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CALCULUS OF VARIATIONS

05-04-2024

Maxima / minima of functions

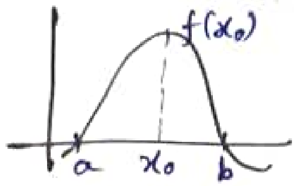
$S \subseteq \mathbb{R}$, $f : S \to \mathbb{R}$

If x_0 ($x_0 \in S$) is a maxima point of f if

$$f(x_0) \geq f(x) \quad \forall x \in S.$$

The point $x_0 \in S$ is minima point of f if

$$f(x_0) \leq f(x) \quad \forall x \in S.$$



Weierstrass Theorem : (Sufficient condition)

$S \subseteq \mathbb{R}$, $f : S \to \mathbb{R}$.

Suppose (i) S is closed and bounded set. $\to [a,b]$ $(-\infty < a, b < \infty)$

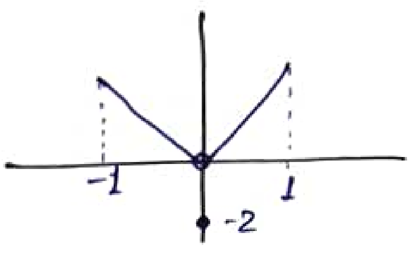
(ii) $f : S \to \mathbb{R}$ is continuous.

Then, f has a maxima / minima point in S.

It gives only sufficient conditions but not necessary.

Eg. $f : [-1, 1] \to \mathbb{R}$.

$$f(x) = \begin{cases} -2, & x = 0 \\ |x|, & x \neq 0. \end{cases}$$

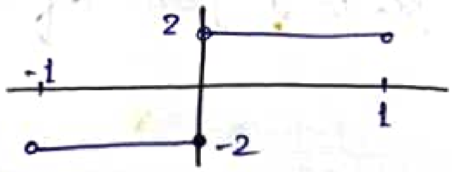


→ f is NOT continuous at $x = 0$.

But f has max at $x = +1, -1$, min at $x = 0$.

Eg. $f : (-1, 1) \to \mathbb{R}$

$$f(x) = \begin{cases} 2, & x > 0 \\ -2, & x \leq 0. \end{cases}$$

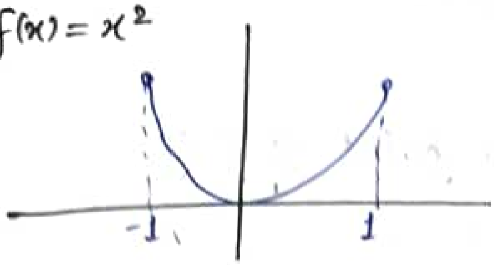


→ conditions (i) & (ii) of weierstrass theorem are not satisfied.

→ f has max 2 and min -2.

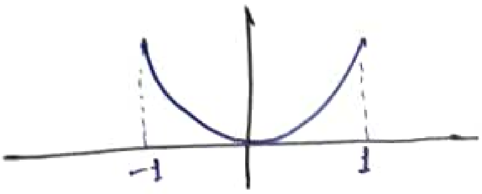
Eg. $f : (-1,1) \to \mathbb{R}$

$f(x) = x^2$



→ No max for f, min at $x=0$

$f : [-1,1] \to \mathbb{R}$



→ max at $-1, 1$, min at $x=0$.

Necessary Point

$S \subseteq \mathbb{R}$, $f : S \to \mathbb{R}$, f is differentiable on S.

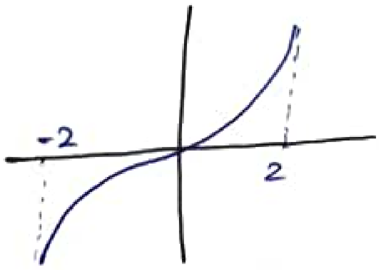
$x_0 \in S$, interior point.

If x_0 is maxima/minima point, then $f'(x_0) = 0$.

→ x is called stationary point.

Converse is NOT true.

Eg. $f : (-2,2) \to \mathbb{R}$, $f(x) = x^3$.



$f'(x_0) = 3x_0^2$

$x_0 = 0 \Rightarrow f'(x_0) = 0$

But x_0 is NOT max/min of f.

→ If $f''(x_0) > 0$, then x_0 is a point of minima.

$f''(x_0) < 0$, then x_0 is a point of maxima.

Multivariable

$f: \mathbb{R} \to \mathbb{R}$

$x_0 \in \mathbb{R}$ and f has all partial derivatives.

If x_0 is a maxima/minima point of f.

$$\left.\frac{\partial f}{\partial x_1}\right|_{x=x_0} = 0, \quad \left.\frac{\partial f}{\partial x_2}\right|_{x=x_0} = 0, \ \dots, \ \left.\frac{\partial f}{\partial x_n}\right|_{x=x_0} = 0.$$

$\underline{n=3}$

$$M = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{bmatrix}$$

↳ Hessiam matrix.

If M is positive definite matrix

$$x^T M x > 0 \ \forall \ x \in \mathbb{R}^n.$$

11-04-2024

Functional

$C[0,1] \to$ space of continuous function.

Let S be a space of functions.

Then, the functional $I: S \to \mathbb{R}$,

$$I(y) = \int_{0\,(x_0)}^{1\,(x_1)} F(x,y)\, dx, \quad \forall\, y \in S.$$

Eg. $I: C[0,1] \to \mathbb{R}$.

$$I(y) = \int_0^1 y\, dx, \quad y \in C[0,1].$$

$y(x)$	$I(y(x))$
x	0.5
x^2	0.333
$\sin x$	0.4597
e^x	1.71
1	1

$I(y)$ attains $\max^m$ at $y = e^x$,

$\min^m$ at $y = x^2$.

→ $C^1[x_0, x_1]$: Space of continuously differentiable function.

Let $I : C^1[x_0, x_1] \longrightarrow \mathbb{R}$

$$I(y) = \int_{x_0}^{x_1} F(x, y, y')\,dx, \quad \forall\, y \in C^1[x_0, x_1],$$

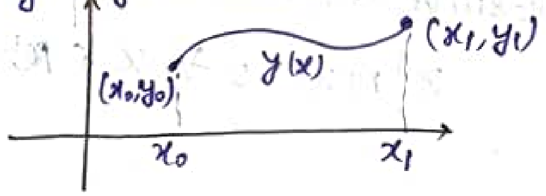
Subject to $y(x_0) = y_0$, $y(x_1) = y_1$.

