) = x|x|$ *is* $\mathcal{C}^1$*-type on* $\mathbb{R}$*, but not of* $\mathcal{C}^2$*-type on* $\mathbb{R}$*.*
6. $f(x, y) = x + \exp(xy)$ *is* $\mathcal{C}^\infty$*-type function.*
7. *Any* $\mathcal{C}^n$*-type function is also a* $\mathcal{C}^{n-1}$*-type function.*

1.5 Few Applications

1.5.1 Line passing though a point, and parallel to a vector

Let $P_0 \in \mathbb{R}^m$ a point and $\bar{v}$ a vector in $\mathbb{R}^m$. It is well known that $< \bar{v} >=: L_{\bar{v}}(\bar{0}) = \{\alpha\bar{v} \in \mathbb{R}^m |\ \alpha \in \mathbb{R}\}$ is a straight line passing though origin parallel to $\bar{v}$ (i.e., having direction as $\bar{v}$). Suppose $L_{\bar{v}}(P_0)$ be the line passing though P_0 and parallel to vector $\bar{v}$. We shall show that $L_{\bar{v}}(P_0) = P_0 + L_{\bar{v}}(\bar{0})$. It is easy to check that for any $\bar{w} \in L_{\bar{v}}(\bar{0})$, the point $P_0 + \bar{w}$ is in $L_{\bar{v}}(P_0)$. Which shows that $P_0 + L_{\bar{v}}(\bar{0}) \subset L_{\bar{v}}(P_0)$. Next one may prove $L_{\bar{v}}(P_0) \subset P_0 + L_{\bar{v}}(\bar{0})$ in the following fashion: suppose P be any point in $L_{\bar{v}}(P_0)$. Then, P_0 being a special point in $L_{\bar{v}}(P_0)$, we have $(P - P_0) || \bar{v}$, i.e., $P = P_0 + \alpha\bar{v}$ for some $\alpha \in \mathbb{R}$. This shows that $L_{\bar{v}}(P_0) \subset P_0 + L_{\bar{v}}(\bar{0}) = \{P_0 + \alpha\bar{v} \in \mathbb{R}^m |\ \alpha \in \mathbb{R}\}$. This completes the description of $L_{\bar{v}}(P_0)$.

Another approach would be to see that any kind of translation of the line $L_{\bar{v}}(\bar{0})$ look like $P^* + L_{\bar{v}}((0, 0))$ for some $P^* \in \mathbb{R}^m$, and which is a straight line. Note that this straight line is not a vector space as it does not contain $\bar{0}$. It is easy to see that this straight line is passing though P^* and is parallel to $\bar{v}$, i.e., to the straight line $L_{\bar{v}}(\bar{0})$; and therefore the straight line passing though P_0 and parallel to $\bar{v}$ will be $P_0 + L_{\bar{v}}((0,0))$.

As $L_{\bar{v}}(P_0) = P_0 + L_{\bar{v}}(\bar{0}) = \{P_0 + \alpha\bar{v} \in \mathbb{R}^m | \ \alpha \in \mathbb{R}\}$, the parametric representation of the line $L_{\bar{v}}(P_0)$, when $m = 3$, i.e., in $\mathbb{R}^3$ would be $(x, y, z) = (x_0 + \alpha v_1, y_0 + \alpha v_2, z_0 + \alpha v_3)$ where $P_0 = (x_0, y_0, z_0)$ and $\bar{v} = (v_1, v_2, v_3)$.

1.5.2 Line passing through a two points

Let P_1 and P_2 be two points in $\mathbb{R}^m$, and $L(P_1, P_2)$ be the line passing though P_1 and P_2. One may check that

$$L(P_1, P_2) = P_1 + L_{P_2 - P_1}(\bar{0}) = L_{P_2 - P_1}(P_1) = \{P_1 + \alpha(P_2 - P_1) \mid \alpha \in \mathbb{R}\}.$$

1.5.3 Plane passing through a point, and perpendicular to a vector

Let $\bar{v}$ be a vector in $\mathbb{R}^3$, and P_0 a point. Let $S_{\bar{v}}(P_0)$ denote the plane passing though P_0 and orthogonal to $\bar{v}$. One may check that

$$S_{\bar{v}}(P_0) = \{P \in \mathbb{R}^3 \mid \ < P - P_0, \bar{v} >= 0\}.$$

Chapter 2

Directional Derivative and Differentiability

2.1 Introduction

Let G be an open set in R and $f : G \longrightarrow \mathbb{R}$ a real valued function of real variable. We call f to be differentiable at $x_0 \in G$ if $\lim\limits_{h\to 0} \dfrac{f(x_0+h)-f(x_0)}{h}$ exists, and the limit is termed as $\frac{d}{dx}(f(x))|_{x_0}$ or $f'(x_0)$. $f'(x_0)$ represents the rate of change of f along x-axis at the point x_0. We already have learned that if f is differentiable at x_0, it also continuous at x_0.

If G be open in $\mathbb{R}^2$ and $f : G \longrightarrow \mathbb{R}$ a scalar field, then we defined the limits $\lim\limits_{h\to 0} \dfrac{f(x_0+h,y_0)-f(x_0,y_0)}{h}$ and $\lim\limits_{h\to 0} \dfrac{f(x_0,y_0+h)-f(x_0,y_0)}{h}$, if exist, partial derivatives of f with respect to x and y respectively at $(x_0,y_0) \in G$. We denoted the value of the limit as $\frac{\partial}{\partial x}(f)|_{(x_0,y_0)}$ and $\frac{\partial}{\partial y}(f)|_{(x_0,y_0)}$ respectively; and they represent the rate of change of f along x-axis and y-axis respectively. It is well known that existence of partial derivatives of a function can not ensure the continuity of the function. For example let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \frac{x^2y}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

One may check that $\frac{\partial}{\partial x}(f)|_{(0,0)}$ and $\frac{\partial}{\partial y}(f)|_{(0,0)}$ exist, but f is not continuous at $(0,0)$.

This tells us the concept of partial differentiability for several variable functions is too weak to be defined as the equivalent concept of differentiability as we have for real variable function.

We further have learned that a scalar field $f : G \longrightarrow \mathbb{R}$ is said to be differentiable at $P_0 = (x_0,y_0) \in G$ if there exists $\alpha, \beta \in \mathbb{R}$ such that

$$\lim_{(h,k)\to(0,0)} \frac{f(x_0+h,y_0+k)-f(x_0,y_0)-\alpha h-\beta k}{\sqrt{h^2+k^2}} = 0.$$

And, in that case it can be verified that $\alpha = \frac{\partial}{\partial x}(f)|_{P_0}$ and $\beta = \frac{\partial}{\partial y}(f)|_{P_0}$.

One should be careful that the mere existence of the partial derivatives at P_0 does not ensure differentiability of the given function. Further, if the partial derivatives of the function at the point P_0 do not exists, the function is not differentiable at P_0. One may check that that if f is differentiable at P_0, the it is continuous at P_0 (Prove it!).

Exercise 2.1.1. *1. Let* $f(x,y) = \begin{cases} x+y & \textit{if } x = 0 \textit{ or } y = 0 \\ 1 & \textit{otherwise} \end{cases}$. *Show that* f *has partial derivatives at* $(0,0)$, *but* f *is not continuous at* $(0,0)$.

2. *Show that the function* $f(x,y) = |x| + |y|$ *for all* $(x,y) \in \mathbb{R}^2$ *is continuous, but partial derivatives of* f *at* $(0,0)$ *does not exists, and therefore* f *is not differentiable at* $(0,0)$.

3. *Let* $f(x,y) = x^2 + y^2$ *for all* $(x,y) \in \mathbb{R}^2$. *Show that partial derivatives of* f *exists at any point of* $\mathbb{R}^2$. *Is* f *differentiable at every point of* $\mathbb{R}^2$*?*

4. *Let* $f(x,y) = \sqrt{x^2+y^2}$ *for all* $(x,y) \in \mathbb{R}^2$. *Show that partial derivatives of* f *does not exists at* $(0,0)$.

5. *Let* $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \textit{if } (x,y) \neq (0,0) \\ 0 & \textit{if } (x,y) = (0,0) \end{cases}$. *Show that* f *has partial derivatives at* $(0,0)$, *but* f *is not continuous at* $(0,0)$.

At this point we ask

Problem 2.1.2. *1. As partial differentiation represents the rate of change of a function along respective axes, can we talk of rate of change of function along any given straight line or, in general, along any given curve? If yes, whether it has any relation with the partial derivatives.*

2. *Under what condition the partial differentiability can replace the concept of differentiability of a scalar field?*

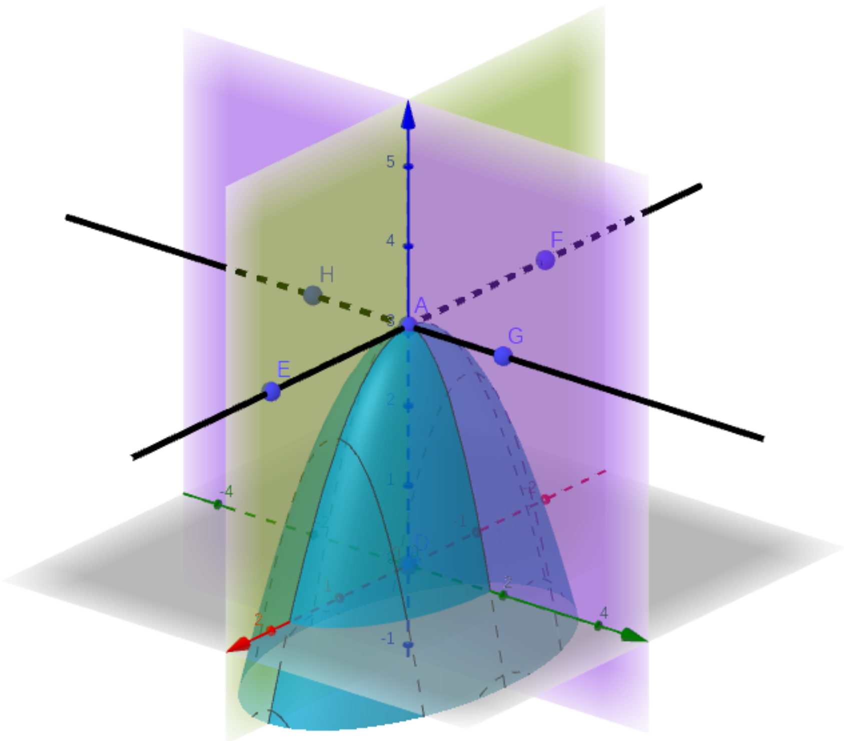
In this chapter we mainly try to get the answer to the above questions.

2.2 Directional Derivative and Differentiability

Let G be open in $\mathbb{R}^3$, $P_0 \in G$, and $f : G \longrightarrow \mathbb{R}$ a scalar field. Suppose $\frac{\partial}{\partial x}(f)|_{P_0}$ exists. If we look at the definition of partial derivatives carefully we shall see that it is the rate of change of f along the line passing though P_0 and parallel to the x-axis, i.e, the vector $e_1 = (1,0,0)$. Note that the parametric form of line $L_{e_1}(P_0)$ is $P_0 + he_1$ where $h \in \mathbb{R}$; and

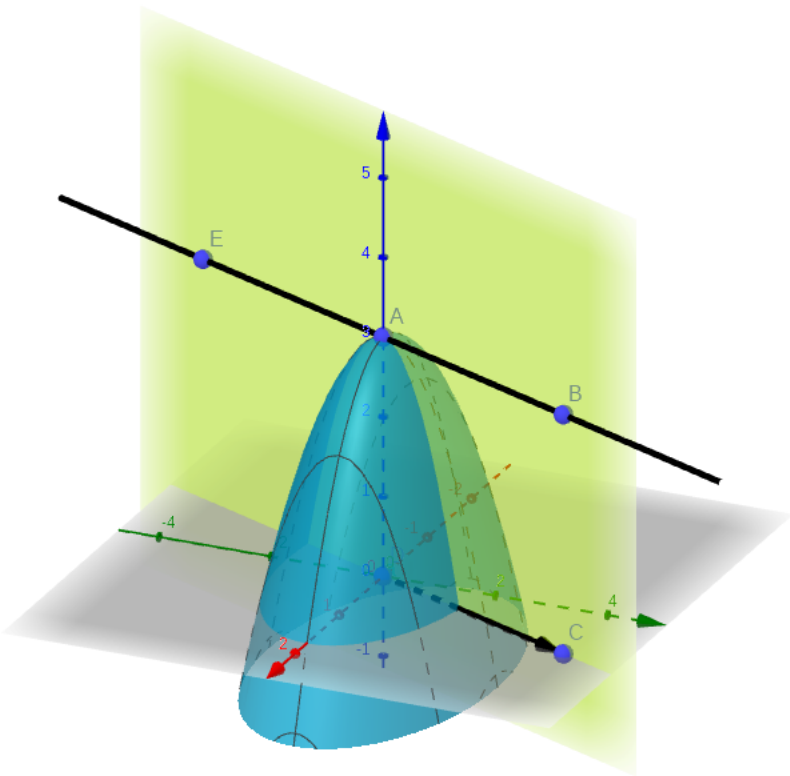
$$\begin{aligned}
\tfrac{\partial}{\partial x}(f)|_{P_0} &= \lim_{h\to 0} \frac{f(x_0+h, y_0, z_0) - f(x_0,y_0,z_0)}{h} \\
&= \lim_{t\to 0} \frac{f((x_0,y_0,z_0) + t(1,0,0)) - f((x_0,y_0,z_0))}{t} \\
&= \lim_{t\to 0} \frac{f(P_0 + te_1) - f(P_0)}{t}
\end{aligned}$$

If the domain of f, i.e., G is a subset of $\mathbb{R}^2$, then the points $(x, y, f(x, y))$ will generate a surface which we call the surface induced from f. In that case the existence of $\frac{\partial}{\partial x}(f)|_{P_0}$ can be seen, geometrically, to be equivalent to the existence of the tangent line to the curve generated as a intersection of the plane induced from f and the plane parallel to xz-plane at the point $P_0 = (x_0, y_0) \in G$.



Now, imitating this we define the rate of change of f along any line passing though P_0 and parallel to a unit vector $\bar{v}$ (i.e., $\|\bar{v}\| = 1$), i.e., along the line $L_{\bar{v}}(P_0)$ as $\lim_{t\to 0} \dfrac{f(P_0 + t\bar{v}) - f(P_0)}{t}$, if the limit exists. We define this quantity the **directional derivative of f at P_0 along the unit vector $\bar{v}$** and is denoted by $D_{\bar{v}}(f)|_{P_0}$, i.e., $D_{\bar{v}}(f)|_{P_0} = \lim_{t\to 0} \dfrac{f(P_0 + t\bar{v}) - f(P_0)}{t}$, if the limit exists.

To understand the geometric equivalence of the existence of directional derivative assume that the domain of f, i.e., G is a subset of $\mathbb{R}^2$. In that case the existence of $D_{\bar{v}}(f)|_{P_0}$ can be seen, geometrically, to be equivalent to the existence of the tangent line to the curve generated as a intersection of the plane induced from f and the plane perpendicular to xy-plane and passing though the line $L_{\bar{v}}(P_0)$ i.e., $P_0 + t\bar{v}$, $t \in \mathbb{R}$, which is the line passing through the point P_0 and parallel to the vector $\bar{v}$.



Exercise 2.2.1. 1. *Show that* $D_{\bar{v}}(f)|_{P_0} = \frac{d}{dt}(f(P_0 + t\bar{v}))|_{t=0}$.

2. *For any direction* $\bar{v}$, *find the directional derivative* $D_{\bar{v}}(f)|_{(0,0)}$ *where* $f(x, y) = x^2 + y^2$.

3. *Let* $f(x, y) = \begin{cases} x + y & \text{if } x = 0 \text{ or } y = 0 \\ 1 & \text{otherwise} \end{cases}$. *Find directions* $\bar{v}$ *such that* $D_{\bar{v}}(f)|_{(0,0)}$ *exists.*

4. *Let* $f(x, y) = \sqrt{x^2 + y^2}$ *for all* $(x, y) \in \mathbb{R}^2$. *Show that* $D_{\bar{v}}(f)|_{(x_0,y_0)}$ *exists for all direction* $\bar{v}$ *if* $(x_0, y_0) \neq (0, 0)$.

It is obvious that if $D_{P_0}(f)|_{\bar{v}}$ exists for all direction $\bar{v}$ (unit vector), then all the partial derivatives of f exists at P_0, as $D_{e_1}(f)|_{P_0} = \frac{\partial}{\partial x}(f)|_{P_0}$, $D_{e_2}(f)|_{P_0} = \frac{\partial}{\partial y}(f)|_{P_0}$, and $D_{e_3}(f)|_{P_0} = \frac{\partial}{\partial z}(f)|_{P_0}$. So, we ask

Problem 2.2.2. *If all the partial derivatives of* f *exists at* P_0, *does* $D_{\bar{v}}(f)|_{P_0}$ *exit for all direction* $\bar{v}$*?*

This question may come to many naturally as we know that any vector $\bar{v}$ can be written as linear combinations of e_1, e_2 and e_3 in $\mathbb{R}^3$, i.e., in $\mathbb{R}^3$ we can reach any point $\bar{v}$ from $\bar{0}$ by moving along x-axis, y-axis and z-axis.

However, if we translate this question geometrically, assuming the domain of f, i.e., G as subset of $\mathbb{R}^2$, we shall see that it asks whether the existence of two tangent lines on the two specific curves generated as intersections of the plane induced from f and the plane parallel to xz-plane and yz-plane at the point $P_0 = (x_0, y_0) \in G$, ensures the tangent lines on the all possible curves generated as intersection of the plane induced from f and the planes perpendicular to the xy-plane and passing though the lines $L_{\bar{v}}(P_0)$ i.e., $P_0 + t\bar{v}$, $t \in \mathbb{R}$, which are the lines passing through the point P_0 and parallel to the vector $\bar{v}$, where $\|\bar{v}\| = 1$. We do not see any reason why existence of tangent of two curves at a point will unconditionally ensure existence of tangents to other curves passing though the same point. In reality, we shall see that this will be ensured by some extra conditions. The following discussion is towards the same.

