MA 311 - Numerical Methods
Tutorial

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SC22B146

(1) Errors: Ratio of errors:

$$|x_0 - x^*| = 9.0579e-03$$ 0.3807
$$|x_1 - x^*| = 3.4483e-03$$ 0.2334
$$|x_2 - x^*| = 8.0996e-04$$ 0.1138
$$|x_3 - x^*| = 9.2205e-05$$ 0.0384
$$|x_4 - x^*| = 3.5416e-06$$ 7.5279×10^{-3}
$$|x_5 - x^*| = 2.6659e-08$$

Ratio of errors, $r_K = \left|\dfrac{x_{K+1} - x^*}{x_K - x^*}\right|$

For linear convergence, $r_K \to c$ as $K \to \infty$
or $r_K \le c$

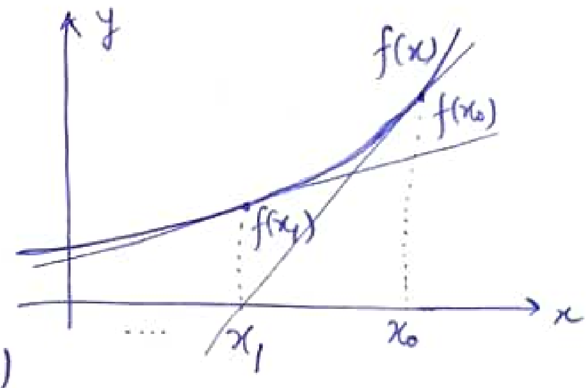
For superlinear convergence,
$r_K \to 0$ as $K \to \infty$

Here, r_K is approaching zero increasingly, hence it has superlinear convergence.

(2) $f(x) = x^2 - e^{-x}$ $\rightarrow$ Root : x^*

Equation of tangent:

$y - f(x_K) + f'(x_K)(x - x_K)$
[At any general x_K]



At $x = x_{K+1}$, $f(x_{K+1}) = 0$.

$\Rightarrow 0 = f(x_K) + f'(x_K)(x_{K+1} - x_K)$

$\Rightarrow x_{K+1} = x_K - \dfrac{f(x_K)}{f'(x_K)}$

$f(x) = x^2 - e^{-x}$
$f'(x) = 2x + e^{-x}$

Given that $x_1 = 0$, [K=1]

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$\Rightarrow x_2 = 0 - \dfrac{(-1)}{1} = 1 = 1e-0$

For K=2, $x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$

$= 1 - \left(\dfrac{1 - e^{-1}}{2 + e^{-1}}\right) = 0.733044 = 7.33044e-1$

For $K=3$, $x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$

$$= 7.33044e-1 - \frac{(7.33044e-1)^2 - \exp(-7.33044e-1)}{2(7.33044e-1) + \exp(-7.33044e-1)}$$

$$= 0.717002$$
$$= 7.17002e-1$$

For $x=4$, $x_5 = x_4 - \frac{f(x_4)}{f'(x_4)}$

$$= 7.17002e-1 - \frac{(7.17002e-1)^2 - \exp(-7.17002e-1)}{2(7.17002e-1) + \exp(-7.17002e-1)}$$

$$= 0.703534$$
$$= 7.03534e-1$$

$\therefore$ Root, $\boxed{x^* \approx 7.03534e-1}$ [As $x_k \longrightarrow x^*$].

③ Given that $g'(x^*)=0$, $g''(x^*)=0$, $g \in C^3(B_\epsilon(x^*))$.

$x_{k+1} = g(x_k)$.

By Taylor's theorem,

$$g(x) = g(x^*) + (x-x^*)\cancel{g'(x^*)} + \frac{(x-x^*)^2}{2}\cancel{g''(x^*)} + \frac{(x-x^*)^3}{6} g'''(c)$$

[c lies b/w x & x^*]

$$\Rightarrow g(x) = g(x^*) + \frac{(x-x^*)^3}{6} g'''(c)$$

[At $x = x_k$]

$$\Rightarrow \frac{|x_{k+1} - x^*|}{|x_k - x^*|^3} \le c$$

[$\because g(x_k) = x_{k+1}$, $\frac{1}{6}g'''(c_k)$ is finite
$g(x^*) = x^*$]

$\Rightarrow x_{k+1} = g(x_k)$ converges cubically to x^*.

