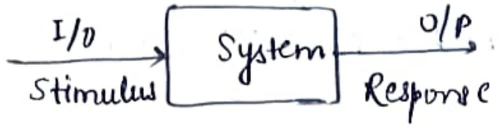
11-08-2025

Feeback

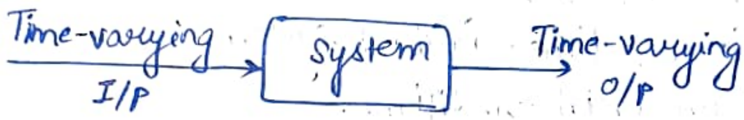
System :



→ open-loop system

YouTube :
- → M. Gopal on Control Systems
- → C.S. RAM (IITM) control systems.

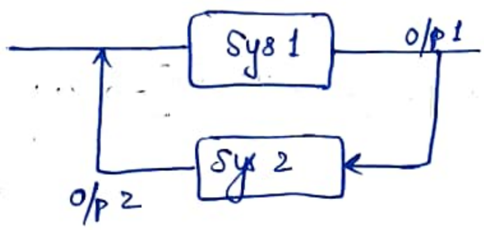
Dynamical system :



Feeback :

↳ Related to dynamical system

↳ Interconnection of two dynamical systems such that their outputs depend on each other.



→ circular argument

→ closed-loop system

Books:
- Norman Nise
  ↳ Control sys
- K. Ogata → Modern control Engg.
  ↳ Digital control.
- Dorf & Bishop
  ↳ Control sys.

Control System

↳ Classical control → Freq. domain analysis

↳ Modern Control
↳ Time-domain / state-space analysis

Frequency-domain analysis :

↳ Transfer Function approach

→ T.F. : $\dfrac{\text{LT of O/P}}{\text{LT of I/P}}$, under the condⁿ of zero initial state.

$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$: SISO system

$y(t) = x(t) * h(t) \Rightarrow Y(s) = X(s) \cdot H(s) \Rightarrow H(s) = \dfrac{Y(s)}{X(s)}$

→ T.F. approach (classical approach) deals with SISO system only.

↳ For LTI system.

→ Convolution is only applicable for LTI system.

State-Space Approach:

↳ Applicable to MIMO system also.

↳ Non-linear, time-variant system.

↳ Non-zero initial conditions.

Frequence Domain Analysis

↳ Routh-Locus plot

↳ Bode plot.

↳ Nyquist plot

↓

All governed by $\boxed{1 + G(s)\, H(s) = 0}$ → characteristic eqn for negative feedback system

14-08-2025

Feedback System

↳ Also known as closed loop system.

Control system performance matrix:

1. Transient response
2. Steady state error.

Compress sensing:

$$f_s < 2 f_m$$

↳ can be constructed.

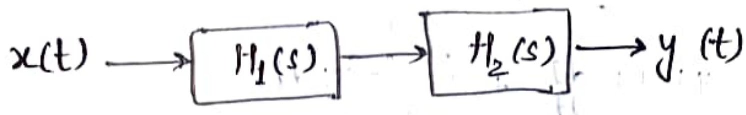
$\boxed{1 + \underbrace{G(s) \cdot H(s)}_{\text{also called openloop gain.}} = 0}$ → R.L, Bode plot, N.P.

$G(s)$: forward path gain.

Simplifying Blocks :

① cascade / Series connection
② Parallel connection
③ Feedback connection.

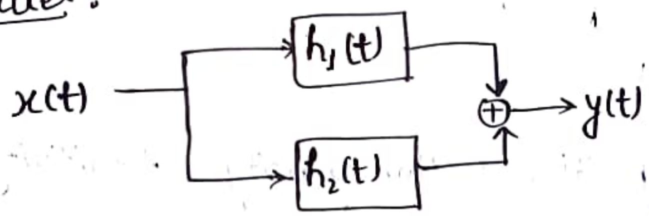
Series :



$$y(t) = x(t) * h_1(t) * h_2(t)$$

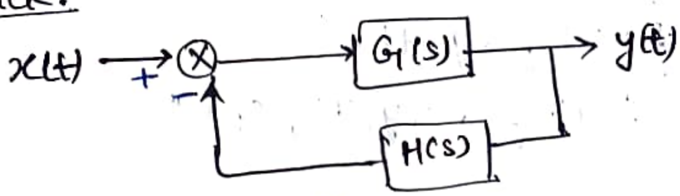
- In Laplace domain, we get into whether a system will decay or diverge (stability).
- In Fourier domain, we do not get such information.

Parallel :



$$y(t) = x(t) * h_1(t) + x(t) * h_2(t)$$

Feedback :

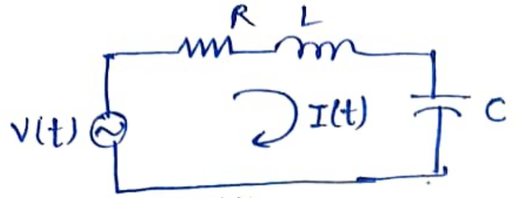


$$T.F. = \frac{G(s)}{1 + G(s) . H(s)}$$

18-08-2025

Some Terminologies Related to Systems

① Linear and Non-linear Systems :



$$V(t) = RI(t) + L\frac{dI}{dt} + \frac{1}{C}\int_{t_0}^{t} I(t)\,dt$$

Follows additivity and homogeneity ⇒ Linear system

Diode → Non-linear (exponential or parabolic)

$$u(t) \longrightarrow \boxed{H(\cdot)} \longrightarrow y(t)$$

$$y(t) = H(u(t))$$

Linearity ≡ superposition = Homogeneity + Additivity

ⓘ Homogeneity /scaling :

$$u_1(t) \longrightarrow \boxed{H(\cdot)} \longrightarrow y_1(t)$$

$$u_2(t) \longrightarrow \boxed{H(\cdot)} \longrightarrow y_2(t)$$

$$c_1 u_1(t) \longrightarrow \boxed{H(\cdot)} \longrightarrow c_1 y_1(t), \quad c_1 \in \mathbb{R}$$

ⓘⓘ Additivity :

$$u_1(t) + u_2(t) \longrightarrow \boxed{H(\cdot)} \longrightarrow y_1(t) + y_2(t)$$

Combining ⓘ and ⓘⓘ,

$$c_1 u_1(t) + c_2 u_2(t) \longrightarrow \boxed{H(\cdot)} \longrightarrow c_1 y_1(t) + c_2 y_2(t).$$

$y = mx + c \longrightarrow$ Affine (Non-linear)

② Causal and Non-Causal Systems

Causal systems : Practical systems

↳ Output at any instant depends on past and present inputs.

$$y(t) = 2x(t) + 4\underbrace{x(t-2)}_{\text{memory}}$$

Non-causal : Also depends on future inputs.

③ SISO and MIMO :

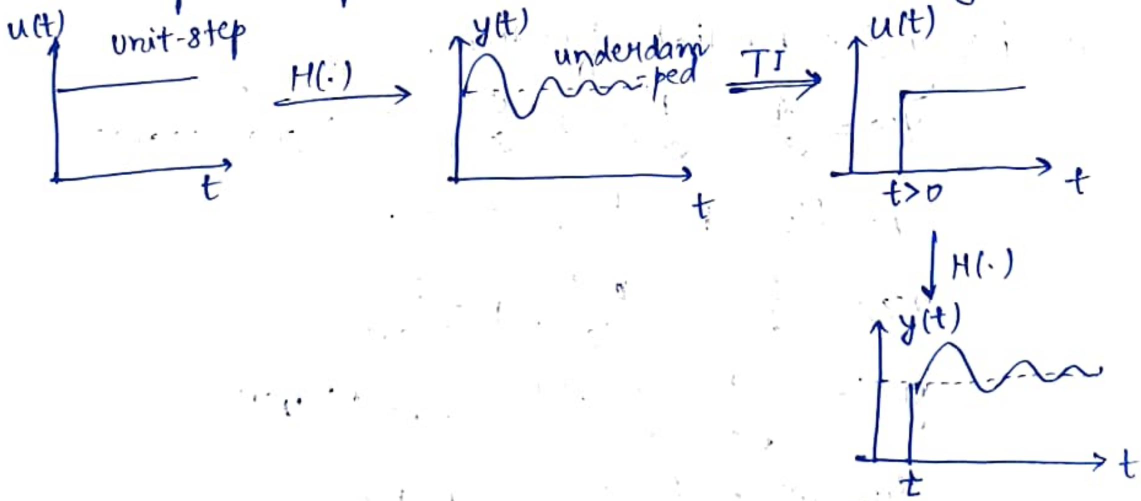
SISO : $\underbrace{\text{Single input}}_{u(t) \in \mathbb{R}}$ and $\underbrace{\text{single output}}_{y(t) \in \mathbb{R}}$

Federated Learning
Edge Computing

MIMO : Multiple input and multiple output.

④ Time Invariant and Time varying :

TI : Input-output relationship never changes with time.



↳ Doesn't depend at which instant i/p is applied; Output pattern/shape remains the same.

28-08-2025

BIBO Stability [Bounded Input Bounded Output]

For any bounded i/p, $u(t)$, i.e., $|u(t)| \leq M < \infty$, $\forall\ t \geq 0$.
The system response, $y(t)$, is bounded,
i.e., $|y(t)| \leq N < \infty$, $\forall\ t > 0$.

$$u(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$y(t) = h(t) * u(t)$
$\mathcal{L}(y(t)) = \mathcal{L}(h(t) * u(t))$
$Y(s) = H(s)U(s)$
$u(t) = \delta(t) \Rightarrow U(s) = 1$
$Y(s) = H(s)$
$\Rightarrow y(t) = h(t)$

- **Need for unit-step function:**
  For analysis of sudden disturbances.

Eg. $\ddot{y}(t) + 5\dot{y}(t) + 6y(t) = u(t)$. Find the unit step response.

Taking LT both side, $U(s) = 1/s$,

$$s^2 Y(s) + 5sY(s) + 6Y(s) = U(s)$$

$$Y(s) = \frac{U(s)}{s^2 + 5s + 6}$$

$$H(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2 + 5s + 6} = \frac{1}{(s+3)(s+2)}$$

$$= \frac{1}{s+2} - \frac{1}{s+3} \rightarrow \text{Poles}: -2, -3$$

$$\therefore Y(s) = \frac{1}{s(s+2)} - \frac{1}{s(s+3)}$$

Use heavy-side partial fraction,

$$Y(s) = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3} = \frac{1}{s(s+2)(s+3)}$$

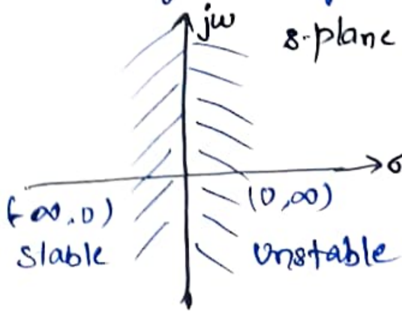
$$A = 1/6,\quad B = \frac{1}{-2},\quad C = \frac{1}{3}.$$

$$Y(s) = \frac{1}{6s} - \frac{1}{2(s+2)} + \frac{1}{3(s+3)}$$

$$y(t) = \left(\frac{1}{6} + \frac{1}{3}e^{-3t} - \frac{1}{2}e^{-2t}\right) u(t).$$

$$\lim_{t\to\infty} y(t) = \frac{1}{6} < \infty \quad : \text{BIBO}$$

- In frequency domain, if all poles lie on the left side of $j\omega$-axis, then system is stable.
- At $j\omega$-axis, system is marginally stable.
- Marginally stable points are unstable system except for some points.



Eg. $\ddot{y}(t) + y(t)$. Find the step response & also consider $u(t)$.

Soln: Take LT both side,

$s^2 Y(s) + Y(s) = U(s)$

$\Rightarrow H(s) = \frac{1}{s^2+1} \rightarrow$ poles : $\pm j \Rightarrow h(t) = \sin(t)$

$\therefore Y(s) = \frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$

$\Rightarrow y(t) = u(t) - \cos(t)\, u(t)$

$= (1-\cos t)\, u(t)$

$\cos, \sin \in [-1, 1]$

↳ Bounded by 0 or 2 times $u(t)$.

Let $u(t) = \sin t \Rightarrow U(s) = \frac{1}{1+s^2}$

$Y(s) = \frac{1}{(s^2+1)^2} \Rightarrow y(t) = t \cdot \sin t$ ← bounded

$\lim_{t\to\infty} y(t) = \infty$. → Unbounded

Let $u(t) = \sin 2t$

$U(s) = \frac{2}{s^2+4}$

$Y(s) = \frac{1}{(s^2+1)} \cdot \frac{2}{(s^2+4)} \Rightarrow y(t) = \overbrace{\cos 2t \pm \sin 2t}^{\text{not exact}}$ ↳ Bounded

Conclusion: This system goes to infinity if input natural frequency is same as the impulse response frequency at which pole occurs.

Eg. $\ddot{y}(t) + \dot{y}(t) = u(t)$. (i) $u(t) = \sin t$ and (ii) $u(t) =$ unit-step.

$s^2 Y(s) + s Y(s) = U(s)$

$\Rightarrow H(s) = \frac{1}{s+s^2} = \frac{1}{s(s+1)}$; Poles : $-1, 0$

$\Rightarrow h(t) = (1-e^{-t})\, u(t)$

(i) $Y(s) = H(s)U(s) = \frac{1}{s(s+1)} \cdot \frac{1}{s^2+1}$

$y(t) = 1 \pm e^{-t} \pm \sin t$

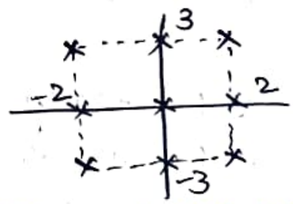
$\lim_{t\to\infty} |y(t)| \le M < \infty$

(ii) $Y(s) = H(s)U(s) = \frac{1}{s(s+1)} \cdot \frac{1}{s} = \frac{1}{s^2(s+1)}$

$y(t) = t \pm e^{-t}$.

$\lim_{t\to\infty} y(t) = \infty$

- If pole is at origin, then DC i/p signal will make the system unstable.



Pole: $-2 \pm 3j$ (stable)

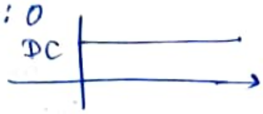
Pole: -2 (stable)

Pole : 2 (Unstable)

Pole: $2 \pm 3j$ (Unstable)

Pole : $3j$

Pole : 0



29-08-2025

BIBO stability:

$$u(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

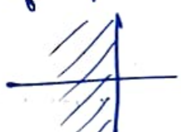
$u(t) \leftrightarrow U(s)$, $h(t) \leftrightarrow H(s)$, $y(t) \leftrightarrow Y(s)$

$$|u(t)| \leq M < \infty, \ \forall t, \forall u$$

$$|y(t)| \leq N < \infty, \ \forall t$$

$$\underbrace{H(s) = \frac{Y(s)}{U(s)}}_{\text{Poles of the TF should strictly lie left side of s-plane}} \longleftrightarrow \int_0^t |h(t)|\, dt < \infty$$

↳ Absolutely integrable (for stability)



$\sigma \in (-\infty, 0)$

⇓

BIBO stable

Taylor series to linearize a system about a point.

Q) Prove that if the real parts of the ~~roots~~ poles are strictly negative, then the LTI system is BIBO stable.

Poles : steady state/Stability
Zeros : Transient response.

Effects of Zeros of TF on Output:

Eg. ① $H_1(s) = \dfrac{1}{(s+1)(s+10)}$, ② $H_2(s) = \dfrac{s+2}{(s+1)(s+10)}$

Analyze unit-step response.

Soln: ① $Y_1(s) = H_1(s)\, U(s) = \dfrac{1}{s(s+1)(s+10)} = \dfrac{A}{s} + \dfrac{B}{(s+1)} + \dfrac{C}{(s+10)}$

$$= \frac{1}{10}\cdot\frac{1}{s} - \frac{1}{9}\cdot\frac{1}{s+1} + \frac{1}{90}\cdot\frac{1}{s+10}$$

$$y_1(t) = \left(\frac{1}{10} - \frac{1}{9}e^{-t} + \frac{1}{90}e^{-10t}\right) u(t)$$

$$\lim_{t\to\infty} y_1(t) = \frac{1}{10} < \infty.$$

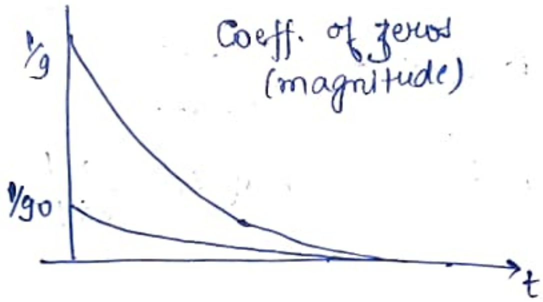
② $Y_2(s) = H_2(s)\,U(s) = \frac{s+2}{s(s+1)(s+10)} = \frac{A}{s} + \frac{B}{(s+1)} + \frac{C}{(s+10)}$

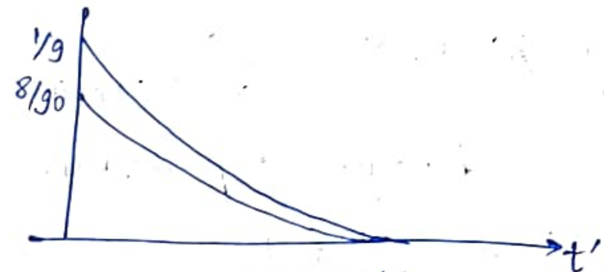
$$= \frac{2}{10}\cdot\frac{1}{s} + \frac{1}{9}\cdot\frac{1}{s+1} - \frac{8}{90}\cdot\frac{1}{s+10}$$

$$= \frac{1}{5}\frac{1}{s} - \frac{1}{9}\frac{1}{s+1} - \frac{8}{90}\cdot\frac{1}{s+10}$$

$$y_2(t) = \left(\frac{1}{5} - \frac{1}{9}e^{-t} - \frac{8}{90}e^{-10}\right)u(t)$$

$$\lim_{t\to\infty} y_2(t) = \frac{1}{5} < \infty.$$





Dominant Poles : Poles closer to the $j\omega$ axis (or origin). → slow decay rate

↳ Less damping : Effects can be observed longer.

$\frac{\cancel{(s+2)}}{\cancel{(s+1)}(s+10)}$ ≈ more towards the dominant pole ∴ Pole-zero cancellation.

$H(s) = \frac{(s-2)}{(s-2)(s+1)(s+10)}$ → ~~3-degree~~ order : 3 → stable a/c to TF → unstable a/c to state-space.

Minimal Realization of State-space

~~Controbatility~~

Controllability : we should be able to move to any point from a point.

↳ Loss in degree of freedom ⇒ May not be able to get.

Observability: we need sensor to observe.

No. of poles : No. of energy storing elements.

Pole-zero cancellation ⇒ Reduction in degree of freedom.

Minimal realization : when poles & zeros don't cancel.

In reality, ~~H~~ H* system is unstable but as we cancel poles & zeros, it becomes stable.

Non-Minimum Phase Zero (Nyquist plot):

Minimum Phase System: All zeros and poles must lie in the left half of s-plane.

Bode plot: Can only represent min^m ~~and non-min^n~~ phase sys.
Nyquist plot: Can represent both.

Non-Minimum Phase system = All pass system × Min^m Phase system.

01-09-2025

Eg. (A) $H_1(s) = \dfrac{s-2}{(s+1)(s+10)}$ (B) $H_2(s) = \dfrac{-(s-1)}{(s+1)^2}$

Analyze step response of both. (C) $H_3(s) = \dfrac{s-1}{(s+1)^2}$

Sol^n: (A) Poles: $-1, -10$; Zeros: 2

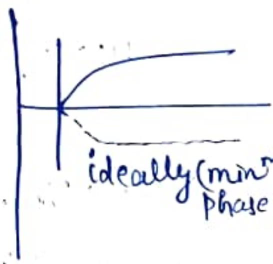
$n = 2;\ m = 1.$

$$H_1(s) = \frac{Y_1(s)}{U(s)} \Rightarrow Y_1(s) = H_1(s)\,U(s) = \frac{s-2}{s(s+1)(s+10)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+10}$$

$$= \left(-\frac{2}{10}\right)\frac{1}{s} + \frac{3}{9}\,\frac{1}{s+1} - \frac{12}{90}\,\frac{1}{s+10}$$

$$y_1(t) = \left(-\frac{2}{10} + \frac{3}{9}e^{-t} - \frac{12}{90}e^{-10t}\right)u(t).$$

$$\lim_{t\to\infty} y_1(t) = -\frac{2}{10} < \infty \rightarrow \text{stable}$$

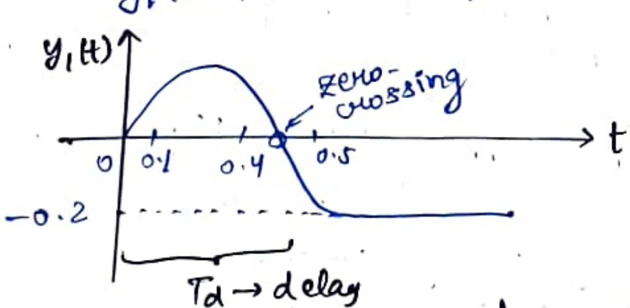


$y_1(0) = 0,\ y_1(5) \approx -0.198$

$y_1(10) \approx -0.2$

$y_1(0.1) \approx 0.05,\ y_1(0.4) \approx 0.019$

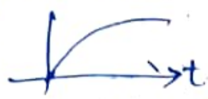
$y_1(0.5) \approx -0.0007.$



$T_d \rightarrow$ delay
↳ determined by mag of zero.

