Indian Institute of Space Science and Technology
Thiruvananthapuram-695 547

Summer Supplementary Examination - July 2013

B. Tech - II Semester

MA121 - Vector Calculus and Differential Equations(2011 & 2012 Batch)

Date : 24/06/2013 | **Time: 9.30 am to 12.30 pm** | **Max. Marks: 100**

SECTION A (Attempt all 10 questions - 10x5= 50 marks.)

1. Let $g_n(x) = \begin{cases} \frac{1}{n} \text{ if } x = 0 \text{ or if } x \text{ is irrational} \\ b + \frac{1}{n} \text{ if } x \text{ is rational and } x = \frac{a}{b},\ b > 0 \end{cases}$.
   Show that $\{g_n\}$ converges uniformly on $[\frac{-1}{2}, \frac{1}{2}]$.

2. Show that $\displaystyle\sum_{0}^{\infty} \left[\frac{x^{2n+1}}{2n+1} - \frac{x^{n+1}}{2n+2}\right]$ converges pointwise but not uniformly on $[0, 1]$.

3. Solve $xy^2 \dfrac{dy}{dx} + y^3 = x \cos x$.

4. Discuss the existence and uniqueness of the solution for the following initial-value problem
$$\begin{aligned} \frac{dy}{dx} &= y^{1/3}, \\ y(0) &= 0. \end{aligned}$$

5. Using method of variation of parameters, find a particular solution of the equation $\dfrac{d^2y}{dx^2} + k^2y = f(x)$, where k is positive constant.

6. Find the general solution of $(D-5)^4 y = e^{5x} \cos x$, here $D \equiv \dfrac{d}{dx}$.

7. Find the arc length function of the curve $c : z^2 = y^3;\ x^2 = y^4, \quad x \geq 0$ with initial point $(0, 0, 0)$; and using the arc length function find the length of the curve from $(0, 0, 0)$ to $(1, 1, 1)$.

8. Let $\vec{F}(x, y, z) = (ye^z + y,\ xe^z + x,\ xye^z + 1)$ be a vector field and C be the union of two curves C_1 & C_2 from $(0, 0, 0)$ to $(1, -1, 1)$ where C_1 & C_2 are given by $C_1 : y = x^2$ from $(0, 0, 0)$ to $(1, -1, 0)$ & $C_2 : x = 1, y = -1$ from $(1, -1, 0)$ to $(-1, 1, 1)$. Find the line integral of $\vec{F}$ along the curve C.

9. Using Green's theorem find the area of the region enclosed by the curves $y^2 = x$ and $x = 1$.

10. Using surface integration find the surface area of the cylinder $x^2 + y^2 = 1, \quad -1 \leq z \leq 1$.

[P.T.O.]

SECTION B (Attempt any 5 questions - 5x10= 50 marks.)

11. (a) Show that $\lim_{n\to\infty} \int_0^1 xe^{-nx^2}dx = 0$ using uniform convergence of sequence of functions.

(b) Show that $\ln 2 = \sum_1^{\infty} \frac{1}{n.2^n} = \frac{1}{2} + \frac{1}{2.2^2} + \frac{1}{3.2^3} + ...$

(Hint : First show that $\sum_1^{\infty} x^{n-1}$ converges uniformly to $\frac{1}{1-x}$ on $[0, \frac{1}{2}]$)

12. (a) Find the general solution of $\frac{d^2y}{dx^2} - f(x)\frac{dy}{dx} + [f(x) - 1]\, y = 0.$

(b) Does eigenvalues and eigenfunctions for the following differential equation exist? If yes, then find all eigenvalues and corresponding eigenfunctions.

$$\frac{d}{dx}\left[x\frac{dy}{dx}\right] + \frac{\lambda}{x}y = 0, \quad \frac{dy}{dx}(1) = 0, \quad \frac{dy}{dx}(e^{2\pi}) = 0.$$

13. (a) Solve

$$(2x^2 + y)dx + (x^2y - x)dy = 0.$$

(b) For the point $x = 0$, find the indicial equation for the following differential equations.

(i) $x^3\frac{d^2y}{dx^2} + (\cos 2x - 1)\frac{dy}{dx} + 2xy = 0.$

(ii) $4x^2\frac{d^2y}{dx^2} + (2x^4 - 5x)\frac{dy}{dx} + (3x^2 + 2)y = 0.$

Also, discuss whether two linearly independent Frobenius series solutions around $x = 0$ exist or do not exist for the above differential equations.(No need to find solutions).

14. State Stoke's Theorem. Verify Stokes theorem for the vector field $\vec{F} = (x, y, 0)$ over the outward oriented hemisphere S: $x^2 + y^2 + z^2 = 1;\ z \geq 0$.

15. State Gauss divergence theorem. Verify Gauss divergence theorem for the vector field $\vec{F}(x, y, z) = (x, y, z)$ over the unit cube $E : 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ 0 \leq z \leq 1$.

16. State Green's theorem. Let G be a region on xy plane given by $1 \leq x^2 + y^2 \leq 2$ and let $\vec{F}$ be a smooth vector field with domain G such that $\text{curl}(\vec{F}) = 0$ (note that we cannot say that $\vec{F}$ is conservative). Show, using Green's theorem, that $\int_{C_1} \vec{F} = \int_{C_2} \vec{F}$ where $C_1 : x^2 + y^2 = 1$ and $C_2 : x^2 + y^2 = 2$.
