11-08-2025

LTI : Linear Time Invariant

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

$y(n) = x(n) * h(n)$: Convolution

$Y(Z) = X(Z) \cdot H(Z)$

$\Rightarrow H(Z) = \dfrac{Y(Z)}{X(Z)}$

Low Pass Filter:

$$H_{LP} = \begin{cases} 1, & 0 \leq w \leq w_c \\ 0, & w > w_c \end{cases}$$

$$h_{LP}(n) = \frac{\sin(\pi n w_c)}{\pi n w_c}$$ (not realisable, not causal)

$h(n) = \delta(n - n_d)$

$H(e^{jw}) = e^{-jn_d w}$

Magnitude response: $|H(e^{jw})| = 1$

Phase response: $\angle H(e^{jw}) = -w n_d$

→ Practical systems always have non-zero phase response.

$$h_{LP}(n) = \frac{\sin \pi (n - n_d) w_c}{\pi (n - n_d) w_c}$$

$$H_{LP}(e^{jw}) = \begin{cases} e^{-jwn_d}, & 0 < w < w_d \\ 0, & w_c < w < \pi \end{cases}$$

Realisable

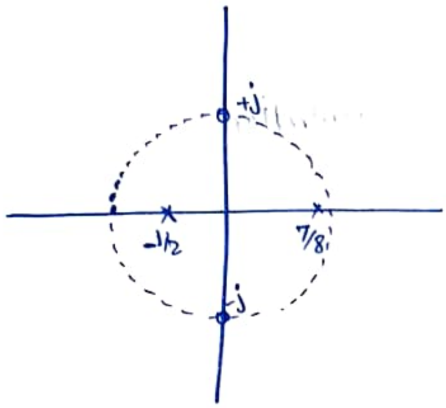
Eg. $y(n) - \frac{3}{8} y(n-1) - \frac{7}{16} y(n-2) = x(n) + x(n-2)$ → difference eqn.

$\Rightarrow Y(Z) - \frac{3}{8} Z^{-1} Y(Z) - \frac{7}{16} Z^{-2} \cdot Y(Z) = X(Z) + Z^{-2} X(Z)$

$\Rightarrow \dfrac{Y(Z)}{X(Z)} = \dfrac{1 + Z^{-2}}{1 - \frac{3}{8} Z^{-1} - \frac{7}{16} Z^{-2}}$

$$= \frac{(1+jz^{-1})(1-jz^{-1})}{(1-\frac{7}{8}z^{-1})(1+\frac{1}{2}z^{-1})}$$

poles: $7/8, -1/2$, zeros: $\pm j$



Discrete System

12-08-2025

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

$$ZT \Rightarrow \sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

$$= \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M} (1-c_k z^{-1})}{\prod_{k=1}^{N} (1-d_k z^{-1})}$$

$N_r \rightarrow c_k \rightarrow$ zeros @ $k=1$ to M
pole @ $z=0$.

$D_r \rightarrow d_k \rightarrow$ zeros @ $k=1$ to M
pole @ $z=0$.

Causality and Stability

Causality:

ROC : $|Z| >$ outside the outermost pole:

"α_{max}" $\Rightarrow$ $|Z| > \alpha_{max}$

Stability (with causality):

All poles should lie inside the unit circle.

$\alpha_{max} < 1$.

Stability (in general):

The ROC should contain the unit circle.

Causality & stability:

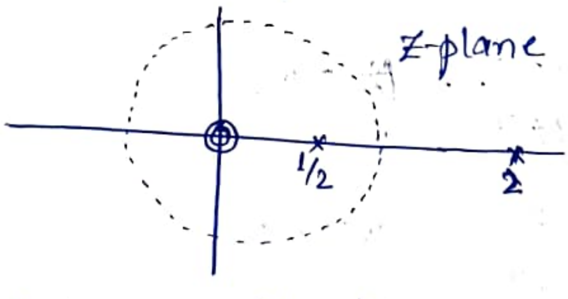
All poles should lie inside the unit circle, and ROC is outside the outermost pole.

Eg. $y(n) - 5/2\, y(n-1) + y(n-2) = x(n)$

$\Rightarrow Y(Z) - 5/2\, Z^{-1} Y(Z) + Z^{-2} Y(Z) = X(Z)$

$\Rightarrow H(Z) = \dfrac{Y(Z)}{X(Z)} = \dfrac{1}{1 - 5/2\, Z^{-1} + Z^{-2}} = \dfrac{2Z^2}{2Z^2 - 5Z + 2} = \dfrac{2Z^2}{2(Z-2)(2Z-1)}$

$= \dfrac{1}{(1-2Z^{-1})(1-1/2\, Z^{-1})}$; zeros : $Z=0$ (two)
Poles : $2, 1/2$



Case-1 : ROC outside the outermost pole ($|Z| > 2$)
↳ Causal & unstable.

Case-2 :

$|Z| < 1/2$ → Non causal, unstable.

Case-3 :

$1/2 < |Z| < 2$ → stable, non-causal.

Inverse Systems

$$h(n) \cdot h_i(n) \longrightarrow \delta(n)$$

$$X(Z) \longrightarrow \underbrace{\boxed{H(Z)} \longrightarrow \boxed{H_i(Z)}}_{G(Z)} \longrightarrow Y(Z) = X(Z)$$

$$G(Z) = H(Z)\,H_i(Z) = 1$$

$$H_i(Z) = \frac{1}{H_o(Z)}$$

$$H_i(Z) = \left(\frac{a_0}{b_0}\right) \frac{\prod_{K=1}^{N} (1 - d_K Z^{-1})}{\prod_{K=1}^{M} (1 - C_K Z^{-1})}$$

→ Not all causal and stable systems have inverse which are causal and stable.

→ while designing the inverse systems, ensure that the poles and zeros of the system is inside the unit circle.

→ The poles of $H_i(Z)$ are the zeros of $H(Z)$, the ROC of $H(Z)$ and $H_i(Z)$ should overlap.

Eg. $H(Z) = \dfrac{1 - 0.5Z^{-1}}{1 - 0.9Z^{-1}}$; $|Z| > 0.9$

$\Rightarrow H_i(Z) = \dfrac{1 - 0.9Z^{-1}}{1 - 0.5Z^{-1}}$, $|Z| > 0.5$

Eg. $H(Z) = \dfrac{Z^{-1} - 0.5}{1 - 0.9Z^{-1}}$, $|Z| > 0.9$

$= \dfrac{-2 + 1.8Z^{-1}}{1 - 2Z^{-1}}$, ROC : $|Z| > 2$

Eg. $h(n) = 2(2)^n u(n) - 1.8\,(2)^{n-1} u(n-1)$.

$$H(Z) = \frac{2}{1-2Z^{-1}} - 1.8 \cdot \frac{1}{2} \frac{2Z^{-1}}{1-2Z^{-1}}$$

$$= \frac{2-1.8Z^{-1}}{1-2Z^{-1}}, \quad |Z| > 2$$

$$H_i(Z) = \frac{1-2Z^{-1}}{2-1.8Z^{-1}}$$

$$= \frac{0.5-Z^{-1}}{1-0.9Z^{-1}}, \quad |Z| > 0.9$$

13-08-2025

Frequency Response of System

Inference from poles & zeros:

$$H(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}$$

Magnitude:

$$|H(e^{j\omega})| = \left|\frac{b_0}{a_0}\right| \frac{\prod_{k=1}^{M} |(1-c_k e^{-j\omega})|}{\prod_{k=1}^{N} |(1-d_k e^{-j\omega})|}$$

Magnitude Squared fn:

$$|H(e^{j\omega})|^2 = H(e^{j\omega}) H^*(e^{j\omega})$$

$$= \left(\frac{b_0}{a_0}\right)^2 \frac{\prod_{k=1}^{M} (1-c_k e^{-j\omega})(1-c_k^* e^{j\omega})}{\prod_{k=1}^{N} (1-d_k e^{-j\omega})(1-d_k^* e^{j\omega})}$$

$$H^*(1/z^*) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M} (1-c_k^* z)}{\prod_{k=1}^{N} (1-d_k^* z)}$$

$$x(n) \longleftrightarrow X(z)$$

$$x^*(-n) \longleftrightarrow X^*(1/z^*)$$

$$P(z) = H(z) H^*(1/z^*)$$

$$(1-c_k z^{-1}) = 1 - \frac{c_k}{z} = \frac{z-c_k}{z} \rightarrow \text{zeros @ } c_k, \text{ poles @ } z=0$$

$$(1-c_k^* z^{+1}) = (-c_k^*)\left(z - \frac{1}{c_k^*}\right) \rightarrow \text{zeros @ } 1/c_k^*$$

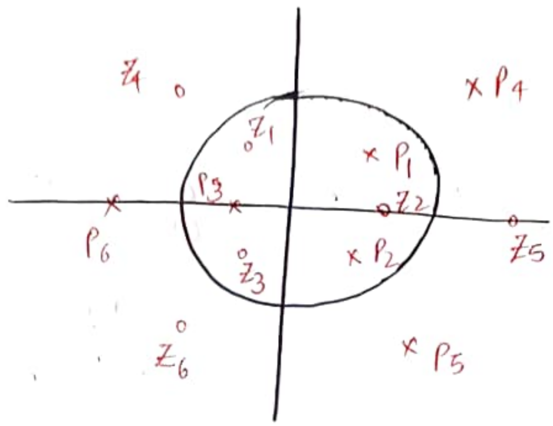
∴ The zeros of $P(z)$ are

$$c_1, \frac{1}{c_1^*}, c_2, \frac{1}{c_2^*}, \dots, c_M, \frac{1}{c_M^*}$$

Similarly, poles of $P(z)$ are

$$d_1, \frac{1}{d_1^*}, d_2, \frac{1}{d_2^*}, \dots, d_N, \frac{1}{d_N^*},$$

in addition to poles & zero at $z=0$.



$$H(Z) = \frac{(1-Z_1 Z^{-1})(1-Z_2 Z^{-1})(1-Z_3 Z^{-1})}{(1-P_1 Z^{-1})(1-P_2 Z^{-1})(1-P_3 Z^{-1})}$$

All Pass Filter / System

↳ used for phase compensation / delay equalizer.

$H(e^{j\omega}) = A$, for all ω

$|H(e^{j\omega})| = A$

Application: Phase compensation

→ (Minimum phase)

$$H_{ap}(Z) = \frac{Z^{-1} - a^*}{1 - aZ^{-1}}$$

$$H_{ap}(e^{j\omega}) = \frac{e^{-j\omega} - a^*}{1 - a\,e^{-j\omega}}$$

$$= \frac{e^{-j\omega}(1 - a^* e^{j\omega})}{(1 - a^* e^{j\omega})^*}$$

$$|H_{ap}(e^{j\omega})| = 1$$

$$\angle H_{ap}(e^{j\omega}) = \measuredangle e^{-j\omega} + \measuredangle(1 - a^* e^{j\omega}) - \measuredangle(1 - a^* e^{j\omega})^*$$

$$\left[\begin{array}{l} a = r e^{j\theta} \\ a^* = r e^{-j\theta} \\ \measuredangle(1 - a^* e^{j\omega}) = \measuredangle(1 - r\cos(\omega-\theta) - j r\sin(\omega-\theta)) \end{array}\right]$$

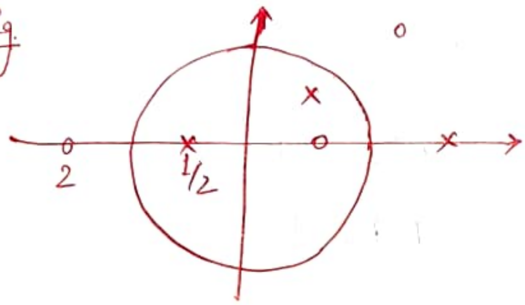
$$= -\omega + 2\tan^{-1}\frac{r\sin(\omega-\theta)}{r\cos(\omega-\theta)}$$

Poles and zeros:

$$1 - az^{-1} \longrightarrow 1 - a/z \rightarrow \frac{z-a}{z} \quad : \quad \text{pole @ } z = a, \ \text{zero @ } z = 0$$

$$z^{-1} - a^* = -a^*\left(z - \frac{1}{a^*}\right) \quad : \quad \text{pole @ } z = 1/a^*, \ \text{zero @ } z = 0.$$

eg.



→ All pass filters always have poles and zeros in pairs.

18-08-2025

Group Delay and Phase Delay

$$y(n) = A\,x(n - n_d)$$

$$\Rightarrow H(e^{j\omega}) = Ae^{-j\omega n_d}$$

$$|H(e^{j\omega})| = A,$$

$$\measuredangle H(e^{j\omega}) = -\omega n_d$$

Phase response of the system: $\phi(\omega) = -\omega n_d$

Phase delay $= -\dfrac{\phi(\omega)}{\omega} \Rightarrow n_d$

Group delay $= -\dfrac{d\,\phi(\omega)}{d\omega} \Rightarrow n_d$

- The group delay and phase delay of LTI system are constant.

Minimum Phase System

→ The system and its inverse are stable and causal.

→ The poles and zeros will lie inside the unit circle.

→ Min^m phase delay among systems having same response.

→ Most of the system can be decomposed into minimum phase system and all-pass filter.

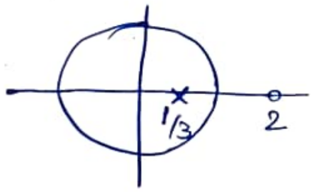
$$H(Z) = H_{min\,phase}(Z) \cdot H_{ap}(Z)$$

$$\boxed{H(z)} \equiv \boxed{H_{min}(Z)} \text{---} \boxed{H_{ap}(z)}$$

$$H(z) = H_1(z)\,(z^{-1} - c^*) \cdot \frac{1 - cz^{-1}}{1 - cz^{-1}}$$

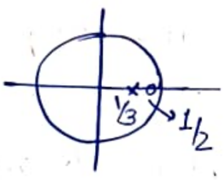
$$= \underbrace{H_1(z)\,(1 - cz^{-1})}_{\text{min.phase}} \ \underbrace{\frac{(z^{-1} - c^*)}{(1 - cz^{-1})}}_{\text{All pass}}$$

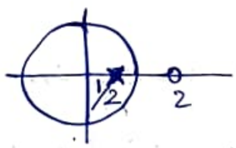
Eg. $H(z) = \dfrac{1 - 2z^{-1}}{1 - \frac{1}{3}z^{-1}}$



$$= \frac{1 - 2z^{-1}}{1 - \frac{1}{3}z^{-1}} \cdot \frac{z^{-1} - 2}{z^{-1} - 2}$$

$$= \underbrace{\frac{z^{-1} - 2}{1 - \frac{1}{3}z^{-1}}}_{\text{Min phase}} \cdot \underbrace{\frac{1 - 2z^{-1}}{z^{-1} - 2}}_{\text{All pass}}$$





$x(n)$ — $\boxed{H_d(z)}$ — $\boxed{H_L(z)}$ — $\hat{x}(n)$

↳ $H_{min} \cdot H_{ap}$

- Poles, zeros cancellation can take place

↳ No change in mag. response; only changes phase response.

Eg. $H(z) = \dfrac{(1+3z^{-1})(1-\frac{1}{2}z^{-1})}{z^{-1}(1+\frac{1}{3}z^{-1})}$

$= \dfrac{(1+3z^{-1})(1-\frac{1}{2}z^{-1})}{z^{-1}(1+\frac{1}{3}z^{-1})} \cdot \dfrac{(\cancel{z^{-1}+3})}{(z^{-1}+3)}$

$\underbrace{(1+\frac{1}{3}z^{-1})}_{\frac{1}{3}(\cancel{z^{-1}+3})}$

if go with min phase, will make it unstable
⇓
Append with All pass
(Delaying all pass, still it remains all pass)

$= 3(1-\frac{1}{2}z^{-1}) \cdot \dfrac{1+3z^{-1}}{z^{-1}(z^{-1}+3)}$

Eg. $H(z) = \underbrace{(1-0.9e^{j0.6\pi}z^{-1})(1-0.9e^{-j0.6\pi}z^{-1})}_{\text{Min phase}} (1-1.25e^{j0.8\pi}z^{-1})$
$(1-1.25e^{-j0.8\pi}z^{-1})$

$(1.25)^2 e^{j0.8\pi}e^{-j0.8\pi}\,[z^{-1}-0.8e^{-j0.8\pi}]\,[z^{-1}-0.8e^{j0.8\pi}]$

scaling   all pass ~~all pass~~

$= (1.25)^2 e^{j0.8\pi}e^{j0.8\pi}\,(z^{-1}-0.8e^{-j0.8\pi})(z^{-1}-0.8e^{j0.8\pi}) \cdot \dfrac{(1-0.8e^{j0.8\pi}z^{-1})(1-0.8e^{-j0.8\pi}z^{-1})}{(1-0.8e^{j0.8\pi}z^{-1})(1-0.8e^{-j0.8\pi}z^{-1})}$

scaling

delay (with all-pass) ~~min phase~~

min phase

~~min phase all pass~~   all-pass

Properties of Minimum phase system:

- Min^m phase lag system ⇒ Min^m phase delay among system having same response.
- Min^m group delay.
- Least delay in energy delivery.

19-08-2025

Linear Phase Systems

$$H(e^{j\omega}) = \underbrace{|H(e^{j\omega})|}_{\text{Magnitude}}\, e^{-j(\omega b + c)}$$

(Linear) Phase: $\phi(\omega) = -\omega b + c$, b, c : constants.

Group delay : $\tau_g \rightarrow$ constant.

$\rightarrow$ For different frequencies, the system delay is const.

IIR : Infinite Impulse Response

FIR : Finite Impulse Response $\rightarrow$ More stable

Four Types of Filters :

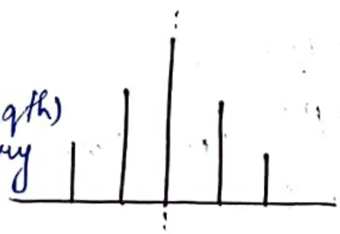
Based on : (of impulses) Length → odd / even
Symmetry → symm. / Anti-symm.

$$H(z) = \sum_{n=0}^{\infty} h(n)\, z^{-n}$$

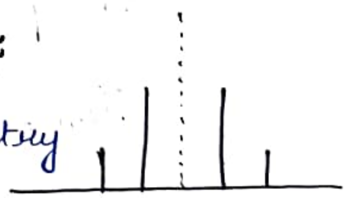
FIR: $H(z) = \sum_{n=0}^{N \rightarrow \text{or } (N-1)} h(n)\, z^{-n}$ [For $n = 0$ to N, N: even $\Rightarrow$ odd sys.]

$\boxed{h(n)}$

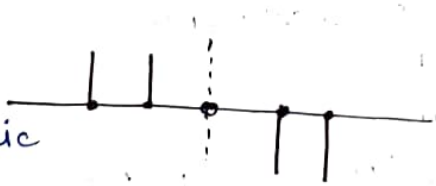
Type-1: Odd (length) Symmetry



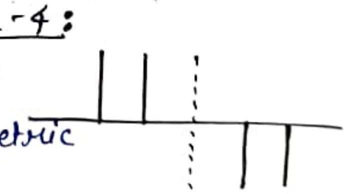
Type-2: even symmetry



Type-3: odd anti-symmetric



Type-4: even anti-symmetric



Symmetric : $h(n) = h(N-n)$

Anti-symmetric : $h(n) = -h(N-n)$

[N=8]

$h(0)$ $h(1)$ $h(2)$ ... $h(8)$

$h(0) = h(8)$

$h(1) = h(7)$

Type-I system:

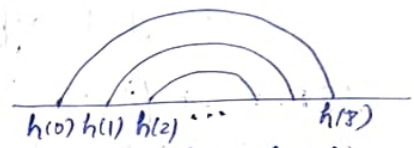
$$H(z) = \sum_{n=0}^{N} h(n)\, z^{-n}$$

$$N = 8 \ (\text{length} = 9) \rightarrow \text{odd}$$

$$\hookrightarrow H(z) = \sum_{n=0}^{8} h(n)\, z^{-n}$$

$$= h(0) + h(1) z^{-1} + h(2) z^{-2} + \dots + h(8) z^{-8}$$

$$= h(0)\,[1 + z^{-8}] +$$
$$h(1)\,[z^{-1} + z^{-7}] +$$
$$h(2)\,[z^{-2} + z^{-6}] +$$
$$h(3)\,[z^{-3} + h^{-5}] +$$
$$h(4)\, z^{-4}$$



$h(0) = h(8)$
$h(1) = h(7)$
$h(2) = h(6)$
$h(3) = h(5)$
$\vdots$

$$= \underbrace{z^{-4}}_{e^{-j\omega 4} = e^{-j\omega N/2}} \Big[h(4) + h(0)\,\underbrace{[z^{+4} + z^{-4}]}_{2\cos 4\omega} + h(1)\,\underbrace{[z^{+3} + z^{-3}]}_{2\cos 3\omega}$$

$$\underbrace{+\, h(2)\,\underbrace{[z^{+2} + z^{-2}]}_{2\cos 2\omega} + h(3)\,\underbrace{[z + z^{-1}]}_{2\cos\omega} \Big]}_{\tilde{H}(\omega)}$$

$[\because z^{-4} = e^{-j\omega 4}]$

$$\boxed{H(e^{j\omega}) = e^{-j\omega N/2}\, \tilde{H}(\omega)} \ ; \ \boxed{\phi(\omega) = -\omega N/2}$$

Group delay, $\tau_g = \frac{N}{2} = 4$.

$$\boxed{\tilde{H}(\omega) = h\left(\tfrac{N}{2}\right) + 2 \sum_{n=1}^{N/2} h\left(\tfrac{N}{2} - n\right) \cos\omega n}$$

In general,

$$H(e^{j\omega}) = e^{-j\omega N/2} \sum_{K=0}^{N/2} a(K) \cos\omega K$$

$$\left[\begin{array}{l} a(0) = h(N/2) \\ a(K) = 2h\left(\frac{N}{2} - K\right), \quad K = 1, 2, \dots, N/2 \end{array}\right]$$

$$= e^{-j\omega N/2} \left[a(0)\cos\omega(0) + a(1)\cos\omega + \dots \right]$$

Type-2 System:

Length → even; symmetric

$$H(z) = \sum_{n=0}^{7} h(n) z^{-n}$$

$$= h(0) + h(1) z^{-1} + \dots + h(7) z^{-7}$$

$$= h(0)[1 + z^{-7}] + h(1)[z^{-1} + z^{-6}] + h(2)[z^{-2} + z^{-5}] + h(3)[z^{-3} + z^{-4}].$$

$$= \underbrace{z^{-7/2}}_{e^{-j\omega 7/2} = e^{-j\omega N/2}} \Big[h(0) \underbrace{[z^{+7/2} + z^{-7/2}]}_{2\cos(7/2\,\omega)} + h(1) \underbrace{[z^{5/2} + z^{-5/2}]}_{2\cos(5/2\,\omega)} + h(2) \underbrace{[z^{3/2} + z^{-3/2}]}_{2\cos(3/2\,\omega)} + h(3) \underbrace{[z^{1/2} + z^{-1/2}]}_{2\cos(1/2\,\omega)} \Big]$$

$$H(e^{j\omega}) = e^{-j\omega N/2}\, \tilde{H}(\omega).$$

Phase: $\phi(\omega) = -\omega N/2$

Group delay, $\tau_g = N/2$ → Not desirable [even, symmetric]

$$\tilde{H}(\omega) = 2 \sum_{n=1}^{\frac{N+1}{2}} h\left[\frac{N+1}{2} - n\right] \cos \omega(n - 1/2)$$

$$H(e^{j\omega}) = e^{-j\omega 7/2} \cdot 2 \cdot \left[h(0)\cos 7/2\,\omega + h(1) \cos 5/2\,\omega + h(2) \cos 3/2\,\omega + h(3) \cos \omega/2 \right]$$

↳ HPF is not possible,
as $\cos \omega(n - 1/2) = 0 \Rightarrow \omega = \pi$.
[Phase is zero for $\omega = \pi$.].

Type-3 System:

Odd, anti-symmetric.

$$H(z) = \sum_{n=0}^{8} h(n) z^{-n}$$

$$H(z) = h(0) + h(1) z^{-1} + \dots + h(8) z^{-8}.$$

$$= h(0)[1 - z^{-8}] + h(1)[z^{-1} - z^{-7}] + h(2)[z^{-2} - z^{-6}] + h(3)[z^{-3} - z^{-5}] + h(4) z^{-4}$$

$$= \underbrace{z^{-4}}_{e^{-j\omega 4} = e^{-j\omega N/4}} \Big[h(0) \underbrace{[z^{+4} - z^{-4}]}_{2j\sin 4\omega} + h(1)[z^{3} - z^{-3}] + h(2)[z^{2} - z^{-2}] + h(3)[z - z^{-1}] + h(4) \Big]$$

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$$= e^{-j4w}\left[2j\{h(0)\sin 4w + h(1)\sin 3w + h(2)\sin 2w + h(3)\sin w\} + h(4)\right]$$

$$= e^{-j4w} \cdot e^{+j\pi/2} \cdot 2\{\tilde{H}(w)\}$$

$$= e^{-j4w} \cdot j\{\tilde{H}(e^{jw})\}$$

↪ purely imaginary with odd symm.

$\tau_g = 4.$

$$\tilde{H}(e^{jw}) = \sum_{K=1}^{N/2} c(K)\sin wK \longrightarrow \text{zero for } w = 0, \pi.$$

$$c(K) = 2h(N/2 - K)$$

- Only able to design band ~~pass~~ pass and band-stop filters.
- HP and LP cannot be designed.

$$H(e^{jw}) = 0, \quad w = 0, \pi.$$

Type-4 system:

Length → even ; anti-symmetric.

$$H(z) = \sum_{n=0}^{7} h(n) z^{-n}$$

$$\begin{bmatrix} h(0) = -h(7) \\ h(1) = -h(6) \\ h(2) = -h(5) \\ h(3) = -h(4) \end{bmatrix}$$

$$= h(0) + h(1)z^{-1} + \dots + z^{-7}h(7)$$

$$= h(0)[1 - z^{-7}] + h(1)[z^{-1} - z^{-6}] + h(2)[z^{-2} - z^{-5}] + h(3)[z^{-3} - z^{-4}]$$

$$= z^{-7/2}\Big[h(0)\underbrace{[z^{+7/2} - z^{-7/2}]}_{2j\sin 7/2 w} + h(1)\underbrace{[z^{5/2} - z^{-5/2}]}_{2j\sin 5/2 w} + h(2)\underbrace{[z^{3/2} - z^{-3/2}]}_{2j\sin \frac{3w}{2}} + h(3)\underbrace{[z^{1/2} - z^{-1/2}]}_{2j\sin 1/2\, w}\Big]$$

$= e^{-j7/2 w} \cdot 2j \cdot [\ \cdot\ ]$

$\tau_g = 7/2$

$H(e^{jw}) = e^{-j7/2 w} \cdot e^{+j\pi/2} \cdot \tilde{H}(e^{jw})$

$\tilde{H}(e^{jw}) = 2 \sum_{n=1}^{\frac{N+1}{2}} h\left(\frac{N+1}{2} - n\right) \cdot \sin w(n - 1/2)$

$w = 0 \Rightarrow$ response is zero $\Rightarrow$ L.P. not possible.

Eg. $h(n) = \begin{cases} 1, & 0 \le n \le 4. \\ 0, & \text{otherwise.} \end{cases}$

$H(z) = \sum_{n=0}^{4} h(n) \cdot z^{-n}$

$= \dfrac{1 - e^{-j5w}}{1 - e^{-jw}}$ $\left[\sum_{n=0}^{N} r^N = \dfrac{1 - r^{N+1}}{1 - r} \right]$

$\hookrightarrow$ odd, symmetric $\Rightarrow$ Linear phase (Type- I)

$1 - e^{-jNw} = e^{-jNw/2} \left[e^{+jwN/2} - e^{-jNw/2} \right]$: Euler Identity

$= e^{-jNw/2} \cdot 2j \sin wN/2$

$\Rightarrow H(z) = \dfrac{e^{-j5w/2} \cdot 2j \sin 5w/2}{e^{-jw/2} \cdot 2j \sin w/2}$

$= e^{-j2w} \dfrac{\sin 5w/2}{\sin w/2}$

$\phi(w) = -2w$

$\Rightarrow \tau_g = 2 \rightarrow$ Type- I.

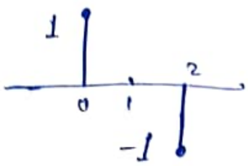
eg. $h(n) = \begin{cases} 1, & 0 \le 1 \le 5 \\ 0, & \text{otherwise.} \end{cases}$

$H(z) = \sum_{n=0}^{5} h(n) \cdot z^{-n}$

$= \dfrac{1 - e^{-j6w}}{1 - e^{-jw}} = \dfrac{e^{-j3w} \cdot 2j \sin 3w}{e^{-jw/2} \cdot 2j \sin w/2} = e^{-j5/2 w} \cdot \dfrac{\sin 3w}{\sin w/2}$

$\hookrightarrow \tau_g = 5/2$

Eg. $h(n) = \delta(n) - \delta(n-2)$



$$H(e^{j\omega}) = 1 - e^{-j\omega 2}$$
$$= e^{-j\omega}\left[e^{j\omega} - e^{-j\omega}\right]$$
$$= e^{-j\omega} \cdot 2j\sin\omega \rightsquigarrow \text{Purely imaginary (odd, anti-symm.) (Type-3)}$$

$\hookrightarrow \tau_g = 1.$ $\hookrightarrow$ 0 for $\omega = 0, \pi$

Eg. $h(n) = \delta(n) - \delta(n-1)$

$$H(e^{j\omega}) = 1 - e^{-j\omega}$$
$$= e^{-j\omega/2}\left[e^{j\omega/2} - e^{-j\omega/2}\right]$$
$$= e^{-j\omega/2} \cdot 2j\sin\omega/2 \longrightarrow 0. \text{ for } \omega = 0.$$

- Except for the phase response, all the four types of systems satisfy linearity constraint.

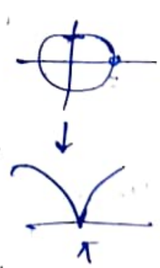
Inferences from poles and zeros:

FIR: Freq. response is primarily determined from zeros.

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}.$$

$\hookrightarrow$ $(N-1)$ zeros.

- If z_0 is a zero, then $\frac{1}{z_0^*}$ is also a zero [For symmetric/anti-symm.]
- If zero is on the unit circle at the zeroth position, then there'll be a notch the response at π.



25-08-2025

Interpretation of Zeros of FIR Linear Phase System

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n} \quad [\text{Type-1 and } 2]$$

FIR → freq. response is primarily determined by the zeros

$$h(n) = h(M-n)$$

$$H(z) = z^{-m} H(z^{-1})$$

If z_0 is a zero, then there is a zero at z_0^{-1}.

$z_0 = re^{j\theta}$

If $h(n)$ is real, each complex zero(s) not on unit circle will be a part of 4 complex zeros.

$$(1-re^{-j\theta_0}z^{-1})(1-re^{-j\theta_1}z^{-1})(1-re^{-j\theta_2}z^{-1})(1-re^{-j\theta_3}z^{-1})$$

Zeros is on unit circle

$$(1-e^{-j\theta}z^{-1})(1-e^{j\theta}z^{-1})$$

If the zero is at $z=\pm 1$,
then $(1 \pm z^{-1})$

Eg. $h(n) = \{1, 2, 1\}$ [odd-symmetric : Type-I]

$$\Rightarrow H(z) = 1 + 2z^{-1} + z^{-2} = z^{-1}(z + 2 + z^{-1}) = \frac{z^2+2z+1}{z^2} = \frac{(z+1)^2}{z^2}$$

↳ Pole : $z = 0$
Zeros : $z = -1, -1$.

$$H(e^{j\omega}) = e^{-j\omega}[2 + 2\cos\omega]$$

$\omega = 0 \longrightarrow |H(e^{j\omega})| = 4$

$\omega = \pi \longrightarrow |H(e^{j\omega})| = 0$.