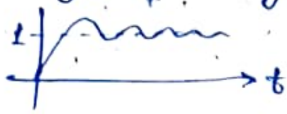
→ Non-min^m phase zero
↳ called as faulty sys.

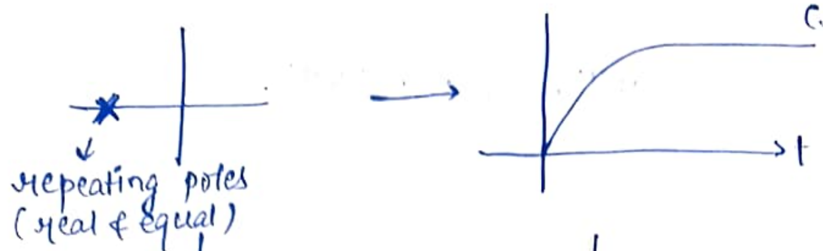
→ first go in the wrong dir^n, then in reverse dir^n.
↳ Should have been in only one-dir^n.



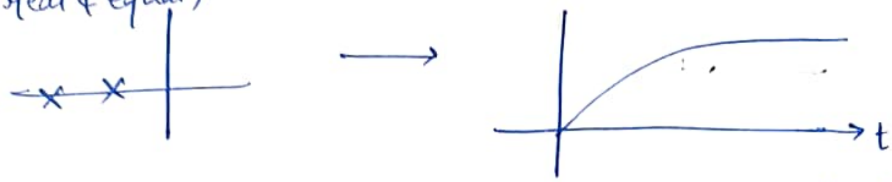
↳ Min^m phase zero

→ Underdamped systems don't have zero-crossings.
↳ Oscillates around '1'.





critically damped
↳ faster



over-damped

→ we design underdamped system with ξ value 0.7 – 0.8.

→ Time-delay in a system might introduce NMP zero.

Eg. $\dot{y}(t) + y(t) = u(t - T_d)$, where $T_d > 0$ is transport delay

$$s\,y(s) + y(s) = e^{-s T_d}\, u(s) \Rightarrow H(s) = \frac{y(s)}{u(s)} = \left(\frac{e^{-T_d s}}{s+1}\right)$$

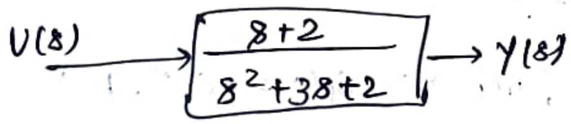
Exponential delay term can be approximated.

① $e^{-T_d s} \approx \dfrac{1}{T_d s + 1} \Rightarrow H(s) = \dfrac{1}{(s+1)(T_d s+1)}$

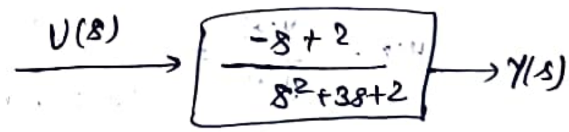
② $e^{-T_d s} = \dfrac{e^{-T_d s/2}}{e^{T_d s/2}} \approx \dfrac{1 - T_d s/2}{1 + T_d s/2} = \dfrac{2 - T_d s}{2 + T_d s}$

... Pade's Approx^m.

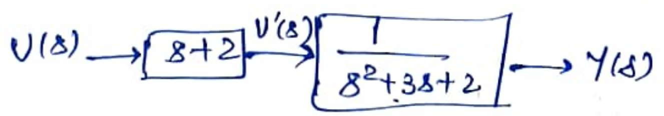
Eg. ①



②



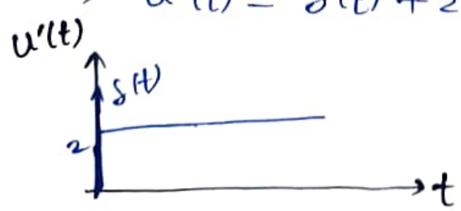
Solⁿ: ①



$$\frac{U'(s)}{U(s)} = s+2 \Rightarrow U'(s) = (s+2)\,U(s) \xrightarrow{ILT} u'(t) = \frac{d}{dt}(u(t)) + 2u(t)$$

Let's consider u(t) to be step input.

$$\Rightarrow u'(t) = \delta(t) + 2u(t)$$

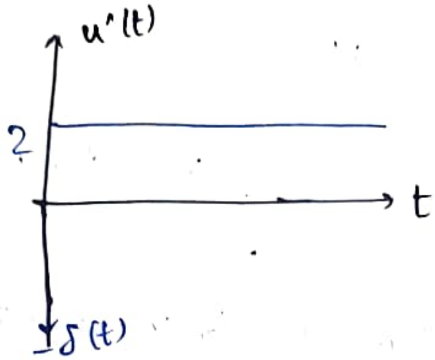


$\delta(t)$ → shock
↳ in +ve dirⁿ.

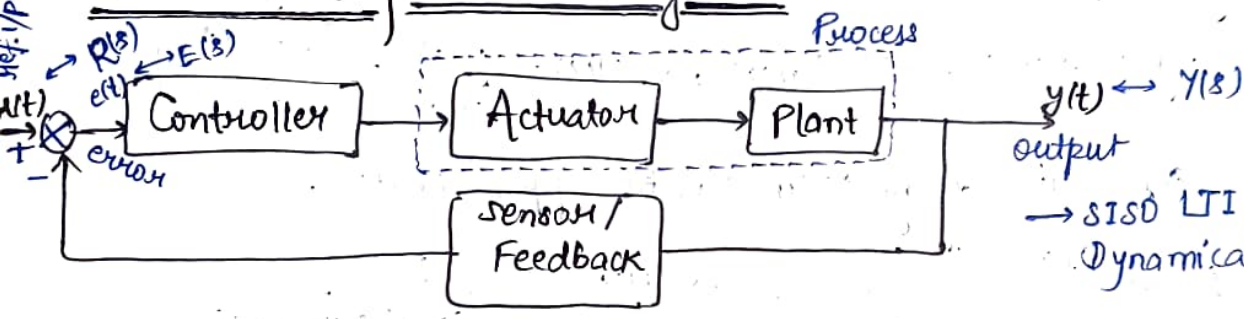
→ Poles: −1, −2
Zeros: −2.

(2) Poles: $-2, -1$
Zeros: $+2$

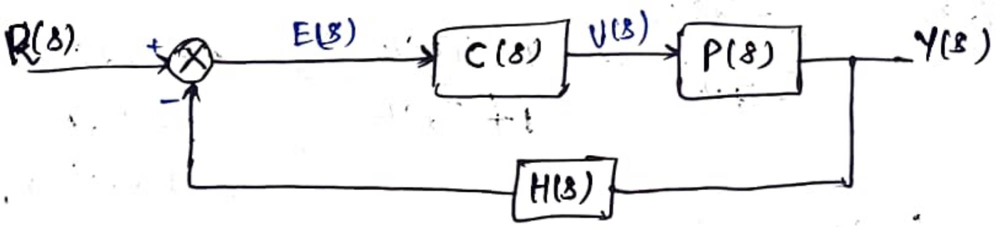
$u(t) = -\delta(t) + 2u(t)$.



Closed Loop Control System



→ SISO LTI (causal) Dynamical system

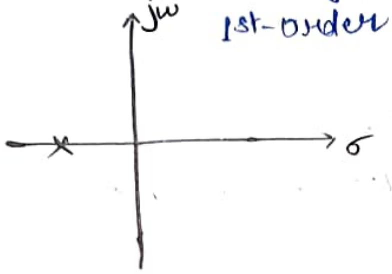


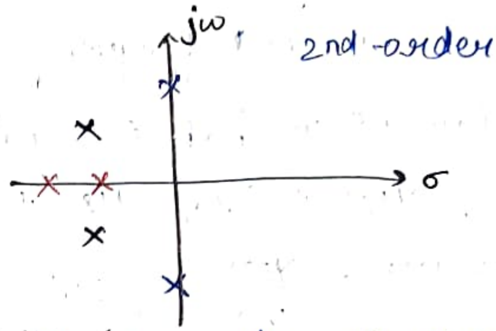
$$H(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$, where $G(s) = C(s).P(s)$

- Full-state feedback.
- SISO LTI (causal) dynamic system.

04-09-2025

- 1st-order system
- 2nd-order system.
- Any higher order system can be studied through 1st and 2nd order system.





- In first order, pole will always be at σ-axis only, and no complex poles, because for complex poles, its conjugate should also exist (2nd order).

1st-order system :

$$T \frac{dy(t)}{dt} + y(t) = u(t), \quad T > 0$$

Taking LT on both sides with ZIC,

$$T s Y(s) + Y(s) = U(s)$$

$$\Rightarrow Y(s) = \frac{U(s)}{1 + Ts}$$

$$= \frac{1}{s} \cdot \frac{1}{sT + 1}$$

$$= \frac{1}{s} - \frac{1}{s + 1/T}$$

$\updownarrow$ ILT

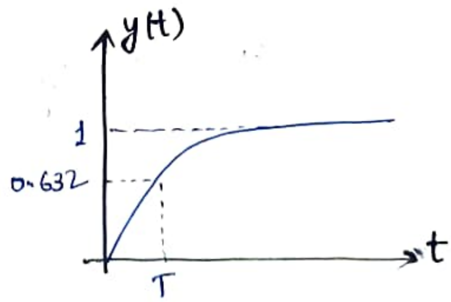
$$y(t) = (1 - e^{-t/T})\, u(t)$$

At $t = 0$, $y(0) = 0$

$t \to \infty$, $y(\infty) = 1$

$t = T$, $y(T) = 0.632$

$= 63.2\%$ of F.V.



2nd-order System

$$\frac{d^2y(t)}{dt^2} + 2\xi\omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = \omega_n^2 u(t),$$

where $y(t)$: o/p,
$u(t)$: i/p

ξ : damping ratio / const. / coeff.
ω_n : natural frequency.

Take LT on both side with $\not{I}C$

$$s^2 Y(s) + 2\xi\omega_n s Y(s) + \omega_n^2 Y(s) = \omega_n^2 U(s)$$

$$\Rightarrow H(s) = \frac{Y(s)}{U(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Poles: $s^2 + 2\xi\omega_n s + \omega_n^2 = 0$

$$\Rightarrow s = \frac{-2\xi\omega_n \pm \sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2}$$

$$= -\xi\omega_n \pm \sqrt{\omega_n^2(\xi^2 - 1)}$$

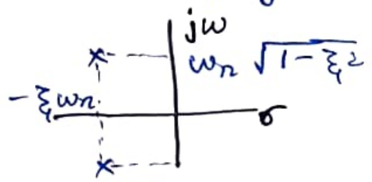
Case: (i) $\xi < 1$:

$s = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$ → complex poles

oscillating with frequency,

$\omega_d = \omega_n\sqrt{1-\xi^2}$ → damping freq.

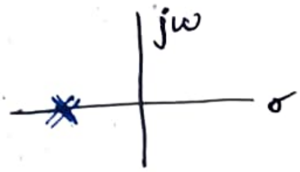
"Underdamped"



Case: (ii) $\xi = 1$:

$s = -\omega_n, -\omega_n$ → Repeating real poles

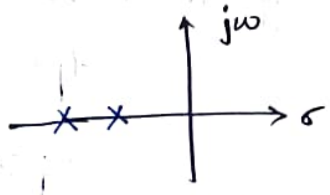
"critically damped"

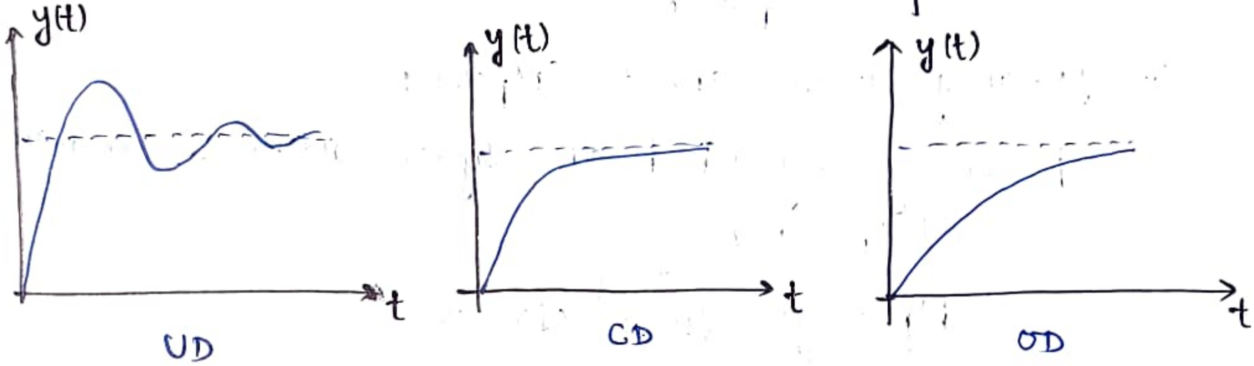


Case-(iii) : $\xi > 1$:

$s = -\xi\omega_n \pm \omega_d$: Non-repeating real poles

"overdamped"





Unit-step response of 2nd order system :

$$H(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

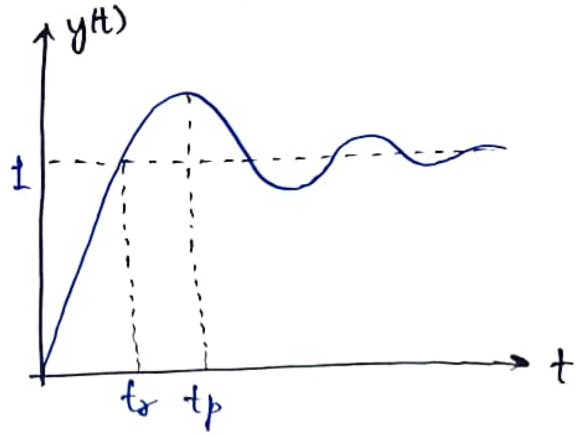
$$\Rightarrow Y(s) = \frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)}$$

$$= \frac{A}{s} + \frac{Bs + C}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$y(t) = 1 - e^{-\xi\omega_n t}\cos(\omega_d t) - \frac{\xi}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t}\sin(\omega_d t)$$

$t = 0,\ y(0) = 1 - 1 - 0 = 0$

$t \to \infty,\ y(\infty) = 1$



(i) Rise time, $t_r = \frac{\pi - \beta}{\omega_d}$

where,

$\beta = \cos^{-1}(\xi)$,

$\omega_d = \omega_n\sqrt{1-\xi^2}$

↳ Time take by the system (step) response to reach 100% of its final value for the 1st time.

(ii) Peak time (t_p) :

: Time taken to reach the maximum value of the system response.

$$t_p = \frac{\pi}{\omega_d}$$

$$t_p > t_r$$

(iii) Max^m peak overshoot ($M_p\%$) :

Percentage by which the system response max^m value exceed the F.V.

$$M_p = e^{-\left(\frac{\xi\pi}{\sqrt{1-\xi^2}}\right)} \times 100\ \%$$

(iv) Setting time (t_s) :

Time taken by the system response to reach FV within a particular band (2% and 5%).

$$t_s = \frac{4}{\xi\omega_n} \quad (\pm 2\%\ \text{band})$$

$$t_s = \frac{3}{\xi\omega_n} \quad (\pm 5\%\ \text{band})$$

11-09-2025

Routh - Hurwitz Stability Criterion

$$H(s) = \frac{G(s)}{1 \pm G(s) H(s)}$$

Poles: $1 + G(s)H(s) = 0$

↳ Poles: soln / roots of this eqn.

General eqn :

$$H(s) = \frac{N(s)}{D(s)}$$

$$D(s) = a_0 s^n + a_1 s^{n-1} + \dots + a_n \quad (n\text{th})$$

Hurwitz polynomial : Polynomial whose roots have negative real parts.

$$e^{j\omega t} \longrightarrow \boxed{\begin{matrix} H(s) \\ h(t) \end{matrix}} \longrightarrow A e^{j(\omega t - \phi)}$$

→ Just scaling and some phase is introduced.

Hurwitz Matrix : Matrix whose eigen value have negative real part.

$$D(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_n$$

$n = 1 \Rightarrow D(s) = a_0 s + a_1 \Rightarrow s = -\frac{a_1}{a_0}$

[a_1 & a_0 have same sign]

$n = 2 \Rightarrow D(s) = a_0 s^2 + a_1 s + a_2$

$$\Rightarrow s = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0 a_2}}{2a_0}$$

$$= \frac{-a_1}{2a_0} \pm \frac{\sqrt{a_1^2 - 4a_0 a_2}}{2a_0}$$

$$Y = A\,x = \lambda x$$

eigen value (λ), eigen vector (x)

transformation (A); (scalar)

$A \in \mathbb{R}^{2\times3}$

$x \in \mathbb{R}^3$

vector has not changed directly, only magnitude changed.

Eigen pair (λx) :

Dirn of linear transformation that never changes from original dirn; but the scaling dirn may change, defined by eigen value.

So, here $\frac{a_1}{2a_0} > 0$ (a_1 and a_0 of same sign)

$\sqrt{a_1^2 - 4a_0a_2} < a_1 \Rightarrow a_0$ & a_2 of same sign.

$\Rightarrow a_0, a_1$ and a_2 are of same sign.

$n=3 \Rightarrow D(s) = a_0s^3 + a_1s^2 + a_2s + a_3$

Let $D_1(s) = s^3 + 2s^2 + 3s + 1$

$\Rightarrow s = -0.43, -0.785 \pm 1.13j$

↳ This is Hurwitz polynomial.

$D_2(s) = s^3 + 22s^2 + s + 1$

$\Rightarrow s = -2.17, 0.09 \pm 1.17j$

↳ Not Hurwitz, +ve real part.

$D_3(s) = s^3 + 2s^2 + 3s$

$\Rightarrow s = 0, -1 \pm \sqrt{2}j$

↳ Not Hurwitz

For $n \geq 3$, All the coefficients have to be non-negative.

↳ This is necessary but not sufficient condition.

Necessary condition: All coefficients must be non-zero and of the same sign.

Sufficient : Routh Array

$$\begin{array}{c|ccc}
s^n & a_0 & a_2 & a_4 \cdots \\
s^{n-1} & a_1 & a_3 & a_5 \cdots \\
s^{n-2} & b_1 = \frac{a_1a_2 - a_0a_3}{a_1} & b_2 & b_3 \cdots \\
\vdots & & & \\
s^0 & & &
\end{array}$$

$\boxed{D(s) = a_0s^n + a_1s^{n-1} + a_2s^{n-2} + a_3s^{n-3} + \ldots + a_n}$

$b_2 = \frac{a_1a_4 - a_0a_5}{a_1}$

* Sufficient cond^n: Along the 1st column, all coefficients are positive.
* If there are sign changes, the polynomial is not Hurwitz & corresponding system is unstable.
* No. of sign changes denote the no. of pole in +ve side of $j\omega$-axis.

Case-II : The 1st element in any row is zero with other element as non-zero.

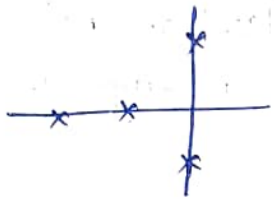
Eg. ① $D(s) = s^4 + 3s^3 + 3s^2 + 3s + 2$ $\Rightarrow$ Roots : $-1, -2, \pm j$

s^4	1	3	2
s^3	3	3	
s^2	$\frac{3\times3 - 3\times1}{3} = 2$	2	
s^1	$0 \to \epsilon$		
s^0	2		

No sign change

$\epsilon \to 0, \epsilon > 0$

→ Marginally stable (due to zero).



Case-II :

$D(s) = s^7 + 6s^6 + 11s^5 + 6s^4 + 4s^3 + 24s^2 + 44s + 24$

s^7	1	11	4	44
s^6	6	6	24	24
s^5	10	0	40	
s^4	6	0	24	
s^3	$0 \to 4$	0		
s^2	$0 \to \epsilon$	24		

→ Entire row becomes zero.
↓
Construct auxiliary eqn (not ϵ) with the row above.

$A(s) = 6s^4 + 24 = 0$

$\Rightarrow A(s) = s^4 + 4 = 0$

$A'(s) = \frac{dA}{ds} = 4s^3$

↳ Replace s^3 by 4.

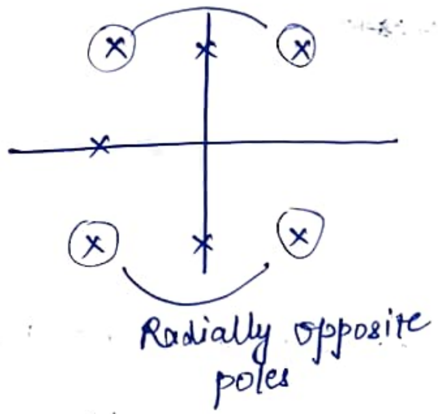
s^1	$-96/\epsilon$
s^0	24

$s^2 \rightarrow s^1$ (1 sign change)
$s^1 \rightarrow s^0$ (Another sign change)
$\Rightarrow$ Unstable system

n = No. of roots/poles = 7
2 sign changes $\Rightarrow$ 2 poles on RHP.

$A(s) = s^2 + 4 = 0 \Rightarrow A'(s) = \frac{dA}{ds} = 4s^3$ coeff. of s^3

$\Rightarrow \underbrace{(s^2+2s+2)}_{i \pm j}\underbrace{(s^2-2s+2)}_{1 \pm j} = 0$ $\Rightarrow$ 4 radially opposite poles ; 1 pole on LHP.