Calculus of Variations (COV)

Find a function $y \in C^1[x_0, x_1]$ such that if maxima/minima $I(y)$ subject to

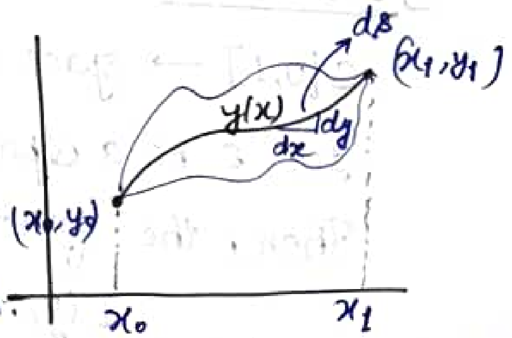
$$y(x_0) = y_0, \; y(x_1) = y_1.$$

The function y is called extremizing function, the corresponding $I(y)$ is called extremum.



Motivation

Find a plane curve joining (x_0, y_0) and (x_1, y_1), having the shortest distance.



[Infinitely many such curves possible]

Let ds be the infinitesimal arc length:

$$ds = \sqrt{dx^2 + dy^2}$$

Total arc length of the curve,

$$I = \int_{x_0}^{x_1} ds \qquad \Bigg| \qquad \frac{ds}{dx} = \sqrt{1 + y'^2}, \quad y' = \frac{dy}{dx}.$$

$$\Rightarrow I(y) = \int_{x_0}^{x_1} \sqrt{1 + y'^2}\,dx$$

Find a minima of $I(y)$ such that

$y(x_0) = y_0$, $y(x_1) = y_1$.

Euler-Lagrange Equation (Necessary condition ~~of~~ for extrema)

Let y be the extremum of

$$I(y) = \int_{x_0}^{x_1} F(x, y, y')\, dx$$

Subject to $y(x_0) = y_0$, $y(x_1) = y_1$.

Then, y solves the E-L eq^n

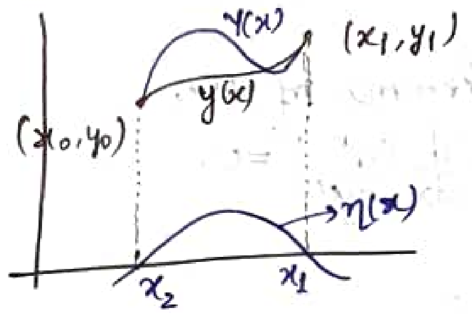
$$\underbrace{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0}_{\text{E-L eq}^n}, \quad y(x_0) = y_0, \quad y(x_1) = y_1.$$

Proof:

Let $Y(x) = y(x) + \epsilon \eta(x)$

where $\eta \in C^1[x_0, x_1]$, such that

$$\eta(x_0) = \eta(x_1) = 0.$$



Since y is an extremum of $I(y)$,
$I(\epsilon)$ will attain extremum at $\epsilon = 0$.

$$\Rightarrow \left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = 0.$$

$$I(Y) = \int_{x_0}^{x_1} F(x, Y, Y')\, dx.$$

$$\Rightarrow \frac{dI}{d\epsilon} = \frac{d}{d\epsilon}\int_{x_0}^{x_1} F(x, Y, Y')\, dx.$$

$$= \int_{x_0}^{x_1}\left[\frac{\partial F}{\partial x}\overbrace{\frac{dx}{d\epsilon}}^{0\ (x \text{ is indep. of } \epsilon)} + \frac{\partial F}{\partial Y}\overbrace{\frac{dY}{d\epsilon}}^{\eta(x)} + \frac{\partial F}{\partial Y'}\overbrace{\frac{dY'}{d\epsilon}}^{\eta'(x)}\right] dx$$

$$= \int_{x_0}^{x_1}\left[\frac{\partial F}{\partial Y}\cdot\eta + \frac{\partial F}{\partial y'}\cdot\eta'\right] dx \qquad \Bigg| \quad Y' = y' + \epsilon\eta' \Rightarrow \frac{dY'}{d\epsilon} = \eta'$$

$$\left[\int_{x_0}^{x_1}\frac{\partial F}{\partial y'}\,d(\eta)=\cancel{\frac{\partial F}{\partial y'}\eta\Big|_{x_0}^{x_1}}^{\,0}-\int_{x_0}^{x_1}\eta\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]$$

$$\Rightarrow 0=\frac{\partial I}{\partial \epsilon}\Big|_{\epsilon=0}=\int_{x_0}^{x_1}\left[\frac{\partial F}{\partial y}\Big|_{\epsilon=0}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\Big|_{\epsilon=0}\right]\eta\,dx$$

$$=\int_{x_0}^{x_1}\underbrace{\left[\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]}_{\text{continuous}}\eta\,dx=0 \qquad \left[\because \text{At } \epsilon=0,\ y_\epsilon=y\right]$$

[
Fundamental Lemma of COV

Let f be a continuous function in $[a,b]$ such that

$$\int_a^b f(x)g(x)\,dx=0$$

$\forall$ ~~where~~ $g\in C^1[a,b]$, $g(a)=g(b)=0$.

Then $f(x)\equiv 0\ \forall\ x\in[a,b]$.
]

By fundamental lemma of COV,

$$\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0.$$

12-04-2024

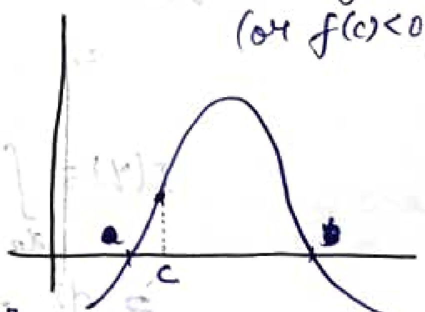
Proof of fundamental lemma:

Suppose $f(c)\neq 0$ for some $c\in(a,b)$. Further, $f(c)>0$ (or $f(c)<0$)

Since f is continuous on $[a,b]$,

$\exists\ [x_1,x_2]\subset(a,b)$

such that $f(x)>0\ \forall\ x\in(x_1,x_2)$.