- when there is zero at ± 1, there will be a notch at π.

$\Rightarrow H(z) = z^{-M} H(z^{-1})$

If $z = -1$; $z = 1$

- If M is even, we have the same identity ; but if M is ~~even~~ odd, $z = -1$ should be a zero.
- $H(z)$ must be a zero at $z = 1$ for both M as even or odd.

$\rightarrow z = -1,$

$$H(-1) = (-1)^{M-1} H(-1)$$

$z = -1$ must be a zero for odd.

Linear Phase System (FIR):

$$H(z) = H_{min}(z)\, H_{max}(z)\, H_{uc}(z)$$

$H_{uc}(z) \rightarrow$ complex zeros only on the unit circle.

$$H_{min}(z) = H_{min}(z^{-1})\, z^{-M_i}$$

$M_i \rightarrow$ No of zeros in $H_{min}(Z)$

Order $= 2M_i + M_o$.

Eg. $H(z) = 1 - 0.5z^{-1}$

$= \dfrac{z - 1/2}{z}$

$H(z) = 1 - 2z^{-1}$

$= \dfrac{z - 2}{z}$

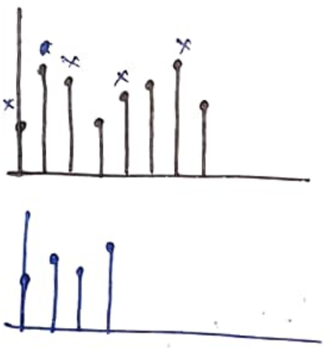
- Same magnitude response, different phase response.

Multi Rate System

Sampling
- Single rate
- Multi-rate
  - Decimation (down sampling)
  - Interpolation (up sampling)

~~Ad~~ Audio
- Broad casting: 32 KHz
- Digital CD: 44.1 KHz
- DAT: 48 KHz

Decimation



$\rightarrow$ Decimator $\rightarrow$

$$x_d = \begin{cases} x[2n], & n = 0, 1, 2, \dots \\ 0, & \text{otherwise} \end{cases}$$

Time Domain Representation of Interpolation (Upsampler)

26-08-2025

$\hookrightarrow$ Inserting zeros

Eg. $x(n) = \{1, 2, 5, 7, 9, 10\}$ $\quad [L=2] \rightarrow$ Interpolation factor

$x_u(n) = \{1, 0, 2, 0, 5, 0, 7, 0, 9, 0, 10, \dots\}$ $\rightarrow$ Insert $(L-1)$ zeros in between.

$$x_u(n) = \begin{cases} x\left(\frac{n}{L}\right), & n = 0, \pm L, \pm 2L \\ 0, & \text{otherwise} \end{cases}$$

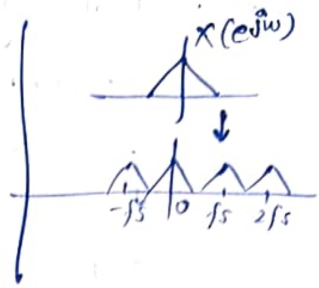
$x(n) \longrightarrow \boxed{\downarrow D} \longrightarrow x_d(n)$

i/p samping freq. : $f_T = \frac{1}{T}$ $\qquad f_T' = \frac{f_T}{D}$

$x(n) \longrightarrow \boxed{\uparrow L} \longrightarrow x_u(n)$

$f_T = \frac{1}{T}$ $\qquad f_T' = L f_T$

Frequency Domain Characteristics of Upsampler :



$$x_u(n) = \begin{cases} x\left(\frac{n}{L}\right), & n = 0, \pm L, \pm 2L \\ 0, & \text{otherwise} \end{cases}$$

For $L=2$,

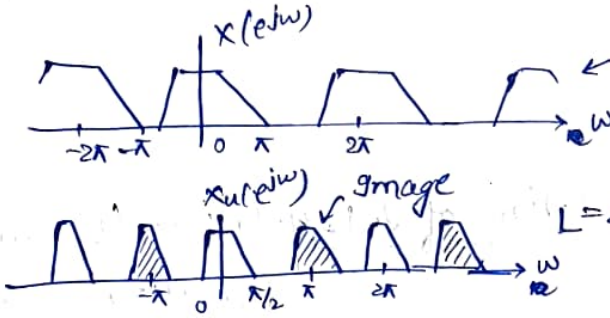
$$X_u(z) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2}\right) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(k)\, z^{-2k} \qquad \left[\frac{n}{L} = k \Rightarrow n = kL\right]$$

$$X_u(z) = X(z^2)$$

In general,

$$X_u(z) = X(z^L)$$



← Already sampled
↳ So repeating by L (=2)

L=2 ↔ compressed (shrunk) & periodically repeating

→ Frequency domain, (L-1) images are introduced.

→ The images can be eliminated by filtering [cutoff freq. $= \frac{\pi}{L}$]

Frequency Domain Characteristics of Downsampler :

$$x_d(n) = \begin{cases} x[Mn], & n = 0, \pm 1, \pm 2, \ldots \\ 0, & \text{otherwise} \end{cases} ; \quad M: \text{Decimation factor}$$

$$X_d(z) = \sum_{n=-\infty}^{\infty} x(Mn)\, z^{-n}$$

$$x_{int}(n) = \begin{cases} x(n), & n = 0, \pm M, \pm 2M \\ 0, & \text{otherwise} \end{cases}$$

intermediate

$$X_d(Z) = \sum_{n=-\infty}^{\infty} x_{int}(Mn)\, Z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x_{int}(K)\, Z^{-K/M} \qquad \begin{bmatrix} Mn = K \\ \Rightarrow n = K/M \end{bmatrix}$$

$$= X_{int}\left(Z^{1/M}\right)$$

$$x_{int}(n) = c(n)\, x(n)$$

$$c(n) = \begin{cases} 1, & n = 0, \pm M, \pm 2M, \ldots \\ 0, & \text{otherwise} \end{cases}$$

$$c(n) = \frac{1}{M} \sum_{K=0}^{M-1} W_M^{Kn}, \quad W_M = e^{-j2\pi/M}$$

$$X_{int}(Z) = \sum_{n=-\infty}^{\infty} c(n)\, x(n)\, Z^{-n}$$

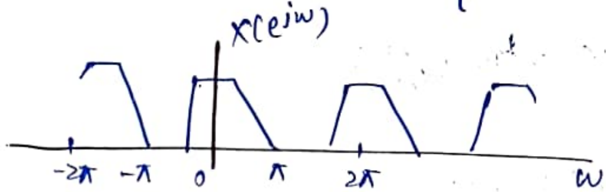
$$= \sum_{n=-\infty}^{\infty} \frac{1}{M} \sum_{K=0}^{M-1} W_M^{Kn}\, x(n)\, Z^{-n}$$

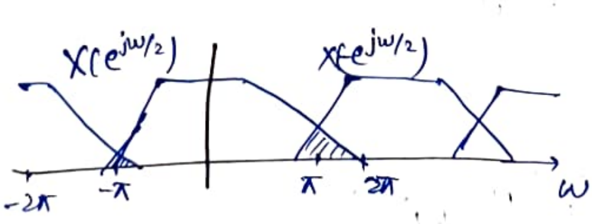
$$= \frac{1}{M} \sum_{K=0}^{M-1} X\left(Z W_M^{-K}\right)$$

$$X_d(Z) = Y(Z) = \frac{1}{M} \sum_{K=0}^{M-1} X\left(Z^{1/M} \cdot W_M^{-K}\right)$$

For $M = 2$,

$$Y(e^{j\omega}) = \frac{1}{2}\left[X(e^{j\omega/2}) + X(-e^{-j\omega/2})\right]$$

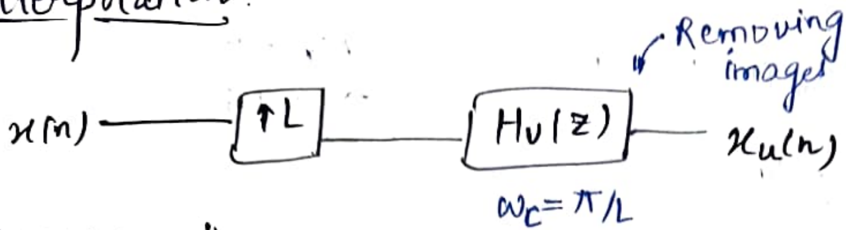




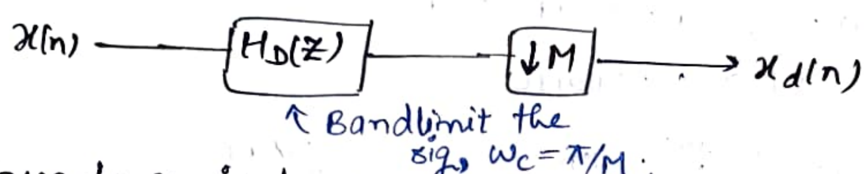
→ The impact of downsampling is aliasing.

→ Aliasing can be overcome with proper "band-limiting" of i/p signal ; $|\omega| \leq \pi/M$.

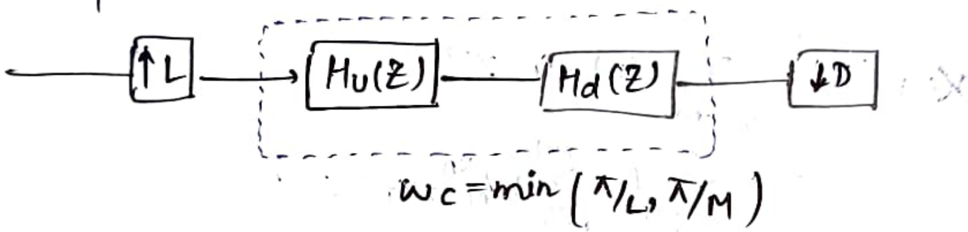
Interpolation:

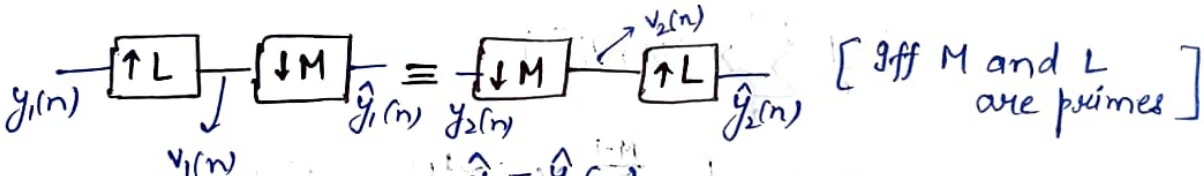


Downsampling:



Cascade equivalence:





$$\hat{y}_1 = \hat{y}_2(n)$$

$$V_1(z) = X(z^L)$$

$$\hat{Y}_1(z) = \frac{1}{M}\sum_{k=0}^{M-1} V_1(z^{1/M} W_M^{-k})$$

$$= \frac{1}{M}\sum_{k=0}^{M-1} X(z^{L/M} \cdot W_M^{-k}).$$

$$V_2(z) = \frac{1}{M}\sum_{k=0}^{M-1} X(z^{1/M} W_M^{-k})$$

$$\hat{Y}_2(z) = V_2(z^L)$$

$$= \frac{1}{M}\sum_{k=0}^{M-1} X(z^{L/M} \cdot W_M^{-kL})$$

$$\therefore \hat{Y}_1(z) = \hat{Y}_2(z).$$

27-08-2025

Eg. $x(n) = \{1, 3, 5, 7, 9, 10, 21, -23, 3, 2\}$

Case I → $L = 2, M = 3$

$x_1 = \{1, 0, 3, 0, 5, 0, 7, 0, \ldots\}$

$x_2 = \{1, 0, 7, 0, \ldots\}$

→ $M = 3, L = 2$

$x_1 = \{1, 7, 21, 2\}$

$x_2 = \{1, 0, 7, 0, 21, 0, 2\}$

Case II → $L = 2, M = 4$

$x_1 = \{1, 0, 3, 0, 5, 0, 7, 0, 9, 0, 10, 0, 21, 0, \ldots\}$

$x_2 = \{1, 5, 9, 21\}$

→ $M = 4, L = 2$

$x_1 = \{1, 9, 3\}$

$x_2 = \{1, 0, 9, 0, 3\}$

↳ Not same (has to be prime)

Spectrum of Sampling Rate converters

$$x(n) \rightarrow \boxed{\uparrow L} \xrightarrow{v(n)} \boxed{h(n)} \xrightarrow{w(n)} \boxed{\downarrow M} \rightarrow y(n)$$

$$V(e^{j\omega}) = X(e^{j\omega L})$$

$$H(e^{j\omega}) = \begin{cases} L, & 0 \text{ to } \omega_c \\ 0, & \text{otherwise} \end{cases}, \quad \omega_c = \min(\pi/L, \pi/M).$$

$$W(e^{j\omega}) = X(e^{j\omega L})\, H(e^{j\omega}) = L\, X(e^{j\omega L})$$

$$Y(e^{j\omega}) = \frac{1}{M} \sum_{k=0}^{M-1} W\left(e^{j\frac{(\omega - 2\pi k)}{M}}\right)$$

Since, the signal is band-limited, the aliasing can be eliminated.

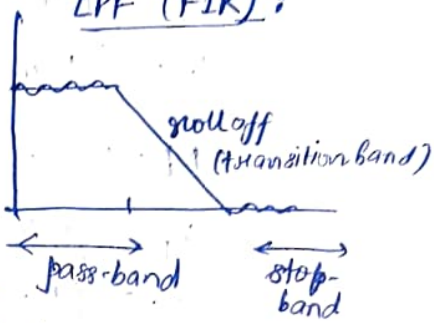
$$\therefore\ Y(e^{j\omega}) = \frac{1}{M} \cancel{\sum_{k=0}^{M-1}} W\left(e^{\frac{j\omega}{M}}\right) \quad (\text{at } k = 0)$$

$\therefore Y(e^{j\omega}) = \frac{L}{M} X(e^{j\omega L/M})$, for $\omega = 0$ to ω_y

$L \rightarrow$ higher
$M \rightarrow$ higher

Impact on the filter design
$\frac{\pi}{M}, \frac{\pi}{L}$.

LPF (FIR):



Filter specifications

$N\uparrow$ (order) $\Rightarrow$ Computational complexity $\uparrow$

Multistage Interpolation/Decimation:

$L = L_1 L_2 L_3 \cdots L_n \Rightarrow \prod_{i=1}^{n} L_i = L$

$D = D_1 D_2 D_3 \cdots D_n \Rightarrow \prod_{i=1}^{n} D_i = D$

$\rightarrow \boxed{\uparrow L_1} \rightarrow \boxed{h_1(n)} \rightarrow \boxed{\uparrow L_2} \rightarrow \boxed{h_2(n)} \rightarrow$

$\rightarrow \boxed{h_1(n)} \rightarrow \boxed{\downarrow D_1} \rightarrow \boxed{h_2(n)} \rightarrow \boxed{\downarrow D_2} \rightarrow$

Passband : $0 \le F \le F_{pc}$

Stopband : $F_{pc} \le F \le F_{sc}$

$F_{sc} \le \frac{F_x}{2D}$ $\rightarrow$ i/p freq.

Transition band : $F_{pc} \le F \le F_i - F_{sc}$

Stop band : $F_i - F_{sc} \le F \le \frac{F_{i-1}}{2}$.

1st ~~stage~~ filter stage : $F_1 = \frac{F_x}{D_1}$.

Passband : $0 \le F \le F_{pc}$

Transition band : $F_{pc} \le F \le F_i - F_{sc}$

Stop band : $F_i - F_{sc} \le F \le f_0/2$.

Eg. Consider a audio s/g with BW $= 4$ KHz that has to be sampled at 8KHz. Suppose we have to isolate freq. component below 80 KHz with a filter with passband specified as $0 \le F \le 75$ Hz and transition band $75 \le F \le 80$ Hz.

[$F_{pc} = 75$ Hz, $F_{sc} = 80$ Hz]

Decimation factor: $D = F_x / 2F_{sc} = \frac{8000}{2 \times 80} = 50$

The filter should have a passband ripple $\underset{(\delta p)}{\delta_1} = 10^{-2}$ and stopband ripple $\underset{(\delta s)}{\delta_2} = 10^{-4}$.

Soln: $N = \frac{-10 \log(\overset{\delta p}{\delta_1} \overset{\delta s}{\delta_2}) - 13}{14.6 \, \Delta f} + 1$: Single-stage approach. [Kaiser Formula]

length of FIR filter

$\Delta f = \frac{f_{sc} - f_{pc}}{f_x} = \frac{80 - 75}{8000} = \frac{5}{8000}$

$\therefore N = \frac{-10 \log(10^{-6}) - 13}{14.6 \left(\frac{5}{8000}\right)}$

$= \frac{60 \log 10 - 13}{\frac{14.6 \times 5}{8000}} = 2575 \times 2 \rightarrow$ High computation complexity

Multi-stage approach:

$D = 50 = 25 \times 2$

$$\underset{f_x}{8000\text{ Hz}} \longrightarrow \underset{D_1 = 25}{\boxed{\downarrow 25}} \xrightarrow{320\text{ Hz}} \underset{D_2 = 2}{\boxed{\downarrow 2}} \longrightarrow 160\text{ Hz}$$

1st stage:

Passband: $0 \le F \le 75$ Hz

Trans. band: $75 \le f \le F_1 - F_{sc}$
$= 320 - 80 = 240$ Hz

$N_1 = \frac{-10 \log_{10}(\overset{(\delta p/2)}{\delta_1'} \overset{(\delta s)}{\delta_2'}) - 13}{\Delta f \times 14.6} + 1$ | $\delta_1' = \delta p/2$, $\delta_2' = \delta s$

$\Delta f = \frac{240 - 75}{8000} = 0.020625$

$\therefore N_1 = \frac{-10 \log(\frac{1}{2} \times 10^{-6}) - 13}{0.020625 \times 14.6}$

≈ 167

2nd stage:

Passband : $0 \le f \le 75$

Transition Band : $75 \le f \le f_2 - f_{sc}$

$= 160 - 80 = 80\ Hz$

$$\therefore N = \frac{-10\log_{10}(\delta_p'' \delta_s'') - 13}{14.6\ \Delta f} + 1$$

$\simeq 220$

$\delta_p'' = \delta_p/2$

$\delta_s'' = \delta_s$

$\Delta f = \frac{80-75}{320} = \frac{5}{320}$

01-09-2025

Filter Design

$\longrightarrow \boxed{h(n)} \longrightarrow \boxed{\downarrow D} \longrightarrow$

$\longrightarrow \boxed{\uparrow L} \longrightarrow \boxed{h(n)} \longrightarrow$

FIR Filter (Type)

↳ Reduce computational complexity

Polyphase Filter

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

Divides an 'N' order filter into L subsections of subfilters that can be realised in parallel. In this decomposition, the subfilters will differ only in phase. $\longrightarrow$ Reduce delays

L=2: $$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

$= h(0) + h(1) z^{-1} + h(2) z^{-2}$

$= h(0) + h(2) z^{-2} + \dots \quad + \quad h(1) z^{-1} + h(3) z^{-3} + \dots$

$= h(0)(z^2)^0 + h(2)(z^2)^{-1} + \dots + z^{-1}\left[h(1)(z^2)^0 + h(3)(z^2)^{-1} + \dots\right]$

$$H(z) = E_0(z^2) + z^{-1} E_1(z^2),$$

$$E_0(z^2) = h(0) + h(2)(z^2)^{-1}$$

$$E_1(z^2) = h(1) + h(3)(z^2)^{-1}$$

$L = 3:$

$$E_0(z^3) = h(0) + h(3)(z^3)^{-1}$$

$$E_1(z^3) = h(1) + h(4)(z^3)^{-1}$$

$$E_2(z^3) = h(2) + h(5)(z^3)^{-1}$$

$$E_0(z) = h(0) + h(2) z^{-2}$$

$$E_1(z) = h(1) + h(3) z^{-2}$$

<u>'L'-section polyphase filter</u>: (if L-order)

$$H(z) = E_0(z^L) + z^{-1}E_1(z^L) + z^{-2}E_2(z^L) + \dots + z^{-(L-1)}E_{L-1}(z^L).$$

$$\boxed{H(z) = \sum_{l=0}^{L-1} E_l(z^L)\, z^{-l}}$$

: Type-1 Realisation
Polyphase Representation

$$H(z) = \sum_{K=-\infty}^{\infty} h(2K) z^{-2K} + z^{-1} \sum_{K=-\infty}^{\infty} h(2K+1) z^{-2K}$$

$$E_0(z) = \sum_{K=-\infty}^{\infty} h(2K) z^{-K}$$

$$E_1(z) = \sum_{K=-\infty}^{\infty} h(2K+1) z^{-K}$$

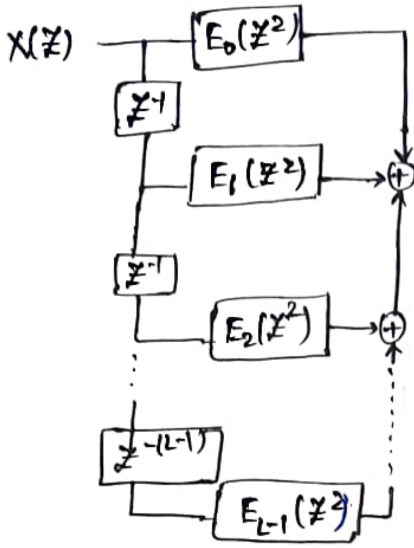
Eg $H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 7z^{-4}$

$$E_0 = 1 + 3z^{-1} + 7z^{-2}$$

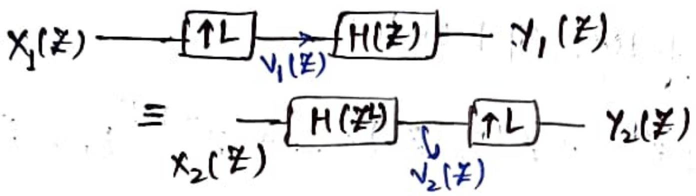
$$E_1 = 2 + 4z^{-1}$$

$$X(z) \longrightarrow \boxed{H(z)} \longrightarrow Y(z)$$

$L = 2:$ $H(z) = E_0(z^2) + z^{-1} E_1(z^2)$



(L-subsections.)

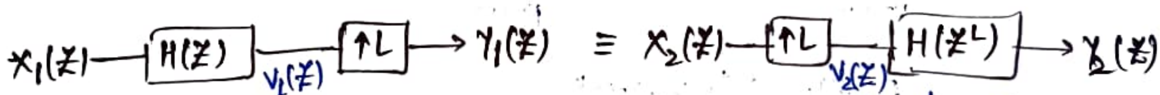


LHS: $V_1(Z) = X_1(Z^L)$

$Y_1(Z) = X_1(Z^L)\,H(Z)$

RHS: $V_2(Z) = X_2(Z)\,H(Z^L)$

$Y_2(Z) = V_2(Z^L) = X_2(Z^L)\,H$



↳ Polyphase Response

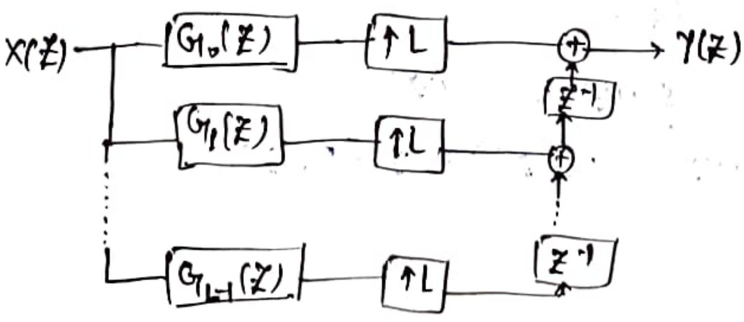
LHS: $V_1(Z) = X_1(Z)H(Z)$

$Y_1(Z) = V_1(Z^L) = X_1(Z^L)H(Z^L)$

RHS: $V_2(Z) = X_2(Z^L)$

$Y_2(Z) = X_2(Z^L)\,H(Z^L)$

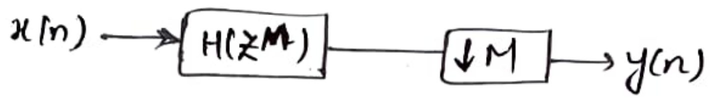
LHS = RHS if $X_1(Z) = X_2(Z)$

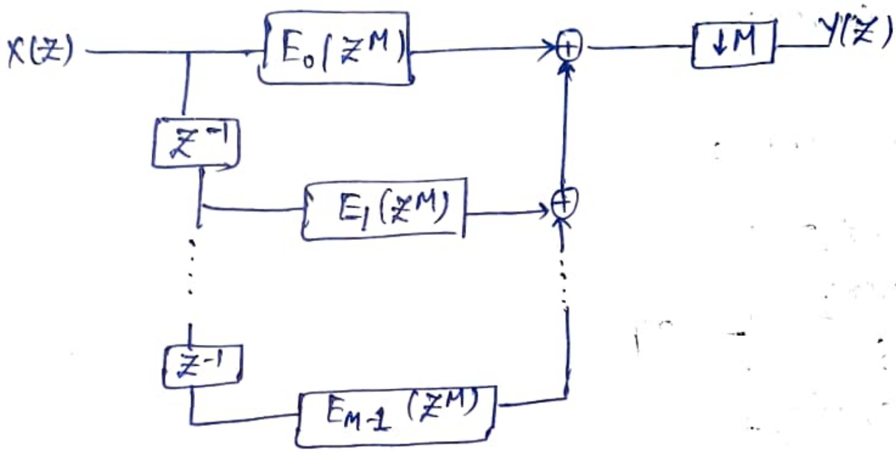


Noble Identity for Decimator

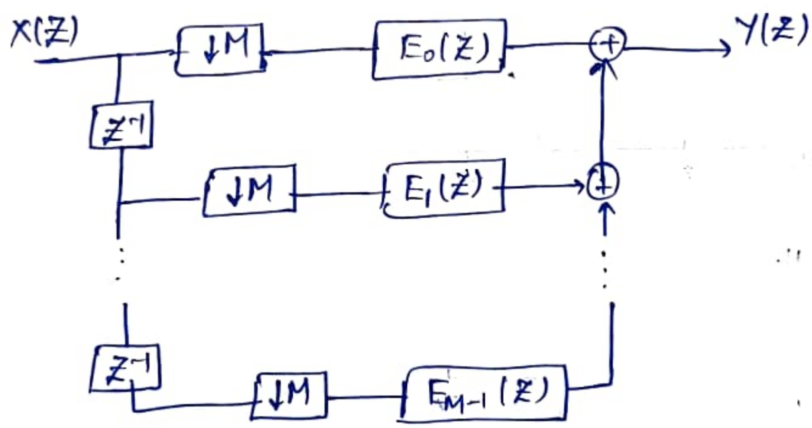
→ computationally efficient

$\equiv$





$\equiv$

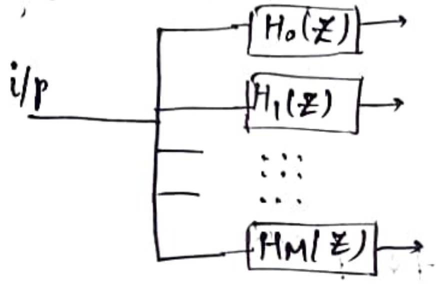


$$H(z) = \sum_{k=0}^{M-1} z^{-k} E_k(z^M) \quad \ldots \text{Type I}$$

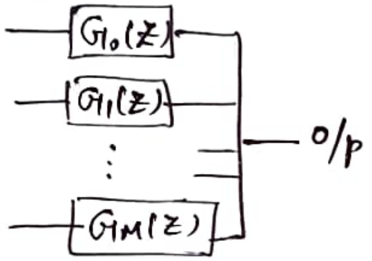
$$\left.\begin{aligned} H(z) &= \sum_{l=0}^{M-1} z^{-(M-l-1)} R_l(z^M) \\ R_l(z^M) &= E_{M-1-l}(z^M), \quad 0 \le l \le M-1 \end{aligned}\right\} : \text{Type II}$$

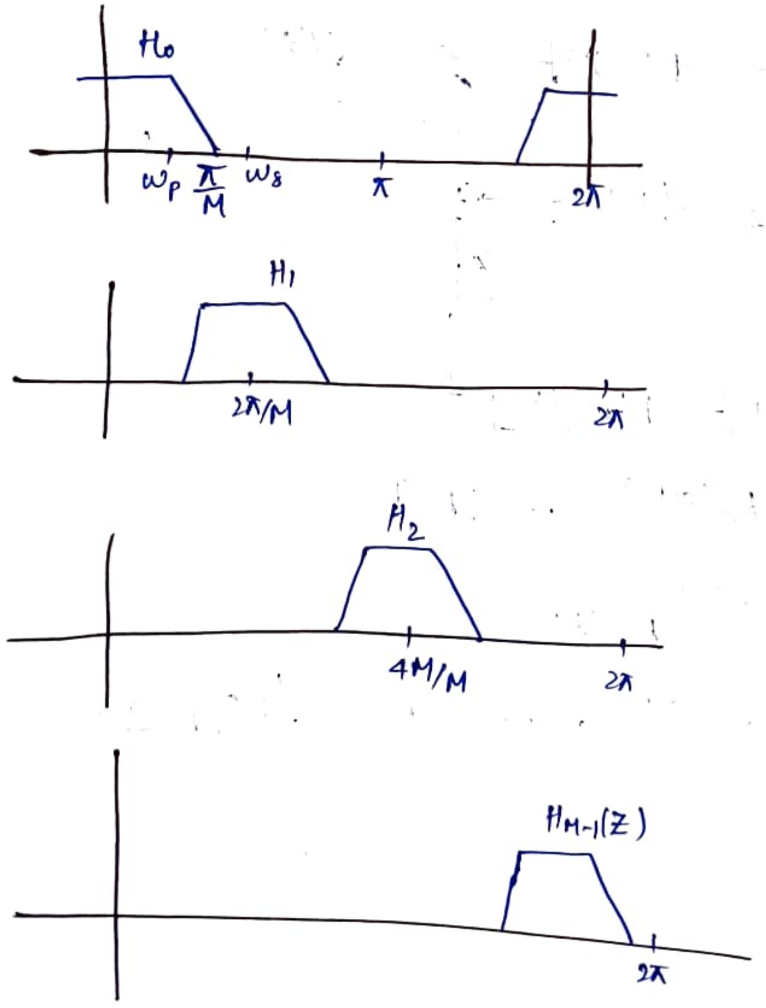
Filter Bank (FB)

Analysis filter bank: common input feeding all the filters.



