फूर्येश्रेण्यः

FOURIER SERIES

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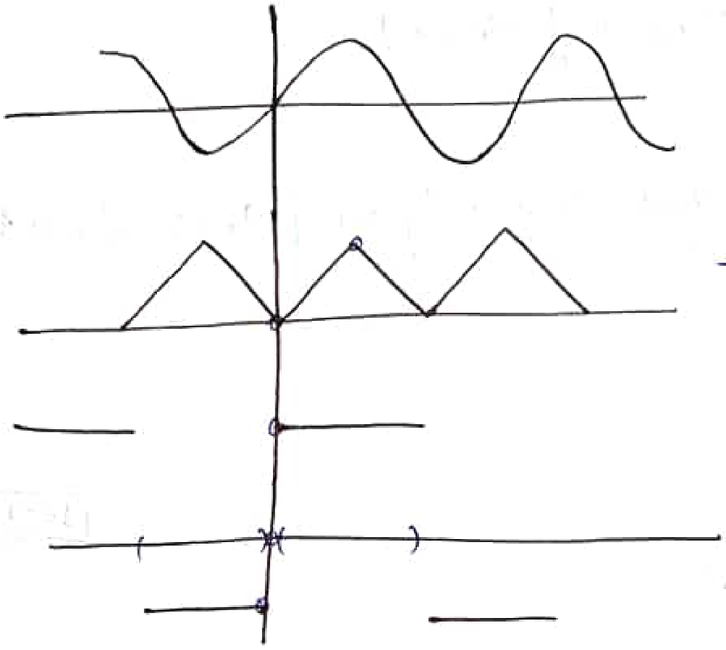
If $f =$ Taylor series of f about x_0 $\forall x \in (x_0 - h, x_0 + h)$ holds,

$f \approx$ polynomial of certain degree

$1 + x + x^2 + \dots$
↳ not a polynomial
↳ polynomial has finite terms.

Signal (waves):



→ Discontinuity

→ Discontinuity
↳ cannot be represented by a single function.

→ Taylor series requires a function to be infinitely differentiable.

Given $f : [-\pi, \pi] \to \mathbb{R}$

Does not tell f.s. of f converges to f

$$\text{Fourier series of } f = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \neq f \quad -\circledast$$

(= only if f converges to the series)

provided a_0, a_n, b_n can be $\underbrace{\text{expressed in terms of } f}_{\text{(using Euler Formula)}}$.

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx \quad (\text{average of } f \text{ over } [-\pi, \pi])$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx, \quad n = 1, 2, 3, \dots$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx, \quad n = 1, 2, 3, \dots$$

$1, \cos x, \cos 2x, \cdots$

$\sin x, \sin 2x, \cdots$

Aim: To represent f interms of Fourier Series of f.

Q) Suppose f is not known. Can we say the trigonometric series (*) will be the F.S. of some function f? $\to$ Yes

$\hookrightarrow$ eg. $1 + \sum \left(\frac{1}{n} \cos nx + \frac{1}{n^2} \sin nx\right)$

Eg. $\sum_{n=2}^{\infty} \frac{\sin nx}{\ln n}$ converges uniformly on $[a, 2\pi - a]$, $a \in (0, \pi)$,

but it is not a F.S.

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$\to$ Infinite differentiability

$\downarrow$

n^{th} term of Taylor series

$f^{n}(x_0) \frac{(x-x_0)^n}{n!}$

$\to$ $f : [-\pi, \pi] \to \mathbb{R}$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx$$

$$\int_{-\pi}^{\pi} |f(x) dx| dx < \infty \to L_1 [-\pi, \pi]$$

$L_n [-\pi, \pi] \to$

$\int_{-\pi}^{\pi} |f(x) dx|^n dx < \infty$

Eg. $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ 4, & 0 < x \le \pi \end{cases}$

$f : [-\pi, \pi] \to \mathbb{R}$

Write F.S. of f $\left[a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \right]$

(Compute a_0, a_n, b_n)

$a_0 = $ avg. f over $[-\pi, \pi] = 2$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} 4 \cos nx \, dx$$

$$= 0$$

$$b_n = \frac{1}{\pi} \int_0^{\pi} 4 \sin nx \, dx \neq 0$$

$$= -\frac{4}{n\pi} \left[\cos nx \right]_0^{\pi}$$

$$= \frac{8}{n\pi} \quad (n \text{ is odd, positive integer})$$

($b_n = 0$ for $n =$ even integer)

$\therefore$ F.S. of f

$$= a_0 + \sum_{n=1}^{\infty} (\cancel{a_n \cos nx}^{\,0} + b_n \sin nx)$$

$$= 2 + \sum_{n=1}^{\infty} \frac{8}{n\pi} \sin nx, \quad n \text{ is odd positive integer}$$

(odd harmonics)

Q) can we represent f in terms of F.S. of f?