Let $P_0 = (x_0, y_0, z_0)$ and $\bar{v} = (v_1, v_2, v_3)$ where $\|\bar{v}\| = 1$. Note that $v_i \neq 0$ for $i = 1, 2, 3$ as $\|\bar{v}\| = 1$. Now, Write $f(P_0 + t\bar{v}) - f(P_0)$ as

$$\begin{aligned} f(P_0 + t(v_1, v_2, v_3)) - f(P_0) = \; & f(P_0 + t(v_1, v_2, v_3)) - f(P_0 + t(v_1, v_2, 0)) + \\ & f(P_0 + t(v_1, v_2, 0)) - f(P_0 + t(v_1, 0, 0)) \\ & + f(P_0 + t(v_1, 0, 0)) - f(P_0 + t(0, 0, 0)). \\ \\ & = f([P_0 + t(v_1, v_2, 0)] + tv_3 e_1) - f(P_0 + t(v_1, v_2, 0)) + \\ & f([P_0 + t(v_1, 0, 0)] + tv_2 e_2) - f(P_0 + t(v_1, 0, 0)) \\ & + f([P_0 + t(0, 0, 0)] + tv_1 e_1) - f(P_0 + t(0, 0, 0)). \end{aligned}$$

And, therefore,

$$\begin{aligned} & \lim_{t \to 0} \frac{f(P_0 + t\bar{v}) - f(P_0)}{t} \\ = & \lim_{t \to 0} \frac{f(P_0 + t(v_1, v_2, 0) + tv_3 e_3) - f(P_0 + t(v_1, v_2, 0))}{t} + \\ & \lim_{t \to 0} \frac{f(P_0 + t(v_1, 0, 0) + tv_2 e_2) - f(P_0 + t(v_1, 0, 0))}{t} + \\ & \lim_{t \to 0} \frac{f(P_0 + t(0, 0, 0) + tv_1 e_1) - f(P_0 + t(0, 0, 0))}{t} \\ = & \lim_{t \to 0} \frac{f(P_0 + t(v_1, v_2, 0) + tv_3 e_3) - f(P_0 + t(v_1, v_2, 0))}{t} \frac{v_3}{v_3} + \\ & \lim_{t \to 0} \frac{f(P_0 + t(v_1, 0, 0) + tv_2 e_2) - f(P_0 + t(v_1, 0, 0))}{t} \frac{v_2}{v_2} + \\ & \lim_{t \to 0} \frac{f(P_0 + t(0, 0, 0) + tv_1 e_1) - f(P_0 + t(0, 0, 0))}{t} \frac{v_1}{v_1} \end{aligned}$$

Now, if partial derivative of f with respect to x exists at P_0, with respect to y at $P_0 + t(v_1, 0, 0)$ and with respect to z at $P_0 + t(v_1, v_2, 0)$, then we can calculate the above limit and we get the following.

$$\lim_{t \to 0} \frac{f(P_0 + t\bar{v}) - f(P_0)}{t} = v_3 \frac{\partial}{\partial z}(f)|_{P_0 + t(v_1, v_2, 0)} + v_2 \frac{\partial}{\partial y}(f)|_{P_0 + t(v_1, 0, 0)} + v_1 \frac{\partial}{\partial x}(f)|_{P_0}.$$

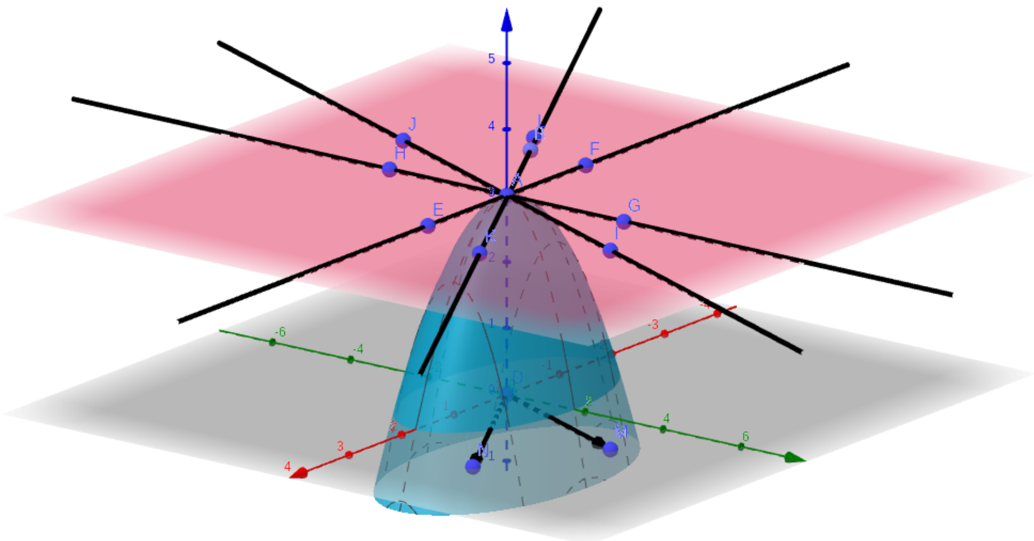
Which will be equal to $v_3 \frac{\partial}{\partial z}(f)|_{P_0} + v_2 \frac{\partial}{\partial y}(f)|_{P_0} + v_1 \frac{\partial}{\partial x}(f)|_{P_0}$ if $\frac{\partial}{\partial z}(f)|_{(x,y,z)}$ and $\frac{\partial}{\partial y}(f)|_{(x,y,z)}$ are continuous at P_0.

Therefore, we have the following:

Theorem 2.2.3. *Let G be open in $\mathbb{R}^3$, $P_0 \in G$ and $f : G \longrightarrow \mathbb{R}$ an scalar field. Suppose partial derivatives of f exists in an open ball around P_0 and, two of the three partial derivatives are continuous at P_0, then for any unit vector $\bar{v}$ in $\mathbb{R}^3$,* *(I)* $D_{\bar{v}}(f)|_{P_0}$ *exists and*

(II) $D_{\bar{v}}(f)|_{P_0} =< \nabla f|_{P_0}, \bar{v} >= v_1 \frac{\partial}{\partial x}(f)|_{P_0} + v_2 \frac{\partial}{\partial y}(f)|_{P_0} + v_3 \frac{\partial}{\partial z}(f)|_{P_0}$.

It is well known that if f is differentiable at P_0, then all the partial derivatives exist at P_0. It is worth exploring whether $D_{\bar{v}}(f)|_{P_0}$ exists for all direction $\bar{v}$ if f is differentiable at P_0. Geometrically, it is something very trivial to understand, however, to prove it we need some tools. before going into the detailed discussion, we first see the geometric view towards it. Assume that G, the domain of f is a subset of $\mathbb{R}^2$. Then, the differentiability of f at $P_0 \in G$ will mean the existence of the tangent plane at the point $(P_0, f(P_0))$ on the surface induced by f. Clearly, the existence of tangent plane means the existence of the tangents ot the all possible curves lying on that surface at the point $(P_0, f(P_0))$; and hence in specific, the existence of the tangent lines on the all possible curves generated as intersection of the plane induced from f and the planes perpendicular to the xy-plane and passing though the lines $L_{\bar{v}}(P_0)$ where $\|\bar{v}\| = 1$, which essentially implies the existence of directional derivatives of f along all possible direction at the point P_0. Therefore, differentiability should imply directional differentiability along all direction.



Now to tackle the situation we need tool which converts the differentiability to something a geometric object. For this we first revisit the definition of differentiability of a real valued fucntion $g : G \longrightarrow \mathbb{R}$ where $G \subset R$ is open. Suppose $a \in G$. We say that g is differentiable at a if

$\lim\limits_{x \to a} \dfrac{g(x) - g(a)}{x - a}$ exists. If this limit exists we call this value to be $g'(a)$, i.e., $g'(a) = \lim\limits_{x \to a} \dfrac{g(x) - g(a)}{x - a}$. Now define a function $g_1 : G \longrightarrow \mathbb{R}$ by

$$g_1(x) = \begin{cases} \frac{f(x)-f(a)}{x-a} & \text{if } x \neq a \\ \alpha & \text{if } x = a \end{cases}$$

Note that if we set $\alpha = g'(a)$, then g_1 is continuous at $x = a$, as g is differentiable at $x = a$. Now suppose it is not known whether g is differentiable at a. If we assume that g_1 is continuous at a, then it is easy to see that g is differentiable at a, and $\alpha = f'(a)$. Thus, we have

Theorem 2.2.4 (Caratheodory's Lemma)**.** *Let $G \subset \mathbb{R}$ be open, $a \in G$, and $f : G \longrightarrow \mathbb{R}$ a function. Then, f is differentiable at a if and only if there exists a function $f_1 : G \longrightarrow \mathbb{R}$ such that*

(I) f_1 is continuous at a, and

(II) $f(x) - f(a) = (x - a)f_1(x)$ for all $x \in G$.

Specifically, $f_1(x) = \dfrac{f(x) - f(a)}{x - a}$ *for* $x \in G\backslash\{a\}$*; and* $f_1(a) = \frac{d}{dx}(f)|_a = f'(a)$.

It can be seen that the above result also holds true for functions of several variables.

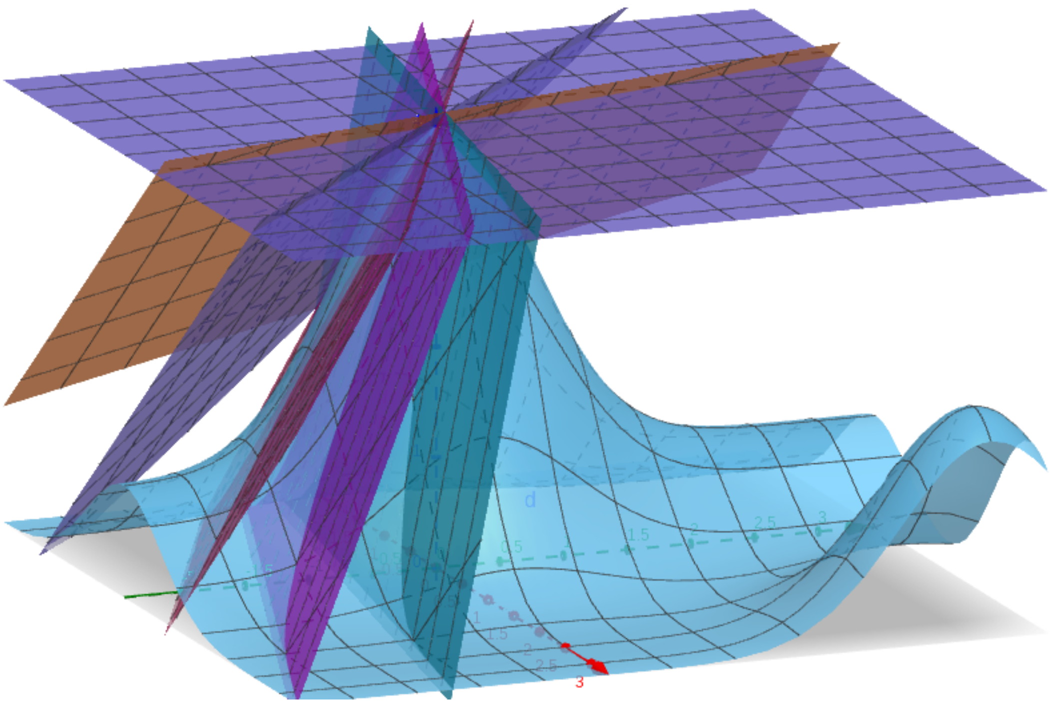
Theorem 2.2.5 (Caratheodory)**.** *Let $G \subset \mathbb{R}^3$ be open, $\bar{a} \in G$, and $f : G \longrightarrow \mathbb{R}$ a function. Then, f is differentiable at $\bar{a}$ if and only if there exists three functions $f_1, f_2, f_3 : G \longrightarrow \mathbb{R}$ such that*

(I) f_1, f_2 and f_3 are continuous at $\bar{a}$, and

(II) $f(x, y, z) - f(a_1, a_2, a_3) = (x - a_1)f_1(x, y, z) + (y - a_2)f_2(x, y, z) + (z - a_3)f_3(x, y, z)$ for all $(x, y, z) \in G$.

Further, it can be seen that $f_1(\bar{a}) = \frac{\partial}{\partial x}(f)|_{\bar{a}}$, $f_2(\bar{a}) = \frac{\partial}{\partial y}(f)|_{\bar{a}}$, *and,* $f_3(\bar{a}) = \frac{\partial}{\partial z}(f)|_{\bar{a}}$.

Proof. Assignment! $\square$



Now, we assume that $f : G \longrightarrow \mathbb{R}$ be differentiable at P_0 where $G \subset \mathbb{R}^3$ is open and $P_0 \in G$. Then by Theorem 2.2.5 there exists functions $f_1, f_2, f_3 : G \longrightarrow \mathbb{R}$ continuous at P_0 such that

$$f(x, y, z) - f(x_0, y_0, z_0) = (x - x_0)f_1(x, y, z) + (y - y_0)f_2(x, y, z) + (z - z_0)f_3(x, y, z)$$

for all $(x, y, z) \in G$. Now, let $\bar{v}$ be any unit vector in $\mathbb{R}^3$ and we choose the points $(x, y, z) \in G$ such that they lie on the straight line $P_0 + t\bar{v}$ where $t \in \mathbb{R}$. Then we get

$$\begin{aligned} f(P_0 + t\bar{v}) \ - \ f(P_0) = \ & ((x_0 + tv_1) \ - \ x_0) \ f_1(P_0 + t\bar{v}) + \\ & ((y_0 + tv_2) \ - \ y_0) \ f_2(P_0 + t\bar{v}) + \\ & ((z_0 + tv_3) \ - \ z_0) \ f_3(P_0 + t\bar{v}) \end{aligned}$$

for all $t \in \mathbb{R}$, i.e.,

$$\begin{aligned} \frac{f(P_0 + t\bar{v}) - f(P_0)}{t} = \ & \frac{((x_0 + tv_1) \ - \ x_0)}{t} f_1(P_0 + t\bar{v}) + \\ & \frac{((y_0 + tv_2) \ - \ y_0)}{t} f_2(P_0 + t\bar{v}) + \\ & \frac{((z_0 + tv_3) \ - \ z_0)}{t} f_3(P_0 + t\bar{v}) \end{aligned}$$

for all $t \in \mathbb{R} \setminus \{0\}$.

Since f_1, f_2 and f_3 are continuous at P_0, as $t \longrightarrow 0$, we see the limits in the right side exists and therefore, the limit in the left side exists, i.e., $D_{\bar{v}}(f)|_{P_0}$ exist. Since $\bar{v}$ was arbitrary, we see that $D_{\bar{v}}(f)|_{P_0}$ exists for all direction $\bar{v}$. Further, from the above equality we get, under limit,

$$D_{\bar{v}}(f)|_{P_0} = v_1 \frac{\partial}{\partial x}(f)|_{P_0} + v_2 \frac{\partial}{\partial y}(f)|_{P_0} + v_3 \frac{\partial}{\partial z}(f)|_{P_0} = < \nabla f|_{P_0}, \bar{v} > .$$

From this we conclude the following

Theorem 2.2.6. *Let $G \subset \mathbb{R}^3$ be open, $P_0 \in G$ and $f : G \longrightarrow \mathbb{R}$ differentiable at P_0, then*
(I) $D_{\bar{v}}(f)|_{P_0}$ exists for all direction $\bar{v}$, and
(II) $D_{\bar{v}}(f)|_{P_0} = < \nabla f|_{P_0}, \bar{v} >$.

Now, if we follow the statements of Theorem 2.2.3 and Theorem 2.2.6, we shall see that two different "hypotheses" gives the same result. So, naturally one asks whether those two "hypotheses" in Theorem 2.2.3 and Theorem 2.2.6 have any relation. Clearly, mere differentiability of a function can not ensure continuity of partial derivatives or even the existence of partial derivatives in neighbourhood points. So, we should check whether the "conditions" of partial differentiations in Theorem 2.2.3 ensure differentiability of the function in the corresponding point. The answer is, in fact, "Yes"! From the technique applied in Theorem 2.2.3, and the technique of forming the increment function f_1 in Theorem 2.2.4 we see a scope/ we sense our intuition of proving differentiability of a scalar field under the condition the partial derivatives are continuous.

Let $G \subset \mathbb{R}^n$ be open and $P_0 = (x_0, y_0, z_0) \in G$. Suppose $f : G \longrightarrow \mathbb{R}$ a scalar field. Then,

$$\begin{aligned} f(x,y,z) - f(x_0,y_0,z_0) = \ & f(x,y,z) - f(x,y,z_0) + \\ & f(x,y,z_0) - f(x,y_0,z_0) + \\ & f(x,y_0,z_0) - f(x_0,y_0,z_0) \end{aligned}$$

Therefore, if we define

$$f_1(x,y,z) = \begin{cases} \frac{f(x,y,z) - f(x,y,z_0)}{x - x_0} & \text{if } z \neq z_0 \\ \frac{\partial}{\partial z}(f)|_{(x,y,z_0)} & \text{if } z = z_0 \end{cases},$$

$$f_2(x,y,z) = \begin{cases} \frac{f(x,y,z_0) - f(x,y_0,z_0)}{y - y_0} & \text{if } y \neq y_0 \\ \frac{\partial}{\partial y}(f)|_{(x,y_0,z_0)} & \text{if } y = y_0 \end{cases} \text{ and}$$

$$f_3(x,y,z) = \begin{cases} \frac{f(x,y_0,z_0) - f(x_0,y_0,z_0)}{x - x_0} & \text{if } x \neq x_0 \\ \frac{\partial}{\partial x}(f)|_{(x_0,y_0,z_0)} & \text{if } x = x_0 \end{cases},$$

provided $\frac{\partial}{\partial z}(f)|_{(x,y,z_0)}$, $\frac{\partial}{\partial y}(f)|_{(x,y_0,z_0)}$ and $\frac{\partial}{\partial x}(f)|_{(x_0,y_0,z_0)}$ exist.

If we further assume that $\frac{\partial}{\partial z}(f)|_{(x,y,z)}$, and $\frac{\partial}{\partial y}(f)|_{(x,y,z)}$ are continuous at P_0, then we see that the functions f_1, f_2 and f_3 are continuous at P_0. And, it is easy to check that

$$f(x,y,z) - f(x_0,y_0,z_0) = (z - z_0) f_1(x,y,z) + (y - y_0) f_2(x,y,z) + (x - x_0) f_1(x,y,z)$$

for all $(x,y,z) \in G$. Therefore, by Theorem 2.2.5, we have f is differentiable at P_0. This observation leads to the following

Theorem 2.2.7. *Let $G \subset \mathbb{R}^3$ be open and $P_0 \in G$. Suppose $f : G \longrightarrow \mathbb{R}$ a scalar field such that all the partial derivatives exists in an open ball around P_0, and two of the three partial derivatives are continuous at P_0, then f is differentiable at P_0.*

Thus, in view of Theorem 2.2.7 and Theorem 2.2.6, we see that Theorem 2.2.3 follows as a corollary; and combining those two results we can write the following for $G \subset \mathbb{R}^3$ open, $P_0 \in G$ and $f : G \longrightarrow \mathbb{R}$:

Remark 2.2.8. (I)

$$A : \left\{ \begin{array}{ll} A1: & \textit{all the partial derivatives of } f \textit{ exist in an open ball araound } P_0 \\ A2: & \textit{two of the three partail derivatibes of } f \textit{are continuous at } P_0 \end{array} \right\}$$

$$\Downarrow$$

$$B : \left\{ \ f \textit{ is differentiable at } P_0 \ \right\}$$

$$\Downarrow$$

$$C : \left\{ \begin{array}{ll} C1: & D_{\bar{v}}(f)|_{P_0} \textit{ exists for all direction } \bar{v} \\ C2: & D_{\bar{v}}(f)|_{P_0} = < \nabla f|_{P_0}, \bar{v} > \end{array} \right\}$$

(II) *If we assume that f is oc $\mathcal{C}^1$-type, then both the conditions $A1$ and $A2$ are satisfied.*

(III) *Since $A \Rightarrow B \Rightarrow C$ follows, contrapositively, we have $(\sim C) \Rightarrow (\sim B) \Rightarrow (\sim A)$. These results will be useful to prove or disprove a scalar field is differentiable.*

Example 2.2.9. *1. Let $f(x, y) = \exp(x + y)$ for all $(x, y) \in \mathbb{R}^2$. Since f is a composition of $\mathcal{C}^\infty$-type function, f is a $\mathcal{C}^\infty$-type function on $\mathbb{R}^2$ (and therefore it satisfies all the conditions of "A" in Remark 2.2.8 or the hypothesis of Theorem 2.2.7), we see that f is differentiable on $\mathbb{R}^2$.*

2. Let $f(x, y) = \sqrt{x^2 + y^2}$ for all $(x, y) \in \mathbb{R}^2$. It can be checked that the partial derivatives of f do not exist at $(0, 0)$, and therefore ("C1" fails to hold in Remark 2.2.8) f is not differentiable at $(0, 0)$.

3. Let $f(x, y) = \begin{cases} \frac{x^2 y}{x^4+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$. It can be shown that for any direction $\bar{v}$, $D_{\bar{v}}(f)|_{(0,0)}$ exits; but $D_{\bar{v}}(f)|_{(0,0)} \neq < \nabla(f)|_{(0,0)}, \bar{v} >$ (i.e., "C2" of Remark 2.2.8 fails to hold) and therefore f is not differentiable at $(0, 0)$.

4. Let $f(x, y) = |x| + |y|$ for all $(x, y) \in \mathbb{R}^2$. It can be seen that $D_{\bar{v}}(f)|_{(0,0)}$ does not exist for any direction $\bar{v}$, and therefore f is not differentiable at $(0, 0)$.