Synthesis filter bank:





Uniform DFT FB :

$h_0(n) \rightarrow$ prototype

$h_K(n) = h_0 W_M^{-Kn}$,

$$W_M^{+Kn} = e^{-j2\pi \frac{Kn}{M}}$$

$$H_K(Z) = \sum_{n=0}^{\infty} h_K(n) Z^{-n}$$

$$= \sum_{n=0}^{\infty} h_0(n) W_M^{+Kn} \cdot Z^{-n}$$

$$= \sum_{n=0}^{\infty} h_0(n) (Z W_M^{+K})^{-n}$$

$\therefore H_K(Z) = H_0(Z W_M^{+K})$: Modulation of prototype

$$\Rightarrow H_K(Z) = H_0\left(e^{j\omega} e^{-j\frac{2\pi K}{M}}\right)$$

$$\Rightarrow H_K(Z) = H_0\left(e^{j(\omega - \frac{2\pi K}{M})}\right)$$

The frequency response of $H_K(Z)$ is obtained by shifting $H_0(Z)$ by $\left(\frac{2\pi K}{M}\right)$.

$$|H_K(Z)| = \left| H_0\left(e^{j(\omega - \frac{2K\pi}{M})}\right)\right|$$

Polyphase representation of UFB :

$$H_0(Z) = \sum_{l=0}^{M-1} Z^{-l} \cdot E_l(Z^M),$$

$$E_l(Z) = \sum_{n=0}^{\infty} e_l(n) Z^{-n}$$

$$H_K(Z) = H_0(Z W_M^K) \qquad [Z \rightarrow Z W_M^K]$$

$$H_K(Z) = \sum_{l=0}^{M-1} (Z W_M^K)^{-l} \cdot E_l((Z W_M^K)^M)$$

$$= \sum_{l=0}^{M-1} Z^{-l} W_M^{-Kl} \cdot E_l(Z^M \cdot \underbrace{W_M^{KM}}_{=1})$$

Phasor

$$\left[\begin{array}{l} W_M^{KM} = 1, \\ 0 \le K \le M-1 \end{array}\right]$$

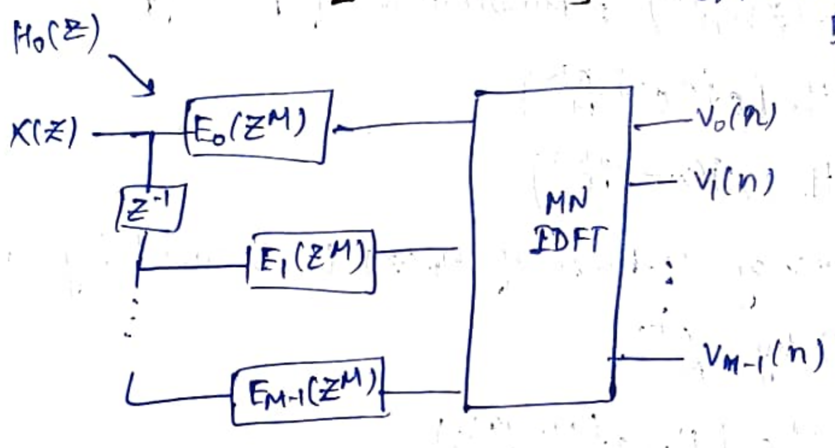
$$H_K(Z) = \begin{bmatrix} 1 & W_M^{-K} & W_M^{-2K} & \cdots & W_M^{-(M-1)K} \end{bmatrix} \begin{bmatrix} Z^0 E_0(Z^M) \\ Z^{-1} E_1(Z^M) \\ \vdots \\ Z^{-(M-1)} E_{M-1}(Z^M) \end{bmatrix}$$

→ DFT matrix

$$\mathbb{D} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & W_M^{+1} & W_M^{2} & & & W_M^{M-1} \\ \vdots & W_M^{2} & W_M^{4} & & & W_M^{2(M-1)} \\ & \vdots & & & & \vdots \\ 1 & & & & & W_M^{(M-1)^2} \end{bmatrix}$$

$$\begin{bmatrix} H_0(z) \\ H_1(z) \\ \vdots \\ H_{M-1}(z) \end{bmatrix} = M\mathbb{D}^{-1} \begin{bmatrix} E_0(z^M) \\ z^{-1}E_1(z^M) \\ \vdots \\ z^{-(M-1)}E_{M-1}(z^M) \end{bmatrix}$$

DFT → radix $\frac{M}{2}\log_2 M$



08-09-2025

DFT → Computational complexity $\}$ $M \times N$

Polyphase → $\frac{M}{2}\log_2 M + N$

Distortions in Filter Banks:

(i) Phase distortion.
(ii) Magnitude distortion
(iii) Aliasing

$M=4$:

$$\left[\begin{array}{c} H_0(z) \\ H_1(z) \\ H_2(z) \\ H_3(z) \end{array}\right] = \overset{M \cdot D^{-1} = 4D^{-1}}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}} \begin{bmatrix} E_0(z^4) \\ z^{-1}E_1(z^4) \\ z^{-2}E_2(z^4) \\ z^{-3}E_3(z^4) \end{bmatrix}$$

$$\begin{bmatrix} E_0(z^4) \\ z^{-1}E_1(z^4) \\ z^{-2}E_2(z^4) \\ z^{-3}E_3(z^4) \\ \vdots \\ z^{-(M-1)}E_{M-1}(z^M) \end{bmatrix} = \frac{1}{M} D \begin{bmatrix} H_0(z) \\ H_1(z) \\ H_2(z) \\ H_3(z) \\ \vdots \\ H_{M-1}(z) \end{bmatrix} = \frac{1}{M} D \begin{bmatrix} H_0(z) \\ H_0(zW_M^1) \\ H_0(zW_M^2) \\ H_0(zW_M^3) \\ \vdots \\ H_0(zW_M^{M-1}) \end{bmatrix}$$

Eg. Let $H(z) = \dfrac{a + bz^{-1}}{1 + cz^{-1}}$, $|c| < 1$.

Find $E_0(z)$, $E_1(z)$ and $E_2(z)$.

Soln: $M=3$

$$\begin{bmatrix} E_0(z^3) \\ z^{-1}E_1(z^3) \\ z^{-2}E_2(z^3) \end{bmatrix} = \frac{1}{3} D \begin{bmatrix} H_0(z) \\ H_0(zW_M^1) \\ H_0(zW_M^2) \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 & 1 \\ 1 & W_3^1 & W_3^2 \\ 1 & W_3^2 & W_3^4 \end{bmatrix}$$

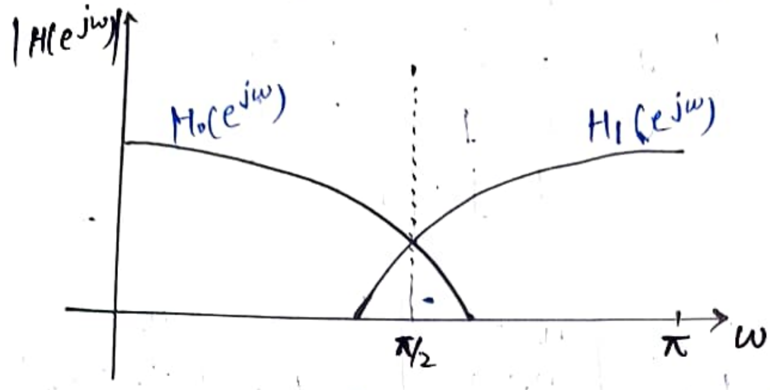
$$E_0(z^3) = \frac{1}{3}\left[H_0(z) + H_0(zW_3^1) + H_0(zW_3^2)\right]$$

$$W_3^1 = e^{-j\frac{2\pi}{3}}$$

$$W_3^2 = e^{-j\frac{4\pi}{3}}$$

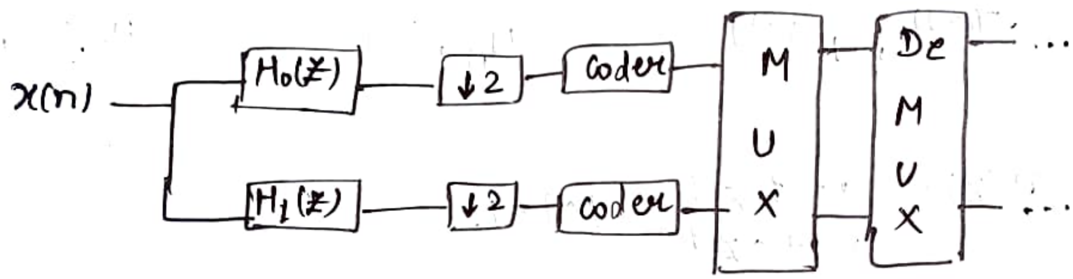
$$\Rightarrow E_0(z^3) = \frac{1}{3}\left[\frac{a + bz^{-1}}{1 + cz^{-1}} + \frac{a + be^{j\frac{2\pi}{3}}z^{-1}}{1 + ce^{j\frac{2\pi}{3}}z^{-1}} + \frac{a + be^{j\frac{4\pi}{3}}z^{-1}}{a + be^{j\frac{4\pi}{3}}z^{-1}}\right]$$

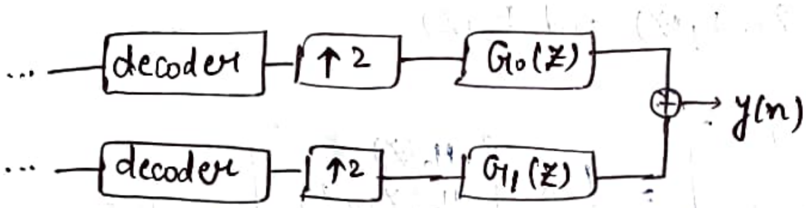
Quadrature Filter Bank: (QMF: Quadrature Mirror Filter)



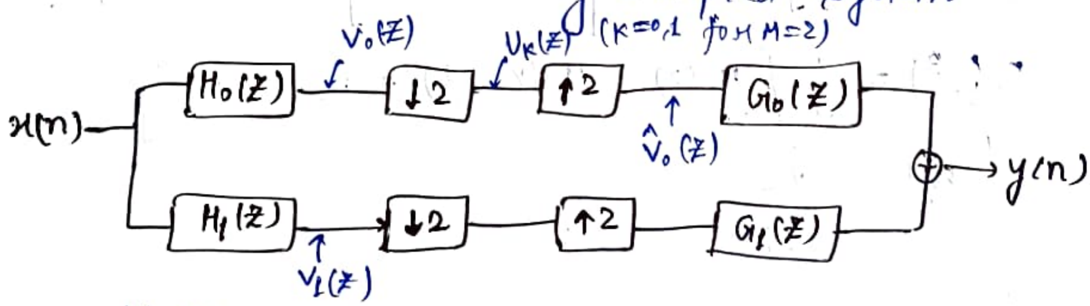
↳ Distortions: Magnitude, phase, aliasing

Application: coder, decoder system.





↳ $M = 2$: Critically sampled system



$$V_0(Z) = X(Z)\,H_0(Z)$$

$$V_K(Z) = X(Z)\,H_K(Z)$$

$$\hat{V}_K(Z) = U_K(Z^2)$$

$$U_K(Z) = \frac{1}{2}\left[V_K(Z^{1/2}) + V_K(-Z^{1/2})\right]$$

$$= \frac{1}{2}\left[H_K(Z^{1/2})X(Z^{1/2}) + H_K(-Z^{1/2})X(-Z^{1/2})\right]$$

$$Y(Z) = G_0(Z)\hat{V}_0(Z) + G_1(Z)\hat{V}_1(Z)$$

$$= \frac{1}{2}\left[H_0(Z)G_0(Z)X(Z) + H_0(-Z)G_0(+Z)X(-Z)\right]$$

$$+ \frac{1}{2}\left[H_1(+Z)G_1(Z)X(Z) + H_1(-Z)G_1(Z)X(-Z)\right]$$

$$Y(Z) = \overbrace{\frac{1}{2}\left[H_0(Z)G_0(Z) + H_1(Z)G_1(Z)\right]}^{\text{distortion term} \;:\; (T(Z))} X(Z)$$

$$+ \underbrace{\frac{1}{2}\left[H_0(-Z)G_0(Z) + H_1(-Z)G_1(Z)\right]}_{\text{aliasing term} \;:\; A(Z)} X(-Z)$$

$$Y(Z) = T(Z)X(Z) + A(Z)X(-Z)$$

Suppose $A(Z) = 0$,

$$Y(Z) = T(Z)X(Z)$$

→ ~~Usa~~ Upsampling and downsampling are time-varying system.

$$H_0(-Z)G_0(Z) + H_1(-Z)G_1(Z) = 0$$

$$Y(Z) = T(Z)X(Z)$$

$$= |T(e^{j\omega})|\, e^{j\phi(\omega)} X(e^{j\omega})$$; $T(Z)$ is made linear phase

$$|T(e^{j\omega})| = k,$$

$$Y(Z) = k X(e^{j\omega})$$

→ $T(Z)$ is made linear phase.

$\phi(\omega) = \alpha\omega + \beta$ → constant group delay.

09-09-2025

(Near) Perfect Reconstruction

$H_0(z) = 1 \qquad G_0(z) = z^{-1}$

$H_1(z) = z^{-1} \qquad G_1(z) = 1$

$$T(z) = \frac{1}{2}\left[H_0(z)G_0(z) + H_1(z)G_1(z)\right]$$

$$= z^{-1}$$

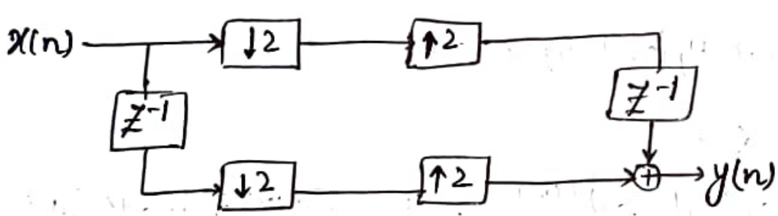
$$A(z) = \frac{1}{2}\left[H_0(-z)G_0(z) + H_1(-z)G_1(z)\right]$$

$$= 0$$

$\therefore Y(z) = z^{-1}X(z)$

$\Rightarrow y(n) = x(n-1)$

$\therefore$ O/p is delayed replica of i/p.



→ structure for perfect reconstruction

Conditions for Near perfect Reconstruction (Generalised)

$$H_0(z)G_0(z) + H_1(z)G_1(z) = 2z^{-l}$$

$\Rightarrow T(z) = z^{-l}$

$\therefore Y(z) = z^{-l}X(z)$

$\therefore y(n) = x(n-l)$

Product Filters:

$P_0(z) = H_0(z)G_0(z) \longrightarrow$ both are LPF

$P_1(z) = H_1(z)G_1(z) \longrightarrow$ both are HPF.

$G_0(z) = H_1(-z)$

$G_1(z) = -H_0(-z)$

$\Rightarrow \quad P_1(Z) = H_1(Z)\, G_1(Z) = H_1(Z) \cdot -H_0(-Z)$

$P_0(Z) = H_0(Z)\, G_0(Z) = H_1(-Z) \cdot H_0(Z)$

$P_0(-Z) = H_0(-Z)\, H_1(Z)$

$\therefore \quad P_1(Z) = -P_0(-Z)$

$Y(Z) = T(Z)\, X(Z) + A(Z)\, X(-Z)$

$A(Z) = \frac{1}{2}\left[H_0(-Z) \cdot H_1(-Z) - H_1(-Z)\, H_0(-Z) \right] = 0$

$T(Z) = P_0(Z) - P_0(-Z) = 2Z^{-l}$

$A(Z) = 0 \quad , \quad T(Z) \rightarrow$ continuous delay.

If $\quad G_0(Z) = H_0(Z)$

$H_1(Z) = G_0(-Z) = H_0(-Z)$

$G_1(Z) = -H_1(Z) = -H_0(-Z)$

$T(Z) = \frac{1}{2}\left[H_0^2(Z) - H_0^2(-Z) \right]$

Computationally Efficient Polyphase structure:

$H_0(Z) = E_0(Z^2) + Z^{-1} E_1(Z^2)$

$H_1(Z) = E_0(Z^2) - Z^{-1} E_1(Z^2)$

Analysis FB:

$$\begin{bmatrix} H_0(Z) \\ H_1(Z) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} E_0(Z^2) \\ Z^{-1} E_1(Z^2) \end{bmatrix}$$

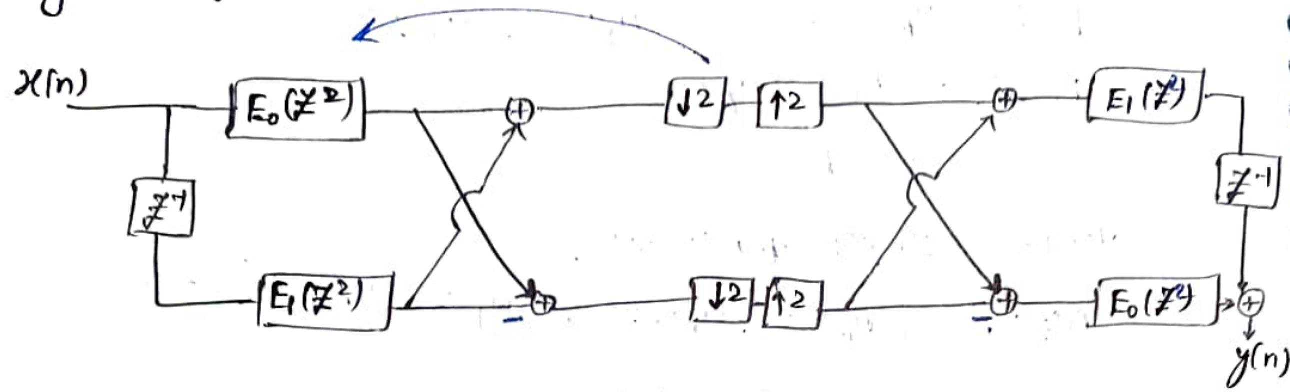
Synthesis FB:

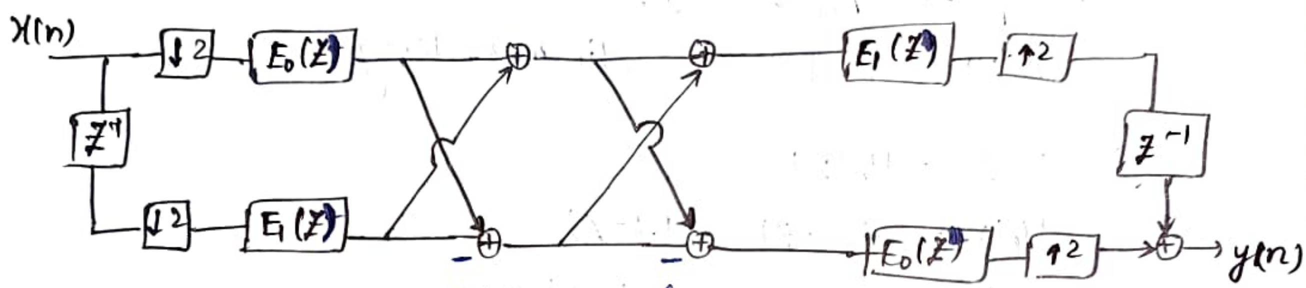
$$\begin{bmatrix} G_0(Z) & G_1(Z) \end{bmatrix} = \begin{bmatrix} Z^{-1} E_1(Z^2) & E_0(Z^2) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$T(Z) = \frac{1}{2}\left[\left[E_0(Z^2) + Z^{-1} E_1(Z^2) \right]^2 - \left[E_0(Z^2) - Z^{-1} E_1(Z^2) \right]^2 \right]$$

$$= 2Z^{-1} E_0(Z^2)\, E_1(Z^2).$$

Analysis & Synthesis FB :





↳ Computationally efficient 2 channel QMF.

Alias-free 'FIR' QMF

$H_0(z) \rightarrow$ linear phase [No phase distortion]

$E_0(z),\ E_1(z)$

$$T(z) = 2z^{-1} E_0(z^2) E_1(z^2)$$

$h_0(n) = h_0(N-n)$ [Type-I, II filter].

$$h_0(e^{j\omega}) = e^{-j\omega N/2} \cdot \hat{H}(\omega).$$

$$T(z) = \frac{1}{2}\left\{ H_0^2(z) - H_0^2(-z) \right\}$$

$$T(e^{j\omega}) = \frac{e^{-jN\omega}}{2} \left\{ |H_0(e^{j\omega})|^2 - (-1)^N |H_0(e^{j(\pi-\omega)})|^2 \right\}$$

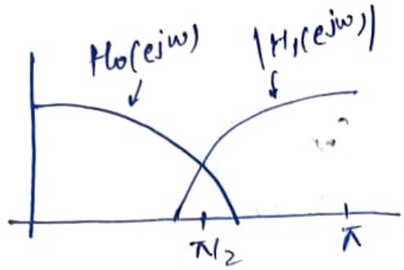
If N is even, $T(e^{j\omega}) = 0$ at $\omega = \pi/2$,

implying severe amplitude distortion.

$\therefore$ N has to be choosen odd.

$$T(e^{j\omega}) = \frac{e^{-jN\omega}}{2} \left\{ |H_0(e^{j\omega})|^2 + \underbrace{|H_0(e^{j(\pi-\omega)})|^2}_{H_1(e^{j\omega})} \right\}$$

$$= \frac{e^{-jN\omega}}{2} \left\{ |H_0(e^{j\omega})|^2 + |H_1(e^{j\omega})|^2 \right\}$$



Power complexity

$$|H_0(e^{jw})|^2 + |H_1(e^{jw})|^2 \cong 1.$$

Alias-free IIR QMF:

$T(z) = 2z^{-1} E_0(z^2) E_1(z^2)$

↳ all pass

$E_0(z) = \frac{1}{2} A_0(z)$

$E_1(z) = \frac{1}{2} A_1(z)$

$H_0(z) = \frac{1}{2}\left[A_0(z^2) + z^{-1} A_1(z^2)\right]$

$H_1(z) = \frac{1}{2}\left[A_0(z^2) - z^{2} A_1(z^2)\right]$

- $A_0(z), A_1(z)$ are "stable" all-pass filter

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Matrix Representation

$$Y(Z) = \frac{1}{2}\begin{pmatrix} G_0(Z) & G_1(Z) \end{pmatrix}\begin{bmatrix} H_0(Z) & H_0(-Z) \\ H_1(Z) & H_1(-Z) \end{bmatrix}\begin{bmatrix} X(Z) \\ X(-Z) \end{bmatrix}$$

$$Y(-Z) = \frac{1}{2}\begin{pmatrix} G_0(-Z) & G_1(-Z) \end{pmatrix}\begin{bmatrix} H_0(Z) & H_0(-Z) \\ H_1(Z) & H_1(-Z) \end{bmatrix}\begin{bmatrix} X(Z) \\ X(-Z) \end{bmatrix}$$

$$\begin{bmatrix} Y(Z) \\ Y(-Z) \end{bmatrix} = \frac{1}{2}\underbrace{\begin{bmatrix} G_0(Z) & G_1(Z) \\ G_0(-Z) & G_1(-Z) \end{bmatrix}}_{G^m(Z)}\underbrace{\begin{bmatrix} H_0(Z) & H_0(-Z) \\ H_1(Z) & H_1(-Z) \end{bmatrix}}_{[H^m(Z)]^t}\begin{bmatrix} X(Z) \\ X(-Z) \end{bmatrix}$$

Synthesis FB — Analysis FB

$$Y(Z) = \frac{1}{2} G^m(Z)\left[H^m(Z)\right]^t\begin{bmatrix} X(Z) \\ X(-Z) \end{bmatrix}$$

For PR:

$$Y(Z) = Z^{-l} X(Z)$$

$$Y(-Z) = (-Z)^{-l} X(-Z)$$

$$G^{(m)}(Z)\left[H^{(m)}(Z)\right]^t = 2\begin{bmatrix} Z^{-l} & 0 \\ 0 & (-Z)^{-l} \end{bmatrix}$$

$\therefore$ By knowing Analysis $H_0(Z)$, $H_1(Z)$, the synthesis filters can be found.

$$G^{(m)}(Z) = 2\begin{bmatrix} Z^{-l} & 0 \\ 0 & (-Z)^{-l} \end{bmatrix}\left[\left[H^{(m)}(Z)\right]^t\right]^{-1}$$

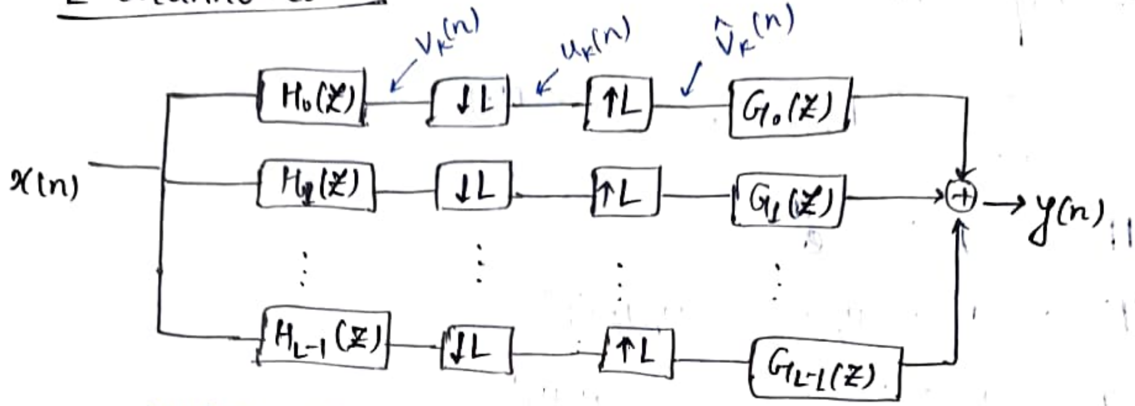
After manipulation, $l \to$ odd +ve no.

$$G_0(Z) = \frac{2Z^{-l}}{\det\left[H^{(m)}(Z)\right]} H_1(-Z) \quad \Rightarrow \quad G_0(Z) = \frac{2}{c} Z^{-(l-K)} H_1(-Z)$$

$$G_1(Z) = \frac{-2}{c} Z^{-(l-K)} H_0(-Z)$$

$$G_1(Z) = \frac{-2Z^{-l}}{\det\left[H^{(m)}(Z)\right]} H_0(-Z)$$

L-channel QMF:



$$V_k(z) = H_k(z)X(z)$$

$$\hat{V}_k(k) = U_k(z^L)$$

$$U_k(z) = \frac{1}{L}\sum_{l=0}^{L-1} H_k(z^{1/L}W_L^l)\, X(z^{1/L}W_L^l)$$

$$Y(z) = \sum_{k=0}^{L-1} G_k(z)\,\hat{V}_k(z)$$

$$\Rightarrow Y(z) = \sum_{k=0}^{L-1} G_k(z)\,U_k(z^L)$$

$$= \sum_{k=0}^{L-1} G_k(z)\left[\frac{1}{L}\sum_{l=0}^{L-1} H_l(zW_L^l)\,X(zW_L^l)\right] \rightarrow$$ Except for $l=0$, all other terms are aliasing terms.

$$= \sum_{k=0}^{L-1} \frac{1}{L}\sum_{l=0}^{L-1} H_k(zW_L^l)\,G_k(z)\,X(zW_L^l)]$$

$$a_l(z) = \frac{1}{L}\sum_{k=0}^{L-1} H_k(zW_L^l)\,G_k(z), \quad 0 \le l \le L-1.$$

$$Y(z) = \sum_{l=0}^{L-1} a_l(z)\,X(zW_L^l)$$

$$= a_0(z)X(z) + \underbrace{\sum_{l=1}^{L-1} a_l(z)\,X(zW_L^l)}_{\text{aliasing term}}$$

$$Y(z) = T(z)X(z).$$

$$Y(z) = T(z)X(z) + \cancel{A(z)X(-z)}$$

- L-channel QMFs can be implemented using polyphase, and they are computationally efficient.

Polyphase Representation :

$$H_k(z) = \sum_{m=0}^{M-1} z^{-m} P_{km}(z^M), \quad 0 \le k \le M-1$$

$$\Rightarrow H(z) = P(z^M)\, a(z)$$

where, $H(z) = [H_0(z) \; H_1(z) \; \dots \; H_M(z)]^t$

$a(z) = [1 \;\; z^{-1} \;\; z^{-2} \;\; \dots \;\; z^{-(M-1)}]^t$

$$P(z) = \begin{bmatrix} P_{0,0}(z) & P_{0,1}(z) & \dots & P_{0,M-1}(z) \\ P_{1,0}(z) & P_{1,1}(z) & \dots & P_{1,M-1}(z) \\ \vdots & \vdots & & \\ P_{M-1,0}(z) & P_{M-1,1}(z) & & P_{M-1,M-1}(z) \end{bmatrix}$$

$$G(z) = z^{-(M-1)} Q(z^M)\, a(z^{-1})$$

$$= [G_0(z) \;\; G_1(z) \;\; \dots \;\; G_{M-1}(z)]^t$$

$$Q(z) = \begin{bmatrix} Q_{0,0}(z) & Q_{0,1}(z) & \dots & Q_{0,M-1}(z) \\ \vdots & & & \\ Q_{M-1,0}(z) & & \dots & Q_{M-1,M-1}(z) \end{bmatrix}$$

$$Q(z)P(z) = c z^{-k} I, \quad I_{M\times M} \text{ identity matrix.}$$

$$\therefore Q(z) = c z^{-k} [P(z)]^{-1}$$

Eg. The polyphase matrix of a 3-channel perfect reconstruction (PR) FIR QMF filter is given

$$P(z^3) = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix}.$$

Find the analysis filter band and the synthesis FB.

Soln:

$$\begin{bmatrix} H_0(z) \\ H_1(z) \\ H_2(z) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \\ z^{-2} \end{bmatrix} \quad : \text{Analysis FB.}$$

$$H(z) = \left[(1 + z^{-1} + 2z^{-1}) \;\; (2 + 3z^{-1} + z^{-2}) \;\; (1 + 2z^{-1} + z^{-2}) \right]^t$$

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Synthesis FB :

$$Q(z) = c\, z^{-K} [P(z)]^{-1}$$

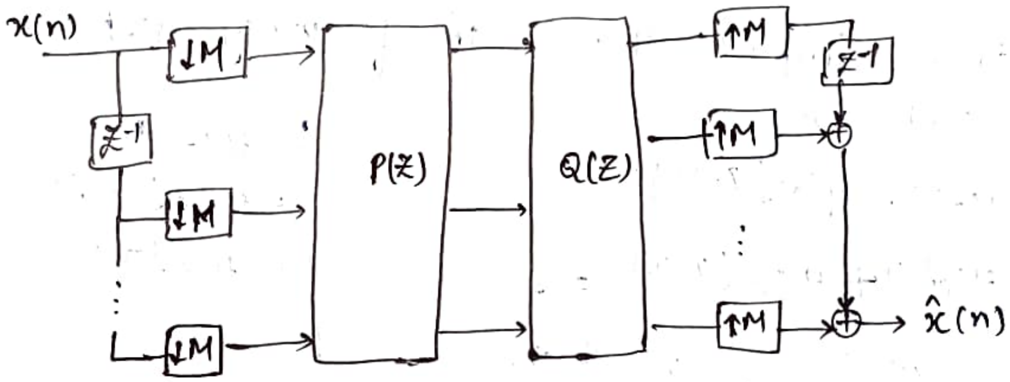
$$[P(z)]^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 3 & -5 \\ -1 & -1 & 3 \\ 1 & -1 & +1 \end{bmatrix}$$

Now, $G(z) = z^{-(m-1)} Q(z^m)\, a(z^{-1})$ ← Analysis & synthesis

$[c=2]$

$$\begin{bmatrix} G_0(z) \\ G_1(z) \\ G_2(z) \end{bmatrix} = z^{-2}\begin{bmatrix} 1 & 3 & -5 \\ -1 & -1 & 3 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ z \\ z^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & -5 \\ -1 & -1 & 3 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} z^{-2} \\ z^{-1} \\ 1 \end{bmatrix}$$



Nyquist Filters

↳ computationally efficient

L=2 : half-band filters

↳ used in decimators and interpolations, for sampling rate alteration by a factor of 2.

