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SC22B146

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Subject

AVD611 - Modern Signal Processing
Tutorial - 3

(1) $H(Z) = \dfrac{1-0.5Z^{-1}}{1-0.75Z^{-1}}$

$r_x(K) = (1/3)^{|K|}$

$$\Rightarrow P_x(Z) = \sum_{m=-\infty}^{\infty} (1/3)^{|m|} Z^{-m}$$

$$= \sum_{m=-\infty}^{-1} (1/3)^{-m} Z^{-m} + \sum_{m=0}^{\infty} (1/3)^m Z^{-m}$$

$$= \frac{1}{1-aZ} - 1 + \frac{1}{1-aZ^{-1}}$$

$$= \frac{1-a^2}{1-a(Z+Z^{-1})+a^2}, \quad a = 1/3.$$

$\therefore$ Output power spectrum,

$$P_y(Z) = H(Z)H^*(Z^{-1})P_x(Z)$$

$$= \frac{1-0.5Z^{-1}}{1-0.75Z^{-1}} \cdot \frac{1-0.5Z}{1-0.75Z} \cdot \frac{1-(1/3)^2}{1-\frac{1}{3}(Z+Z^{-1})+(\frac{1}{3})^2}$$

$$= \frac{1.25-0.5(Z+Z^{-1})}{1.5625-0.75(Z+Z^{-1})} \cdot \frac{8/9}{1-\frac{1}{3}(Z+Z^{-1})+\frac{1}{9}}$$

$$= \frac{8}{9} \frac{1.25-0.5(Z+Z^{-1})}{1.5625-0.75(Z+Z^{-1})} \frac{1}{\frac{10}{9}-\frac{1}{3}(Z+Z^{-1})}$$

$$= \frac{8[1.25-0.5(Z+Z^{-1})]}{1.5625-0.75(Z+Z^{-1})} \cdot \frac{1}{10-3(Z+Z^{-1})}$$

Now, $$P_y(Z) = \frac{\frac{8}{9}[1.25-0.5(Z+Z^{-1})]}{(1-0.75Z^{-1})(1-0.75Z)(1-\frac{1}{3}Z^{-1})(1-\frac{1}{3}Z)}$$

$\downarrow$ IZT $$= \frac{2/3}{(1-0.75Z^{-1})(1-0.75Z)} + \frac{2/9}{(1-1/3Z^{-1})(1-1/3Z)}$$

$$\Rightarrow r_y(K) = \frac{2}{3}(0.75)^{|K|} + \frac{2}{9}\left(\frac{1}{3}\right)^{|K|}$$

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Now, $$P_x(Z) = \frac{1-\frac{1}{9}}{1-\frac{1}{3}(Z+Z^{-1})+\frac{1}{9}} = \frac{8/9}{(1-\frac{1}{3}Z^{-1})(1-\frac{1}{3}Z)}$$

As $$P_{xy}(Z) = H(Z)P_x(Z)$$
$$= \frac{1-0.5Z^{-1}}{1-0.75Z^{-1}} \cdot \frac{8/9}{(1-\frac{1}{3}Z^{-1})(1-\frac{1}{3}Z)}$$
$$= \frac{8/9(1-0.5Z^{-1})}{(1-0.75Z^{-1})(1-\frac{1}{3}Z^{-1})(1-\frac{1}{3}Z)}$$
$$= \frac{4/3}{(1-0.75Z^{-1})(1-\frac{1}{3}Z)} + \frac{-4/9}{(1-\frac{1}{3}Z^{-1})(1-\frac{1}{3}Z)}$$

IZT

$$r_{xy}(K) = \begin{cases} \frac{4}{3}(0.75)^K, & K \geq 0 \\ \frac{4}{3}(\frac{1}{3})^{-K}, & K<0 \end{cases}, \quad -\frac{4}{9}(\frac{1}{3})^{|K|}$$

$$= \begin{cases} \frac{4}{3}(0.75)^K - \frac{4}{9}(\frac{1}{3})^K, & K \geq 0 \\ \frac{4}{3}(\frac{1}{3})^{-K} - \frac{4}{9}(\frac{1}{3})^{-K}, & K<0 \end{cases}$$

2 (i) $$r_x(K) = \delta(K) + 2(0.5)^{|K|}$$
$$ZT \Rightarrow P_x(Z) = 1 + 2\left[\frac{1-(0.5)^2}{1-0.5(Z+Z^{-1})+0.5^2}\right]$$
$$= 1 + \frac{2(0.75)}{1.25-0.5(Z+Z^{-1})}$$

