EE250: Control Systems
Solutions : Tutorial 6

24 February 2020

Q1. **Sol.** The open loop transfer function is given by $G(s)H(s) = \dfrac{1+4s}{s^2(1+s)(1+2s)}$

$c_1 : s = j\omega$ where $\omega : 0^+$ to $+\infty$
$c_2 : s = Re^{j\theta}$ where R$\rightarrow \infty$ and $\theta : +90^o$ to 0^o to -90^o
$c_3 : s = -j\omega$ where $\omega : -\infty$ to 0^+
$c_4 : s = \epsilon e^{j\phi}$ where $\epsilon \rightarrow 0$ and $\phi : -90^o$ to 0^o to 90^o

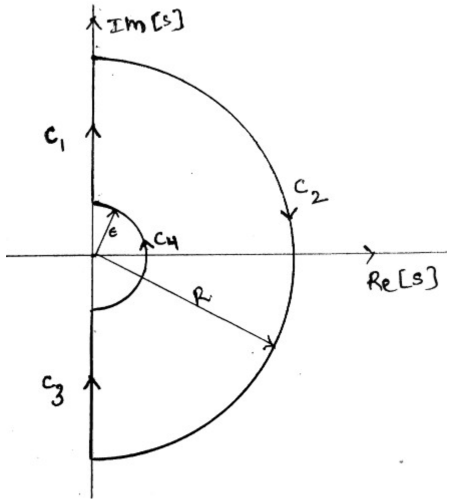

Figure 1: Nyquist Contour

Mapping of c_1:

$$\begin{aligned}
G(j\omega)H(j\omega) &= \frac{1+4j\omega}{(j\omega)^2(1+j\omega)(1+2j\omega)} \\
|GH| &= \frac{\sqrt{1+16\omega^2}}{\omega^2\sqrt{1+\omega^2}\sqrt{1+4\omega^2}} \\
\angle GH &= -\pi + tan^{-1}4\omega - tan^{-1}\omega - tan^{-1}2\omega
\end{aligned}$$

1. when $\omega = 0 \implies |GH| = \infty$ and $\angle GH = -\pi$
2. when $\omega = \infty \implies |GH| = 0$ and $\angle GH = -\pi + \frac{\pi}{2} - \frac{\pi}{2} - \frac{\pi}{2} = -\frac{3\pi}{2}$
3. when $\omega = 0.01 \implies \angle GH = -179.43^o$
   Need to check real axis crossing point
   $\implies -\pi + tan^{-1}4\omega - tan^{-1}\omega - tan^{-1}2\omega = -\pi \implies tan^{-1}4\omega - tan^{-1}\dfrac{\omega+2\omega}{1-2\omega^2} = 0$

$$\implies tan^{-1}\frac{4\omega - \frac{3\omega}{1-2\omega^2}}{1+4\omega\frac{3\omega}{1-2\omega^2}} = 0 \implies tan^{-1}\frac{4\omega - 8\omega^3 - 3\omega}{1-2\omega^2+12\omega^2} = 0$$

$\implies \omega - 8\omega^3 = 0 \implies \omega(1 - 8\omega^2) = 0 \implies \omega = 0$ or $\omega^2 = \frac{1}{8} \implies \omega = 0$
$\implies \omega = \pm 0.35 rad/sec$

For $\omega = 0.35$

$$|GH| = \frac{\sqrt{2.96}}{0.1225 \times \sqrt{1.1225} \times \sqrt{1.50}} = 10.67$$

$\therefore$ Mapping of c_1 and c_3 (mirror image) is as shown in Figure 2 below.

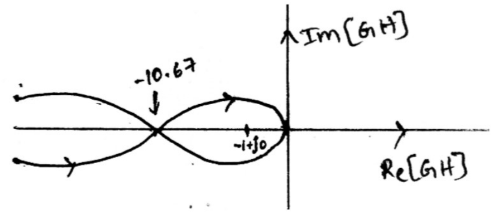

Figure 2: Nyquist Plot corresponding to c_1 and c_2

Mapping of c_3 is ignored.

