INDIAN INSTITUTE OF SPACE SCIENCE AND TECHNOLOGY
THIRUVANANTHAPURAM 695 547

B.Tech End Semester Examination - April 2013

MA121 - Vector Calculus and Differential Equations

Time: 9.30 am - 12.30 pm **Date: 22/04/2013** **Max. Marks: 100**

SECTION A (Attempt all 10 questions - 10x5= 50 marks.)

1. Find the interval on which $\sum \frac{x}{n(1+nx^2)}$; $x \in \mathbb{R}$ converges uniformly.

2. Check the uniform convergence of $\left\{\frac{nx}{1+n^2x^2}\right\}$ on $[0, \alpha)$.

3. Find an interval in which the following IVP has a unique solution:

$$\frac{dy}{dx} = e^y, \ y(0) = 0, \ R := \{(x, y) : |x| \le 3, |y| \le 4\}.$$

   Also, find minimum no. of 'n' such that the error in Picard approximation does not exceed by 0.5.

4. Find the general solution of $\frac{d^2y}{dx^2} - xf(x)\frac{dy}{dx} + f(x)y = 0$.

5. Discuss whether two linearly independent Frobenius series solutions around $x = 0$ exist or do not exist for the following equations: (No need to find solutions).

   (i) $2x^2\frac{d^2y}{dx^2} + x(x+1)\frac{dy}{dx} - (\cos x)y = 0$.

   (ii) $x^4\frac{d^2y}{dx^2} - (x^2\sin x)\frac{dy}{dx} + 2(1-\cos x)y = 0$.

6. Find the first three terms of the Legendre series of the following function:

$$f(x) = \begin{cases} 0 & \text{if } \ -1 \le x < 0, \\ x & \text{if } \ \ 0 \le x \le 1. \end{cases}$$

7. Let G be the union of two regions G_1 and G_2 given by $G_1 : x^2 + y^2 \le 4, x \ge 0$ and $G_2 : \frac{x^2}{9} + \frac{y^2}{4} \le 1$, $x \le 0$. Using Green's theorem find the area of G.

8. Define directional derivative of a function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ at a point P_0 along a vector $\vec{v}$. Find the directional derivative of $f(x, y, z) = ze^{xy}$ at the point $P_0 = (1, 1, 1)$ along the vector $\vec{v} = (0, 4, 3)$.

9. Define arc length function of a curve. Find the arc length function of the curve $C : y = x^2, z^2 = y$, $z \ge 0$ with initial point $(0, 0, 0)$.

10. Let $\vec{F}= (ye^z + ze^x, xe^z + e^y, xye^z + e^x + 1)$ be a vector field and C be a curve joining straight line segments $(0,1,0) \longrightarrow (0,0,1) \longrightarrow (1,0,0)$ and oriented accordingly. Evaluate $\int_C \vec{F}\,.d\vec{r}$. Is the integral independent of the path? Justify your answer.

SECTION B (Attempt any 5 questions - 5x10= 50 marks.)

11. (a) $f_n(x) = \dfrac{1}{1 + x^2 + \frac{x^4}{n}}$, Find $\lim_{n\to\alpha} \int_0^1 f_n(x)dx$.

    (b) $\dfrac{d}{dx}\sum_1^\alpha \dfrac{\cos nx}{n^3} = -\sum_1^\alpha \dfrac{\sin nx}{n^2}$ on $\mathbb{R}$ say True/False, Justify your answer.

12. (a) Solve $(f(y))^2\dfrac{dx}{dy} + 3f(y)f'(y)x = f'(y)$.

    Here, $f'(y) = \dfrac{d}{dx}f(y)$ and $f(y) \neq 0\,\forall\, y \in \mathbb{R}$.

    (b) Find all eigen values and eigen functions if exist of the following differential equation.

$$\frac{d^2y}{dx^2} + \lambda y = 0, \quad y(0) - y(\pi) = 0, \quad \frac{dy}{dx}(0) - \frac{dy}{dx}(\pi) = 0.$$

13. Find the general solution around $x = 0$ of the following differential equation

$$4x^2\frac{d^2y}{dx^2} - 8x^2\frac{dy}{dx} + (4x^2 + 1)y = 0.$$

14. Verify Stokes theorem for the vector field $\vec{F}= (-y, z, x^2)$ over the outward oriented cylindrical surface $S : x^2 + y^2 = 1,\ 0 \leq z \leq h$ where h is a constant.

15. Using Gauss divergence theorem find the volume of solid upper hemisphere of radius 1 with center $(0,0,0)$.

16. Let C_1 and C_2 be two positively oriented circles in xy-plane given by $C_1 : x^2 + y^2 = 1$ and $C_2 : x^2+y^2 = 4$. Suppose $\vec{F}= \left(\dfrac{-y}{x^2 + y^2}, \dfrac{x}{x^2 + y^2}\right)$ be a vector field. Using Green's theorem for simply connected domains show that $\int_{C_1} \vec{F}\,.d\vec{r} = \int_{C_2} \vec{F}.d\vec{r}$. Further show that $\int_{C_2} \vec{F}\,.d\vec{r} = 2\pi$ for any positively oriented simple smooth loop C around the point $(0,0)$.

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