↳ Apart from "n=0", all even indexed coefficients of half band filter are zero.

$$X(z) \longrightarrow \boxed{\uparrow L} \longrightarrow \boxed{H(z)} \longrightarrow Y(z)$$

$$Y(z) = X(z^L)\, H(z)$$

$$H(z) = E_0(z^2) + z^{-1} E_1(z^2)$$

$$Y(z) = [E_0(z^2) + z^{-1} E_1(z^2)]\, X(z^L)$$

$$= [\alpha + z^{-1} E_1(z^2)]\, X(z^L)$$

$$Y(Z) = \alpha X(Z^L) + Z^{-1} X(Z^L) E_1(Z^2)$$

$L=2$: $Y(Z) = \alpha X(Z^2) + Z^{-1} X(Z^2) E_1(Z^2)$;

$$y(2n) = \begin{cases} \alpha, & n=0 \\ 0, & \text{elsewhere} \end{cases}$$

$$y(2n) = \alpha x(n)$$

Let $\alpha = 1 \Rightarrow y(2n) = x(n)$

- The input samples appear at the output without distortion, whereas the other samples are determined by the interpolators.

$$H(Z) + H(-Z) = 2\alpha$$

Due to even impulse response, the frequency response is real and even.

$$H(-Z) = H(e^{-j(\pi-\omega)})$$

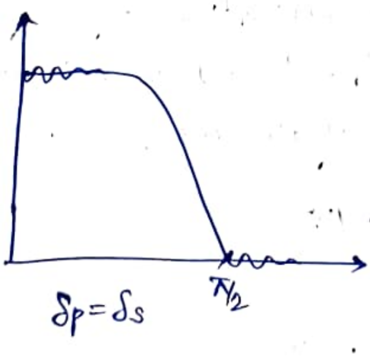
$$H(e^{j\omega}) + H(e^{-j(\pi-\omega)}) = 1$$

$$\omega = \frac{\pi}{2} - \theta, \text{ for all } \theta.$$

$H(-e^{+j\omega})$
$= H(e^{-j\pi} e^{+j\omega})$
$= H(e^{-j(\pi-\omega)})$

$$H\{e^{j(\frac{\pi}{2}-\theta)}\} + H\{e^{j(\frac{\pi}{2}+\theta)}\} < 1$$

This ensures symmetry around $\pi/2$.



$\delta_p = \delta_s$

Check whether the given filter is half-band.

$$H(z) = -1 + 9z^{-2} + 16z^{-3} + 9z^{-4} - z^{-6}$$

↳ Not half-band filter.

$E_0(z^2)$

$E_L(z^{-L})$

$$H(-z) = -1 + 9z^{-2} - 16z^{-3} + 9z^{-4} - z^{-6}$$

$$H(z) + H(-z)$$

$$= -2 + 18z^{-2} + 18z^{-4} - 2z^{-6}$$

$$\left.\begin{aligned} H_1(z) &= 1 + z^{-3} \\ H_2(z) &= z + 1 + z^{-1} \\ H_3(z) &= 1 + z^{-1} + z^{-3} \end{aligned}\right\} \Rightarrow \text{Half-band} \checkmark$$

For $L=2$,

$$H(z) = c + z^{-1} E_1(z^2)$$

<u>L-Nyquist Filter:</u>

$$H(z) = c + z^{-1} E_1(z^L) + z^{-2} E_2(z^L) + \dots + z^{-(L-1)} E_{L-1}(z^L)$$

$$X(z) \longrightarrow \boxed{\uparrow L} \longrightarrow \boxed{H(z)} \longrightarrow Y(z)$$

$$Y(z) = X(z^L)\, H(z)$$

$$= cX(z^L) + \sum_{l=1}^{L-1} z^{-l} E_l(z^L) X(z^L)$$

$$h(ln) = \begin{cases} c, & n=0 \\ 0, & \text{elsewhere.} \end{cases}$$

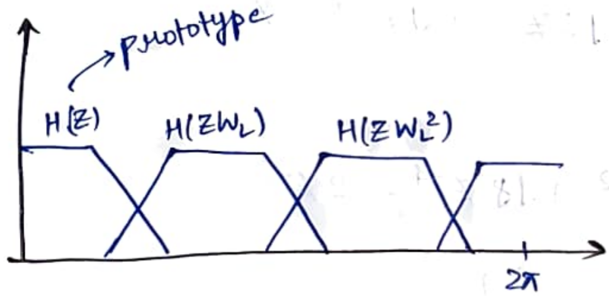
$$\sum_{k=0}^{L-1} H(zW^k) = c + z^{-1} E_1(z^L) + \dots + z^{-(L-1)} E_{L-1}(z^L)$$

$$= c + z^{-1} W^{-1} E_1(z^L W^L) + \dots + z^{-(L-1)} W^{-(L-1)} E_{L-1}(z^L W^L)$$

$$= C + Z^{-1} W^{-(L-1)} E_1 (Z^L W^{(L-1)L}) + \dots$$

$$+ Z^{-(L-1)} W^{-(L-1)(L-1)} E_{L-1} (Z^L W^{(L-1)L})$$

$$= C + Z^{-1} \left(\cancel{1 + W^{-1} + \dots + W^{(L-1)}} \right)^{0} E_1$$



16-09-2025

Time-Frequency Analysis of Signals

Time-frequency Resolution:

Eg. Wavelets satisfy this.

FT: $X(\omega) = \int_{-\infty}^{\infty} x(t)\, \underbrace{e^{-j\omega t}}_{\text{basis } f^n \text{ (global)}}\, dt$

↓

only freq. resolution (not time)

Fourier Series

↓

Applicable for periodic signals; Should satisfy Dirichlet's condition.

At what time, what freq?

↳ FT cannot tell!

→ Chirp Signals → Non-stationary signals

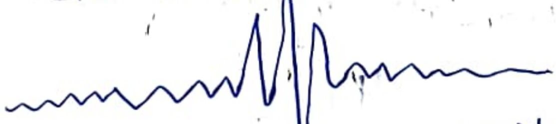
↳ characteristics change with time.

Short-time Fourier Transform (STFT): To analyse a small section of signal.

$$\int_{-\infty}^{\infty} x(t_1, t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, \underbrace{w(t - t_1)}_{\text{window}}\, e^{-j\omega t}\, dt$$

- Fourier analysis is based on an idefinitely long cosine wave of a specific freq.
- Wavelet Analysis is based on a short duration wavelet of a specific center freq.
- Wavelet f^n can be dilated and translated.



STFT:

- Window f^n is finite, so freq. resolution decreases.
- we can know what freq bands exist (not freq.) in the time interval.

Expanded wavelet

Resolve low freq.

Bad time resolution

Shrunken wavelet

Resolve high freq.

Good time resolution

Wavelet Transform

↳ Local basis functions

Continuous Wavelet Transform (CWT)

$$\underset{\text{Wavelet transform}}{W(a,b)} = \int_{-\infty}^{\infty} f(t)\, \underbrace{\psi_{a,b}}_{\text{mother wavelet}}(t)\, dt$$

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\,\psi\left(\frac{t-b}{a}\right)$$

to satisfy the power constraint

a : scaling parameter

b : translation parameter

The scaled version of $\psi(t)$:

$\psi(t/a)$

Eg. Basis $\psi(t) = \cos\omega t$

$\psi(t) = \cos\left(\frac{\omega t}{a}\right)$, $a > 1$

Localized → extract the features.

The translated version of $\psi(t)$ is $\psi(t-b)$

$$\psi\left(\frac{t-b}{a}\right)$$

Admissibility condition: Conditions to be a wavelet.

$$C_\psi = \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{\omega}\, d\omega < \infty$$

- $\hat{\psi}(\omega)$ is the Fourier transform.
- This condition ensures $\hat{\psi}(\omega) \to 0$ as $\omega \to 0$.

- Zero mean: $\int_{-\infty}^{\infty} \psi(t)\, dt = 0$

The wavelets have symmetry which satisfy zero mean condition.

- Unit energy: $\int_{-\infty}^{\infty} |\psi(t)|^2 dt = 1$

- Vanishing Moments:

Determines the ability to represent polynomial signal and detect features like edges and ~~sign~~ singularities.

$$\int_{-\infty}^{\infty} t^K \psi(t)\, dt = 0, \quad K = 0, 1, \ldots, N-1$$

- Wavelets is orthogonal to all polynomials of degree $< N$. $(N-1)$
- It ignores polynomial components of a signal.

$N=1 \rightarrow$ ignores constant terms
$N=2 \rightarrow$ ignores linear terms
$N=3$.

$$W(a,b) = \int f(t) \frac{1}{\sqrt{a}} \psi\left(\frac{t-b}{a}\right) dt$$

$a = 2^j$

$b = k 2^j$

j, k: integers representing translation & dilation.

DW:

$$= \int f(t) \cdot 2^{j/2} \psi(2^j t - k)\, dt.$$

Regularity: A wavelet has regularity 'r' if it is r-times continuously differentiable.

$$\psi(t) \in C^r$$

- 'r' derivatives of $\psi(t)$ & $\varphi(t)$ are continuous.
- wavelets with more vanishing moments are more regular.

→ Mother wavelet.

22-09-2025

CWT:

$$W(a,b) = \int f(t) \frac{1}{\sqrt{a}} \psi\left(\frac{t-b}{a}\right) dt$$

- Linearity
- Scaling
- Regularity

- **Symmetry:**

Some wavelets satisfy all properties while some don't.

Discretizing: DWT

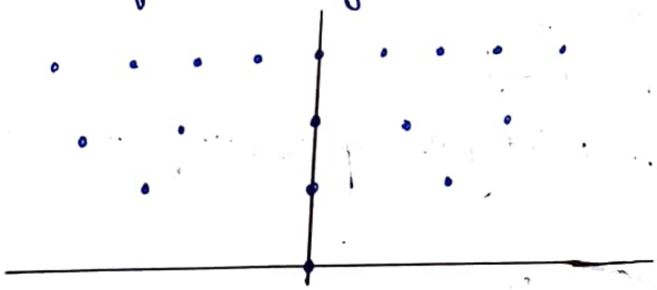
$$a = 2^{j}$$
$$b = k 2^{-j}$$

$$\psi_{j,k}(t) = 2^{j/2} \psi(2^{j} t - k)$$

$j=k=0$: $\psi_{0,0}(t) = \psi(t)$: Mother wavelet

$$\psi_{0,1}(t) = \psi(t-1)$$
$$\psi_{1,0}(t) = \sqrt{2}\, \psi(\sqrt{2} t)$$

The wavelets for all integers j and k produce orthogonal basis.



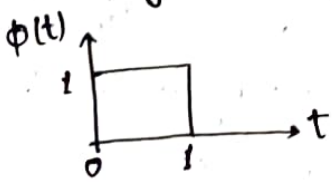
→ Dyadic transform

Z-transform → cont. applied for discrete signals

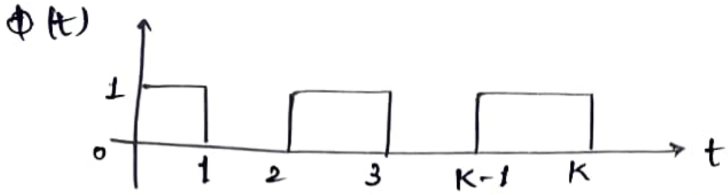
DTFT → cont.

DFT → discrete

Haar Scaling Function



$$\phi(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & \text{otherwise.} \end{cases}$$



$\phi(t)$ is non-zero b/w K and $K+1$.

The domain function is $[0,1]$ time-limited and finite energy.

$\phi(t-m), \phi(t-n)$

$m \neq n$: then ~~orthogon~~ orthonormal.

$$\int_{-\infty}^{\infty} |f(t)|^2 dt$$

$$\int_{-\infty}^{\infty} \phi(t-m)\,\phi(t-n)\,dt = \delta_{m-n}$$

If $m=n$, $\int_{-\infty}^{\infty} \phi(t)\,\phi(t)\,dt = 1$: Orthonormal.

$\phi(t-1), \phi(t-2), \dots, \phi(t-k)$ are translations of $\phi(t)$.

$\phi(t)$ & $\phi(t-1)$ are orthogonal.

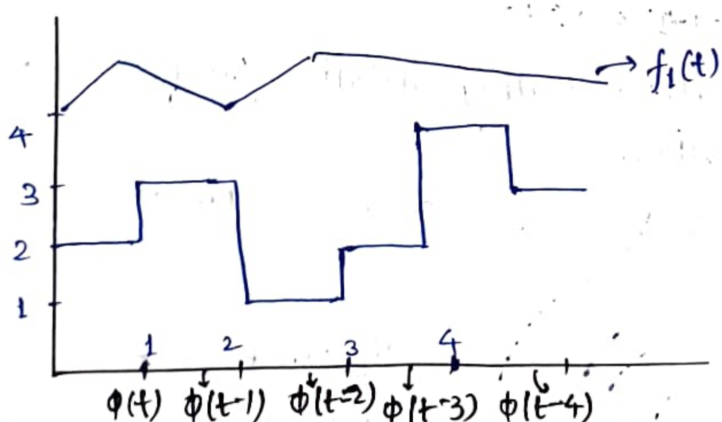
Function Space V_0 :

Consider a set of orthogonal basis f^n $[\phi(t+1), \phi(t), \phi(t-1), \dots]$ which are translates of a single f^n $\phi(t)$.

Let V_0 be the space.

$$V_0 = \operatorname*{span}_{K}\left[\overline{\phi(t-K)}\right] \Rightarrow f(t) = \sum_{K=-\infty}^{\infty} a_K\,\phi(t-K)$$

↳ piecewise constant



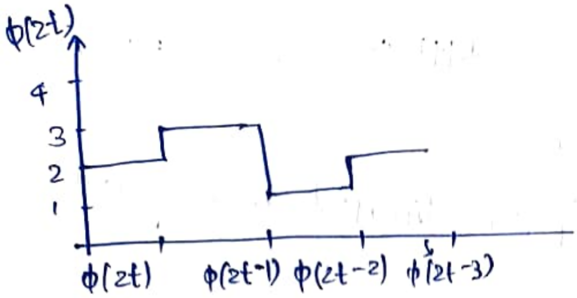
$$f(t) = 2\phi(1) + 3\phi(t-1) + 2\phi(t-2) + 5\,\phi(t-3) + 4\,\phi(t-4) + \dots$$

Scale function by 2,

$$[\dots, \phi(2t+1), \phi(2t), \phi(2t-1), \phi(2t-2), \dots]$$

Compressed by 2

They are non-overlapping & orthogonal to each other.



$$f(t) = 2\{\phi(2t) + 2\phi(2t-1)\} + 3\{\phi(2t-2) + \phi(2t-3)\}$$

$$V_1 = \underset{K}{\text{span}}\left[\phi(2t-K)\right]$$

$$f^n\left[\phi(2t-K),\ K \in \mathbb{N}\right]$$

$$f(t) = \sum_{-\infty}^{\infty} a_K\, \phi(2t-K)$$

$$V_2 = \underset{K}{\text{span}}\{\phi(2^2 t - K)\}$$

$\vdots$

$$V_j = \underset{K}{\text{span}}\{\phi(2^j t - K)\}$$

The scaling f^n we choose should be nested.

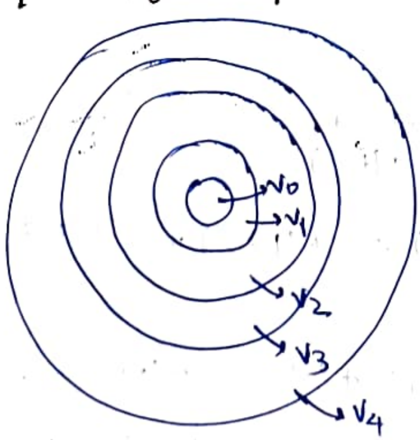
Nested space.

$V_0 \subset V_1$, V_1 is a finer space than V_0.

$$\phi(t) = \phi(2t) + \phi(2t-1)$$
$$\phi(t-1) = \phi(2t-2) + \phi(2t-3)$$

This relation is called scaling relation / refinement / dilation relation.

$$V_{-1} \subset V_0 \subset V_1 \subset \ldots \subset V_\infty. \quad \text{[Nested vector]}$$



V_1 can be expressed as V_0 but V_0 can't be expressed as V_1.

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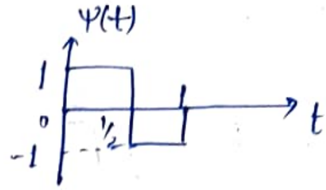
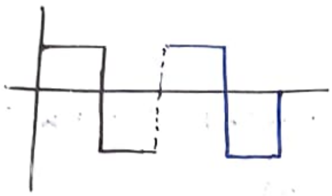
Wavelet Function:

$$\psi(t) = \begin{cases} 1, & 0 \le t \le 1/2 \\ -1, & \frac{1}{2} \le t \le 1 \\ 0, & \text{elsewhere} \end{cases}$$



$\psi(t-k),\ k \in \mathbb{N} \longrightarrow$ orthonormal set of basis



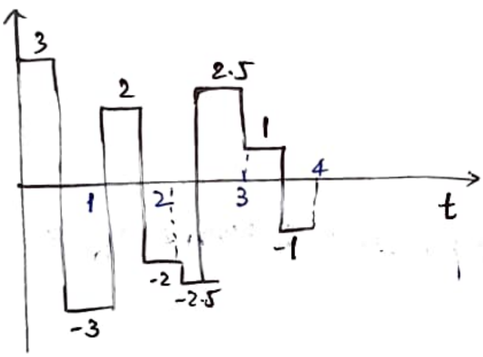
$$\int_{-\infty}^{\infty} \psi(t)\, \psi(t-1)\, dt = 0$$

$$\int_{-\infty}^{\infty} \psi(t-m)\, \psi(t-n)\, dt = \delta_{m-n}$$

$$m=n; \int_{-\infty}^{\infty} \psi(t)\, \psi(t)\, dt = 1$$

$$W_0 = \underset{k}{\text{span}} \left\{ \overline{\psi(t-k)} \right\}$$

Eg.



$$f(t) = 3\psi(t) + 2\psi(t-1) - 2.5\,\psi(t-2) + 1.\psi(t-3)$$

$$\psi(t) = \phi(2t) - \phi(2t-1)$$

$$\psi(2t) = \begin{cases} 1, & 0 \le t \le 1/4 \\ -1, & \frac{1}{4} \le t \le 1/2 \end{cases}$$

$$W_1 = \underset{k}{\text{span}} \left\{ \overline{\psi(2t-k)} \right\}$$

Basis $\{\psi(2t-k),\ k \in \mathbb{N}\}$

$W_0 \not\subset W_1$ but $W_0 \perp W_1$ (orthogonal).

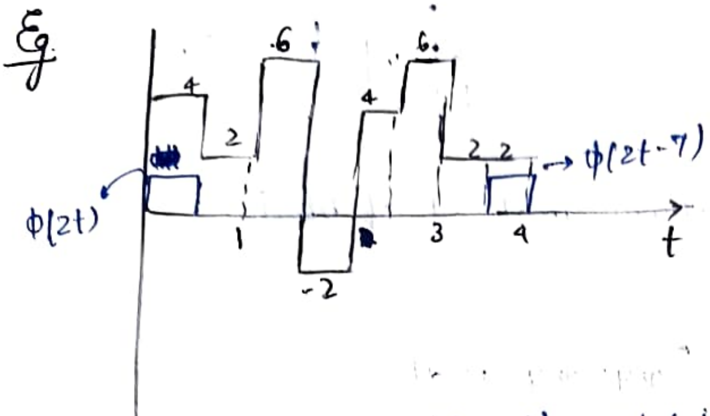
$$\int \psi(t)\, \psi(2t)\, dt = 0$$

$$\int \psi(t)\, \psi(2t-1)\, dt = 0$$

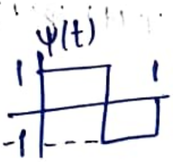
$$W_0 \perp W_1 \perp W_2 \ldots$$

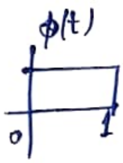
→ The space spaned by wavelet f^ns ~~are~~ is orthogonal among themselves.

Eg.



$$f(t) = 4\phi(2t) + 2\phi(2t-1) + 6\phi(2t-2) - 2\phi(2t-3) + 4\phi(2t-4) + 6\phi(2t-5) + 2\phi(2t-6) + 2\phi(2t-7)$$





$$4\phi(2t) + 2\phi(2t-1) = \frac{4+2}{2}\phi(t) + \frac{4-2}{2}\psi(t)$$

$$= \underset{\sim}{3}\phi(t) + \underset{\sim}{1}\psi(t)$$

$(3, 1)$

→ A signal in V_1 span can be expressed in terms of V_0 and W_0 space.

$$V_1 = V_0 \oplus \underset{\longrightarrow \text{complementary } f^n}{W_0}$$

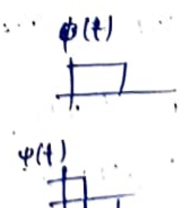
- W_0 is called the complementary space of V_0.

Coeff: 4 2 6 -2 4 6 2 2 → 3 2 5 ... V_0; 1 4 -1 ... W_0

Orthogonality between $\phi(t)$ and $\psi(t)$:

$$V_0 \perp W_0$$

scaling f^n ← V_0

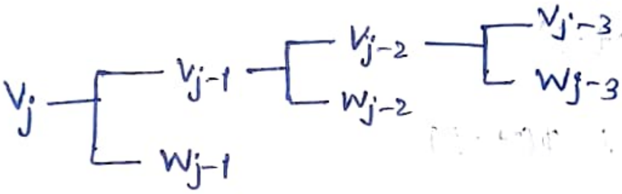


$$\langle \phi(t), \psi(t) \rangle$$

$$= \int_{-\infty}^{\infty} \phi(t)\psi(t)\,dt$$

$$= \int_0^{1/2} 1 \cdot 1\,dt + \int_{1/2}^{1} (-1)(1)\,dt = 0$$

$$V_1 = V_0 \oplus W_0$$
$$V_2 = V_1 \oplus W_1$$
$$\vdots$$
$$V_j = V_{j-1} \oplus W_{j-1}$$
$$V_{j-1} = V_{j-2} \oplus W_{j-2}$$
$$V_j = W_{j-1} \oplus W_{j-2} \oplus V_{j-2}$$
$$V_j = W_{j-1} \oplus W_{j-2} \oplus W_{j-3} \oplus \cdots \oplus W_0 \oplus V_0$$

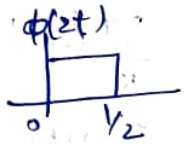


: wavelet decomposition

Normalization of Haar basis f^n at different scales:

$$\int \phi(t)\,\phi(t)\,dt = 1$$
$$\int \phi(t-k)\,\phi(t-k) = 1$$

$$\int \phi(2t)\,\phi(2t)\,dt = 1/2$$



$$\int \sqrt{2}\,\phi(2t).\,\sqrt{2}\,\phi(2t)\,dt = 1$$
$$V_1 = \underset{k}{\text{span}} \overline{\{\sqrt{2}\,\phi(2t-k)\}}$$
$$\int \phi(4t)\,\phi(4t)\,dt = 1/4$$
$$V_2 = \underset{k}{\text{span}} \overline{\{2\phi(2^2 t - k)\}}$$
$$\vdots$$
$$V_j = \underset{k}{\text{span}} \overline{\{2^{j/2}\phi(2^j t - k)\}} \to \phi_{j,k}$$

K : translation
j : scaling

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V_0 basis : $\{ \dots \phi_{0,-1}(t), \phi_{0,0}(t), \phi_{0,1}(t), \dots \}$

$\vdots$

V_j basis : $\{ \dots \phi_{j,-1}(t), \phi_{j,0}(t), \phi_{j,1}(t), \dots \}$

W_j basis : $\{ \dots \psi_{j,-1}(t), \psi_{j,0}(t), \psi_{j,1}(t), \dots \}$

$$\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k)$$

$$\phi(t) = \phi(2t) + \phi(2t-1)$$
$$\psi(t) = \phi(2t) - \phi(2t-1)$$

$$\phi(t) = \frac{1}{\sqrt{2}}\sqrt{2}\,\phi(2t) + \frac{1}{\sqrt{2}}\sqrt{2}\,\phi(2t-1)$$
$$= \frac{1}{\sqrt{2}}\phi_{1,0}(t) + \frac{1}{\sqrt{2}}\phi_{1,1}(t)$$

Similarly, $\psi(t) = \frac{1}{\sqrt{2}}\sqrt{2}\,\phi(2t) - \frac{1}{\sqrt{2}}\sqrt{2}\,\phi(2t-1)$
$$= \frac{1}{\sqrt{2}}\phi_{1,0}(t) - \frac{1}{\sqrt{2}}\phi_{1,1}(t)$$

$$\phi(t) = \sum_k h(k)\,\phi_{1,k}(t) = \sum_k h(k)\sqrt{2}\,\phi(2t-k)$$
$$\psi(t) = \sum_k g(t)\,\phi_{1,k}(t) = \sum_k g(t)\sqrt{2}\,\phi(2t-k)$$

$h(0) = h(1) = \frac{1}{\sqrt{2}}$
$g(0) = \frac{1}{\sqrt{2}}, g(1) = \frac{-1}{\sqrt{2}}$

Eg. Haar

$$\phi(t) = \sum_{k=0}^{1} h(k)\,\phi_{1,k}(t)$$
$$\psi(t) = \sum_{k=0}^{1} g(k)\,\phi_{1,k}(t)$$

$h(0) = h(1) = \frac{1}{\sqrt{2}}$; $g(0) = \frac{1}{\sqrt{2}}$ | $g(1) = -\frac{1}{\sqrt{2}}$

$\{h(k), k \in \mathbb{N}\}$: LPF

$\{g(k), k \in \mathbb{N}\}$: HPF

Support of Wavelet System:

Basically defines the range of values for which the scaling and wavelet functions are defined.

Eg. Haar, $\phi(t)$, $\psi(t)$, $[0, 1]$

Refinement Relation for orthogonal wavelets:

$$\phi(t) = \sum_{K=0}^{N-1} h(K)\sqrt{2}\,\phi(2t-K) \qquad [\text{Unknown}: h(K)]$$

$$= \sum_{K=0}^{N-1} C_K\,\phi(2t-K), \quad C_K = h(K)\sqrt{2}$$

$$\psi(t) = \sum_{K=0}^{N-1} C_K'\,\phi(2t-K), \quad C_K' = g(K)\sqrt{2}$$

$$\underbrace{\int_{-\infty}^{\infty}\phi(t)\,dt}_{1} = \int_{-\infty}^{\infty}\sum_{K=0}^{N-1} h(K)\sqrt{2}\,\phi(2t-K)\,dt$$

$$= \sum_{K=0}^{N-1} h(K)\sqrt{2}\underbrace{\int_{-\infty}^{\infty}\phi(2t-K)\,dt}_{1/2}$$

$$\Rightarrow 1 = \sum_{K=0}^{N-1} h(K)\sqrt{2}\cdot\frac{1}{2}$$

$$\Rightarrow \sum_{K=0}^{N-1} h(K) = \sqrt{2}$$

Restrictions on Filter Coefficients:

1. Unit area under scaling function,

   $\int \phi(t)\,dt = 1$; Eg. Haar scaling.

2. Orthonormality of translation of scaling function
3. Orthonormality of scaling ~~and~~ of wavelet functions.
4. Approximation (smoothness).

DWT and Relation to Filter Bank (FB)

$$\phi(t) = \sum_{n=0}^{N-1} h(n)\sqrt{2}\,\phi(2t-n)$$

$$= \sum_{n=0}^{N-1} h(n)\,\phi_{1,n}$$

$N=2$:

$$\phi(t) = h(0)\sqrt{2}\,\phi(2t) + h(1)\sqrt{2}\,\phi(2t-1)$$

$$= h(0)\,h_{1,0}(t) + h(1)\,\phi_{1,1}(t)$$

For Haar: $h(0)$ & $h(1)$: filter coeff.

$$h(0) = 1/\sqrt{2},\ h(1) = 1/\sqrt{2}.$$

for higher scales $(N>2)$;

$$\phi(2^j t - k) = \sum_{n=0}^{N-1} h(n)\sqrt{2}\,\phi(2(2^j t - k) - n)$$

$$= \sum_{n=0}^{N-1} h(n)\sqrt{2}\,(2^{j+1}t - 2k - n)$$

Let $m = 2k + n$.

$$\phi(2^j t - k) = \sum_{m=2k}^{2k+N-1} h(m-2k)\,\sqrt{2}\,\phi(2^{j+1}t - m)$$

$n=0 \leftarrow m=2k$

$$\psi(2^j t - k) = \sum_{m=2k}^{2k+N-h} g(m-2k)\cdot\sqrt{2}\cdot\phi(2^{j+1}t - m)$$

Eg. $j=0,\ k=3,\ N=2.$

$$\phi(t-3) = \sum_{m=6}^{7=N+6-1} h(m-6)\sqrt{2}\,\phi(2^1 t - m)$$

$$= h(0)\sqrt{2}\,\phi(2t-6) + h(1)\sqrt{2}\,\phi(2t-7)$$

If Haar: $h(0) = 1/\sqrt{2},\ h(1) = 1/\sqrt{2}$,

$$\phi(t-3) = \phi(2t-6) + \phi(2t-7).$$

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DWT:

$$f(t) = \sum_{k=-\infty}^{\infty} c_j(k)\, 2^{j/2} \phi(2^j t - k) + \sum_{k=-\infty}^{\infty} d_j(k)\, 2^{j/2} \psi(2^j t - k)$$

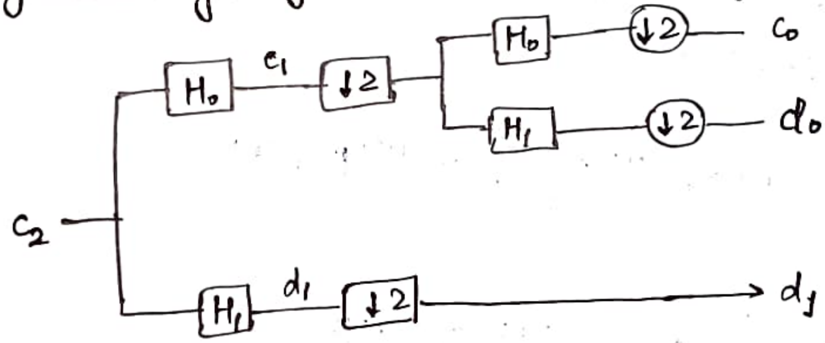
$$c_j(k) = \sum_{m=-\infty}^{\infty} c_{j+1}(m)\, h_0(m-2k)$$

$$d_j(k) = \sum_{m=-\infty}^{\infty} c_{j+1}(m)\, h_1(m-2k)$$

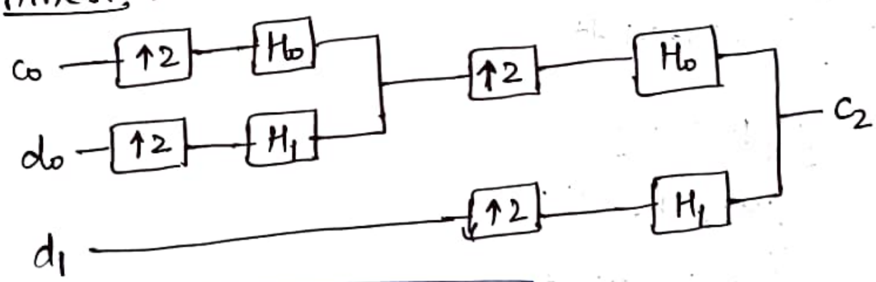
$h_0(k) \rightarrow$ LPF, $h_1(k) \rightarrow$ HPF

$$h_1(k) = (-1)^k h_0(N-1-k).$$

Analysis using dyadic subband coding:



Synthesis:



$$\boxed{\begin{aligned} f(k) &= 2^{j/2} c_j(k) \\ \Rightarrow c_j(k) &= 2^{-j/2} f(k) \end{aligned}}$$

Eg. Problem:

Use Haar wavelet and expand the signal $x(t) = e^{-5t}$, $0 \le t \le 1$.

using (a) $\phi(2t-k)$ (b) $\phi(t)$ & $\psi(t-k)$.

soln: $f(t) = \sum_{k=-\infty}^{\infty} c_j(k)\, 2^{j/2} \phi(2^j t - k)$

$c_j(k) = \langle f(t), \phi_{j,k}(t) \rangle$
$= \int f(t)\, 2^{j/2} \phi(2^j t - k)\, dt$

ⓐ $j=1,\ k=0$:

$c_1(0) = \langle f(t), \phi_{1,0}(t) \rangle$
$= \int_0^{1/2} f(t) \sqrt{2}\, \phi(2t-0)\, dt$
$= \int_0^{1/2} e^{-5t} \sqrt{2}\, \phi(2t)\, dt$
$= \sqrt{2} \int_0^{1/2} e^{-5t}\, dt$
$= \sqrt{2} \left[\frac{e^{-5t}}{-5}\right]_0^{1/2}$
$= -\frac{\sqrt{2}}{5}\left[e^{-5/2} - 1\right]$

$c_1(0) = \frac{\sqrt{2}}{5}(1 - e^{-2.5})$

$j=1,\ k=1$:
$c_1(1) = \langle f(t), \phi_{1,1}(t) \rangle$
$= \int_{1/2}^{1} e^{-5t} \sqrt{2}\, \phi(2t-1)\, dt$
$= \sqrt{2} \int_{1/2}^{1} e^{-5t}\, dt$
$= \frac{\sqrt{2}}{-5}\left[e^{-5} - e^{-2.5}\right]$

$c_1(1) = \frac{\sqrt{2}}{5}(e^{-2.5} - e^{-5})$

ⓑ $f(t) = \sum_{k=-\infty}^{\infty} c_0(k)\, \phi_{0,k}(t) + \sum_{k=-\infty}^{\infty} d_0(k)\, \psi_{0,k}(t)$

$j=0,\ k=0$:
$c_0(0) = \langle f(t), \phi_{0,0}(t) \rangle$
$= \int_0^1 e^{-5t} \phi(t)\, dt$
$= \frac{1}{5}[1 - e^{-5}]$

$d_0(0) = \langle f(t), \psi_{0,0}(t) \rangle$
$= \int_0^1 e^{-5t} \psi(t)\, dt$
$= \int_0^{1/2} e^{-5t} \psi(t)\, dt + \int_{1/2}^{1} e^{-5t} \psi(t)\, dt$
$= \frac{1}{5}(1 - e^{-2.5}) - \frac{1}{5}(e^{-2.5} - e^{-5})$
$= \frac{1}{5}(1 - 2e^{-2.5} - e^{-5})$

$f(t) = c_0(0)\, \phi(t) + d_0(0)\, \psi(t)$
$= \frac{1}{5}\left[(1 - e^{-5})\, \phi(t) + (1 - 2e^{-2.5} - e^{-5})\, \psi(t)\right]$

Eg For the given samples $[4, 2, -1, 0]$.
Using Haar wavelet, determine the coefficients.