Eg. of functions:

① $f \in L_1 [-\pi, \pi]$ ✓

② $f \in L_1 [0, \pi]$

③ $f \in L_1 [-1, 1]$ → by using transformation ✓

or $L_1 [-l, l]$

$L_1 [a, b]$

Eg. Suppose $f \in L_1[-l, l]$, Q what is the F.S. of f?

Let $x \in [-l, l]$

Define $g : [-\pi, \pi] \to \mathbb{R}$

$$g(y) = f\left(\frac{ly}{\pi}\right)$$

Then, $g\left(\frac{\pi x}{l}\right) = f(x)$, $x \in [-l, l]$.

$\therefore$ F.S. of $f(x) =$ F.S. of $g\left(\frac{\pi x}{l}\right)$

$$= a_0 + \sum \left(a_n \cos n\left(\frac{\pi x}{l}\right) + b_n \sin n\left(\frac{\pi x}{l}\right)\right)$$

where

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} g\left(\frac{n\pi}{l}\right) d\left(\frac{n\pi}{l}\right)$$

$$= \frac{1}{2l} \int_{-l}^{l} f(x)\, d(x)$$

$= $ avg. of f over $[-l, l]$.

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g\left(\frac{\pi x}{l}\right) \cos n\left(\frac{\pi x}{l}\right) d\left(\frac{\pi x}{l}\right)$$

$$= \frac{1}{l} \int_{-l}^{l} f(x) \cos n\left(\frac{\pi x}{l}\right) dx$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin n\left(\frac{\pi x}{l}\right) dx$$

Eg. $f \in L_1[a, b]$

By analogy,

amplitude — frequency:

$$\boxed{\text{discrete frequency spectrum: } \omega_n = \frac{2\pi n}{T}}$$

$$\text{F.S. of } f = a_0 + \sum a_n \cos n\left(\frac{2\pi x}{T}\right) + b_n \sin n\left(\frac{2\pi x}{T}\right)$$

$[T = b - a]$

where

$$a_0 = \text{avg. of } f \text{ over } [a, b] = \frac{1}{T} \int_a^b f(x)\, dx$$

$$a_n = \frac{1}{T/2} \int_a^b f(x) \cos n\left(\frac{2\pi x}{T}\right) dx$$

$$b_n = \frac{1}{T/2} \int_a^b f(x) \sin n\left(\frac{2\pi x}{T}\right) dx$$

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Eg. Suppose $f \in L_1[-l, l]$ and assume that f is even function.

$$\text{F.S. of } f = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos n\frac{\pi}{l}x + b_n \sin n\frac{\pi}{l}x \right)$$

where, $b_n = \frac{1}{l} \int_{-l}^{l} \underbrace{\underbrace{f(x)}_{\text{even}} \times \underbrace{\sin \frac{n\pi}{l} x}_{\text{odd}}}_{\text{odd}} \, dx$ $\quad \Big| \; \frac{2n\pi}{T}$

$\Rightarrow b_n = 0$

$$\therefore \boxed{\text{F.S. of } f = a_0 + \sum_{n=1}^{\infty} a_n \cos n\frac{\pi}{l}x} \quad \text{(Fourier cosine series)}$$

If f is odd function,

$$b_n = \frac{1}{l} \int_{-l}^{l} \underbrace{\underbrace{f(x)}_{\text{odd}} \times \underbrace{\sin\left(\frac{n\pi}{l}x\right)}_{\text{odd}}}_{\text{even}} dx \quad \Bigg\} \quad a_0 = 0, \; a_n = 0$$

$\neq 0$

$$\therefore \boxed{\text{F.S. of } f = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x} \quad \text{(Fourier Sine Series)}$$

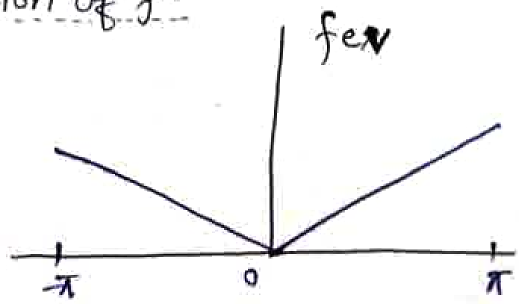
Eg $f(x) = \cos x$, $x \in [-\pi, \pi] \rightarrow$ even f^n $\rightarrow b_n = 0$

Eg. $f(x) = \sin x$, $x \in [-\pi, \pi]$. $\rightarrow$ odd f^n

Q) Suppose f is neither odd nor even function.
Can we find F. cosine or F. sine ?

Eg. $f(x) = x$, $x \in [0, \pi]$.

