साङ्ख्यविधयश्च

NUMERICAL METHODS

(11-09-2024)

Numerical Methods

- Solving scalar non-linear equations
- Solving first-order linear ODEs numerically
- Numerical Integration + Implementation

Scalar Non-linear Equations

$$x^2 - e^x = 0$$

$$x^3 + m\cos x = 0$$

$$x^5 + 5x^4 + 12x^3 + 10x^2 + 9x + 11 = 0$$

① Choose initial x_0.

Define iterative procedure $x_{k+1} = f(x_k)$.

$$\{x_0, x_1, x_2, \dots\}$$

$x_n \approx x$, x_n, n is large

$f(x) = 0$, x is root

$\Rightarrow x - g(x) = 0$

$\Rightarrow x = g(x)$, x is fixed point

$$\downarrow$$

$$x_{k+1} = g(x_k)$$

$$x_1 = g(x_0)$$
$$x_2 = g(x_1)$$
$$x_3 = g(x_2)$$
$$\vdots$$

$|x_n - x_{n-1}| = tol$ ($= 10^{-5}$, say)

$x_1, x_2, x_3, \dots$

Cauchy sequence: $|x_n - x_m| < \epsilon$ $\forall\, n, m > N$

$\Downarrow$

converges

Let I be the interval.

$g(I) \subseteq I$, $g : I \to I$ ↳ self map

$x, y \in I$, $x \neq y$.

$|g(x) - g(y)| \leq \alpha |x - y|$, $0 \leq \alpha < 1$.

↳ Contraction

: Contraction mapping

Then, $x_{K+1} = g(x_K)$ converges,
for any $x_0 \in I$.

x^* is fixed point $\Rightarrow g(x^*) = x^* \Rightarrow f(x^*)=0$
$\Rightarrow x^*$ is the root of the equation.

Eg. $g(x) = x^2$, $x_0 = \frac{1}{4}$. $\Rightarrow$ Fixed point : $x = 0, 1$ $[x^2 = x]$

$\cancel{g(x_k) = x_{k+1}^2}$

$\cancel{g(x_0) = x_1^2}$

$x_{K+1} = g(x_K) = x_K^2$

$x_1 = x_0^2 = \frac{1}{16} = 0.0625$

$x_2 = (0.0625)^2 = 0.00390625$

$x_3 = 1.5259e-05$

$\vdots$

$x_9 = 5.5627e-309$

$x_{10} = 0$ [Limit of floating point no. $= 10^{-309}$]

$\Downarrow$

converges to 0.

Machine epsilon

Choose x_0 such that it converges to 1.

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Result: Let I be closed and bounded interval.
g is continuous on I, $g(I) \subseteq I$.

$$x \neq y, \quad |g(x) - g(y)| \leq \alpha |x-y|, \quad 0 \leq \alpha < 1$$
$\hookrightarrow$ contraction

$$x_{k+1} = g(x_k), \text{ then } x_k \longrightarrow x^* \text{ (unique)}$$

$$|x_k - x_{k-1}| = |g(x_{k-1}) - g(x_{k-2})| \leq \alpha |x_{k-1} - x_{k-2}|$$
$$= \alpha |g(x_{k-2}) - g(x_{k-3})|$$
$$\leq \alpha^2 |x_{k-2} - x_{k-3}|$$
$$\cdots \leq \alpha^{k-1} |x_1 - x_0|$$

$\{x_k\} \in I \quad (\because g(I) \subseteq I)$

$x_k \longrightarrow x^* \in I$ [converges]

As $x_k = g(x_{k-1})$,

$g(x_k) \longrightarrow x^*$

As g is continuous on I,

$g(x_k) \longrightarrow g(x^*) \; (= x^*)$ [Apply $g(\cdot)$ on both side]

$g(x_k) \longrightarrow x_1^*$
$g(x_k) \longrightarrow x_2^*$

$\Rightarrow x_1^* = x_2^* \Longrightarrow x^*$ is a unique point.