(ii) $$r_x(K) = \begin{cases} 10-|K|, & |K|<0 \\ 0, & \text{otherwise} \end{cases}$$

triangular → convolution of two rectangular sequences

Triangular autocorrelation → autocorr. of rectangular sequence

∴ $P_x(e^{j\omega}) = |W(e^{j\omega})|^2$, $W(e^{j\omega})$ being DTFT of some $w(n)$ sequence

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$$W(e^{j\omega}) = \sum_{n=0}^{N-1} e^{-j\omega n} = e^{-j\omega(N-1)/2} \cdot \frac{\sin(N\omega/2)}{\sin(\omega/2)}$$

$$\Rightarrow |W(e^{j\omega})|^2 = \left|\frac{\sin(N\omega/2)}{\sin(\omega/2)}\right|^2$$

for $N = 10$,

$$P_x(e^{j\omega}) = \left|\frac{\sin(10\omega/2)}{\sin(\omega/2)}\right|^2.$$

(3) $$P_x(Z) = \frac{-2Z^2 + 5Z - 2}{3Z^2 + 10Z + 3}$$

$$= \frac{-2 + 5Z^{-1} - 2Z^{-2}}{3 + 10Z^{-1} + 3Z^{-2}}$$

$$= \frac{-2 + 5Z^{-1} - 2Z^{-2}}{(1 + 3Z^{-1})(3 + Z^{-1})}$$

$$= \frac{-\frac{1}{2} + \frac{1}{2}Z^{-1}}{1 + 3Z^{-1}} + \frac{-\frac{1}{2} - \frac{1}{2}Z^{-1}}{3 + Z^{-1}}$$

$$= -\frac{1}{2}\,\frac{1 - Z^{-1}}{1 + 3Z^{-1}} - \frac{1}{2}\,\frac{1 + Z^{-1}}{3 + Z^{-1}}$$

$$\Rightarrow r_x(k) = -\frac{1}{2}\left[(-3)^k u(k) - (-3)^{k-1}u(k-1) + \left(-\frac{1}{3}\right)^k u(k) + \left(-\frac{1}{3}\right)^{k-1} u(k-1)\right]$$

(4) $$r_x(k) = 10\left(\tfrac{1}{2}\right)^{|k|} + 3\left(\tfrac{1}{2}\right)^{|k-1|} + 3\left(\tfrac{1}{2}\right)^{|k+1|}$$

$r_y(k) = \delta(k)$ [white]

$$P_x(Z) = 10\left[\frac{1 - \frac{1}{4}}{1 - \frac{1}{2}(Z + Z^{-1}) + \frac{1}{4}}\right] + \frac{3Z^{-1}\left(\frac{3}{4}\right)}{\frac{5}{4} - \frac{1}{2}(Z + Z^{-1})} + \frac{3Z\left(\frac{3}{4}\right)}{\frac{5}{4} - \frac{1}{2}(Z + Z^{-1})}$$

$$= \frac{3}{4}\,\frac{10 + 3(Z + Z^{-1})}{\frac{5}{4} - \frac{1}{2}(Z + Z^{-1})}$$

$$= \frac{\frac{3}{4}(10Z + 3Z^2 + 3)}{1.25Z - \frac{1}{2}(Z^2 + 1)}$$ [Multiply N^r & D^r by Z]

$$= -\frac{3}{2}\,\frac{3Z^2 + 10Z + 3}{Z^2 - 2.5Z + 1}$$

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$$= \frac{K(3z^2+10z+3)}{(z+2)(z+1/2)} \quad (K>0)$$

$$= A(z)\,A(z^{-1}), \text{ where } A(z) = \sqrt{K}\,\frac{3z+1}{z-1/2}$$

$\therefore$ whitening filter $[y(n) = h_w(n) * x(n)]$,

$$H_w(z) = \frac{1}{A(z)}$$

$$= \frac{(z-1/2)}{\sqrt{K}(3z+1)}$$

choose $K=1 \Rightarrow H_w(z) = \frac{z-1/2}{3z+1}$
(o/p variance $=1$)

(5) $x(n) = 4v(n) - 2v(n-1) + v(n-2)$, $\sigma_v^2 = 1$

$r_x(k) = \mathbb{E}[x(n)x(n+k)]$

$\Rightarrow r_x(0) = \mathbb{E}[x^2(n)] = \mathbb{E}[\{4v(n) - 2v(n-1) + v(n-2)\}^2]$