Even extension of f:



$f_{ev} : [-\pi, \pi] \to \mathbb{R}$

$f_{ev}(x) = x, \quad x \in [0, \pi]$

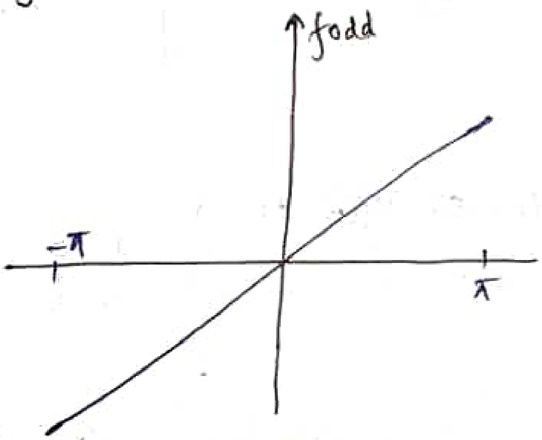
$f_{ev}(x) = -x, \quad x \in [-\pi, 0]$

$\therefore \ f_{ev}(x) = |x|, \quad x \in [-\pi, \pi].$

F.S. of f_{ev} is called as Fourier cosine Series of f.

We want F. sine Series of $f(x) = x, \quad x \in [0, \pi]$

$f_{odd} : [-\pi, \pi] \to \mathbb{R}$



such that $f_{odd}(x) = x, \quad x \in [-\pi, \pi]$.

Dirichlet Pointwise Convergence Theorem

↳ "Periodic extension" of a function f.

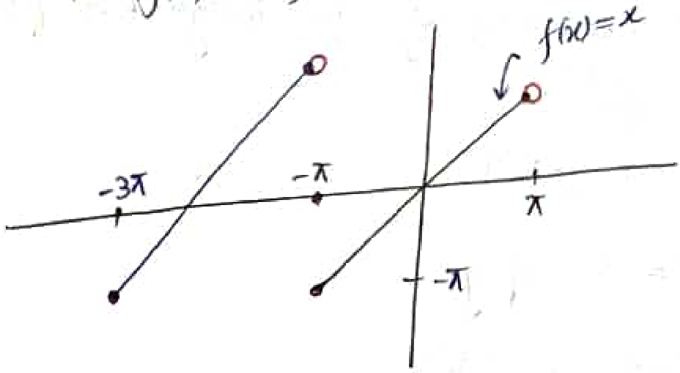
• Periodic f^n must be defined on $-\infty$ to ∞.

$$\boxed{\begin{array}{l}\textbf{Periodic function:} \\ f:\mathbb{R}\to\mathbb{R} \text{ is periodic of period } T \text{ if} \\ \qquad f(x+T)=f(x), \quad x\in\mathbb{R}.\end{array}}$$

Aim: $f:[-l,l]\to\mathbb{R}$

and we want to define "periodic extension" of f with period $2l$.

→ $f(x)=x, \quad x\in[-\pi,\pi]$



I $\quad f(-\pi)=f(\pi)=-\pi$

II $\quad f(-\pi)=f(\pi)=\pi$.

$g:\mathbb{R}\to\mathbb{R}$

$g(x)=f(x), \quad x\in[-l,l)$ or $x\in(-l,l]$ or $x\in(-l,l)$

redefine $x=l/-l$.

$g(x+2l)=g(x)$

\# $g(-l)=f(-l)$

& $g(x+2l)=g(x)$

$\Rightarrow g(-l+2l)=g(-l)$

$\Rightarrow g(l)=g(-l)$

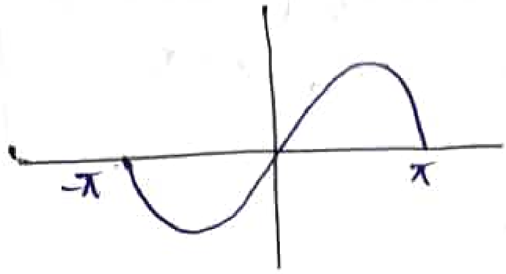
20-11-2023

Suppose f is a 2π-periodic function.

F.S. of f = F.S. of f on $[-\pi, \pi]$

Eg $f(x) = \sin x$, $x \in [-\pi, \pi]$.

$f(x) = \sin x$, $x \in \mathbb{R}$



Aim: To represent a function on $[-\pi, \pi]$ as F.S. of f.

Dirichlet point-wise convergence theorem:

Let g be a 2π-periodic extension of an integrable function f on $[-\pi, \pi]$.

g must be P-S (piece-wise smooth) on $[-\pi, \pi]$.

$\Rightarrow$ F.S. of f = F.S. of g

Eg $f(x) = \sin x$, $x \in [-\pi, \pi]$

g is 2π-periodic extension of f.

$g(x) = \sin x$, $x \in \mathbb{R}$.

F.S. of f = F.S. of g $= \frac{1}{2}\left(g(x^+) + g(x^-)\right)$ $\forall x \in \mathbb{R}$

where, $g(x^+)$ = RHL of g at x.