$\therefore g(x^*) = x^*$.

Eg: $x^2 - x - 2 = 0$

$g(x) = x^2 - 2$ | $x_{k+1} = x_k^2 - 2 \; (= g(x_k))$

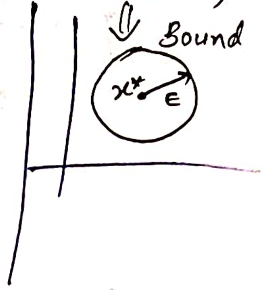
$x_0 = 2.1$ (close to the root) ($x^* = 2$)

x_k	$\|x_k - 2\|$
2.1	1.0e-01 = 0.1
2.41	4.1e-01 = 0.41
3.8081	1.8081
12.5016	10.5016
⋮	⋮
2.3804e+04	2.3802e+04

Result: Suppose x^* is a fixed point of g, $g \in C^1(B_\epsilon(x^*))$ and $|g'(x^*)| < 1$ in $B_\epsilon(x^*)$.

Then x_0 is chosen close to x^*,

$$x_{k+1} = g(x_k) \text{ converges to } x^*$$



$x^* = 2$

$g'(x) = 2x$

$g'(2) = 4 \not< 1$

$\Downarrow$

That's why $|x_k - 2|$ is divergent.

Now, take

$g(x) = \sqrt{x+2}$

$$\left[\begin{array}{l} g(x) = x = \sqrt{x+2} \\ \Rightarrow x^2 = x+2 \end{array}\right]$$

$\cancel{x_{k+1}}$ $g'(x) = \dfrac{1}{2\sqrt{x+2}}$

$\Rightarrow g'(2) = \dfrac{1}{4} < 1 \Rightarrow$ converges.

x_k	$\lvert x_k - 2 \rvert$
2.1	1.0 e−01
2.02484	2.484 e−02
⋮	⋮
2.0001	9.6854e−05

$$\left[\, x_{k+1} = \sqrt{x_k + 2} \,\right]$$

error reduces by a factor of $\frac{1}{4}$ each time.

$\frac{1}{4} = |g'(x^*)|$

Now, for

$g(x) = 1 + \dfrac{2}{x}$

$g'(x) = -\dfrac{2}{x^2}$

$|g'(2)| = \left|\dfrac{-2}{4}\right| = \dfrac{1}{2} < 1 \Rightarrow$ converges

x_k	$\lvert x_k - 2 \rvert$
2.1	1.0 e-01
1.95238	4.762 e-02
2.02439	2.439 e-02
$\vdots$	

$\downarrow$ reduces by $\frac{1}{2}$

$$x_{k+1} = 1 + \frac{2}{x_k}$$

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Eg. $g(x) = 1 + \frac{2}{x}$

$x_0 = 2.1, \quad x^* = 2$

$g(x) = x$

x_k	$\lvert x_k - x^* \rvert$
2.1	1.0 E-01
1.9524	4.7619 E-02
$\vdots$	

$\downarrow$ converges

$$|g'(x^*)| = \left|-\frac{2}{x^2}\right|_{x^* = 2} = \frac{1}{2} < 1$$

Error cuts roughly by 1/2.

Eg. $g(x) = \dfrac{x^2 + 2}{2x - 1}$

$$x_{k+1} = \frac{x_k^2 + 2}{2x_k - 1}$$

$x_0 = 2.1, \quad x^* = 2$

$g(x) = x$

$$\Rightarrow \frac{x^2 + 2}{2x - 1} = x$$

$$\Rightarrow x^2 + 2 = 2x^2 - x$$

$$\Rightarrow x^2 - x - 2 = 0 \Rightarrow x(x-2) + (x-2) = 0 \Rightarrow x = 2, -1$$

x_k	$\lvert x_k - 2 \rvert$
2.1	1.0 E-01
2.0031	3.1250 E-03

$\downarrow$ converges (Quadratic)