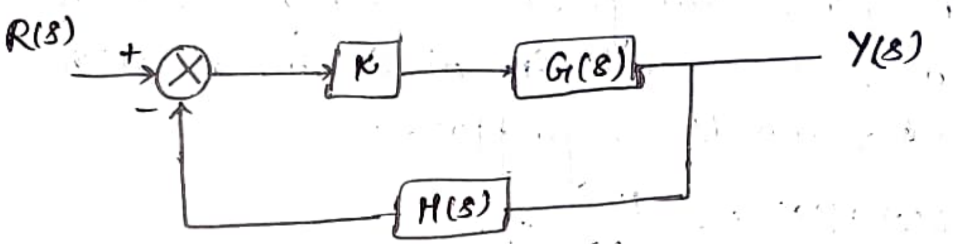
All zeros in a row, ensure atleast one pair of poles on the Img-axis.

ROOT LOCUS.

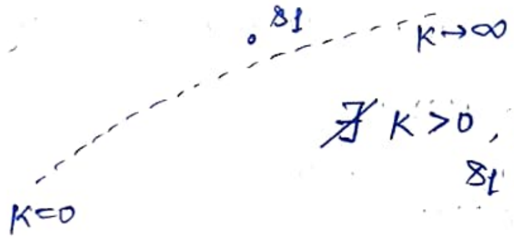
Closed Loop TF:

$$H(s) = \frac{G(s)}{1 + G(s)H(s)}$$

Characteristic eqn: $\boxed{1 + G(s)H(s) = 0} \Rightarrow$ soln/roots are C.L poles



$$CLTF = \frac{KG(s)}{1 + KG(s)H(s)}$$



$\nexists K > 0$, for which s_1 is CL pole.

$1 + G(s)H(s) = 0$: charac. eqn

$G(s)H(s)$: OL gain/TF

$G(s)H(s) = -1$: OLTF

$|G(s)H(s)| = 1$ ... Magnitude Criteria

$\angle G(s)H(s) = \pm 180^\circ (2K+1), \quad K = 0, 1, \ldots$

(odd multiples of $\pm 180^\circ$)

... Angle criteria

OLTF: $$G(s)H(s) = \frac{K(s+Z_1)(s+Z_2)\ldots(s+Z_m)}{(s+P_1)(s+P_2)\ldots(s+P_n)}$$

m : no. of zeros

n : no. of poles

How would the CL poles change as we vary $K \geq 0$?

↪ Answer lies in RL representation.

CL char. eqn: $1 + G(s)\,H(s) = 1 + \frac{K(s+Z_1)(s+Z_2)\ldots(s+Z_m)}{(s+P_1)(s+P_2)\ldots(s+P_n)} = 0$

Branch : Trajectories of RL

- n = no. of branches. (= no. of OL poles)
- Branches always start from pole locations, and terminate at zeros or move towards asymptotes.
- n branches ~~always~~ start from poles, terminate at 'm' zeros and ~~(m-n)~~ $(n-m)$ branches moves towards asymptotes.

Eg. OLTF : $G(s)\,H(s) = \frac{K}{(s+3)(s-2)}$.

$n = 2$; $P_1 = -3$, $P_2 = 2$

$m = 0$.

- 2 branches; 2 branches towards asymptotes.

CL char. eqn : $1 + G(s)\,H(s) = 0$

$\Rightarrow 1 + \frac{K}{(s+3)(s-2)} = 0$

$\Rightarrow (s+3)(s-2) + K = 0$

$K = 0 \Rightarrow s = -3, 2$

Eg. $G(s)\,H(s) = \frac{K(s+1)}{(s+3)(s-2)}$

CLCE: $1 + G(s)\,H(s) = 0$

$\Rightarrow 1 + \frac{K(s+1)}{(s+3)(s-2)} = 0$

$\Rightarrow (s+3)(s-2) + K(s+1) = 0$

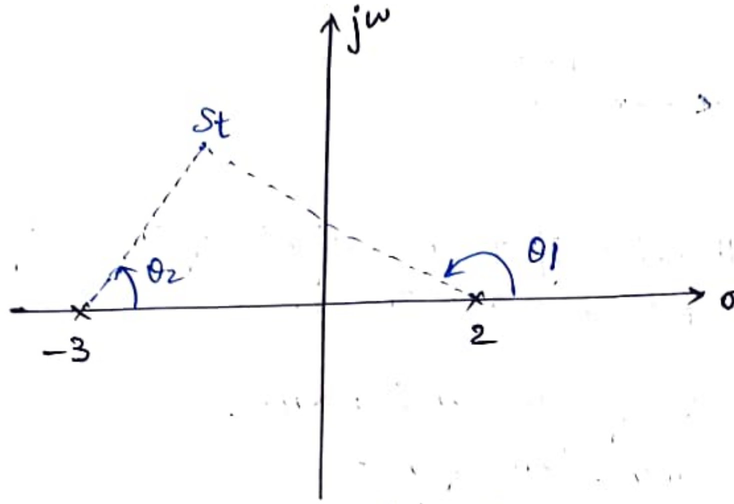
$\Rightarrow \frac{1}{K}(s+3)(s-2) + (s+1) = 0$

If $K \rightarrow \infty$, $s = -1$.

How to check if a testpoint 's_t' lies in RL?

↳ 's_t' is a CL pole for a certain $K>0$ value.

Eg $$G(s)\,H(s) = \frac{K}{(s+3)(s-2)}$$



$$|G(s_t)\,H(s_t)| = \left|\frac{K}{(s_t+3)(s_t-2)}\right| = 1 \quad \text{(mag. criteria)}$$

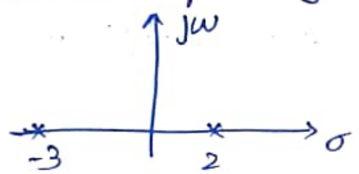
whether 's_t' satisfies this.

$$\angle G(s_t)\,H(s_t) = \angle K - \angle(s_t+3) - \angle(s_t-2)$$
$$= 0^\circ - \theta_1 - \theta_2$$
$$= -(\theta_1+\theta_2) = \pm 180^\circ (2K+1)$$

Whether 's_t' satisfies this.

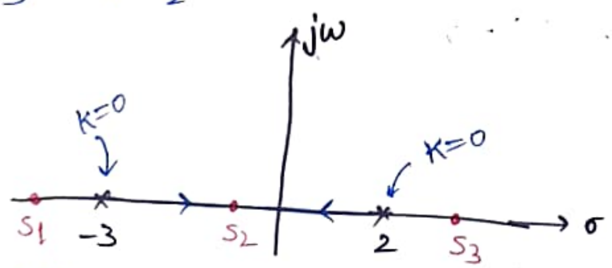
Steps:

① Locate the OL poles and zeros.



$n=2 \Rightarrow 2$ Branches

$m=0 \Rightarrow (n-m)=2$ branches towards asymptotes.



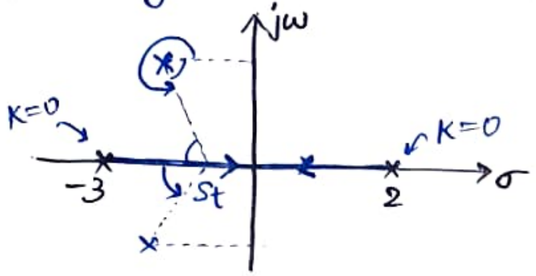
For s_1: $(-\infty,-3)$: 360° ✗

For s_2: $[-3,2]$: 180° ✓ → Part of RL

For s_3: $(2,\infty)$: 0° ✗

12-09-2025

② Locate the RL on the real axis.
↳ Determined by real OL poles, and zeros.
(others get cancelled to 360°/0°.)



- RL branch will exist if to the right of the test point, odd no. of poles + zeros are there.

③ Find the asymptotes of root loci ($n > m$).
n : no. of branches / no. of poles
m : ~~no~~ termination of branches.
$n-m$: no. of asymptotes.

$$\angle \text{asymptotes} = \pm \frac{180^\circ (2K+1)}{n-m}, \quad K = 0, 1, \ldots$$

Centroid : Intersection of asymptotes with real axis.

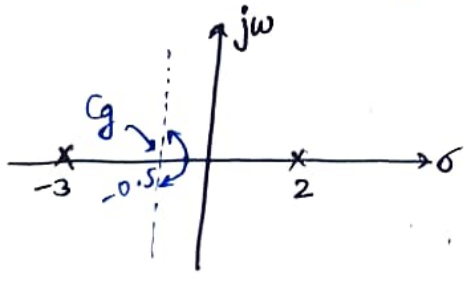
$$\hookrightarrow C_g = \frac{\sum \text{Real parts of poles} - \sum \text{Real parts of zeros}}{n-m}$$

Eg. $G(s)H(s) = \dfrac{K}{(s+3)(s-2)}$

$$\theta_{asym.} = \frac{\pm 180^\circ (2K+1)}{2} = \pm 90^\circ (2K+1), \; K = 0, 1, \ldots$$

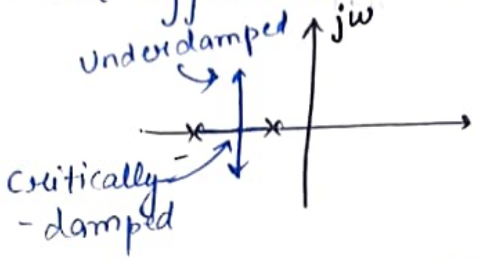
$$= \pm 90^\circ$$

$$C_g = \frac{(-3+2) - 0}{2} = -0.5$$



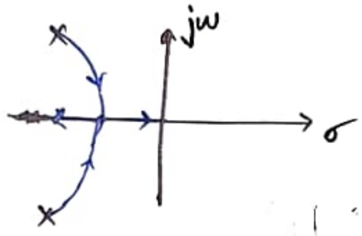
④ Determine breakaway (BA) / break-in (BI) points.

- BA applicable to two consecutive poles with no zero in b/w.



: Breaks away from real axis.

- BI points applicable to two consecutive zeros with no poles in b/w.



: Breaks into the real axis.

Eg. $P_1(s) = (s+2)(s+3) \Rightarrow P_1'(s) = (s+2) + (s+3)$

$P_2(s) = (s+2)^2(s+3) \Rightarrow P_2'(s) = 2(s+2)(s+3) + (s+2)^2$

→ Solⁿ: $-2, -2, -3$

→ Solⁿ: $-2, -3$

$P_1'(s)|_{s=-2,-3} \neq 0$

$P_2'(s)|_{s=-2} = 0$

- At BA / BI points, double solutions exist and hence the derivative becomes zero.

Eg. $G(s)H(s) = \dfrac{K}{(s+3)(s-2)} = \dfrac{KA(s)}{B(s)}$

CL char. eqⁿ: $1 + G(s)H(s) = 0$

$\Rightarrow 1 + \dfrac{KA(s)}{B(s)} = 0 \triangleq P_{CL}(s)$

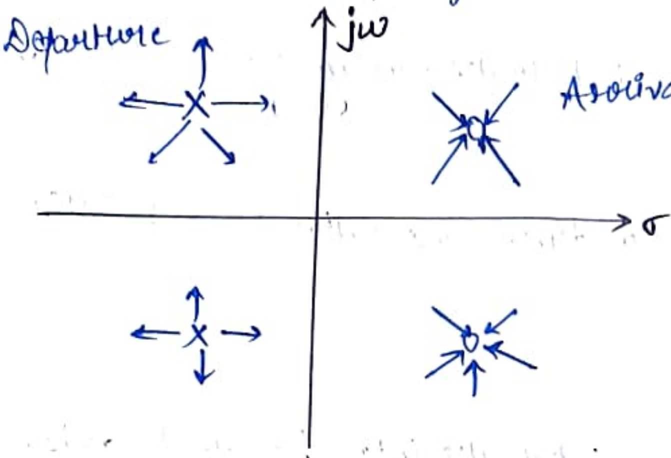
$\dfrac{d\,P_{CL}(s)}{ds} = K\dfrac{B(s)A'(s) - A(s)B'(s)}{(B(s))^2} = 0 \quad (K>0)$

$P_{CL}'(s) = B(s)A'(s) - A(s)B'(s) = 0$

↳ solⁿ will give s_{BA}/s_{BI} point.

15-09-2025

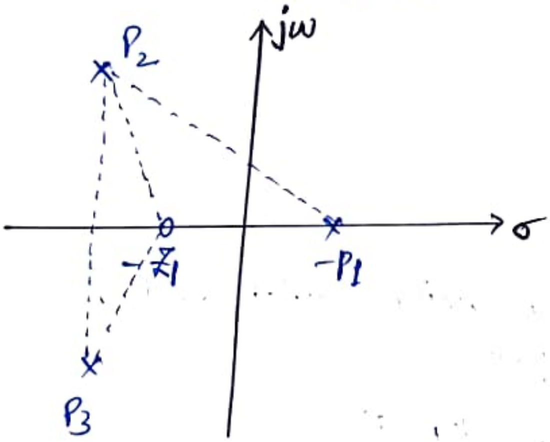
⑤ Find the angle of departure / angle of arrival.
complex poles complex zeros



Eg. $$G(s)H(s) = \frac{K(s+Z_1)}{(s+P_1)(s+P_2)(s+P_3)}$$

where P_2 and P_3 are complex poles.

$Z_1 > 0$, $P_1 < 0$.



$\angle$Departure $= 180° -$ (sum of angles made by vectors from other OL poles) + (sum of angles made by vectors from other OL zeros to this pole)

$\angle$Arrival $= 180° -$ (sum of angles made by vectors from other OL zeros) + (sum of angles made by vectors from other OL poles to this zero).

$\angle$ Departure from P_2:

$\angle G(s)H(s) = \pm 180°(2K+1)$, $K = 0, 1, \ldots$

$$\underbrace{\pm 180^\circ(2K+1)}_{\text{Take } 180^\circ} = \underbrace{\angle K}_{0^\circ} + \angle s+z_1 - (\angle s+p_1 + \underbrace{\angle s+p_2}_{\theta_{dep}^{-p_2}} + \angle s+p_3)$$

$$\Rightarrow \theta_{dep}^{-p_2} = 180^\circ + \angle -p_2+z_1 - (\angle -p_2+p_1 + \angle -p_2+p_3)$$

[Replace s with $-p_2$]

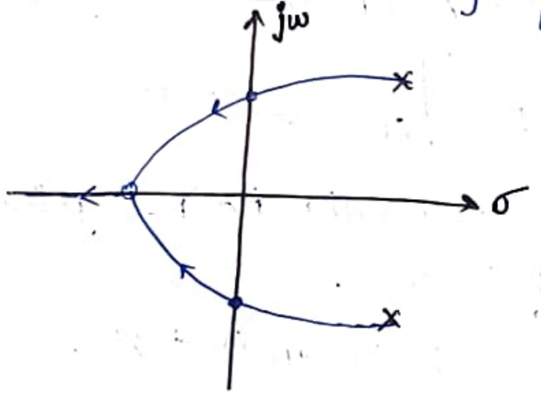
Since RL is always symmetric about axes:

Thus, $\theta_{dep}^{-p_2} = -\theta_{dep}^{-p_3}$.

Eg. $G(s)H(s) = \dfrac{K}{(s+3)(s-2)}$ → No θ_{dep}, no θ_{arr}.

⑥ Determine the cross-over points.

↳ Tells about the stability.



Put $s=j\omega$ in char. eqn. and find ω and K values.

Char. eqn: $1 + G(s)H(s) = 0$

$$\Rightarrow 1 + \frac{K}{(s+3)(s-2)} = 0$$

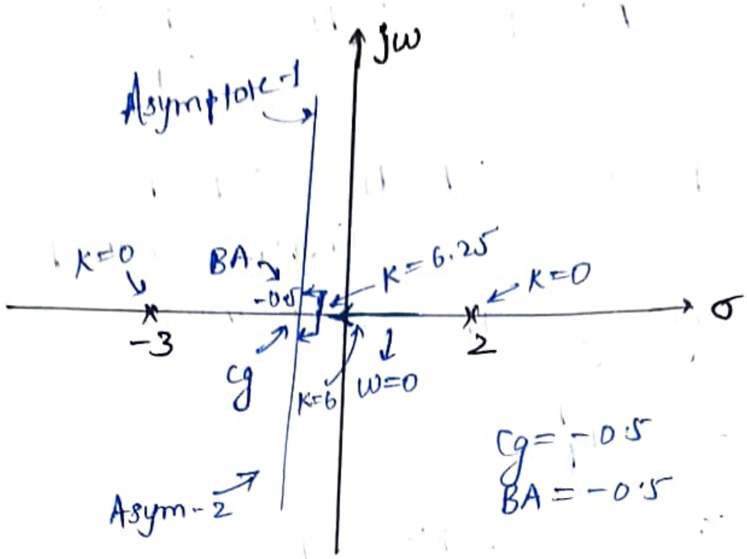
$$\Rightarrow s^2+s-6+K=0$$

$$s=j\omega \Rightarrow -\omega^2 + j\omega + (K-6) = 0$$

Comparing Real and Im parts

$$\Rightarrow K-6-\omega^2=0, \quad \omega=0$$

$$\Rightarrow K=6.$$



$$1 + G(s)H(s) = \frac{KA(s)}{B(s)} + 1 = 0 \Rightarrow K = -\frac{B(s)}{A(s)}$$

BA point : $A'(s)\,B(s) - A(s)\,B'(s) = 0 \quad \left[\frac{dK}{ds} = 0\right]$

$$K = -\frac{(s+3)(s-2)}{1}\bigg|_{s_b = -0.5} = 6.25 > 0$$

If $K \in \mathbb{R}^+$, then s_b is a valid BA/BI point, provided s_b itself is real.

For stability, $K > 6$: CL system is stable.

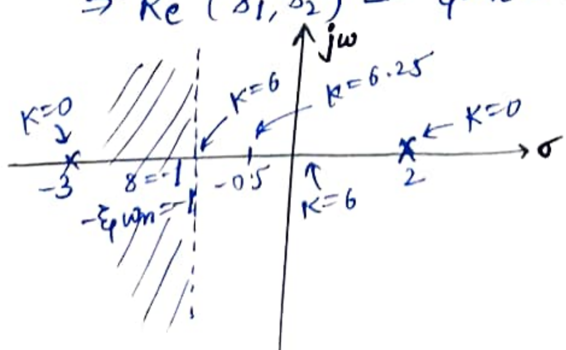
Q Find the range of various values (parameters) for which the previous system has max^m settling time (in 2% band) of 4 s, considering the system to be underdamped.

Sol^n: $t_s \le 4s \Rightarrow \frac{4}{\xi\omega_n} \le 4 \Rightarrow \xi\omega_n \ge 1 \Rightarrow -\xi\omega_n \le -1.$

Underdamped $\Rightarrow \xi < 1$

$\Rightarrow s_{1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$

$\Rightarrow Re(s_1, s_2) = -\xi\omega_n \le -1.$



Char. eqn: $1 + \dfrac{1}{(s+3)(s-2)} = 0$

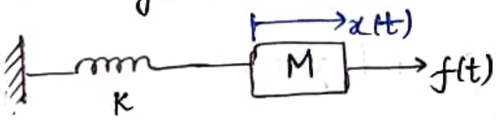
$\Rightarrow s^2 + s - 6 + K = 0$

Put $s = 1 \Rightarrow 1 - 1 - 6 + K = 0$

$\Rightarrow K = 6$

$\boxed{17\text{-}09\text{-}2025}$

Eg. Consider the arrangement shown. Design a suitable controller in $-$ve feedback to stabalize the CL system. Consider $M = 1$ kg, $K = 1$ N/m. Sketch RL.



Governing eqn of the mass-spring system:

$$M \frac{d^2x(t)}{dt^2} + Kx(t) = f(t)$$

Take LT on both side with ZIC,

$$M s^2 X(s) + KX(s) = F(s)$$

$$\Rightarrow \frac{X(s)}{F(s)} = \cancel{\frac{1}{s^2+1}} \frac{1}{Ms^2+K} = \frac{1}{s^2+1} \quad \text{(OLTF)}$$

We have to design a PD controller.

PID:

P : Present
I : Past
D : Future

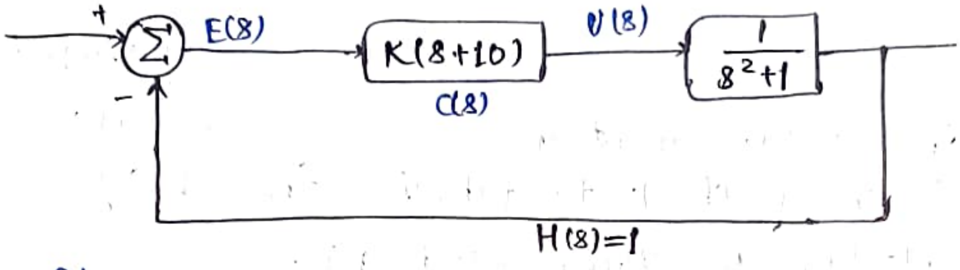
$$\propto \left(\frac{dx}{dt}\right) \xrightarrow{LT} sX(s)$$

PD: $\dfrac{U(s)}{E(s)} = \underbrace{K_P}_{P} + \underbrace{sK_D}_{D}$

$C(s) = (K_P + K_D s)$

Let $K_p = 20$, $K_D = 2$.

$$C(s) = K_p\left(1 + s\frac{K_D}{K_p}\right)$$
$$= K(s+10)$$

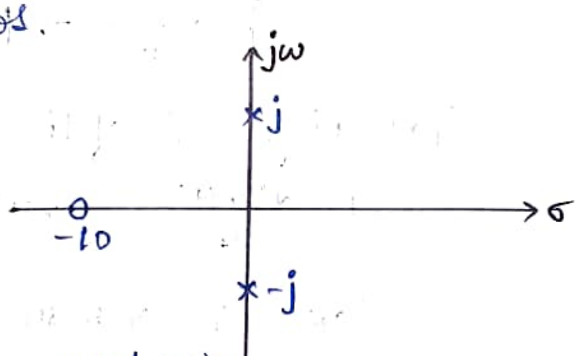


RL :

Step-1 : Locate OL poles and zeros.

$$G(s) = \frac{K(s+10)}{s^2+1}$$

$$G(s)\,H(s) = \frac{K(s+10)}{s^2+1}$$



Step-2 : Locate the RL on the real axis.