Mapping of c_4:

$$G(\epsilon e^{j\phi})H(\epsilon e^{j\phi}) = \frac{1 + 4\epsilon e^{j\phi}}{\epsilon^2 e^{2j\phi}(1 + \epsilon e^{j\phi})(1 + 2\epsilon e^{j\phi})} \approx \frac{1}{\epsilon^2 e^{2j\phi}}$$

since $\epsilon \to 0$
$\therefore$ radius $\to \infty$ and angle:-2ϕ; π to $\frac{\pi}{2}$ to 0 to -$\frac{\pi}{2}$ to $-\pi$.
Complete Nyquist plot is as shown below

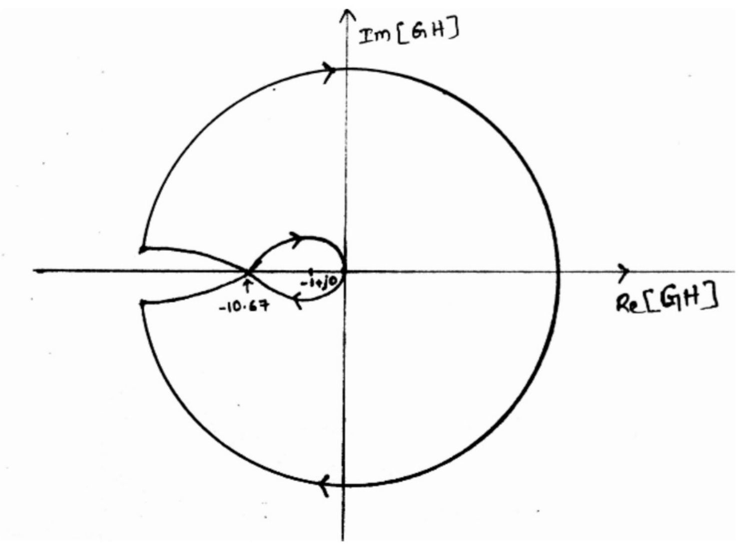

Figure 3: Nyquist Plot

form Nyquist plot $\implies$ N=2 and P=0

$$Z = N + P = 2 + 0 = 2$$

$\therefore$ Closed loop system is unstable with 2 R.H.S poles.

Q2. **Sol** Now, if a proportional gain $K > 0$ is added to the forward path, the sinusoidal magnitude of the system will be modified as

$$|GH| = \frac{K\sqrt{1+16\omega^2}}{\omega^2\sqrt{1+\omega^2}\sqrt{1+4\omega^2}}$$

The Nyquist plot will cross the negative real axis at $-10.67K$. If the critical point, i.e., $-1+j0$ point lies in the left of this crossing point, the closed loop system will be stable since the Nyquist plot in that case does not encircle the critical point. Thus, the stable range of K is

$$-10.67K > -1 \quad or,\ K < 1/10.67 \Rightarrow K < 0.094$$

Let's take $K = 0.05$. The gain at $\omega = 0.35$ becomes $10.67 \times 0.05 = 0.5335$. Thus, the gain margin is $1/0.5335 = 1.87$. In dB, it is 5.44 dB. For phase margin equate $|GH|$ to 1. Find the gain crossover frequency ω_g. Calculate the phase angle (in degree) at ω_g. Add 180^o to get the PM in degree. Complete this exercise.

Q3. **Sol** The open-loop transfer function is given by

$$G(s)H(s) = L(s) = \frac{K(1+0.5s)(s+1)}{(1+10s)(s-1)}$$

We have to determine the range of values of K for which the system is stable.

Step 1. Draw the Nyquist contour as given in Figure 4.
Find the magnitude and phase of $L(j\omega)$ at $\omega = 0$ and $\omega = \infty$.