$g(x^-)$ = LHL of g at x.

(1) f is continuous at $x \Rightarrow$ F.S. of $f = f(x)$, $x \in (-\pi, \pi)$

(2) where F.S. of f converges outside $[-\pi, \pi]$.

Eg. $f(x) = \begin{cases} 0, & -\pi \le x < 0 \\ 4, & 0 < x \le \pi. \end{cases}$

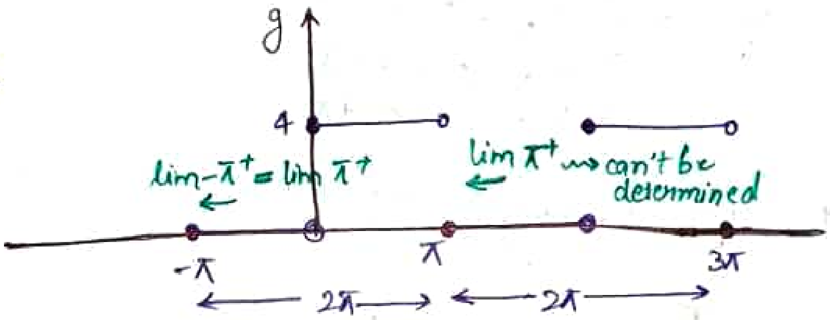
$$2 + \sum \frac{8}{n\pi} \sin nx$$
(n: odd positive)

Steps: ① Define periodic extension of f as 'g'.
② Check if g is P-S on $[-\pi, \pi]$.

$g(x+2\pi) = g(x)$.

$g(x) = f(x)$, $x \in [-\pi, \pi)$.

g:



① Let $x \in (-\pi, 0)$.

$$\text{F.S. of } f = \frac{1}{2}\left(g(x^+) + g(x^-)\right)$$
$$= \frac{1}{2}\left(f(x^+) + f(x^-)\right)$$
$$= \frac{1}{2}\left(f(x) + f(x)\right)$$
$$= f(x) = 0$$

② Let $x \in (0, \pi)$.

$$\text{F.S. of } f = f(x) = 4$$

③ Let $x = 0$, $\text{F.S. of } f = \frac{1}{2}\left(g(0^+) + g(0^-)\right)$

$$= \frac{1}{2}\left(f(0^+) + f(0^-)\right)$$
$$= \frac{1}{2}(4 + 0) = 2$$

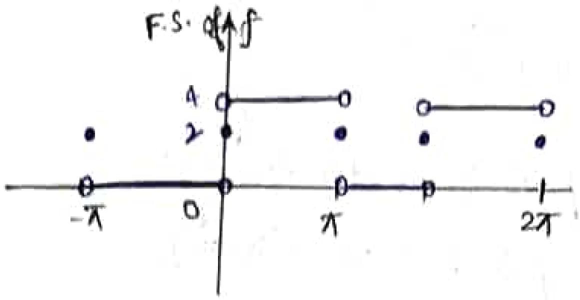
④ Let $x = \pi$.

$$\text{F.S. of } f = \frac{1}{2}\left(g(\pi^+) + g(\pi^-)\right)$$
$$= \frac{1}{2}\left(f(-\pi^+) + f(\pi^-)\right)$$
$$= \frac{1}{2}(0 + 4) = 2$$

⑤ Let $x = -\pi$.

F.S. of $f = 2$.

F.S. of f:



F.S. of $f = f$, except $x = \pm\pi, 0$

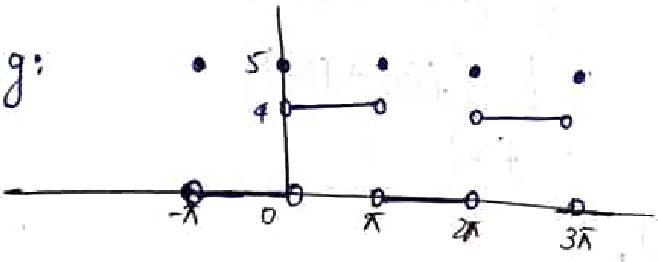
F.S. of $f = g$, except $x = \pi, 3\pi, 2\pi$.

$$\therefore \text{ F.S. of } f = \begin{cases} 0, & -\pi < x < 0 \\ 4, & 0 < x < \pi \\ 2, & x = 0, \pm\pi \end{cases}$$

F.S. of $f = f$, except $x = 0, \pm\pi$.