$$g(x) = \frac{x^2+2}{2x-1}$$

$$g'(x) = \frac{(2x-1)\,(2x) - (x^2+2)\,(2x-1)}{(2x-1)}$$

$$g'(2) = \frac{(3)\,(6) - (6)(3)}{(2x-1)} = 0$$

↳ Error reduces by a large amount

Rate of convergence

$$\frac{|x_{k+1} - x^*|}{|x_k - x^*|} \leq \underbrace{c}_{\text{constant}} \longrightarrow \text{Linear} \quad [\text{Ratio of errors}]$$

$$\frac{|x_{k+1} - x^*|}{|x_k - x^*|^2} \leq c \longrightarrow \text{Quadratic}$$

 # Result: If $g'(x^*) = 0 \Rightarrow$ Quadratic convergence

Result: Suppose x^* is a fixed point of $g(x)$, $g'(x^*) = 0$, $g \in C^2(B_\epsilon(x^*))$, then

$$x_{k+1} = g(x_k) \text{ converges quadratically to } x^*.$$

Proof: $g(x) = g(x^*) + (x - x^*)\,\cancel{g'(x^*)}^{0} + \frac{(x-x^*)^2}{2} g''(c)$, c lies b/w x and x^*

$$\Rightarrow g(x) = \frac{(x-x^*)^2}{2} g''(c_k) + g(x^*), \quad c_k \text{ lies b/w } x \text{ and } x^*.$$

$$\Rightarrow \frac{x_{k+1} - x^*}{(x_k - x^*)^2} = \frac{1}{2}\underbrace{g''(c_k)}_{\text{finite (as } g \in C^2)} \quad [\because g(x^*) = x^*]$$

$$\Rightarrow \frac{|x_{k+1} - x^*|}{|x_k - x^*|^2} \leq c \quad \text{as } k \to \infty.$$

Newton's Method

$f(x) = 0 \longrightarrow$ Root : x^*

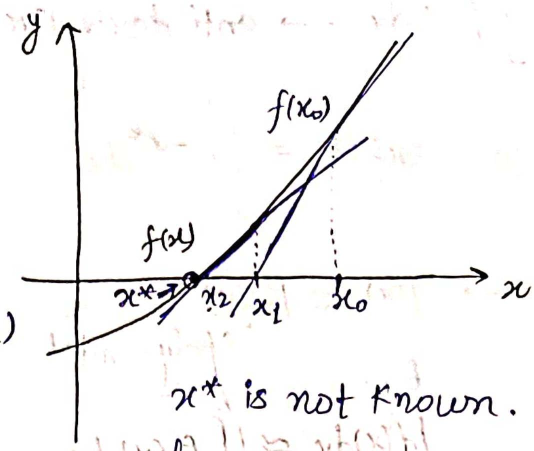
$x_k \xrightarrow{\text{approx.}} x^*$

$y = f(x_k) + (x - x_k) f'(x_k)$

$(x_{k+1}, 0)$

$0 = f(x_k) + (x_{k+1} - x_k) f'(x_k)$

$$\Rightarrow \boxed{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}}$$



x^* is not known.

$|x_{k+1} - x_k| \neq 0 \rightarrow$ very-very small

Result : $f \in C^3(B_\epsilon(x^*))$.

x^* is a root of f, $f'(x^*) \neq 0$.

Then, Newton's method converges quadratically.

\# $\underset{\neq 0}{f'(x^*)} \Rightarrow x^*$ is a simple root.