Exercise 2.2.10. *1. $f(x, y) = \begin{cases} \frac{x^2 y^2}{x^4+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$. Show f_x and f_y exists in a neighborhood $(0, 0)$, f_x is continuous at $(0, 0)$, f_y is not continuous at $(0, 0)$. Conclude that f is differentiable at $(0, 0)$.*

2. Let $f(x, y) = |xy|$ for all $(x, y \in \mathbb{R}^2)$. Using definition show that f is differentiable at $(0, 0)$.

3. let $f(x, y) = \sqrt{|xy|}$ for all $(x, y) \in \mathbb{R}^2$. Show that f is not differentiable at $(0, 0)$. [Hint: Use contrapositive part of Theorem 2.2.6]

4. $f(x, y) = \begin{cases} \frac{x^3 y}{x^4+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$. Using definition show that f fails to be differentiable at $(0, 0)$, though $D_{\bar{v}}(f)|_{(0,0)} = < \nabla(f)|_{(0,0)}, \bar{v} >$ for any direction $\bar{v}$.

5. $f(x, y) = \begin{cases} \frac{x^2 y}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$. Show that f is continuous at $(0, 0)$; $D_{\bar{v}}(f)|_{P_0}$ exists for all direction $\bar{v}$; but f is not differentiable at $(0, 0)$. [Hint: Use contrapositive part of Theorem 2.2.6]

Remark 2.2.11. *Let $G \subset \mathbb{R}^2$ be open, $P_0 \in G$, and $f : G \longrightarrow \mathbb{R}$ a scalar field. Let f be such that for any direction $\bar{v}$ we have*

$$D_{\bar{v}}(f)|_{P_0} = < \nabla(f)|_{P_0}, \bar{v} > = ||\nabla(f)|_{P_0}||\ ||\bar{v}||\ \cos(\theta) = ||\nabla(f)|_{P_0}||\ \cos(\theta).$$

Where θ is the angle inscribed by $\nabla(f)|_{P_0}$ and $\bar{v}$ with $\bar{0}$. Then we have

1. $D_{\bar{v}}(f)|_{P_0}$ is maximum if $\cos(\theta) = 1$, i.e., if $\bar{v} = \frac{\nabla(f)|_{P_0}}{||\nabla(f)|_{P_0}||}$.

2. $D_{\bar{v}}(f)|_{P_0}$ is mimimum if $\cos(\theta) = -1$, i.e., $\bar{v} = -\frac{\nabla(f)|_{P_0}}{||\nabla(f)|_{P_0}||}$.

3. $D_{\bar{v}}(f)|_{P_0} = 0$ if $\cos(\theta) = \pi/2$, i.e., $\bar{v} = \pm(\frac{\partial}{\partial y}(f)|_{P_0}, -\frac{\partial}{\partial x}(f)|_{P_0})$.

We now list down all the pathological examples.

Example 2.2.12. 1. *Partial differentiability at a point does not imply continuity at that point:*

Let $f(x,y) = \begin{cases} x+y & \text{if } x=0 \text{ or } y=0 \\ 1 & \text{otherwise} \end{cases}$

f *has partial derivatives at* $(0,0)$*, but* f *is not continuous at* $(0,0)$.

2. *Continuity at a point does not imply partial differentiability at that point, and therefore, does not imply differentiability at that point:*

   The function $f(x,y) = |x|+|y|$ *for all* $(x,y) \in \mathbb{R}^2$ *is continuous, but partial derivatives of* f *at* $(0,0)$ *does not exists, and therefore* f *is not differentiable at* $(0,0)$. *One may consider the following function too.* $f(x,y) = \sqrt{x^2+y^2}$ *for all* $(x,y) \in \mathbb{R}^2$ *(check at* $(0,0)$*).*

3. *Existence of partial derivatives at a point do not guarantee existence of all the directional derivative at the same point:*

   Let $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$.

   f *has partial derivatives at* $(0,0)$*, at the same time none of directional derivative at* $(0,0)$ *exist except for* e_1 *and* e_2. *Here* f *is also not continuous at* $(0,0)$.

4. *Existence of all the directional derivative at a point neither imply that the directional derivatives at that point can be written as a linear combination of partial derivatives at that point, nor imply continuity at that point:*

   Let $f(x,y) = \begin{cases} \frac{x^2y}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$.

   For any direction $\bar{v}$, $D_{\bar{v}}(f)|_{(0,0)}$ *exits; but* $D_{\bar{v}}(f)|_{(0,0)} \neq < \nabla(f)|_{(0,0)}, \bar{v} >$ *and therefore* f *is not differentiable at* $(0,0)$. *One may check that* f *is not continuous at* $(0,0)$

5. *Existence of all the directional derivatives at a point along with continuity at that point does not imply that directional derivatives at that point can be written as a linear combination of partial derivatives at that point, and consequently differentiability at that point is also not implied:*

   Let $f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$.

   f *is continuous at* $(0,0)$ *and* $D_{\bar{v}}(f)|_{P_0}$ *exists for all direction* $\bar{v}$, $D_{\bar{v}}(f)|_{(0,0)} \neq < \nabla(f)|_{(0,0)}, \bar{v} >$ *and therefore* f *is not differentiable at* $(0,0)$.

6. *Existence of all the directional derivatives at a point along with the expression directional derivatives at that point in terms of a linear combination of partial derivatives at that point does not imply differentiability at that point:*

   Let $f(x,y) = \begin{cases} \frac{x^3y}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

f *fails to be differentiable at* $(0,0)$*, though* $D_{\bar{v}}(f)|_{(0,0)}$ *exists for all direction* $\bar{v}$ *and* $D_{\bar{v}}(f)|_{(0,0)} = < \nabla(f)|_{(0,0)}, \bar{v} >$ *for any direction* $\bar{v}$*.*

7. *A function which differntiable at a point and having partial derivatives in a neighborhood of the point: However the partial derivatives are not continuous at that point:*

   Let $f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

   f *is differentiable at* $(0,0)$ *and* f_x, f_y *exist in a neighborhood of* $(0,0)$*. However,* f_x *and* f_y *are not continuous at* $(0,0)$*.*

8. *A function which differntiable at a point and having partial derivatives in a neighborhood of the point with the property that only one of the partial partial derivatives is continuous at that point:*

   Let $f(x,y) = \begin{cases} \frac{x^2y^2}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$

   f_x *and* f_y *exists in a neighborhood* $(0,0)$*. Though* f_y *is not continuous at* $(0,0)$*,* f_x *is continuous at* $(0,0)$*, and therefore,* f *is differentiable at* $(0,0)$*.*

2.3 Application: Level sets

Let $U \subset \mathbb{R}^n$, $P_0 \in U$, and $f : U \longrightarrow \mathbb{R}$ a scalar field. For each $c \in \mathbb{R}$, the level set of f of the level c is donoted by $S_c(f)$ (or by $L_c(f)$), and is defined by

$$S_c(f) := \{P \in U \mid f(P) = c\}.$$

When $n = 2$, we call level sets to be level curves, as geometrically those represent a curves; and when $n = 3$, we call level sets to be level surfaces as geometrically those represent surfaces.

Example 2.3.1. 1. *Let* $f(x,y) = x^2+y^2$ *for all* $(x,y) \in \mathbb{R}^2$*. Then for each* $c > 0$*, the level curve* $S_c(f)$ *is the circle* $x^2+y^2 = c$*.*

2. *Let* $f(x,y,z) = x^2+y^2+z^2$ *for all* $(x,y,z) \in \mathbb{R}^3$*. Then for each* $c > 0$*, the level surface* $S_c(f)$ *is the sphere* $x^2+y^2+z^2 = c$*.*

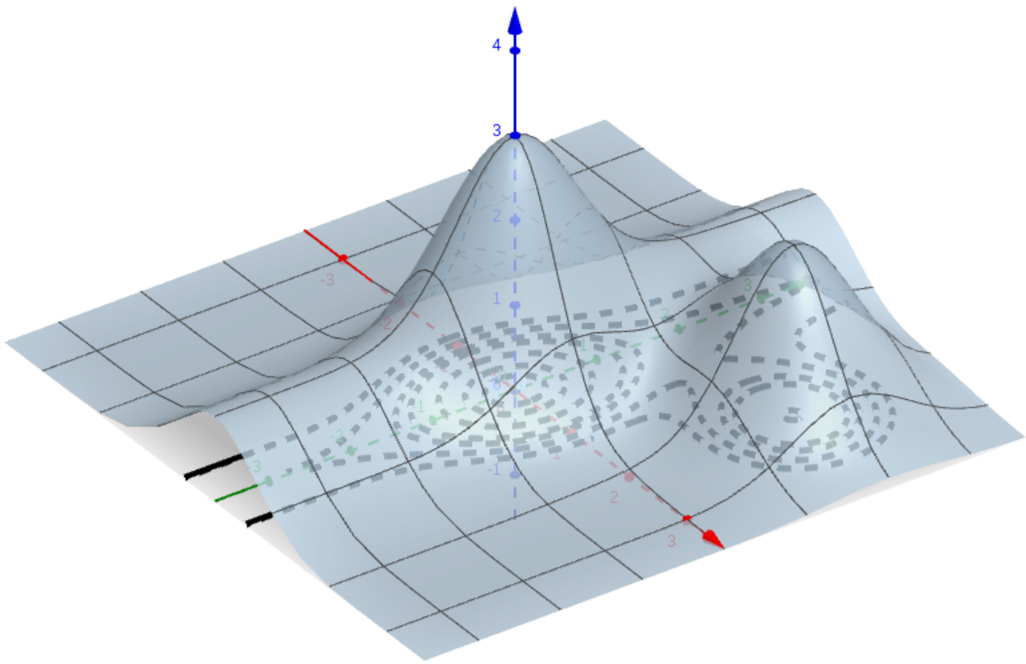
Let $n = 3$, and suppose that the scalar field f is smooth. Consider the level surface $S_c(f)$. Since $S_c(f)$ is a surface, it is union of curves. Let $\gamma(t) = (x(t), y(t), z(t))$ for all $t \in I \subset \mathbb{R}$ be any smooth curve on $S_c(f)$ passing though the point P_0. Then $f(\gamma(t)) = c$. From which, on differentiating with respect to t, we get

$$\frac{\partial}{\partial x}(f)|_{P_0}\frac{d}{dt}(x)|_{t_0} + \frac{\partial}{\partial y}(f)|_{P_0}\frac{d}{dt}(y)|_{t_0} + \frac{\partial}{\partial z}(f)|_{P_0}\frac{d}{dt}(z)|_{t_0} = 0 \text{ , i.e.,}$$

$$< \nabla(f)|_{P_0}, \gamma'(t)|_{t_0} >= 0,$$

where $\gamma(t_0) = P_0$, $\gamma'(t)|_{t=t_0} = (x'(t), y'(t), z'(t))_{t=t_0}$ is tangent to the curve γ at the point P_0.

This shows that $\nabla(f)|_{P_0}$ is normal to the curve γ at the point P_0. Since P_0 is arbitrary, we see that **at any point, the gradient of a scalar field is normal to its corresponding level set at that point, i.e., $\nabla(f)|_{P_0} \perp S_c(f)$ at** P_0.



Exercise 2.3.2. *1. Draw the level surfaces of* $f(x, y, z) = x^2 + y^2 - z^2$ *for all* $(x, y, z) \in \mathbb{R}^3$.

2. *Draw the level curves of* $f(x, y) = x^2 - y^2$ *for all* $(x, y) \in \mathbb{R}^2$.

3. *Let* $h(x, y) = 2e^{-x^2} + e^{-3y^2}$ *denote the height on a mountain at position* (x, y). *In what direction from* $(1, 0)$ *should one begin walking in order to climb the fastest?*

4. *Find a unit normal vector to the following surfaces at the specified point (i)* $x^2 + y^2 + z^2 = 9$ *at* $(0, \sqrt{3}, \sqrt{3})$, *(ii)* $x^3y^3 + y - z + 2 = 0$ *at* $(0, 0, 2)$, *(iii)* $z = 1/(x^2 + y^2)$ *at* $(1, 1, 1/2)$.

5. *Suppose that a particle is ejected from the surface* $x^2 + y^2 - z^2 = -1$ *at the point* $(1, 1, \sqrt{3})$ *along the normal directed toward the* xy *plane to the surface at time* $t = 0$ *with a speed of* 10 *units per second. When and where does it cross the* xy *plane?*

6. *Starting from* $(1, 1)$*, in which direction should one travel in order to obtain the most rapid rate of decrease of the function* $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *defined by* $f(x, y) := (x + y - 2)^2 + (3x - y - 6)^2$*?*

7. *About how much will the function* $f(x, y) := ln\sqrt{x^2 + y^2}$ *change if the point* (x, y) *is moved from* $(3, 4)$ *a distance* 0.1 *unit straight toward* $(3, 6)$*?*

2.4 Using polar coordinets

A discussion on proving continuity or differentiability using polar coordinets:

Let $G \subseteq \mathbb{R}^2$ be open, $P_0 \in G$ and $f : G \longrightarrow \mathbb{R}$ a function. Suppose we need to check continuity of f at P_0. For example $f(x, y) = \begin{cases} \dfrac{x^2 y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{otherwise} \end{cases}$ and we need to check continuity of f at $(0, 0)$.

Sometimes it becomes difficult to check continuity of such a function using Cartesian system and in that case we take help of polar system which sometimes makes the calculations easy. However, there are many cases where we do grave mistakes while doing calculations using polar coordinate system banking on our "intuition" and on which we need to be very careful.

Here we shall discuss a certain aspect of calculations using polar system. Specifically, we shall discuss the "validity" of use of polar system while checking continuity of a function given in terms of Cartesian system, as we do many of such calculations without understanding what actually we are doing.

For example, consider the previous example and we check continuity of f using polar system: Take $x = r\cos(t)$ and $y = r\sin(t)$. Now,

$$\begin{aligned} \lim_{(x,y)\longrightarrow(0,0)} f(x, y) &= \lim_{r\longrightarrow 0} \frac{r^3 \cos^2(t)\sin(t)}{t^2} \\ &= 0 \\ &= f(0, 0), \text{ proving continuity of } f \text{ at } (0, 0). \end{aligned}$$

Here +2 level or college level students rarely ask the validity of the calculation while using this special transformation. Note that here also we have done without justifying the calculation. Here remains a grave chance of doing mistakes. Before going into analysis of the calculation, we shall discuss a few definitions and results for proving a function continuous via some transformation.

Definition 2.4.1. *Let $G_1, G_2 \subseteq \mathbb{R}^2$. We call G_1 and G_2 are homeomorphic (diffeomorphic) if there exists a function $\phi : G_1 \longrightarrow G_2$ such that*

(I) *ϕ is a bijection.*

(II) *ϕ and ϕ^{-1} are continuous (differentiable).*

Further, in this case the function ϕ is called a homeomorphism (diffeomorphism) of G_1 onto G_2.

We now state a result which helps to judge continuity of a function via a transformation.

Theorem 2.4.2. *Let $A, B \subseteq \mathbb{R}^2$ be open, $\phi : A \longrightarrow B$ a homeomorphism (diffeomorphism) such that $\phi(q_0) = p_0$ and $f : B \longrightarrow \mathbb{R}$ a function. Then, f is* *continuous (differentiable) at* p_0 *if and only if* f *is continuous (differentiable) at* q_0 *via the transformation* ϕ*, i.e.,* $f \circ \phi$ *is continuous (differentiable) at* q_0.

In view of our example, note that the transformation $x = r\cos(t)$ and $y = r\sin(t)$ is actually a homemorphism of $(0, \infty) \times [0, 2\pi)$ onto $\mathbb{R}^2 \backslash \{(0, 0)\}$ given by $\phi : (r, t) \mapsto (r\cos(t), r\sin(t))$. Since $(0, 0)$ is not in the range of of the function ϕ, we can not use the above theorem to justify the use the homeomorphism (Cartesian to polar and vise versa) to prove continuity of f!!

However note the following property of homeomorphism, which is useful handling limit or sequence of a function via a homeomorphism.

Theorem 2.4.3. *Let* $A, B \subseteq \mathbb{R}^2$ *be open,* $\phi : A \longrightarrow B$ *a homeomorphism. Then, a sequence* $\{p_n\}$ *in* B *is converges to a point* p_0 *in* B *if and only of the corresponding sequence* $\{\phi^{-1}(p_n)\}$ *in* A *is also converges to the point* $\phi^{-1}(p_0)$ *belonging to* A.

We should understand if $p_0 \in \mathbb{R}^2$ is such that $p_0 \notin B$ and the sequence $\{p_n\}$ in B converges to p_0, then we can not say that the corresponding sequence $\phi^{-1}(p_n)$ converges to some point. It is because, in general, the homeomorphism does not preserve Cauchy sequence (distance!). For example The space $\mathbb{R}$ is homeomorphic to $(0, \infty)$ via the homeomorphism $x \mapsto e^x$. It is known that $\{1/n\}$ is a Cauchy sequence in $(0, \infty)$, but the corresponding sequence $\{\ln(1/n)\}$ is not a Cauchy sequence in $\mathbb{R}$.

Coming back to our example we see that though there are many sequence of points converging to $(0, 0)$, we can not use Theorem 2.4.3 via Cartesian-polar homeomorphism due to the point $(0, 0)$. However, let $\{(x_n, y_n)\} \in \mathbb{R}^2 \backslash \{(0, 0)\}$ be such that it converges to $(0, 0)$ and $(x_n, y_n) \neq (0, 0)$ for all $n \in \mathbb{N}$. Since the Cartesian-polar homeomorphism $((r, t) \mapsto (r\cos(t), r\sin(t)))$ is valid for all points $(x, y) \neq (0, 0)$, due its bijective property we have $(x_n, y_n) = (r_n\cos(t_n), r_n\sin(t_n))$ for of exactly one (r_n, t_n), and therefore, for the sequence $\{(x_n, y_n)\}$ we can correspond to the sequence $\{(r_n, t_n)\}$ element by element. Now, one can see (using projection maps, properties of real number, etc) that $\{(x_n, y_n)\} \longrightarrow (0, 0)$ if and only if $\{x_n\} \longrightarrow 0$ and $\{y_n\} \longrightarrow 0$ if and only if $\{x_n^2 + y_n^2\} \longrightarrow 0$ if and only if $\{\sqrt{x_n^2 + y_n^2}\} \longrightarrow 0$, i.e., $\{r_n\} \longrightarrow 0$. Note that $r_n \neq 0$.

So, for a function f defined on $\mathbb{R}^2 \backslash \{(0, 0)\}$ we have

$$\begin{aligned} \lim_{(x_n, y_n) \longrightarrow (0,0)} f(x_n, y_n) &= \lim_{r_n \longrightarrow 0} f(x_n, y_n) \\ &= \lim_{r_n \longrightarrow 0} f(r_n\cos(t_n), r_n\sin(t_n)). \end{aligned}$$

This equality of limit we could write because of the element by element correspondence between the sequences $\{(x_n, y_n)\}$ and $\{(r_n, t_n)\}$ and due to the eqivalence of the statements "$(x_n, y_n) \longrightarrow (0, 0)$" and "$r_n \longrightarrow 0$". Exactly via this argument one can see that the calculation and conclusion in our example is correct. However, it would be nice if we can write it "nicely"; "indicating" all the "crucial steps" as follows:

$$f(x, y) = \begin{cases} \dfrac{x^2 y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ \\ 0 & \text{otherwise.} \end{cases}$$

We shall check continuity of f at $(0,0)$, i.e., we shall check whether

$$\lim_{(x,y)\longrightarrow(0,0)} f(x,y) = f(0,0).$$

Consider the transformation $\phi : (x,y) = (r\cos(t), r\sin(t))$ of $(0,\infty)\times[0,2\pi)$ onto $\mathbb{R}^2\backslash\{0,0\}$. Since $(x,y) \longrightarrow (0,0)$ if and only if $r \longrightarrow 0$ and the transformation ϕ is bijective, we have $\lim\limits_{(x,y)\longrightarrow(0,0)} f(x,y) = \lim\limits_{r\longrightarrow 0} f(r\cos(t), r\sin(t))$. Now,

$$\begin{aligned}
\lim_{r\longrightarrow 0} f(r\cos(t), r\sin(t)) &= \lim_{r\longrightarrow 0} \frac{r^3\cos^2(t)\sin(t)}{r^2(\cos^(t) + \sin^2(t))} \\
&= \lim_{r\longrightarrow 0} r\cos^2(t)\sin(t) \\
&= 0.
\end{aligned}$$

Therefore, $\lim\limits_{(x,y)\longrightarrow(0,0)} f(x,y) = 0 = f(0,0)$. This shows that f is continuous at $(0,0)$.