④ Given that $x_0 = -1$, $x_1 = -1/3$, $x_2 = 1/3$, $x_3 = 1$

$$w_k = \int_{-1}^{1} L_k(x)\,dx \;;\; w_2 = w_1 \;;\; w_3 = w_0.$$

$$L_0 = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}$$

weight, $w_0 = \int_{-1}^{1} L_0(x)\,dx = \int_{-1}^{1} \frac{(x+1/3)(x-1/3)(x-1)}{(-1+1/3)(-1-1/3)(-1-1)}\,dx$

$$= \cancel{\frac{1}{(-1.27781)}}\left[\int_{-1}^{1} (x+1/3)(x-1/3)(x-1)\,dx\right]\frac{1}{(-\frac{2}{3})(-\frac{4}{3})(-2)}$$

$$= -\frac{9}{16}\int_{-1}^{1}(x+1/3)(x-1/3)(x-1)\,dx$$

$$= -\frac{9}{16}\int_{-1}^{1}\left(x^2-\frac{1}{9}\right)(x-1)\,dx$$

$$= -\frac{9}{16}\int_{-1}^{1}\left(x^3-x^2-\frac{1}{9}x+\frac{1}{9}\right)dx$$

$$= -\frac{9}{16}\left[\frac{x^4}{4}-\frac{x^3}{3}-\frac{1}{9}\frac{x^2}{2}+\frac{1}{9}x\right]_{-1}^{1}$$

$$= -\frac{9}{16}\left[\cancelto{0}{\frac{1^4-(-1)^4}{4}}-\frac{1^3-(-1)^3}{3}-\cancelto{0}{\frac{1}{9}\frac{1^2-(-1)^2}{2}}+\frac{1}{9}(1-(-1))\right]$$

$$= -\frac{9}{16}\left[-\frac{2}{3}+\frac{2}{9}\right]$$

$$= -\frac{9}{16}\left[-\frac{4}{9}\right] = \frac{1}{4}.$$

$$w_1 = \int_{-1}^{1} L_1(x)\,dx = \int_{-1}^{1}\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\,dx$$

$$= \int_{-1}^{1}\frac{(x+1)(x-1/3)(x-1)}{(-1/3+1)(-1/3-1/3)(-1/3-1)}\,dx$$

$$= \frac{1}{(\frac{2}{3})(-\frac{2}{3})(-\frac{4}{3})}\int_{-1}^{1}(x^2-1)(x-1/3)\,dx$$

$$= \frac{27}{16}\int_{-1}^{1}\left(x^3-\frac{1}{3}x^2-x+\frac{1}{3}\right)dx$$

$$= \frac{27}{16}\left[\frac{x^4}{4}-\frac{1}{3}\frac{x^3}{3}-\frac{x^2}{2}+\frac{1}{3}x\right]_{-1}^{1}$$

$$= \frac{27}{16}\left[0-\frac{1}{3}\frac{2}{3}-0+\frac{1}{3}(2)\right]$$

$$= \frac{27}{16}\left[\frac{2}{3}-\frac{2}{9}\right]$$

$$= \frac{27}{16}\left[\frac{4}{9}\right] = \frac{3}{4}.$$

Now, by simpson's $3/8$th rule,

$$Q(f) = \sum_{k} w_k f(x_k)$$

$$= w_0 f(x_0) + w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3).$$

$$= \frac{1}{4} f(-1) + \frac{3}{4} f(-1/3) + \frac{3}{4} f(1/3) + \frac{1}{4} f(1)$$

$$= \frac{1}{4}\left[f(-1) + 3f(-1/3) + 3f(1/3) + f(1)\right].$$

5. By simpson's $1/3$rd rule,

$$Q(f) = \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right], \text{ where } h = \frac{b-a}{3}.$$

$$x_0 = x_1 - h$$
$$x_2 = x_0 + h$$

Let f be a polynomial of degree 'n'.

Then,

$$\int_a^b f(x)dx = \int_a^b f(x_1 + (x - x_1))\, dx$$

$$= \int_a^b \left[f(x_1) + (x-x_1)f'(x_1) + \frac{(x-x_1)^2}{2} f''(x_1) + \frac{(x-x_1)^3}{6} f'''(x_1) + \dots + \frac{(x-x_1)^{(n)}}{(n)!} f^{(n)}(\xi_2)\right] dx$$

$$= 2h f(x_1) + \cancel{\frac{(x-x_1)^2}{2}\Big|_a^b}^{\,0} f'(x_1) + \frac{(x-x_1)^3}{6}\Big|_a^b f''(x_1) + \cancel{\frac{(x-x_1)^4}{24}\Big|_a^b}^{\,0} f'''(x_1) + \frac{(x-x_1)^{n+1}}{(n+1)!}\Big|_a^b f^{(n)}(\xi_2).$$

$$= 2h f(x_1) + \frac{h^3}{3} f''(x_1) + \dots + \frac{2h^{n+1}}{(n+1)!} f^{(n)}(\xi_2).$$