- Count the no. of poles and zeros right of the chosen point.

$(-\infty, 10]$: Part of RL ✓

$[-10, \infty)$: Not part of RL ✗

Step-3 : Determine the asymptotes of RL.

Here, $n_p = 2$, $m_z = 1$

No. of asymptotes $= n_p - m_z = 2 - 1 = 1$

$$\angle\text{asymptotes} = \pm\frac{180^\circ(2K+1)}{n_p - m_z}$$

$$= \pm 180^\circ \quad (K=0)$$

only 2 poles

$$\text{centroid},\ C_g = \frac{\sum Re(\text{poles}) - \sum Re(\text{zeros})}{n_p - m_z}$$

$$= \frac{0-(-10)}{1} = 10.$$

Step-4: Determine BA/BI points.

$$-A(s)B'(s) + A'(s)B(s)\Big|_{s=s_b} = 0$$

where $A(s) = (s+10)$
$B(s) = (s^2+1)$

$\Rightarrow -(s+10).2s + 1.(s^2+1) = 0$

$\Rightarrow -s^2 + 1 - 20s = 0$

$\Rightarrow s^2 + 20s - 1 = 0$

$\Rightarrow s = -20.05$ ✓, 0.05 ✗ → not part of RL

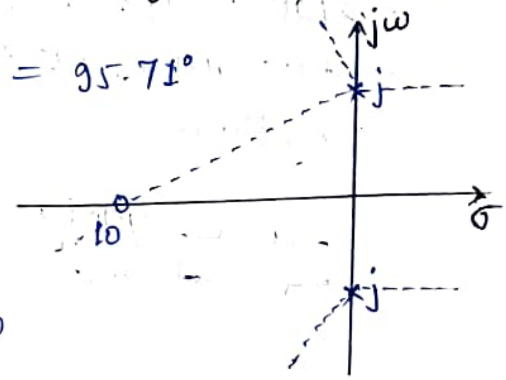
char. eqn: $1 + \dfrac{K(A(s_b))}{B(s_b)} = 0$

$\Rightarrow K = -\dfrac{B(s_b)}{A(s_b)} = 40.1$

Step-5: Angle of departure

$\angle\theta_{dep}^{+j} = 180^\circ - \sum\angle$ made by other OL poles to this pole
$+ \sum\angle$ made by other OL zeros to this pole.

$= 180^\circ - 90^\circ + \tan^{-1}(1/10) = 95.71^\circ$



$\angle\theta_{dep}^{-j} = -95.71^\circ$

Step-6: Find the cross-over point.

CL char. eqn: $1 + \dfrac{K(s+10)}{s^2+1} = 0$

$\Rightarrow s^2 + Ks + 1 + 10K = 0$

Put $s = j\omega$

$\Rightarrow -\omega^2 + jK\omega + 1 + 10K = 0$

$\Rightarrow (1 + 10K - \omega^2) + jK\omega = 0$

Compare real & img. part:

$K\omega = 0$ | $1 + 10K - \omega^2 = 0$

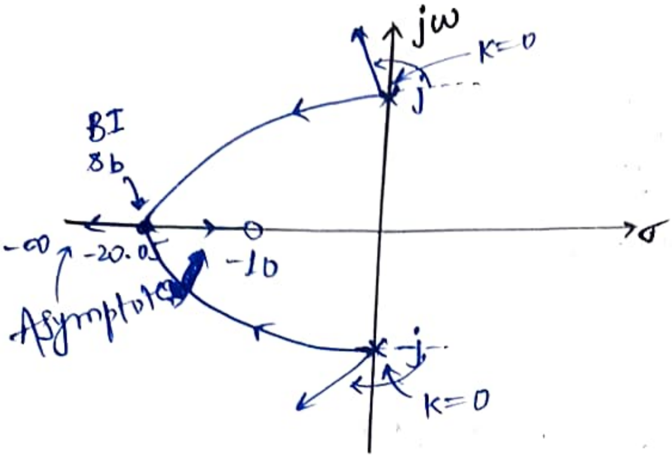
$\Rightarrow K = 0$
or $\omega = 0$

If $K = 0$, $\omega^2 = 1 \Rightarrow \omega = \pm 1$ [$K=0$ → not valid]

If $\omega = 0$, $K = -1/10$, but $K < 0$ not possible.

So, there are no crossover point except the pure imaginary poles.

19-09-2025

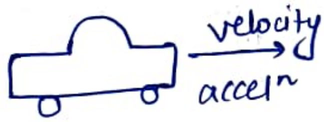


$$A'(s)\,B(s) - A(s)\,B'(s)\Big|_{s=s_{BA}/s_{BI}} = 0$$

STATE SPACE

Space: Collection of points in certain dimension.
↳ Collection (set) of similar entities (points).

- State is related to dynamic systems.
- State represents the evolution characteristics of a dynamical systems.



→ Dynamical system is represented by mathematical equation.
↳ ~~Sat~~ States

State Variables: Variables which can completely describe the evolution of a dynamical system.

↳ Minm no. of variables that completely characterize a dynamical system.

State Space: Collection/set of minm no. of variables that completely characterize a dynamical system.

$$y = \dot{x} = \frac{dx}{dt} = f(x)$$

$f: \mathbb{R}^n \to \mathbb{R}^m$ ⇝ May or may not be linear, TI, SISO.
↳ fn/mapping

$(n \neq m,\ n, m > 0)$

$y = Ax$ (Linear), $x, y \in \mathbb{R}$.

- We'll deal with only linear control system.
- Use Taylor Series expansion to linearize a system.

State-Space Representation:

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

$x \in \mathbb{R}^n$: state vector

$y \in \mathbb{R}^m$: output vector, $u \in \mathbb{R}^p$: input vector

$A \in \mathbb{R}^{n\times m}$: state matrix (or system matrix)

$B \in \mathbb{R}^{n\times p}$: Input matrix

$C \in \mathbb{R}^{m\times n}$: Output matrix,

A, B : system matrix ~~State~~

- Sensor : Observability
- Actuator : Controllability : Move from one state to another in finite time.

$u = Kx$, $K \in \mathbb{R}^{p\times n}$

controller, controller gain

$$\dot{x} = \underbrace{(A + BK)}_{\text{linear}} x$$

Eg. $H(s) = \frac{\cancel{(s-2)}}{\cancel{(s-2)}(s+3)(s+4)}$ → Pole-zero cancellation

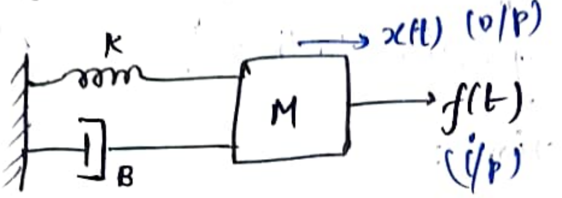
$\lambda(A) = \{2, -3, -4\}$ → Doesn't lie

22-09-2025

State-Space Representation

- Non-linear system
- Time varying
- MIMO
- Non-zero initial condition
- No pole-zero cancellation
- Internal state-based approach.

Mechanical system: Mass-Spring Damped (MSD) system



MSD eqn of motion: $M\ddot{x}(t) + B\dot{x}(t) + Kx(t) = f(t)$ ... (I)

[$M \neq 0$]

K: Spring constant (Nm^{-1})

B: Damping coefficient ($Nm^{-1}s$)

M: Mass (Kg)

f(t): Force (N) : Input

x(t): Displacement (m) : Output

Take LT on both side with ZIC,

$$Ms^2X(s) + BsX(s) + KX(s) = F(s)$$

$$\Rightarrow T.F. = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K}$$

Let $x_1(t) = x(t)$, [choose the variables]

$x_2(t) = \dot{x}(t) = \dot{x}_1(t)$

(I) $\Rightarrow M\dot{x}_2(t) + Bx_2(t) + Kx_1(t) = f(t)$

$$\Rightarrow \dot{x}_2(t) = -\frac{B}{M}x_2(t) - \frac{K}{M}x_1(t) + \frac{1}{M}f(t)$$

$$\dot{x}_1(t) = x_2(t)$$

$$\boxed{\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned}}$$

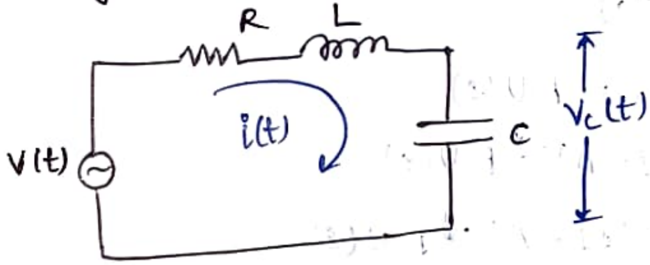
$$x = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -K/M & -B/M \end{bmatrix}}_{\text{state matrix}} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1/M \end{bmatrix}}_{\text{I/P vector}} f(t)$$

$$y = x_1(t) = [1 \quad 0]\, x$$

$x(k+1) = A x(t) + B u(k)$
$y(k) = C x(k)$

<u>Electrical system</u> : Series RLC circuit



$L\dfrac{di}{dt} \to x_1$

$C\dfrac{dv_c}{dt} \to x_2$

SS to TF :

$$\dot{x} = Ax + Bu \quad \cdots ①$$
$$y = Cx \quad \cdots ②$$

Taking LT on both with ZIC for ① & ②,

$$sX(s) = AX(s) + BU(s)$$
$$Y(s) = C\,X(s)$$
$$\Rightarrow (sI - A)\,X(s) = BU(s)$$
$$\Rightarrow X(s) = (sI - A)^{-1} BU(s)$$
$$\therefore Y(s) = C(sI - A)^{-1} BU(s)$$
$$TF = \frac{Y(s)}{U(s)} = C(sI - A)^{-1} B$$
$$P(s) = (sI - A)^{-1} \text{ : State transition matrix.}$$

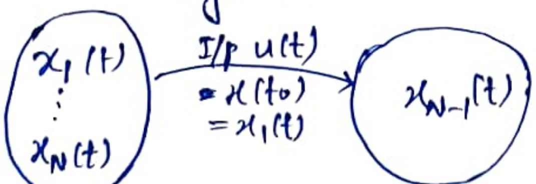
$Ax = \lambda x$

$$\Rightarrow (A - \lambda I)x = 0 \Rightarrow (\lambda I - A)x = 0$$

Controllability and Observability:

Controllability: If there exists an input to a system that can drive every state variable to a desired state value in finite time, then the system is called controllable.

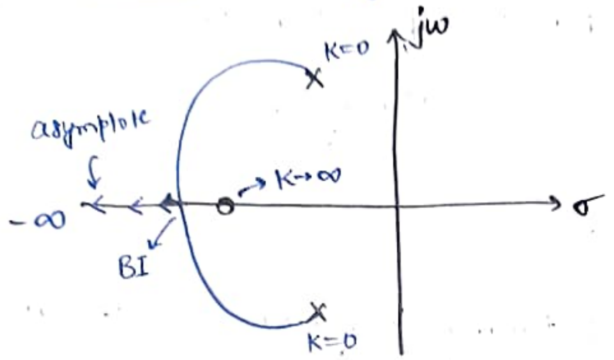
Observability: If any state vector $x(t_0)$ can be found (estimated) from measured o/p $y(t)$ over a finite interval time from t_0, then the system is called observable.



: controllable.

25-09-2025

Controllability and Observability:



~~Obs~~ **Controllability:** Certain input value which can steer the state value.

OR

state values through actuation (A,B).

Observability: O/P, state (C,D)

(A,B,C) : System tuple/matrices

- Ogata Book
- Nise Book

Controllability matrix:

$$\bar{C} = [B \;\; AB \;\; A^2B \;\; \dots \;\; A^{n-1}B]$$

n: Rank of A

Observability matrix:

$$\bar{O} = [C^T \;\; A^TC^T \;\; \dots \;\; (A^{n-1})^T C^T]$$

$$= \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}^T$$

$C = [2 \;\; 1]$, $A = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}$, $C^T = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$CA = [9 \;\; 7]$

$\Rightarrow A^TC^T = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \end{bmatrix}$

$\bar{O} = \begin{bmatrix} 2 & 1 \\ 9 & 7 \end{bmatrix}^T$

Both are the same

$\bar{O} = \begin{bmatrix} 2 & 9 \\ 1 & 7 \end{bmatrix}$

Eg. $\dot{x} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} u$, $x \in \mathbb{R}$.

Find controllability.

$\bar{C} = [B \;\; AB] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$\det(\bar{C}) = 0$ [There is atleast one eigen-value $= 0$] $\Downarrow$

rank $(\bar{C}) = 1 \rightarrow$ Rank-deficient matrix. controllability is not there.

$|\bar{C}| = 0 \Rightarrow$ linear dependency on rows-columns $\Rightarrow$ Atleast one $\lambda = 0$.
$\Leftrightarrow$ Determinant is zero $\Leftrightarrow$ Rank-deficient matrix.

→ If one of the row/column is zero, it always becomes a rank-deficient matrix.
$\Rightarrow$ Not controllable.

MATLAB

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$ → Find transfer function in MATLAB. [ss2tf]

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

→ 1 and -1 are eigen values.

Eg. $$\dot{x} = \underbrace{\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u.$$ Find observability and controllability.

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

$$\bar{O} = \begin{bmatrix} C^T & A^T C^T \end{bmatrix} = \begin{bmatrix} 0 & \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ 1 & \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$|\bar{O}| = 1 \Rightarrow$ observable.
| det$(\bar{O}) = 1$
Rank$(\bar{O}) = 2$

$$\bar{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & \begin{bmatrix} -1 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 1 & \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

$|\bar{C}| \neq 0 \Rightarrow$ controllable.
| det$(\bar{C}) = 1$.
Rank$(\bar{C}) = 2$

→ Both controllable and observable system $\Leftrightarrow$ Minimum Realization of the system

↳ No pole-zero cancellation.

→ Practical control systems are designed to be minimal realisation system.

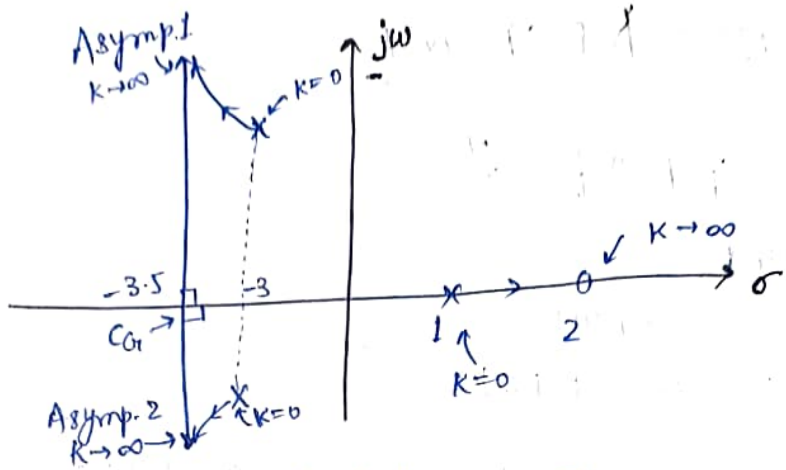
Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow$ spectrum $= \{1, -1\}$ → eigen values
value should be -ve; o/w unstable sys.

$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

→ system is not controllable, but stablizable.
→ stablisable is a subset of controllable.
↳ we can put an actuator at that position to make the sys. stable, still system might be uncontrollable.

26-09-2025

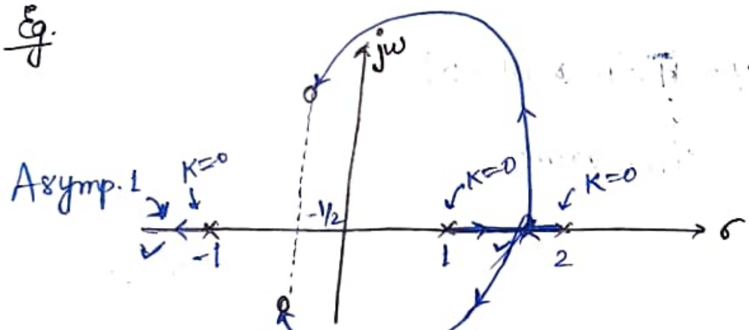
Eg.



$$\text{centroid} = \frac{-3-3+1-2}{2} = -3.5$$

$$\angle asy. = \frac{\pm 180^\circ(2K+1)}{2} = \pm 90^\circ$$

Eg.



Draw the root locus.

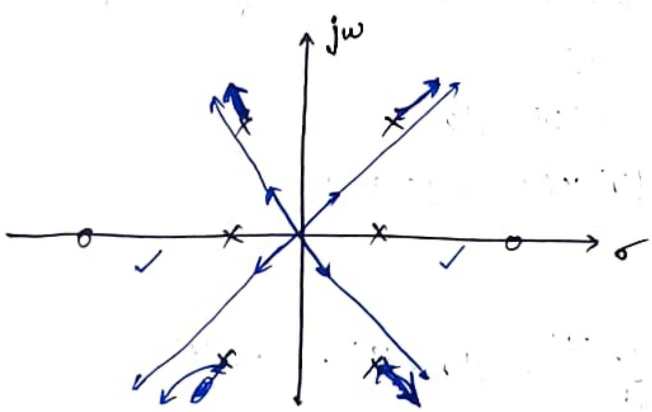
$$\angle asy. = \pm\frac{180^\circ(2K+1)}{n-m}$$

$$= \pm 180^\circ$$

$$C_g = \frac{-1+1+2+1/2+1/2}{1}$$

$$= 3$$

Eg.



$n-m=4$ (asymptotes)

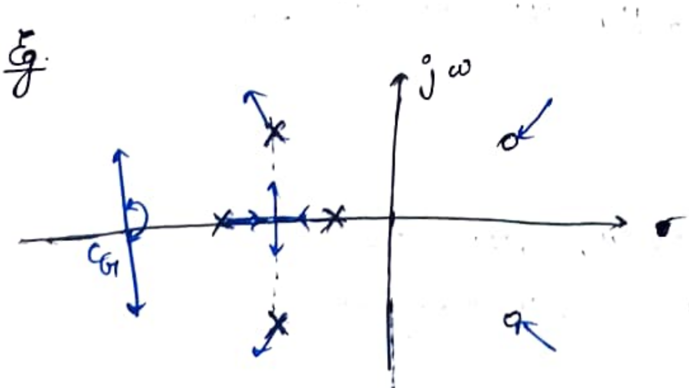
$\pm\frac{180^\circ(2K+1)}{(n-m)}$, $K=0,1,2,\ldots$

$= \pm 45^\circ(2\times 0+1), = \pm 45^\circ$

$\pm 45^\circ(2\times 1+1) = \pm 135^\circ$

$C_G = 0$

Eg.



$n-m=2$ asymptotes

$\pm\frac{180^\circ}{2}(2K+1) = \pm 90^\circ$

FREQUENCY RESPONSE.

$$U(s) \longrightarrow \boxed{P(s)} \longrightarrow Y(s)$$

LTI stable $P(s)$

$$U(s) = U_0 \sin\omega t$$
$$Y(s) = A \sin(\omega t + \phi)$$

$e^{j\omega t} = \cos\omega t + j\sin\omega t$: eigen f^n of LTI system

$Ax = \lambda x$, x : eigen vector
λ : eigen value

$$\sin\omega t \longrightarrow \boxed{H} \longrightarrow y = \underset{\substack{\uparrow \\ \text{eigen value}}}{H} \sin(\omega t + \phi)$$

$\boxed{03\text{-}10\text{-}2025}$

Eg. Given TF, $\dfrac{X(s)}{F(s)} = \dfrac{15}{(s+10)(s+11)}$. find the corresponding differential eq^{n}s. Also what is the order of the system.

Soln: Given, $\dfrac{X(s)}{F(s)} = \dfrac{15}{s^2 + 21s + 110}$

$$\Rightarrow (s^2 + 21s + 110)\,X(s) = 15\,F(s)$$

Taking ILT,

$$\frac{d^2x(t)}{dt^2} + 21\frac{dx(t)}{dt} + 110\,x(t) = 15 f(t).$$

$$\Rightarrow \ddot{x}(t) + 21\dot{x}(t) + 110\,x(t) = 15 f(t) \quad \ldots (1)$$

Let $x_1(t) \triangleq x(t)$
$x_2(t) \triangleq \dot{x}(t)$

$$(1) \Rightarrow \dot{x}_2(t) + 21 x_2(t) + 110\,x_1(t) = 15 f(t)$$
$$\Rightarrow \dot{x}_2(t) = -110\,x_1(t) - 21\,x_2(t) + 15 f(t)$$

$\dot{x}_1(t) = 0 \cdot x_1(t) + 1 \cdot x_2(t)$

$$\dot{x} \triangleq \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -110 & -21 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 15 \end{bmatrix} f(t).$$

$$y(t) = [1 \ 0]\, x(t) = [1 \ 0] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\bar{C} = [B \quad AB \quad \dots \quad A^{n-1}B]$$

$$= \left[\begin{matrix} 0 \\ 15 \end{matrix} \quad \begin{bmatrix} 0 & 1 \\ -110 & -21 \end{bmatrix} \begin{bmatrix} 0 \\ 15 \end{bmatrix} \right]$$

$$= \begin{bmatrix} 0 & 15 \\ 15 & -35 \end{bmatrix}$$

$|\bar{C}| \neq 0 \Rightarrow$ controllable.

$$|\bar{O}| = [C^T \quad A^T C^T \quad \dots] = \left[\begin{matrix} 1 \\ 0 \end{matrix} \quad \begin{bmatrix} 0 & -110 \\ 1 & -21 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow |\bar{O}| \neq 0 \Rightarrow$ observable. identity (not singular)

Eg. Consider a unity (−ve) f/b system where plant TF is $\frac{4s+2}{s^2-s}$. Find if $s=-2$ is a closed loop solution/root of the system or not.