Chapter 3

Arc Length Function

3.1 Introduction

If L is a straight line segment (finite) or a finite union of straight line segments in $\mathbb{R}^3$, it is well known to us how to measure the length of the segment or totality of of the length of the segments. In this chapter we shall drive a tool to measure the length of a segment of a parametric curve.

3.2 Parametric Curves

Definition 3.2.1. *Let* I *be an interval. Any continuous function* $\gamma : I \longrightarrow \mathbb{R}^3$ *is called a* ***parametric curve****. Note that for each* $t \in I$*, we have* $\gamma(t) = (x(t), y(t), z(t)) \in \mathbb{R}^3$*. Let* $C := \gamma(I)$ *(the image of* γ*). Then,* C *is a curve which has a parametric representation given by the function* γ*. We shall use the notation* $\{\gamma\}$ *to denote the* ***trace*** *of* γ*, and is defined by* $\{\gamma\} = \{\gamma(t) \mid t \in I\} \subset \mathbb{R}^3$ *along with an ordering induced from the ordering in* I *(i.e., if* $\preceq$ *is the induced ordering on* $\{\gamma\}$*, then* $\preceq$ *is defined by* $\gamma(t_1) \preceq \gamma(t_2)$ *iff* $t_1 \leq t_2$*). This ordering on the points of* $\gamma(I)$ *is called* ***orientation*** *of* γ*. If* $I = [a, b]$*, then* $\gamma(a)$ *is called the* ***initial point*** *of* γ*, and* $\gamma(b)$ *is called the* ***final point*** *of* γ*. If initial point and final point of a parametric curve are the same point, we call the curve to be a* ***loop/closed curve****. A closed curve* γ *on* $[a, b]$ *is called* ***simple*** *if* γ *is one-to-one on* $[a, b)$*. Let* $C' \subset \mathbb{R}^3$ *be a set of points. If* C' *can be seen as a trace of a parametric curve say,* $\delta : I \longrightarrow \mathbb{R}^3$*, we say that* C *is a parametric curve with a* ***parametrisation*** δ*, and write* $C' = \{\delta : I \longrightarrow \mathbb{R}^3\}$*.*

Definition 3.2.2. *We call a parametric curve to be* $\mathcal{C}^n$***-type/***$\mathcal{C}^\infty$***-type/smooth*** *(*$n \geq 1$*) if it has a parametrisation which is* $\mathcal{C}^n/\mathcal{C}^\infty$*/smooth. If the domain of the curve is a closed interval, then, in this case, it will mean that the curve has an extension in an open interval containing the closed interval and we are taking about* $\mathcal{C}^n$*-ness/*$\mathcal{C}^\infty$*-ness/smoothness of the extension of the curve without mentioning the extension.*

The following example will clarify the terminologies.

Example 3.2.3. *1.* $C =: \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1,\ z = 5\}$ *is a curve which has the following parametric representations. Let* $\gamma(t) = (\cos t, \sin t, 5)$

where $t \in [0, 2\pi]$*, then we have* $\gamma(I) = C$*.* γ *is a simple loop as* $\gamma(0) = \gamma(2\pi)$ *and* γ *is one-to-one in* $[0, 2\pi)$*. Again, for* $\delta(t) = (\cos 2\pi t, \sin 2\pi t, 5)$ *where* $t \in [0, 1]$*, we have* $\gamma(I) = C$*.* δ *is one-to-one in* $[0, 1)$*.*

2. *Figure eight,* $x^4 = a^2(x^2 - y^2)$ *in* $\mathbb{R}^2$ *is a non-simple parametric loop given by* $\gamma(t) = (a \sin t, a \sin t \cos t)$ *where* $t \in [0, 2\pi]$*.*

In first example it can be seen that though $\gamma \neq \delta$, we see that $\{\delta(t) \mid t \in [0, 1]\} = \{\gamma(t) \mid t \in [0, 2\pi]\} = \{(x, y, z) \mid x^2 + y^2 = 1.\ z = 5\} = C$. This shows that the curve C has two parametrizations given by δ and γ.

Remark 3.2.4 (Curves as weighted intervals).

- *A continuous function* $f : [a, b] \longrightarrow \mathbb{R}$ *is nothing but a weight on the interval* $[a, b]$*! It "bends" the st. line segment* $\{(x, 0) \mid x \in [a, b]\}$ *to a curve* $C := \{(x, f(x)) \mid x \in [a, b]\}$*!!*
- *Is length of* $[a, b]$ = *length of* C*? How to measure? (Shall do it!)*
- *What is* $\int_a^b f(x)\ dx$*?* ***Answer:*** *Weighted length of* $[a, b]$ *(= area under the curve* C*)!!*
- *Is* $\int_a^b f(x)\ dx$ = *"Length" of curve* C*?* ***Answer:*** *Not in general! Can you give an example? Answer: Right angled triangle!*
- $\gamma : [a, b] \longrightarrow \mathbb{R}^3$ *viewed as a weighted interval/line segment; i.e., each point* x *of the segment* $[a, b]$ *is given a weight* $\gamma(x) \in \mathbb{R}^3$*!*

Question 1. *How many parametrization is possible for a parametric curves?*

Answer: Infinitely many!

3.3 Re-parametrization of parametric curves

Now we shall show that a parametric curve has infinitely many parametrisation.

Idea: For a curve $C = \gamma : [a, b] \longrightarrow \mathbb{R}^3$, generate composite function $[c, d] \xrightarrow{\theta} [a, b] \xrightarrow{\gamma} C$ such that (geometry is unchanged) (1) The orientation is preserved, i.e., $0 \mapsto a$, $1 \mapsto b$ and intermediate points accordingly **(what happens otherwise?)**, (2) Simple-ness of γ is preserved, i.e., θ is one-one. (3) Continuity of γ is preserved, i.e., θ is continuous, (4) $\mathcal{C}^n$-ness of γ, if exists, is preserved, i.e., θ has to be $\mathcal{C}^n$-type if γ is $\mathcal{C}^n$-type.

Note: (1) & (2) implies θ is strictly increasing and onto.

Examples of re-parametrization: Can take linear function! (but not that nice!!)

Suppose $[a, b]$ be an intervals in $\mathbb{R}$. Define a maps $\Phi_{[a,b]} : [a, b] \longrightarrow [0, 1]$, and $\Psi_{[a,b]} : [0, 1] \longrightarrow [a, b]$ by

$\Phi_{[a,b]}(t) = \dfrac{b-t}{b-a}$ for all $t \in [a,b]$ (this function scales the interval $[a,b]$ onto $[0,1]$)

and

$\Psi_{[a,b]}(s) = bs + a(1-s)$ for all $s \in [0,1]$ (this function scales the interval $[0,1]$ onto $[a,b]$).

Note that $\Phi_{[a,b]}(a) = 0$, $\Phi_{[a,b]}(b) = 1$, $\Psi_{[a,b]}(0) = a$, $\Psi_{[a,b]}(1) = b$, $\Phi_{[a,b]} \circ \Psi_{[a,b]}(s) = s$ for all $s \in [0,1]$, $\Psi_{[a,b]} \circ \Phi_{[a,b]}(t) = t$ for all $t \in [a,b]$, (i.e., $\Phi_{[a,b]} = \Psi^{-1}_{[a,b]}$, and $\Psi_{[a,b]} = \Phi^{-1}_{[a,b]}$) and both the functions are infinitely many differentiable . Now, for given any pair of interval $[a,b]$ and $[c,d]$, we shall define a map $\mathcal{T}^{[a,b]}_{[c,d]} : [c,d] \longrightarrow [a,b]$, using these properties of Φ and Ψ as follows

$$[c,d] \xrightarrow{\quad\Phi_{[c,d]}\quad} [0,1] \xrightarrow{\quad\Psi_{[a,b]}\quad} [a,b], \text{ i.e.,}$$

$\mathcal{T}^{[a,b]}_{[c,d]}(t) = \Psi_{[a,b]} \circ \Phi_{[c,d]}(t)$ for all $t \in [c,d]$. Now for any parametric curve $\gamma : [a,b] \longrightarrow \mathbb{R}^3$, consider the map $\gamma \circ \mathcal{T}^{[a,b]}_{[c,d]} : [c,d] \longrightarrow \mathbb{R}^3$. Check that $\{\gamma\} = \{\gamma \circ \mathcal{T}^{[a,b]}_{[c,d]}\}$, and therefore both γ and $\gamma \circ \mathcal{T}^{[a,b]}_{[c,d]}$ represent the same parametric curve. If we assume that $\{\gamma\} = C$, then the above discussion shows the functions $\Psi_{[a,b]}$ and $\Phi_{[c,d]} : [c,d]$ induce linear change of parameter from any interval $[a,b]$ to any interval $[c,d]$ and that C has infinitely many parametrization. The developed theory will show that it is enough to consider one favorable parametrisation of a parametric curve, and study the topics under discussion.

One can see that for $\gamma(t) = (\cos t, \sin t, 5)$ where $t \in [0, 2\pi]$ and $\delta(t) = (\cos 2\pi t, \sin 2\pi t, 5)$ where $t \in [0,1]$, $\gamma = \delta \circ \Phi_{[0,2\pi]}$ and $\delta = \gamma \circ \Psi_{[0,2\pi]}$. One can check that $\Psi_{[a,b]}$ and $\Phi_{[a,b]}$ are inverse of each other, i.e., $\Psi_{[a,b]} \circ \Phi_{[a,b]} = \mathrm{id}_{[a,b]}$ and $\Phi_{[a,b]} \circ \Psi_{[a,b]} = \mathrm{id}_{[0,1]}$.

3.4 Oppositely oriented parametric curve

Let $[a,b] \subset \mathbb{R}$. Define a map $\overline{\mathrm{id}} : [a,b] \longrightarrow [a,b]$ by $\overline{\mathrm{id}}(x) = a + b - x$ for all $x \in [a,b]$. Note that the function $\overline{\mathrm{id}}$ reverses the order of the elements of the interval $[a,b]$, i.e., it "flips" the interval; and therefore, for any parametric curve $\gamma : [a,b] \longrightarrow \mathbb{R}^3$, the function $\gamma \circ \overline{\mathrm{id}} : [a,b] \longrightarrow \mathbb{R}^3$ represents again a parametric curve such that $\gamma([a,b]) = \gamma \circ \overline{\mathrm{id}}([a,b])$, and the ordering of $\{\gamma \circ \overline{\mathrm{id}}\}$ is the reverse of the ordering of $\{\gamma\}$, i.e., γ and $\gamma \circ \overline{\mathrm{id}}$ represents the same curve (set of points), but the orientation of the curves are opposite to each other; specifically, $\gamma \circ \overline{\mathrm{id}}(t) = \gamma(a + b - t)$ for all $t \in [a,b]$.

Definition 3.4.1. *For any parametric curve $\gamma : [a,b] \longrightarrow \mathbb{R}^3$, the corresponding parametric curve $\gamma^{-1} : [a,b] \longrightarrow \mathbb{R}^3$ defined by $\gamma^{-1}(t) = \gamma \circ \overline{\mathrm{id}}(t) = \gamma(a+b-t)$ for all $t \in [a,b]$ is called the* ***inverse*** *of γ.*

Exercise 3.4.2. 1. *Show that if a parametric curve is $\mathcal{C}^n$-type/$\mathcal{C}^\infty$-type/smooth, then the inverse of the parametric curve is also $\mathcal{C}^n$-type/$\mathcal{C}^\infty$-type/smooth ($n \geq 0$).*

2. *Show that* $\tau_{[c,d]}^{[a,b]}$ *is a smooth function, and conclude that* $\gamma \circ \tau_{[c,d]}^{[a,b]}$ *is smooth if* γ *is smooth.*

3.5 Joining of two parametric curves

Definition 3.5.1. *Let* $\gamma_1 : [a, b] \longrightarrow \mathbb{R}^3$ *and* $\gamma_2 : [c, d] \longrightarrow \mathbb{R}^3$ *be two parametric curves such that the final point of* γ_1 *is the initial point of* γ_2*, i.e.,* $\gamma_1(b) = \gamma_2(c)$*, then the **join** or **addition** of* γ_1 *and* γ_2*, denoted by* $\gamma_1 * \gamma_2$ *is defined by*

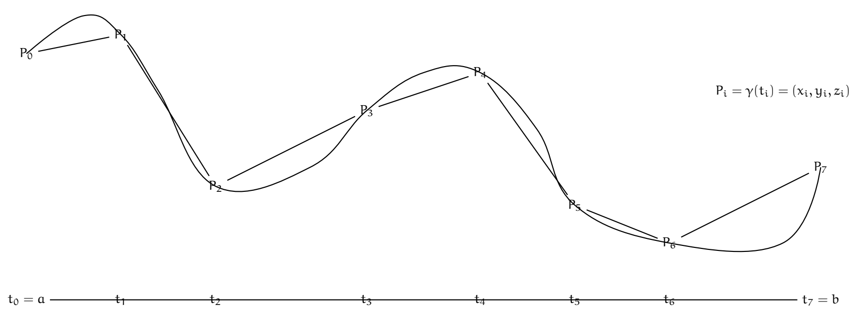
$$\gamma_1 * \gamma_2(t) = \begin{cases} \gamma_1 \circ \mathcal{T}_{[0,\frac{1}{2}]}^{[a,b]}(t) & \text{if } t \in [0,\ \frac{1}{2}] \\ \\ \gamma_2 \circ \mathcal{T}_{[\frac{1}{2},1]}^{[c,d]}(t) & \text{if } t \in [\frac{1}{2},\ 1] \end{cases}$$

Note that $\gamma_1 \circ \gamma_2$ *is also a parametric curve.*

3.6 Arc Lengh of Parametric Curves

Let $\gamma : [a, b] \longrightarrow \mathbb{R}^3$ is a parametric curve. Then the curve (image of γ) $\gamma([a, b])$ is a compact set as γ is continuous. This does not mean that the length of the curve has to be finite! (Ex: space filling curve!)

Question 2. *When is the length of* γ *finite?*



We wish to calculate the length of the curve, if possible. For this we imitate the theory of Riemann integration to calculate area under a curve defined in an finite interval. Let $\mathcal{P}([a, b])$ be the set of all partitions of $[a, b]$. We first take a partition $P_n : \{a = t_0, t_1, t_2, \cdots, t_n = b\}$ from $\mathcal{P}([a, b])$ and, though linear approximation find an approximate value $\ell_{P_n}(\gamma)$ of the length of the curve γ for the chosen partition P_n as

$$\ell_{P_n}(\gamma) = \sum_{i=1}^{n} \|\gamma(t_i) - \gamma(t_{i-1})\|.$$

If $\lim_{n\to\infty} \ell_{P_n}(\gamma)$ exists we say that length of γ exists, and the value of the limit is defined as the length of γ, and is denoted by $\ell(\gamma)$; i.e., $\ell(\gamma) = \lim_{n\to\infty} \ell_{P_n}(\gamma)$, if the limit exists.

If we let $||P_n|| := \max_n |t_n - t_{n-1}|$, we see that $\lim_{n\to\infty} \ell_{P_n}(\gamma) = \lim_{||P_n||\to 0} \ell_{P_n}(\gamma)$.

Let $\mathcal{P}_{eq}([a,b]) = \{P \in \mathcal{P}([a,b]) \mid \text{points in } P \text{ are equidistant}\}$. As can be seen that $\lim_{||P||\to 0} \ell_P(\gamma) = \lim_{||P||\to 0} \{\ell_P(\gamma) \mid P \in \mathcal{P}_{eq}([a,b])\}$, for the context under discussion it is enough to take partitions of $[a,b]$ from $\mathcal{P}_{eq}([a,b])$. Note that any partition in $\mathcal{P}_{eq}([a,b])$ is uniquely determined by the number of points in it; and therefore, instead of notation $\ell_{P_n}(\gamma)$ it is enough to use the notation $\ell_n(\gamma)$ to indicate that we have taken a partition from $\mathcal{P}_{eq}([a,b])$ having $n+1$ equidistant points. Thus, we shall write $\ell(\gamma) = \lim_{n\to\infty} \ell_n(\gamma)$, if the limit exists.

Let $P \in \mathcal{P}_{eq}([a,b])$ with $n+1$ points. Assume that $||P|| = \Delta t$. Clearly, it implies that for any two consecutive points t_i and t_{i+1} from P we have $t_{i+1} = t_i + \Delta t$. We see that $\lim_{n\to\infty} \ell_P(\gamma) = \lim_{||P||\to 0} \ell_n(\gamma) = \lim_{\Delta t\to 0} \ell_n(\gamma)$. If we write $\gamma(t) = (x(t), y(t), z(t))$, then we see that

$$
\begin{aligned}
\lim_{\Delta t\to 0} \ell_n(\gamma) &= \lim_{n\to 0} \ell_n(\gamma) \\
&= \lim_{n\to 0} \sum_{i=1}^{n} \lim_{\Delta t\to 0} \sqrt{(x(t_i)-x(t_{i-1}))^2 + (y(t_i)-y(t_{i-1}))^2 + (z(t_i)-z(t_{i-1}))^2} \\
&= \lim_{n\to 0} \sum_{i=1}^{n} \lim_{\Delta t\to 0} \frac{\sqrt{(x(t_i)-x(t_{i-1}))^2 + (y(t_i)-y(t_{i-1}))^2 + (z(t_i)-z(t_{i-1}))^2}}{\Delta t} \Delta t \\
&= \lim_{n\to 0} \sum_{i=1}^{n} \lim_{\Delta t\to 0} \sqrt{\frac{(x(t_i)-x(t_{i-1}))^2 + (y(t_i)-y(t_{i-1}))^2 + (z(t_i)-z(t_{i-1}))^2}{\Delta t^2}} \Delta t \\
&= \lim_{n\to 0} \sum_{i=1}^{n} \lim_{\Delta t\to 0} \sqrt{\left(\frac{x(t_i)-x(t_{i-1})}{\Delta t}\right)^2 + \left(\frac{y(t_i)-y(t_{i-1})}{\Delta t}\right)^2 + \left(\frac{z(t_i)-z(t_{i-1})}{\Delta t}\right)^2} \Delta t \\
&= \lim_{n\to 0} \sum_{i=1}^{n} \sqrt{\left(\frac{d}{dt}(x(t))|_{t_{i-1}}\right)^2 + \left(\frac{d}{dt}(y(t))|_{t_{i-1}}\right)^2 + \left(\frac{d}{dt}(z(t))|_{t_{i-1}}\right)^2} \Delta t \\
&\qquad \text{if } \gamma \text{ is differentiable, i.e., if } \gamma'(t) \text{ exists} \\
&= \lim_{n\to 0} \sum_{i=1}^{n} ||\gamma'(t)||_{t_{i-1}} \Delta t \\
&= \int_a^b ||\gamma'(t)|| dt \\
&\qquad \text{if } \gamma' \text{ is continuous, i.e., if } \gamma \text{ is } \mathcal{C}^1\text{-type}
\end{aligned}
$$

Thus, we get the following theorem

Theorem 3.6.1. *Let $\gamma : [a,b] \longrightarrow \mathbb{R}^3$ be a $\mathcal{C}^1$-type curve. Then, $\ell(\gamma)$ exists, and $\ell(\gamma) = \int_a^b ||\gamma'(t)||\ dt$ or simply $\ell(\gamma) = \int_a^b ||\gamma'||$*

Exercise 3.6.2. *1. Let $\gamma : [a,b] \longrightarrow \mathbb{R}^3$ be a $\mathcal{C}^1$-type curve. Show that $\gamma \circ \mathcal{T}_{[c,d]}^{[a,b]} : [c,d] \longrightarrow \mathbb{R}^3$ is also $\mathcal{C}^1$-type curve; and $\ell(\gamma) = \ell(\gamma \circ \mathcal{T}_{[c,d]}^{[a,b]})$.*

2. Show that arc length of $\mathcal{C}^1$-type parametric curve does not change under change of parameter.

Remark 3.6.3 ("Proper" bending of st. line to make it a curve). *We now again return to the discussion that a curve $\gamma[a,b] \longrightarrow \mathbb{R}^3$ is nothing but the interval $\underline{[a,b]}$ with a given weight γ, which again can be realized as the st. line segment $\overline{(a,0),\ (b,0)}$ with weight γ and the weight is such that it is "bending" the st. line to be a curve $C = \gamma(I)$. In that case, if the "bending" is "proper", the st. line will not stretch we shall have $\ell(\overline{(a,0),\ (b,0)}) = \ell(C)$. However, we have seen that it is not in general true that $\ell(\overline{(a,0),\ (b,0)}) = \ell(C) = \ell(\gamma)$ and that the length of the st. line varies of the parametric representation of the curve C. So, naturally a question raises that can we have a parametrization of C such that the length of the parametric interval is same as length of C. From our previous discussion we know such parametrisation is possible. However, our interest is more than that!! We aim to measure how much the st. line segment is bent to be the curve which is no more a st. line; in a sense we want to measure the degree of "bend-ness" of a curve with respect to a st. line. Before that we again review some of the concepts.*

3.7 What is "dx" in "$\int f(x)\ dx$" and "$\int_a^b f(x)\ dx$"?

Recall that $\int f(x)\ dx$ refers to an anti-derivative of the funtion $f(x)$ with respect to the variable x. This means it is assumed that $\int f(x)\ dx$ exists and is equal to a function $F(x)$ defined as a function whose "**d**ifferentiation with respect to x" gives the function $f(x)$. Here dx stands for "**d**ifferentiation with respect to x". From that perspective one can see that

$$F(x) = \int f(x)\ dx = \int \frac{d}{dx}F(x)\ dx = \int 1\ dF(x),$$

as $\frac{d}{dF(x)}F(x) = 1$. However, such sense to the definite integral $\int_a^b f(x)\ dx$ is not there directly. The sense builds up only from the Fundamental Theorem of Integral Calculus (FTIC) which states as follows.