10-10-2024

$$\int_a^b f(x)\,dx \longrightarrow \text{anti-derivative} \quad \left[\frac{d}{dx}(g(x)) = f(x)\right]$$

Eg. $$\text{erf}(t) = \int_0^t e^{-x^2}\,dx$$

$$\longrightarrow f(x) \approx P_n(x)$$
$\rightarrow$ polynomial

$$\int_a^b f(x)\,dx \approx \int_a^b P_n(x)\,dx$$

Interpolation

x_i:	5	7	8	9	11
$f(x_i)$:	25	3	?	35	12

$\leftarrow$ nodes

? $\downarrow$ Calculate using given values

$\downarrow$ different ways

Lagrange Interpolation:

$$P_n(x) = \sum_{K=0}^{n} f(x_K)\, L_K(x)$$

$\downarrow$ $[a, b]$

$a = x_0 \quad x_1 \quad x_2 \quad \cdots \quad x_n = b$

$$P_n(x_i) = \sum_{K=0}^{n} f(x_i)\, L_K(x_i)$$
$$= \sum_{K=0}^{n} f(x_i)\, \delta_{Ki}$$
$$= f(x_i)$$

$$\left[\because L_K(x_i) = \delta_{Ki} \quad \sum_{K=0}^{n} f(x_i)\delta_{Ki} = f(x_i)\right]$$

Eg. $x_0 = a$ —— $x_1 = b$

$$P_n(x) = f(x_0)\, L_0(x) + f(x_1)\, L_1(x)$$

$$\left.\begin{aligned} L_0(x) &= \frac{x - x_1}{x_0 - x_1} \\ L_1(x) &= \frac{x - x_0}{x_1 - x_0} \end{aligned}\right\} \quad \left[\begin{aligned} &\text{Derive using } L_0(x) = ax + b, \\ &L_0(x_0) = 1,\ L_0(x_1) = 0 \\ &L_1(x_0) = 0,\ L_1(x_1) = 1. \end{aligned}\right]$$

Eg.

$x_0 = a \quad x_1 = \frac{a+b}{2} \quad x_2 = b$

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$$ → quadratic (3 points) [(n+1) points ⇓ n-degree polynomial]

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$P_n(x) = f(x_0)\,L_0(x) + f(x_1)\,L_1(x) + f(x_2)\,L_2(x).$$

→ $\int_a^b f(x)\,dx$

$$\approx \int_a^b P_n(x)\,dx = \int_a^b \left(\sum_{K=0}^{n} f(x_K)\,L_K(x) \right) dx$$

$$\approx \sum_{K=0}^{n} f(x_K) \left(\int_a^b L_K(x)\,dx \right)$$

$$\approx \sum_{K=0}^{n} \underbrace{\omega_K}_{\text{weights}} . \underbrace{f(x_K)}_{\text{points}}$$

→ For $f(x) = 1 \Rightarrow P_n(x) = 1.$

$$\int_a^b P_n(x)\,dx = \sum_{K=0}^{n} \omega_K f(x_K) = \sum_{K=0}^{n} \omega_K = b-a$$

① <u>Mid-point Rule</u>: $[a, b]$

Points : $\frac{a+b}{2}$

weights : $b-a$

Quadrature, $Q(f) = (b-a)\, f\left(\frac{a+b}{2}\right)$ [Numerical integral] Exact of order 1

$I(f) = \int_a^b f(x)\,dx$ [Exact integral]

Eg. $f(x) = \alpha x + \beta$.

$$Q(f) = (b-a)\left[\alpha\left(\frac{a+b}{2}\right) + \beta\right]$$

$$I(f) = \int_a^b (\alpha x + \beta)\,dx$$

$$= \alpha \frac{x^2}{2}\Big|_a^b + \beta x\Big|_a^b$$

$$= \alpha\left(\frac{b^2-a^2}{2}\right) + \beta(b-a)$$

$$= (b-a)\left[\alpha\frac{(a+b)}{2} + \beta\right].$$

$$= Q(f).$$

Trapezoidal Rule:

Points: $x_0 = a, \quad x_1 = b$

$$w_0 = \int_a^b L_0(x)\,dx = \int_a^b \left(\frac{x - x_1}{x_0 - x_1}\right) dx.$$

$$= \frac{\frac{x^2}{2} - x_1 . x}{(x_0 - x_1)}\Bigg|_a^b$$

$$= \left[\frac{b^2-a^2}{2} - (b-a)x_1\right]\frac{1}{x_0 - x_1} = \frac{b-a}{x_0 - x_1}\left[\frac{b+a}{2} - x_1\right]$$

$$= \frac{b-a}{2}$$

$$w_1 = \int_a^b L_1(x)\,dx = \left[\frac{b^2-a^2}{2} - (b-a)x_0\right]\frac{1}{x_1 - x_0}.$$

$$= \frac{b-a}{x_1 - x_0}\left[\frac{b+a}{2} - x_0\right].$$

$$= \frac{b-a}{2}$$

$$\therefore\ Q(f) = \frac{b-a}{2}\left[f(a) + f(b)\right]$$

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Simpson's one-third Rule:

Points: $x_0 = a,\ x_1 = \frac{a+b}{2},\ x_2 = b$

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$P_2(x) = f(x_0)L_0(x) + f(x_1)L_1(x) + f(x_2)L_2(x)$$

weights:

$$w_0 = \int_a^b L_0(x)\,dx = \frac{1}{(x_0-x_1)(x_0-x_2)} \int_a^b (x-x_1)(x-x_2)\,dx$$

$$\int_{a-x_1}^{b-x_1} y(y+x_1-x_2)\,dy$$

$x-x_1 = y$
$\Rightarrow x = y + x_1$
$x - x_2 = y + x_1 - x_2$

$$\int_{a-x_1}^{b-x_1} \left[y^2 - y(x_1 - x_2)\right] dy$$

$$\frac{1}{(x_0-x_1)(x_0-x_2)} \left[\frac{(b-x_1)^3 - (a-x_1)^3}{3} - \left[\frac{(b-x_1)^2 - (a-x_1)^2}{2}\right](x_1-x_2) \right]$$

$$= \frac{1}{\cancel{(x_0-a)(x_0-b)}\ \left(\frac{a-b}{2}\right)(a-b)} \left[\frac{\left(b-\frac{a+b}{2}\right)^3 - \left(a-\frac{a+b}{2}\right)^3}{3} - \left(\frac{a-b}{2}\right)\left(\frac{\left(\frac{b-a}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2}{2}\right) \right]$$

$$\omega_0 = \int_a^b L_0(x)\,dx = \frac{b-a}{6}$$

$$\omega_1 = \int_a^b L_1(x)\,dx = \frac{2}{3}(b-a)$$

$$\omega_2 = \int_a^b L_2(x)\,dx = \frac{b-a}{6}$$

$x_0 = a,\ x_1 = \frac{a+b}{2},\ x_2 = b$

$= x_0 + h \quad = x_1 + h$

$\quad\quad = x_0 + 2h$

$$\left[\begin{array}{l} x_1 - x_0 = \left(\frac{b-a}{2}\right) \\ x_2 - x_0 = \left(\frac{b-a}{2}\right) \end{array}\right.$$

$$\left\{ h = \left(\frac{b-a}{2}\right)\right.$$

$$Q(f) = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$= \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Simpson's $3/8^{th}$ Rule:

$x_0 = a,\ x_1 = x_0 + h,\ x_2 = x_0 + 2h,\ x_3 = x_0 + 3h;\ h = \frac{a-b}{3}$

$$Q(f) = \frac{b-a}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right]$$

Eg. Approximate $\int_1^3 \ln x\,dx$ using mid-point, trapezoidal and simpson's $1/3^{rd}$ rule.

Exact $= \left[x\ln x - x\right]_1^3$

$= 3\ln 3 - 2$

$= 1.29583687$

Mid-point rule:

$(b-a)\,f\left(\frac{a+b}{2}\right)$

$= 2f(2) = 2\ln 2$

$= 1.38629436$

Trapezoidal rule:

$\ln 1 + \ln 3 = 1.098612$

Simpson's Rule:

$\frac{2}{6}\left[\ln(1) + 4\ln\left(\frac{4}{2}\right) + \cancel{\ldots} + f(3)\right] = 1.29040034.$

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→ Increasing the power of the polynomial increases the error.