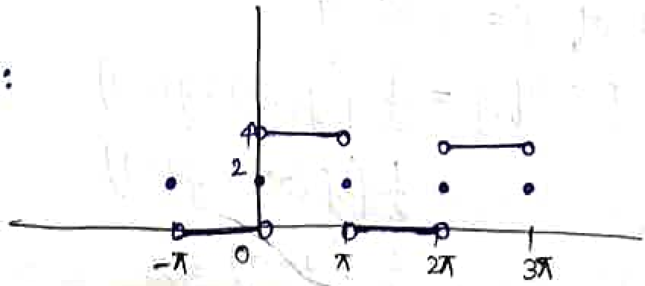
Eg. $f(x) = \begin{cases} \cancel{10}\,5, & x = -\pi, \pi, 0 \\ 0, & -\pi < x < 0 \\ 4, & 0 < x < \pi. \end{cases}$ | 2π-periodic extension of f, called as g.

g:

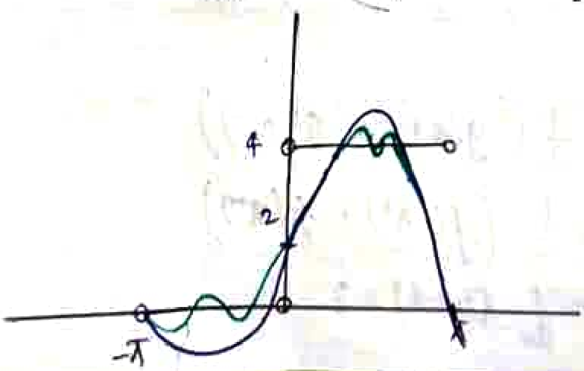


→ changes

F.S. of f:



→ Doesn't change (Same as the previous f)



$2 + \sum \frac{8}{n\pi} \sin nx$

(n: odd)

- Piecewise Continuous (P-C) on $[a,b]$.

$$a = x_0 < x_1 < \dots < x_{n-1} < x_n = b.$$

① f' must be continuous on (x_{i-1}, x_i).

② $\underbrace{f'(x_{i-1}^+)}_{\text{L.H.L of } f \text{ at } x_{i-1}}$ and $\underbrace{f'(x_i^-)}_{\text{RH limit of } f \text{ at } x_i}$ both exist. $\longrightarrow \lim_{h\to 0} \frac{f(x_i-h)-f(x)}{h} \to X$

$= \lim_{h\to 0} f(x_{i-1}-h)$ $\quad = \lim_{h\to 0} f(x_i+h)$

$[-\pi,\pi] = [-\pi,0] \cup [0,\pi]$

$(-\pi,0)$

$(0,\pi)$

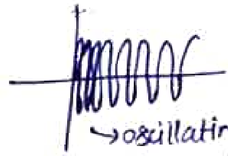
$$f' = \begin{cases} 0, & -\pi < x < 0 \\ 0, & 0 < x < \pi. \end{cases}$$

Eg $f(x) = \begin{cases} x\sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ on $[-1,1]$. $\longrightarrow$ P.C.

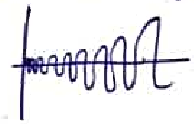
Check P.S.

Eg $f(x) = \begin{cases} \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ on $[-1,1]$. $\to$ P.C.

$\sin\frac{1}{x}$:



$x\sin\frac{1}{x}$:



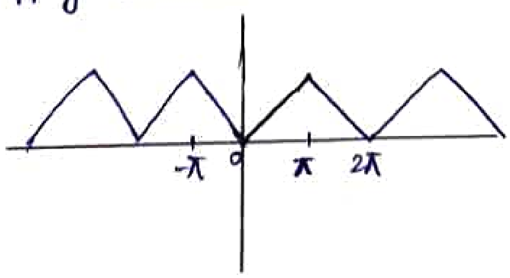
Eg $f(x) = x$, $x \in [0, \cancel{\pi}^{\,1}]$.

Discuss pointwise convergence of Fourier cosine series of f.

<u>Steps</u>: ① Define fex.

② Define g which is $\cancel{2\pi}^{\,2}$-periodic extension of fex.

③ Apply Dirichlet theorem.



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① consider

$$f_{ev} : [-\pi, \pi] \longrightarrow \mathbb{R}$$

$$f_{ev}(x) = |x|$$

② Consider 2π-periodic extension of f_{ev}, and call it as "g".

③ Apply Dirichlet theorem.