Theorem 3.7.1 (FTIC). *Let $f[a,b] \longrightarrow \mathbb{R}$ be a continuous function. Then the the function f has a primitive on $[a,b]$, i.e., there exits a function $F : [a,b] \longrightarrow$ such that $\frac{d}{dx}(F(x)) = f(x)$ for all $x \in [a,b]$. Further,*

(I) $\int_a^b f(x)\ dx = F(b) - F(a)$

(II) $F(x) = \int_a^x f(t)\ dt$ *for all* $x \in [a,b]$

Note that this theorem guarantees that if a function $f : [a,b] \longrightarrow \mathbb{R}$ is continuous, then the differential equation $\frac{dy}{dx} = f(x)$ has a solution in the interval $[a,b]$. It is not that every differential equation $\frac{dy}{dx} = g(x)$ has a solution!!

In Theorem 3.7.1, from $\mathsf{F}(x) = \displaystyle\int_a^x f(t)\ dt$ for all $x \in [a, b]$ and $\frac{d}{dx}(F(x)) = f(x)$ for all $x \in [a, b]$ we have

$$F(x) = \int_a^x f(t)\ dt = \int_a^x \frac{d}{dt}(F(t))\ dt = \int_a^x 1\ dF(t)$$

for all $x \in [a, b]$, as as $\frac{d}{dF(x)}F(x) = 1$ from Theorem 3.7.1!! Hence, from this perspective we have the following.

$$\int_a^b f(t)\ dt = \int_a^b \frac{d}{dt}(F(t))\ dt = \int_a^b 1\ dF(t)$$

Question 3. *What is* $F(a)$*??*

The above expression itself refer to a change of variable/parameter from t to $F(t)$. So, now we look back the arc-length formula for a $\mathcal{C}^1$-type parametric curve $\gamma : [a, b] \longrightarrow \mathbb{R}^3$ and ask what kind of change of parameter as discussed above will give us something like:

$$\ell(\gamma) = \int_a^b ||\gamma'(t)||\ dt = \int_a^b 1\ d??$$

Since γ is $\mathcal{C}^1$-type, $||\gamma'(t)||$ is continuous for all $t \in [a, b]$ and hence we can apply Theorem 3.7.1 to get a function $s : [a, b] \longrightarrow \mathbb{R}$ such that $s(x) = \displaystyle\int_a^x ||\gamma'(t)||\ dt$ and $\dfrac{d}{dx}s(x) = ||\gamma'(x)||$ for all $x \in [a, b]$. So, in that case we shall have the following.

$$\ell(\gamma) = \int_a^b ||\gamma'(t)||\ dt = \int_a^x \frac{d}{dt}s(t)\ dt = \int_a^b 1\ ds(t)$$

.

Here as if a certain kind of change of parameter is happening and which we should understand in full detail, to know know the effect of this change of parameter to the curve γ and what exactly is the change of parameter function, which of course is originating from the function $s(x)$, $x \in [a, b]$.

3.8 Arc Length Function of Parametric Curves

Let $\gamma : [a, b] \longrightarrow \mathbb{R}^3$ be a $\mathcal{C}^1$-type curve. It is well know that the length of the curve γ from $\gamma(a)$ to $\gamma(b)$ is given by $\ell(\gamma) = \displaystyle\int_a^b ||\gamma'(t)||\ dt$; and therefore for any $x \in [a, b]$, the length of the curve γ from $\gamma(a)$ ot $\gamma(x)$ is give by $\displaystyle\int_a^x ||\gamma'(t)||\ dt$. This defines a map $s : [a, b] \longrightarrow \mathbb{R}$ by $s(x) = \displaystyle\int_a^x ||\gamma'(t)||\ dt$. This function is called the arc length function of the curve γ. We shall study the properties of this function after we observe few examples.

Example 3.8.1. *Let* $\gamma(t) = (\cos(2\pi t),\ \sin(2\pi t))$ *for all* $t \in [0, 1]$*. Clearly* γ *is a* $\mathcal{C}^1$*-type curve, and* $||\gamma'(t)|| = ||(-2\pi\sin(2\pi t),\ 2\pi\cos(2\pi t))|| = 2\pi$ *for all* $t \in [0, 1]$*. Therefore, the arc length function of* γ *is given by*

$$\begin{aligned} s(x) &= \int_0^x \|\gamma(t)\| dt \\ &= \int_0^x 2\pi dt \\ &= 2\pi x \end{aligned}$$

We know that the curve $\{\gamma\}$ *has another parametrisation* $\delta : [0, 2\pi] \longrightarrow \mathbb{R}^3$ *defined by* $\delta(t) = (\cos(t),\ \sin(t))$ *for all* $t \in [0,\ 2\pi]$*. For this parametrisation it can be seen that* $s(x) = x$ *for all* $x \in [0, 2\pi]$*.*

Remark 3.8.2. *The above example clearly shows that the different parametrisation to the same curve gives different arc length function. So, in view of the discussion in 3.7 as we see that*

$$s(x) = \int_a^x \|\gamma'(t)\|\ dt = \int_a^x \frac{d}{dt} s(t)\ dt = \int_a^x 1\ ds(t)$$

where $1 = \frac{d}{ds} s$*, and therefore, one naturally ask whether there is any method of parametrization for which given any curve the arc length function is will be* $s(x)$ *will be such that* $\frac{d}{dx} s(x) = \|\eta'(x)\| = 1$ *in the domain of the parameterization* I*. Such parametrization method, if exixts, will be sort of a universal parametrization method. Suppose* $I = [c, d]$*. Since* $\int_I \|\eta'(t)\| dt = \ell(C)$*, and* $s(x) = \int_c^x \|\eta'(t)\| dt$ *it is clear that* $c = 0$ *and* $d = \ell(C)$*. This observation along with the above equality which also talks about certain change of parameter which is depending on the function* $s(x)$*, gives us a possible candidate for that "universal parametrization":* $s^{-1} : [0, \ell(C)] \longrightarrow I$*, if exists, since the arc length function* s *is a surjective function from the domain of the parametrised curve* C *to the interval* $[0, \ell(C)]$*. So, we ask whether the function* s^{-1} *can be used as a change of parameter converting domain of the parametric curve to* $[0, \ell(C)]$ *. The following properties of the arc length function will help us to get the answer to this question.*

Before going into the properties let us recall a fact: ***"Any real valued bijective continuous function defined on a closed interval has continuous inverse. Further, if the function is smooth, the inverse is also smooth."*** *(Prove it!!!)*

Theorem 3.8.3. *Let* $s : [a, b] \longrightarrow \mathbb{R}$ *be the arc length function of the* $\mathcal{C}^1$*-type curve* $\gamma : [a, b] \longrightarrow \mathbb{R}^3$*. Then the following holds*

(I) $s(x) \geq 0$ *for all* $x \in [a, b]$*,* $s(a) = 0$*, and* $s(b) = \ell(\gamma)$*.*

(II) s *is non-decreasing on* $[a, b]$*.*

(III) $s \not\equiv 0$ *if and only* $\gamma'(t) \neq \bar{0}$ *for some* $t \in [a, b]$*, i.e.,* $s \equiv 0$ *if and only if* $\gamma' = \bar{0}$ *on* $[a, b]$*.*

(IV) *If* $\gamma'(t) \neq \bar{0}$ *for all* $t \in [a, b]$ *(i.e.,* γ *is smooth), then* $s(x) = 0$ *if and only if* $x = a$*.*

(V) *If* $\gamma'(t) \neq \bar{0}$ *for all* $t \in [a, b]$ *(i.e.,* γ *is smooth), then* s *is strictly increasing on* $[a, b]$.

(VI) s *is continuous on* $[a, b]$.

(VII) s *is differentiable on* $[a, b]$.

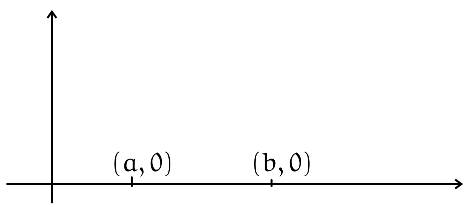
(VIII) s *is* $\mathcal{C}^1$*-type on* $[a, b]$*, and* $\frac{d}{dx}s(x) = ||\gamma'(x)||$ *for all* $x \in [a, b]$.

(IX) s *is smooth if* γ *is smooth.*

Proof. Assignment! □

Definition 3.8.4. *Suppose* $s : [a, b] \longrightarrow \mathbb{R}$ *be the arc length function of the* $\mathcal{C}^1$*-type curve* $\gamma : [a, b] \longrightarrow \mathbb{R}^3$*. Then due to (VIII) of Theorem 3.8.3 we write* $\displaystyle\int_a^b ||\gamma'(t)||dt$ *as* $\displaystyle\int_a^b ds(t)$ *or* $\displaystyle\int_0^{\ell(\gamma)} ds$*, as* $\frac{d}{dx}s(x) = ||\gamma'(x)||$.

3.9 Need of "correct" parametrization



$\gamma(t) = (a + t, 0)\ \forall t \in [0, b - a]$ is a parametrization of the line segment. γ is $\mathcal{C}^1$-type $\Longrightarrow$ arc length function is

$$s_\gamma(x) = \int_0^x ||\gamma'(t)||dt = \int_0^x ||(1, 0)||dt = \int_0^x 1 dt = x \quad \forall x \in [0, b - a]$$

Note that in two calculations of s_γ and s_δ, we have $\dfrac{d}{dx}(s_\gamma) = 1 \longrightarrow$ rate of change is unit!

For $\delta : [0, 1] \longrightarrow \mathbb{R}^2$ given by $\delta(t) = (\cos 2\pi t,\ \sin 2\pi t)$ for all $t \in [0, 1]$, one has $s_\delta(x) = 2\pi x$ for all $x \in [0, 1]$. So, $\dfrac{d}{dx}(s_\delta) = 2\pi \longrightarrow$ rate of change not unit, but is constant! The rate can be changed to 1 by a linear change of parameter! This we already have discussed.

However, for $\eta(t) = (t^2, t, \frac{4}{3}t^{3/2})\ \ \forall t \in [0, 4]$, $\quad s_\eta(x) = x + x^2\ \ \forall x \in [0, 4]$, $\ell(\eta) = 20$. $\dfrac{d}{dx}(s_\eta) = 1 + 2x \longrightarrow$ rate of change is not constant! Here we can not change the rate to 1 by a linear change of parameter, because the rate is non-constant, and only a nonlinear change of parameter function will make this non-linear rate to linear, provided $1 + 2x \neq 0$ for all $x \in [0, 4]$; but that is true!!

Remark 3.9.1. *1.* $\frac{d}{dx}(s(x)) = ||\gamma'(x)||$ *indicates*

- *rate of measurement of the length of the curve*
- *the stretching factor of a st. line segment (parameter interval) corresponding to the weight* γ *on the st. line segment which bent the st. line segment to a curve.*
- *speed of a particle moving on the curve having position vector* $\gamma(x)$.

2. $s(x)$ *and* $\frac{d}{dx}(s(x))$ *changes as the parametrization changes.*

3. If $\frac{d}{dx}(s(x)) = 1(= \|\gamma'(x)\|)$*, it gives a sense*

- *that the length is measured being on the curve itself, as in that case the equality* $\ell(\gamma) = \int_a^b \|\gamma'(t)\|\ dt$ *will be transformed to* $\ell(\gamma) = \int_0^{\ell(\gamma)} 1\ dt$.
- *that if a particle has position vector* $\gamma(t)$*, then the particle is with velocity* $\gamma'(t)$ *such that the speed* $\|\gamma'(t)\|$ *is unit (constant).*
- *a st. line segment is bent to be a curve with no stretching factor.*

4. Any parametrization method for which, for any given curve γ *with length* ℓ *if we have* $\frac{d}{dx}(s(x)) = 1(= \|\gamma'(x)\|)$*, then the parametrization*

- *indicates a uniform/universal parametrization*
- *gives a method which induce no stretching to the parameter interval or the corresponding st. line segment to make it a curve, i.e., it gives a method to "bend a st. line segment* $[0, \ell]$ *is properly without stretching" to get a curve of length* ℓ *.*
- *generates measurement method which for which if a particle is moving with unit speed, then it would take always the same time across the curves of same length, be it a st. line or a curve, if the curve's representation is the particle's position vector.*
- *conceptualize the measurement method of curvature by comparing a st. line segment and a curve made from the st. line segment ("how much curved") by capturing only the "(rate of) change of direction" on the curve which can be measured by measuring the rate of change of velocity keeping speed is unit, where the method does not get affected by representation of the curve.*

3.10 Arc length parametrization

$\gamma : [a, b] \longrightarrow \mathbb{R}^3$ is $\mathcal{C}^1$-type with length $\ell(\gamma)$. Needed "correct" parametrization of γ as where the parameter u varies from 0 to $\ell(\gamma)$, and in that case we shall have $\ell(\gamma) = \int_a^b \|\gamma'(t)\|\ dt = \int_a^b \frac{d}{dt} s_\gamma(t)\ dt = \int_0^{\ell(\gamma)} du$. In case the parametrization is given by a composition function $\gamma \circ \theta$, then we should have the following.

(1): $[0, \ell(\gamma)] \xrightarrow{\theta} [a, b] \xrightarrow{\gamma} \mathbb{R}^3$; (2): θ is strictly increasing; (3): θ is surjective; (4): θ is $\mathcal{C}^1$-type; (5): θ is smooth if γ is smooth; and (6): $\frac{d}{dx}(\theta(x)) = \frac{1}{\|\gamma'(x)\|}$ as $\frac{d}{dx} s_\gamma(x) = \|\gamma'(x)\|$, $s_\gamma(x) = \int_a^x \|\gamma'(t)\|\ dt$ and $\ell(\gamma) = \int_a^b \|\gamma'(t)\|\ dt$.

Look back at $s_\gamma(x)$ itself! $s_\gamma^{-1} : [0, \ell(\gamma)] \longrightarrow [a, b]$ is such a function if γ is smooth!! Let us look at the following corollary which is immediate from Theorem 3.8.3.

Corollary 3.10.1. *Let* $\gamma : [a, b] \longrightarrow \mathbb{R}^3$ *be smooth (*$\mathcal{C}^1$*-type and* $\|\gamma'\| \neq 0$*). Then (1):* $s_\gamma(x) \geq 0$ *for all* $x \in [a, b]$*,* $s_\gamma(a) = 0$*, and* $s_\gamma(b) = \ell(\gamma)$*; (2):*

s_γ is strictly increasing on $[a,b]$; (3): s_γ is continuous on $[a,b]$; (4): s_γ is differentiable on $[a,b]$; (5): s_γ is $\mathcal{C}^1$-type on $[a,b]$, and $\frac{d}{dx}s_\gamma(x) = ||\gamma'(x)||$ for all $x \in [a,b]$; and (6): s_γ is smooth if γ is smooth.

Note that

(A): (1) + (2) $\implies$ s_γ is strictly increasing and surjective, and therefore, bijective, i.e., $s_\gamma^{-1} : [0, \ell(\gamma)] \longrightarrow [a,b]$ exists.

(B): (3) + (A) $\implies$ s_γ^{-1} is continuous (bijective continuous function on compact interval is homeomorphism)

(C): (4) + (B) $\implies$ s_γ^{-1} is diff. (real-differentiable homeomorphic maps are diffeomorphic); and from (5) we have $\frac{d}{dx}s_\gamma^{-1}(u) = \frac{1}{||\gamma'(u)||}$

So, the correct parametrization will be given by the composite function $[0,\ell(\gamma)] \xrightarrow{s_\gamma^{-1}} [a,b] \xrightarrow{\gamma} \mathbb{R}^3$, i.e., $\gamma \circ s^{-1} : [0,\ell(C)] \longrightarrow \mathbb{R}^3$ **which is called parametrization of the curve γ by its arc length function, or the arc-length parametrization of the curve γ.** Check that $||\frac{d}{dt}\gamma \circ s_\gamma^{-1}(u)|| = ||\frac{d}{du}\gamma(s_\gamma^{-1}(u))|| = 1$. and therefore for $\gamma \circ s^{-1}$ the arc length function is $s(x) = \displaystyle\int_0^x ||\frac{d}{du}(\gamma \circ s^{-1}(u))||du = \int_0^x 1\ du = x$ for all $x \in [0,\ell(\gamma)]$. Generally we use the notation $\gamma(s)$, $s \in [0,\ell(\gamma)]$ to mean that γ is parametrized by arc-length function, i.e., the function $\gamma \circ s_\gamma^{-1}(u)$ is written in short as $\gamma(s)$.

The following result is an exercise.

Theorem 3.10.2 (Formula to check arc-length parametrization). *Let $\gamma : [a,b] \longrightarrow \mathbb{R}^3$ be a parametric representation of a smooth curve C. Then, γ is the arc-length parametrization of C if and only if $||\gamma'(t)|| = 1$ for all $t \in [a,b]$.*

Example 3.10.3. *Let $\gamma(t) = (t^2, t, \frac{4}{3}t^{3/2})$ $\forall t \in [0,4]$. Since the component functions of γ are $\mathcal{C}^1$-type, γ is $\mathcal{C}^1$-type and hence the arc length function of γ is given by $s_\gamma(x) = \displaystyle\int_0^x ||\gamma'(t)||\ dt$ for all $x \in [0,4]$, i.e., $s_\gamma(x) = x + x^2$ $\forall x \in [0,4]$.*

Note that $\ell(\gamma) = 20$. $\dfrac{d}{dx}(s_\gamma) = 1 + 2x = ||\gamma'(x)|| \neq 1$, and therefore, we see that γ is not the arc-length parametrization of the corresponding curve.

Since $||\gamma'(x)|| = 1 + 2x$ for all $x \in [0,4]$, $||\gamma'(x)|| \neq 0$ for all $x \in [0,4]$, and therefore, γ is smooth. Since γ is smooth, we can have the arc-length parametrisation of γ.