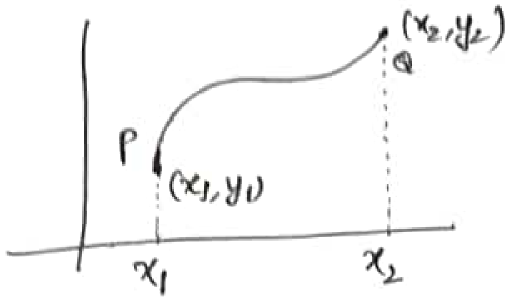
Let $g(x)=\begin{cases}(x-x_1)(x_2-x), & \forall\ x\in[x_1,x_2]\\ 0, & \forall\ x\notin[x_1,x_2].\end{cases}$

$$\int_a^b f(x)g(x)\,dx=\cancel{\int_a^{x_1}fg}^{\,0}+\int_{x_1}^{x_2}fg+\cancel{\int_{x_2}^{b}fg}^{\,0}$$

>0 [as $f\cdot g>0$ in (x_1,x_2)]

It is a contradiction.
Hence, $f(c)=0 \ \forall c \in (a,b)$.

Eg.



Find a curve joining P and Q having shortest distance.

$$I(y) = \int_{x_1}^{x_2} \underbrace{\sqrt{1+y'^2}}_{F}\, dx \quad , \quad F(x,y,y') = \sqrt{1+y'^2}.$$

$\to$ Find y' which minimizes $I(y)$.

Soln: Let y be the minimizer of $I(y)$.

By E-L eqn;

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0.$$

As $\frac{\partial F}{\partial y} = 0$ $\quad (\because$ F is indep. of $y)$

$$\Rightarrow \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\Rightarrow \frac{\partial F}{\partial y'} = c \Rightarrow \frac{y'}{\sqrt{1+y'^2}} = c.$$

$$\Rightarrow \frac{y'^2}{c^2} = 1+y'^2$$

$$\Rightarrow y'^2\left(\frac{1}{c^2}-1\right) = 1 \Rightarrow y' = \sqrt{\frac{c^2}{1-c^2}} = m.$$

$$\Rightarrow y = mx+b$$

As $y_1 = y(x_1)$, $y(x_2) = y_2$

$$\Rightarrow y_1 = mx_1 + b$$
$$y_2 = mx_2 + b$$

$$\Rightarrow m = \frac{y_1-y_2}{x_1-x_2}.$$

Eg. Find the extremum of $\int_{x_0}^{x_1} \frac{y'^2}{x^3}\,dx$.

Sol^n: let $F(x,y,y') = \frac{y'^2}{x^3}$.

As $\frac{\partial F}{\partial y} = 0$

$$\Rightarrow \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\Rightarrow \frac{d}{dx}\left(\frac{2y'}{x^3}\right) = 0$$

$$\Rightarrow \frac{2y'}{x^3} = c$$

OR

$$\frac{2x^3 y'' - 3x^2(2y')}{x^6} = 0$$

$$\Rightarrow xy'' = 3y'$$

$$\Rightarrow \int \frac{y''}{y'} = \int \frac{3}{x} + c$$

$$\Rightarrow \ln(y') = 3\ln(x) + c.$$

$$\Rightarrow \ln\left(\frac{y'}{x^3}\right) = c \Rightarrow \frac{y'}{x^3} = e^c = c_1$$

$$\Rightarrow y' = c_1 x^3$$

$$y = c_1 \frac{x^4}{4} + c_2$$

OR, $y = Ax^4 + B$

$\hookrightarrow$ Find A, B.

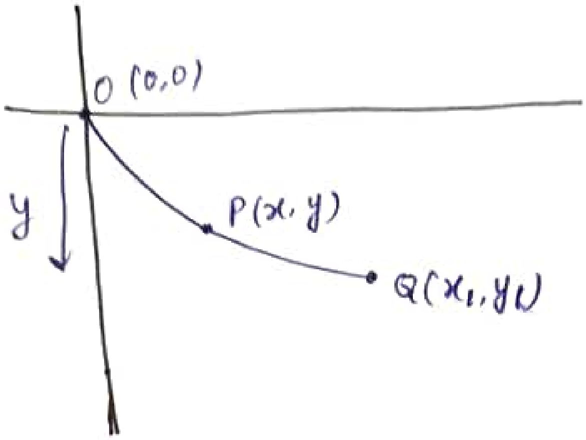
Eg Find the extremum of:

$$\int_0^1 (y'^2 + 12xy)\,dx,$$

$$y(0) = 0,\ y(1) = 1.$$

Brachistochrone Problem

↳ Shortestest time



A particle is moving under gravity with its mass.
Find the curve for the shortest time from O to Q.

17-04-2024

Conservation of energy,

$$\frac{1}{2}mv^2 = mgy$$

Arc length, $OP = s$

and, $v = \frac{ds}{dt}$

$$\left(\frac{ds}{dt}\right)^2 = 2gy$$

$$\frac{ds}{dt} = \sqrt{2gy}$$

(Total) time taken by mass,

$$T = \int_0^{x_1} dt$$

$$= \int_0^{x_1} \frac{ds}{\sqrt{2gy}}$$

$$\Rightarrow T(y) = \int_0^{x_1} \frac{\sqrt{1+y'^2}\,dx}{\sqrt{2gy}}$$

$$\therefore F(x, y, y') = \frac{\sqrt{1+y'^2}}{\sqrt{y}}$$

E-L eqn: $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

By Beltrami identity: $F - y'\frac{\partial F}{\partial y'} = c.$

$$\Rightarrow \frac{\sqrt{1+y'^2}}{\sqrt{y}} - \frac{y'}{\sqrt{y}}\frac{y'}{\sqrt{1+y'^2}} = c$$

$$\Rightarrow \frac{1}{\sqrt{y}} \frac{(\sqrt{1+y'^2})^2 - (y')^2}{\sqrt{1+y'^2}} = c$$

$$\Rightarrow \frac{1}{\sqrt{y}\sqrt{1+y'^2}} = c$$

$$\Rightarrow \sqrt{y}\sqrt{1+y'^2} = \frac{1}{c} = \sqrt{a}$$

$$\Rightarrow y(1+y'^2) = a$$

$$\Rightarrow \frac{dy}{dx} = \sqrt{\frac{a-y}{y}} \Rightarrow \sqrt{\frac{y}{a-y}}\, dy = dx$$

$$\Rightarrow x = \int \frac{\sqrt{y}}{\sqrt{a-y}}\, dy + c_1$$

Curve passes through $(0,0) \Rightarrow c_1 = 0$.