$\Downarrow$ source of error

Triangle rule: a , b

Trapezoidal rule: a, $\frac{a+b}{2}$, b

} The points are equidistant (same 'h').

$\Downarrow$

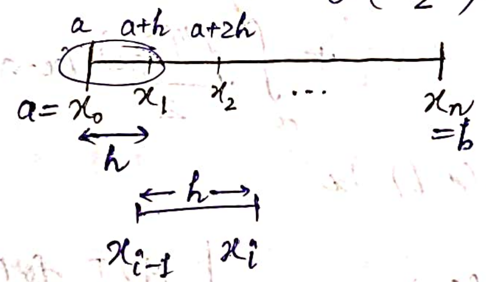
Use unequally spaced points.

(Legendary Polynomial) $\Rightarrow$ complex.

OR

Composite Midpoint Rule

Midpoint rule: $(b-a) f\left(\frac{a+b}{2}\right)$.



$\Rightarrow$ Make partitions and apply mid-point rule on the partitions.

(Equally spaced)

$$h f\left(a+\frac{h}{2}\right) + h f\left(a+3\left(\frac{h}{2}\right)\right) + \dots$$

$[a, a+h], [a+h, a+2h], \dots, [b-h, b]$ → l intervals

$$h \sum_{K=0}^{l-1} f\left(a + (2K+1)\frac{h}{2}\right)$$

Composite Trapezoidal Rule (CTR)

$$\left(\frac{b-a}{2}\right)\left(f(a)+f(b)\right) = \frac{h}{2}\left(f(a)+f(b)\right)$$

$$\frac{h}{2}\left(f(a)+f(a+h)\right) + \frac{h}{2}\left(f(a+h)+f(a+2h)\right) + \dots$$

$$= \frac{h}{2}\left[f(a) + 2\sum_{K=1}^{l-1} f(a+h) + f(b)\right]$$

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Error

Trapezoidal Rule: $\frac{h}{2}\left[f(a)+f(b)\right]$

Result: $f \in C^{n+1}([a,b])$, $\{x_i\}_{i=0}^{n}$

$$E(x) = f(x) - P_n(x) = \frac{f^{n+1}(\xi_x)}{(n+1)!}\prod_{i=0}^{n}(x-x_i)\ ; \quad \xi \in [a,b]$$

$\downarrow$ error

Trapezoidal Rule:

$a \quad b$ $\rightarrow$ Linear polynomial

$$\int_a^b f(x)\,dx = \int_a^b \Big[\underbrace{f(a)\left(\frac{x-b}{a-b}\right) + f(b)\left(\frac{x-a}{b-a}\right)}_{P_1(x)}\Big]dx \leadsto \frac{h}{2}\left[f(a)+f(b)\right]$$

$$+ \int_a^b \frac{f''(\xi_x)}{2}(x-a)(x-b)\,dx$$

$$\therefore E(x) = \int_a^b \frac{f''(\xi_x)}{2}(x-a)(x-b)\,dx$$

$$= \frac{f''(\xi)}{2}\int_a^b (x-a)(x-b)\,dx$$

$$= \frac{f''(\xi)}{2}\left[\frac{x^3}{3} - \frac{x^2}{2}(a+b) + abx\right]_a^b$$

$$= \frac{f''(\xi)}{2}\left[\frac{b^3-a^3}{3} - \frac{(b^2-a^2)(a+b)}{2} + ab(b-a)\right]$$