To find the arc-length parametrization of γ We need to find the function s_γ^{-1}. Let $y = s_\gamma(x) = x + x^2$, then $s_\gamma^{-1}(y) = x$. Now, $y + 1/4 = x^2 + 2.\frac{1}{2}.x + 1/4 = (x+1/2)^2$. So, $x = \sqrt{y + 1/4} - 1/2$ which makes sense as $s_\gamma^{-1}(x) = y \geq 0$. Now, replacing the value of x in $s_\gamma^{-1}(y)$ we have $s_\gamma^{-1}(y) = \sqrt{y+1/4} - 1/2$. Therefore, the arc-length parametrization of γ is given by

$$\begin{aligned}\gamma \circ s_\gamma^{-1}(u) &= \gamma(s_\gamma^{-1}(u))\\ &= \gamma(\sqrt{u+1/4}-1/2)\\ &= ((\sqrt{u+1/4}-1/2)^2,\ \sqrt{u+1/4}-1/2,\ \tfrac{4}{3}(\sqrt{u+1/4}-1/2)^{3/2})\end{aligned}$$

for all $u \in [0, 20]$, *where* $20 = \ell(\gamma)$.

3.11 Curvature of a curve

The concept of curvature "how much a given curve is bent" can be felt correctly by the rate of change of direction of a particle moving on the curve. As this is meant to compare among the curves, of course while measuring it we should make sure that the particle is moving across the curves with a constant speed; and since we should fix a unit for measurement, we make the speed to be unit, otherwise a constant multiple will propagate in the measurement of curvature. So, to be precise, the measurement of the curvature will be given by the measurement of the rate of change of direction a particle of a particle moving on the curve with unit speed. Since velocity of a particle moving with a unit speed itself gives the direction, in this case the rate of change of velocity is same as rate of change of direction. Note that in this case for a curve C of length $\ell(C)$, the particle takes time $\ell(C)$ to traverse the curve.

Let $\gamma : [a, b] \longrightarrow \mathbb{R}^3$ is the given curve whose curvature is to be measured. Then, for each $t \in [a, b]$, the vector $\gamma(t)$ itself denote the position vector of the particle. So, if we want to use the function γ itself to measure the rate of change of velocity, we need to make sure that

- to re-parametrize γ to $\gamma \circ \theta$ for some function θ with parameter interval $[0, \ell(\gamma)]$ so that $\gamma \circ \theta$ represents position vector of the particle which will traverse the curve.
- the velocity of the particle, i.e., $\frac{d}{du}\gamma \circ \theta(u)$ where $u \in [0, \ell(\gamma)]$ is such that $\|\frac{d}{du}(\gamma \circ \theta(u))\| = 1$ for all $u \in [0, \ell(\gamma)]$.

The above mentioned properties tell us that γ has to be a **smooth** curve, as the tangent is unit throughout! Further, it tells us that we should choose **arc-length parametrization** of the curve γ to re-parametrize according to our need.

Let $\gamma \circ s_\gamma^{-1}(u)$, where $u \in [0, \ell(\gamma)]$ denote the re-parametrisation of γ by arc-length function. Then the tangent $\mathbb{T}(u) = \frac{d}{du}(\gamma \circ s_\gamma^{-1})(u)$ is such that $\|\mathbb{T}(u)\| = 1$, and the **curvature** of γ, denoted by κ_γ, is defined by $\kappa_\gamma = \|\frac{d}{du}(\mathbb{T}(u))\|$.

Theorem 3.11.1. *(Working formula for curvature) Let* $\gamma : [a, b] \longrightarrow \mathbb{R}^3$ *be a parametric representation of a* ***smooth*** *curve* C. *Then, the curvature of* γ *is given by*

$$\kappa_\gamma = \|\frac{d}{du}\mathbb{T}(u)\| = \frac{\|\frac{t}{dt}\mathbb{T}(t)\|}{\|\gamma'(t)\|}$$

where $\mathbb{T}(t) := \dfrac{\gamma'(t)}{\|\gamma'(t)\|}$.

Proof. Let the arc length function of γ is given by $s(x) = \displaystyle\int_a^x \|\gamma'(t)\|\ dt$. Since γ is smooth, s^{-1} exists and we have $[0, \ell(C)] \xrightarrow{s^{-1}} [a, b] \xrightarrow{\gamma} C$. Let $t \in [a, b]$, then there exists $u \in [0, \ell(C)]$ such that $s^{-1}(u) = t$ and $s(t) = u$.

Let $T(t) := \gamma'(t)$ denote the tangent at the point $\gamma(t)$ on the curve. Note that $T(t)$ it is not the unit tangent. The unit tangent is denoted by $\mathbb{T}(t)$ at the point $\gamma(t)$ and given by $\mathbb{T}(t) := \dfrac{\gamma'(t)}{\|\gamma'(t)\|} = \dfrac{T(t)}{\|\gamma'(t)\|}$ at the point $\gamma(t)$. Now,

$$\begin{aligned}
T(t) = \tfrac{d}{dt}\gamma(t) \quad &= \tfrac{d}{dt}\gamma(s^{-1}(u)) \\
&= \tfrac{d}{du}\gamma(s^{-1}(u))\tfrac{d}{dt}u \\
&= \tfrac{d}{du}\gamma(s^{-1}(u))\tfrac{d}{dt}s(t) \\
&= \mathbb{T}(u)\|\gamma'(t)\|, \qquad\quad \text{as } s'(x) = \|\gamma'(x)\|
\end{aligned}$$

This shows that $\mathbb{T}(u) = \dfrac{T(t)}{\|\gamma'(t)\|} = \mathbb{T}(t)$. for all $t \in [a, b]$ and for all $u \in [0, \ell(C)]$ (unit tangent is unique!)

So,

$$\begin{aligned}
\tfrac{d}{dt}\mathbb{T}(t) \quad &= \tfrac{d}{dt}\mathbb{T}(u) \\
&= \tfrac{d}{du}\mathbb{T}(u)\tfrac{d}{dt}(u) \\
&= \tfrac{d}{du}\mathbb{T}(u)\tfrac{d}{dt}(s(t)) \\
&= \tfrac{d}{du}\mathbb{T}(u)\|\gamma'(t)\|
\end{aligned}$$

This shows that we have $\|\frac{d}{dt}\mathbb{T}(t)\| = \kappa_\gamma\|\gamma'(t)\|$, i.e., $\kappa_\gamma = \dfrac{\|\frac{t}{dt}\mathbb{T}(t)\|}{\|\gamma'(t)\|}$. □

Definition 3.11.2. *The **radius of curvature** of the curve γ is defined by* $R_\gamma := \dfrac{1}{\kappa_\gamma}$.

3.12 Examples

Example 3.12.1. *Let $C : x^2 = y^3,\ 9z^2 = 4y$, with initial point $(0, 0, 0)$ and final point $(8, 4, 4/3)$. We shall find the curvature of C, if possible. For this we need to find a parametric equation to the curve and the arc-length function of it and we should see whether the curve is smooth.*

Let $\gamma(t) = (t^3, t^2, \frac{2}{3}t)$ for all $t \in [0, 2]$. Check that γ represents C. Since the component functions of γ are polynomial, γ is $\mathcal{C}^1$-type and $\gamma'(t) = (3t^2, 2t, 2/3) \neq \vec{0}$, we see that γ is smooth. Now the arc-length function of γ is given by $s(x) = \displaystyle\int_0^x \|\gamma'(t)\|\ dt$. As, $\|\gamma'(t)\| = \sqrt{9t^4 + 4t^2 + 4/9} = \sqrt{(3t^2 + 2/3)^2} = 3t^2 + 2/3$, as $\|\gamma'(t)\|$ is non-negative and $t \geq 0$. Therefore, $s(x) = x^3 + \frac{2}{3}x$ for all $x \in [0, 2]$.

Since $\|\gamma'(t)\| \neq 1$*, we see that* γ *is not the arc-length parametrization of the curve* C*.*

The curvature of γ *is given by* $\kappa = \dfrac{\|\frac{d}{dt}\mathbb{T}(t)\|}{\|\gamma'(t)\|}$ *where* $\mathbb{T}(t) = \dfrac{\gamma'(t)}{\|\gamma'(t)\|}$.

Now, $\mathbb{T}(t) = \dfrac{(3t^2, 2t, 2/3)}{3t^2 + 2/3}$. *Therefore,* $\dfrac{d}{dt}\mathbb{T}(t) = \dfrac{(4t, -6t^2 + 4/3, -4t)}{(3t^2 + 2/3)^2}$.

Therefore, $\kappa = \dfrac{\|(4t, -6t^2 + 4/3, -4t)\|}{(3t^2 + 2/3)^3} = \dfrac{\sqrt{16t^2 + (6t^2 - 4/3)^2 + 16t^2}}{(3t^2 + 2/3)^3} = \dfrac{6t^2 + 4/3}{(3t^2 + 2/3)^3}$.

For more examples look at

https://sites.und.edu/timothy.prescott/apex/web/apex.Ch12.S5.html
http://spot.pcc.edu/math/APEXCalculus/sec_curvature.html
https://math.libretexts.org/Bookshelves/Calculus/Book%3A_Calculus_(OpenStax)/13%3A_Vector-Valued_Functions/13.3%3A_Arc_Length_and_Curvature
https://ltcconline.net/greenl/courses/202/vectorFunctions/curvat.htm
https://www.whitman.edu/mathematics/calculus_online/section13.03.html

Exercise 3.12.2. *1. pareametrise the curve* $C : x^2 = y^3, y = z^2$*,* $z \geq 0$*,* $x \geq 0$ *and thereby find the arc length function of the curve* C*.*

2. Let $\gamma(t) = (a\cos(t),\ a\sin(t),\ bt)$ *for all* $t \in [0, 2\pi]$*. Find arc length function of the curve* γ*; and also parametrize the curve by arc length function, if possible.*

3. Show that for a $\mathcal{C}^1$*-type parametric curve* γ *on a finite interval, we have* $\ell(\gamma) = \ell(\gamma^{-1})$*.*

4. Show that for any smooth increasing function $\phi : [c, d] \longrightarrow [a, b]$ *and for any* $\mathcal{C}^1$*-type curve* $\gamma : [a, b] \longrightarrow \mathbb{R}^3$*, we have* $\{\gamma \circ \phi\} = \{\gamma\}$*, and* $\ell(\gamma) = \ell(\gamma \circ \phi)$*.*

Remark 3.12.3. *It is well known that* $\int_a^b dx = \int_a^b 1\ dx$ = *the length of the line segment* $\overline{(a,0),\ (b,0)}$ *whose parametric equations is give by* $\bar{x}(t) = (t,0)$ *for all* $t \in [a,b]$. *Note that in this case the arc length function of the curve* $\bar{x}$ *is given by* $s(x) = x - a$ *for all* $x \in [a,b]$, *and* $s'(x) = 1 = \frac{d}{dx}(x)$ *for all* $x \in [a,b]$. *Now, for any parametric curve* $\gamma : [a,b] \longrightarrow \mathbb{R}$, *the length of the curve is* $\int_a^b \|\gamma'(t)\| dt$ *which does not look like* $\int_a^b 1\ dx$, *rather it is weighted, i.e., instead of the fixed unit weight* 1 *at each point* t, *a weight* $\|\gamma'(t)\|$ *being carried by each point* t. *So, one may ask what should replace "*dx*", and which will look like "*dx*" so that* $\int_a^b$ "dx'' = *length of the curve. Now, recall that if* $\frac{d}{dx}(f(x)) = g(x)$, *we write it in terms of language of integration/ symbol of integration as* $\int g(x)dx = f(x)$ *or* $\int f'(x)dx = f(x)$, *or sometimes in short* $\int d(f(x)) = f(x)$, *or simply* $\int df = f$*; and therefore, in the light of (VIII) it is clear that "*dx*" should be replaced by "*ds*" so that* $\int ds = s$, *and hence* $\int_a^b ds = s(b) - s(a)$ = *length of the curve.*

Further, if $\gamma'(t) \neq \bar{0}$ *for all* $t \in [a,b]$, *clearly, by (V)th property we see that* s *has its inverse* $s^1 : [0, \ell(\gamma)] \longrightarrow [a,b]$, *and (VIII)th property ensures* $\mathcal{C}^1$*-typeness of it. So, we consider the* $\mathcal{C}^1$*-type curve* $\gamma \circ s^{-1} : [0, \ell(\gamma)] \longrightarrow \mathbb{R}^3$. *It can be checked that* $\{\gamma\} = \{\gamma \circ s^{-1}\}$, *and* $\|\frac{d}{dt}(\gamma \circ s^{-1}(t))\| = 1$, *and therefore for* $\gamma \circ s^{-1}$ *the arc length function is* $s(x) = \int_0^x \|\frac{d}{dt}(\gamma \circ s^{-1}(t))\| dt = \int_0^x 1\ dt = x$ *for all* $x \in [0, \ell(\gamma)]$. ***The parametrisation*** $\gamma \circ s^{-1}$ ***is called parametrisation of the curve*** γ ***by its arc length function, or the arc-length parametrisation of the curve*** γ***.***

Exercise 3.12.4.

1. *pareametrise the curve* $C : x^2 = y^3, y = z^2,\ z \geq 0,\ x \geq 0$ *and thereby find the arc length function of the curve* C.

2. *Let* $\gamma(t) = (a\cos(t),\ a\sin(t),\ bt)$ *for all* $t \in [0, 2\pi]$. *Find arc length function of the curve* γ*; and also parametrize the curve by arc length function, if possible.*

3. *Show that for a* $\mathcal{C}^1$*-type parametric curve* γ *on a finite interval, we have* $\ell(\gamma) = \ell(\gamma^{-1})$.

4. *Show that for any smooth increasing function* $\phi : [c,d] \longrightarrow [a,b]$ *and for any* $\mathcal{C}^1$*-type curve* $\gamma : [a,b] \longrightarrow \mathbb{R}^3$, *we have* $\{\gamma \circ \phi\} = \{\gamma\}$, *and* $\ell(\gamma) = \ell(\gamma \circ \phi)$.

Chapter 4

Line Integration

4.1 Introduction

In this chapter we shall study theory of integration of vector fields along a curve aiming to classify a certain kind of vector fields called conservative vector fields.

4.2 Line Integration of Scalar Fields

Let $G \subset \mathbb{R}^3$ and $f : G \longrightarrow \mathbb{R}^3$ a scalar field. Suppose $\gamma : [a, b] \longrightarrow \mathbb{R}^3$ is a parametric curve such that $\mathrm{Im}(\gamma) \subset G$. We define line integration of f along the length of γ as the limit $\lim_{n\to\infty} \sum_{i=1}^{n} f(\gamma(\xi_i))\, \overline{\gamma(t_i),\ \gamma(t_{i-1})} = \lim_{n\to\infty} \sum_{i=1}^{n} f(\gamma(\xi_i))\, \|\gamma(t_i) - \gamma(t_{i-1})\|$, if exists; where $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ is an equidistant partition of $[a, b]$ and ξ_i is an arbitrary point in $[t_{i-1}, t_i]$; and the limiting value is denoted by $\int_\gamma f$ or $\int_\gamma f\ ds$. As in the previous chapter, one may check that the limit will exists if f is continuous, and γ is $\mathcal{C}^1$-type, which we state as a theorem below.

Theorem 4.2.1. *Let* $G \subset \mathbb{R}^3$ *and* $f : G \longrightarrow \mathbb{R}^3$ *a continuous scalar field and* $\gamma : [a, b] \longrightarrow \mathbb{R}^3$ *a* $\mathcal{C}^1$*-type curve such that Im*$(\gamma) \subset G$*. Then* $\int_\gamma f$ *exists, and* $\int_\gamma f = \int_a^b f(\gamma(t))\, \|\gamma'(t)\|\ dt = \int_a^b f(\gamma(t))\ ds.$

Exercise 4.2.2. *Let* $G \subset \mathbb{R}^3$ *and* $f : G \longrightarrow \mathbb{R}^3$ *a continuous scalar field and* $\gamma : [a, b] \longrightarrow \mathbb{R}^3$ *a* $\mathcal{C}^1$*-type curve such that Im*$(\gamma) \subset G$

1. *Show that* $\int_\gamma f = \int_{\gamma^{-1}} f.$
2. *If* $\phi : [c, d] \longrightarrow [a, b]$ *is a smooth increasing function, then show that* $\int_\gamma f = \int_{\gamma\circ\phi} f$*, i.e.,* $\int_\gamma f$ *is independent of parametrisation of* γ.

Remark 4.2.3. *Let* γ *be a piece-wise* $\mathcal{C}^r$*-type/smooth* $(r \geq 0)$ *parametric curve, i.e.,* γ *is a* $\mathcal{C}^r$*-type curve expect finitely many points, say* m*-many points, then* *clearly γ can be realized as a union (or addition) of $m+1$ parametric curves which are $\mathcal{C}^r$-type/smooth, i.e., there exists $\gamma_1, \gamma_2, \cdots, \gamma_{m+1}$ $\mathcal{C}^m$-type/smooth parametric curves such that $\gamma = \gamma_1 * \gamma_2 * \cdots * \gamma_{m+1}$.*

Now, we shall define line integration of a continuous scalar field along the length of a piece-wise $\mathcal{C}^1$-type curve.

Definition 4.2.4. *Let $G \subset \mathbb{R}^3$, $f : G \longrightarrow \mathbb{R}^3$ a scalar field, and γ a piecewise continuous parametric curve, and suppose $\gamma = \gamma_1 * \gamma_2 * \cdots * \gamma_s$ where each γ_i is continuous. Then we define $\displaystyle\int_\gamma f = \sum_{i=1}^{s} \int_{\gamma_i} f$.*

4.3 Line Integration of Vector Fields

Let $G \subset \mathbb{R}^3$ and $\bar{F} : G \longrightarrow \mathbb{R}^3$ a vector field. Suppose $\gamma : [a, b] \longrightarrow \mathbb{R}^3$ a smooth curve such that $\mathrm{Im}(\gamma) \subset G$. Since γ is smooth, at each point of γ we have a unit tangent $\bar{T}(t)$ defined by $\bar{T}(t) = \dfrac{\gamma'(t)}{||\gamma'(t)||}$. We define the line integration of the vector field $\bar{F}$ along γ by the line integration of the scalar field $< \bar{F}(\gamma(t)),\ \bar{T}(t) >$ along the length of γ; and denoted by $\displaystyle\int_\gamma \bar{F}$ or by $\displaystyle\int_\gamma \bar{F}.d\bar{s}$, i.e., $\displaystyle\int_\gamma \bar{F}.d\bar{s} = \int_\gamma < \bar{F}(\gamma(t)),\ \bar{T}(t) > \ ds$. Note that if $\bar{F}$ is continuous, then the function $< \bar{F}(\gamma(t)),\ \bar{T}(t) >$ is also continuous, and therefore by Theorem 4.2.1, we have

$$\begin{aligned}
\int_\gamma \bar{F}.d\bar{s} &= \int_\gamma < \bar{F}(\gamma(t)),\ \bar{T}(t) > \ ds \\
&= \int_a^b < \bar{F}(\gamma(t)),\ \bar{T}(t) > ||\gamma'(t)||\ dt \\
&= \int_a^b < \bar{F}(\gamma(t)),\ \frac{\gamma'(t)}{||\gamma'(t)||} > ||\gamma'(t)||\ dt \\
&= \int_a^b \frac{< \bar{F}(\gamma(t)),\ \gamma'(t) >}{||\gamma'(t)||} ||\gamma'(t)||\ dt \\
&= \int_a^b < \bar{F}(\gamma(t)),\ \gamma'(t) > \ dt, \text{ as } \gamma'(t) \neq \bar{0}.
\end{aligned}$$

Thus we have the following theorem.