Take $y = a\sin^2\theta \Rightarrow dy = 2a\sin\theta\cos\theta\, d\theta$, $0<\theta<\pi/2$.

$$x = \int_0^\theta \sqrt{\frac{a\sin^2\theta}{a-a\sin^2\theta}} \cdot 2a\sin\theta\cos\theta\, d\theta.$$

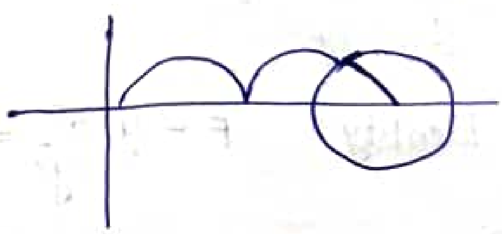
$$= \int_0^\theta 2a\sin^2\theta\, d\theta$$

$$= 2a\int_0^\theta \left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= a\left(\theta - \frac{\sin 2\theta}{2}\right) = \frac{a}{2}(2\theta - \sin 2\theta)$$

Take $b = \frac{a}{2}$, $2\theta = \phi$

$$\Rightarrow \left.\begin{aligned} x &= b(\phi - \sin\phi), \\ y &= \frac{a}{2}(1-\cos 2\theta) \\ &= b(1-\cos\phi) \end{aligned}\right\} \text{cycloid}$$



→ E-L eqⁿ:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad , \quad F = F(x, y, y')$$

$$\frac{dF}{dx} = \frac{\partial F}{\partial x}\cancel{\frac{dx}{dx}} + \frac{\partial F}{\partial y}\underbrace{\frac{dy}{dx}}_{y'} + \frac{\partial F}{\partial y'}\underbrace{\left(\frac{dy'}{dx}\right)}_{y''} \quad \text{-- (1)}$$

$$\frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right) = y'\underbrace{\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)}\ + \frac{\partial F}{\partial y'}y''$$

$$= y'\frac{\partial F}{\partial y} + \frac{\partial F}{\partial y'}y'' \quad \text{-- (2)}$$

$$(1) - (2) \Rightarrow \boxed{\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial x} = 0}$$

↳ Equivalent form of E-L eqⁿ

Case: (i) F is independent of x.

$$\frac{\partial F}{\partial x} = 0 \Rightarrow \frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) = 0$$

$$\Rightarrow \boxed{F - y'\frac{\partial F}{\partial y'} = c} \quad \text{... Beltrami identity.}$$

(ii) F is independent of y.

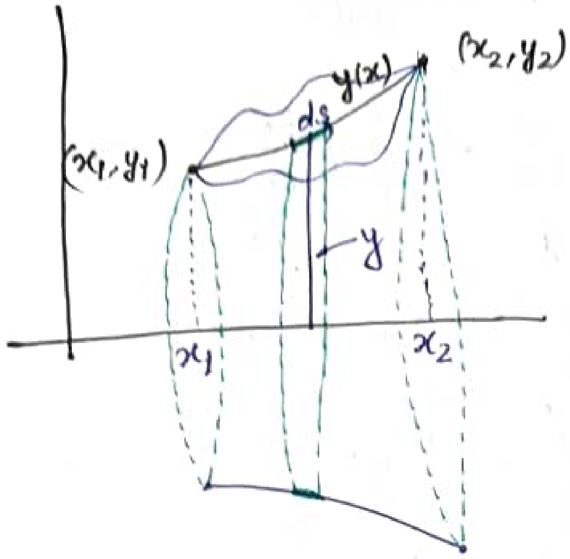
$$\frac{\partial F}{\partial y} = 0 \Rightarrow \frac{\partial F}{\partial y'} = c$$

(iii) F is independent of y'.

$$\frac{\partial f}{\partial y'} = 0 \Rightarrow \text{E-L eq}^n\text{:}$$

$$\boxed{\frac{\partial F}{\partial y} = 0}$$

Minimal Surface Area of Revolution



What should be the curve, $y(x)$, so that it generates minm surface area when rotated about x-axis?

ds : small strip on $y(x)$.

Surface area generated by $ds = 2\pi y \, ds$

$$\text{Total surface area} = \int_{x_1}^{x_2} 2\pi y \, ds$$

$$I(y) = \int_{x_1}^{x_2} 2\pi y \sqrt{1+y'^2} \, dx .$$

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$$F(x, y, y') = y \sqrt{1+y'^2}$$

Beltrami - identity:

$$F - y' \frac{\partial F}{\partial y'} = c$$

$$\Rightarrow y\sqrt{1+y'^2} - y' \frac{y y'}{\sqrt{1+y'^2}} = c \Rightarrow \frac{y(1+y'^2 - y'^2)}{\sqrt{1+y'^2}} = c$$

$$\Rightarrow y' = \frac{\sqrt{y^2 - c^2}}{c}$$

$$\Rightarrow \int \frac{dy}{\sqrt{y^2 - c^2}} = \int \frac{dx}{c} + c_1$$

$$\Rightarrow \cosh^{-1}\left(\frac{y}{c}\right) = \frac{x}{c} + c_1$$

$$= \frac{x + a}{c}$$

$$\Rightarrow y = c \cosh\left(\frac{x+a}{c}\right) \longrightarrow \text{Catenary}$$

Isoperimetric Problem

↳ Max/min $I(y)$ subject to

$$y(x_1)=y_1,\ y(x_2)=y_2,$$

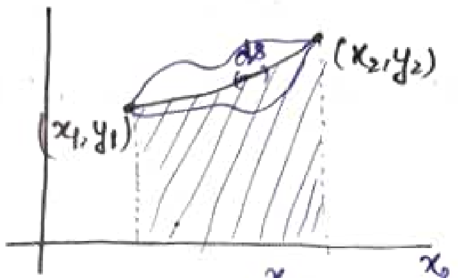
along with an integral constraint.

