INDIAN INSTITUTE OF SPACE SCIENCE AND TECHNOLOGY
THIRUVANANTHAPURAM 695 547

End Semester Examination - May 2023

B.Tech- IV Semester

MA221 - Integral Transforms, PDE and Calculus of Variations

Date: 01/05/2023 **Time: 09.30 am - 12.30 pm** **Max. Marks: 50**

PART - A (Answer all questions) - 2.5 x 10 = 25 Marks.

1. Compute the convolution $(f * g)(t)$ for the following pair of functions:

$$f(x) = \begin{cases} e^x, & \text{for } 0 \leq x \leq \frac{1}{2}, \\ 0, & \text{otherwise} \end{cases} \quad \text{and } g(x) = \begin{cases} e^{2x}, & \text{for } \frac{-1}{4} \leq x \leq \frac{1}{4}, \\ 0, & \text{otherwise.} \end{cases}$$

2. Using the convolution theorem,find the inverse fourier transform of

$$\frac{1}{(20 + 9iw - w^2)}.$$

3. Using the convolution theorem, find the inverse Laplace transform of

$$\frac{e^{-\pi s}}{(s^2 + s - 6)}.$$

4. Find the general solution of the following PDE.

$$(x^2 - yz)\frac{\partial z}{\partial x} + (y^2 - xz)\frac{\partial z}{\partial y} = z^2 - xy$$

5. Solve the PDE $\frac{\partial z}{\partial x} + z\frac{\partial z}{\partial y} = z, x \neq 1$ with $z(0, y) = -y$.

6. For all x, y, reduce the following equation to a canonical form $u_{xx} + xu_{yy} = 0,\ x \neq 0$

7. Solve the following PDE.

$$(D^2 - DD' - 2D'^2)z = (2x^2 + xy - y^2)\sin xy - \cos xy,$$

where $D \equiv \frac{\partial}{\partial x}$ and $D' \equiv \frac{\partial}{\partial y}$.

8. Show that the extremal of the functional $I(y)$ with $y(1) = 0$ and $y(2) = 1$ is a circle, where

$$I(y) = \int_1^2 \sqrt{\frac{1 + (y')^2}{x^2}}\,dx.$$

9. Derive the Euler-Lagrange equation satisfied by the extremal of the functional

$$I(z) = \iint_\Omega \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dxdy,$$

where $\Omega \subset \mathbb{R}^2$, subject to the condition on the boundary $\partial\Omega$ given by $z(x, y) = g(x, y), (x, y) \in \partial\Omega$.

10. Find the curve $\bar{y}$ that extremizes the functional $I(y)$ under the conditions $y(0) = 0, y'(0) = 1, y(\pi/4) = y'(\pi/4) = \frac{1}{\sqrt{2}}$, where

$$I(y) = \int_0^{\pi/4} \left((y'')^2 - y^2 + x^2\right) dx.$$

PART - B (Attempt Any 5 questions) - 25 Marks.

11. (a) Determine Laplace transform of $x^\gamma, -1 < \gamma < 0$. [2.5]

(b) Let $k > 0$. Determine the Fourier transform of the function $f : \mathbb{R} \to \mathbb{R}$ defined by, $f(x) = e^{-kx^2}$. [2.5]

12. Using Laplace transform to solve the following initial value problem.

$$x\frac{d^2y}{dx^2} - 3\frac{dy}{dx} = -4x^2,\ x > 0, \text{wiith } y(0) = 0,\ \frac{dy}{dx}(0) = B.$$

where B is a constant.

13. (a) Does the following PDE has a unique solution? if yes ,then find it and if not,justify your answer.

$$x(y^2 + u)\frac{\partial u}{\partial x} - y(x^2 + u)\frac{\partial u}{\partial y} = (x^2 - y^2)u$$

with the initial data $u = 1$ or $x + y = 0$. [2]

(b) Find a complete and singular solution of $(p^2 + q^2)y = qz$, where $p = \dfrac{\partial z}{\partial x}$ and $q = \dfrac{\partial z}{\partial y}$ [3]

14. (i) Find the solution of the following problem

$u_{tt} - u_{xx} = h,\ 0 < x < 1,\ t > 0,\ h = \text{constant}$

$u(x, 0) = x(1 - x),\ 0 \leq x < 1,$

$u_t(x, 0) = 0,\ 0 \leq x < 1,$

$u(0, t) = t,\ u(1, t) = \sin t,\ t \geq 0$ [3]

(ii) Give an example of a PDE which has

(a) infinity may solution,

(b) No solution.

Justify your answer. [2]

15. (a) Let $S = \{y \in C^1[0, 1] : y(0) = 0,\ y(1) = 1\}$. Find the extremum $\bar{y} \in S$ of the functional $I(y) = \int_0^1 (y' - 1)^2 dx$ subject to the conditions $y(0) = 0, y(1) = 1$. Also show that the solution curve $\bar{y} \in S$ indeed minimizes the functional $I(y)$, that is, $I(y) \geq I(\bar{y}), \forall y \in S$. [2]

(b) Using the method of Lagrange multipliers, determine the shape of a rope of length l and constant linear density ρ that is suspended between two fixed points (a, y_a) and (b, y_b). (Hint: find a curve y that minimizes the potential energy and constraint by the length l). [3]

16. (a) Explain the Rayleigh-Ritz method for finding the extremum of the variational problem [2]

$$\text{Extermize } I(y) = \int_{x_1}^{x_2} F(x, y, y')dx, \; y(x_1) = y_1, \; y(x_2) = y_2.$$

(b) Derive the variational formulation of the BVP: $y'' - y = 0, y(0) = y(1) = 1$, and using the Rayleigh-Ritz method, find the approximate solution. [3]

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