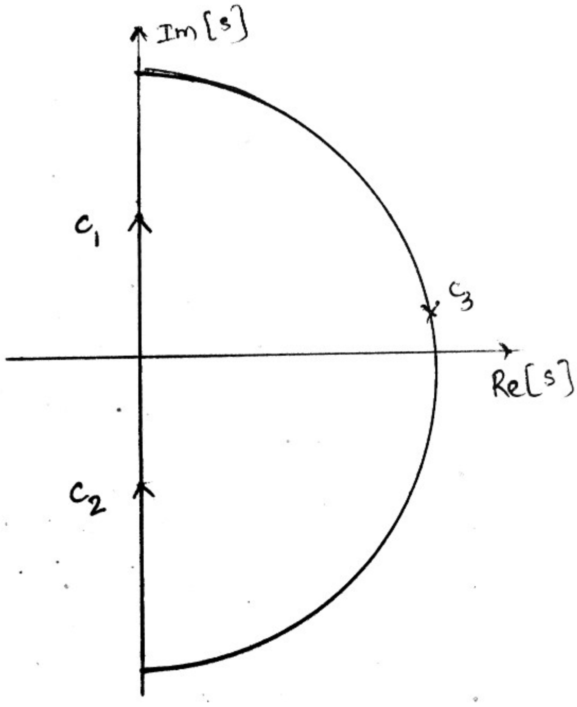

Figure 4: Nyquist contour

$$\begin{aligned} L(j\omega) &= \frac{K(1+j0.5\omega)(1+j\omega)}{(1+j10\omega)(j\omega-1)} \\ &= \frac{K\sqrt{1+0.25\omega^2}}{\sqrt{1+100\omega^2}}\angle\left[\tan^{-1}(0.5\omega)+\tan^{-1}(\omega)-tan^{-1}(10\omega)-180^\circ+\tan^{-1}(w)\right] \end{aligned}$$

$$\begin{array}{lll} \text{At } \omega = +\infty & |L| = \frac{K}{20} & \angle L = 90 + 90 - 90 - 180 + 90 = 0^\circ \\ \text{At } \omega = 0 & |L| = \frac{K}{20} & \angle L = -180^\circ \\ \text{At } \omega = -\infty & |L| = \frac{K}{20} & \angle L = -90 - 90 + 90 - 180 - 90 = 0^\circ \end{array}$$

Step 2. Find the point where $L(j\omega)$ intersects the real axis. For this, equate the imaginary part of $L(j\omega)$ to zero.

$$\begin{aligned} Im[L(j\omega)] &= \frac{(7.5\omega - 19.5\omega^3)}{1 + 101\omega^2 + 100\omega^4} = 0 \\ \text{or,} \quad & \omega(7.5 - 19.5\omega^2) = 0; \quad \text{this gives} \\ \omega &= 0.62 \ \text{ and } \ |L| = 0.167K \end{aligned}$$

Also, you can equate the phase angle to π (as done in class) to get the same solution. With these 3 points, it is not possible to draw the nyquist plot. The sense of direction of $L(s)$-plot can not be guessed. This is very crucial for applying Nyquist criterion to this problem. Hence we need at least one more point. Lets compute the magnitude and phase of $L(j\omega)$ at $\omega = 0.1$ rad/sec. This gives,

$$L(j\omega) = 0.708K\angle 150^\circ$$

The contour in s-plane is given in Figure 4. Since we are moving in clockwise direction along the contour in the s-plane, the corresponding $L(s) - plot$ will have a direction from $\omega = 0^+$ towards $\omega = +\infty$ point i.e c_1. The contour from $\omega = -\infty$ to $\omega = 0^-$ would be a mirror image of the plot obtained for positive frequencies. The complete Nyquist plot is shown in Figure 5.

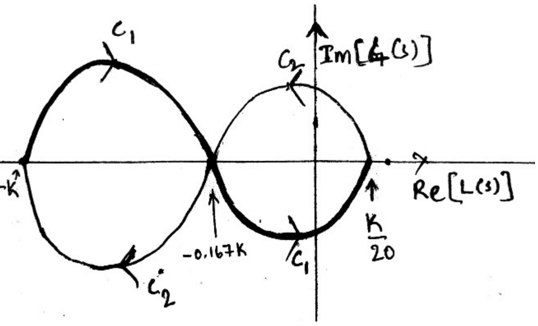

Figure 5: Nyquist Plot

Consider the following two cases:

- $0.167K > 1$ or $K > 5.988$. In this case the critical point $-1 + j0$ would lie in the right hand side loop. Hence, the number of counter clockwise encirclement $N = -1$. Because the point is encircled in clockwise direction only once. Also, the number of poles of $L(s)$ lying on right-half of s-plane $P = 1$. This gives

$$Z = N + P = -1 + 1 = 0$$

  Hence, the system is **stable** when $K > 6$.

- $0.167K < 1$ or $K < 5.988$. In this case, the critical point lies inside the left hand loop. The point $-1 + j0$ is encircled by the $L(s)$-plot in CCW direction once. This gives $N = +1$. Hence, the number roots lying on right-half s-plane is

$$Z = N + P = 1 + 1 = 2$$

  Hence, the system is **unstable** when $K < 6$.