Theorem 4.3.1. *Let $G \subset \mathbb{R}^3$ and $\bar{F} : G \longrightarrow \mathbb{R}^3$ a continuous vector field. Suppose $\gamma : [a, b] \longrightarrow \mathbb{R}^3$ a smooth curve such that Im$(\gamma) \subset G$. Then $\displaystyle\int_\gamma \bar{F}$ exists, and $\displaystyle\int_\gamma \bar{F} = \int_a^b < \bar{F}(\gamma(t)),\ \gamma'(t) > \ dt$.*

Exercise 4.3.2. *Let $G \subset \mathbb{R}^3$ and $\bar{F} : G \longrightarrow \mathbb{R}^3$ a continuous vector field and $\gamma : [a, b] \longrightarrow \mathbb{R}^3$ a smooth curve such that Im$(\gamma) \subset G$*

1. *Show that* $\int_\gamma \bar{F} = -\int_{\gamma^{-1}} \bar{F}$.

2. *If* $\phi : [c, d] \longrightarrow [a, b]$ *is a smooth increasing function, then show that* $\int_\gamma \bar{F} = \int_{\gamma\circ\phi} \bar{F}$, *i.e.,* $\int_\gamma \bar{F}$ *is independent of parametrisation of* γ.

Definition 4.3.3. *Let* $G \subset \mathbb{R}^3$, $\bar{F} : G \longrightarrow \mathbb{R}^3$ *a vector field, and* γ *a piecewise continuous parametric curve, and suppose* $\gamma = \gamma_1 * \gamma_2 * \cdots * \gamma_s$ *where each* γ_i *is continuous. Then we define* $\int_\gamma \bar{F} = \sum_{i=1}^{s} \int_{\gamma_i} \bar{F}$.

Example 4.3.4. 1. *Let* $\bar{F}(x, y) = (x^2, y^2)$ *for all* $(x, y) \in \mathbb{R}^2$, *and* $\gamma(t) = (r\cos(t),\ r\sin(t))$ *for all* $t \in [0, 2\pi]$. *Clearly* $\bar{F}$ *is continuous and* γ *is* $\mathcal{C}^1$*-type, and check that* $\gamma'(t) \neq 0$ *for all* $t \in [0, 2\pi]$ *(i.e.,* γ *is smooth), therefore,* $\int_\gamma \bar{F}$ *exists and,*

$$\begin{aligned}
\int_\gamma \bar{F} &= \int_0^{2\pi} < \bar{F}(\gamma(t)),\ \gamma'(t) > \ dt \\
&= r^3 \int_0^{2\pi} < (\cos(t)^2,\ \sin(t)^2),\ (-\sin(t), \cos(t)) > \ dt \\
&= r^3 \int_0^{2\pi} < (\cos(t)^2,\ \sin(t)^2),\ (-\sin(t), \cos(t)) > \ dt \\
&= 0
\end{aligned}$$

2. *Let* $\bar{F}(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\right)$ *for all* $(x, y) \in \mathbb{R}^2 \backslash (0, 0)$, *and* $\gamma(t) = (r\cos(t),\ r\sin(t))$ *for all* $t \in [0, 2\pi]$. *Clearly* $\bar{F}$ *is continuous and* γ *is* $\mathcal{C}^1$*-type, and therefore,* $\int_\gamma \bar{F}$ *exists and,*

$$\begin{aligned}
\int_\gamma \bar{F} &= \int_0^{2\pi} < \bar{F}(\gamma(t)),\ \gamma'(t) > \ dt \\
&= \int_0^{2\pi} < (-\sin(t),\ \cos(t)),\ (-\sin(t), \cos(t)) > \ dt \\
&= \int_0^{2\pi} 1 \ dt \\
&= 2\pi
\end{aligned}$$

3. *Let* $\bar{F}(x, y) = (f_1(x), f_2(y))$ *for all* $(x, y) \in \mathbb{R}^2$ *where* f_1, f_2 *are continuous functions such that* $g_1' = f_1$, *and* $g_2' = f_2$; *and* $\gamma(t) = (r\cos(t),\ r\sin(t))$ *for all* $t \in [0, 2\pi]$. *Clearly* $\bar{F}$ *is continuous and* γ *is* $\mathcal{C}^1$*-type, and therefore,* $\int_\gamma \bar{F}$ *exists and,*

$$\int_\gamma \bar{F} = \int_0^{2\pi} < \bar{F}(\gamma(t)),\ \gamma'(t) > \ dt$$
$$= 0(\textit{Check!!})$$

Since we know that for a $\mathcal{C}^1$-type scalar field f, the vector field $\nabla(f)$ is continuous, and directional derivatives $D_{\bar{v}}(f)|_P$ exists for all direction $\bar{v}$, and for all point P; and $D_{\bar{v}}(f)|_P =< \nabla(f)|_P,\ \bar{v} >$. And since for a continuous vector field $\bar{F}$ in $\mathbb{R}^3$ and a smooth parametric curve $\gamma : [a, b] \longrightarrow \mathbb{R}^3$, we have $\displaystyle\int_\gamma \bar{F} = \int_a^b < \bar{F}(\gamma(t)),\ \gamma'(t) > \ dt$, one naturally ask what happens if $< \bar{F}(\gamma(t)),\ \gamma'(t) >$ represents a directional derivatives of a $\mathcal{C}^1$ scalar field, i.e., we are asking what happens to $\displaystyle\int_\gamma \bar{F}$, if $\bar{F} = \nabla(f)$ for some $\mathcal{C}^1$-type scalar field f. Note that thought $\gamma'(t)$ is not a unit vector, we can choose a parametrisation of the given curve such that the derivative of it a unit vector (e.g., consider $\gamma \circ s^{-1}$).

$$\text{So, } \int_\gamma \nabla(f) = \int_a^b < \nabla(f)(\gamma(t)),\ \gamma'(t) > \ dt = \int_a^b D_{\gamma'(t)}(f)|_{\gamma(t)}\ dt$$
$$= \int_a^b < \nabla f(\gamma(t)),\ \gamma'(t) > \ dt$$
$$= \int_a^b \left(\frac{\partial}{\partial x} f|_{\gamma(t)} \frac{d}{dt} x(t) + \frac{\partial}{\partial y} f|_{\gamma(t)} \frac{d}{dt} y(t) + \frac{\partial}{\partial z} f|_{\gamma(t)} \frac{d}{dt} z(t) \right) \ dt$$
$$= \int_a^b \frac{d}{dt} f(\gamma(t))$$
$$= f(\gamma(b)) - f(\gamma(a)).$$

Thus, $\displaystyle\int_a^b D_{\gamma'(t)}(f)|_{\gamma(t)}\ dt = \int_\gamma \nabla(f) = f(\gamma(b)) - f(\gamma(a))$. This leads to the following theorem called the **Fundamental Theorem of Line Integration**.

Theorem 4.3.5. *Let $G \subset \mathbb{R}^3$ and $\bar{F} : G \longrightarrow \mathbb{R}^3$ a continuous vector field. If $\bar{F} = \nabla(f)$ for some scalar field $f : G \longrightarrow \mathbb{R}$, then for any smooth parametric curve γ with initial point A and final point B we have $\displaystyle\int_\gamma \bar{F} = f(B) - f(A)$.*

4.4 Conservative Vector Fields

As any vector field which is of the type $\nabla(f)$ is special, as it is known from previous result; and for which we shall set a definition for it.

Definition 4.4.1. *Let $G \subset \mathbb{R}^3$, and $\bar{F} : G \longrightarrow \mathbb{R}^3$ a vector field. $\bar{F}$ is called **conservative vector field (CVF)** if there exists a scalar field $f : G \longrightarrow \mathbb{R}$ such that $\bar{F}(\bar{x}) = \nabla(f(\bar{x}))$ for all $\bar{x} \in G$, or simply $\bar{F} = \nabla(f)$ on G. The function f is called of the **gradient function** of $\bar{F}$.*

Thus, in view of Theorem 4.3.5 we can say that line integration of conservative vector field along a path does not depend upon the path, but upon the end points of the curve. If the end points of the curve are the same point, i.e., the curve is a loop, then clearly the value of the line integration is zero.

Definition 4.4.2. *Let $\bar{F} : G \longrightarrow \mathbb{R}^3$ a continuous vector field. We shall say that line integration of $\bar{F}$ is path independent, if $\int_{\gamma_1} \bar{F} = \int_{\gamma_2} \bar{F}$ for any two smooth curves γ_1 and γ_2 with same initial and final points.*

Clearly, by Theorem 4.3.5, **line integration of a continuous conservative vector field is path independent.** Now, suppose $\bar{F}$ is a general continuous vector field (non necessarily conservative). If line integration of F is path independent, for any smooth loop γ, we shall have $\int_{\gamma} \bar{F} = \int_{\gamma^{-1}} \bar{F}$, as γ and γ^{-1} has same initial and final points, and therefore, since by the properties of line integration of vector fields $\int_{\gamma^{-1}} \bar{F} = -\int_{\gamma} \bar{F}$, we get $\int_{\gamma} \bar{F} = -\int_{\gamma} \bar{F}$, i.e., $\int_{\gamma} \bar{F} = 0$. So, naturally one asks whether the converse is true, i.e., suppose $\bar{F}$ is a continuous vector field such that $\int_{\gamma} \bar{F} = 0$ for all smooth loop γ, is then the line integration of $\bar{F}$ path independent? The answer is, in fact, "yes"!. We describe it in the theorem below.

Theorem 4.4.3. *Let $G \subset \mathbb{R}^3$ and $\bar{F} : G \longrightarrow \mathbb{R}^3$ a continuous vector field. Then the following are equivalent*

(I) *Line integration of $\bar{F}$ is independent of path.*

(II) $\int_{\gamma} \bar{F} = 0$ *for all smooth loop γ in G.*

Proof. (I) $\Longrightarrow$ (II)**:** Clear from the discussion above.

(II) $\Longrightarrow$ (I)**:** Let γ_1 and γ_2 are two paths in G with same initial point A and final point B. We shall show that $\int_{\gamma_1} \bar{F} = \int_{\gamma_2} \bar{F}$. Consider $\gamma = \gamma_1 * \gamma_2^{-1}$. Clearly, γ is a loop with initial and final point A, and therefore, by hypothesis, $\int_{\gamma} F = 0$. From which we see

$$
\begin{aligned}
0 &= \int_{\gamma} \bar{F} \\
&= \int_{\gamma_1 * \gamma_2^{-1}} \bar{F} \\
&= \int_{\gamma_1} \bar{F} + \int_{\gamma_2^{-1}} \bar{F} \qquad . \\
&= \int_{\gamma_1} \bar{F} + \left(-\int_{\gamma_2} \bar{F}\right) \\
&= \int_{\gamma_1} \bar{F} - \int_{\gamma_2} \bar{F}
\end{aligned}
$$

This shows that $\int_{\gamma_1} \bar{F} = \int_{\gamma_2} \bar{F}$, which completes the proof. $\square$

From Theorem 4.3.5 and Theorem 4.4.3 we see the following implications

$$A : \left\{ \ \bar{F} \text{ is a continuous CVF } \right\}$$
$$\Downarrow$$
$$B : \left\{ \text{ Line integration of } F \text{ is path independent } \right\}$$
$$\Updownarrow$$
$$C : \left\{ \int_{\gamma} \bar{F} = 0 \text{ for all smooth loop } \gamma \right\}$$

At this point we should check whether $B \implies A$ (same as $C \implies A$) holds true. In that case we shall be able to describe the conservative vector fields in terms of its properties of line integration. In the theorem below, called **1st Structure Theorem of CVF**, we shall see that, in fact, $B \implies A$ **holds under certain conditions**; and from which we get the structure of conservative vector fields in terms of theory of line integration.

4.5 1st Structure Theorem of CVFs

Definition 4.5.1. $G \subset \mathbb{R}^3$ *is called path-connected, if for every two points* A *and* B *in* G*, there exists a path (continuous)* $\gamma : [0, 1] \longrightarrow G$ *from* A *to* B*, i.e.,* $\gamma(0) = A$*, and* $\gamma(1) = B$.

It can be seen that if G *is open and path-connected, then every two points in* G *can be connected by a piece-wise smooth path.*

Theorem 4.5.2. *Let* $G \subset \mathbb{R}^3$ *be open and path-connected.* $\bar{F} : G \longrightarrow \mathbb{R}^3$ *a continuous vector field. Then the following are equivalent*

(I) *Line integration of* $\bar{F}$ *is independent of path.*

(II) $\int_{\gamma} \bar{F} = 0$ *for all smooth loop* γ *in* G.

(III) F *is a CVF.*

Proof. We already have completed "(I) $\Leftrightarrow$ (II) $\Longleftarrow$ (III)". So, to complete the proof we need to show "(II) $\implies$ (III)". So we suppose that (II) holds or equivalently (I) holds. Shall show that (III) holds, though the proof is out of the syllabus.

Step 1: Fix a point A in G and take a variable point $P = (x, y, z)$ in G. Since G is open and path connected, there exists a polygon γ_P (piece-wise smooth path) from A to P. Consider $\int_{\gamma_P} \bar{F}$.

Step 2: Since line integration of $\bar{F}$ is independent of path, for another piece-wise smooth path δ_P from A to P, we have $\int_{\delta_P} \bar{F} = \int_{\gamma_P} \bar{F}$. Therefore, we can define a function $f : G \longrightarrow \mathbb{R}$ by $f(P) = \int_{\gamma_P} \bar{F}$

Step 3: It can be shown that partial derivatives of f exists, and $\nabla(f) = \bar{F}$ on G.

This completes the proof. □

Remark 4.5.3. *Note that we can not prove a vector field to be conservative using Theorem 4.5.2; rather it will be extremely helpful to disprove a VF is conservative, if the VF satisfies all the hypotheses of the theorem.*

Example 4.5.4. *Let* $\bar{F}(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$ *for all* $(x, y) \in \mathbb{R}^2 \backslash (0, 0)$. *Note that the domain of* $\bar{F}$ *is path-connected, and* $\bar{F}$ *is continuous. We have seen that* $\int_\gamma \bar{F} = 2\pi \neq 0$ *for any circle* γ *around* $(0, 0)$, *and therefore by 1st Structure Theorem of CVF we get* $\bar{F}$ *is not conservative!*

Since computation line integration of a vector field becomes extremely easy if the vector field is conservative, we, for sure, want to build up a tool which affirmatively detects a vector is conservative, if it is so.

Example 4.5.5. *Let* $\bar{F}(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$ *for all* $(x, y) \in \mathbb{R}^2$ *except the "non negative part of* x*-axis". We shall see that* $\bar{F}$ *is conservative, but right now we do not have any tool to detect it.*

One approach would be trying to find out a scalar field, on using some algorithm, for which we can write the vector field as a gradient of the scalar field. There are disadvantages of this approach, which will be discussing after we discuss the algorithm. Suppose $\bar{F} : G \longrightarrow \mathbb{R}^3$ a conservative vector field. We shall try to find $f : G \longrightarrow \mathbb{R}$ such that $\bar{F} = \nabla(f)$ on G.

Step 1: Suppose $\bar{F} = (F_1, F_2, F_3)$, and suppose we have found $f : G \longrightarrow \mathbb{R}$ such that $\bar{F} = \nabla(f)$. In that case we have three equations.

$$(1) \begin{cases} \frac{\partial}{\partial x} f = F_1 & --(1a) \\ \frac{\partial}{\partial y} f = F_2 & --(1b) \\ \frac{\partial}{\partial z} f = F_3 & --(1c) \end{cases}$$

Step 2: From (1a) we get

$$f = \int_x F_1 + \phi_1(y, z) \quad --(2)$$

where ϕ is an arbitrary function of y and z.

Step 3: Differentiating (2) with respect to y and comparing with (1b) we get

$$\frac{\partial}{\partial y}\phi_1(y,z) = F_2 - \frac{\partial}{\partial y}\int_x F_1,$$

and therefore,

$$\phi_1(y,z) = \int_y \left(F_2 - \frac{\partial}{\partial y}\int_x F_1\right) + \phi_2(z) \quad --(3)$$

Plugging in the value of ϕ_1 from (3) in (2) we get

$$f = \left[\int_x F_1 + \int_y \left(F_2 - \frac{\partial}{\partial y}\int_x F_1\right)\right] + \phi_2(z) \quad --(4)$$

where ϕ_2 is an arbitrary function of z.

Step 4: Again differentiating (4) with respect to z and comparing with (1c) we get

$$\phi_2(z) = \int_z \left\{F_3 - \frac{\partial}{\partial z}\left[\int_x F_1 + \int_y \left(F_2 - \frac{\partial}{\partial y}\int_x F_1\right)\right]\right\} + c \quad --(5)$$

where c is an arbitrary pure constant. Plugging in the value of ϕ_2 from (5) in (4) we get the function f, i.e.,

$$f = \left[\int_x F_1 + \int_y \left(F_2 - \frac{\partial}{\partial y}\int_x F_1\right)\right] + \int_z \left\{F_3 - \frac{\partial}{\partial z}\left[\int_x F_1 + \int_y \left(F_2 - \frac{\partial}{\partial y}\int_x F_1\right)\right]\right\} + c$$

Since we do not have, so far, a definitive tool which tells us a given vector field is conservative, we need check whether $\nabla(f) = \bar{F}$ holds after finding f in this process.

The process in short:

Given a CVF $\bar{F} = (F_1, F_2, F_3) : G \longrightarrow \mathbb{R}^3$ to find $f : G \longrightarrow \mathbb{R}$ such that $\bar{F} = \nabla(f)$, i.e.,

$$(1)\begin{cases} \frac{\partial}{\partial x} f = F_1 & --(1a) \\ \frac{\partial}{\partial y} f = F_2 & --(1b) \\ \frac{\partial}{\partial z} f = F_3 & --(1c) \end{cases}$$

Step-I: Integrate F_1 (x-component) with respect to x and get f_1. Write $f_\mathcal{I} = f_1$. $f_\mathcal{I}$ is an intermediate function to get f, which we shall keep updating. Go to the next step.

Step-II: Differentiate $f_\mathcal{I}$ with respect to y to get $(f_\mathcal{I})_y$ and look at the difference $F_2 - (f_\mathcal{I})_y$.

If the difference is zero, i.e., $(f_\mathcal{I})_y$ matches with the y component of F, i.e., F_2, go to the next step.

If the difference is non-zero, integrate the difference $F_2 - (f_\mathcal{I})_y$ with respect to y and get f_2. Update $f_\mathcal{I}$ as $f_\mathcal{I} = f_\mathcal{I} + f_2 = f_1 + f_2$. Now look that you have $(f_\mathcal{I})x = F_1$ and $(f_\mathcal{I})_y = F_2$. Now go to the next step.

Step-III: Differentiate $f_{\mathcal{J}}$ with respect to z to get $(f_{\mathcal{J}})_z$ and look at the difference $F_3 - (f_{\mathcal{J}})_y$.

If the difference is zero, i.e., $(f_{\mathcal{J}})_z$ matches with the z component of F, i.e., F_3, we set $f = f_{\mathcal{J}}$. Check that we have achieved $\nabla f = F$. The process stops.

If the difference is non-zero, integrate the difference $F_3 - (f_{\mathcal{J}})_z$ with respect to z and get f_3. Update $f_{\mathcal{J}}$ as $f_{\mathcal{J}} = f_{\mathcal{J}} + f_3 = f_1 + f_2 + f_3$. Now set $f = f_{\mathcal{J}}$. Check that we have achieved $\nabla f = F$. The process stops.

Example 4.5.6. *1. Let* $F(x, y, z) = (\sin(z)e^x,\ y^2,\ \cos(z)e^x + z^3)$. $F_1 = \sin(z)e^x$, $F_2 = y^2$, *and* $F_3 = \cos(z)e^x + z^3$.

So, $f = \displaystyle\int_x \sin(x)e^x + \phi_1(y, z) = \sin(z)e^x + \phi_(y, z)$.

We know that $\frac{\partial}{\partial y}\phi_1 = F_2 - \frac{\partial}{\partial y}\sin(z)e^x = y^2$, *and therefore* $\phi_1 = \displaystyle\int_y y^2 + \phi_1(y, z) = 1/3z^3 + \phi_2(z)$.

Thus, $f = \sin(z)e^x + 1/3y^3 + \phi_2(z)$.