Soln: $c_j(K) = 2^{-j/2} f(K)$

$j=2$: $c_2(K) = 2^{-1} f(K)$

$= \frac{1}{2}[4, 2, -1, 0]$

$= [2, 1, -0.5, 0]$

$h_0 = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right], \quad h_1 = \left[\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right]$

$$f_2(t) = \sum_{K=-\infty}^{\infty} c_j(K)\, 2^{j/2}\, \phi(2^j t - K)$$

$= 4\phi(4t) + 2\phi(4t-1) - 1.\phi(4t-2) + 0.\phi(4t-3)$

$j=1$:
$$c_1(K) = \sum_{m=-\infty}^{\infty} c_2(m)\, h_0(m-2K)$$

$$d_1(K) = \sum_{m=-\infty}^{\infty} c_2(m)\, h_1(m-2K)$$

$K=0$:
$c_1(0) = c_2(0) h_0(0) + c_2(1) h_0(1)$

$= 2\left(\frac{1}{\sqrt{2}}\right) + 1\left(\frac{1}{\sqrt{2}}\right) = \frac{3}{\sqrt{2}}$

$K=1$: $c_1(1) = c_2(2) h_0(0) + c_2(3) h_0(1)$

$= -\frac{1}{2}\left(\frac{1}{\sqrt{2}}\right) + 0\left(\frac{1}{\sqrt{2}}\right) = \frac{-1}{2\sqrt{2}}$

$K=0$: $d_1(0) = \sum_{m=\infty} c_2(m)\, h_1(m)$

$= c_2(0) h_1(0) + c_2(1) h_1(1)$

$= 2\left(\frac{1}{\sqrt{2}}\right) + 1\left(-\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$

$K=1$: $d_1(1) = \sum_{m=-\infty}^{\infty} c_2(m)\, h_1(m-2)$

$= c_2(2) h_1(0) + c_2(3) h_1(1)$

$= \frac{-1}{2}\left(\frac{1}{\sqrt{2}}\right) + 0 = \frac{-1}{2\sqrt{2}}$

$j=0$:

$K=0$: $c_0(0) = c_1(0)\,h_0(0) + c_1(1)\,h_0(1)$

$$= \frac{3}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{-1}{2\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{3}{2} - \frac{1}{4} = \frac{5}{4}$$

$d_0(0) = c_1(0)\,h_1(0) + c_1(1)\,h_1(1)$

$$= \frac{3}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{-1}{2\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}}\right)$$

$$= \frac{3}{2} + \frac{1}{4} = \frac{7}{4}$$

$\therefore\ W_2(K) = [c_0(0)\ \ d_0(0)\ \ d_1(0)\ \ d_1(1)]$

$$= \left[5/4,\ \ 7/4\ \ \ \frac{-1}{\sqrt{2}}\ \ \ \frac{-1}{2\sqrt{2}}\right]$$

Then, the function can be expanded into one scaling and 3 wavelet functions.

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$$C_0(K) = \sum_{m=-\infty}^{\infty} c_1(m)\, h_0(m-2K)$$

$$d_0(K) = \sum_{m=-\infty}^{\infty} c_1(m)\, h_1(m-2K)$$

$$C_0(0) = \sum_{m=-\infty}^{\infty} c_1(m)\, h_0(m) = c_0(0)\, h_0(0) + c_1(1)\, h_0(1)$$
$$= \frac{3\sqrt{2}}{2} \cdot \frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{2}} \times \frac{1}{\sqrt{2}} = 5/4$$

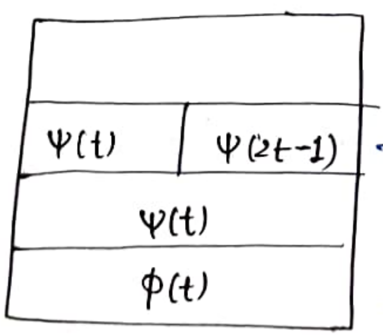
$$d_0(0) = \sum_{m=-\infty}^{\infty} c_1(m)\, h_1(m) = c_1(0)\, h_1(0) + c_1(1)\, h_1(1)$$
$$= \frac{3\sqrt{2}}{2} \cdot \frac{1}{\sqrt{2}} + \left(-\frac{1}{2\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right) = \frac{7}{4}$$

$$W_2(K) = \left[\, C_0(0) \quad d_0(0) \quad d_1(0) \quad d_1(1) \,\right]$$
$$= \left[\, \frac{5}{4} \quad \frac{7}{4} \quad \frac{1}{\sqrt{2}} \quad -\frac{1}{2\sqrt{2}} \,\right].$$

$$f(t) = f_2(t)$$
$$= \sum_{K=-\infty}^{\infty} c_0(K)\, \phi(t) + \sum_{j=0}^{1} \sum_{K=-\infty}^{\infty} d_j(K)\, 2^{j/2}\, \psi(2^j t - K)$$
$$= \frac{5}{4}\phi(t) + \frac{7}{4}\psi(t) + \psi(2t) - \frac{1}{2}\psi(2t-1)$$

$f(t) = f_J(t)$, J: level can be chosen.

$$f_J(t) = \sum_{K=-\infty}^{\infty} C_0(t)\, \phi(t-K) + \sum_{j=0}^{J} \sum_{K=-\infty}^{\infty} d_j(K)\, 2^{j/2}\, \psi(2^j t - K)$$



→ good time resolution

} frequency resolution

↳ Time frequency plane ⇒ MRA (Multi-Resolution Analysis)

Eg. Given the wavelet coefficient :

$$[c_0(0), d_0(0), d_1(0), d_1(1)]$$
$$= [5/4, 7/4, 1/\sqrt{2}, -1/2\sqrt{2}]$$

Determine the samples.

Solⁿ:

$$c_1(K) = \sum_{m=-\infty}^{\infty} c_0(m)\, h_0(K-2m) + \sum_{m=-\infty}^{\infty} d_0(m)\, h_1(K-2m)$$

$$\Rightarrow \quad c_1(0) = \sum_{m=-\infty}^{\infty} c_0(m)\, h_0(-2m) + \sum_{m=-\infty}^{\infty} d_0(m)\, h_1(-2m)$$
$$= c_0(0)\, h_0(0) + d_0(0)\, h_1(0)$$
$$= \frac{5}{4}\frac{1}{\sqrt{2}} + \frac{7}{4}\frac{1}{\sqrt{2}} = \frac{3}{\sqrt{2}}$$

$$c_1(1) = \sum_{m=-\infty}^{\infty} c_0(m)\, h_0(1-2m) + \sum_{m=-\infty}^{\infty} d_0(m)\, h_1(1-2m)$$
$$= c_0(0)\, h_0(1) + d_0(0)\, h_1(1)$$
$$= \frac{5}{4}\frac{1}{\sqrt{2}} + \frac{7}{4}\left(\frac{-1}{\sqrt{2}}\right)$$
$$= -\frac{1}{2\sqrt{2}}$$

$$c_2(K) = \sum_{m=-\infty}^{\infty} c_1(m)\, h_0(K-2m) + \sum_{m=-\infty}^{\infty} d_1(m)\, h_1(K-2m)$$

$$c_2(0) = \sum_{m=-\infty}^{\infty} c_1(m)\, h_0(-2m) + \sum_{m=-\infty}^{\infty} d_1(m)\, h_1(-2m)$$
$$= c_1(0)\, h_0(0) + d_1(0)\, h_1(0)$$
$$= \frac{3}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = 2$$

$$c_2(1) = \sum_{m=\infty}^{\infty} c_1(m)\, h_0(1-2m) + \sum_{m=-\infty}^{\infty} d_1(m)\, h_1(1-2m)$$
$$= c_1(0)\, h_0(1) + d_1(0)\, h_1(1)$$
$$= \frac{3}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(\frac{-1}{\sqrt{2}}\right) = 1$$

$c_2(3) = 0$

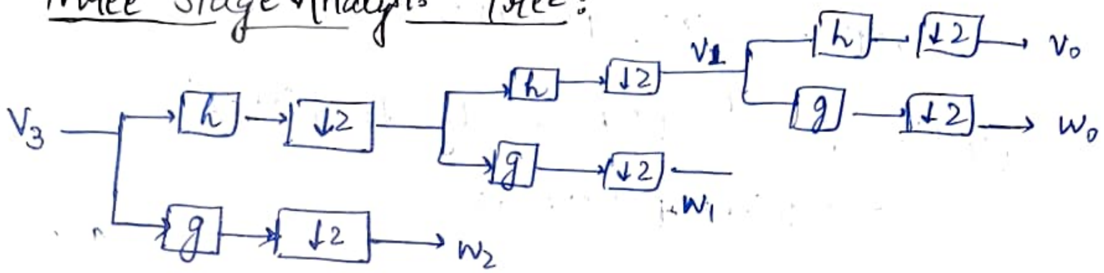
$\vdots$

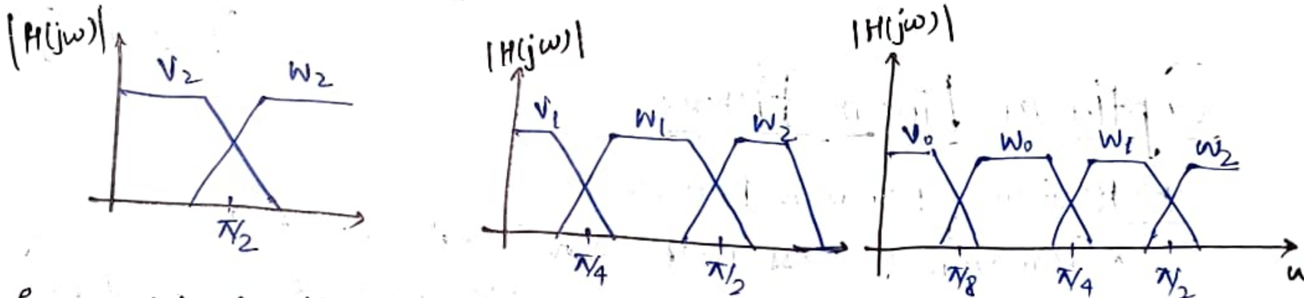
$$c_2(2) = \sum_{m=-\infty}^{\infty} c_1(m)\, h_0(2-2m) + \sum_{m=-\infty}^{\infty} d_1(m)\, h_1(2-2m)$$
$$= c_1(1)\, h_0(0) + d_1(1)\, h_1(0) = -\frac{1}{2\sqrt{2}}\frac{1}{\sqrt{2}} + \left(-\frac{1}{2\sqrt{2}}\right)\frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}}$$

$$f(K) = 2^{j/2}\,[2 \quad 1 \quad -0.5 \quad 0]$$
$$= [4 \quad 2 \quad -1 \quad 0] \quad (\text{for } j=2)$$

Frequency Response

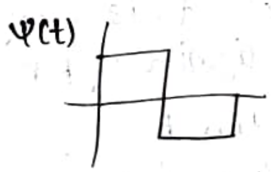
Three stage Analysis Tree:



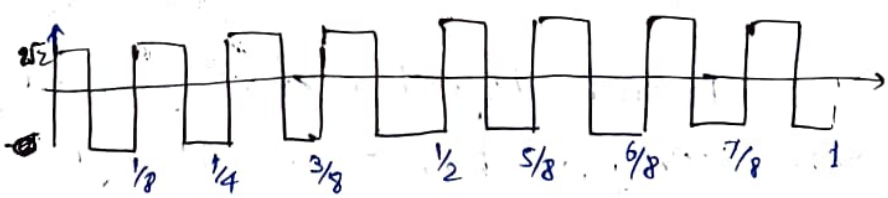


Eg. sketch the Haar mother wavelet families for 4 different scales $j=0,1,2,3$ for a period of 1 second.

$j=2$: $\psi_{2,K} = 2\psi(4t-K)$, $k=0,1,2,3$



$j=3$: $\psi_{3,K} = 2\sqrt{2}\,\psi(8t-K)$, $K=0,1,2,3,4,\dots,7$



$j=0$: $\psi_{0,K} = \psi_{0,K} = \psi(t-K)$, $K=0$

$j=1$: $\psi_{1,K} = \sqrt{2}\,\psi(2t-K)$, $K=0,1$

Eg Approximate the following f^n with Haar scaling f^n, with $j=1$.

$$f(t) = \begin{cases} 2, & 0 < t \le 0.5 \\ 1, & 0.5 < t \le 1 \end{cases}$$

Coeff:

(2) $\phi(2t)$ + (1) $\phi(2t)$

Solⁿ: $f_K(t) = \sum_{K=-\infty}^{\infty} c_j(K)\, 2^{j/2} \phi(2^j t - K)$

$c_1(0) = \langle f(t), \phi(2t) \rangle = \sqrt{2}$

$c_1(1) = \langle f(t), \phi(2t-1) \rangle = 1/\sqrt{2}$

$f_1(t) = \sqrt{2} \cdot 2^{1/2} \phi(2t) + \frac{1}{\sqrt{2}} \sqrt{2}\, \phi(2t-1)$

$= 2\phi(2t) + \phi(2t-1)$

↑ Midsem

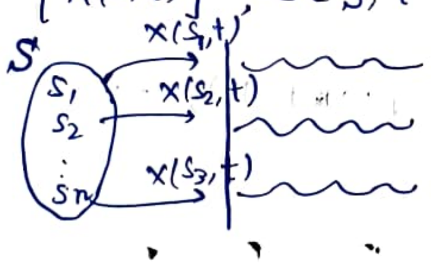
16-10-2025

Optimum Linear Filters

A Review of Random Process :

- A random process maps each sample point to a waveform.
- A random process on the sample space S can be defined as an indexed family of random variables.

$\{X(s,t)\}$, $s \in S$, $t \in \Gamma$ (an indexed set of denoting time)



$s_0 \in S$, $X(s_0,t)$ is a single realization of RP.

The random process is normally denoted as $\{X(t)\}$.

- A discrete RP can be defined as $\{x(n)\}$, $n \in \Gamma$.
- To describe $\{x(n)\}$, we have to use CDF of RV at different instant n.

$x(n_1), x(n_2), \ldots, x(n_K)$ represent K-joint RVs.

Thus, a RP can be described by joint CDF.

$F_{x(n_1)x(n_2)\ldots}(x_1, x_2, \ldots, x_K)$ $\forall K \in \mathbb{N}$ and $\forall n_K \in \Gamma$.

Moments of RP:

Mean: $E\{X(n)\} = \mu_x(n)\ \forall n \rightarrow$ constant.

Autocorrelation: $R_x(n_1, n_2) = E\{X(n_1)X(n_2)\}\ \forall n_1, n_2$.

Autocovariance:

$$C_x(n_1, n_2) = E\{X(n_1) - \mu_x(n_1)\}\, E\{X(n_2) - \mu_x(n_2)\}$$
$$= R_x(n_1, n_2) - \mu_x(n_1)\mu_x(n_2).$$

Stationary RP:

A RP $\{X(n)\}$ is called **strict sense stationary** (SSS) if its probability structure is invariant with time.

In terms of joint CDF,

$$F_{X(n_1)X(n_2)\ldots}(x_1, x_2, \ldots, x_K) = F_{X(n_1+h)X(n_2+h)\ldots X(n_K+h)}(x_1, x_2, \ldots x_K)$$
$$\forall K \in N \ \&\ \forall h, n_1, n_2, \ldots, n_K \in \Gamma$$

- Analysis of SSS process is very complex, so we generally look for weak form of stationarity, i.e., **Wide sense stationary** (WSS) RP.
- A RP is WSS if $\forall h, n_1, n_2$.

① $E[x(n)] = E[x(n+h)]$ = constant.

② $R_x(n_1, n_2) = R_x[n_1 + h, n_2 + h]$

If $h = -n_1$,

$R_x(n_1, n_2) = R_x(0, n_2 - n_1)\ \forall n_1, n_2 \rightarrow f^n$ of lag only.

③ Covariance: $C_K(0) < \infty$.

Properties of Autocorrelation (AC):

① $R_x(K) = E[x(n+K)\, x^*(n)] = E[x^*(n)\, x(n+K)]$

Lag $= R_x(-K)$

If it is real, $R_x(K) = R_x(-K)$.

② The AC of a WSS process at lag $K=0$ is equal to the mean square value of the process.

$$R_x(0) = E\{|X(n)|^2\} \geq 0.$$

Proof: cauchy-schwarz inequality

$$\|\langle x(n), X(n+K)\rangle\| \leq \|X(n)\| . \|x(n+K)\|$$

$\therefore \mathbb{E}\{x(n)\,x(n+k)\} \leq \sqrt{\mathbb{E}[x^2(n)]}\,\sqrt{\mathbb{E}[x^2(n+k)]}$

$|R_x(K)| \leq \sqrt{R_x(0)}\,\sqrt{R_x(0)} = R_x(0) \rightarrow R_x(K)$ is min^m at $K=0$.

③ The magnitude of the AC sequence of a WSS RP at lag k is upper bounded by its value at lag $k=0$.

$$R_x(0) \geq |R_x(K)|$$

Cross Correlation: CC (R_{xy}) of two discrete RP $x(n)$ & $y(n)$:

$$R_{xy}(n,k) = \mathbb{E}[x(n)\,y(n+k)]$$

$x(n)$ & $y(n)$ are jointly wss if they are individually WSS, $R_{xy}(n,k)$ is of a f^n of time lag k.

Symmetry property: $R_{xy}(K) = \mathbb{E}[x(n)\,y(n+k)]$

$= \mathbb{E}[y(n+k)\,x(n)] = R_{xy}(-k)$

Wiener ~~Weiner~~ Khinchin Theorem:

$R_x(m) \longrightarrow S_x(\omega)$ [DTFT pair]

$R_x(m)$: Autocorrelation; $S_x(\omega)$: Power spectral Density (PSD)

$$S_x(\omega) = +\sum_{m=-\infty}^{\infty} R_x(m)\,e^{-j\omega n}$$

$$R_x(m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_x(\omega)\,e^{+j\omega m}\,d\omega$$

ZT: $S_x(Z) = \sum_{m=-\infty}^{\infty} R_x(m)\,Z^{-m}$

Response of LTI system:

$$\underset{\text{WSS}}{x(n)} \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

$y(n) = x(n) * h(n)$

$\mathbb{E}[x(n)] = \mu_x$

$R_x(m) = \mathbb{E}[x(n)\,x(n+m)]$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\,x(n-k)$$

$$\mathbb{E}[y(n)] = \mu_y = \mathbb{E}\Big[\sum_{k=-\infty}^{\infty} h(k)\,x(n-k)\Big] = \sum_{k=-\infty}^{\infty} h(k)\,\mathbb{E}[x(n-k)]$$

$\mathbb{E}[x(n-k)] \rightarrow \mu_x$

$$= \underbrace{\sum_{k=-\infty}^{\infty} h(k)}_{H(0)}\mu_x = H(0)\,\mu_x$$

$$H(\omega) = \sum_{k=-\infty}^{\infty} h(k)\,e^{-j\omega n} \Rightarrow 1$$

$$\omega = 0 \Rightarrow \sum_{k=-\infty}^{\infty} h(k) = H(0).$$

21-10-2025

Cross-correlation

$$x(n) \rightarrow \boxed{h(n)} \rightarrow y(n)$$

$$\mathbb{E}[x(n)\, y(n+m)] = R_{xy}(m)$$

$$R_{xy}(m) = R_x(m) * h(m)$$

Proof: $$y(n+m) = \sum_{k=-\infty}^{\infty} h(k)\, x(n+m-k)$$

$$R_{xy}(m) = E[x(n)\, y(n+m)]$$

$$= E\left[x(n) \sum_{K} h(k)\, x(n+m-k)\right]$$

$$\cancel{= E[x}$$

$$= \sum_{k=-\infty}^{\infty} h(k)\, R_x(m-k)$$

$$\Rightarrow R_{xy}(m) = R_x(m) * h(m)$$

Similarly,

$$E[y(n)\, y(n+m)]$$

$$= R_y = h(-m) * R_{xy}(m)$$

$$= h(-m) * R_x(m) * h(m)$$

→ $x(n)$ and $y(n)$ are WSS.

$$\mu_y = \mu_x H(0)$$

$$Y(\omega) = H(\omega)\, X(\omega).$$
$$Y(Z) = H(Z)\, X(Z)$$

$$S_y(\omega) = |H(\omega)|^2 S_x(\omega).$$

$$S_Y(Z) = |H(Z)|^2 S_X(Z)$$

$$= H(Z)\, H(Z^{-1})\, S_X(Z)$$

Eg. $R_X(m) = a^{|m|}$, $a < 1$

Find PSD.

Soln: $$S_X(Z) = \sum_{m=-\infty}^{\infty} a^{|m|} z^{-m}$$

$$= \sum_{m=-\infty}^{-1} a^{-m} z^{-m} + \sum_{m=0}^{\infty} a^{m} z^{-m}$$

$$= \sum_{m=0}^{\infty} a^{m} z^{m} - 1 + \sum_{m=0}^{\infty} (az^{-1})^{m}$$

$$= \frac{1}{1-az} - 1 + \frac{1}{1-az^{-1}}$$

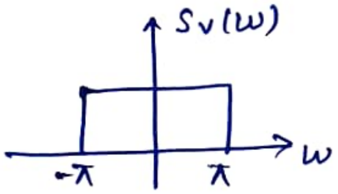
$$= \frac{1-a^2}{1-a(z+z^{-1})+a^2}$$

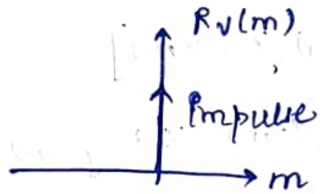
White Noise :

A discrete time process $v(n)$ is called white noise,

iff $\quad S_v(w) = \sigma^2, \quad -\pi \le w \le \pi$

- PSD is uniform for all frequencies.





$$R_v(m) = \sigma_v^2\, \delta(m)$$

Properties of white noise :

ⓐ $E[v(n)] = 0$: Zero mean

ⓑ The samples are uncorrelated.

covariance = Autocorrelation (∵ mean is zero)

ⓒ If the samples follow Gaussian distribution, it is called Gaussian white noise.

Response of LTI system to white noise:

$$v(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

$R_v(m) = \sigma_v^2 \delta(m)$; $\quad S_v(w) = \sigma^2$; $|w| \le \pi$

$R_y(m) = h(m) * h(-m) * R_v(m)$

$\updownarrow$

$S_y(z) = \sigma_v^2 \, |H(z)|^2$

$= \sigma_v^2 \, H(z) H(z^{-1})$

- When white noise is given as input to LTI system, the output is WSS.

Eg. $H(z) = \dfrac{1}{1 - a z^{-1}}, \quad \sigma^2 = 2$

$S_y(z) = 2 \cdot \dfrac{1}{1 - a z^{-1}} \cdot \dfrac{1}{1 - a z}$

$\Rightarrow R_y(m) = 2 \cdot \dfrac{1}{1 - a^2} \cdot a^{|m|}$

PSD of a filtered white noise :

$S_y(z) = \sigma_v^2 \, H(z) \, H(z^{-1})$ ... Spectral factorization

Spectrum Factorization Theorem:

If $S_x(z)$ and $\ln S_x(z)$ are analytical ~~factorizations~~ functions of z in an annular region $p < |z| < 1/p$, then

$S_y(z) = \sigma_v^2 \, H_c(z) \, H_a(z)$

$= \sigma_v^2 \, H_c(z) \, H_c(z^{-1})$

$H_c(z) \rightarrow$ causal min-phase

$H_a(z) \rightarrow$ anticausal.

22-10-2025

Whitening of WSS Process:

$H(Z)$ is a minimum phase filter $\Rightarrow$ an inverse filter exists.

$$x(n) \longrightarrow \boxed{\frac{1}{H_c(Z)}} \longrightarrow v(n)$$

Proof: $\ln S_x(Z)$ is an analytic, ROC: $p < |Z| < 1/p$.

Laurent Series expansion:

$$\ln S_x(Z) = \sum_{K=-\infty}^{\infty} c(K) Z^{-K}$$

$$c(K) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln S_x(w) e^{jwK} dw$$

↳ K^{th}-order cepstral coefficients

→ for real signal, $c(k) = c(-k)$

$$c(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln S_x(w) dw$$

$$S_x(Z) = e^{\sum_{K=-\infty}^{\infty} c(K) Z^{-K}}$$

$$= \underbrace{e^{\sum_{K=-\infty}^{-1} c(K) Z^{-K}}}_{\substack{\text{anti-causal} \\ H_c(Z^{-1})}} \cdot \underbrace{e^{c(0)}}_{\substack{\text{constant} \\ \sigma_v^2}} \cdot \underbrace{e^{\sum_{K=1}^{\infty} c(K) Z^{-K}}}_{\substack{\text{causal} \\ H_c(Z)}}$$

$$= \sigma_v^2 H_c(Z^{-1}) H_c(Z)$$

$$H_c(Z) = e^{\sum_{K=1}^{\infty} c(K) Z^{-K}}, \quad |Z| > p$$

$$= 1 + h_c(1) Z^{-1} + h_c(2) Z^{-2} + \dots$$

$$\left[\because h_c(0) = \lim_{Z \to \infty} H_c(Z) = 1\right]$$

Similarly,

$$H_a(Z) = e^{\sum_{K=-\infty}^{-1} c(K) Z^{-K}}$$

$$= H_c(Z^{-1}), \quad |Z| < 1/p.$$

$$\therefore S_x(Z) = \sigma_v^2 H_c(Z) H_c(Z^{-1})$$

- $S_x(Z)$ can be factorized into a minimum phase $H_c(Z)$ and maximum phase $H_c(Z^{-1})$.

Sufficient condition for spectral factorization:

$$\int_{-\pi}^{\pi} |\ln S_x(\omega)|\, d\omega < \infty \quad : \text{Poly-Weiner Criterion.}$$

Consequence of spectral factorization:

$$v(n) \longrightarrow \underset{H_c(Z)}{\boxed{h(n)}} \longrightarrow x(n)$$

* Innovative representation of WSS process

Whitening of a WSS process:

If $H_c(Z)$ is a minimum phase filter then the corresponding inverse filter exists.

$$x(n) \longrightarrow \boxed{\frac{1}{H_c(Z)}} \longrightarrow v(n)$$

Eg. $S_x(\omega) = 5 - 4\cos\omega$

$$= 5 - 2(Z + Z^{-1})$$

Factorizing into $S_x(Z) = \sigma_v^2 H_c(Z) H_c(Z^{-1})$

$$= 4\left(1 + \tfrac{1}{2}Z^{-1}\right)\left(1 - \tfrac{1}{2}Z\right)$$

$\sigma_v^2 = 4$

$$v \longrightarrow \boxed{H_c(Z) = 1 - \tfrac{1}{2}Z^{-1}} \longrightarrow x(n)$$

$$x(n) \longrightarrow \boxed{\frac{1}{H_c(Z)} = \frac{1}{1 - \frac{1}{2}Z^{-1}}} \longrightarrow v(n)$$

Wold's decomposition

Any WSS signal can be decomposed as a sum of two mutually orthogonal process.

$$x(n) = \underbrace{x_p(n)}_{\text{prediction process}} + \underbrace{x_r(n)}_{\text{required process}}$$

$x_r(n) \rightarrow$ can be expressed as the o/p of LTI using a white noise sequence of i/p.

$x_p(n) \rightarrow$ predictable process is the process to be predicted from its own part with zero prediction error.

$$E\{x_r(m)\, x_p^*(m)\} = 0.$$

Optimum Linear filter (Eg. wiener filter)

$$x(n) \longrightarrow \oplus \longrightarrow y(n) = x(n) + v(n)$$

$x(n)$ ← original signal

$v(n)$ → $\oplus$

$v(n)$ ← white Gaussian noise with mean 0, σ^2

$y(n)$ ← observed signal

$y(n) = y(n), y(n-1), \ldots, y(1)$

$$R_x(m) = \mathbb{E}[x(n)\,x(n+m)]$$

$$r_{xy} = \mathbb{E}[x(n)\,\mathbf{y}(n)] = \begin{bmatrix} \mathbb{E}[x(n)\,y(n)] \\ \mathbb{E}[x(n)\,y(n-1)] \\ \vdots \\ \mathbb{E}[x(n)\,y(1)] \end{bmatrix} = \begin{bmatrix} R_{xy}(0) \\ R_{xy}(-1) \\ \vdots \\ \end{bmatrix}$$

Autocorrelation matrix:

$$R_y = \mathbb{E}[\mathbf{y}(n)\,\mathbf{y}'(n)]$$

$$= \begin{bmatrix} R_y(0) & R_y(1) & \cdots & R_y(n-1) \\ R_y(1) & R_y(0) & \cdots & R_y(n-2) \\ \vdots & & & \vdots \\ R_y(n-1) & & \cdots & R_y(0) \end{bmatrix}$$

Optimum filtering / Weiner filtering

$$x(n) \longrightarrow \oplus \xrightarrow{y(n)} \boxed{\text{Linear filter}} \longrightarrow \hat{x}(n)$$

$v(n)$ → $\oplus$

$$[\,y(n-M+1) \ldots y(n) \ldots y(n+N)\,]$$

$$\hat{x}(n) = \sum_{\substack{i=-\infty \\ i=-N}}^{\substack{M-1 \\ \infty}} h(i)\,y(n-i)$$

The problem is to find an optimal set of filter coefficients.

$h(-N),\ h(-N+1)\ \ldots\ h(0);\ h(1), \ldots h(M-1)$

↳ Non-causal

Mean Square Error

$$\mathbb{E}\left[x(n) - \hat{x}(n)\right]^2 \longrightarrow \text{minimum}$$

$$= \mathbb{E}\left[x(n) - \sum_{i=-N}^{M-1} h(i)\, y(n-i)\right]^2$$

Minimize w.r.t. $h(-N), h(-N+1), \ldots, h(M-1)$.

Minimum Mean Square Error (MMSE):

- Linear MMSE : The error function will be linear.

$$\begin{array}{ccccc} y(n-M+1) & \cdots & y(n) & \cdots & y(n+N) \\ \hline n-M+1 & \cdots & n & \cdots & n+N \end{array}$$

Applications:

① Optimal Smoothening : $N>0$

② optimal filtering : $N=0$

③ Optimal prediction : $N<0$.