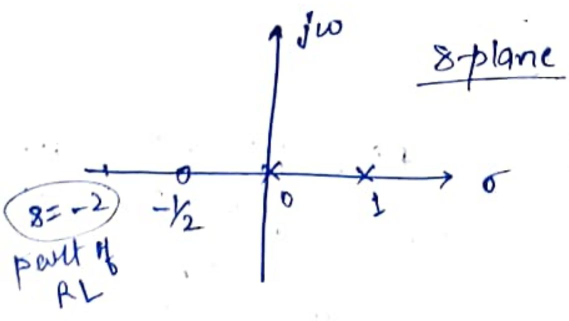
Soln: $G(s).H(s) = \frac{4s+2}{s(s-1)}$

$\frac{GH}{1+GH}$
↳ pole: $1+GH=0$

$1 + G(s)H(s) = 0 \Rightarrow G(s)H(s) = -1$

$\Rightarrow |G(s)H(s)| = 1$

$\angle G(s)H(s) = \pm 180^\circ (2K+1), \quad K=0,1,\dots$



$$G(s)H(s)\Big|_{-2} = \frac{4(-2)+2}{(-2)(-2-1)} = -1$$

$1 + G(s)H(s) = 0$

Eg. Consider OLTF with unity (-ve) F/B be $G(s) = \dfrac{K}{s^3+4s^2+8s}$, $K \geq 0$.

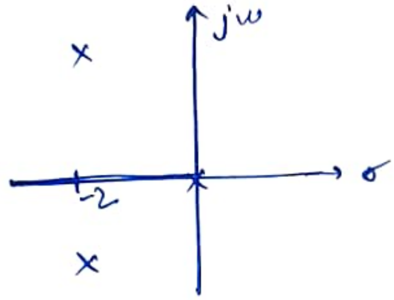
(i) How many branches does the CL system have in RL?

(ii) Find the region of the real axis that lies on RL.

Ans: 3 (3 poles)

Soln:

$$s^3 + 4s^2 + 8s = 0$$
$$\Rightarrow s(s^2+4s+8) = 0$$
$$\Rightarrow s = 0,\ -2 \pm j2$$



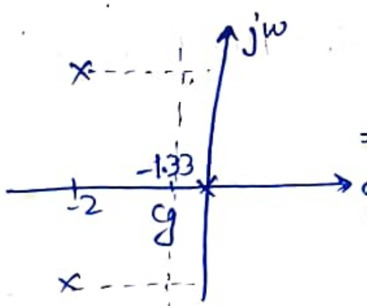
Ans: $(0, -\infty)$.

(iii) Find if there're breakaway and BI points in RL.

Soln:

$$\angle \text{asymptote} = \pm \frac{180^\circ (2k+1)}{3} = \pm 60^\circ, \pm 180^\circ$$

$$\sigma = \frac{0 + (-2) + (-2) - 0}{3} = -1.33$$



$B(s)A'(s) - A(s)B'(s) = 0$

$\Rightarrow 3s^2 + 8s + 8 = 0$

$\Rightarrow s = -\frac{4}{3} \pm 0.9428j$

$\left.\dfrac{dK}{ds}\right|_{s_b \in \mathbb{R}} = 0 \rightarrow$ BA / BI pt.

Maxm / Minm pts. for s if $s \in \mathbb{C}$

(iv) Find the range of K-values for which the CL system is stable.

[Using R-H / Crossover pt.]

Soln:

$$1 + \frac{K}{s^3+4s^2+8s} = 0 \Rightarrow s^3 + 4s^2 + 8s + K = 0$$

$s = j\omega \Rightarrow -j\omega^3 - 4\omega^2 + 8j\omega + K = 0$

$\Rightarrow -\omega^3 + 8\omega = 0 \quad | \quad K - 4\omega^2 = 0$

$\Rightarrow \omega = 0, \pm 2\sqrt{2}$

If $\omega = 0 \Rightarrow K = 0$

If $\omega = \pm 2\sqrt{2} \Rightarrow K = 32$.

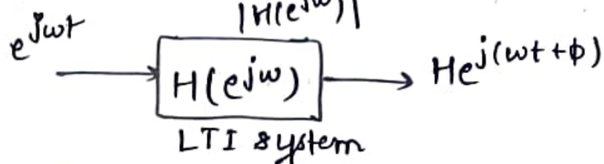
16-10-2025

- Pole placement (controller Design) → controllability
- observable Design → Observability

Frequency Domain Analysis

LTI system : eigen-function = $e^{j\omega t}$

ϕ : phase shift
A : magnitude / gain



① Magnitude
② Phase

DTFT → $H(e^{j\omega t})$
CTFT → $H(e^{j\omega})$
DFT → $H(k)$

How can we visualise $H(j\omega)$?

- Bode plot
- Nyquist plot

① Bode plot
ⓐ Gain vs. frequency

ⓑ Phase vs. frequency

semilog : y - log
x - linear
Equal emphasis on low & high freq.
↳ but zero freq. cannot be represented.

② Nyquist plot
$Re\{H(j\omega)\}$ vs. $Im\{H(j\omega)\}$ in the complex plane.

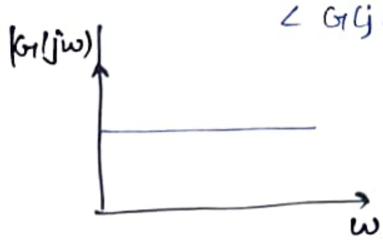
Constituent element / block :

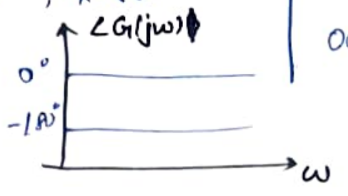
K, $1/s$, s, $\frac{1}{1+sT}$, $Ts+1$, $\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$, $\frac{s^2+2\xi\omega_n s+\omega_n^2}{\omega_n^2}$.

Eg. $G(s) = K$, $K \in \mathbb{R}$

$G(j\omega) = k$, $|G(j\omega)|\ (dB) = 20\log_{10}|K|\ dB$

$\angle G(j\omega) = \begin{cases} 0^\circ, & K>0 \\ -180^\circ, & K<0 \end{cases}$





Decade → 10 fold change in freq.
Octave → 2 fold change in freq.

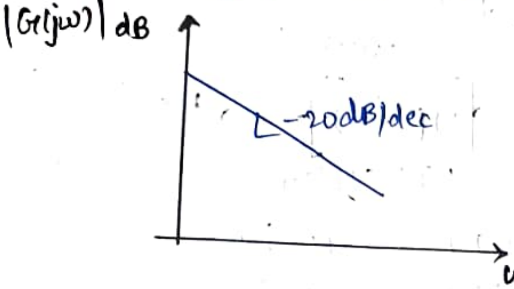
23-10-2025

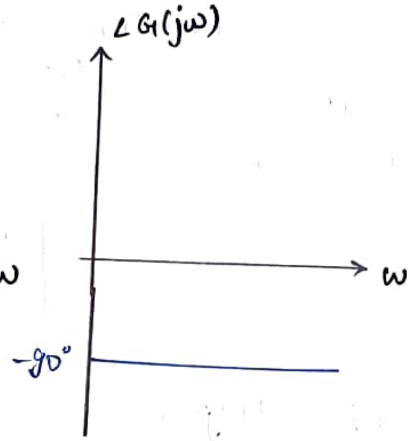
Eg. $G(s) = \frac{1}{s}$

$G(j\omega) = \frac{1}{j\omega}$

$|G(j\omega)| = \frac{1}{\omega}$

$\angle G(j\omega) = -\tan^{-1}\infty = -90^\circ$

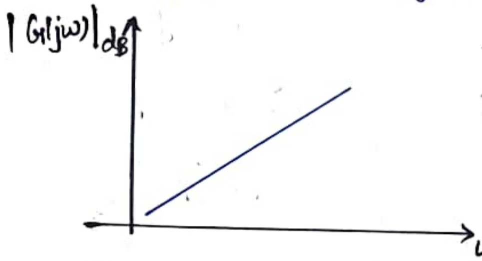


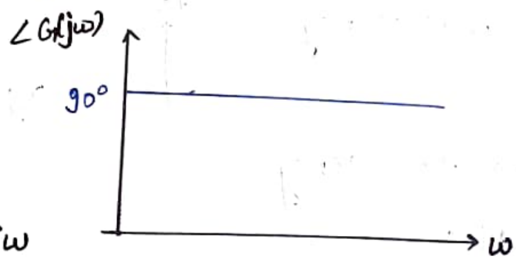


Eg. $G(s) = s$

$G(j\omega) = j\omega$

$|G(j\omega)| = \omega, \quad \angle G(j\omega) = 90^\circ$





- At ω, $|G(j\omega)| = 20\log_{10}\left(\frac{1}{\omega}\right)$

10ω, $|G(j\omega)| = 20\log_{10}\left(\frac{1}{10\omega}\right)$

$= -20 - 20\log(\omega)$

2ω, $|G(j\omega)| = -20\log(2\omega)$

$= -20\log 2 - 20\log_{10}\omega$

$= -6 - 20\log_{10}\omega$

Eg. $G(s) = \frac{1}{1+sT}, \quad T > 0$

$G(j\omega) = \frac{1}{1+j\omega T}$

$|G(j\omega)| = \frac{1}{\sqrt{1+\omega^2T^2}}$

$\angle G(j\omega) = -\tan^{-1}(\omega T)$

At $\omega \ll \frac{1}{T}$,

$|G(j\omega)| = 1, \quad |G(j\omega)|_{dB} = 0$

(Low-frequency asymptote)

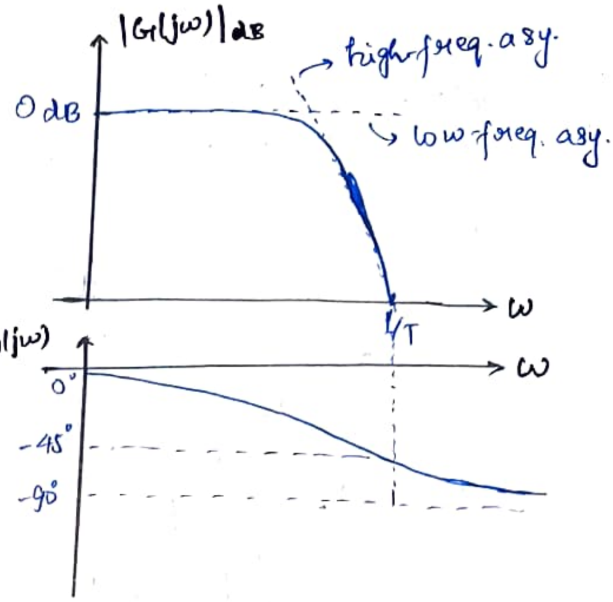
At $\omega \gg \frac{1}{T}$,

$|G(j\omega)| \approx \frac{1}{\omega T}$

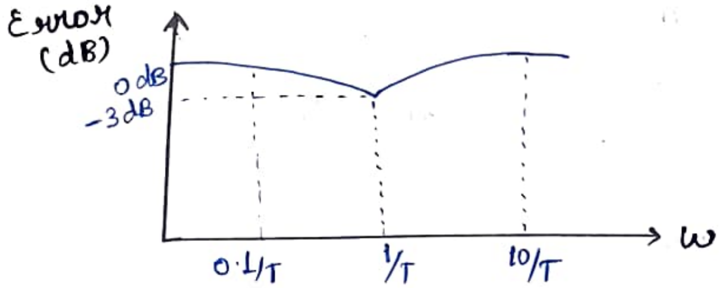
$|G(j\omega)|_{dB} = -20 \log \omega T \ dB$

(High-frequency asymptote)

↪ Asymptotic approach

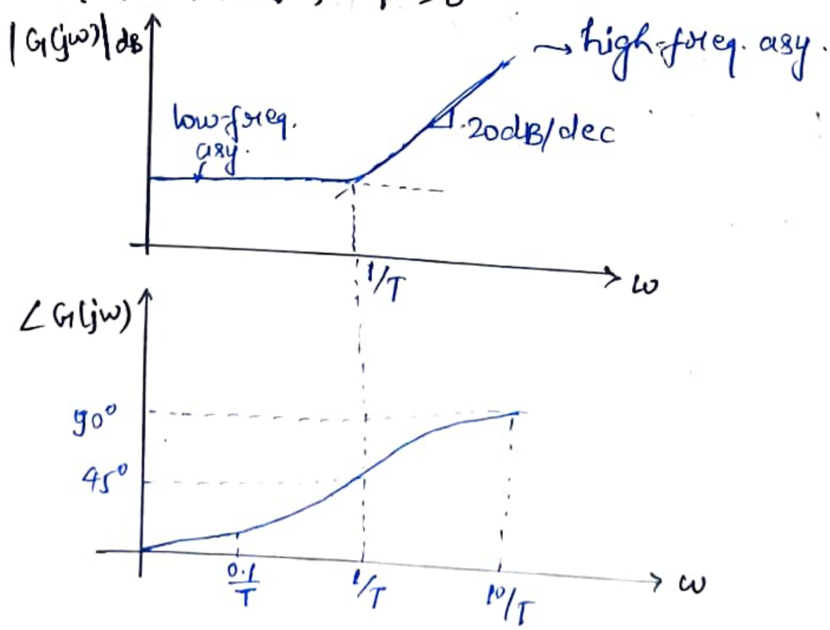


ω	$\lvert G(j\omega) \rvert_{dB}$		Asymptotic value (dB)
$1/T$	-3.01	← 3.01 dB →	0
$2/T$	-6.99	← 0.97 dB →	-6.02
$0.5/T$	-0.97	← ~1 dB →	0
$10/T$	-20.04	← 0.04 dB →	-20
$0.1/T$	-0.04	← 0.04 dB →	0



$\omega = 1/T$: corner freq., 3-dB cutoff freq.

Eg. $G(s) = 1 + sT, \quad T > 0$



Eg. $G(s) = \dfrac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$

$G(s) = \dfrac{1}{\frac{s^2}{\omega_n^2} + \frac{2\xi s}{\omega_n} + 1}$

$G(j\omega) = \dfrac{1}{1 - \frac{\omega^2}{\omega_n^2} + 2j\frac{\xi\omega}{\omega_n}}$

$|G(j\omega)| = \dfrac{1}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(\frac{2\xi\omega}{\omega_n}\right)^2}}$

$\angle G(j\omega) = -\tan^{-1}\left(\dfrac{2\xi\,\omega/\omega_n}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right)$

$|G(j\omega)|_{dB} = -20\log_{10}\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(\frac{2\xi\omega}{\omega_n}\right)^2}$ dB

If $\omega \ll \omega_n$, $|G(j\omega)|_{dB} \approx 0\,dB$ (low-freq. asy.)

$\omega \gg \omega_n$, $|G(j\omega)|_{dB} = -40\log\left(\frac{\omega}{\omega_n}\right)$ dB (high-freq. asy.)

At $\omega = \omega_n$,

$|G(j\omega)|_{dB} = -20\log_{10} 2\xi$

under-damped system, $\xi = 0.5$:

$|G(j\omega)|_{dB} \simeq 0\,dB.$

Eg. Find the freq. at which $|G(j\omega)|$ is max.

$|G(j\omega)| = \dfrac{1}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(\frac{2\xi\omega}{\omega_n}\right)^2}}$

Let $g(\omega) = \left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(\frac{2\xi\omega}{\omega_n}\right)^2$

$\therefore g'(\omega) = 0$

$\Rightarrow \omega = 0$ or $\omega_n\sqrt{1-2\xi^2}$

27-10-2025

$$g'(\omega) = 2\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)\left(-\frac{2\omega}{\omega_n}\right)\left(\frac{1}{\omega_n}\right) + 2\left(\frac{2\xi\omega}{\omega_n}\right)\left(\frac{2\xi}{\omega_n}\right)$$

$g'(\omega) = 0 \Rightarrow$

$$\left[2\xi^2 - \left(1-\frac{\omega^2}{\omega_n^2}\right)\right]\frac{\omega}{\omega_n^2} = 0$$

$$\Rightarrow \left[2\xi^2 - \left(1-\frac{\omega^2}{\omega_n^2}\right)\right]\omega = 0 \qquad [\because \omega_n \neq 0]$$

Either, $\omega = 0$

or, $2\xi^2 = 1 - \frac{\omega^2}{\omega_n^2}$

$\omega_R = \omega_n\sqrt{1-2\cdot\xi^2}$ (Resonance)

ω_R exist if $0 < \xi < 1/\sqrt{2}$.

$\omega_d = \omega_n\sqrt{1-\xi^2}$ (damped)

$\therefore \omega_d > \omega_R$

Resonance freq. : Freq. of max gain of T.F.

Minimum and Non-Minimum Phase System

Consider systems : $G_1 = \frac{1+sT}{1+sT_1}$, $T, T_1 > 0$ → MP, stable

$G_2 = \frac{1-sT}{1+sT_1}$, $T, T_1 > 0$ → NMP, stable

first go in wrong dirn then gets rectified

Frequency response of the systems,

$$G_1(j\omega) = \frac{1+j(T\omega)}{1+j(T_1\omega)}$$

$$G_2(j\omega) = \frac{1-j(T\omega)}{1+j(T_1\omega)}$$

Hyper-imaginary no.
j, j_1, j_2
(different axes)

$$\Rightarrow |G_1(j\omega)| = |G_2(j\omega)| = \sqrt{\frac{1+\omega^2T^2}{1+\omega^2T_1^2}}$$

$$\angle G_1(j\omega) = \tan^{-1}(\omega T) - \tan^{-1}(\omega T_1)$$

$$\angle G_2(j\omega) = -\tan^{-1}(\omega T) - \tan^{-1}(\omega T_1)$$

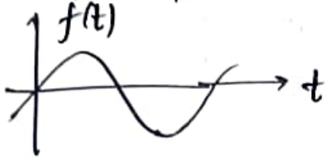
$\omega = 0 \Rightarrow \angle G_1 = \angle G_2 = 0$

$\omega \to \infty \Rightarrow \angle G_1 = 0^\circ, \angle G_2 = -180^\circ$. → NMP system

↳ same mag. response, diff. phase response.

Eg. $G_1(s) = \frac{s-2}{s+4}$, $G_2(s) = \frac{s+2}{s+4}$.

Frequency Response of Undamped Systems

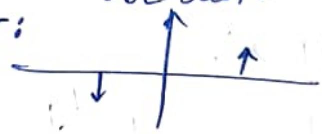


→ Redundant information

↓

Use basis

FT:



⇒ compressed sensing

Need not satisfy : $f_s \geq 2f_m$
[Nyquist-Shanon Theorem]
$f_s \leq 2f_m$

$\lambda(A) = 0$ → Redundant rows & columns

$\xi = 0$ (Undamped System) :

$$G(s) = \frac{\omega_n^2}{s^2+\omega_n^2} \Rightarrow G(j\omega) = \frac{\omega_n^2}{(j\omega)^2+\omega_n^2} = \frac{1}{1-\omega^2/\omega_n^2}$$

$$\Rightarrow |G(j\omega)| = \frac{1}{\left|1-\frac{\omega^2}{\omega_n^2}\right|}$$

$\omega \to \omega_n \Rightarrow |G(j\omega)| \to \infty$. ⇒ Don't take freq. as natural freq.