M.V.T. for integrals:

$$\int_a^b f(x)\,g(x)\,dx = g(\xi)\int_a^b f(x)\,dx, \quad \xi \in [a,b]$$

$$= \frac{f''(\xi)(b-a)}{2}\left[\frac{(a^2+b^2+ab)}{3} - \frac{(a+b)^2}{2} + ab\right]$$

$$= \frac{f''(\xi)\,(h)}{2}\left[\frac{2a^2+2b^2+2ab-3a^2-3b^2-6ab+6ab}{6}\right]$$

$$= \frac{f''(\xi)\cdot h}{2}\left[\frac{a^2+b^2-2ab}{6}\right]$$

$$= -f''(\xi)\frac{h^3}{12}$$

$$|E(x)| \approx ch^3$$

Simpson's 1/3rd Rule :

$$\frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$\int_a^b f(x)\,dx = \int_a^b f(x_1 + (x - x_1))\,dx$$

$$= \int_a^b \left[f(x_1) + (x - x_1) f'(x_1) + \frac{(x-x_1)^2}{2} f''(x_1) + \frac{(x-x_1)^3}{6} f'''(x_1) + \frac{(x-x_1)^4}{24} f^{IV}(\xi) \right] dx \qquad (\xi \in [a,b])$$

$$= 2h f(x_1) + \cancelto{0}{\left(\frac{(x-x_1)^2}{2}\right)_a^b} f'(x_1) + \left(\frac{(x-x_1)^3}{6}\right)_a^b f''(x_1) + \cancelto{0}{\left(\frac{(x-x_1)^4}{24}\right)_a^b} f'''(x_1) + \left(\frac{(x-x_1)^5}{120}\right)_a^b f^{IV}(\xi)$$

$$= 2h f(x_1) + \frac{h^3}{3} f''(x_1) + \frac{h^5}{60} f^{IV}(\xi_2)$$

$$\left[f''(x_1) = \frac{f(x_0) - 2f(x_1) + f(x_2)}{h^2} \underbrace{- \frac{h^2}{12} f^{IV}(\xi_1)}_{\text{error}} \right]$$

$$\left.\begin{aligned} f(x_0) &= f(x_1 - h) \\ f(x_2) &= f(x_1 + h) \end{aligned}\right\} \text{expand using taylor's series}$$

$$= 2h f(x_1) + \frac{h^3}{3}\left[\frac{f(x_0) - 2f(x_1) + f(x_2)}{h^2} - \frac{h^2}{12} f^{IV}(\xi_1)\right] + \frac{h^5}{60} f^{IV}(\xi_2)$$

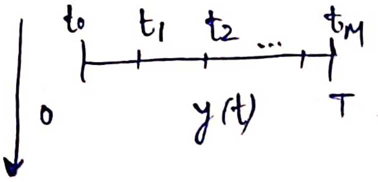
$$= \underbrace{\frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]}_{\text{Simpson's rule}} - \underbrace{\frac{h^5}{90} f^{IV}(\xi_1)}_{\text{error}}$$

$$\left[f^{IV}(\xi) = \max{}^m\left(f^{IV}(\xi_1), f^{IV}(\xi_2)\right) \right]$$

↳ Better estimation

23-10-2024

$y'(t) = f(t, y(t)), \quad t \in [0, T]$



Approximate soln:

$y_0 \approx y(t_0)$

$\{(t_0, y_0), (t_1, y_1), \ldots, (t_M, y_M)\}$

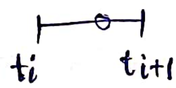
$[\underset{\text{approx.}}{y_0} \approx \underset{\text{exact}}{y(t_0)}]$

$\approx y(t_0)$ ~exact

$\neq y(t_M)$

$$\int_{t_i}^{t_{i+1}} y'(t)\,dt = \int_{t_i}^{t_{i+1}} f(t, y(t))\,dt$$

Reimann integral:



$\Delta t_i \cdot f(t_i^*)$

$y(t_{i+1}) - y(t_i) = \Delta t_i \cdot f(t_i, y(t_i))$

$y_{i+1} - y_i = \Delta t \cdot f(t_i, y_i)$ $\quad [\because t_1 - t_0 = t_2 - t_1 = \ldots = t_M - t_{M-1}]$