Wiener hopf equation

$$\mathbb{E}[e^2(n)] = \mathbb{E}\left[x(n) - \hat{x}(n)\right]^2$$

$$= \mathbb{E}\left[x(n) - \sum_{i=-N}^{M-1} h(i)\, y(n-i)\right]^2$$

We have to minimize $\mathbb{E}\{e^2(n)\}$ w.r.t. each $h(i)$ to get optimal estimation.

$$\frac{\partial\, \mathbb{E}\{e^2(n)\}}{\partial h(j)} = 0, \quad j = -N, \ldots, 0, \ldots, M-1.$$

$\mathbb{E}$ and $\frac{\partial}{\partial h(j)}$ can be interchanged.

$$\Rightarrow \mathbb{E}\left[e(n) \cdot y(n-j)\right] = 0 \qquad \text{[orthogonal]}$$

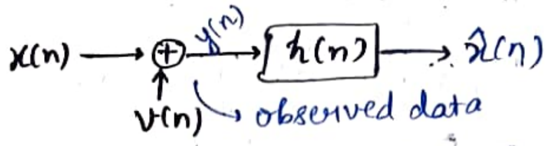
$$\Rightarrow \mathbb{E}\left[(x(n) - \hat{x}(n))\, y(n-j)\right] = 0$$

$$\Rightarrow \mathbb{E}\Big[\underbrace{\Big\{x(n) - \sum_{i=-N}^{M-1} h(i)\, y(n-i)\Big\}}_{e(n)}\, y(n-j)\Big] = 0 \quad : \text{optimality condition for LMMSE}$$

$$\Rightarrow R_{xy}(j) = \sum_{i=-N}^{M-1} h(i)\, R_y(j-i), \quad j = -N, \ldots, M-1.$$

27-10-2025

FIR Wiener Filter



$$\hat{x}(n) = \sum_{i=0}^{M-1} h(i)\, y(n-i)$$

$$\mathbb{E}\left[x(n) - \sum_{i=0}^{M-1} h(i)\, y(n-i)\right] y(n-j) = 0 \rightarrow \text{optimal condition}$$

$$\Rightarrow \sum_{i=0}^{M-1} h(i)\, R_y(j-i) = R_{xy}(j)$$

- There are a set of M normal equations.

Matrix form of the Wiener Hopf eqn:

$$\underset{\substack{\downarrow \\ \text{observed data}}}{R_y}\, h = \underset{\substack{\downarrow \\ \text{desired data}}}{r_{xy}} \quad (M \times N)$$

(auto corr. matrix)

$$\Rightarrow h = R_y^{-1} r_{xy}$$

→ Difficulty : finding inverse.

$$R_y = \begin{bmatrix} R_y(0) & R_y(1) & \cdots & R_y(M-1) \\ R_y(1) & & \ddots & \vdots \\ \vdots & & & \\ R_y(M-1) & & \cdots & \end{bmatrix}$$

$$h = \begin{bmatrix} h(0) \\ h(1) \\ \vdots \\ h(M-1) \end{bmatrix}, \quad r_{xy} = \begin{bmatrix} R_{xy}(0) \\ R_{xy}(1) \\ \vdots \\ \end{bmatrix}$$

Problem:

$x(n) = d(n) + v(n)$

observed signal, desired signal, noise (zero mean; uncorrelated with $d(n)$)

Design the filter.

$h = R_x^{-1} \gamma_{dx}$

$\gamma_{dx} = \mathbb{E}[d(n)\, x(n-k)]$

$= \mathbb{E}[d(n) \{ d(n-k) + v(n-k) \}]$

As $\mathbb{E}[d(n)\, v(n-k)] = 0$ ($\because$ uncorrelated)

$\therefore \gamma_{dx} = \gamma_d(k)$

$R_x(k) = \mathbb{E}[x(n+k)\, x(n)]$

$= \mathbb{E}[(d(n+k) + v(n+k))\,(d(n) + v(n))]$

$= \gamma_d(k) + \gamma_v(k)$

$\therefore \boxed{h = [\gamma_d(k) + \gamma_v(k)]^{-1} \gamma_d(k)}$ → Wiener hopf eq^n

Problem:

Let $d(n) \to AR(1) \rightsquigarrow$ random

$\gamma_d(k) = \alpha^{|k|}, \quad 0 < \alpha < 1 \quad [\alpha = 0.8]$

$d(n) \to$ desired, $x(n) \to$ observed

$v(n) \to \sigma_v^2$ (uncorrelated) $\quad [\sigma_v^2 = 1]$

Design a 1st order FIR Wiener filter to remove noise.

Soln: 1st order → 2 coefficients $(h(0), h(1))$

$h = R_{obs}^{-1} \cdot \gamma_{obs., des.}$

$$\begin{bmatrix} h(0) \\ h(1) \end{bmatrix} = \begin{bmatrix} \gamma_x(0) & \gamma_x(1) \\ \gamma_x(1) & \gamma_d(0) \end{bmatrix}^{-1} \begin{bmatrix} \gamma_{dx}(0) \\ \gamma_{dx}(1) \end{bmatrix}$$

$\gamma_{dx} = \gamma_d$

$\gamma_{dx}(0) = ? \to 1$

$\gamma_{dx}(1) = ? \to \alpha$

$\gamma_x(k) = \gamma_d(k) + \gamma_v(k)$

$= \alpha^{|k|} + \sigma_v^2 \delta(k)$

$\Rightarrow \gamma_x(0) = 1 + \sigma_v^2$

$\gamma_x(1) = \alpha$

$$\begin{bmatrix} h(0) \\ h(1) \end{bmatrix} = \begin{bmatrix} 1+\sigma_v^2 & \alpha \\ \alpha & 1+\sigma_v^2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ \alpha \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0.8 \\ 0.8 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0.8 \end{bmatrix} = \begin{bmatrix} 2 & -0.8 \\ -0.8 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.8 \end{bmatrix} \frac{1}{4-0.64}$$

$$= \begin{bmatrix} 0.4048 \\ 0.23 \end{bmatrix}$$

$\Rightarrow h(0) = 0.4048$

$h(1) = 0.23$

$$E_{min} = \gamma_d(0) - h(0)\gamma_{dx}(0) - h(1)\gamma_{dx}(1) + h(0)^2\gamma_x(0) + 2h(0)h(1)\gamma_x(1) + h(1)^2\gamma_x(0)$$

$$= 0.4045$$

IIR Wiener Filter

Eg. Noise filtering by Non-Causal IIR filter

Consider a case of carrier signal in the presence of white Gaussian noise.

$$y(n) = x(n) + v(n)$$

obs. des.

$$R_y(n) = E[y(n)\,y(n+m)]$$

$$= E[(x(n+m) + v(n+m))(x(n) + v(n))]$$

$$= R_x(m) + R_v(m) \quad [\because E[x(n+m)v(n)] = 0]$$

($\because$ uncorrelated)

$$S_y(w) = S_x(w) + S_v(w)$$

$$R_{xy}(m) = E[x(n+m)\,y(n)]$$

$$= E[x(n+m)(x(n) + v(n))]$$

$$= \gamma_x(m)$$

28-10-2025

$$H(w) = \frac{S_{xy}(w)}{S_y(w)} = \frac{S_x(w)}{S_x(w) + S_v(w)} = \frac{S_x(w)}{S_v(w)\left[1 + \frac{S_x(w)}{S_v(w)}\right]}$$

$$\begin{bmatrix} S_y(w) = S_x(w) + S_v(w) \\ S_{xy}(w) = S_x(w) \end{bmatrix}$$

$$SNR = \frac{S_x(w)}{S_v(w)} \to \infty \Rightarrow \text{Pass without attenuation}$$

$$H(w) = \frac{\cancel{S_x(w)}}{\cancel{S_x(w)}\left[1 + \frac{S_v(w)}{S_x(w)}\right]}$$

$$\approx 1$$

Eg. Consider a signal $y(n) = x(n) + v(n)$. $v(n)$ is GWN with $\sigma_v^2 = 1$, $x(n) = 0.8\,x(n-1) + w(n)$, where $w(n)$ is zero mean WGN with $\sigma_w^2 = 0.68$. The signal and noise are uncorrelated. Find the optimal non-causal Wiener Filter to estimate $x(n)$.

Soln:

$$w(n) \longrightarrow \boxed{\frac{1}{1-0.8z^{-1}}} \longrightarrow x(n)$$

$$\frac{X(z)}{W(z)} = \frac{1}{1-0.8z^{-1}}$$

$$S_x(z) = \frac{\cancel{S_w(z)} \to \sigma_w^2 = 0.68}{(1-0.8z^{-1})(1-0.8z)}$$

$$S_y(z) = S_x(z) + \cancel{S_v(z)} \to \sigma_v^2 = 1$$

$$= \frac{0.68}{(1-0.8z^{-1})(1-0.8z)} + 1$$

$$= \frac{0.68 + (1-0.8z^{-1})(1-0.8z)}{(1-0.8z^{-1})(1-0.8z)}$$

$$= \frac{2.32 - 0.8(z+z^{-1})}{(1-0.8z^{-1})(1-0.8z)}$$

$$= \frac{2(1-0.4z^{-1})(1-0.4z)}{(1-0.8z^{-1})(1-0.8z)}$$

$$H(z) = \frac{S_{xy}(z)}{S_x(z) + S_v(z)}$$

$$= \frac{S_x(z)}{S_x(z) + S_v(z)}$$

$$= \frac{0.68}{(1-0.8z^{-1})(1-0.8z)} \cdot \frac{(1-0.8z^{-1})(1-0.8z)}{2(1-0.4z^{-1})(1-0.4z)}$$

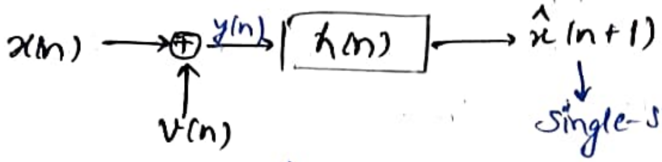
$$= \frac{0.34}{(1-0.4z^{-1})(1-0.4z)}$$

$$= \frac{A}{1-0.4z} + \frac{B}{1-0.4z^{-1}}$$

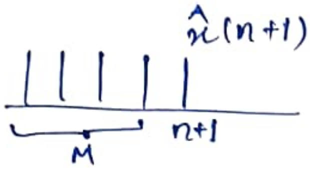
$$= \frac{0.404}{1-0.4z} + \frac{0.404}{1-0.4z^{-1}}$$

$$h(n) = 0.404\,(0.4)^{n}u(n) + 0.404\,(0.4)^{-n}u(-n-1).$$

Linear Predictor FIR



Single-step Prediction



$y(n) = x(n) + v(n)$

Noise is uncorrelated.

$h = R_y^{-1} r_{xy}$

obs. des. obs.

$\hat{x}(n+1)$

$R_y(K) = \cancel{E[y(n)}$

$= [y(n) y(n-K)]$

$= [\cancel{x(n)} (x(n)+v(n))(x(n-K)+v(n-K))]$

$\Rightarrow R_y(K) = R_x(K) + r_v(K)$

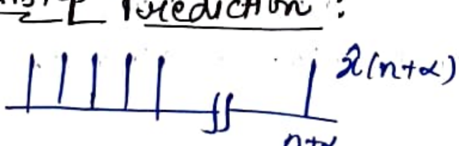
$r_{dy}(K) = E[\hat{x}(n+1)[x(n-K)+v(n-K)]] \quad [\equiv d(n)\, y(n-K)]$

$= r_x(K+1)$

Single-step:

$$\begin{bmatrix} h(0) \\ \vdots \\ h(M-1) \end{bmatrix} = \begin{bmatrix} r_y(0) & r_y(1) & \cdots & r_y(M-1) \\ r_y(1) & r_y(0) & & \\ \vdots & & & \\ r_y(M-1) & & \cdots & r_y(0) \end{bmatrix}^{-1} \begin{bmatrix} r_x(1) \\ r_x(2) \\ \vdots \\ r_x(M) \end{bmatrix}$$

Multistep Prediction:



$h = R_y^{-1} r_{xy}$

$r_{dy} = r_x(K+\alpha)$

$$\begin{bmatrix} h(0) \\ h(1) \\ \vdots \\ h(M-1) \end{bmatrix} = \begin{bmatrix} r_y(0) & r_y(1) & \cdots & r_y(M-1) \\ r_y(1) & & & \\ \vdots & & & \\ r_y(M-1) & & \cdots & r_y(0) \end{bmatrix}^{-1} \begin{bmatrix} r_x(\alpha) \\ r_x(\alpha+1) \\ \vdots \\ r_x(\alpha+M-1) \end{bmatrix}$$

MMSE : $E_{min} = \gamma_x(0) - \sum_{K=0}^{M-1} h(K)\gamma_x(\alpha+K)$

Eg Design a 1^{st} order one-step predictor for IIR process given as $d(n)$ with AC f^n $\gamma_d(K) = \alpha^{|K|}$, $0<\alpha<1$, $[\alpha = 0.8]$, which has a noise with σ_v^2 ~~vari~~ variance $[\sigma_v^2 = 1]$.

$$\begin{bmatrix} h(0) \\ h(1) \end{bmatrix} = \begin{bmatrix} \gamma_y(0) & \gamma_y(1) \\ \gamma_y(1) & \gamma_y(0) \end{bmatrix}^{-1} \begin{bmatrix} \gamma_d(1) \\ \gamma_d(2) \end{bmatrix}$$

$\gamma_y(K) = R_x(K) + R_v(K)$ $\qquad [R_x = R_d]$

$\Rightarrow \gamma_y(0) = 1 + \sigma_v^2$

$\gamma_y(1) = \alpha$

$\cancel{\gamma_y(2) = \alpha^2 + \sigma_v^2}$

$\gamma_d(0) = 1$

$\gamma_d(1) = \alpha$

$$\therefore \begin{bmatrix} h(0) \\ h(1) \end{bmatrix} = \begin{bmatrix} 1+\sigma_v^2 & \alpha \\ \alpha & 1+\sigma_v^2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ \alpha \end{bmatrix}$$

29-10-2025

Wiener IIR Filter

$$x(n) \longrightarrow \oplus \xrightarrow{\;y(n)\;} \boxed{h(n)} \longrightarrow \hat{x}(n)$$

$$\uparrow v(n)$$

$$\hat{x}(n) = \sum_{i=0}^{\infty} h(i)\, y(n-i)$$

$$e^2(n) = E\left[(x(n) - \hat{x}(n))^2 \right]$$

Orthogonality principle:

$$E\{ e(n)\, y(n) \} = 0$$

$$\Rightarrow E\left[\left(x(n) - \sum_{i=0}^{\infty} h(i)\, y(n-i) \right) y(n-j) \right] = 0$$

$$\Rightarrow \sum_{i=0}^{\infty} h(i)\, R_y(j-i) = R_{xy}(j)$$

$h(i) \to$ one-sided

$R_y(j) \to$ two-sided

Wiener suggested spectral factorization.

$$S_y(z) = \sigma_v^2\, H_c(z)\, H_c(z^{-1})$$

Input has to be white. [Whitening process]

Obs. signal

$$y(n) \longrightarrow \boxed{H_1(z) = \frac{1}{H_c(z)}} \longrightarrow v(n)$$

whitening filter

Innovation process:

$$v(n) \longrightarrow \boxed{H_2(z)} \longrightarrow x(n)$$

$$\begin{matrix} Y(z) \\ y(n) \end{matrix} \longrightarrow \underbrace{\boxed{H_1(z)} \longrightarrow \boxed{H_2(z)}} \longrightarrow \begin{matrix} \hat{X}(z) \\ \hat{x}(n) \end{matrix}$$

$\underline{H_2(z)}$:

$$\hat{x}(n) = \sum_{i=0}^{\infty} h_2(i)\, v(n-i)$$

(causality)

MSE : $\mathbb{E}\left[\left(x(n)-\sum_{i=0}^{\infty} h_2(i)\, v(n-i)\right)^2\right]$

Orthogonality principle :

$$\mathbb{E}\left[\underbrace{\left(x(n)-\sum_{i=0}^{\infty} h_2(i)\, v(n-i)\right)}_{e(n)} v(n-j)\right], \quad j=0,1,\dots,\infty$$

$$R_{xv}(j)=\sum_{i=0}^{\infty} h_2(i)\, R_v(j-i)$$

$$\left\{R_v(n)=\sigma^2 \delta(n) \quad [\text{Constant PSD}]\right\}$$

$$\Rightarrow R_{xv}(j)=\sum_{i=0}^{\infty} h_2(i)\sigma_v^2\delta(j-i)$$

$$i=j: \; h_2(j)\sigma_v^2 = R_{xv}(j)$$

$$\Rightarrow h_2(j)=\frac{R_{xv}(j)}{\sigma_v^2}$$

$$\Rightarrow H_2(z)=\frac{S_{xv}(z)}{\sigma_v^2}$$

$\underline{H_1(z)}$:

$$v(n)=\sum_{i=0}^{\infty} h_1(n).\, y(n-i)$$

$$R_{xv}(j)=\mathbb{E}\left[x(n).\, v(n-j)\right]$$

$$=\mathbb{E}\left[\sum_{i=0}^{\infty} h_1(i)\, x(n)\, y(n-i-j)\right]$$

$$=\sum_{i=0}^{\infty} h_1(i)\, R_{xy}(i+j) \Rightarrow \frac{H_1(z^{-1})}{\cancel{H_c(z^{-1})}} S_{xy}(z)=S_{xv}(z)$$

$$\Rightarrow S_{xv}(z)=\frac{1}{H_c(z^{-1})} S_{xy}(z) \qquad \left[\because H_1(z^{-1})=\frac{1}{H_c(z^{-1})}\right]$$

$$\therefore H_2(z)=\frac{S_{xv}(z)}{\sigma_v^2}=\frac{1}{H_c(z^{-1})}\,\frac{S_{xy}(z)}{\sigma_v^2}$$

$$\therefore H_0(z)=H_1(z)\,H_2(z)$$

$$=\frac{1}{H_c(z)}\left[\frac{1}{\sigma_v^2}\,\frac{S_{xy}(z)}{H_c(z^{-1})}\right]$$

Mean Square Error :

$$\mathbb{E}[e^2(n)]$$

$$= \mathbb{E}\left[e(n)\left\{x(n) - \sum_{i=0}^{\infty} h(i)\, y(n-i)\right\}\right],$$

where $\mathbb{E}[e(n)\, x(n)] = 0$

$$\mathbb{E}\left[\left[x(n) - \sum_{i=0}^{\infty} h(i)\, y(n-i)\right] x(n)\right] = 0$$

$$= R_x(0) - \sum_{i=0}^{\infty} h(i)\, R_{yx}(i)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} S_x(\omega)\, d\omega - \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega)\, S_{xy}^{*}(\omega)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(S_x(\omega) - H(\omega)\, S_{xy}^{*}(\omega)\right) d\omega$$

$$= \frac{1}{2\pi}\oint \left[S_x(Z) - H(Z)\, S_{xy}(Z^{-1})\right] Z^{-1}\, dZ$$

Eg. $y(n) = x(n) + v(n)$ → WGN ; $\sigma_v^2 = 1$

$x(n) = 0.8\, x(n-1) + w(n)$ → WGN; $\sigma_w^2 = 0.68$

Design optimal causal W.F. [Noise is uncorrelated.]

Soln:

$$w(n) \longrightarrow \boxed{\frac{1}{1-0.8Z^{-1}}} \longrightarrow x(n)$$

$$S_x(Z) = \sigma_w^2\, H_c(Z)\, H_c(Z^{-1})$$

$$= \frac{0.68}{(1-0.8Z^{-1})(1-0.8Z)}$$

$$S_y(Z) = S_x(Z) + S_v(Z)$$

$S_v(Z)$ → $\sigma_v^2 = 1$

$$= \frac{0.68}{(1-0.8Z^{-1})(1-0.8Z)} + 1$$

$$= \frac{2\,(1-0.4Z^{-1})(1-0.4Z)}{(1-0.8Z^{-1})(1-0.8Z)} \qquad [\sigma^2 H_c(Z)\, H_c(Z^{-1})]$$

$$H_c(Z) = \frac{1-0.4Z^{-1}}{1-0.8Z^{-1}}$$

Noise and signal are uncorrelated.

$$S_{xy}(Z) = S_x(Z)$$

$$H(Z) = \frac{1}{\sigma_v^2 H_c(Z)} \left[\frac{S_{xy}(Z)}{H_c(Z^{-1})} \right]_+$$ → consider the causal term only

$$= \frac{1}{2} \frac{(1-0.8Z^{-1})}{(1-0.4Z^{-1})} \left[\frac{0.68/(1-0.8Z^{-1})(1-0.8Z)}{\frac{1-0.4Z}{1-0.8Z}} \right]_+$$

$$= \frac{1}{2} \frac{1-0.8Z^{-1}}{1-0.4Z^{-1}} \left[\frac{0.68}{(1-0.8Z^{-1})(1-0.4Z)} \right]_+$$

$$= \frac{0.68}{2(1-0.4Z^{-1})}$$

$$\Rightarrow h(n) = 0.34 \, (0.4)^n \, u(n).$$

03-11-2025

Special Random Process

① ARMA (p,q) → Auto Regressive Moving Average process

$p \to$ poles
$q \to$ zeros

when we filter the white noise with a linear shift-invariant filter, the output process will be a WSS.

$$H(z) = \frac{B_q(z)}{A_p(z)} = \frac{\sum_{k=0}^{q} b_q(k)\, z^{-k}}{1 + \sum_{k=1}^{p} a_p(k)\, z^{-k}}$$

$$\underset{\text{white noise}}{v(n)} \longrightarrow \boxed{h(n)} \longrightarrow x(n)$$

Filter: Shift-invariant, stable

$$P_v(z) = \sigma_v^2$$

$$P_x(z) = \frac{\sigma_v^2\, B_q(z)\, B_q^*(1/z^*)}{A_p(z)\, A_p^*(1/z^*)}$$

$$P_x(e^{j\omega}) = \frac{\sigma_v^2\, |B_q(e^{j\omega})|^2}{|A_q(e^{j\omega})|^2}$$

- A process having a power spectrum of this form is known as ARMA (p,q) process.

② $$x(n) + \sum_{k=1}^{p} a_k\, x(n-k) = \sum_{k=0}^{q} b_k\, v(n-k)$$

For AR $(p,0)$,
$b_0 = 1, \quad b_k = 0 \quad \forall\, k > 0$

$$H(z) = \frac{1}{A_q(z)\, A_q^*(1/z^*)}$$

$$x(n) + \sum_{k=1}^{p} a_k\, x(n-k) = v(n) \to \text{AR}(p,0)$$

③ MA process:

$a_k = 0, \ \forall\, k \geq 1$

$$x(n) = \sum_{k=0}^{q} w(n-k)$$

$H(z) = B(z)$
MA $(0,q)$

Yule walker equation:

$$v(n) \longrightarrow \boxed{h(n)} \longrightarrow x(n)$$

$$x(n) + \sum_{l=1}^{p} a_p(l)\, x(n-l) = \sum_{l=0}^{q} b_q(l)\, v(n-l)$$

Multiply both side with $x^*(n-k)$ and take expectation.

$$r_x(k) + \sum_{l=1}^{p} a_p(l)\, r_x(k-l) = \sum_{l=0}^{q} b_q(l)\, \underbrace{E\left[v(n-l)\, x^*(n-k)\right]}_{r_{vx}(k-l)}$$

$$x(n) = h(n) * v(n) = \sum_{m=-\infty}^{\infty} v(m)\, h(n-m)$$

$$E\left[v(n-l)\, x^*(n-k)\right]$$

$$= E\left[\sum_{m=-\infty}^{\infty} v(n-l)\, v^*(m)\, h^*(n-k-m)\right]$$

$$= \sum_{m=-\infty}^{\infty} \underbrace{E\left[v(n-l)\, v^*(m)\right] h^*(n-k-m)}_{\sigma_v^2 h^*(l-k)}$$

As $E\left[v(n-l)\, v^*(m)\right] = \sigma_v^2\, \delta(n-l-m)$

$$\therefore\ r_x(k) + \sum_{k=1}^{p} a_p(l)\, r_x(k-l) = \sigma_v^2 \underbrace{\sum_{l=0}^{q} b_q(l)\, h^*(l-k)}_{c_q(k)}$$ ... Yule Walker eqn

where, $c_q(k) = \sum_{l=k}^{q} b_q(l)\, h^*(l-k)$

$$= \sum_{l=0}^{q-k} b_q(l+k)\, h^*(l)$$

If the filter is causal,

If $k > q$, $c_q(k) = 0$

$$\therefore\ r_x(k) + \sum_{l=1}^{p} a_p(l)\, r_x(k-l) = \begin{cases} \sigma_v^2\, c_q(k), & 0 \le k \le q \\ 0, & k > q \end{cases}$$

$$r_x(k) = -\sum_{l=1}^{p} a_p(l)\, r_x(k-l)$$

$$\begin{bmatrix} r_x(0) & r_x(-1) & \cdots & r_x(-p) \\ r_x(1) & r_x(0) & \cdots & r_x(-p+1) \\ \vdots & & & r_x(q-p+1) \\ r_x(q+p) & & & r_x(q) \end{bmatrix} \begin{bmatrix} 1 \\ a_p(1) \\ a_p(2) \\ \vdots \\ a_p(p) \end{bmatrix} = \sigma_v^2 \begin{bmatrix} c_q(0) \\ c_q(1) \\ \vdots \\ c_q(q) \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

$q=0 \rightarrow$ AR process
$p=0 \rightarrow$ MA process

AR Process ($q=0$):

$$H(Z) = \frac{B_q(Z)}{A_p(Z)} = \frac{\sum_{K=0}^{q} b_q(K) Z^{-K}}{1 + \sum_{K=+1}^{p} a_p(K) Z^{-K}}$$

$$q=0 \Rightarrow \quad H(Z) = \frac{b(0)}{1 + \sum_{K=1}^{p} a_p(K) Z^{-K}}, \quad P_v(Z) = \sigma_v^2$$

$\text{ARMA}(p,0) \longrightarrow \text{AR}(p)$

$$P_x(Z) = P_v(Z) \frac{|b(0)|^2}{A_p(Z) A_p^*(1/Z^*)}$$

$$P_x(e^{j\omega}) = \underbrace{P_v(e^{j\omega})}_{\sigma_v^2} \frac{|b(0)|^2}{|A_p(e^{j\omega})|^2}$$

$$r_x(K) + \sum_{l=1}^{p} a_p(l) r_x(K-l) = \sigma_v^2 |b(0)|^2 \delta(K)$$

$$c_0(0) = b(0) b^*(0) = |b(0)|^2, \quad K \geq 0$$

$$\begin{bmatrix} r_x(0) & r_x(-1) & \cdots & r_x(-p) \\ r_x(1) & \cdots & & r_x(-p+1) \\ \vdots & & & \\ r_x(p) & \cdots & & r_x(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_p(1) \\ \vdots \\ a_p(p) \end{bmatrix} = \sigma_v^2 |b(0)|^2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

Problem:

Consider first two AR values of a real-valued AR(1) process.
Assume $\sigma_v^2 = 1$.

$$r_x(-k) = r_x(k) \quad \text{[property]}$$

$$\begin{bmatrix} r_x(0) & r_x(1) \\ r_x(1) & r_x(0) \end{bmatrix} \begin{bmatrix} 1 \\ a(1) \end{bmatrix} = \sigma_v^2 |b(0)|^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow r_x(0) + r_x(1)\, a(1) = \sigma_v^2 |b(0)|^2$$

$$r_x(1) + a(1)\, r_x(0) = 0$$

$$\Rightarrow a(1) = -\frac{r_x(1)}{r_x(0)},$$

$$|b(0)|^2 = \frac{r_x^2(0) - r_x^2(1)}{r_x(0)}$$

$$\therefore H(z) = \frac{b(0)}{1 + a(1) z^{-1}}$$

$$\Rightarrow H(z) = \frac{\sqrt{r_x(0)\left(r_x^2(0) - r_x^2(1)\right)}}{r_x(0) - r_x(1) z^{-1}}$$

04-11-2025

MA process : MA(0,q)

ARMA(p,q) → p=0

$$H(Z) = \sum_{K=0}^{q} b_q(K) Z^{-K}$$

$v(n) \longrightarrow \boxed{h(n)} \longrightarrow x(n)$

$$P_x(Z) = \sigma_v^2 B_q(Z) B_q^*(1/Z^*)$$

$$P_x(e^{jw}) = \sigma_v^2 |B_q(e^{jw})|^2$$

$P_x(Z) \longrightarrow 2q$ zeroes & no poles (except at $Z=0$ or $Z=\infty$)

$$C_q(K) = \sum_{l=0}^{q-K} b_q(l+K) b_q^*(l)$$

$$= \sigma_v^2 \sum_{l=0}^{q-|K|} b_q(l+|K|) b_q^*(l)$$

Eg. Suppose that we determine the zeros for $q=2$.

$$r_x(K) = \sigma_v^2 \sum_{l=0}^{q-|K|} b_q(l+|K|) b_q^*(l)$$

$K=0 \Rightarrow r_x(0) = \sigma_v^2 \sum_{l=0}^{q} b_q(l) b_q^*(l)$ [Take $\sigma_v^2 = 1$]

$= \sum_{l=0}^{2} b_q(l) b_q^*(l)$ | b → filter coeff.

$= b_2(0) b_2^*(0) + b_2(1) b_2^*(1) + b_2(2) b_2^*(2)$

$= |b_2(0)|^2 + |b_2(1)|^2 + |b_2(2)|^2$

$K=1 \Rightarrow r_x(1) = b_2(1) b_2^*(0) + b_2(2) b_2^*(1)$

$K=2 \Rightarrow r_x(2) = b_2(2) b_2^*(0)$

Eg. Consider a MA(q) process that is generated by the difference eqn $y(n) = \sum_{K=0}^{q} b(K) w(n-K)$,

$w(n)$ is zero mean white noise with σ_w^2 variance.

Find :

(i) Unit sample response of the filter that generates $y(n)$ from $w(n)$.

(ii) autocorrelation of $y(n)$.

(iii) power spectrum of $y(n)$.

Soln: (i) $H(Z) = \sum_{K=0}^{q} b(K) Z^{-K}$

$$h(n) = \sum_{K=0}^{q} b(K)\, \delta(n-K)$$

(ii) $r_y(K) = \delta(K) * h(K) * h(-K)$

$$r_y(K) = \sigma_w^2 \sum_{l=0}^{q-|K|} b_q(l+|K|)\, b_q^*(l)$$

(iii) $P_y(e^{j\omega}) = \sigma_w^2 B_q(e^{j\omega}) B_q^*(e^{-j\omega})$

Eg. Let $x(n)$ be a random process that is generated by filtering white noise $w(n)$ with first order linear shift invariant filter having system eqn.

$$H(Z) = \frac{1}{1-0.25Z^{-1}}$$

Consider $\sigma_w^2 = 1$. Find output power spectrum of $x(n)$.

Soln:
$$P_x(Z) = \sigma_w^2 H(Z) H(Z^{-1})$$
$$= \frac{1}{1-0.25Z^{-1}} \cdot \frac{1}{1-0.25Z}$$
$$= \frac{Z^{-1}}{(1-0.25Z^{-1})(Z^{-1}-0.25)}$$
$$= \frac{16/15}{1-0.25Z^{-1}} + \frac{4/15}{Z^{-1}-0.25}$$
$$= \frac{16/15}{1-0.25Z^{-1}} - \frac{16/15}{1-4Z^{-1}}$$
$$R_x(n) = \frac{16}{15}(1/4)^n u(n) + \frac{16}{15}(4)^n u(-n-1).$$

Eg. The o/p power spectrum is given by $P_x(e^{j\omega}) = \frac{5+4\cos 2\omega}{10+6\cos\omega}$.
Determine a stable causal filter to generate this.