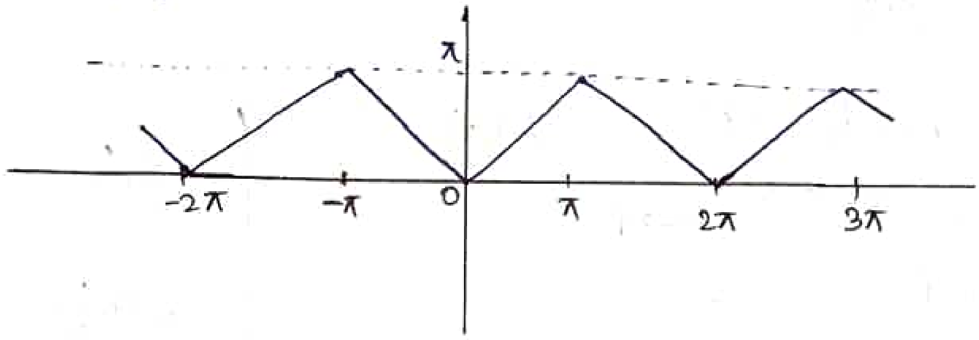
F. cosine series of f = F.S. of f_{ev}

$= F.S.$ of g

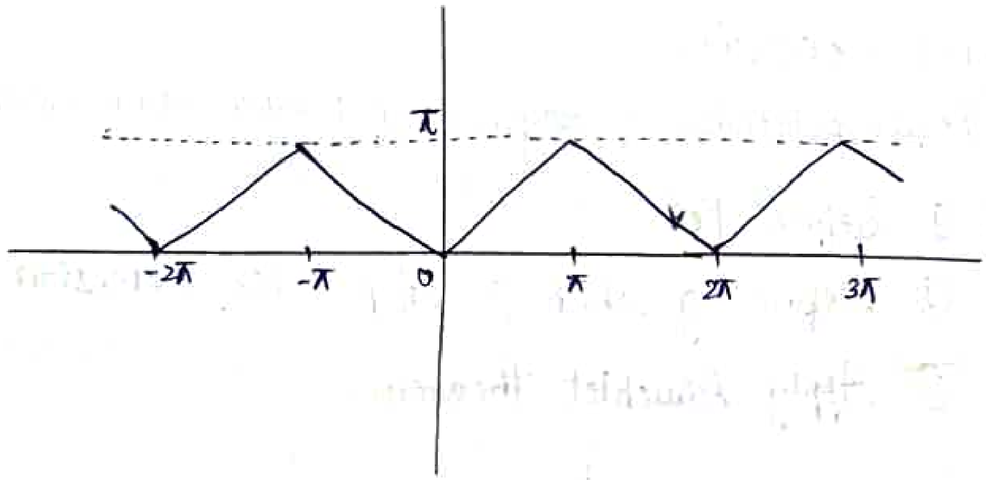
$= \frac{1}{2}\left(g(x^+) + g(x^-)\right)$

f_{ev} is same as g on $[-\pi, \pi]$.

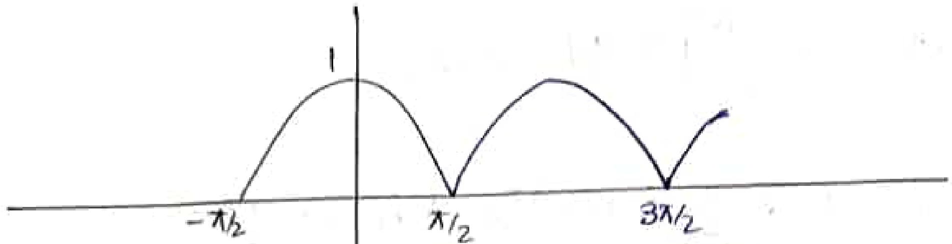
Graph of g:



Graph of F. cosine series of f : $[-\pi, 3\pi]$

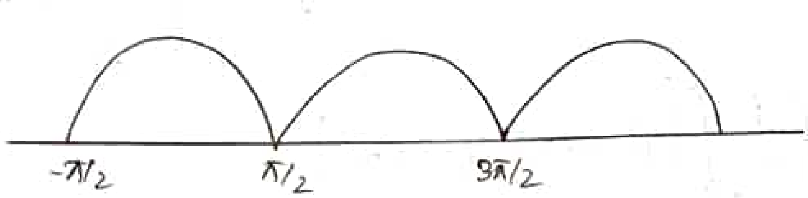


Eg. $\cos x$, $x \in [-\pi/2, \pi/2]$.



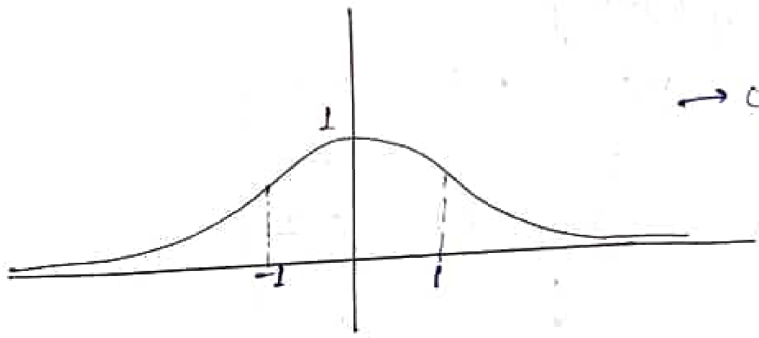
If F.S. converges here $\Rightarrow$ then F.S. will converge on the same profile outside $\longrightarrow$ F.S. is periodic