Again, $\frac{\partial}{\partial z}\phi_2 = F_3 - \frac{\partial}{\partial z}\sin(z)e^x + 1/3y^3 = z^3$, *and therefore* $\phi_2 = 1/4z^4 + c$ *where* c *is an arbitrary constant. Which gives us*

$$f = \sin(z)e^x + 1/3y^3 + 1/4z^4 + c.$$

In this case we see that $\nabla(f) = \bar{F}$ *on the domain of* $\bar{F}$, *and therefore* $\bar{F}$ *is a conservative vector field.*

2. Let $\bar{F}(x, y) = (e^y, x)$. *Following the above process we see that* $f = xe^y + xy - e^y + c$, *but it is easy to note that* $\nabla(f) \neq \bar{F}$.

Let $G \subset \mathbb{R}^2$ be open and $\bar{F} = (F_1, F_2) : G \longrightarrow \mathbb{R}^2$ a conservative vector field. Then there exists $f : G \longrightarrow \mathbb{R}$ such that $\nabla(f) = \bar{F}$ in G, i.e, $\frac{\partial}{\partial x}f = F_1$ and $\frac{\partial}{\partial y}f = F_2$; and therefore $\frac{\partial^2}{\partial x \partial y}f = \frac{\partial}{\partial x}F_2$, and $\frac{\partial^2}{\partial y \partial x}f = \frac{\partial}{\partial y}F_1$. Thus, if $\frac{\partial^2}{\partial x \partial y}f = \frac{\partial^2}{\partial y \partial x}f$, we shall have $\frac{\partial}{\partial x}F_2 = \frac{\partial}{\partial y}F_1$.

Fact: (Schwarz's theorem or Clairaut's theorem, or Young's theorem) Let G be an open set in $\mathbb{R}^2$ and $P_0 \in G$. Suppose $f : G \longrightarrow \mathbb{R}$ be such that (i) f_x and f_y exists in G, and (ii) either f_{xy} or f_{yx} exists and is continuous at P_0, then both $f_{xy}(P_0)$ and $f_{yx}(P_0)$ exist and $f_{xy}(P_0) = f_{yx}(P_0)$.

i.e., in particular, if f is $\mathcal{C}^2$-type, then all the partial derivatives of f of second order commute; and therefore for a $\mathcal{C}^1$-conservative vector field $\bar{F} = (F_1, F_2)$, we have, by our previous discussion, $\frac{\partial}{\partial x}F_2 = \frac{\partial}{\partial y}F_1$ on G.

Theorem 4.5.7. *Let* G *be an open set in* $\mathbb{R}^2$ *and* $P_0 \in G$. *Suppose* $\bar{F} = (F_1, F_2) : G \longrightarrow \mathbb{R}^2$ *be a* $\mathcal{C}^1$*-type conservative vector field, then* $\frac{\partial}{\partial x}F_2 = \frac{\partial}{\partial y}F_1$ *on* G.

This result will help us quite a bit to quickly detect non-CVF which are $\mathcal{C}^1$-type.

Example 4.5.8. *Consider the previous examples*

1. $\bar{F}(x, y) = \left(\dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \right)$ *for all* $(x, y) \in \mathbb{R}^2 \backslash (0, 0)$. *It can be seen, in this example, that dom*$(\bar{F})$ *is open,* $\bar{F}$ *is* $\mathcal{C}^1$*-type, and* $\frac{\partial}{\partial x} F_2 - \frac{\partial}{\partial y} F_1 = 0$*; but from this we can not have any conclusion.*

2. $\bar{F} = (e^y, x)$. *In this case it is easy to see that* $\frac{\partial}{\partial x} F_2 - \frac{\partial}{\partial y} F_1 \neq 0$*, and therefore* $\bar{F}$ *is not conservative.*

4.6 Fundamental Theorem in $\mathbb{R}^2$: Green's Theorem

We can realize the fundamental theorem of line integration or fundamental theorem integral calculus as evaluation of an integration of the "differentiation" of a function over a "one dimensional" bounded region with "no hole" by evaluation of the function at the "boundary" points. Naturally one asks what could be its higher dimensional analogue! Essentially it asks about formulation of a possible version of fundamental theorem of integration for higher dimensional cases. Green's theorem gives answer to this question for the case "dimension" is two.

Definition 4.6.1. $\mathsf{G} \subset \mathbb{R}^3$ *is called simply connected if* G *is path-connected and "hole-less"(:every loop in* G *can be continuously deformed, being in* G*, to a point in* G*).*

Theorem 4.6.2. *Let* $\mathsf{G} \subset \mathbb{R}^2$ *be a simply connected bounded set with a piecewise smooth boundary* $\partial \mathsf{G}$ *which is oriented positively (anti-clock), and* $\bar{\mathsf{F}} = (\mathsf{F}_1, \mathsf{F}_2) : \mathsf{G} \longrightarrow \mathbb{R}^2$ *a* $\mathcal{C}^1$*-type vector field. Then* $\iint_{\mathsf{G}} (\frac{\partial}{\partial x}\mathsf{F}_2 - \frac{\partial}{\partial y}\mathsf{F}_1) = \int_{\partial \mathsf{G}} \bar{\mathsf{F}}.$

Note that, the integral $\int_{\partial G} \bar{F}$ sometimes is written as $\int_{\partial G} (F_1 \ dx + F_2 \ dy)$ with the understanding that $\partial G := \gamma(t) = (x(t), \ y(t))$ for $t \in I$, leading to $\gamma'(t) = (x'(t), \ y'(t))$, and therefore,

$$\int_{\partial G} \bar{F} = \int_{\partial G} < (F_1(\gamma(t)), F_2(\gamma(t))), \ (x'(t), y'(t)) > \ dt$$

$$= \textstyle\int_{\partial G} (F_1(\gamma(t)) \ x'(t) + F_2(\gamma(t)) \ y'(t)) \ dt$$

which symbolically written as $\int_{\partial G} (F_1 \ dx + F_2 \ dy)$.

4.6.1 Application: 2nd Strure Theorem of CVFs in $\mathbb{R}^2$

Let $G \subset \mathbb{R}^2$ be open and simply connected, and $\bar{F} = (F_1, \ F_2) : G \longrightarrow \mathbb{R}^2$ a $\mathcal{C}^1$-type vector field. Suppose $\frac{\partial}{\partial x}F_2 - \frac{\partial}{\partial y}F_1 = 0$. We take any loop C, oriented positively in G, and consider the region G_C enclosed by C in G. Clearly $\partial G_C = C$. Since G is simply connected, G_C is also simply connected, and therefore, we can apply Green's theorem on G_C for the function $\bar{F}$, and get

$$0 = \iint_G \left(\frac{\partial}{\partial x}F_2 - \frac{\partial}{\partial y}F_1 \right) \ dx \ dy = \int_{\partial G_C} \bar{F} = \int_C \bar{F}.$$

This shows that line integration of $\bar{F}$ along any loop vanishes and which, from 1st Structure Theorem of CVF we see that $\bar{F}$ is conservative.

Theorem 4.6.3. *Let* $G \subset \mathbb{R}^2$ *be open and simply connected, and* $\bar{F} = (F_1, \ F_2) : G \longrightarrow \mathbb{R}^2$ *a* $\mathcal{C}^1$*-type vector field. Then the following are equivalent*

1. $\bar{F}$ *is a conservative vector field.*
2. $\frac{\partial}{\partial x}F_2 - \frac{\partial}{\partial y}F_1 = 0.$

3. *Line Integration of* $\bar{F}$ *is path independent.*

4. $\displaystyle\int_C \bar{F} = 0$ *for all loop* C.

Remark 4.6.4. 1. *Note that one never achieves 1st structure theorem of CVF from the 2nd structure theorem, and vice versa.*

2. *Suppose* $\bar{F} = (F_1, F_2) : G \longrightarrow \mathbb{R}^2$ *is a* $\mathcal{C}^1$*-type vector field with an open domain* G. *Suppose* $\frac{\partial}{\partial x}F_2 - \frac{\partial}{\partial y}F_1 = 0$. *Suppose* C *be a loop in* G. *Consider the region* G_C *in* G *enclosed by the loop* C. *If* G_C *has no hole, by Green's theorem,* $\displaystyle\int_C \bar{F} = 0$*; and if* G_C *has a hole (due to hole in* G*), then* $\displaystyle\int_C \bar{F}$ *may not be zero. This shows that for* $\mathcal{C}^1$*-type vector fields having "curl" zero, only the holes in its domain creates obstructions in its way to be CVF.*

4.6.2 Application: Extended Green's Theorem

Theorem 4.6.5 (**Extended Green's Theorem**)**.** *Let* $\bar{F} = (F_1, F_2)$ *be a* $\mathcal{C}^1$ *type vector field with an open domain* $G \subset \mathbb{R}^2$ *having* n *holes* $H_1, H_2, \cdots, H_n$. *Suppose* C_1 *is a loop in* G *such that the region of* G *entangled by* C *contains all the holes. Further suppose* $C_2, C_3, \cdots, C_{n+1}$ *are loops in* G *such that the region of* G *entangled by* C_i *contains only one hole* H_i *for all* $i = 1, 2, \cdots, n$*, and the entangled region is inside the region entangled by* C_1 *(i.e.,* C_i*'s are smaller loops than* C_1*). Orientation of all the loops are positive. Then*

$$\iint_G \left(\frac{\partial}{\partial x}F_2 - \frac{\partial}{\partial y}F_1\right)\, dx\, dy = \int_{C_1} \bar{F} - \sum_{i=2}^{n+1} \int_{C_i} \bar{F}.$$

Proof. Look at Apostol's Calculus Vol-II □

Remark 4.6.6. *As a consequence of this we see that if* $\frac{\partial}{\partial x}F_2 - \frac{\partial}{\partial y}F_1 = 0$*, and if* C_1 *and* C_2 *are any two loops around a hole in* G*, then* $0 = \displaystyle\iint_G \left(\frac{\partial}{\partial x}F_2 - \frac{\partial}{\partial y}F_1\right) dx\, dy = \int_{C_1} \bar{F} - \int_{C_2} \bar{F}$*, i.e.,* $\displaystyle\int_{C_1} \bar{F} = \int_{C_2} \bar{F}$.

Example 4.6.7. *Let* $\bar{F}(x, y) = \left(\dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2}\right)$ *for all* $(x, y) \in G := \mathbb{R}^2 \setminus (0, 0)$. *Suppose* C *be a loop in* G. *Then* $\displaystyle\int_C \bar{F} = 0$ *or* 2π*:*

We all know that for this vector field $\frac{\partial}{\partial x}F_2 - \frac{\partial}{\partial y}F_1 = 0$*, and* $\bar{F}$ *is* $\mathcal{C}^1$*-type.*

Case-I: Suppose C *be such that the region entangled by it in* G *contains the hole (i.e., missing* $(0, 0)$*), i.e.,* C *is around* $(0, 0)$*, then* $\displaystyle\int_C \bar{F} = \int_{C^*} \bar{F}$ *where* C^* *is any another loop around* $(0, 0)$. *We choose* C^* *to be* $x^2 + y^2 = r^2$ *where* r *is such that* C^* *is inside the region entangled by* C. *We know that* $\displaystyle\int_{C^*} \bar{F} = 2\pi$*, and therefore we have* $\displaystyle\int_C \bar{F} = 2\pi$.

Case-II: Suppose the region entangled by C *in* G *do not contain the hole. Then, we can apply Green's theorem and get* $\displaystyle\int_C \bar{F} = 0$.

Example 4.6.8. *Let* $\bar{F}(x,y) = \left(\dfrac{1}{x^2}e^y,\ -\dfrac{1}{x}e^y\right)$ *for all* $(x,y) \in \mathbb{R}^2\setminus\ y-\text{axis}$. *Let* γ *be a path from* $(1,1)$ *to* $(10,30)$. *We shall calculate* $\displaystyle\int_\gamma \bar{F}$.

Since dom$(\bar{F}) = \mathbb{R}^2\setminus y-\text{axis}$, *it is not path-connected. But the domain has two path-connected components* G_L, *and* G_R, *where* $G_L = \{(x,y) \mid y < 0\}$, *and* $G_R = \{(x,y) \mid y > 0\}$. *Since the points* $(1,1)$ *&* $(10,30)$ *lies in* G_R, *and* γ *is path from* $(1,1)$ *to* $(10,30)$, *the path* γ *lies on* G_R; *and therefore, we consider the function* $\bar{F}$ *on* G_R *for the computation of* $\displaystyle\int_\gamma \bar{F}$. *We see that* $\frac{\partial}{\partial x}F_2 - \frac{\partial}{\partial y}F_1 = 0$. *Since* G_R *is open and simply connected, and* $\bar{F}$ *is* $\mathcal{C}^1$*-type, by 2nd structure theorem of CVF we get* $\bar{F} : G_R \longrightarrow \mathbb{R}^2$ *is conservative on* G_R, *and hence* $\displaystyle\int_\gamma \bar{F} = f((10,30)) - f((1,1))$ *where* $\nabla(f) = \bar{F}$ *on* G_R. *It is easy to see that* $f = -\dfrac{1}{x}e^y$, *and therefore* $\displaystyle\int_\gamma \bar{F} = -\frac{1}{10}e^{30} + e$.

Exercise 4.6.9. *Let* $\bar{F}(x,y) = \left(\dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2}\right)$ *for all* $(x,y) \in G := \mathbb{R}^2\setminus(0,0)$. *Let* γ *be a path from* $(1,1)$ *to* $(10,30)$. *Calculate* $\displaystyle\int_\gamma \bar{F}$.

[Hint: Use the technique applied in the previous example.]

4.6.3 Application: Recognizing CVF in $\mathbb{R}^2$

Suppose $G \subset \mathbb{R}^2$ be open and $\bar{F} = (F_1, F_2) : G \longrightarrow \mathbb{R}^2$ a continuous vector field.

■ If $\bar{F}$ is $\mathcal{C}^1$-type, we check $\frac{\partial}{\partial x}F_2 - \frac{\partial}{\partial y}F_1$.

♣ If $\frac{\partial}{\partial x}F_2 - \frac{\partial}{\partial y}F_1 \neq 0$ we conclude that $\bar{F}$ is not a CVF.

♣ If $\frac{\partial}{\partial x}F_2 - \frac{\partial}{\partial y}F_1 = 0$, we check whether dom$(\bar{F})$.

♦ If dom$(\bar{F})$ is simply connected, we conclude that $\bar{F}$ is a CVF.

♦ If dom$(\bar{F})$ has holes, around each hole we take loops C for which we can compute $\displaystyle\int_C \bar{F}$.

▲ If $\displaystyle\int_C \bar{F} = 0$ for each hole, we conclude that $\bar{F}$ is a CVF.

▲ If $\displaystyle\int_C \bar{F} \neq 0$ for some hole, we conclude that $\bar{F}$ is not a CVF.

■ If $\bar{F}$ is not of $\mathcal{C}^1$-type, but continuous, we try to get $f : \text{dom}(\bar{F}) \longrightarrow \mathbb{R}$ such that $\nabla(f) = \bar{F}$ on dom$(\bar{F})$ by solving equation "$\nabla(f) = \bar{F}$".

♣ If this method gives affirmative answer, for sure, $\bar{F}$ is a CVF.

♣ If this method fails, we try to find a loop C such that $\displaystyle\int_C \bar{F} \neq 0$.

4.6.4 Application: Finding Area enclosed by a loop

Let C be a piecewise smooth loop in $\mathbb{R}^2$ and G be the region in $\mathbb{R}^2$ enclosed by the loop C, i.e., $C = \partial G$. We shall find Area(G) using Green's theorem. We know that Area$(G) = \iint_G 1 \ dx \ dy$ which we can write as $\int_C \bar{F}$, on using Green's theorem if we get suitable $\mathcal{C}^1$-type vector field $\bar{F}$ on G and orient $C = \partial C$ positively.

So, we orient C positively and choose $\bar{F}$ for which $\frac{\partial}{\partial x} F_2 - \frac{\partial}{\partial y} F_1 = 1$. It is to be noted that there are infinite number of such $\bar{F}$, e.g., $(0, \ x)$; $(-y, \ 0)$; $(-y/2, \ x/2)$; $(-y/n, \ (n-1)x/n)$, etc. We choose $\bar{F} = (0, x)$, which is $\mathcal{C}^1$-type. Since G is simply connected, $C = \partial G$ is positively oriented, and $\bar{F}$ is of $\mathcal{C}^1$-type, we get applying Green's theorem

$$\text{Area}(G) = \iint_G 1 \ dx \ dy = \iint_G (\frac{\partial}{\partial x} F_2 - \frac{\partial}{\partial y} G_2) \ dx \ dy = \int_{\partial G} \bar{F} = \int_C (0, x).$$

Suppose C has a parametric form $\gamma(t) = (x(t), y(t))$ for all $t \in [a, b]$. Then,

$$\text{Area}(G) = \int_a^b < (0, x(t), \ (x'(t), y'(t))) > \ dt = \int_a^b x(t) \ y'(t) \ dt.$$

Example 4.6.10. *We shall find the area enclosed by* $C : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, *where* $a, b > 0$ *by Green's theorem.*

Let G *be the region enclosed by* C*On Choosing* $F = (0, x)$ *we see that*

$$\textit{Area}(G) = \iint_G 1 \ dx \ dy = \iint_G (\frac{\partial}{\partial x} F_2 - \frac{\partial}{\partial y} G_2) \ dx \ dy = \int_{\partial G} \bar{F} = \int_C (0, x).$$

We parametrize C *as* $C : \gamma(t) = (a\cos(t), \ b\sin(t))$ *for all* $t \in [0, 2\pi]$. *And therefore we have,*

$$\textit{Area}(G) = \int_0^{2\pi} < (0, a\cos(t)), \ (-a\sin(t), b\cos(t)) > \ dt = \int_0^{2\pi} ab\cos^2(t) \ dt = \pi ab.$$

Chapter 5

Surface Integration

We completed our discussion on the structure of conservative vector field in $\mathbb{R}^2$ and we now aim to understand their structure in $\mathbb{R}^3$. As the related theory depends upon surface integration of vector fields, we first study parametric surfaces and a few properties of surface integration of vector fields.

Definition 5.0.1. *Let $\mathcal{S} \subset \mathbb{R}^3$. $\mathcal{S}$ is called a parametric surface if there exists a continuous function $\bar{r} : D \longrightarrow \mathbb{R}^3$ where $D \subseteq \mathbb{R}$ such that $\mathcal{S} = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y, z) = \bar{r}(u, v),\ (u, v) \in D\}$.*

We now go though some examples of parametric surfaces.

Example 5.0.2. *For $a > 0$, let $aS^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = a^2\}$. Then aS^2 is a parametric surface, as for $\bar{r}(u, v) = (a \cos u \cos v, a \sin u \cos(v), a \sin v)$ where $(u, v) \in [0, 2\pi] \times [0, \pi]$ we can see that $aS^2 = \{\bar{r}(u, v) \in \mathbb{R}^3 \mid (u, v) \in [0, 2\pi] \times [0, \pi]\}$. Obviously $\bar{r}$ is a continuous function, as the component functions are the product of trigonometric functions.*

Example 5.0.3. $D^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1,\ z = 0\}$ *is a parametric surface as* $D^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x = u \cos v,\ y = u \sin v,\ z = 0,\ v \in [0, 2\pi],\ u \in [0, 1]\}$*, i.e., if we take* $\bar{r}(u, v) = (u \cos v, u \sin v, 0))$ *where* $(u, v) \in [0, 1] \times [0, 2\pi]$*, then* $\bar{r}$ *is continuous and it represents* D^2*.*

Example 5.0.4. *Let $H = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1, -\infty < z < \infty\}$. Then H is a parametric surface and it is represented by the function $\bar{r}(u, v) = (\cos u, \sin u, v)$ where $(u, v) \in [0, 2\pi] \times (-\infty, \infty)$. Clearly $\bar{r}$ is a continuous function.*

The function $\bar{r}$ is called a parametric representation of the surface or a parametrization of the surface. It can be shown that a parametric surface has infinitely many parametrization.

Exercise 5.0.5. *Show that a right circular cone is a parametric surface.*

Definition 5.0.6. *Let $\mathcal{S}$ be a parametric surface with parametrisation $\bar{r}(u, v)$ where $(u, v) \in D$.*