Soln:

$$P_x(e^{j\omega}) = \frac{5+4\cos 2\omega}{10+6\cos\omega}$$

$$P_x(Z) = \frac{5+4\left[\frac{Z^2+Z^{-2}}{2}\right]}{10+6\left[\frac{Z+Z^{-1}}{2}\right]}$$

$$= \frac{5+2(Z^2+Z^{-2})}{10+3(Z+Z^{-1})}$$

$$= \frac{\underbrace{(2Z^2+1)}\,\underbrace{(2Z^{-1}+1)}}{\underbrace{(3Z+1)}_{H(Z)}\,\underbrace{(3Z^{-1}+1)}_{H(Z^{-1})}}$$

$$H(Z) = \frac{2Z^2+1}{3Z+1}$$

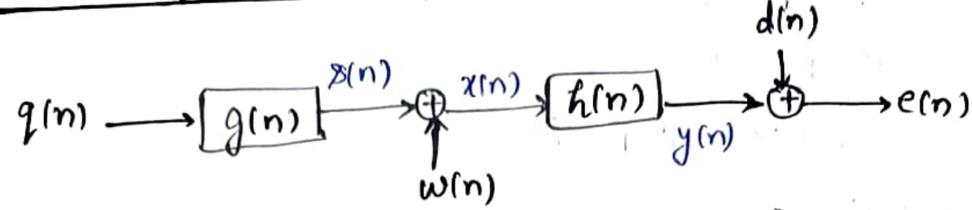
$$= \frac{2Z^2}{3Z}\frac{\left[1+\frac{1}{2Z^2}\right]}{\left[1+\frac{1}{3Z}\right]}$$

$$= \frac{2Z}{3}\left[\frac{1+\frac{1}{2}Z^{-2}}{1+\frac{1}{3}Z^{-1}}\right]$$

delay or advance (doesn't alter the system)

10-11-2025

Deconvolution Filter



$$s(n) = g(n) * q(n)$$

$$y(n) = x(n) * h(n)$$

$$= \sum_{l=-\infty}^{\infty} h(l)\, x(n-l)$$

$$e(n) = d(n) - y(n)$$

$$= d(n) - \sum_{l=-\infty}^{\infty} h(l)\, x(n-l)$$

Orthogonality principle:

$$\mathbb{E}\left[e(n)\, y(n-k)\right] = 0$$

$$\mathbb{E}\left[\left(d(n) - y(n)\right) y(n-k)\right] = 0$$

$$r_{dx}(K) = \sum_{l=-\infty}^{\infty} h(l)\, r_x(K-l)$$

$$= h(K) * r_x(K)$$

$$\Rightarrow P_{dx}(Z) = H(Z)\, P_x(Z)$$

$$\Rightarrow H(Z) = \frac{P_{dx}(Z)}{P_x(Z)}$$

$$r_x(K) = r_s(K) + r_w(K)$$

$$\Rightarrow P_x(Z) = P_s(Z) + P_w(Z)$$

$$\therefore\ H(e^{j\omega}) = \frac{P_{dx}(e^{j\omega})}{P_s(e^{j\omega}) + P_w(e^{j\omega})}$$

$$P_s(e^{j\omega}) = |G(e^{j\omega})|^2\, P_q(e^{j\omega})$$

$$r_{dx}(K) = \mathbb{E}\left[d(n)\, x(n-k)\right] = r_q(K)$$

$$= \mathbb{E}\left[d(n)\left(s(n-k) + w(n-k)\right)\right]$$

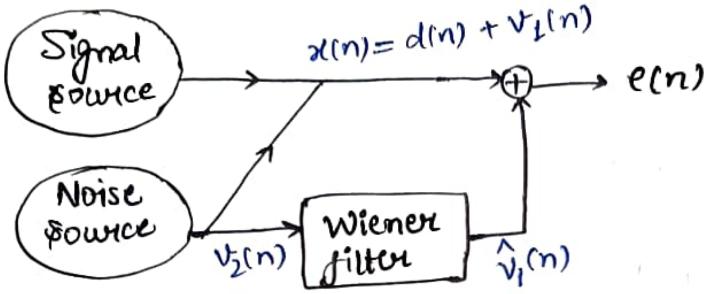
$$= r_{ds}(K) \qquad \left[\because\ \underbrace{\mathbb{E}\{d(n)\, w(n-k)\}}_{=0}\right]$$

$$= g(-K) * r_q(K)$$

$$\therefore \quad P_{dx}(e^{j\omega}) = G(e^{j\omega}) P_q(e^{j\omega})$$

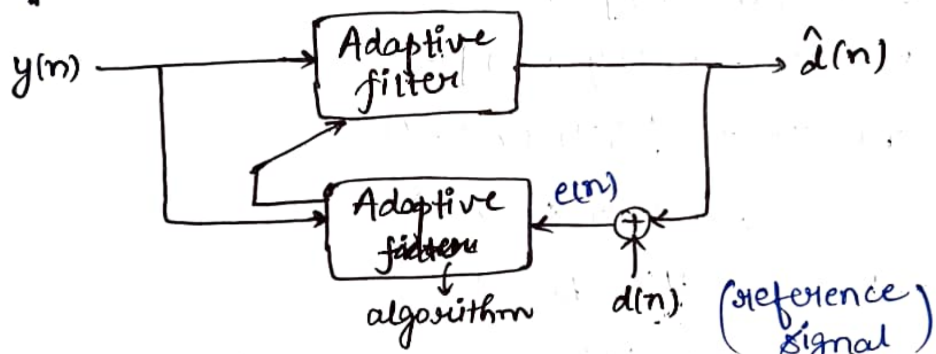
$$H(e^{j\omega}) = \frac{|G(e^{j\omega})|^2 P_q(e^{j\omega})}{P_w(e^{j\omega}) + |G(e^{j\omega})|^2 P_q(e^{j\omega})}$$

Noise Cancellation:



$$h = R_{v_2}^{-1} \, r_{v_1 v_2}$$

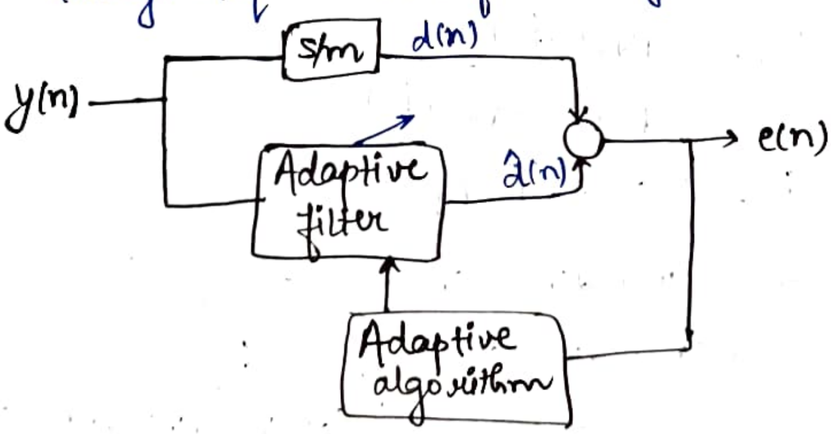
Adaptive Filter:



- error b/w $\hat{d}(n)$ + $d(n)$ is minimized by changing the coefficients of the adaptive filter.

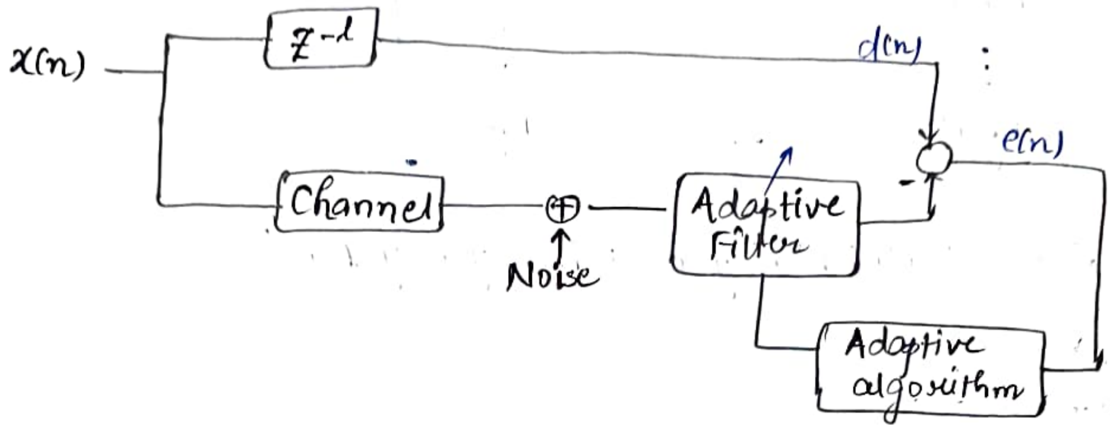
Limitations:

- Always require a reference signal.



11-11-2025

Channel Equalization



FIR wiener Filter: Steepest Descent Algorithm.

Assume signal is WSS.

Goal: To estimate $d(n)$ using FIR wiener filter of length M.

Filter coefficient,

$$h_i(n), i=0, 1, \dots, M-1.$$

$$E[e^2(n)] = E[(d(n) - \hat{d}(n))^2]$$

$$= E\left[\left(d(n) - \sum_{i=0}^{M-1} h_i(n)\, y(n-i)\right)^2\right]$$

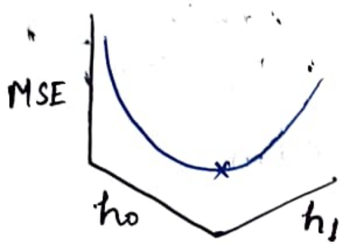
$$y(n) = \begin{bmatrix} y(n) \\ y(n-1) \\ \vdots \\ y(n-M+1) \end{bmatrix}$$

$$E[e^2(n)] = R_d(0) - 2h(n)\, r_{dy} + h(n)\, R_y\, h'(n)$$

$$r_{dy} = \begin{bmatrix} R_{dy}(0) \\ R_{dy}(1) \\ \vdots \\ R_{dy}(M-1) \end{bmatrix}$$

$$R_y = \begin{bmatrix} R_y(0) & R_y(1) & \cdots & R_y(M-1) \\ R_y(1) & & & \\ \vdots & & & \vdots \\ R_y(M-1) & & \cdots & R_y(0) \end{bmatrix}$$

$$\mathbb{E}\left[d^2(n) - 2d(n)h'(n)y(n) + h'(n)y(n)y'h(n)\right]$$



Gradient,

$$\nabla \mathbb{E}\{e^2(n)\} = \begin{bmatrix} \dfrac{\partial \mathbb{E}[e^2(n)]}{\partial h_0} \\ \vdots \\ \dfrac{\partial \mathbb{E}[e^2(n)]}{\partial h_{M-1}} \end{bmatrix}$$

$$= -2\, r_{dy} + 2 R_y h(n)$$

By setting $\nabla \mathbb{E}\{e^2(n)\} = 0$

$$R_y h_{opt.} = r_{dy} \qquad \text{[wiener hopf eq}^n\text{]}$$

To bring adaptability, bring in iteration.

SDA iteration :

The direction is in maximum decrease of the function.

$$h(n+1) = h(n) + \frac{\mu}{2}\left[-\nabla \mathbb{E}\{e^2(n)\}\right]$$

μ : step size parameter

2 : scaling

$$\Rightarrow h(n+1) = h(n) + \mu\underbrace{\left(r_{dy} - R_y h(n)\right)}_{\text{cost fn}}$$

Convergence of SDA:

$$h(n+1) = h(n) + \mu(r_{dy} - R_y h(n))$$
$$= h(n)[I - \mu R_y] + \mu r_{dy}$$

$I_{M\times M} \rightarrow$ Identity matrix

$$\begin{bmatrix} h_0(n+1) \\ h_1(n+1) \\ \vdots \\ h_{M-1}(n+1) \end{bmatrix} = \begin{bmatrix} 1-\mu R_y(0) & -R_y(1) & \cdots & -R_y(M-1) \\ -R_y(1) & 1-\mu R_y(0) & \cdots & -R_y(M-2) \\ \vdots & & & \vdots \\ -R_y(M-1) & & \cdots & 1-\mu R_y(0) \end{bmatrix} \begin{bmatrix} h_0(n) \\ h_1(n) \\ \vdots \\ h_{M-1}(n) \end{bmatrix}$$

$$+ \mu \begin{bmatrix} R_{dy}(0) \\ R_{dy}(1) \\ \vdots \\ R_{dy}(M-1) \end{bmatrix}$$

- Symmetric non-singular matrix
- It can be diagonalized.

$$R_y = Q \Lambda Q'$$

Q : orthogonal matrix
Λ : diagonal matrix

$$I \rightarrow QQ' \rightarrow Q'Q$$

$$h(n+1) = (QQ' - \mu Q \Lambda Q') h(n) + \mu r_{dy}$$

Multiply both side by Q'.

$$Q'(h(n+1)) = (I - \mu\Lambda) Q' h(n) + \mu Q' r_{dy}$$

Define a new variable,

$$\bar{h}(n) = Q' h(n), \quad \bar{r}_{dy} = Q' r_{dy}$$
$$\bar{h}(n+1) = Q' h(n+1)$$
$$\bar{h}(n+1) = (I - \mu\Lambda)\bar{h}(n) + \mu \bar{r}_{dy}$$

$$\Rightarrow \bar{h}(n+1) = \begin{bmatrix} 1-\mu\lambda_1 & 0 & 0 & 0 & \cdots \\ 0 & 1-\mu\lambda_2 & \cdots & & \\ \vdots & & & & \\ & & & & 1-\mu\lambda_m \end{bmatrix} \bar{h}(n) + \mu \bar{\gamma}_{dy}$$

$$\overline{h_i(n+1)} = (1-\mu\lambda_i)\, \overline{h_i}(n) + \mu \bar{\gamma}_{dy}, \quad i = 0, 1, \dots, M-1.$$

The convergence condition is given by

$$|1-\mu\lambda_i| < 1$$

$$\Rightarrow -1 < 1-\mu\lambda_i < 1$$

$$\Rightarrow 0 < \mu < \frac{2}{\lambda_{max}}, \quad i = 1, 2, \dots, M$$

$$\overline{h_i(n+1)} = (1-\mu\lambda_i)\, \overline{h_i(n)} + \mu \overline{\gamma_{dy}(i)}$$

12-11-2025

Simpler Condition for Convergence:

$$\lambda_{max} > \lambda_1 + \lambda_2 + \dots + \lambda_{\cancel{max}\, n}$$

$$\cong \text{Trace of } R_y \ (\text{Trace}(R_y))$$

$$0 < \mu < \frac{2}{\text{Trace}(R_y)}$$

$$\cong \frac{2}{M R_y(0)}$$

The SDA converges to Wiener filter.

$$\lim_{n\to\infty} h(n) = R_y^{-1} \gamma_{dy}$$

Rate of Convergence:

The rate of convergence depends on the eigen value spread in the AC matrix.

R_y, condition no., $k = \dfrac{\lambda_{max}}{\lambda_{min}}$, $\lambda_{max}, \lambda_{min}$: max, min eigen value.

If $K=1 \Rightarrow$ fastest convergence of the system

Eg. $R_y = \begin{bmatrix} 21 & 16 \\ 16 & 21 \end{bmatrix}$, $r_{dy} = \begin{bmatrix} 20 \\ 16 \end{bmatrix}$.

Determine a 2-tap FIR Wiener filter using SDA.

Soln: Take $h(0) = \begin{bmatrix} h_0(0) \\ h_1(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Update eqn: $h(n+1) = h(n) + \mu \left[r_{dy} - R_y h(n) \right]$

Compute the eigen values of R_y

$\Rightarrow \lambda_1 = 37, \; \lambda_2 = 5$

$0 < \mu < 2/\lambda_{max}$

Choose $\mu = \frac{2}{\lambda_{max}} = 0.02$

$$h(1) = \begin{bmatrix} h_0(1) \\ h_1(1) \end{bmatrix} = \begin{bmatrix} h_0(0) \\ h_1(0) \end{bmatrix} + 0.02 \left[\overset{r_{dy}}{\begin{bmatrix} 20 \\ 16 \end{bmatrix}} - \overset{R_y \quad h(n)}{\begin{bmatrix} 21 & 16 \\ 16 & 21 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}} \right]$$

$$= \begin{bmatrix} 0.40 \\ 0.32 \end{bmatrix}$$

$$h(2) = \begin{bmatrix} h_0(2) \\ h_1(2) \end{bmatrix} = \begin{bmatrix} 0.40 \\ 0.32 \end{bmatrix} + 0.02 \left[\begin{bmatrix} 20 \\ 16 \end{bmatrix} - \begin{bmatrix} 21 & 16 \\ 16 & 21 \end{bmatrix} \begin{bmatrix} 0.40 \\ 0.32 \end{bmatrix} \right]$$

$$= \begin{bmatrix} 0.40 \\ 0.32 \end{bmatrix} + 0.02 \left[\begin{bmatrix} 20 \\ 16 \end{bmatrix} - \begin{bmatrix} 13.52 \\ 13.12 \end{bmatrix} \right]$$

$$= \begin{bmatrix} 0.40 \\ 0.32 \end{bmatrix} + \begin{bmatrix} 0.1296 \\ 0.0576 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5296 \\ 0.3776 \end{bmatrix}$$

LMS $\rightarrow$ Least Mean Square

SDA $\rightarrow$ cost fn $\rightarrow$ $\mathbb{E}[e^2(n)]$

cost function $\rightarrow$ $e^2(n)$ $\rightsquigarrow$ directly depends on observed data

$$e^2(n) = \left[d(n) - \sum_{i=0}^{M-1} h_i(n)\, y(n-i)\right]^2$$

$\nabla e^2(n)$ $\rightarrow$ stochastic gradient

$$e(n) = d(n) - \sum_{i=0}^{M-1} h_i(n)\, y(n-i)$$

$$\frac{\partial e(n)}{\partial h_i(n)} = -y(n-i), \quad i = 0, 1, \ldots, M-1.$$

$$\nabla e^2(n) \longrightarrow 2e(n) \begin{bmatrix} \frac{\partial e(n)}{\partial h_0} \\ \frac{\partial e(n)}{\partial h_1} \\ \vdots \\ \frac{\partial e(n)}{\partial h_{M-1}} \end{bmatrix}$$

$$= 2e(n) \begin{bmatrix} -y(n) \\ -y(n-1) \\ \vdots \\ -y(n-M+1) \end{bmatrix}$$

Update eqn:

$$h(n+1) = h(n) + \mu\,[e(n)\,y(n)]$$

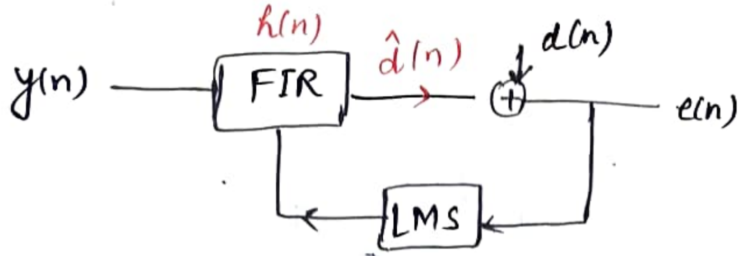
<u>LMS:</u>

i/p $\rightarrow$ signal $(y(n))$; desired signal ; filter length ; stepsize

$\hat{d}(n) = h(n)\, y(n)$

Estimation error : $d(n) - \hat{d}(n)$

Update eqn : $h(n+1) = h(n) + \mu\,[e(n)\,y(n)]$



Independance Assumption :

- Filter h(n) is independent of future data, y(n+1).

$$h(n) = h(n-1) + \mu\, e(n-1)\, y(n-1)$$

$$\mathbb{E}[h(n)\,y(n)] = \mathbb{E}[h(n)]\,\mathbb{E}[y(n)]$$

$$\therefore\ \mathbb{E}[h(n+1)] = \mathbb{E}[h(n)] + \mu\,\mathbb{E}[y(n)]\,\mathbb{E}[e(n)]$$

$$= \mathbb{E}[h(n)] + \mu\,\mathbb{E}[y(n)][d(n) - h(n)\,y(n)]$$

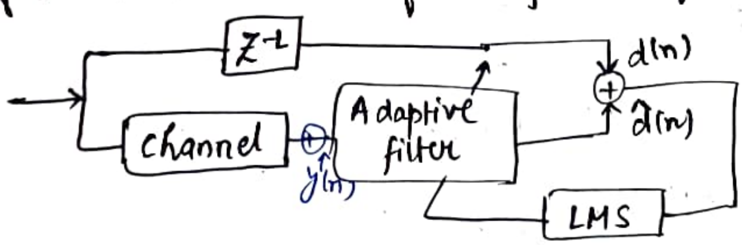
$$= \mathbb{E}[h(n)] + \mu r_{dy} - \mu\,\mathbb{E}[y(n)\,y'(n)]\,h(n)$$

$$= \mathbb{E}[h(n)] + \mu r_{dy} - \mu R_y\,\mathbb{E}[h(n)]$$

Hence, the mean value of the filter coefficients satisfy SDA.

Eg. The input to a communication channel is a sequence $x(n) = 0.8\,x(n-1) + w(n)$, $w(n)$ is a zero mean unity variance white noise. The channel transfer f^n is $H(z) = z^{-1} - 0.5z^{-2}$, and is affected by WGN of variance 1. Find the FIR Wiener filter of length 2 for channel equalization. Choose a value of μ and write LMS filter update eqn

Soln:



17-11-2025

AR

$x(n) = 0.8\,x(n-1) + w(n)$.

$$\begin{bmatrix} r_x(0) & r_x(1) & r_x(2) & r_x(3) \\ r_x(1) & r_x(0) & r_x(1) & r_x(2) \\ r_x(2) & r_x(1) & r_x(0) & r_x(1) \\ r_x(3) & r_x(2) & r_x(1) & r_x(0) \end{bmatrix} \begin{bmatrix} 1 \\ -0.8 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} r_x(0) - 0.8\,r_x(1) &= 1 \\ r_x(1) - 0.8\,r_x(0) &= 0 \\ r_x(2) - 0.8\,r_x(1) &= 0 \\ r_x(3) - 0.8\,r_x(2) &= 0 \end{aligned}\right\} \Rightarrow \text{solving, } r_x(0) = 2.78,\ r_x(1) = 2.22,\ r_x(2) = 1.78,\ r_x(3) = 1.42.$$

$y(n) = x(n-1) - 0.5\,x(n-2) + v(n)$

$R_y(m) = \mathbb{E}[y(n)\,y(n-m)]$

$= \mathbb{E}\{(x(n-1) - 0.5\,x(n-2) + v(n))\,(x(n-1-m) - 0.5\,x(n-2-m) + v(n-m))\}$

$= R_x(m) + 0.25\,R_x(m) - 0.5\,R_x(m+1) - 0.5\,R_x(m-1) + R_v(m)$.

$\Rightarrow R_y(0) = R_x(0) + 0.25\,R_x(0) - 0.5\,R_x(1) - 0.5\,R_x(-1) + R_v(0)$

$R_y(1) = R_x(1) + 0.25\,R_x(1) - 0.5\,R_x(2) - 0.5\,R_x(0) + 0$

$r_{xy}(m) = \mathbb{E}[x(n)\,y(n-m)]$.

$= \mathbb{E}[x(n)\,\{x(n-1-m) - 0.5\,x(n-2-m) + v(n-m)\}]$

$R_{xy}(m) = R_x(m+1) - 0.5\,R_x(m+2)$

$R_{xy}(0) = R_x(1) - 0.5\,R_x(2) = 1.33$

$R_{xy}(1) = R_x(2) - 0.5\,R_x(3) = 1.07$

$$R_y = \begin{bmatrix} R_y(0) & R_y(1) \\ R_y(1) & R_y(0) \end{bmatrix} = \begin{bmatrix} 2.255 & 0.49 \\ 0.49 & 2.255 \end{bmatrix}$$

Step size: $\mu < \dfrac{2}{trace(R_y)} = \dfrac{2}{4.5} \approx 0.44$

λ_1, λ_2:

- Wiener filter

$$\begin{bmatrix} R_y(0) & R_y(1) \\ R_y(1) & R_y(0) \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} = \begin{bmatrix} R_{xy}(0) \\ R_{xy}(1) \end{bmatrix}$$

Eg. An AR(2) process is defined by the difference equation $x(n) = x(n-1) - 0.6x(n-2) + w(n)$, $w(n)$ is white noise with variance 1. Use Yule walker equation to solve for auto-correlation $r_{xx}(0)$, $r_{xx}(1)$ and $r_{xx}(2)$.

Soln:

$$\begin{bmatrix} r_x(0) & r_x(1) & r_x(2) \\ r_x(1) & r_x(0) & r_x(1) \\ r_x(2) & r_x(1) & r_x(0) \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow r_x(0) = 2.5$

$r_x(1) = 1.6$

$r_x(2) = 0.064$

NLMS (Normalized LMS)

$$0 < \mu < \frac{2}{\text{Trace}(R_y)} = \frac{2}{M R_y(0)} = \frac{2}{M E[y^2(n)]}$$

$$0 < \mu < \frac{2}{\sum_{i=0}^{M-1} y^2(n-i)} = \frac{2}{\|y(n)\|^2}$$

$$\left[\frac{1}{M} \sum_{i=0}^{M} y^2(n)\right] \rightarrow E[y^2(n)]$$

$$\mu = \beta \frac{1}{\|y(n)\|^2}$$

$0 < \beta < 2$: Control parameter for misadjustment factor

$\therefore$ Update eqn:

$$h(n+1) = h(n) + \beta \frac{1}{\|y(n)\|^2} e(n) y(n)$$

$$\downarrow$$

$$h(n) + \beta \frac{1}{\gamma + \|y(n)\|^2} e(n) y(n)$$

[γ is introduced to prevent the term from approaching ∞ when $y(n) = 0$]

- Parameters: β, γ, order

Types of LMS:

→ Block LMS

→ Signed error LMS

→ Sign sign LMS

18-11-2025

Recursive Least Squares (RLS)

→ LMS → has slow convergence
  ↳ eigenvalues spread ~~by~~ of AC matrix
  [Application dependent adaptive filter]

→ It does not depend on the eigen values spread of AC matrix
→ ~~com~~ Highly complex.
→ step size is avoided.

Forgetting factor: Give higher importance (weightage) to the nearby data and less to far-off data.

$$h(n) = \begin{cases} h_0(n) \\ h_1(n) \\ \vdots \\ h_{M-1}(n) \end{cases}$$

$$\xi(n) = \sum_{K=0}^{n} \lambda^{n-K} e^2(K)$$

$\lambda \rightarrow$ forgetting factor.

$$= \sum_{K=0}^{n} \lambda^{n-K} [\, d(K) - \underbrace{y'(n)}_{\text{obs. data}} h(n)]^2$$

$0 \leq \lambda \leq 1.$

$\lambda \approx 0.99 \longrightarrow$ is effective in tracking the local stationary.

Minimize $\xi(n) = \sum_{K=0}^{n} \lambda^{n-K} [\, d(K) - y'(K)\, h(n)]^2$

w.r.t. $h(n)$,

$$\frac{\partial \xi(n)}{\partial h(n)} = 0$$

$$= 2 \sum_{K=0}^{n} \lambda^{n-K} [\, d(K) - y'(K) h(n)]\, y(K) = 0$$

$$\therefore \sum_{K=0}^{n} \lambda^{n-K} y(K)\, y'(K)\, h(n) = \sum_{K=0}^{n} \lambda^{n-K} d(K)\, y(K)$$

... Normal eqⁿ for LS estimate.

~~Normal eqⁿ for LS~~

$$\left[\hat{R}_y(n)\, h(n) = \hat{\gamma}_{dy}(n) \right]$$

$$\hat{R}_y(n) = \sum_{K=0}^{n} \lambda^{n-K} y(K)\, y'(K) \qquad [AC\ R_y]$$

Similarly,

$$\hat{\gamma}_{dy}(n) = \sum_{K=0}^{n} \lambda^{n-K} d(K)\, y(K) \qquad \left[\text{cross corr } \gamma_{dy}(n)\right]$$

$$\hat{R}_y(n)\, h(n) = \hat{\gamma}_{dy}(n)$$

$$\Rightarrow h(n) = \hat{R}_y(n)^{-1}\, \hat{\gamma}_{dy}(n)$$

$\Rightarrow$ "Matrix inversion is required."

$$\hat{R}_y(n) = \sum_{K=0}^{N-1} \lambda^{n-K} y(K)\, y'(K) + y(n)\, y'(n)$$

$$= \lambda \sum_{K=0}^{N-1} \lambda^{n-1-K} y(K)\, y'(K) + y(n)\, y'(n)$$

$$\hat{R}_y(n) = \sum_{n=0}^{n} \lambda^{n-K} y(n)\, y'(n)$$

$$\hat{R}_y(n) = \lambda\, \hat{R}_y(n-1) + y(n)\, y'(n)$$

Similarly,

$$\hat{\gamma}_{dy}(n) = \lambda\, \hat{\gamma}_{dy}(n-1) + d(n)\, y(n)$$

$$\therefore h(n) = \left[\hat{R}_y(n)\right]^{-1} \hat{\gamma}_{dy}(n)$$

$$= \lambda \left[\underbrace{\hat{R}_y(n-1)}_{A} + \underbrace{y(n)}_{U}\, \underbrace{y'(n)}_{V} \right]^{-1} \hat{\gamma}_{dy}(n)$$

<u>Matrix inversion lemma:</u>

$$[A + UV]^{-1} = A^{-1} - \frac{A^{-1} U V' A^{-1}}{1 + V' A^{-1} U},$$

provided $1 + V' A^{-1} U \neq 0$

A : Non-singular matrix (M×M)
U, V : column vectors.

~~Hello~~

Shermon-Morrison Formula

$$R_y^{-1}(n) = [\lambda \hat{R}_y(n-1) + y(n) y'(n)]^{-1}$$

$$= \frac{1}{\lambda} \hat{R}_y^{-1}[n-1] - \frac{\frac{1}{\lambda}\hat{R}_y^{-1}[n-1]\, y(n) y'(n) \frac{1}{\lambda}\hat{R}_y(n-1)}{1 + y'(n)\frac{1}{\lambda}\hat{R}_y^{-1}(n-1)\, y(n)}$$

$$= \frac{1}{\lambda} R_y^{-1}[n-1] - \frac{\hat{R}_y^{-1}[n-1]\, y(n) y'[n]\, \hat{R}_y^{-1}[n-1]}{\lambda[\lambda + y'(n)\, \hat{R}_y^{-1}[n-1]\, y[n]]}$$

Let $P[n] = R_y^{-1}[n]$.

$$\Rightarrow P[n] = \frac{1}{\lambda} P[n-1] - \frac{P[n-1] y[n] y'[n] P[n-1]}{\lambda[\lambda + y[n] P[n-1] y[n]]}$$

Let $g(n) = \dfrac{P[n-1]\, y[n]}{\lambda + y'[n] P[n-1] y[n]}$

$$\Rightarrow P(n) = \frac{1}{\lambda}[P(n-1) - g(n) y'(n) P(n-1)] \quad \ldots (A)$$

$$\Rightarrow \boxed{g(n) = P(n) y(n)}$$

$$\lambda g(n) = P(n-1) y(n) - g(n) y'(n) P(n-1) y(n)$$

$$\Rightarrow \lambda P(n) = P(n-1) - g(n) y'(n) P(n-1)$$

Multiply both side by $y(n)$.

$$\Rightarrow \lambda P(n) y(n) = P(n-1) y(n) - g(n) y'(n) P(n-1) y(n)$$
$$= \lambda g(n)$$

Filter update

$$h(n) = \hat{R}_y^{-1} \hat{\gamma}_{dy}(n)$$
$$= P(n)[\lambda \hat{\gamma}_{dy}(n-1) + d(n) y(n)]$$
$$= \lambda P(n) \hat{\gamma}_{dy}(n-1) + d(n) y(n) P(n)$$

$$= \lambda \left[\frac{1}{\lambda} P(n-1) - \frac{1}{\lambda} g(n) y'(n) P(n-1) \right] \hat{\gamma}_{dy}(n-1)$$
$$+ d(n) \underbrace{P(n) y(n)}_{g(n)}$$

$$= \underbrace{P(n-1)\hat{\gamma}_{dy}(n-1)}_{h(n-1)} - g(n) y'(n) \underbrace{P(n-1)\hat{\gamma}_{dy}(n-1)}_{h(n-1)}$$
$$+ d(n) g(n)$$

$$= h(n-1) - g(n) y'(n) h(n-1) + g(n) d(n)$$

$$\therefore h(n) = h(n-1) + g(n)\left[d(n) - y'(n) h(n-1) \right] \rightarrow \text{Recursive relation}$$

Frequency Estimation Tech

→ Eigen decomposition of autocorrelation matrix
  ↳ Signal subspace
  ↳ Noise subspace.