Book

↳ L. Komzsik, Advanced COV for engineers

↳ CR Wylie and L.C. Barrett Advanced Engineering Maths. (for problems)

Dido's Problem



$$\text{Arc length} = \int_{x_1}^{x_2} ds = \int_{x_1}^{x_2} \sqrt{1+y'^2}\, dx = L$$

↳ fixed

Area under the curve,

$$I(y) = \int_{x_1}^{x_2} y(x)\, dx$$

Maximise $I(y)$ subject to $y(x_1)=y_1,\ y(x_2)=y_2,$

and $\int_{x_1}^{x_2} \sqrt{1+y'^2}\, dx = L.$

General Problem:

Extremize $I(y) = \int_{x_1}^{x_2} F(x,y,y')\, dx$

subject to $\int_{x_1}^{x_2} G(x,y,y')\, dx = L,$

↳ L depends on y.

and, $y(x_1)=y_1,\ y(x_2)=y_2.$

Lagrange Method:

Define $H(x,y,y') = F(x,y,y') + \lambda G(x,y,y')$, λ is a parameter (multiplier).

↳ constant

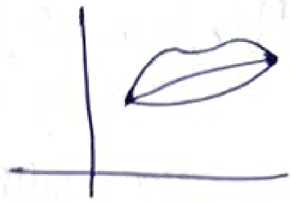
Extremize $\tilde{I}(y) = \int_{x_1}^{x_2} H(x,y,y')\, dx$

subject to $y(x_1)=y_1,\ y(x_2)=y_2.$

$$\Rightarrow H(x,y,y') = F(x,y,y') + \lambda\sqrt{1+y'^2}$$

$$= y + \lambda\sqrt{1+y'^2}$$

Suppose y is an extremum of $\tilde{I}(y)$.



E-L eqn: $\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0,$

$y(x_0) = y_0, \; y(x_1) = y_1.$

$\Rightarrow \lambda \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 1 \Rightarrow \lambda \frac{y'}{\sqrt{1+y'^2}} = x + a$

$\Rightarrow \lambda^2 y'^2 = (1+y'^2)(x+a)^2 \Rightarrow y'^2(\lambda^2 - (x+a)^2) = (x+a)^2$

$\Rightarrow y' = \frac{x+a}{\sqrt{\lambda^2 - (x+a)^2}}$

$\Rightarrow \int dy = \int \frac{x+a}{\sqrt{\lambda^2 - (x+a)^2}} dx + b$

$\left[\begin{array}{l} \text{Take } u = \lambda^2 - (x+a)^2 \\ \Rightarrow du = -2(x+a)\,dx \end{array}\right]$

$\Rightarrow y = -\frac{1}{2}\int \frac{du}{\sqrt{u}} + b$

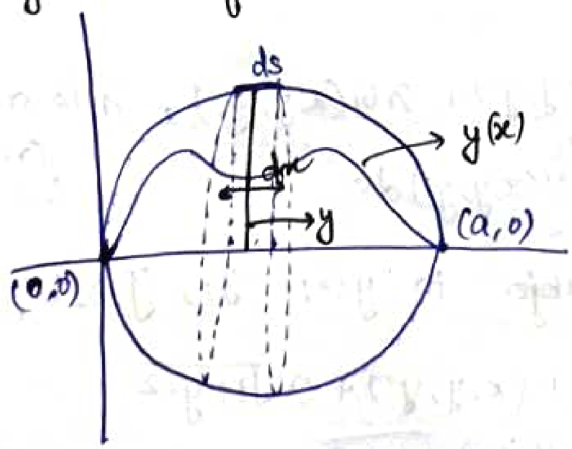
$\Rightarrow y = -\sqrt{\lambda^2 - (x+a)^2} + b$

$\Rightarrow (y-b)^2 = \lambda^2 - (x+a)^2$

$\Rightarrow (x+a)^2 + (y-b)^2 = \lambda^2 \longrightarrow$ circle

Eg. Show that the sphere is a solid figure of revolution which for a given surface area has maxm volume ~~and~~ enclosed.

Soln:



$y(0) = 0$
$y(a) = 0$

Surface area of disk $= 2\pi y\, ds$

Volume of the circular disk $= \pi y^2 dx$

Total surface area,

$$S = \int_0^a 2\pi y\, ds$$

$$= \int_0^a \underbrace{2\pi y \sqrt{1+y'^2}}_{G(x,y,y')} dx \quad , \quad y(0)=0, \quad y(a)=0$$

Total volume,

$$I(y) = \int_0^a \underbrace{\pi y^2}_{F(x,y,y')} dx$$

Maximize $I(y)$ subject to

$$\int_0^a 2\pi y\sqrt{1+y'^2}\, dx = S,$$

$$y(0)=0, \quad y(a)=0$$

$$H(x,y,y') = F(x,y,y') + \lambda G(x,y,y')$$

$$= \pi y^2 + \lambda 2\pi y\sqrt{1+y'^2}$$

E-L eqn: $\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0$

→ H is independent of x

Beltrami identity,

$$H - y'\frac{\partial H}{\partial y'} = c$$

$$\left(\pi y^2 + 2\pi y\lambda\sqrt{1+y'^2}\right) - y'\frac{2\pi\lambda y y'}{\sqrt{1+y'^2}} = c$$