Eg. $\cos x$, $x \in \mathbb{R}$



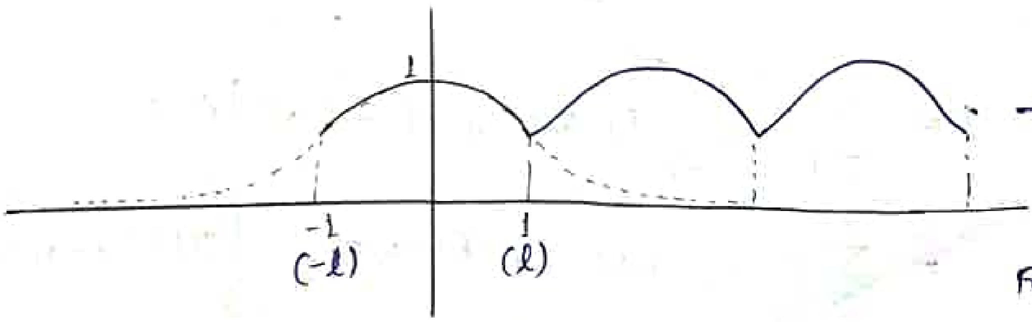
Eg. $f(x) = e^{-x^2}$ $\longrightarrow$ $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

$\rightsquigarrow$ Improper integral



$\rightarrow$ cannot be represented by F.S.

$f(x) = e^{-x^2}$, $x \in [-1, 1]$ $\rightsquigarrow$ F.S. possible ✓



$\rightarrow$ cannot be represented as F.S. but as Fourier integral.

Aim: To check what happens to F.S. representation as $l \to \infty$.

$f : [-l, l] \to \mathbb{R}$ and $\int_{-\infty}^{\infty} |f(x)|\,dx < \infty$.

$$\text{F.S. of } f = a_0 + \underbrace{\sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{l}x + b_n \sin \frac{n\pi}{l}x}_{I}.$$

$$a_0 = \frac{1}{2l} \int_{-l}^{l} f(x)\,dx \le \underbrace{\frac{1}{2l} \underbrace{\int_{-\infty}^{\infty} |f(x)|\,dx}_{\text{~~fixed~~ finite}}}_{\to 0 \text{ as } l \to \infty.}$$

$$I = \sum_{n=1}^{\infty} \left(\frac{1}{l} \int_{-l}^{l} f(t) \cos \frac{n\pi}{l} t \,dt \right) \cos \frac{n\pi}{l} x + \left(\frac{1}{l} \int_{-l}^{l} f(t) \sin \frac{n\pi}{l} t\,dt \right) \sin \frac{n\pi}{l} x$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{l} \int_{-l}^{l} f(t) \cos \frac{n\pi}{l} (t-x)\,dt \right)$$

$\boxed{\# I \text{ as } l \to \infty \,/\, \text{as } \Delta w \to 0}$

$w_n = n\frac{\pi}{l}$

(discrete frequency)

	$n=0$	$n=1$	$n=2$
$l = \pi$	0	1	2
$l = 10\pi$	0	0.1	0.2

$\downarrow$ ∞ $\qquad \downarrow$ $[0, \infty) \leftarrow w$

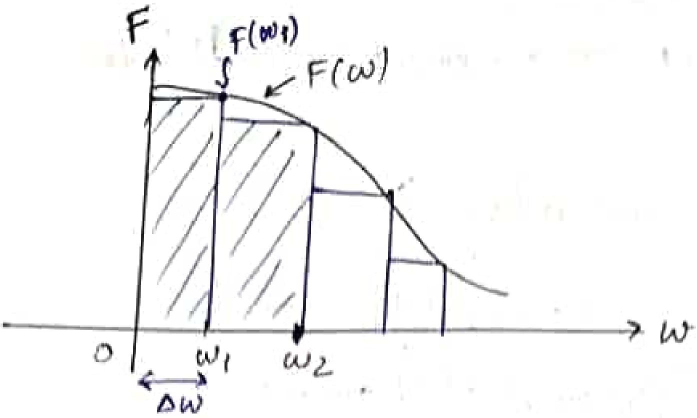
$\Delta w = \frac{\pi}{l}$.

$$I = \sum_{n=1}^{\infty} \left(\frac{1}{\pi} \int_{-l}^{l} f(t) \cos w_n (t-x)\,dt \right) \Delta w$$

$$= \sum_{n=1}^{\infty} F(w_n)\Delta w, \quad F(w_n) = \frac{1}{\pi} \int_{-l}^{l} f(t) \cos w_n (t-x)\,dt.$$

$$\lim_{\Delta w \to 0} I = \lim_{\Delta w \to 0} \sum_{n=1}^{\infty} F(w_n)\Delta w$$

$$= \int_0^{\infty} F(w)\,dw.$$



F.S. of f as $l \to \infty$

$$= \int_0^\infty F(\omega)\,d\omega$$

$$= \int_0^\infty \left(\frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos\omega(t-x)\,dt \right) d\omega$$