$\Rightarrow \boxed{y_{i+1} = y_i + \Delta t \cdot f(t_i, y_i)}$: Forward Euler

$\Rightarrow \dfrac{y_{i+1} - y_i}{\Delta t} = f(t_i, y_i) \quad - (1)$

$y'(t_i) = f(t_i, y(t_i))$

$\downarrow$

$$\frac{y(t_{i+1}) - y(t_i)}{\Delta t} - \frac{\Delta t}{2} y''(t_i^*) = f(t_i, y(t_i)) \quad - (2)$$

(2) − (1) $\Rightarrow$

$$\frac{e_{i+1} - e_i}{\Delta t} - \frac{\Delta t}{2} y''(t_i^*) = f(t_i, y(t_i)) - f(t_i, y_i)$$

$[M = Max\, |y''(t)|, \; t \in [0, T]]$

$y(t_i + \Delta t) = y(t_i) + \Delta t\, y'(t_i) + \frac{\Delta t^2}{2} y''(t_i^*),$

$t_i^* \in [t_i, t_{i+1}]$

$e_i = y(t_i) - y_i$: error

$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$

for $L > 0 \Rightarrow$ then f is Lipchitz

$$e_{i+1} = e_i + \frac{\Delta t^2}{2} y''(t_i^*) + \left(f(t_i, y(t_i)) - f(t_i, y_i)\right)\Delta t$$

$\downarrow$ Lipchitis condn

$$|e_{i+1}| \le |e_i| + M\frac{\Delta t^2}{2} + \Delta t.L\,|y(t_i)-(y_i)|$$

$$\le |e_i| + M\frac{\Delta t^2}{2} + \Delta t.L\,|e_i|$$

$$\le (1+L\Delta t)\,|e_i| + M\frac{\Delta t^2}{2}$$

$$\boxed{\begin{array}{l}\{x_i\},\ a\&b:\text{non-negative}\\ x_{i+1} \le a x_i + b\\ \Rightarrow x_{i+1} \le a^i x_0 + \dfrac{a^i-1}{a-1}.b\,.\end{array}}$$

$$|e_{i+1}| \le (1+L\Delta t)^i |e_0| + \frac{(1+L\Delta t)^i - 1}{(1+L\Delta t)-1}\left(M\frac{\Delta t^2}{2}\right)$$

$$\le \cancel{(1+L\Delta t)^i |e_0|}^{\,0} + \frac{(1+L\Delta t)^i}{L\cancel{\Delta t}}\left(\frac{M\Delta t^{\cancel{2}}}{2}\right)$$

$e_0 = y(t_0) - y_0$ initial condn given $= 0$ (no approximation)

$$\Rightarrow |e_{i+1}| \le \frac{e^{LT}}{2L} M\Delta t \rightarrow \text{(no 'i' this side)}$$

$t_0, t_1, t_2, \dots, t_N$

$$(1+L\Delta t)^i \le e^{iL\Delta t} \le e^{NL\Delta t} \le e^{LT}$$

(replace i by N)

Take maxm over i both the side,

$$\Rightarrow \max_{1\le i\le N}\{|e_i|\} \le \frac{e^{LT}}{2L} M\Delta t$$

Other method:

$$y(t_{i+1}) - y(t_i) = \int_{t_i}^{t_{i+1}} f(t, y(t))\,dt$$

$\downarrow$ approx. by trapezoidal rule

: Heün's Method

$$y_{i+1} - y_i = \frac{\Delta t}{2}\left[f(t_i, y_i) + f(t_{i+1}, y_{i+1})\right]$$

$\downarrow$ forward euler

$$= \frac{\Delta t}{2}\left[f(t_i, y_i) + f\left(t_{i+1}, y_i + \Delta t. f(t_i, y_i)\right)\right]$$