$y(0)=0 \Rightarrow c=0$

$$\therefore \cancel{\pi} y^{\cancel{2}}\sqrt{1+y'^2} + 2\cancel{\pi}\cancel{y}\lambda(1+y'^2) = 2\cancel{\pi}\lambda \cancel{y} y'^2$$

$$\Rightarrow y^2(1+y'^2) = 4\lambda^2$$

$$\Rightarrow y'^2 = \frac{4\lambda^2}{y^2} - 1$$

$$\Rightarrow y' = \frac{\sqrt{4\lambda^2 - y^2}}{y}$$

$$\Rightarrow \int \frac{y}{\sqrt{4\lambda^2-y^2}}\,dy = \int dx + K$$

$$\Rightarrow -\sqrt{4\lambda^2-y^2} = x + K$$

$$\Rightarrow (x+K)^2 + y^2 = 4\lambda^2 \longrightarrow \text{circle.}$$

Problems with Higher Derivatives

Extremize $I(y) = \int_{x_0}^{x_1} F(x, y, y', y'')\,dx$,

$$y(x_0) = y_0,\ y(x_1) = y_1,\ y'(x_0) = y_2,\ y'(x_1) = y_3.$$

E-Poisson Equation:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0.$$

Eg. Extremize $I(y) = \int_{x_0}^{x_1} (y^2 - (y'')^2)\,dx$

$$\frac{\partial F}{\partial y} = 2y,\ \frac{\partial F}{\partial y'} = 0,\ \frac{\partial F}{\partial y''} = -2y''$$

$$\therefore\ 2y - \frac{d^2}{dx^2}(2y'') = 0$$

$$\Rightarrow\ y^{(iv)} - y = 0$$

Functional with two dependent variable

$$I(u, v) = \int_{x_0}^{x_1} F(x, u, v, u', v')\,dx$$

$$u(x_0) = u_0,\ u(x_1) = u_1$$

$$v(x_0) = v_0,\ v(x_1) = v_1.$$

E-L eqn: $\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) = 0$

$$\frac{\partial F}{\partial v} - \frac{d}{dx}\left(\frac{\partial F}{\partial v'}\right) = 0.$$

2 independent variable $\longrightarrow$ PDE.

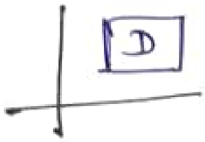
Functionals with two independent variable

Let x and y be independent variables.

$z = z(x,y)$ be the dependent variable.

Extremize $$I(z) = \int_{y_0}^{y_1}\int_{x_0}^{x_1} F(x,y;\, z, z_x, z_y)\,dx\,dy \quad (*)$$

$$= \iint_D F(x,y,z,z_x,z_y)\,dx\,dy$$



where z is prescribed on ∂D, boundary of D.

Eg Find a function ϕ whose mean square value of the magnitude of the gradient over a region D is minimum.

$\phi = \phi(x,y)$

$\nabla\phi = (\phi_x, \phi_y)$ → gradient of ϕ → vector

$|\nabla\phi|^2 = \phi_x^2 + \phi_y^2$

$$I(\phi) = \iint_D (\phi_x^2 + \phi_y^2)\,dx\,dy,$$

where value of ϕ on the boundary of D is given.

E-L eqn for two independent variables:

Suppose z is an extremum of $(*)$

Then, solve the E-L eqn,

$$\frac{\partial F}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_y}\right) = 0$$

$\therefore F(x,y,\phi,\phi_x,\phi_y) = \phi_x^2 + \phi_y^2$

$$\frac{\partial F}{\partial z} = 0,\quad \frac{\partial F}{\partial \phi_x} = 2\phi_x,\quad \frac{\partial F}{\partial \phi_y} = 2\phi_y$$

$$\therefore\ -\frac{\partial}{\partial x}(2\phi_x) - \frac{\partial}{\partial y}(2\phi_y) = 0$$

$\Rightarrow \phi_{xx} + \phi_{yy} = 0$, ϕ on the boundary.

19-04-2024

Variation of Functionals

$y(x)$, change in $y(x+\Delta x)$

Variation of $y(x)$ is $y + \underbrace{\epsilon\eta(x)}$,

η: differentiable function

ϵ: parameter.

Denote $\delta y = \epsilon\eta(x)$

$\delta y' = \epsilon\eta'(x)$

$y \to y + \delta y$, $y' \longrightarrow y' + \delta y'$

Fixed x, $F(x, y, y') \longrightarrow F(x, y+\delta y, y'+\delta y')$

Change in F,

$$\Delta F = F(x, y+\delta y, y'+\delta y') - F(x, y, y')$$

Taylor series in two variables:

$$f(x+h, y+h) = f(x,y) + \frac{f_x}{1!}h + \frac{f_y}{1!}k + \frac{1}{2!}\left[f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2\right] + \dots$$

$$f(x+h) = f(x) + \frac{f'(x)}{1!}h + \frac{f''(x)}{2!}h^2 + \underbrace{\dots}_{\text{very small}} \quad [\because h \text{ is very small.}]$$

x is a stationary point, $f'(x_0) = 0$.

$$f(x_0+h) - f(x_0) \approx \frac{f'(x_0)}{2!}h^2$$

$\geq 0 \Rightarrow x_0$ is a minimum point.

$$\Delta F = F(x, y+\delta y, y'+\delta y') - F(x, y, y')$$

$$= F(x,y,y') + \underbrace{\left(\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'\right)}_{\delta F} + \overbrace{\frac{1}{2!}\left[\frac{\partial^2 F}{\partial y^2}(\delta y)^2 + \frac{2\partial^2 F}{\partial y\,\partial y'}\delta y\,\delta y' + \frac{\partial^2 F}{\partial y'^2}(\delta y')^2\right]}^{\delta^2 F} + \dots - F(x,y,y')$$

First variation of F,

$$\delta F = \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'$$

$$D = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} ; \quad D_y = \begin{bmatrix} \delta y \\ \delta y' \end{bmatrix}$$

Second variation of F,

$$\delta^2 F = \frac{1}{2}\left[\frac{\partial^2 F}{\partial y^2}(\delta y)^2 + \frac{2\partial^2 F}{\partial y \partial y'}\delta y\,\delta y' + \frac{\partial^2 F}{\partial y'^2}(\delta y')^2\right]$$

↳ Variation is analagous to derivative in calculus.

(i) $\delta(F_1 \pm F_2) = \delta F_1 \pm \delta F_2$

(ii) $\delta(F_1 F_2) = F_1\delta F_2 + F_2 \delta F_1$

(iii) $\delta\left(\frac{F_1}{F_2}\right) = \frac{F_2\delta F_1 - F_1 \delta F_2}{F_2^{\,2}}$

(iv) $\delta(F^n) = nF^{n-1}\delta(F)$.