$= \sigma_v^2(16+4+1)$

$= 21\sigma_v^2 = 21$

$r_x(1) = \mathbb{E}[x(n)x(n+1)]$

$= \sigma_v^2[4(-2)+(-2)1] = -10$

$r_x(2) = 4\sigma_v^2 = 4$, $r_x(3) = 0$

Now,
$$\begin{bmatrix} r_x(0) & r_x(1) \\ r_x(1) & r_x(0) \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} r_x(2) \\ r_x(3) \end{bmatrix}$$

$$\Rightarrow \begin{pmatrix} 21 & -10 \\ -10 & 21 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

$\Rightarrow 21h_1 - 10h_2 = 4$,

$-10h_1 + 21h_2 = 0 \Rightarrow h_1 = 21/10\, h_2$

$\Rightarrow h_1 = \frac{84}{341}$, $h_2 = \frac{40}{341}$

$$MMSE = r_x(0) - [r_x(2) - r_x(3)]\begin{bmatrix} r_x(0) & r_x(1) \\ r_x(1) & r_x(0) \end{bmatrix}^{-1}\begin{bmatrix} r_x(2) \\ r_x(3) \end{bmatrix}$$

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$$= 21 - \begin{bmatrix} 4 & 0 \end{bmatrix} \begin{bmatrix} 21 & -10 \\ -10 & 21 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$= 21 - \begin{bmatrix} 4 & 0 \end{bmatrix} \frac{1}{341} \begin{pmatrix} 21 & 10 \\ 10 & 21 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

$$= 21 - \frac{1}{341} \begin{pmatrix} 4 & 0 \end{pmatrix} \begin{pmatrix} 84 \\ 40 \end{pmatrix}$$

$$= 21 - \frac{1}{341} \times 336 = \frac{6825}{341} \approx 20.01.$$

6 (a) $y(n) = x(n) + 0.8x(n-1) + v(n)$, $\sigma_v^2 = 1$

$$r_x = [4 \quad 2 \quad 1 \quad 0.5]^T$$

$$= 4(0.5)^{|k|}$$

$$\Rightarrow P_x(Z) = 4 \frac{(1 - 0.25)}{1 - \frac{1}{2}(Z + Z^{-1}) + \frac{1}{4}}$$

$$= \frac{3}{1.25 - 0.5(Z + Z^{-1})}$$

Also, $Y(Z) = X(Z) + 0.8Z^{-1}X(Z) + \cancel{\sigma_v^2} \, 1 = (1 + 0.8Z^{-1})X(Z) + 1$

$$H(Z) = \frac{P_{xy}(Z)}{P_y(Z)} = \frac{P_x(Z)}{P_y(Z)}$$

$$= \frac{3}{1.25 - 0.5(Z + Z^{-1})} \cdot \frac{1}{|(1 + 0.8Z^{-1})|^2 P_x(Z) + 1}$$

$$= \frac{\cancel{3}}{1.25 - 0.5(Z + Z^{-1})} \cdot \frac{1}{\frac{\cancel{3}(1 + 0.8Z^{-1})(1 + 0.8Z)}{1.25 - 0.5(Z + Z^{-1})} + 1}$$

$$= \frac{1.25 - 0.5(Z + Z^{-1})}{(1 + 0.8Z^{-1})(1 + 0.8Z) + 1.25 - 0.5(Z + Z^{-1})}$$

$$= \frac{1.25 - 0.5(Z + Z^{-1})}{1.64 + 0.8(Z + Z^{-1}) + 1.25 - 0.5(Z + Z^{-1})}$$

$$= \frac{1.25 - 0.5(Z + Z^{-1})}{2.89 - 0.3(Z + Z^{-1})}$$

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(b) $H(z) = \dfrac{1.25z - 0.5z^2 - 0.5}{2.89z - 0.3z^2 - 0.3}$

$= \dfrac{0.5z^2 - 1.25z + 0.5}{0.3z^2 - 2.89z + 0.3}$

$= \dfrac{0.5\,(z-2)(z-1/2)}{0.3\,(z-0.1)(z-9.5)}$

$= 5/3\,\dfrac{(1-2z^{-1})(1-1/2\,z^{-1})}{(1-0.1z^{-1})\,(1-9.5z^{-1})}$

$H_{causal}(z) = \dfrac{5}{3}\dfrac{(1-1/2\,z^{-1})\,(1-2z^{-1})}{(1-0.1z^{-1})}$