30-10-2025

Bode Plot : [Fourier Transform]

Eg: $G(S) = \dfrac{S}{(S+1)(S+10)}$

Convert into time constant form.

$$G(S) = \frac{S/10}{(1+S)(1+S/10)}$$

$$G(j\omega) = \frac{j\omega/10}{(1+j\omega)(1+j\omega/10)} = \frac{j\omega/10}{1-\frac{\omega^2}{10}+\frac{11}{10}j\omega}$$

$$|G(j\omega)| = \frac{\omega/10}{\sqrt{\left(1-\frac{\omega^2}{10}\right)^2+\left(\frac{11}{10}\omega\right)^2}}$$

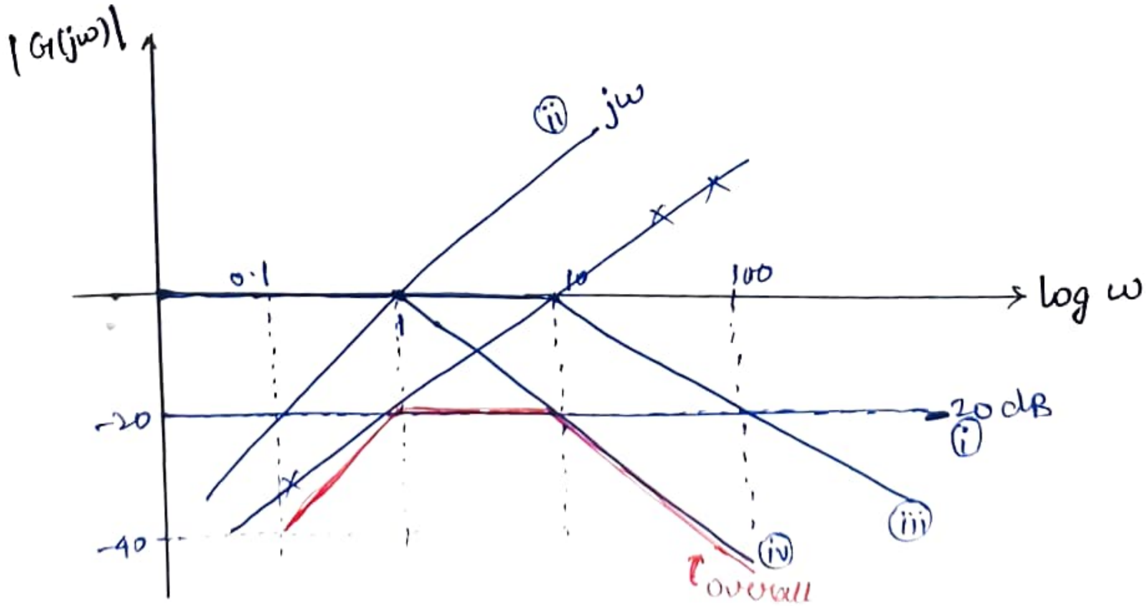
$$G(j\omega) = (0.1)(j\omega)\left(\frac{1}{1+j\omega}\right)\left(\frac{1}{1+0.1j\omega}\right)$$

(i) 0.1 : $20\log(0.1) = -20$ dB / No slope

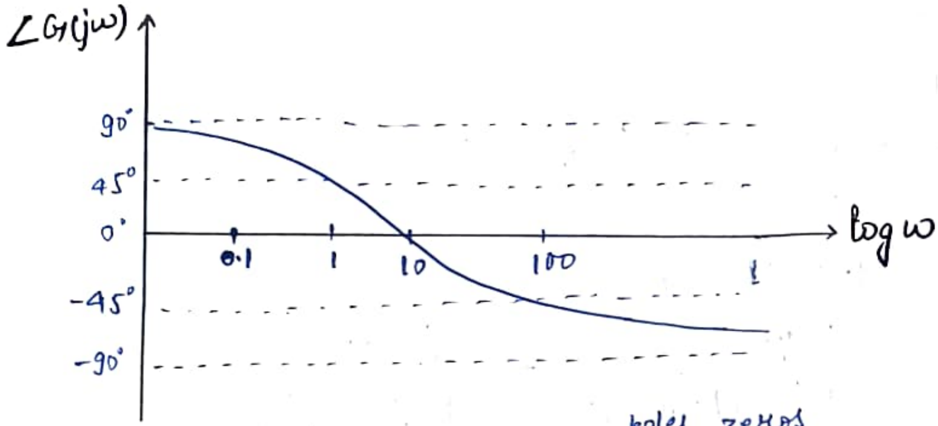
(ii) $j\omega$: $20\log(\omega)$ / $+20$ dB/dec

(iii) $\dfrac{1}{1+j\frac{\omega}{10}}$: $-20\log\sqrt{1+\left(\frac{\omega}{10}\right)^2}$ / -20 dB/dec

(iv) $\dfrac{1}{1+j\omega}$: $-20\log\sqrt{1-\omega^2}$ / -20 dB/dec.



$\angle G(j\omega) = 0 + 90^\circ - \tan^{-1}(\omega) - \tan^{-1}(0.1\omega)$



- For minimum phase, $-90\,(\underset{\text{poles}}{n} - \underset{\text{zeros}}{m})$

03-11-2025

Lead and Lag Compensators.

$$G(s) = \frac{s+z}{s+p}$$

$$\angle G(s) = \angle s+z - \angle s+p \begin{cases} +ve : \phi - \text{lead} : \text{Lead compensator} \\ -ve : \phi - \text{lag} : \text{Lag compensator} \end{cases}$$

$$G_c(s) = \frac{s+\frac{1}{\alpha T}}{s+\frac{1}{\beta T}} \Rightarrow G_c(j\omega) = \frac{j\omega + \frac{1}{\alpha T}}{j\omega + \frac{1}{\beta T}}$$

$$\Rightarrow \angle G_c(j\omega) = \tan^{-1}(\alpha \omega T) - \tan^{-1}(\beta \omega T) > 0$$

$\alpha = 1, \beta = 2$

$$\Rightarrow \angle G_c(j\omega) = \tan^{-1}(\omega T) - \tan^{-1}(2\omega T) < 0$$

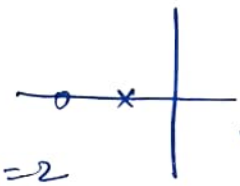
If $\alpha < \beta \Rightarrow$ Lag compensator.

$\alpha = 2, \beta = 1$

$$\Rightarrow \angle G_c(j\omega) = \tan^{-1}(2\omega T) - \tan^{-1}(\omega T) > 0°$$

If $\alpha > \beta \Rightarrow$ Lead compensator.

Eg.



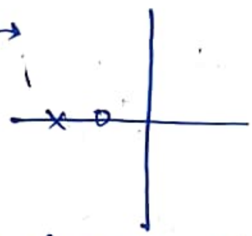
$\rightarrow$ (Lag compensator)

$\alpha = 1, \beta = 2$

$z_1 = \frac{1}{T}$, $p_1 = \frac{1}{2T}$, $T > 0$

Eg. $\alpha = 2, \beta = 1$ (Lead) $\rightarrow$

$z_1 = \frac{1}{2T}$, $p_1 = \frac{1}{T}$

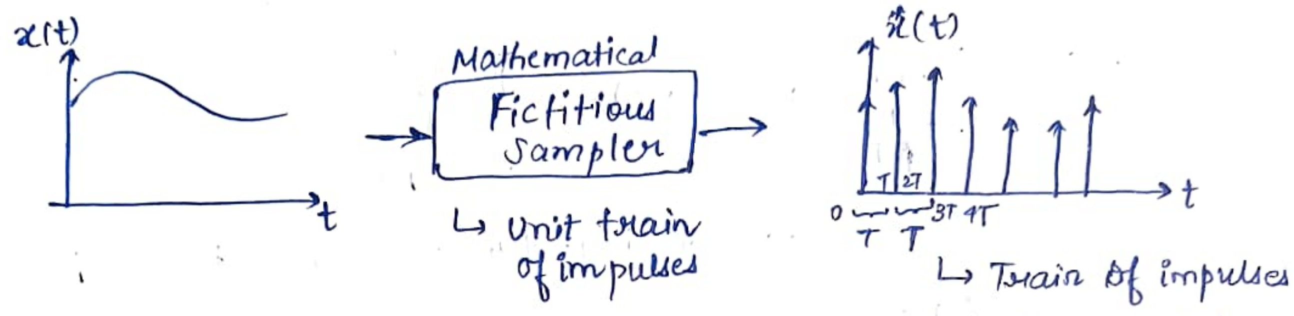


check how increasing GM changes BW.

Discrete-Time Control System

$x(t)$: Continuous-time / analog signal

Sampling : $f_s \geq 2 f_m$, f_m : max^m freq. content in signal.



T: sampling period

Let $x^*(t)$ denote the impulse-sampled o/p in a sequence of impulses.

Train of impulses, $x^*(t)$, is given by

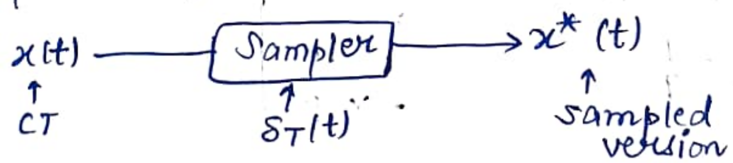
$$x^*(t) = \sum_{K=0}^{\infty} x(KT)\,\delta(t-KT) \quad \dots (1)$$

$$x^*(t) = x(0)\,\delta(t) + x(T)\,\delta(t-T) + \dots + x(KT)\,\delta(t-KT) + \dots \quad \dots (2)$$

Define a train of unit impulses,

$$\delta_T(t) = \sum_{K=0}^{\infty} \delta(t-KT) \quad \dots (3)$$

The sampler o/p is the product of the CT signal and the unit impulse sequence.



Consider the LT of (2).

$$X^*(s) = \mathcal{L}[x^*(t)]$$

$$= \mathcal{L}\left[x(0)\delta(t) + x(T)\,\delta(t-T) + \dots + x(KT)\,\delta(t-KT) + \dots\right]$$

$$= x(0) + x(T)e^{-sT} + \dots + x(KT)e^{-KsT} + \dots$$

$$= \sum_{K=0}^{\infty} x(KT)\, e^{-KsT}$$

Consider $Z = e^{sT}$ or $s = \frac{1}{T}\ln(Z)$

$$Z(x(n)) = \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

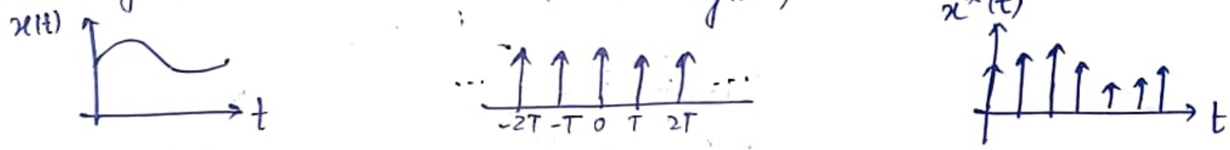
$$x^*(t) = x(KT), \quad K = 0, 1, \ldots$$

$$X^*(s)\Big|_{s=\frac{1}{T}\ln(Z)} = \sum_{K=0}^{\infty} x(KT)\, Z^{-K}$$

$$= X(Z) \xleftrightarrow{Z} Z(x^*(t)) = Z(x^*(KT)).$$

$$\underbrace{x(t)}_{\text{Baseband/ modulating Signal}} \longrightarrow \boxed{\begin{array}{c}\text{Modulator}\\ \text{(Train of unit impulses)}\end{array}} \longrightarrow \underbrace{x^*(t) = x(KT)}_{\text{Sampled o/p or modulated signal}}$$

$\uparrow$ $\delta(t)$ (carrier signal)



Q) For any LTI systems, apart from complex exponentials/ Sinusoids can there be any other basis functions?

↳ Yes, infinitely many.

↳ But for LTI systems, only exponentials are the eigenfunctions.

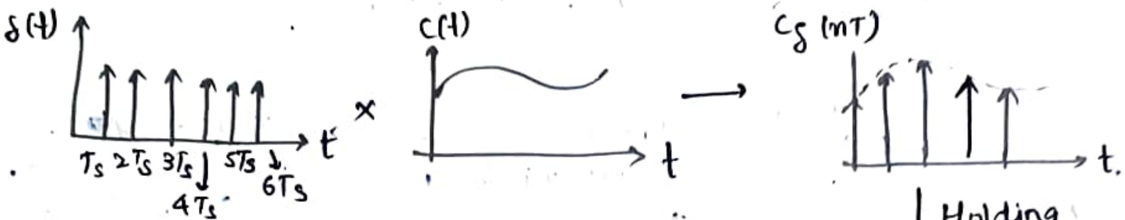
06-11-2025

Digital Control System :

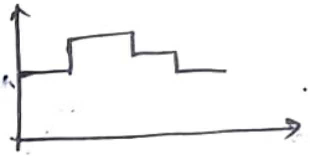
S/H : Sample and Hold

ZOH : Zero order Hold

$e(t) \longrightarrow$ sampler $\longrightarrow$ Holding $\longrightarrow$ ADC



↓ Holding
Aperture duration



← Quantization

$\hat{Q} = \frac{\text{Full-scale}}{2^n}$, n: no. of bits

ZOH (Zero Order Hold) :

$y(t) = x(n)$, $KT_s \le n \le (K+1)T_s$,
$K = 0, 1, 2, 3, \ldots$

↑ Hold ckt. eqn

→ Federated learning

→ Edge computing.

07-11-2025

Derivation of Pulse Transfer Function

↳ LTI

$R(s) \longrightarrow \boxed{G(s)} \longrightarrow C(s)$

$r(t)$ $\qquad\qquad\qquad$ $c(t)$

$G(s) = \frac{C(s)}{R(s)}$, with ZIC.

- Feedback control / ~~Feedback~~ Feedforward control.

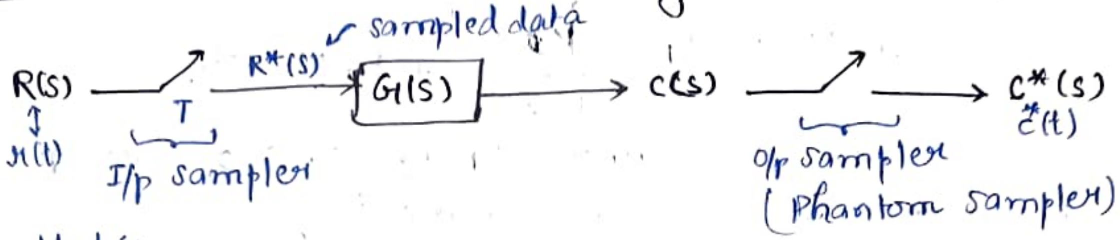
Unmodelled dynamics $\longrightarrow f(x(t))$

$\|f(x,t)\| < M < 0$

↑ uncertainity, bounded disturbance

↳ $\|s(t)\| < N < \infty$.

*OGATA: Discrete-Time control System. [Book]



Sampled i/p,

$$r^*(t) = \sum_{K=0}^{\infty} r(KT)\,\delta(t-KT) \quad \dots ①$$

$$c(t) = \sum_{K=0}^{\infty} r(KT)\,g(t-KT) \quad \dots ②$$

we have,

$$c(z) = \sum_{n=0}^{\infty} c(nT)\,z^{-n} \quad \dots ③$$

Put ③ in ② with $t = nT$.

$$\Rightarrow c(nT) = \sum_{K=0}^{\infty} r(KT)\,g(nT-KT) \quad \dots ④$$

Put ④ in ③,

$$\Rightarrow c(z) = \sum_{n=0}^{\infty}\sum_{K=0}^{\infty} r(KT)\,g((n-K)T)\,z^{-n} \quad \dots ⑤$$

Let $m \triangleq n-k$. Then,

$$c(z) = \sum_{m+K=0}^{\infty}\sum_{K=0}^{\infty} r(KT)\,g(mT)\,z^{-(m+k)}$$

$$= \sum_{m=0}^{\infty} g(mT)\,z^{-m} \sum_{K=0}^{\infty} r(KT)\,z^{-K}$$

[$K \geq 0$, $m \leq 0 \Rightarrow m = 0$
$g(mT)$ doesn't exist for $m<0$]

$$\therefore C(z) = G(z).\,R(z)$$

Sampled O/P ZT   Pulse TF   Sampling I/P ZT

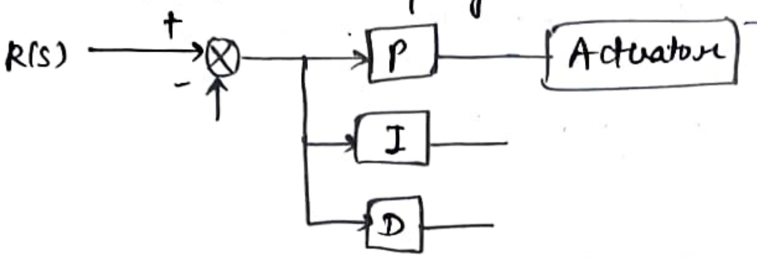
10-11-2025

Prove that the pulse TF of a digital PID controller is

$$G_{PID}(z) = K_P + \frac{K_I}{1-z^{-1}} + K_D(1-z^{-1}),$$

where K_P, K_I & $K_D \geq 0$ are proportional, integral and derivative gain, respectively.

Consider 'T' to be sampling time.



saturation can happen due to accumulation of error

PID controller : $K_P\, e(t) + K_I \int e(t)\,dt + K_D \frac{de(t)}{dt}$

- → Makes system sluggish
- (lag also) ← Integrator
- → error also accumulates
- → anti windup design
- → for steady state response.
- → lags the process

- → can introduce damping
- → for transient response design
- → Noise can be amplified through derivative controller
- → ↑ the BW
- → Speeds up the process.

Realization of Digital Filter and Digital Controller

↳ Software OR Hardware

⇓ Software → Computer

⇓ Hardware → DSP processor. (Adder, multiplier, delay (shift register))

→ DC is a part of DF. (special type)

DF: A sophisticated algorithm that takes some sequence as i/p and provides o/p in a desired type of sequence.

→ The general form of pulse TF b/w o/p $Y(z)$ and i/p $X(z)$ is

$$G(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_m z^{-m}}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_n z^{-n}}, \quad n \geq m$$

$Y(z) = G(z)\,X(z)$,

where a_i and b_i's are real coefficients.

(i) Direct Programming (ii) Standard Programming

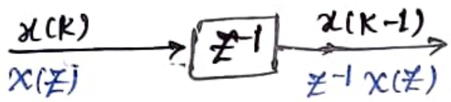
a_i and b_i as multiplier in filter representation:

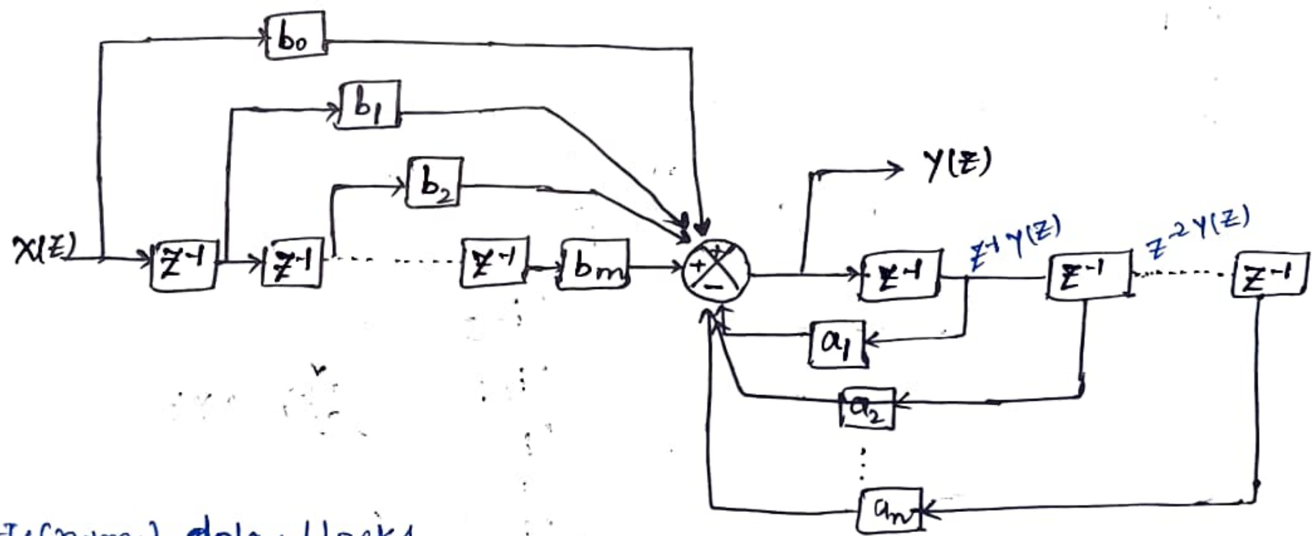
Direct structure
- ↳ Direct Form - I
- ↳ Direct Form - II / Canonical form / standard programming

Direct Programming (Form-I)

$$G(Z) = \frac{Y(Z)}{X(Z)} = \frac{b_0 + b_1 Z^{-1} + b_2 Z^{-2} + \ldots + b_m Z^{-m}}{1 + a_1 Z^{-1} + a_2 Z^{-2} + \ldots + a_n Z^{-n}}$$

$$Y(Z) = -a_1 Z^{-1} Y(Z) - a_2 Z^{-2} Y(Z) - \ldots - a_n Z^{-n} Y(Z) + b_0 X(Z) + b_1 Z^{-1} X(Z) + \ldots + b_m Z^{-m} X(Z)$$





DF-I: $(n+m)$ delay blocks

DF-II: max (n, m) blocks. (std/canonical)

Standard Programming / Canonical Structure / DF-II

$$\frac{Y(Z)}{X(Z)} = \frac{Y(Z)}{H(Z)} \cdot \frac{H(Z)}{X(Z)} = (b_0 + b_1 Z^{-1} + b_2 Z^{-2} + \ldots + b_m Z^{-m}) \cdot \frac{1}{1 + a_1 Z + a_2 Z^{-2} + \ldots + a_n Z^{-n}}$$

where

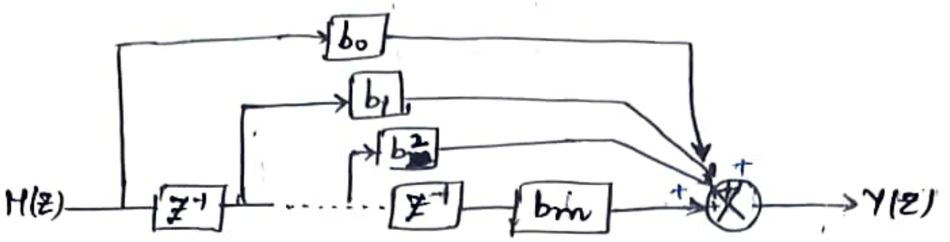
$$\frac{Y(Z)}{H(Z)} = b_0 + b_1 Z^{-1} + b_2 Z^{-2} + \ldots + b_m Z^{-m} \quad \ldots (1)$$

$$\frac{H(Z)}{X(Z)} = \frac{1}{1 + a_1 Z^{-1} + a_2 Z^{-2} + \ldots + a_n Z^{-n}} \quad \ldots (2)$$