\# larger

$S_1 = \lim_{l \to \infty} \int_{-l}^{l} f(x)\,dx$ ✓

$\Uparrow$ ✱

$S_2 = \lim_{\substack{l \to \infty \\ m \to \infty}} \int_{-l}^{m} f(x)\,dx$ ✗

$S_1 \supset S_2$

$e \in S_2 \Rightarrow e \in S_1$

$\boxed{23\text{-}11\text{-}2023}$

$\rightarrow \int_{-\infty}^{\infty} |f(x)|\,dx < \infty.$

$[-l, l]$:

F.S. of $f \xrightarrow[\text{as } l \to \infty]{}$ F.I. of f

$$\text{F.I. of } f = \int_0^\infty \left(\frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos\omega(t-x)\,dt \right) d\omega$$

Fourier Integral

If $f =$ F.I. of f,

then $f = (F_T^{-1} \circ F_T)(f)$ ← Map

\# $f \xrightarrow{F_T} F_T(f)$, $F_T(f) \xrightarrow{F_T^{-1}} f$

$X = A^{-1}A(X)$

As F.I. of $f = \int_0^\infty \left(\frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos\omega(t-x)\,dt \right) d\omega$,

$$\text{F.I. of } f = \int_0^\infty \left(a(\omega) \cos\omega x + b(\omega) \sin(\omega x) \right) d\omega$$

$$= \int_{\omega=0}^{\infty} \left(\frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \left(\cos t x \cos \omega x + \sin \omega t \sin \omega x \, dt \right) \right) d\omega$$

$$= \int_{0}^{\infty} \left(\frac{1}{\pi} \underbrace{\int_{-\infty}^{\infty} \left(f(t) \cos \omega t \, dt \right)}_{a(\omega)} \right) \cos \omega x$$

$a(\omega) \rightarrow$ Integral depends on 'ω'

$$+ \int_{0}^{\infty} \left(\frac{1}{\pi} \underbrace{\int_{-\infty}^{\infty} \left(f(t) \sin \omega t \, dt \right)}_{b(\omega)} \right) \sin \omega x$$

$$a(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos \omega x \, dx$$

$$b(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin \omega x \, dx$$

$\therefore$ F.I. of $f = \displaystyle\int_{0}^{\infty} \left(a(\omega) \cos \omega x + b(\omega) \sin \omega x \right) d\omega$,

where $a(\omega) = \dfrac{1}{\pi} \displaystyle\int_{-\infty}^{\infty} f(x) \cos \omega x \, dx$

$b(\omega) = \dfrac{1}{\pi} \displaystyle\int_{-\infty} f(x) \sin \omega x \, dx$

Eg. $f(x) = \begin{cases} 1, & -1 \leq x \leq 1 \\ 0, & \text{elsewhere} \end{cases}$

F.I. of $f = \displaystyle\int_{0}^{\infty} \frac{2}{\pi} \frac{\sin \omega}{\omega} \cos \omega x \, d\omega \overset{?}{=} f(x)$

$b(\omega) = 0$

$$a(\omega) = \frac{1}{\pi} \int_{-1}^{1} 1 \cdot \cos \omega x \, dx$$

$$= \frac{1}{\pi} \left. \frac{\sin \omega x}{\omega} \right|_{-1}^{1}$$

$$= \frac{2 \sin \omega x}{\pi \omega}$$

F.I. Theorem

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty.$$

f must be piece-wise smooth (P-S) on every finite interval in $(-\infty, \infty)$.

$$\text{F.I. of } f = \frac{1}{2}\left(f(x^+) + f(x^-)\right) \quad \forall x \in \mathbb{R}$$

$$\text{F.I. of } f = \begin{cases} 1, & x \in (-1,1) \\ 1/2, & x = \pm 1 \\ 0, & \text{elsewhere.} \end{cases}$$

$= f$ except at $x = \pm 1$.

Eg. $\displaystyle\int_0^{\infty} \frac{\sin x}{x}\, dx = ?$

Put $x = 0$,

$$\int_0^{\infty} \frac{2}{\pi} \frac{\sin \omega}{\omega}\, d\omega = 1$$

$$\Rightarrow \int_0^{\infty} \frac{\sin \omega}{\omega}\, d\omega = \frac{\pi}{2}$$

Eg. Prove that $\displaystyle\int_0^{\infty} \frac{2}{\pi} \frac{\sin \omega}{\omega} \cos \omega\, d\omega = \begin{cases} 1, & x \in (-1,1) \\ 1/2, & x = \pm 1 \\ 0, & \text{elsewhere.} \end{cases}$