(7) $x(n) = x(n-1) - 0.6x(n-2) + w(n)$, σ_w^2

Yule walker eqn:

$$\begin{pmatrix} \gamma_x(0) & \gamma_x(1) & \gamma_x(2) \\ \gamma_x(1) & \gamma_x(0) & \gamma_x(1) \\ \gamma_x(2) & \gamma_x(1) & \gamma_x(0) \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0.6 \end{pmatrix} = \begin{pmatrix} \sigma_w^2 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \gamma_x(0) - \gamma_x(1) + 0.6\,\gamma_x(2) = \sigma_w^2$

$\gamma_x(1) - \gamma_x(0) + 0.6\,\gamma_x(1) = 0 \Rightarrow 1.6\gamma_x(1) = \gamma_x(0)$

$\gamma_x(2) - \gamma_x(1) + 0.6\gamma_x(0) = 0 \Rightarrow \gamma_x(2) = 0.04\,\gamma_x(1)$

$\Rightarrow 1.6\gamma_x(1) - \gamma_x(1) + 0.024\,\gamma_x(1) = \sigma_w^2$

$\Rightarrow 0.624\,\gamma_x(1) = \sigma_w^2$

$\Rightarrow \gamma_x(1) = 1.60\,\sigma_w^2$

$\gamma_x(0) = 2.56\,\sigma_w^2$

$\gamma_x(2) = 0.064\,\sigma_w^2$.

(8) $x(n) = s(n) + w(n)$

$s(n) = 0.4\,s(n-1) + v(n)$, $\sigma_w^2 = 1$, $\sigma_v^2 = 0.49$

(a) $\gamma_s(m) = E\,[\,s(n)\,s(n+m)\,]$

$= E\,[\{0.4\,s(n-1) + v(n)\}\{0.4\,s(n+m-1) + v(n+m)\}]$

$= 0.16\,\gamma_s(m) + \gamma_v(m)$

$\Rightarrow 0.84\,\gamma_s(m) = \gamma_v(m)$

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$$\Rightarrow r_s(m) = 1.19\, r_v(m)$$
$$= 1.19 \times 0.49\, \delta(m)$$
$$= 0.58\, \delta(m).$$

Also, $r_x(m) = r_s(m) + r_w(m)$
$$= 0.58\,\delta(m) + \delta(m)$$
$$= 1.58\,\delta(m).$$

(b) Wiener filter, $h = R_x^{-1} r_{sx}$

$$\begin{pmatrix} h_0 \\ h_1 \end{pmatrix} = \begin{pmatrix} r_x(0) & r_x(1) \\ r_x(1) & r_x(0) \end{pmatrix} \begin{pmatrix} r_{sx}(0) \\ r_{sx}(1) \end{pmatrix}$$
$$= \begin{pmatrix} 0.58 & 0 \\ 0 & 1.58 \end{pmatrix} \begin{pmatrix} 0.58 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0.916 \\ 0 \end{pmatrix}$$

(c) $$\text{MMSE} = r_s(0) - \begin{bmatrix} r_{sx}(0) & r_{sx}(1) \end{bmatrix} \begin{bmatrix} r_x(0) & r_x(1) \\ r_x(1) & r_x(0) \end{bmatrix}^{-1} \begin{bmatrix} r_{xs}(0) \\ r_{sx}(1) \end{bmatrix}$$
$$= 0.58 - \begin{bmatrix} 0.58 & 0 \end{bmatrix} \begin{bmatrix} 1.58 & 0 \\ 0 & 1.58 \end{bmatrix}^{-1} \begin{bmatrix} 0.58 \\ 0 \end{bmatrix}$$
$$= 0.58 - \begin{bmatrix} 0.58 & 0 \end{bmatrix} \frac{1}{2.49} \begin{bmatrix} 1.58 & 0 \\ 0 & 1.58 \end{bmatrix} \begin{bmatrix} 0.58 \\ 0 \end{bmatrix}$$
$$= 0.58 - \frac{1}{2.49} \begin{bmatrix} 0.58 & 0 \end{bmatrix} \begin{bmatrix} 0.916 \\ 0 \end{bmatrix}$$
$$= 0.58 - \frac{1}{2.49} \begin{bmatrix} 0.531 \\ 0 \end{bmatrix}$$
$$= 0.366$$