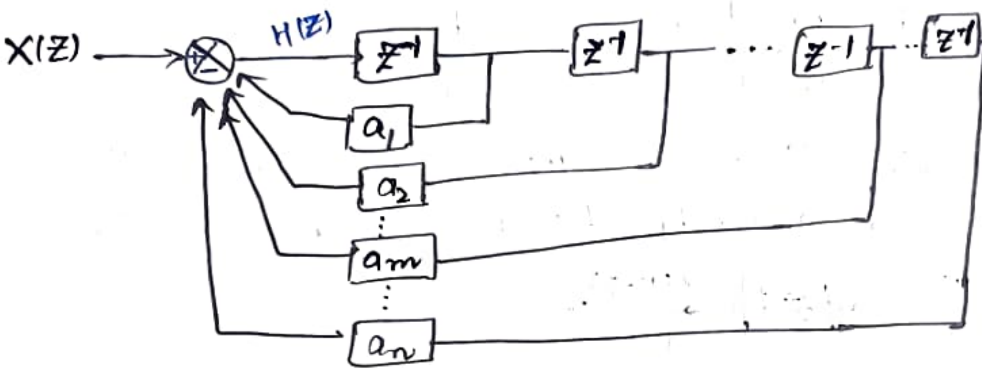
① ⇒ $Y(z) = b_0 H(z) + b_1 z^{-1} H(z) + b_2 z^{-2} H(z) + \dots + b_m z^{-m} H(z)$ ...③

② ⇒ $H(z) = X(z) - a_1 z^{-1} H(z) - a_2 z^{-2} H(z) - \dots - a_n z^{-n} H(z)$ ...④

③ :



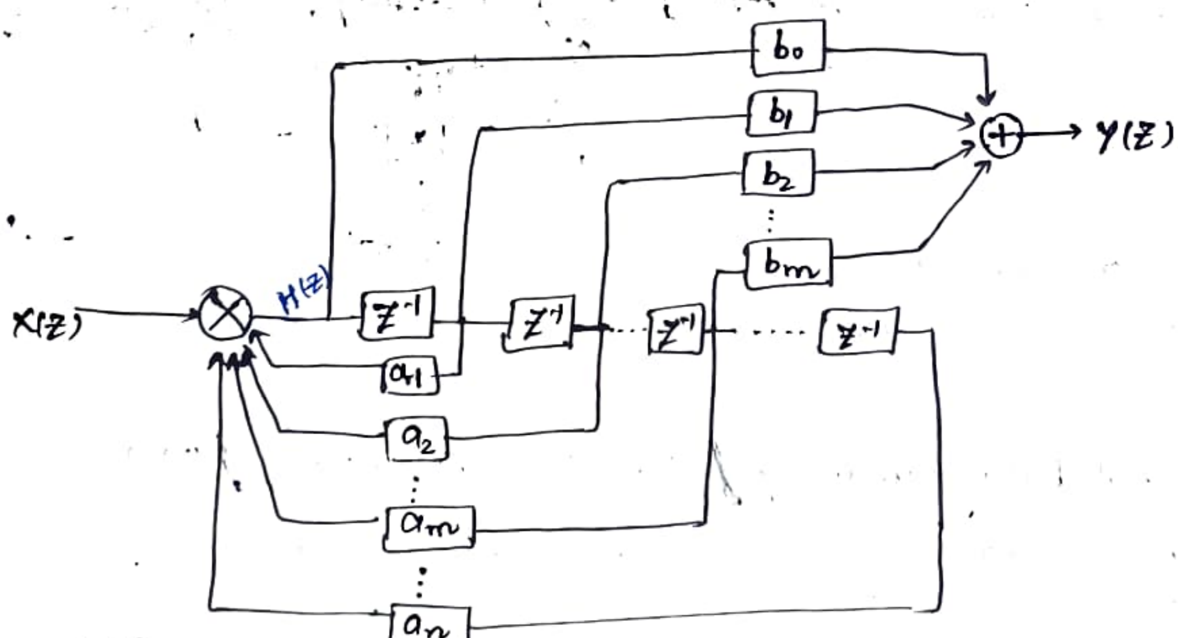
④ :



13-11-2025

PROAKIS, DSP

* Pole placement
* Observer Design
* Digital twin : Real time replica of the system modelled.

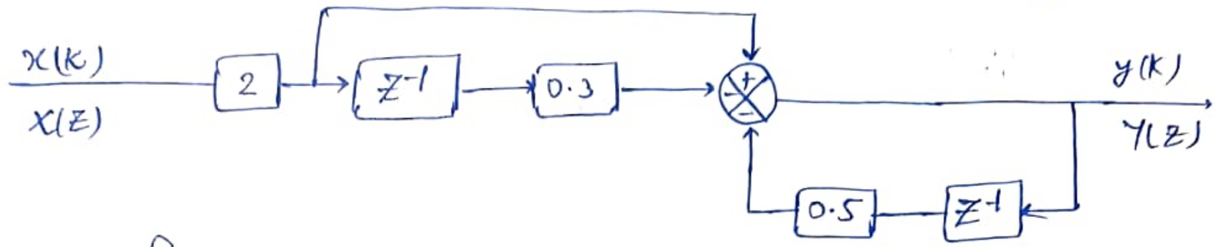


14-11-2025

Eg. Obtain the block diagrams in direct and standard program for the following pulse T.F:

$$\frac{Y(Z)}{X(Z)} = \frac{2-0.6Z^{-1}}{1-0.5Z^{-1}}$$

Sol^n: $Y(Z) = -0.5Z^{-1}Y(Z) + 2X(Z) - 0.6Z^{-1}X(Z)$ [~~Stand.~~ Direct Prog.]

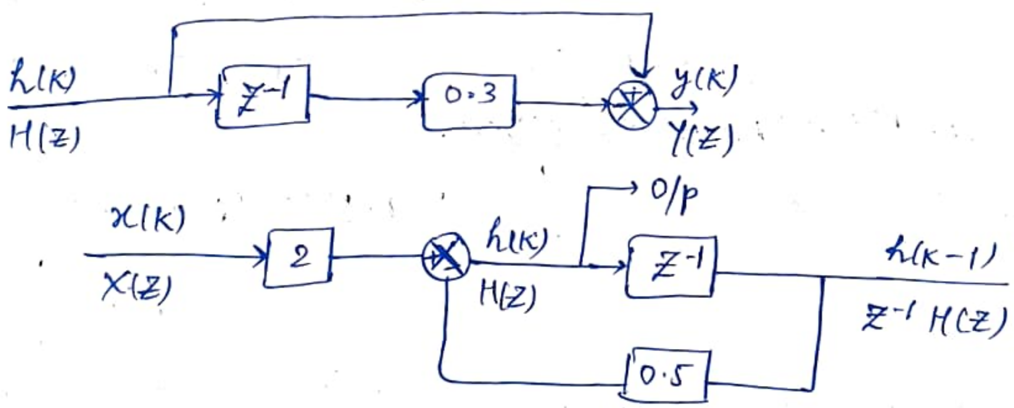


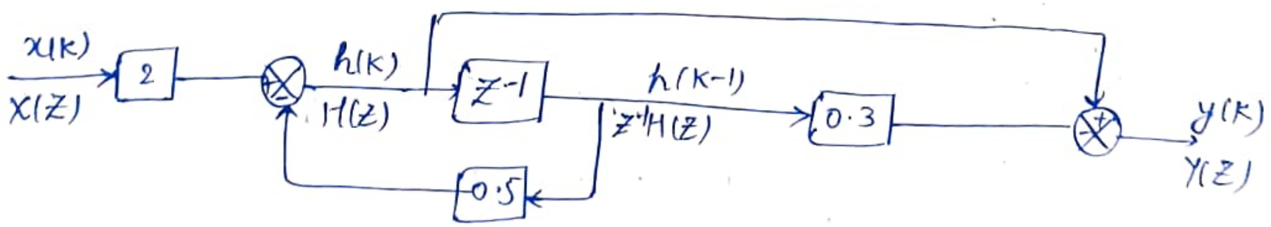
Standard Program :

$$\frac{Y(Z)}{X(Z)} = \frac{Y(Z)}{H(Z)} \cdot \frac{H(Z)}{X(Z)} = (1-0.3Z^{-1})\frac{2}{1+0.5Z^{-1}}$$

where, $\frac{Y(Z)}{H(Z)} = (1-0.3Z^{-1})$, $\frac{H(Z)}{X(Z)} = \frac{2}{1+0.5Z^{-1}}$

$\Rightarrow Y(Z) = H(Z) - 0.3Z^{-1}H(Z)$ | $\Rightarrow H(Z) = -0.5Z^{-1}H(S) + 2X(Z)$





Sources of error in Digital Filters:

① Quantization error

② Accumulation error

$$y(K) = \underbrace{2y(K-1)} + \underbrace{3y(K-2)} + \dots + 2x(K) + 3x(K-1) + \dots$$

③ Coefficient Quantization error

$$\frac{Y(Z)}{X(Z)} = \frac{b_0 x^m + b_1 x^{m-1} + \dots}{1 + a_0 x^n + \dots}$$

IIR Filter — Bilinear Transformation.

IIR / FIR

$$G(Z) = \frac{Y(Z)}{X(Z)} = \frac{b_0 + b_1 Z^{-1} + \dots + b_m Z^{-m}}{1 + a_1 Z^{-1} + a_2 Z^{-2} + \dots + a_n Z^{-n}}, \quad n \geq m \quad \dots ①$$

Difference eqn:

$$y(K) = -a_1 y(K-1) - a_2 y(K-2) - \dots - a_n y(K-n)$$
$$+ b_0 x(K) + b_1 x(K-1) + \dots + b_m x(K-m) \quad \dots ②$$

If $\exists\, i \in \{1, 2, \dots, n\}$ such that $a_i \neq 0$, then the filter is called IIR Filter / Recursive filter.

If $\forall i,\ a_i = 0,\ i \in \{1, 2, \dots, n\} \Rightarrow$ FIR Filter / Non-Recursive filter / Moving-average filter.

Let $g(KT)$ be the finite impulse / weighting sequence of the FIR filter.

If $x(KT)$ is the sequence applied to this filter, the o/p $y(KT)$ is

$$y(KT) = \sum_{h=0}^{K} g(hT)\, x(KT - hT),$$

i.e., o/p is convolution of i/p sequence and weighting seq.

$$= g(0)x(KT) + g(T)x((K-1)T) + \dots + g(KT)x(0) \quad \dots Ⓐ$$

When 'K' is large, we need to store more no. of past inputs.

$\Rightarrow$ Memory requirement becomes large.

- Instead we keep track of past 'N' i/p samples including the current sample.

Let's process 'N' immediate past i/p samples, $x((K-1)T)$, $x((K-2)T)$, ..., $x((K-N)T)$ and current sample $x(KT)$, i.e.,

$$y(KT) = g(0)\,x(KT) + g(T)\,x((K-1)T) + \dots + g(NT)\,x((K-N)T) \quad \dots \text{(B)}$$

$$ZT \Rightarrow \boxed{Y(Z) = g(0)\,X(Z) + g(T)\,Z^{-1}X(Z) + \dots + g(NT)\,Z^{-N}X(Z)}$$

↳ Moving Average / Non-recursive.

17-11-2025

Nyquist Plot

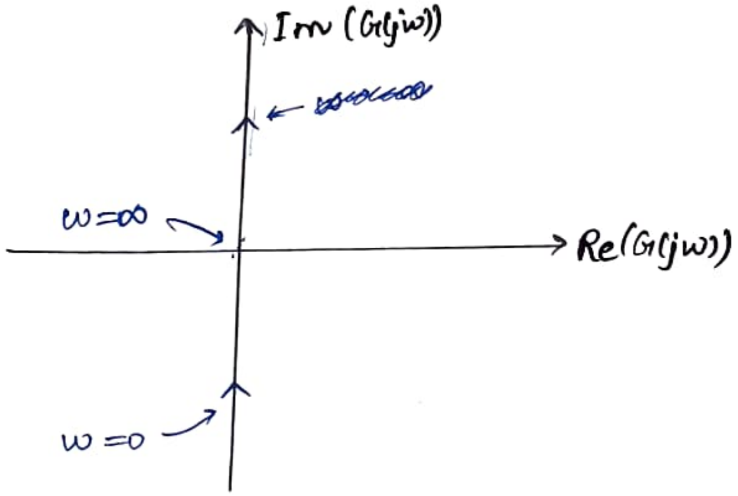
↳ Re{G(jω) H(jω)} vs Im{G(jω) H(jω)} as ω is varied.

Eg. $G(s) = \frac{1}{s}$.

$$G(j\omega) = \frac{1}{j\omega} = \frac{j}{j^2\omega} = 0 + j\left(-\frac{1}{\omega}\right) = \frac{1}{\omega}\angle -90^\circ$$

$= 1/\omega$

ω	Re(G(jω))	Im(G(jω))	|G(jω)|	∠G(jω)
0	0	−∞	∞	−90°
∞	0	0	0	−90°



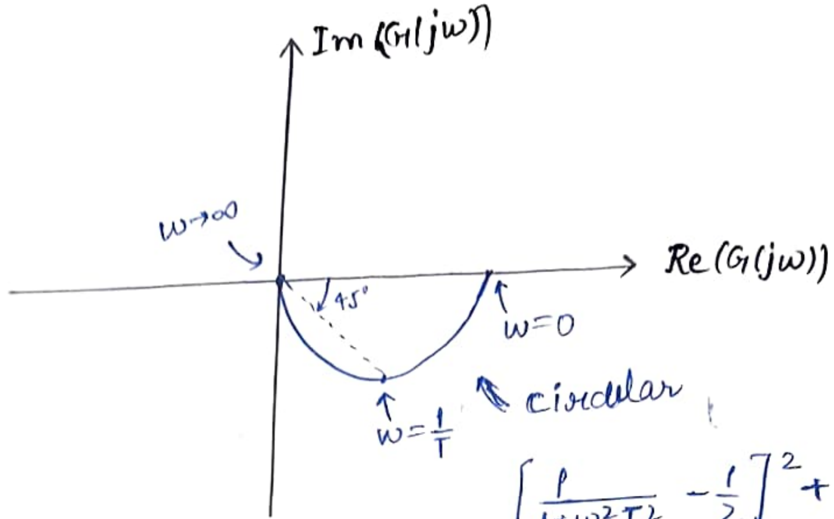
Eg. $G(s) = \frac{1}{1+sT}$

$$G(j\omega) = \frac{1}{1+j\omega T} = \frac{1-j\omega T}{1^2+\omega^2T^2} = \frac{1}{1+\omega^2T^2} - \frac{j\omega T}{1+\omega^2T^2}$$

$$|G(j\omega)| = \frac{1}{\sqrt{1+\omega^2T^2}}$$

$$\angle G(j\omega) = -\tan^{-1}(\omega T)$$

ω	Re(G(jω))	Im(G(jω))	|G(jω)|	∠G(jω)
0	1	0	1	0
1/T	1/2	−1/2	$1/\sqrt{2}$	−45°
∞	0	0	0	−90°



$$\left[\frac{1}{1+\omega^2T^2}-\frac{1}{2}\right]^2+\left[\frac{-\omega^2}{1+\omega^2T^2}\right]=\left(\frac{1}{2}\right)^2$$

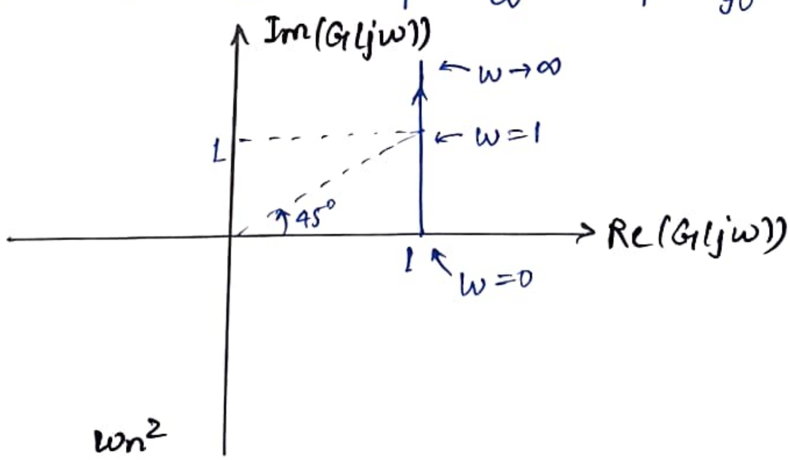
Eg. $G(s) = 1+sT$

$G(j\omega) = 1+j\omega T$

$|G(j\omega)| = \sqrt{1+\omega^2T^2}$

$\angle G(j\omega) = \tan^{-1}(\omega T)$

ω	$Re(G(j\omega))$	$Im(G(j\omega))$	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
0	1	0	1	0
$1/T$	1	1	$\sqrt{2}$	$\tan^{-1}(1) = 45°$
∞	1	∞	∞	90°



Eg. $G(s) = \dfrac{\omega_n^2}{s^2+2j\omega_n s\xi+\omega_n^2}$

$$G(j\omega) = \frac{1}{\left(1-\frac{\omega^2}{\omega_n^2}\right)+j\left(\frac{2\xi\omega}{\omega_n}\right)} = \frac{1-\omega^2/\omega_n^2}{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(\frac{2\xi\omega}{\omega_n}\right)^2}$$

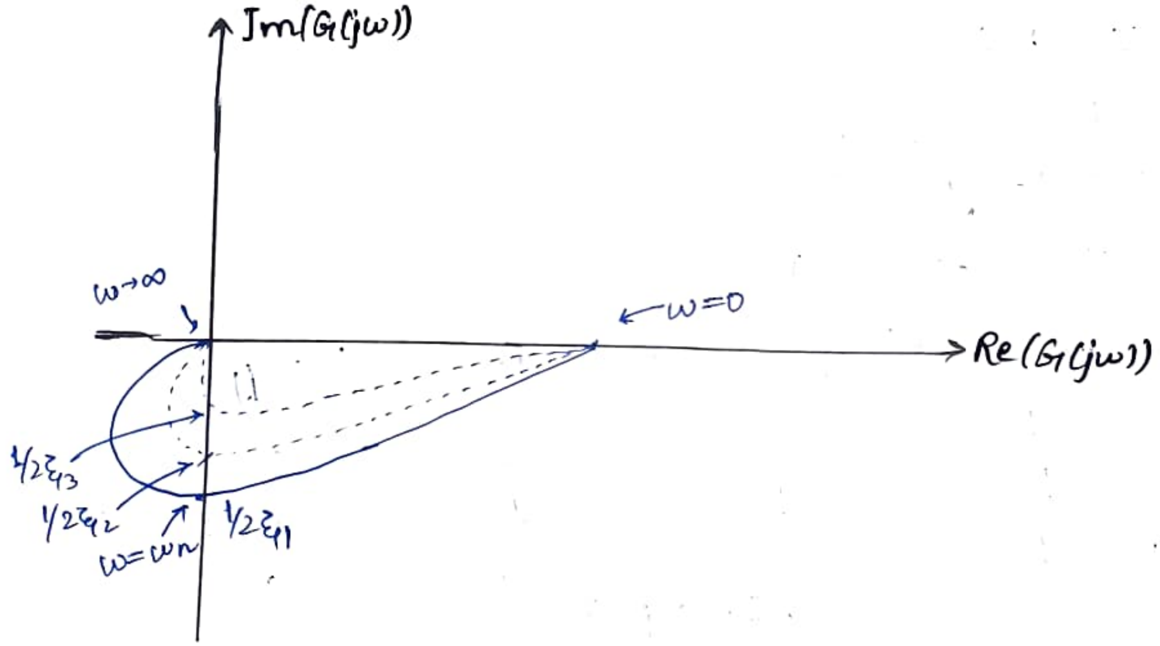
$$+j\left[\frac{-2\xi\omega/\omega_n}{(1-\omega^2/\omega_n^2)^2+(2\xi\omega/\omega_n)^2}\right]$$

$$G(j\omega) = \frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(\frac{2\xi\omega}{\omega_n}\right)^2}} \angle -\tan^{-1}\left(\frac{2\xi\omega/\omega_n}{1-\omega^2/\omega_n^2}\right)$$

$$G = \frac{1}{a+jb}$$

$$\Rightarrow G = \frac{1}{\sqrt{a^2+b^2}} \angle -\tan^{-1}\left(\frac{b}{a}\right)$$

ω	$Re(G(j\omega))$	$Im(G(j\omega))$	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
0	1	0	1	0
ω_n	0	$-1/2\xi_1$	$1/2\xi_1$	$-90°$
∞	0	0	0	$-180°$



20-11-2025

Nyquist Stability

CL char. eqⁿ: $F(s) \triangleq 1 + G(s)H(s) = 0$

Let $G(s)H(s) = \dfrac{n_o(s)}{d_o(s)}$ (OL).