Eg. (i) $\delta(y^2) = 2y\delta y$

(ii) $\delta(y'^2) = 2y'\delta y'$

(iii) $\delta(xy) = x\delta y$ $\quad$ ($\because x$ is fixed)

(iv) $\delta(x^2) = 0$.

Eg. $I(y) = \int_{x_0}^{x_1} F(x,y,y')\,dx$. Find δI.

$$\delta I = \delta\left[\int_{x_0}^{x_1} F(x,y,y')\,dx\right] = \int_{x_0}^{x_1}\delta F(x,y,y')\,dx$$

$$= \int_{x_0}^{x_1}\left[\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'\right]dx$$

[→ If x_0 is an extremum of $f(x) \Rightarrow f'(x_0) = 0$.

Eg. If y^* is an extremum of $I(y)$

$\Rightarrow \delta I(y^*) = 0$.]

$$\int_{x_0}^{x_1}\frac{\partial F}{\partial y'}\delta y'\,dx = \int_{x_0}^{x_1}\left(\frac{\partial F}{\partial y'}\cdot\frac{d(\delta y)}{dx}\right)dx$$

$$= \cancel{\frac{\partial F}{\partial y'}\delta y\Big|_{x_0}^{x_1}}^{\,0} - \int_{x_0}^{x_1} \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\delta y\, dx$$

$$\left[\begin{aligned} \delta y &= \epsilon\eta \\ \delta y' &= \epsilon\eta' \\ &= \epsilon\frac{d\eta}{dx}\end{aligned}\right]$$

$$\therefore\ \delta I(y) = \int_{x_0}^{x_1} \underbrace{\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]}_{0}\delta y\, dx$$

[η is chosen such that it vanishes at boundaries]

(since y is an extremum of $I(y)$)

$$\Rightarrow \boxed{\delta I(y) = 0}.$$

Legendre Test for Extremum

Let y be the extremum of $I(y)$.

(i) $\delta I(y) = 0$ by E-L equation.

(ii) $\delta^2 I(y) > 0 \Rightarrow y$ is a minimum of $I(y)$.

(iii) $\delta^2 I(y) < 0 \Rightarrow y$ is a maximum of $I(y)$.

25-04-2024

Reduction of BVP into Variational Formulation

Eg. ① $y'' - y + x = 0, \quad \ldots ①$
$y(0) = y(1) = 0.$

Multiplying ① by $\delta y \ (= \epsilon\eta)$

Integrate over $(0,1)$: η is a diff. f^n, $\eta \in C^1[x_1, x_2]$
$\eta[x_1] = \eta[x_2] = 0.$

$$\int_0^1 y''\delta y\, dx - \int_0^1 y\,\delta y\, dx + \int_0^1 x\,\delta y\, dx = 0$$

Integrate by parts:

$$\int_0^1 y''\delta y\, dx = \cancel{y'\delta y\Big|_0^1}^{\,0} - \int_0^1 y' \underbrace{\frac{d}{dx}\delta y}_{=\delta y'}\, dx$$

$$\left[\because \int_a^b u'v\,dx = uv\Big|_a^b - \int_a^b uv'\,dx\right]$$

$$y\delta y = \delta(y^2/2)$$

$$y'\delta y' = \delta\left(\frac{y'^2}{2}\right)$$

$$-\int_0^1 y'\delta y'\, dx = -\int_0^1 \delta\left(\frac{y'^2}{2}\right) dx$$

$$-\int_0^1 y\,\delta y\, dx = -\int_0^1 \delta\left(\frac{y^2}{2}\right) dx$$

$$\int_0^1 x\,\delta y\,dx = \int_0^1 \delta(xy)\,dx \qquad \left[\because \delta\left[\int_{x_1}^{x_2} F(x,y,y')\,dx\right] = \int_{x_1}^{x_2} \delta F\,dx\right]$$

independent variable $\quad (\because \delta(x)=0)$

Combining,

$$\int_0^1 \left[-\delta\left(\frac{y'^2}{2}\right) - \delta\left(\frac{y^2}{2}\right) + \delta(xy)\right] dx = 0$$

$$\Rightarrow \delta\left[\underbrace{\int_0^1 (-y'^2 - y^2 + 2xy)\,dx}_{I(y)}\right] = 0$$

$$\Rightarrow \delta(I(y)) = 0$$

Extremize $I(y) = \int_0^1 (-y'^2 - y^2 + 2xy)\,dx$,

subject to $y(0) = y(1) = 0$.

Suppose there exists an extremum.

Then, the extremum solves E-L eqn

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\frac{\partial F}{\partial y'} = -2y', \quad \frac{\partial F}{\partial y} = -2y + 2x$$

$$\therefore -2y + 2x - \frac{d}{dx}(-2y') = 0$$

$$\left.\begin{aligned} y'' - y + x &= 0 \\ y(0) = y(1) &= 0 \end{aligned}\right\} \text{Given BVP}$$

$$\delta y = \epsilon\eta$$

$$Y = y + \delta y$$

② $\frac{d}{dx}\left(K(x)\frac{dy}{dx}\right) + \rho\omega^2 y + p(x) = 0,$

$y(0) = y(L) = 0.$

Variational Formulation:

Multiply by δy and integrate in $(0, L)$.

$$\int_0^L \frac{d}{dx}\left(K\frac{dy}{dx}\right)\delta y\, dx = \cancel{K(x)\frac{dy}{dx}\delta y\Big|_0^L}^{\,0} - \int_0^L K(x)\underbrace{\frac{dy}{dx}\frac{d}{dx}\delta y}_{\delta(y'^2/2)}\, dx$$

$$\int_0^L \rho\omega^2 y\,\delta y\, dx = \int_0^L \rho\omega^2\,\delta(y^2/2)\, dx$$

$$\int_0^L \left[-\delta\left(K(x)\frac{y'^2}{2}\right) + \delta\left(\rho\omega^2\frac{y^2}{2}\right) + \delta(p(x)y)\right] dx$$

$$= \delta\Big[\underbrace{\int_0^L \left[-K(x)y'^2 + \rho\omega^2 y^2 + 2py\right] dx}_{I(y)}$$

Extremize $I(y)$ subject to $y(0) = y(L) = 0$.

③ $\frac{d}{dx}\left(x\frac{dy}{dx}\right) + y = x, \quad \ldots ①$

$y(0) = 0,\ y(1) = 1.$

Find the VF of ①.