$$= 2h f(x_1) + \frac{h^3}{3}\left[\frac{f(x_0) - 2f(x_1) + f(x_2)}{h^2} - \frac{h^2}{12} f^{(4)}(\xi_1)\right] + \frac{2h^{n+1}}{(n+1)!} f^{(n)}(\xi_2).$$

for $n=3$,

$$\int_a^b f(x)dx = 2h f(x_1) + \frac{h^3}{3}\left[\frac{f(x_0) - 2f(x_1) + f(x_2)}{h^2}\right] - \cancel{\frac{h^5}{90} f^{IV}(\xi)}^{\,0}$$

$$= \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right] \rightsquigarrow \text{Exact Simpson's rule.}$$

6. Second-order Taylor expansion of $f(x)$ at $\left(\frac{a+b}{2}\right)$.

$$f(x) = f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right)\left(x-\left(\frac{a+b}{2}\right)\right) + \frac{\left(x-\left(\frac{a+b}{2}\right)\right)^2}{2} f''(\xi_1).$$

$$\Rightarrow \int_a^b f(x) = f\left(\frac{a+b}{2}\right)(b-a) + \cancel{\left.\frac{\left(x-\left(\frac{a+b}{2}\right)\right)^2}{2}\right|_a^b}^{\,0} f'\left(\frac{a+b}{2}\right) + \left.\frac{\left(x-\left(\frac{a+b}{2}\right)\right)^3}{6}\right|_a^b f''(\xi_1).$$

$$\Rightarrow I(f) = f\left(\frac{a+b}{2}\right)(b-a) + \left.\frac{\left(x-\left(\frac{a+b}{2}\right)\right)^3}{6}\right|_a^b f''(\xi_1).$$

and Quadrature,

$$Q(f) = (b-a) f\left(\frac{a+b}{2}\right)$$

Error,

$$I(f) - Q(f) = \left.\frac{\left(x-\left(\frac{a+b}{2}\right)\right)^3}{6}\right|_a^b f''(\xi_1)$$

$$= \frac{\left(b-\left(\frac{a+b}{2}\right)\right)^3 - \left(a-\left(\frac{a+b}{2}\right)\right)^3}{6} f''(\xi_1).$$

$$= \frac{\left(\frac{b-a}{2}\right)^3 - \left(\frac{a-b}{2}\right)^3}{6} \cdot f''(\xi_1).$$

$$= \frac{(b-a)^3}{24} \cdot f''(\xi_1).$$

$$= \frac{(b-a)^3}{24} \cdot f''(c), \text{ for some } c \in (a,b).$$

Trapezoidal Rule:

By Taylor expansion,

$$f(x) = f(a) + f'(a)\frac{(x-a)}{1} + f''(a)\frac{(x-a)^2}{2} + \dots + f^m(a)\frac{(x-a)^m}{m!} + f^{m+1}(c)\frac{(x-a)^{m+1}}{(m+1)!}$$

$P_m(x) \rightarrow$ polynomial of degree $\leq m$. $[c \in [a,b]]$

$$\therefore f(x) = P_1(x) + \frac{f''(c)(x-a)(x-b)}{2!}$$

$$\Rightarrow \boxed{f(x) - P_1(x) = \frac{f''(c)(x-a)(x-b)}{2}}$$

Integrate both side,

$$\int_a^b f(x)dx - \int_a^b P_1(x)dx = \int_a^b \frac{f''(c)(x-a)(x-b)}{2} dx$$

$$\Rightarrow \int_a^b f(x)dx - Q(f) = \frac{f''(c)}{2}\int_a^b (x-a)(x-b)dx$$

$$= \frac{f''(c)}{2}\left[\frac{x^3}{3} - \frac{x^2}{2}(a+b) + abx\right]_a^b$$

$$= \frac{f''(c)}{2}\left[\frac{b^3-a^3}{3} - \frac{b^2-a^2}{2}(a+b) + ab(b-a)\right]$$

$$= \frac{f''(c)}{2}(b-a)\left[\frac{a^2+b^2+ab}{3} - \frac{(a+b)^2}{2} + ab\right]$$

$$= \frac{f''(c).h}{2}\left[\frac{2a^2+2b^2+2ab-3a^2-3b^2-6ab+6ab}{6}\right]$$

$$= -\frac{f''(c)h}{2}\left[\frac{a^2+b^2-2ab}{6}\right]$$

$$= -\frac{f''(c)h}{2}\frac{(a-b)^2}{6}$$

$$= -f''(c)\frac{h^3}{12} = -f''\frac{(b-a)^3}{12}.$$

7 ~~$y(t) = -1.2y + 7e^{-0.3t}, \; y(0)=3,$~~

$y'(t) = e^t(\sin(t) + \cos(t)) + 2t, \quad y(0)=1, \quad 0 \le t \le 2$

True solⁿ: $y(t) = e^t \sin(t) + t^2 + 1.$

$\Delta t = 1/10, 1/20, 1/40, 1/80, 1/160.$

$\Rightarrow y'(t) = f(t, y(t)) = e^t \cos t + e^t \sin t + 2t.$

Euler's method:

$\Delta t = 1/10 = 0.1$

$$y_{i+1} = y_i + \Delta t . f(t_i, y(t_i))$$
$$= 1 + (0.1).f(0,1)$$
$$= 1 + 0.1 \times 1$$
$$= 1.1$$

$f(0,1) = y'(0) = 1$

Exact: $y(0.1) = e^{0.1}\sin(0.1) + (0.1)^2 + 1 = 1.1203$

$\Delta t = 1/20 = 0.05$

$y_{i+1} = y_i + \Delta t(f(0,1))$ Exact: $y(0.05) = \cancel{1.0034}\ 1.0550$

$= 1 + 0.05(1)$

$= 1.05$

$\Delta t = 1/40 = 0.025$

$y_{i+1} = 1 + 0.025(1)$

$= 1.025$ Exact: $y(0.025) = 1.0263$

$\Delta t = 1/80 = 0.0125$

$y_{i+1} = 1 + 0.0125(1)$

$= 1.0125$ Exact: $y(0.0125) = 1.0128$

$\Delta t = 1/160 = 0.00625$

$y_{i+1} = 1 + 0.00625(1)$ Exact: $y(0.00625) = 1.0063$

$= 1.00625$

Trapezoidal Method:

$$y_{i+1} = y_i + \frac{\Delta t}{2}\left[f(t_i, y_i) + f(t_{i+1}, y_{i+1})\right]$$

$\Delta t = 1/10 = 0.1$

$y_{i+1} = 1 + \frac{0.1}{2}[1 + 1.4099]$

$= 1 + 0.05[2.4099]$

$= 1.1205$

$f(0.1, 1.1) = e^{0.1}\cos(0.1) + e^{0.1}\sin(0.1) + 2(0.1)$
$= 1.4099$

$\Delta t = 1/20 = 0.05$

$y_{i+1} = 1 + \frac{0.05}{2}[\cancel{2.4099}\ 2.20249] = \cancel{1.0602}\ 1.05506$

$f(0.05, 1.05) = 1.2049$

$\Delta t = 1/40 = 0.025$

$y_{i+1} = 1 + \frac{0.025}{2}[2.\cancel{4099}\,1006] = 1.0263$

$f(0.025, 1.025) = 1.1006$

$\Delta t = 1/80 = 0.0125$

$y_{i+1} = 1 + \frac{0.0125}{2}[2.\cancel{4099}\,0502] = 1.01208$

$f(0.0125, 1.0125) = 1.0502$

$\Delta t = 1/160 = 0.00625$

$y_{i+1} = 1 + \frac{0.00625}{2}[2.\cancel{4099}\,0251] = 1.0063$

$f(0.00625, 1.00625) = 1.02541$

Δt	Euler's error	Trapezoidal's error	Euler Error rate	Trap. Error rate
$\frac{1}{10}$	0.0203	0.0002	–	–
$\frac{1}{20}$	0.005	0.00006	4	8
$\frac{1}{40}$	0.0013	~~0.0038~~ 0	4	8
$\frac{1}{80}$	0.0003	0~~.0023~~ 0	4	8
$\frac{1}{160}$	0.00005	~~0.0012~~ 0	4	8

Trapezoidal errate rate is reducing two times faster than euler's rate.

⑧ $y'(t) = -1.2y + 7e^{-0.3t}$, $y(0)=3$, $0 \leq t \leq 2.5 = T$

True soln: $y(t) = \frac{70}{9}e^{-0.3t} - \frac{43}{9}e^{-1.2t}$

Using python code, [maximum error]

Δt	Euler Error	Trapezoidal error	Heun's error	Convergence rate: Euler Error rate	Trap. error rate	Heun's error rate
$\frac{T}{10}$	0.26104	0.02689	0.02689	2	4.5	4.5
$\frac{T}{20}$	0.1204	0.00592	0.00592	2	4.25	4.25
$\frac{T}{40}$	0.05804	0.00139	0.00139	2	4.1	4.1
$\frac{T}{80}$	0.028515	0.00337	0.000337	2	4	4