Then, $F(s) = 1 + G(s)H(s)$

$$= 1 + \frac{n_o(s)}{d_o(s)} = \frac{d_o(s) + n_o(s)}{d_o(s)}$$

We have (-ve feedback),

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} = \frac{n_o'(s)/d_o'(s)}{\dfrac{n_o(s) + d_o(s)}{d_o(s)}}$$

Pole of $F(s) \longrightarrow$ OL poles

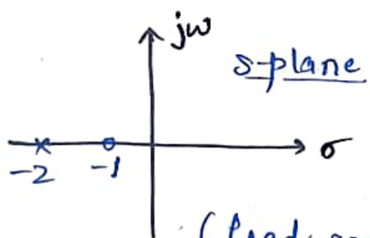
Zeros of $F(s) \longrightarrow$ CL poles

For ~~F~~ $H(s) = 1$,

$$\frac{Y(s)}{R(s)} = \frac{n_o(s)}{n_o(s) + d_o(s)}$$

Mapping Theorem:

Let's consider $G(s) = \dfrac{s+1}{s+2}$



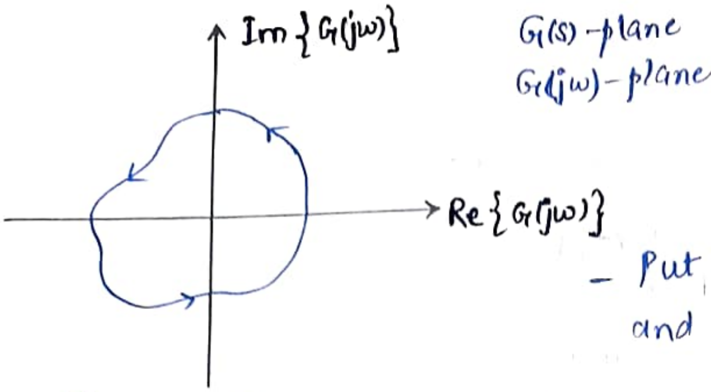
(Lead compensator)

Let $s = -1 + j$ in $G(s)$.

$$G(j\omega) = \frac{-1+j+1}{-1+j+2} = \frac{j}{j+1} = \frac{j(1-j)}{2} = \frac{1}{2} + j\frac{1}{2}$$

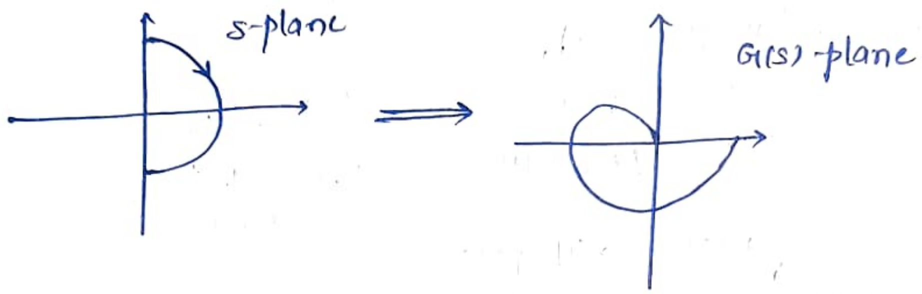
$Re\{G(j\omega)\} = 1/2$

$Im\{G(j\omega)\} = 1/2$.



- Put any random value of s and we will get a contour.

Mapping theorem : Collection of point forms a contour.

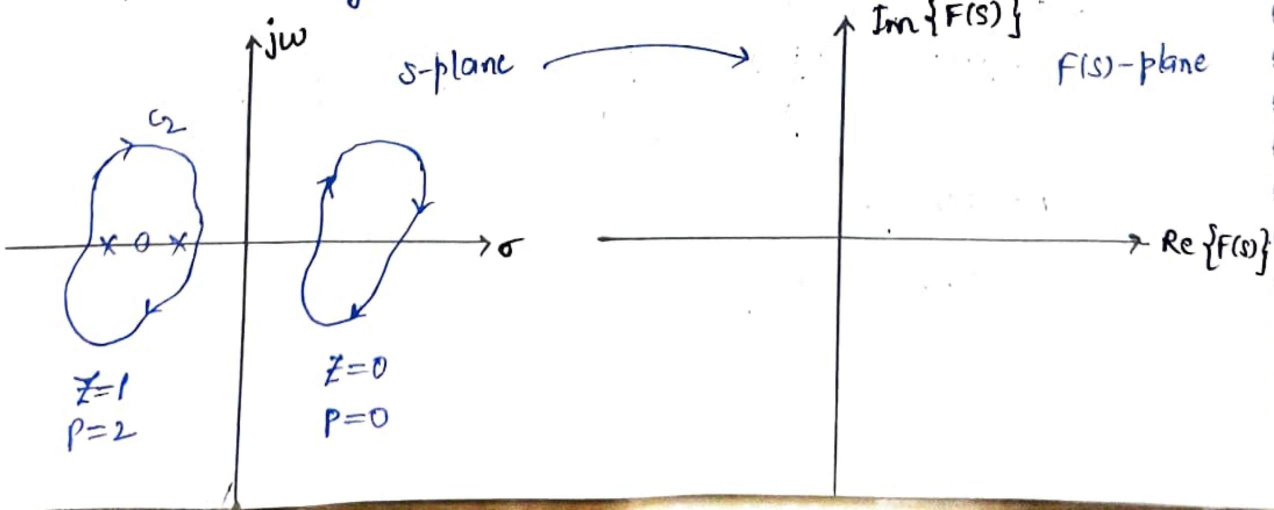


Let F(s) be a ratio of two polynomials in s. Let a closed contour in the s-plane contains 'Z' zeros and 'P' poles of F(s) without passing over any of the poles and zeros of F(s). Now, this closed contour in the s-plane is mapped into closed contour in F(s)-plane. The total no. of CW encircdements (N_c) of the origin by the contour in F(s) plane is equal to

$N_c = Z - P$,

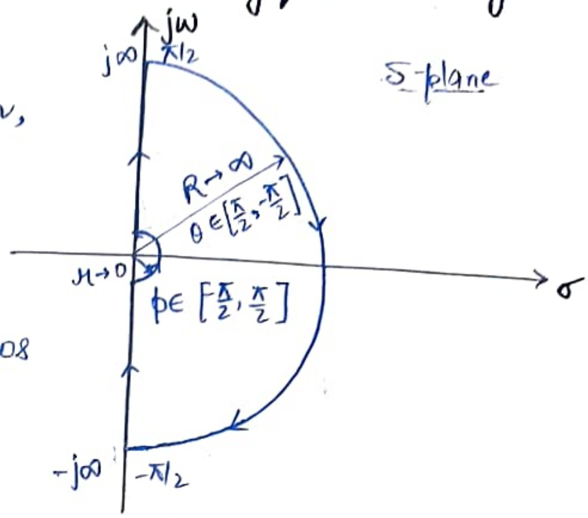
Z : No. of zeros of F(s) within closed contour in s-plane.
P : No. of poles of F(s) within closed contour in s-plane.
N_c : No. of CW encircdements of the origin.

For stability, $Z=0 \Rightarrow N_c = -P$



Application of Mapping Theorem to determine Nyquist Stability:

- circle of infinite radius.
- If there is any pole @ origin, then draw a small circle.



Assumptions:

(i) None of the OL poles and zeros lie in RHP.

(ii) $\lim_{s\to\infty} G(s)H(s)$ is either 0 or a finite constant.

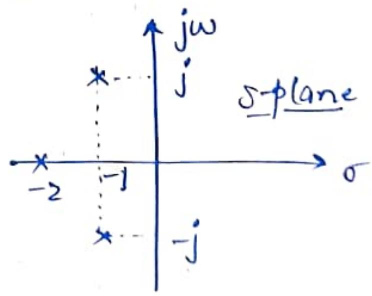
Nyquist stability criteria: Consider OLTF $G(s)H(s)$ does not have any poles or zeros on img.-axis. If the OLTF $G(s)H(s)$ has 'K' poles in RH s-plane and if $\lim_{s\to\infty} G(s)H(s)$ is either 0 or a finite non-zero constant, then for stability of CL system, the locus of $G(j\omega)H(j\omega)$ as 'ω' varies from $-\infty$ to ∞ must encircle the $-1+j0$ point 'K' times in ACW dirn.

Eg. Consider a system with unity -ve F/B of

$$G(s) = \frac{K}{(s+2)(s^2+2s+2)}$$

Comment about CL stability using nyquist.

Soln: Poles: $s=-2$, $s=-1\pm j$



Since no OL poles in RHs-plane, thus nyquist contour should not encircle $-1+j0$ point in ACW dirn.

$$N_c = Z - P = 0 \text{ for stability}$$

Let $K=1$. $$G(j\omega) = \frac{1}{(j\omega+2)(-\omega^2+2j\omega+2)}$$

$$Re\{G(j\omega)\} = \frac{4(1-\omega^2)}{16(1-\omega^2)^2+\omega^2(6-\omega^2)^2}$$

$$Im\{G(j\omega)\} = \frac{-\omega^2(6-\omega^2)}{16(1-\omega^2)^2+\omega^2(6-\omega^2)^2}$$

Set $Re\{G(j\omega)\} = 0$ to find the intersection of the curve with imaginary axis.

$$\frac{4(1-\omega^2)}{16(1-\omega^2)^2 + \omega^2(6-\omega^2)^2} = 0$$

$$\Rightarrow \omega^2 = 1 \Rightarrow \omega = \pm 1.$$

Put this value in $Im\{G(j\omega)\}$, we have

$$\frac{-\omega(6-\omega^2)}{16(1-\omega^2)^2 + \omega^2(6-\omega^2)^2} = \frac{-1(6-1)}{16(1-1)^2 + 1(6-1)^2} = -\frac{5}{25} = -\frac{1}{5}$$

For $\omega = -1$, $-\frac{\omega(6-\omega^2)}{16(1-\omega^2)^2 + \omega^2(6-\omega^2)^2} = -\frac{(-1)(6-1)}{16(1-1)^2 + 1(6-1)^2}$

$$= 5/25 = 1/5.$$

For $\omega = +1$, $-\frac{1(6-1)}{16(1-1)^2 + 1(6-1)^2} = -\frac{1 \times 5}{25} = -1/5.$

Again set $Im\{G(j\omega)\} = 0$ to find the intersection of the curve with real axis.

$$-\frac{\omega(6-\omega^2)}{16(1-\omega^2)^2 + \omega^2(6-\omega^2)^2} = 0$$

$$\Rightarrow \omega(6-\omega^2) = 0$$

$$\Rightarrow \omega = 0 \text{ or } \omega^2 = 6 \Rightarrow \omega = \pm\sqrt{6}.$$

Put these values in $Re\{G(j\omega)\}$,

$\omega = 0 \Rightarrow \frac{4(1-0^2)}{16(1-0^2)^2} = \frac{4}{16} = \frac{1}{4}$

$\omega = \sqrt{6} \Rightarrow \frac{4(1-6)}{16(1-6)^2 + 6(6-6)^2} = \frac{4(-5)}{16 \times 25} = -\frac{1}{20}.$

$\omega = -\sqrt{6} \Rightarrow \frac{4(1-6)}{16(1-6)^2 + 6(6-6)^2} = -1/20.$

Also put $\omega = 0$ in $Re\{G(j\omega)\}$

$$\Rightarrow \frac{4(1-0)}{16(1-0)^2 + 0} = 1/4.$$

~~∴ At ω=0,~~
Put $\omega = 0$ in $Im\{G(j\omega)\}$, $\Rightarrow -\frac{0(6-0)}{16(1-0)^2 + 0(6-0)^2} = 0.$

$\therefore$ At $\omega=0$, $(Re(G(j\omega)), Im(G(j\omega))) = (1/4, 0)$.

Put $\omega \to \infty$ in $Re\{G(j\omega)\}$,

$$\lim_{\omega\to\infty} \frac{4\left(\frac{1}{\omega^6}-\frac{1}{\omega^4}\right)}{16\left(\frac{1}{\omega^3}-\frac{1}{\omega}\right)^2+\left(\frac{6}{\omega^2}-1\right)^2} \quad \text{[Dividing N}^r \text{ \& D}^r \text{ by } \omega^6\text{]}$$

$$= \frac{4(0-0)}{16(0-0)^2+(0-1)^2} = 0$$

Put $\omega \to \infty$ in $Im\{G(j\omega)\}$,

$$\lim_{\omega\to\infty} - \frac{\left(\frac{6}{\omega^5}-\frac{1}{\omega^3}\right)}{16\left(\frac{1}{\omega^3}-\frac{1}{\omega}\right)^2+\left(\frac{6}{\omega^2}-1\right)^2} = 0$$

$\therefore$ $\omega=0 \Rightarrow Re\{G(j\omega)\}=1/4$, $Im\{G(j\omega)\}=0$

$\omega=+1 \Rightarrow Re\{G(j\omega)\}=0$, $Im\{G(j\omega)\}=-1/5$

$\omega=-1 \Rightarrow Re\{G(j\omega)\}=0$, $Im\{G(j\omega)\}=1/5$

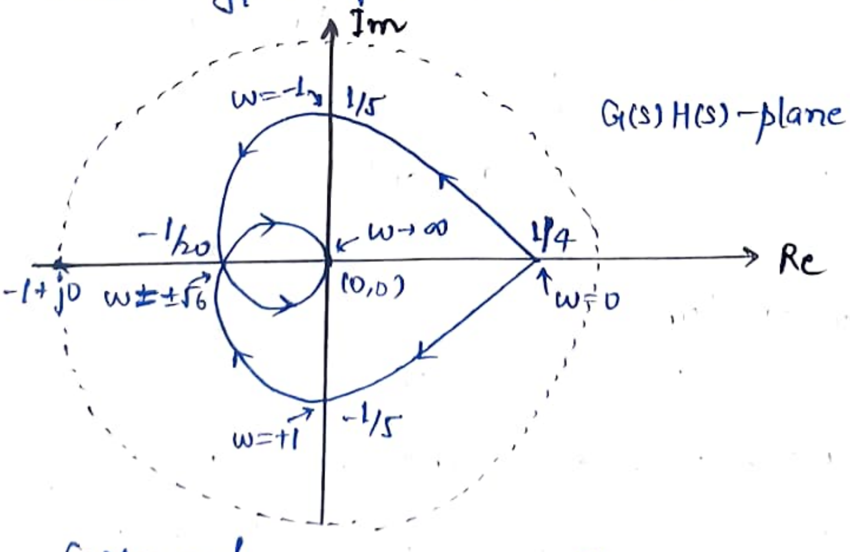
$\omega=\pm\sqrt{6} \Rightarrow Re\{G(j\omega)\}=-1/20$, $Im\{G(j\omega)\}=0$

$\omega\to\infty \Rightarrow Re\{G(j\omega)\}=0$, $Im\{G(j\omega)\}=0$.

24-11-2025

ω_{gc} : 'ω' where nyquist plot intersects the unit circle.

ω_{pc} : 'ω' where nyquist plot intersects the -ve real axis.



$$GM = \frac{1}{|G(j\omega)H(j\omega)|_{\omega_{pc}}} = \frac{1}{|-1/20|} = 20.$$

$GM(dB) = 20\log_{10} 20 = 26\ dB > 0$

$PM \to \infty$

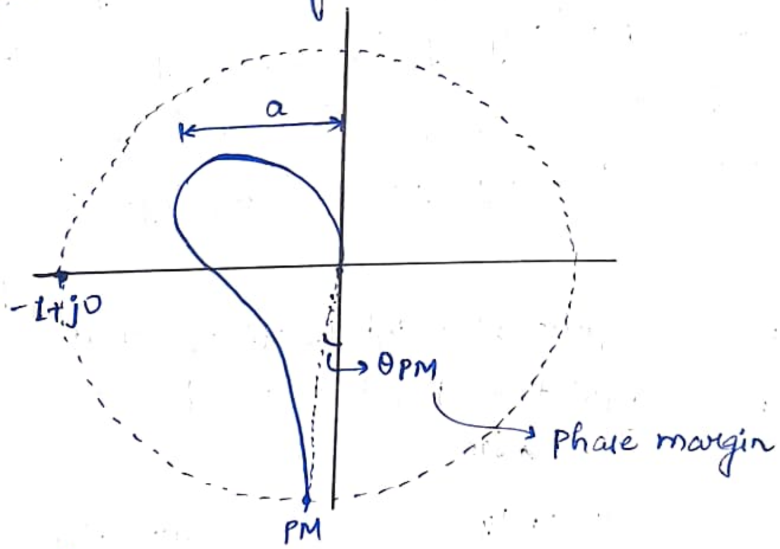
Since there are no OL poles in RHP, thus ~~the~~ the nyquist contour should not encircle the $(-1+j0)$ point.

However, for $K=1$, the contour intersects the real axis at $-1/20$ for $\omega = \pm\sqrt{6}$. Thus, for $K<20$, the contour does not encircle the $(-1+j0)$ point and hence the system is stable for $K<20$. For $K>20$, the contour encircles the $(-1+j0)$ point in ACW dirn and hence the system becomes unstable.

For $K=20$, the system is marginally stable with freq. of oscillation $\omega = \sqrt{6}$ rad/sec.

Root Locus : K varies

Nyquist plot: ω varies from 0 to ∞.



$GM = \frac{1}{|a|} \rightarrow -20\log|a|$ (dB)

<u>Filter / Estimator / observer:</u>

- Filter and estimator are the same thing.

Kalman Filter : certain unwanted part rejected.

Estimator $\Rightarrow$ Estimate the o/p (in statistical sense)

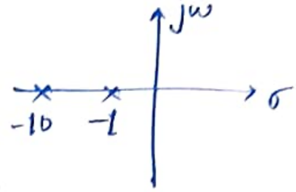
$$y = Hx + \eta$$

$\uparrow$ o/p $\quad$ $\uparrow$ I/p $\quad$ $\uparrow$ noise

Observer $\Rightarrow$ Deterministic | # Book by Dorf & Bishop

Eg. Sketch the nyquist plot for the unity f/b control system with $G(s) = \dfrac{100}{(s+1)(0.1s+1)}$ and comment about stability.

Soln: Poles: $-1, -10$



$$G(j\omega) = \frac{100}{(j\omega+1)(0.1j\omega+1)} \qquad [H(s)=1]$$

$$= \frac{100}{-0.1\omega^2 + 1.1j\omega + 1}$$

$$= \frac{100}{(1-0.1\omega^2) + 1.1j\omega}$$

$$= \frac{100}{(1-0.1\omega^2)+1.1j\omega} \times \frac{(1-0.1\omega^2) - 1.1j\omega}{(1-0.1\omega^2) - 1.1j\omega}$$

$$= \frac{(100-10\omega^2) - 110j\omega}{(1-0.1\omega^2)^2 + (1.1\omega)^2}$$

$$= \frac{10(10-\omega^2)}{(1-0.1\omega^2)^2+(1.1\omega)^2} + j\left(-\frac{110\omega}{(1-0.1\omega^2)^2+(1.1\omega)^2}\right)$$

$$= \frac{10(10-\omega^2)}{(1+1.01\omega^2)+0.01\omega^4} - j\left(\frac{110\omega}{(1+1.01\omega^2)+0.01\omega^4}\right)$$

$$\therefore Re\{G(j\omega)\} = \frac{10(10-\omega^2)}{1+1.01\omega^2+0.01\omega^4}$$

$$Im\{G(j\omega)\} = -\frac{110\omega}{1+1.01\omega^2+0.01\omega^4}$$

$$Im\{G(j\omega)\} = 0 \Rightarrow -\frac{110\omega}{1+1.01\omega^2+0.01\omega^4} = 0 \Rightarrow \omega = 0$$

Put $\omega=0$ in $Re\{G(j\omega)\} \Rightarrow \dfrac{10(10-0)}{1+0+0} = 100$

$$Re\{G(j\omega)\} = 0 \Rightarrow \frac{10(10-\omega^2)}{1+1.01\omega^2+0.01\omega^4} = 0 \Rightarrow \omega = \pm\sqrt{10}$$

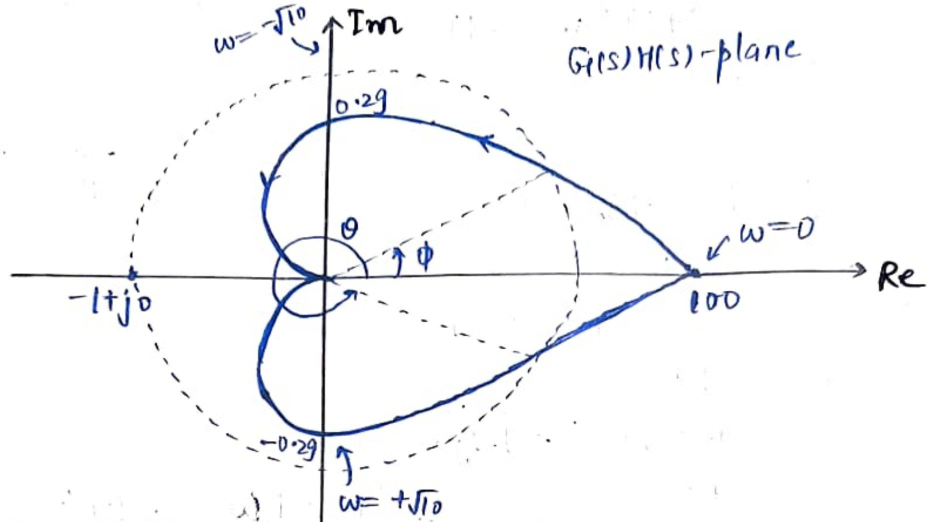
Put $\omega = \pm\sqrt{10}$ in $Im\{G(j\omega)\} \Rightarrow -\dfrac{1.1(\pm\sqrt{10})}{1+0.01\times 10+0.01\times 100}$

$$= \frac{\pm 3.48}{1+10.1+1} = \mp 0.29$$

Put $\omega=0$ in $Im\{G(j\omega)\} \Rightarrow Im\{G(j\omega)\}\big|_{\omega=0} = 0$

Put $\omega \to \infty$ in $Re\{G(j\omega)\} \Rightarrow \lim\limits_{\omega \to \infty} \dfrac{10\left(\frac{1}{\omega^4} - \frac{1}{\omega^2}\right)}{\frac{1}{\omega^4} + 1.01\left(\frac{1}{\omega^2}\right) + 0.01} = 0$

$\omega \to \infty$ in $Im\{G(j\omega)\} \Rightarrow \lim\limits_{\omega \to \infty} \dfrac{-1.1/\omega^3}{\frac{1}{\omega^4} + 1.01 \frac{1}{\omega^2} + 0.01} = 0$



Always stable as '0' OL RHP poles and zero encirclements of $(-1+j0)$ point.

As NP intersects the real axis @ origin, thus

$$GM = 1/0 \to \infty \Rightarrow GM(dB) \to \infty$$

<u>Note</u>: If NP intersects the real-axis at 'a' s.t. $|a| < 1$, then

$$GM = \frac{1}{|a|} \Rightarrow GM = -20\log|a| > 0.$$

On the other hand, we find ϕ or θ, i.e., the angle made by the origin and the unit circle intersecting point.

Thus, $PM = \theta - 180^\circ > 0$

Since, $\theta = 360^\circ - \phi \Rightarrow -180^\circ + \theta = 180^\circ - \phi > 0^\circ$