Multiply by δy and integrate in $(0,1)$.

$$\int_0^1 \frac{d}{dx}\left(x\frac{dy}{dx}\right)\delta y\, dx + \int_0^1 y\,\delta y\, dx = \int_0^1 x\,\delta y\, dx$$

$$= \cancel{x\frac{dy}{dx}\delta y\Big|_0^1}^{\,0} - \int_0^1 x\frac{dy}{dx}\delta y'\, dx + \int_0^1 \delta\left(\frac{y^2}{2}\right) dx = \int_0^1 \delta(xy)\, dx$$

$$= \int_0^1 \left[-\delta\left(\frac{xy'^2}{2}\right) + \delta(y^2/2) - \delta(xy)\right] dx$$

$$= \delta\left[\int_0^1 (-xy'^2 + y^2 - 2xy)\, dx\right] = 0$$

Extremize $I(y) = \int_0^1 (-xy'^2 + y^2 - 2xy)\, dx,$

subject to $y(0) = 0,\ y(1) = 1.$

26-04-2024

Rayleigh - Ritz Method

Approximation Method:

Eg. $\frac{dy}{dx} + y = \sin x$, $x \in (0,1)$

$u \in \mathbb{R}^3$, (e_1, e_2, e_3) : basis f^n

$c_1, c_2, c_3 \rightarrow$ constants

$u = c_1 e_1 + c_2 e_2 + c_3 e_3$

Let $C^1[x_1, x_2]$ be the space of all one-time differentiable f^ns.

Extremize $I(y) = \int_{x_0}^{x_1} F(x, y, y')\, dx$,

subject to $y(x_0) = y_0$, $y(x_1) = y_1$.

Let $y \in C^1[x_1, x_2]$ be the solution of V.P. (Variational Problem).

Let $\{\phi_1(x), \phi_2(x), \ldots, \phi_n(x), \ldots\}$ be the basis of $C^1[x_1, x_2]$.

Let $\bar{y}$ be the approximate sol^n of y,

$$\bar{y} = \sum_{i=0}^{n} c_i \phi_i(x), \quad c_i : \text{constants}.$$

Basis functions satisfy the boundary conditions (BC) of the V.P.

Extremize $I(\bar{y}) = \int_{x_0}^{x_1} F\left(x, \sum_{i=0}^{n} c_i \phi_i, \sum_{i=0}^{n} c_i \phi_i'(x)\right) dx$

Extremize $I(\bar{y}) = \max/\min_{c_0, c_1, c_2, \ldots c_n} I(c_0, c_1, c_2, \ldots, c_n)$

By calculus, $\frac{\partial I}{\partial c_i} = 0$, $i = 0, 1, 2, \ldots, n$.

Q. Find the approximation sol^n of the BVP, using R-R method.

$$y'' - y + x = 0, \quad y(0) = y(1) = 0$$

Sol^n: V.P. : $I(y) = \int_0^1 (2xy - y^2 - y'^2)\, dx$,

$$y(0) = y(1) = 0.$$

Approximate soln: $\bar{y}(x) = c_0 + c_1 x + c_2 x^2$ (good enough!)

$$\bar{y}(0) = c_0 = 0$$

$$\bar{y}(1) = c_1 + c_2 = 0$$

$$\Rightarrow c_2 = -c_1$$

$$\bar{y}(x) = c_1 x - c_1 x^2$$

$$= c_1 x(1-x)$$

$$I(\bar{y}) = \int_0^1 (2x\bar{y} - \bar{y}^2 - \bar{y}'^2)\,dx$$

$$= \int_0^1 \left[2x(c_1 x(1-x)) - (c_1 x(1-x))^2 - (c_1(1-2x))^2\right] dx$$

$$= \frac{c_1}{6} - \frac{11}{30} c_1^2$$

$$\frac{\partial I}{\partial c_1} = 0 \Rightarrow \frac{1}{6} - \frac{11 \times 2c_1}{30} = 0$$

$$\Rightarrow c_1 = \frac{5}{22}$$

$$\therefore \bar{y}(x) = \frac{5}{22} x - \frac{5}{22} x^2$$

Exact soln:

$$y(x) = x - \frac{e^x - e^{-x}}{e - e^{-1}}$$

x	$\bar{y}(x)$	$y(x)$
0.25	0.043	0.035
0.50	0.057	0.057
0.75	0.43	0.05

Deflection of a Beam

$$\begin{cases} E.I.\ \dfrac{d^2y}{dx^2} - M(x) = 0, & M.I: \text{Rigidity of the beam} \\ & M(x): \text{Momentum} \\ y(0) = y(L) = 0 \end{cases}$$

Soln: Variational Formulation:

$$I(y) = \int \left[\frac{EI}{2}\left(\frac{dy}{dx}\right)^2 + M_0 y\right] dx \quad , \quad \text{taking } M(x) = M_0 : \text{constant}$$

Approx. soln:

$$\bar{y}(x) = c_0 + c_1 x + c_2 x^2$$

$$\bar{y}(0) = c_0 = 0$$

$$\bar{y}(L) = c_1 L + c_2 L = 0$$

$$\Rightarrow c_1 = -c_2 L$$

$$\bar{y}(x) = -c_2 L x + c_2 x^2$$

$$= c_2 x (x - L)$$

$$\therefore\ I(\bar{y}) = \int_0^L \left(\frac{EL}{2}\bar{y}^2 + M_0 \bar{y}\right) dx$$

$$= \int_0^L \left[\frac{EL}{2}\left(c_2(2x-L)\right)^2 + M_0 c_2 x (x-L)\right] dx$$

$$= \frac{EL}{2} \cdot \frac{c_2^2 L^3}{3} + M_0 \left(-\frac{c_2 L^3}{6}\right)$$

$$\frac{\partial I}{\partial c_2} = 0 \Rightarrow \frac{2 c_2}{2} \frac{EI \cancel{L^3}}{3} - \frac{M_0 \cancel{L^3}}{6} = 0$$

$$\Rightarrow c_2 = \frac{M_0}{2EI}$$

$$\therefore\ \boxed{\bar{y}(x) = \frac{M_0}{2EI} x (x - L)}$$