Harmonic Model:

$$x(n) = \sum_{i=1}^{P} A_i e^{j \underbrace{w_i(n)}_{\text{discrete freq.}}} + w(n) \quad \rightarrow A_i, w_i : \text{to be estimated}$$

A_i : complex amplitude $\Rightarrow |A_i| e^{j\phi_i}$

ϕ : uniformly distributed $[0, 2\pi]$

$w(n)$: zero mean additive white noise ↳ σ_w^2

$$x(n) = [x(n), x(n+1), \ldots, x(n+M-1)]^T$$

Consider only one complex exponential.

$$x(n) = \begin{bmatrix} A_1 e^{jw_1 n} + w(n) \\ A_1 e^{jw_1(n+1)} + w(n+1) \\ \vdots \quad \vdots \\ A_1 e^{jw_1(n+M-1)} + w(n+M-1) \end{bmatrix}$$

$V(w_1)$ $\quad$ $w(n)$

$$= A_1 e^{jw_1 n} \begin{bmatrix} 1 \\ e^{jw_1} \\ e^{j2w_1} \\ \vdots \\ e^{j(M-1)w_1} \end{bmatrix} + \begin{bmatrix} w(n) \\ w(n+1) \\ \vdots \\ w(n+M-1) \end{bmatrix}$$

$$x(n) = A_1 V(\omega_1) e^{j\omega_1 n} + w(n)$$
$$= s(n) + w(n)$$

$$V(\omega_1) = \begin{bmatrix} 1 & e^{j\omega_1} & e^{j2\omega_1} & \cdots & e^{j(M-1)\omega_1} \end{bmatrix}^T$$

$$r_x(k) = \mathbb{E}\left[x(n)\, x^*(n-k) \right]$$ [considering noise & signal uncorrelated]

$$= \mathbb{E}\left\{ (A_1 e^{j\omega_1 n} + w(n)) \cdot (A_1^* e^{-j\omega_1 (n-k)} + w^*(n-k)) \right\}$$
$$= \mathbb{E}\left\{ A_1 e^{j\omega_1 n} A_1^* e^{-j\omega_1(n-k)} + w(n)\, w^*(n-k) \right\}$$
$$= |A_1|^2 e^{j\omega_1 k} + \sigma_w^2 \delta(k) + \mathbb{E}\left\{ w(n) A_1^* e^{-j\omega_1(n-k)} \right\} \to 0$$
$$= |A_1|^2 e^{j\omega_1 k} + \sigma_w^2 \delta(k)$$
$$= P_1 e^{j\omega_1 k} + \sigma_w^2 \delta(k),$$

where $|A_1|^2$: power $\to P_1$ (power of complex signal)

$$r_x(k) = r_x^*(-k)$$

$$R_x = \mathbb{E}\left\{ x(n)\, x^H(n) \right\}$$

$$= \begin{bmatrix} r_x(0) & r_x^*(1) & \cdots & r_x^*(M-1) \\ r_x(1) & r_x(0) & \cdots & r_x^*(M-2) \\ \vdots & & & \vdots \\ r_x(M-1) & & \cdots & r_x(0) \end{bmatrix}$$

$$= \begin{bmatrix} P_1 + \sigma_w^2 & P_1 e^{-j\omega_1} & \cdots & P_1 e^{-j\omega_1(M-1)} \\ P_1 e^{j\omega_1} & P_1 + \sigma_w^2 & \cdots & \\ \vdots & & & \vdots \\ P_1 e^{j(M-1)\omega_1} & & \cdots & P_1 + \sigma_w^2 \end{bmatrix}$$

$$= \underbrace{P_1 \begin{bmatrix} 1 & e^{-j\omega_1} & \cdots & e^{-j(M-1)\omega_1} \\ e^{j\omega_1} & & & \\ \vdots & & & \\ e^{j\omega_1(M-1)} & & & \end{bmatrix}}_{R_s} + \underbrace{\begin{bmatrix} \sigma_w^2 & 0 & 0 & 0 \\ 0 & \sigma_w^2 & & \\ 0 & & \ddots & \\ \vdots & & & \\ 0 & & & \sigma_w^2 \end{bmatrix}}_{R_w}$$

$\therefore R_x = R_s + R_w$

1st column of R_s:

$V(\omega_1) = [1 \quad e^{j\omega_1} \quad e^{j2\omega_1} \quad \dots \quad e^{j(M-1)\omega_1}]^T$

q_1

$R_s = P_1 q_1 q_1^H$

→ Rank of R_s is ~~only~~ one, it has only one non-zero eigen-value.

Multiply both-side q_1,

$$\Rightarrow R_s q_1 = P_1 q_1 q_1^H q_1$$
$$= P_1 q_1 (q_1^H q_1)$$
$$= M P_1 q_1$$

$\lambda_1^s = MP_1$

$\lambda_i^s = 0 \quad , \quad i = 2, 3, \dots, M$

$R_s \rightarrow$ hermitian

$q_2, q_3, \dots \rightarrow$ orthogonal to q_1.

$q_i^H q_1 = 0 \, , \quad i = 2, 3, \dots, M$

$V^H q_i = 0 \, , \quad i = 2, 3, \dots, M$

$$R_s q_i = \lambda_i^s q_i = (R_s + R_w) q_i$$
$$= \lambda_s q_i + \sigma_w^2 I q_i$$
$$= (\lambda_i + \sigma_w^2) q_i$$

$$\lambda = \lambda_i^s + \sigma_w^2 = \begin{cases} \lambda_1^s + \sigma_w^2, & \text{for } i=1 \\ \lambda_i^s + \sigma_w^2, & i = 2, 3, \dots, M \end{cases}$$

$$\lambda_i = \begin{cases} MP_1 + \sigma_w^2, & i=1 \\ \sigma_w^2, & i = 2, 3, \dots, M-1 \end{cases}$$

$\lambda_{max} = MP_1 + \sigma_w^2$

And the remaining eigenvalues are σ_w^2.

$$R_x = \sum_{i=1}^{M} \lambda_i q_i q_i^H$$

$$= \lambda_1 q_1 q_1^H + \sum_{i=2}^{M} \lambda_i q_i q_i^H$$

$$= (MP_1 + \sigma_w^2) Q_s Q_s^H + \sigma_w^2 Q_w Q_w^H$$

$$Q_s = [q_1]$$

$$Q_w = [q_2 \; q_3 \ldots q_M]$$

Eg. 2 complex exp. with white noise.

$$x(n) = A_1 e^{jw_1 n} + A_2 e^{jw_2 n} + w(n)$$

$$= \sum_{i=1}^{2} A_i e^{jw_i n} + w(n)$$

$$r_x(K) = P_1 e^{jw_1 n} + P_2 e^{jw_2 n} + \sigma_w^2 \delta(K)$$

$$R_x = R_s + R_w.$$

$$R_w = \sigma_w^2 I$$

$$V(w_1)\; q_1 \; [1 \;\; e^{jw_1} \;\; e^{j2w_1} \ldots e^{j(M-1)w_1}]^T$$

$$V(w_2)\; q_2 \; [1 \;\; e^{jw_2} \;\; e^{j2w_2} \ldots e^{j(M-1)w_2}]^T$$

$$R_s = P_1 q_1 q_1^H + P_2 q_2 q_2^H$$

Rank of $R_s \rightarrow 2$

$$\lambda_i = \begin{cases} MP_i + \sigma_w^2, & i = 1, 2 \\ \sigma_w^2, & i = 3, \ldots, M \end{cases}$$

$$R_x = \sum_{i=1}^{M} \lambda_i q_i q_i^H$$

$$= \sum_{i=1}^{2} \lambda_i q_i q_i^H + \sum_{i=3}^{M} \lambda_i q_i q_i^H$$

$$= \sum_{i=1}^{2} (MP_i + \sigma_w^2) q_i q_i^H + \sum_{i=3}^{M} \sigma_w^2 q_i q_i^H$$

$$= Q_s \lambda_s Q_s + \sigma_w^2 Q_w Q_w^H,$$

$$Q_s = [q_1 \quad q_2]$$

$$Q_w = [q_3 \quad q_4 \quad \cdots \quad q_M]$$

$$\lambda_s = \begin{bmatrix} MP_1 + \sigma_w^2 & \\ & MP_2 + \sigma_w^2 \end{bmatrix}$$

- The columns of Q_s and Q_w are the noise/signal eigen vectors.

$$P_s = Q_s Q_s^H \qquad P_w = Q_w Q_w^H$$

$$P_s Q_w = 0 \qquad P_w Q_s = 0$$

$$P_s V(\omega_i) = V(\omega_i)$$

$$P_w V(\omega_i) = 0.$$

eg. If P complex exponentials are there,

$$x(n) = \sum_{i=1}^{P} A_i e^{j\omega_i n} + w(n)$$

Pisarenko Harmonic Decomposition

Length of the time window,

$$M = P+1;$$

one greater than the no. of complex exponentials.

$$x(n) = \sum_{i=1}^{P} A_i e^{j\omega_i n} + w(n)$$

$$= s(n) + w(n)$$

$$V(\omega) = [1 \quad e^{j\omega} \quad e^{j2\omega} \quad \cdots \quad e^{jP\omega}]^T$$

$$R_x q_i = (R_s + R_w) q_i$$

$$\lambda_i = \begin{cases} MP_i + \sigma_w^2, & i = 1, 2, \ldots P \\ \sigma_w^2, & i = M = P+1 \end{cases}$$

$$R_x = \sum_{i=1}^{M} \lambda_i q_i q_i^H$$

$$= \sum_{i=1}^{P} \lambda_i q_i q_i^H + \sum_{i=P+1}^{M} \lambda_i q_i q_i^H$$

$\underline{i=M}$

$$= \sum_{i=1}^{P} \lambda_i q_i q_i^H + \sigma_w^2 q_M q_M^H$$

$$= Q_s \Lambda_s Q_s^H + \sigma_w^2 Q_s Q_w^H$$

$$Q_s = [q_1 \; q_2 \; \cdots \; q_P]$$

$$Q_w = [q_m]$$

<u>Note</u>:- The noise subspace is a single eigen vector.

- The least (minimum) eigenvalue leads the eigen vector (noise vector).

- q_m is orthogonal to each of the signal.

$$q_i^H q_M = 0, \quad i = 1, 2, \ldots, P$$

$$q_i = v(\omega_i)$$

The frequency estimation f^n is

$$\hat{P}_{PHD}(e^{j\omega}) = \frac{1}{|v^H(\omega) q_M|^2}$$

$$= \frac{1}{|Q_M(e^{j\omega})|^2}$$

The peaks of $\hat{P}_{PHD}(e^{j\omega})$ the freq. estimate

$$Q_M(e^{j\omega}) = v^H(\omega) q_M$$

$$= \underbrace{\sum_{K=0}^{M-1} q_M(K) e^{-jK\omega}}$$

DTFT of noise eigen vector (q_M)

$$Q_M(z) = \sum_{K=0}^{M-1} q_M(K) z^{-K}$$

$$= \prod_{K=1}^{M-1} (1 - e^{j\omega_K} z^{-1})$$

- ~~P~~ roots of the polynomial are the ~~eigen~~ frequencies ω_K of the P complex exponentials.

→ Pseudo spectrum as it does not provide information about power, i.e., $|A_i|^2$.

Eg. $(\lambda_1, \lambda_2, \ldots, \lambda_{M-1})$

$\lambda_P \rightarrow$ signal + noise power

$([\lambda_1 - \sigma_w^2 \rightarrow P_1])$

Eg. The 1st two AC values of a random process consisting of P=1 [i.e., one complex signal] in white noise is given as

$$r_x(0) = 3$$
$$r_x(1) = 2 + j2$$

Use PHD to find: ⓐ freq., ⓑ power of exp. signal.

Soln: $x(n) = A_1 e^{j\omega_1 n} + w(n)$

$M = P+1, \quad P=1$

$$R_x = \begin{bmatrix} r_x(0) & r_x^*(1) \\ r_x(1) & r_x(0) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2-j2 \\ 2+j2 & 3 \end{bmatrix}$$

Step-1: eigen value
Step-2: eigen vector

$$\lambda_1 = 3 + 2\sqrt{2}$$
$$\lambda_2 = 3 - 2\sqrt{2}$$

$$q_1 = \begin{bmatrix} 1 \\ \sqrt{2} + j\sqrt{2} \end{bmatrix}; \quad q_2 = \begin{bmatrix} 1 \\ \sqrt{2} - j\sqrt{2} \end{bmatrix}$$

$$\lambda_{max} = 3 + 2\sqrt{2} = MP_1 + \sigma_w^2 \quad \ldots (1)$$
$$\lambda_{min} = 3 - 2\sqrt{2} = \sigma_w^2 \quad \ldots (2)$$

The power of the complex exp. signal,

$$P_1 = \frac{1}{M}[\lambda_{max} - \lambda_{min}]$$
$$= 2\sqrt{2}$$

$$Q_2(Z) = \sum_{K=0}^{1} q_2(K) Z^{-K}$$

$$= 1 + \left(-\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right) Z^{-1} = 0$$

$$\Rightarrow Z = 1/\sqrt{2} + j\, 1/\sqrt{2}$$

$$= e^{j\pi/4}$$

$$\omega_1 = \pi/4.$$

Multiple Signal Classification (MUSIC)

↳ Improved version of PHD

$M > P+1$

∴ The dimension of the noise subspace is greater than one.

$$x(n) = \sum_{i=1}^{P} A_i e^{j\omega_i n} + w(n)$$

$$= \sum_{i=1}^{P} A_i V(\omega_i) e^{j\omega_i n} + w(n)$$

$$= S(n) + w(n)$$

$$V(\omega) = [1 \quad e^{j\omega} \quad e^{j2\omega} \quad \dots \quad e^{j(M-1)\omega}]$$

$$\lambda_i = \begin{cases} MP_i + \sigma_w^2, & i = 1, 2, \dots, P \\ \sigma_w^2, & i = P+1, \dots, M \end{cases}$$

$$R_x = Q_s \Lambda_s Q_s^H + \sigma_w^2 Q_w Q_w^H$$

$$q_i^H q_j = V^H(\omega_i) q_j \quad , \quad q_j : \text{noise vector}$$

$$= \sum_{K=0}^{M-1} q_j(K) e^{jK\omega_i}, \quad j = P+1, \dots, M$$

$$\hat{P}_{Music}(e^{j\omega}) = \frac{1}{|V^H(\omega) q_j|^2}$$

$$= \frac{1}{|Q_j(e^{j\omega})|^2}$$

Power

$\lambda_1 - \sigma_w^2$

$\lambda_2 - \sigma_w^2$

⋮

Non-Parametric Estimation (PSD)

$x_n \rightarrow$ Let θ be a parameter that is estimated from the random variable $(n = 1, 2, \ldots, N)$

$\hat{\theta} \rightarrow$ Estimated value

$B \rightarrow$ Bias

$$B = \theta - \underbrace{\mathbb{E}[\hat{\theta}]}$$

↳ Mean of the estimated value

If $B=0 \rightarrow$ the estimate is unbiased.

$B \neq 0 \rightarrow$ the estimate is biased.

$$\lim_{n\to\infty} \mathbb{E}[\hat{\theta}] = \theta$$

$\rightarrow$ asymptotically unbiased

Variance:

$$\text{var}[\hat{\theta}] = \mathbb{E}[\,|\hat{\theta} - \mathbb{E}[\hat{\theta}]|^2]$$

$$\lim_{n\to\infty} \text{var}(\hat{\theta}) = \lim_{n\to\infty} \mathbb{E}\{|\hat{\theta} - \mathbb{E}(\hat{\theta})|^2\}$$

Mean Square Error:

$$\lim_{N\to\infty} \mathbb{E}[|\hat{\theta} - \theta|^2] = 0$$

$$\mathbb{E}[\{\hat{\theta} - \mathbb{E}[\hat{\theta}] + \underbrace{\mathbb{E}[\hat{\theta}] - \theta}_{\text{Bias}}\}^2] = 0,$$

then MSE equals variance.

Consistency:

(Variance & bias) $\rightarrow 0$ as $N \rightarrow \infty$,

then it is a consistent estimator.

Frequency Resolution:

$$\Delta f$$

Quality factor:

$$Q = \frac{\mathbb{E}[\hat{P}_{per}(e^{j\omega})]^2}{\text{var}[\hat{P}_{per}(e^{j\omega})]}$$

Variability:

$V = 1/Q$

Figure of Merit $= V \times \Delta f$

[24-11-2025]

Periodogram:

$x(n) \longrightarrow$ WSS "ergodic" random process.

Then,

$$P_x(e^{j\omega}) = \sum_{m=-\infty}^{\infty} r_x(m)\, e^{-j\omega m}$$

$$\hat{r}_{xx}(m) = \frac{1}{N} \sum_{n=0}^{N-|m|-1} x^*(n)\, x(n+m), \quad 0 \le m \le N-1$$

mean of $\hat{r}_{xx}(m)$,

$$\mathbb{E}[\hat{r}_{xx}(m)] = \frac{1}{N} \sum_{n=0}^{N-|m|-1} \mathbb{E}[x^*(n)\, x(n+m)]$$

$$= \frac{N-|m|}{N}\, \hat{r}_{xx}(m)$$

$$= \left[1 - \frac{|m|}{N}\right] \hat{r}_{xx}(m)$$

$\therefore \frac{|m|}{N}$ is the bias.

$N \to \infty$, $\mathbb{E}[\hat{r}_{xx}(m)] = \hat{r}_{xx}(m)$.

$\therefore$ Asymptotically unbiased.

Variance:

Book: Jenkins & Watts (authors)

$$\mathrm{Var}[\hat{r}_{xx}(m)] \cong \frac{1}{N} \sum_{n=-\infty}^{\infty} \left[|\hat{r}_{xx}(m)|^2 + \hat{r}_{xx}(n-m)\, \hat{r}_{xx}(n+m) \right]$$

$$= \frac{N}{(N-|m|)^2} \sum_{n=-\infty}^{\infty} \left[|\hat{r}_{xx}(m)|^2 + \hat{r}_{xx}(n-m)\, \hat{r}_{xx}(n+m) \right]$$

$N \to \infty$, variance $\to 0 \Rightarrow$ Asymptotically unbiased (consistent)

$$\hat{P}_{er}(f) = \sum_{-(N-1)}^{N-1} \hat{\gamma}_{xx}(m)\, e^{-j2\pi f m} \longrightarrow$$ "Periodogram"

[Rectangular]

→ Spectral leakage

→ Windows

Eg. Bartlett window, Hamming / Hanning

Mean and Variance of Periodogram

$$P_x(f) = \frac{1}{N}\left| \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi f n} \right|^2$$

$$= \frac{1}{N} |X(f)|^2$$

$$\mathbb{E}[P_x(f)] = \mathbb{E}\left[\sum_{m=-(N-1)}^{N-1} \gamma_{xx}(m)\, e^{-j2\pi f m} \right]$$

$$= \sum_{m=-(N-1)}^{N-1} \mathbb{E}[\gamma_{xx}(m)]\, e^{-j2\pi f m}$$

$$= \sum_{m=-(N-1)}^{N-1} \underbrace{\left[\frac{N-|m|}{N} \hat{\gamma}_{xx}(m)\right]}_{\vartheta_{xx}(m)} e^{-j2\pi f m}$$

$$= \sum_{m=-\infty}^{\infty} \vartheta_{xx}(m)\, e^{-j2\pi f m}$$

$$\left[W_B(m) = 1 - \frac{|m|}{N} \right]$$

Window : Bartlett window

$$= \int_{-1/2}^{1/2} \Gamma_{xx}(\alpha)\, W_B(f-\alpha)\, d\alpha$$

$$\lim_{N\to\infty} \mathbb{E}\left[\sum_{-(N-1)}^{N-1} \gamma_{xx}(m) e^{-j2\pi f m} \right] = \Gamma_{xx}(f) = S_{xx}(f)$$

Variance:

$$\mathrm{var}[P_x(f)] = \Gamma_{xx}^2(f) \left[1 + \left(\frac{\sin(2\pi f N)}{N \sin 2\pi f} \right)^2 \right]$$

But variance of $P_x(f)$ does not decay to zero as $N \to \infty$.

$$\lim_{N\to\infty} \mathrm{var}[\hat{P}_x(f)] = \Gamma_{xx}^2(f)$$

∴ not a consistent estimator

<u>Modified Periodogram</u>: PSD estimator that involves a window function to a time-domain signal before computing the DFT.

↳ Non-rectangular window

$$\hat{P}_M(e^{j\omega}) = \frac{1}{NU}\left|\sum_{n=-\infty}^{\infty} x(n)w(n)e^{-j\omega n}\right|^2$$

$$U = \frac{1}{N}\sum_{n=0}^{N-1} w^2(n) \rightarrow \text{constant.}$$

The constant is used so that $P_M(e^{j\omega})$ is asymptotically unbiased.

<u>Modified Periodogram Bias</u>:

$$E[\hat{P}_M(e^{j\omega})] = \frac{1}{U} E[\hat{P}_{per}(e^{j\omega})] = \frac{1}{2\pi NU}[P_x(e^{j\omega}) \times |W(e^{j\omega})|^2]$$

$W(e^{j\omega}) \rightarrow$ DTFT of $U - \frac{1}{N}\sum_{n=0}^{N-1} w^2(n)$

$$\frac{1}{NU}\sum_{n=0}^{N} w^2(n) = \frac{1}{2\pi NU}\int_{-\pi}^{\pi}|W(e^{j\omega})|^2\,d\omega = 1.$$

$$\lim_{N\to\infty}\frac{1}{NU}W(e^{j\omega}) = 2\pi\delta(\omega)$$

∴ Modified periodogram is asymptotically unbiased.

If $U=1 \Rightarrow$ Modified per. reduces to periodogram.

<u>Variance</u>:

$$\lim_{N\to\infty} \text{var}[\hat{P}_M(e^{j\omega})] \cong P_x^2(e^{j\omega})$$

Since variance does not go to zero as $N \rightarrow \infty$, the modified per. is also not a consistent estimator.

→ Use of non-rectangular window offers no benefit in variance, but tradeoff frequency resolution.

Modified Periodogram :

↳ uses different window function.

Window	Sidelobe	3 dB BW
Rectangular	−13dB	0.9/N
Bartlett	−26dB	1.3/N
Hanning	−31 dB	1.45/N
Hamming	−42dB	1.3/N
Blackman	−56 dB	1.7/N

→ windows only provide tradeoff for frequency resolution and not on variance.

Averaged Periodogram [Bartlett method]

① Split data into non-overlapping signal segment of length 'L'.

$$x_i(n) = x(n+iL), \quad 0 \le n \le L-1, \quad 0 \le i \le M-1$$

② compute periodogram of each signal segment.

$$P_{x,i}(e^{j\omega}) = \frac{1}{L}|X_i(e^{j\omega})|^2$$

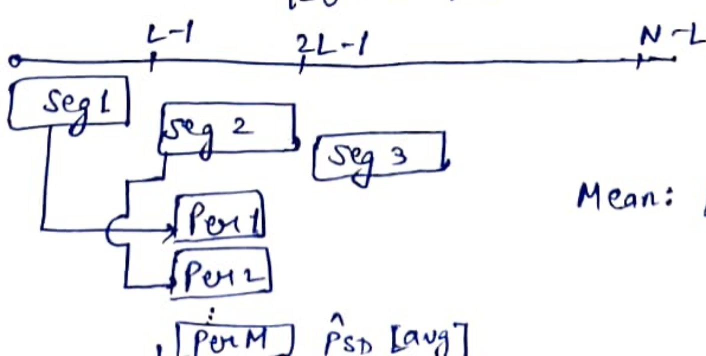
$$= \frac{1}{L}\left|\sum_{i=0}^{L-1} x_i(n)\, e^{-j\omega n}\right|^2$$

$$= \frac{1}{L}\left|\sum_{i=0}^{L-1} x(n+iL)\, e^{-j\omega n}\right|^2$$

③ Average the M periodograms.

$$P_B(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} P_{x,i}(e^{j\omega})$$

$$= \frac{1}{M}\sum_{i=0}^{M-1}\left|\frac{1}{L}\sum_{i=0}^{L-1} x(n+iL)\, e^{-j\omega n}\right|^2$$



Mean: $\mathbb{E}[P_B(e^{j\omega})]$

Mean of $\hat{P}_B(e^{j\omega}) = \mathbb{E}[\hat{P}_B(e^{j\omega})]$

$$= \frac{1}{M}\sum_{i=0}^{M-1} \mathbb{E}[P_{x,i}(e^{j\omega})]$$

$$= \mathbb{E}[P_x(e^{j\omega})] \qquad \left[\because \frac{1}{M}\cdot M\,\mathbb{E}[P_{x,i}(e^{j\omega})]\right]$$

$$= \frac{1}{2\pi L}\{P_{x,i}(e^{j\omega}) * W_B(e^{j\omega})\}$$

$\hookrightarrow$ It is a biased estimator.

$\therefore P_B(e^{j\omega})$ is a biased estimate of $P_x(e^{j\omega})$

$$\lim_{L\to\infty} W_B(e^{j\omega}) = 2\pi\delta(\omega)$$

$\therefore$ The periodogram is asymptotically unbiased.

Variance,

$$\mathrm{var}[\hat{P}_B(e^{j\omega})] = \frac{1}{M}\mathrm{var}[P_{x,i}(e^{j\omega})]$$

$$\cong \frac{1}{M}P_x^2(e^{j\omega})$$

$M\to\infty \Rightarrow \mathrm{var}[\hat{P}_B(e^{j\omega})] \to 0$

$\therefore P_B(e^{j\omega}) \to$ consistent estimator.

$M\uparrow, L\downarrow$

$$\Delta f = \frac{0.9}{L} \Rightarrow M\times\frac{0.9}{L},$$

i.e., M times larger than periodogram.

$$Q = \frac{\mathbb{E}[\hat{P}_B(e^{j\omega})]}{\mathrm{var}[\hat{P}_B(e^{j\omega})]}$$

$$= \frac{P_x^2(e^{j\omega})}{\frac{1}{M}P_x^2(e^{j\omega})} = M$$

Figure of merit,

$$M = \frac{1}{Q}\Delta f$$

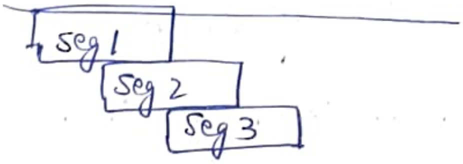
Welch Method [Averaged Modified Periodogram].

① Allow the signal to overlap.

② Use of non-rectangular window.

$x(n)$, $\quad 0 \le n \le N-1$

$x_i(n)$ $\quad$ [D samples] $\quad$ {Length of each segment : L}



$$x_i(n) = x(n+iD), \quad 0 \le n \le L-1$$

$D < L$

$N = L + D(M-1)$ $\quad$ * overlap b/w $x_i(n)$ & $x_{i+1}(n)$ is $L-D$

M s/g segments cover entire $x(n)$

$D = L/2$ $\quad$ [50% overlap]

$$M = \frac{2N}{L} - 1$$

→ Variance is reduced by 2 as the no. of samples are doubled.

$$P_M(e^{j\omega}) = P_{x,i}(e^{j\omega})$$

$$= \frac{1}{LU} |X_i(e^{j\omega})|^2$$

$$= \frac{1}{LU} \left| \sum_{i=0}^{L-1} x_i(n) w(n) e^{-j\omega n} \right|^2$$

$$= \frac{1}{LU} \left| \sum_{i=0}^{L-1} x(n+iD) w(n) e^{-j\omega n} \right|^2$$

$$U = \frac{1}{L} \sum_{n=0}^{L-1} w^2(n)$$

$$P_W(e^{j\omega}) = \frac{1}{M} \sum_{i=0}^{M-1} P_M(e^{j\omega})$$

$$= \frac{1}{M} \sum_{i=0}^{M-1} \left[\frac{1}{LU} \left| \sum_{n=0}^{L-1} x(n+iD) w(n) e^{-j\omega n} \right|^2 \right]$$

$\hat{\theta} = E[\theta]$

Mean,

$$E[\hat{P}_W(e^{j\omega})] = \frac{1}{M} \sum_{i=0}^{M-1} E[\hat{P}_W(e^{j\omega})]$$

$= \frac{1}{M} \cdot M \; \mathbb{E}\left[\hat{P}_M(e^{j\omega})\right]$

→ asymptotically unbiased w.r.t. mean

Variance,

$$\text{var}\left[\hat{P}_W(e^{j\omega})\right] \cong \frac{9}{8M} P_x^2(e^{j\omega})$$

Eg, 50% $\text{var}\left[\hat{P}_W(e^{j\omega})\right] \cong \frac{9}{16M} P_x^2(e^{j\omega})$

Frequency resolution → choice of window.

→ For a given M, variance of welch is larger than Bartlett by a factor 9/8.

For a fixed N & L, 50% overlap, the no. of segments is doubled.

$$\text{var}\left[\hat{P}_W(e^{j\omega})\right] \approx \frac{1}{8}, P_x^2(e^{j\omega})$$

Freq. resolution → window dependent.

If $D = L$,

$M = N/L$ → Bartlett method.

Blackman – Tukey Periodogram Smoothing

→ Compute $\hat{r}_x(m)$ of the signal.

→ Apply suitable window to estimate AC $\hat{r}_x\{w(n)\}$.

→ Take DTFT to obtain PSD.

$$\hat{P}_M(e^{j\omega}) = \sum_{m=-(L-1)}^{L-1} \hat{r}_x(m)\, w(m)\, e^{-j\omega m}$$

$L << N$ {Reduce variance by L/N}

Frequency resolution is also doubled.

$$\hat{P}_{BT}(e^{j\omega}) = \frac{1}{2\pi} P_{Per}(e^{j\omega}) * W(e^{j\omega})$$

$$= \frac{1}{2\pi} P_x(e^{j\omega}) * W_B(e^{j\omega}) * W(e^{j\omega})$$

$$= \sum_{K=-M}^{M} r_x(K) \underbrace{W_B(K)\, w(K)}\, e^{-j\omega}$$

$$\approx \frac{1}{2\pi} P_x(e^{j\omega}) * W_{BT}(e^{j\omega})$$

$$\mathbb{E}\left[\hat{P}_{BT}(e^{j\omega})\right] = P_x(e^{j\omega})$$

Variance,

$$\mathbb{E}\left[\hat{P}^2_{BT}(e^{j\omega})\right] = \mathrm{var}\left[P_x^2(e^{j\omega}) \cdot \underbrace{\frac{1}{N} \sum_{m=-(M-1)}^{(M-1)} W^2(m)}_{\text{depending on the choice of window}}\right]$$

$$= \begin{cases} 2M/N & \{\text{rect.}\} \\ 2M/3N & \{\text{Bartlett}\} \end{cases}$$

$$N \to \infty \Rightarrow \mathrm{Var} \to 0$$

Estimate	Quality Factor
Bartlett	$1.11\, N\Delta f$
Welch (50% overlap)	$1.39\, \Delta f N$
Blackman-Tukey	$2.34\, \Delta f N$

$$QF \longrightarrow 1.11\, \Delta f N$$

$$\Delta f \longrightarrow \frac{QF}{N}$$