Assignment-1

(1.) ⓐ $y' + \frac{y}{x^2} = 2xe^{1/x}$ → linear differential eqⁿ.

$$IF = e^{\int \frac{1}{x^2}dx} = e^{-\frac{1}{x}}$$

$$\therefore e^{-\frac{1}{x}} y = \int e^{-\frac{1}{x}} (2xe^{1/x})\, dx$$

$$\Rightarrow y e^{-1/x} = \int 2x\, dx$$

$$\Rightarrow y e^{-1/x} = x^2 + c$$

$$\therefore y = x^2 e^{1/x} + c e^{1/x}, \quad c: \text{constant}.$$

ⓑ $y' + 3y = \sin x$

$$IF = e^{\int 3dx} = e^{3x}$$

$$\therefore y \cdot e^{3x} = \int e^{3x} \sin x\, dx$$

let $I = \int e^{3x} \sin x\, dx = -e^{3x}\cos x + \int e^{3x} (3)(\cos x)\, dx$

$$= -e^{3x}\cos x + 3\left[e^{3x}\sin x - 3\int e^{3x}\sin x\, dx\right]$$

$$\Rightarrow I = -e^{3x}\cos x + 3e^{3x}\sin x - 9I$$

$$\Rightarrow I = \frac{e^{3x}}{10}(3\sin x - \cos x)$$

$$\therefore y \cdot e^{3x} = \frac{e^{3x}}{10}(3\sin x - \cos x) + c$$

$$\Rightarrow y = \frac{1}{10}(3\sin x - \cos x) + ce^{-3x}$$

② ⓐ $y'' + 2ky' + k^2 y = 0, \; y_1(x) = e^{-2x}$

$p(x) = 2k$

let $U(x) = \frac{1}{y_1^2} e^{-\int p dx} = \frac{1}{e^{-4x}} e^{-\int 2k dx} = \frac{e^{-2kx}}{e^{-4x}} = e^{4x - 2kx}$

so, $u(x) = \int U(x) dx = \int e^{(4x - 2kx)}\, dx = \frac{e^{4x-2kx}}{4-2k}$

$$\therefore y_2 = u(x)\, y_1(x) = \frac{e^{(4-2k)x}}{4-2k} e^{-2x} = \frac{e^{(2-2k)x}}{4-2k}$$

Hence, the basis is $\{y_1, y_2\} \equiv \left\{ e^{-2x}, \frac{e^{(2-2k)x}}{4-2k} \right\}$.

(b) $x^2y'' + xy' - 4y = 0,\ y_1(x) = x^2$

$\Rightarrow y'' + \frac{1}{x}y' - \frac{4}{x^2}y = 0$

Here, $p(x) = \frac{1}{x}$

let $U(x) = \frac{1}{y_1^2}e^{-\int p\,dx} = \frac{1}{x^4}e^{-\int \frac{1}{x}dx} = \frac{1}{x^4}e^{-\ln x} = \frac{1}{x^4}\cdot\frac{1}{x} = \frac{1}{x^5}$

So, $u(x) = \int U(x)dx = \int \frac{1}{x^5}dx = -\frac{1}{4x^4}$

$\therefore y_2 = y_1 . u(x) = x^2\left(-\frac{1}{4x^4}\right) = -\frac{1}{4x^2}$

Hence, basis is $\left\{x^2, -\frac{1}{4x^2}\right\}$.

(3) (a) $y^{(v)} - 7y''' + 12y' = 0$

characteristic eqn : $\lambda^5 - 7\lambda^3 + 12\lambda = 0$

$\Rightarrow \lambda = 0,\ \lambda^4 - 7\lambda^2 + 12\lambda = 0$

$\Rightarrow \lambda = 0,\ (\lambda^2 - 4)(\lambda^2 - 3) = 0$

$\Rightarrow \lambda = 0,\ \lambda = \pm 2,\ \lambda = \pm\sqrt{3}$.

$\therefore y = c_1 + c_2e^{2x} + c_3e^{-2x} + c_4e^{\sqrt{3}x} + c_5e^{-\sqrt{3}x}$

(b) $y^{(iv)} - 8y''' + 26y'' - 40y' + 25y = 0$.

Char. eqn: $\lambda^4 - 8\lambda^3 + 26\lambda^2 - 40\lambda + 25 = 0$

$\Rightarrow (\lambda^2 - 4\lambda + 5)^2 = 0$

$\Rightarrow \lambda = 2 \pm i,\ 2 \pm i \rightarrow$ multiple root (Complex)

$\lambda = \frac{4 \pm \sqrt{16 - 20}}{2} = \frac{4 \pm \sqrt{-4}}{2} = 2 \pm i$

$\therefore y = c_1e^{2x}\cos x + c_2e^{2x}\sin x + xc_3e^{2x}\sin x + xc_4e^{2x}\cos x$

$\Rightarrow y = e^{2x}\cos x(c_1 + xc_4) + e^{2x}\sin x(c_2 + xc_3)$.

(c) $16y^{(iv)} - 8y'' + y = 0$.

Char. eqn : $16\lambda^4 - 8\lambda^2 + 1 = 0$

$\lambda^2 = t \Rightarrow 16t^2 - 8t + 1 = 0$

$\Rightarrow t = \lambda^2 = \frac{8 \pm \sqrt{64 - 64}}{32}$

$\Rightarrow \lambda^2 = \frac{1}{4}$

$\Rightarrow \lambda = \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}$

$\therefore y(x) = c_1e^{x/2} + c_2xe^{x/2} + c_3e^{-x/2} + c_4xe^{-x/2}$

$\Rightarrow y(x) = e^{x/2}(c_1 + c_2x) + e^{-x/2}(c_3 + c_4x)$

$y'''xy$ char. e

(a) $y''' + y' = 2x^2 + 4\sin x$

char. eqn : $\lambda^3 + \lambda = 0$

$\Rightarrow \lambda = 0,\ \lambda^2 = -1$

$\Rightarrow \lambda = 0,\ \lambda = \pm i$

$y_h(x) = C_1 + C_2\cos x + C_3\sin x$

and, $y_p(x) = A_0 + A_1 x + A_2 x^2 + A_3\cos x + A_4\sin x$

$\Rightarrow y_p'(x) = A_1 + 2xA_2 + (-A_3\sin x) + A_4\cos x$

$\Rightarrow y_p''(x) = 2A_2 - A_3\cos x - A_4\sin x$

$\Rightarrow y_p'''(x) = A_3\sin x - A_4\cos x$

$\therefore y_p''' + y_p' = A_1 + 2xA_2 = 2x^2 + 4\sin x$

Try with $y_p(x) = A_0 x + A_1 x^2 + A_2 x^3 + A_3 x\cos x + A_4 x\sin x$ —(1)

$\Rightarrow y_p'(x) = A_0 + 2A_1 x + 3A_2 x^2 + A_3\cos x - A_3 x\sin x + A_4\sin x + A_4 x\cos x$

$\Rightarrow y_p''(x) = 2A_1 + 6A_2 x - A_3\sin x - A_3\sin x - A_3 x\cos x + A_4\cos x + A_4\cos x - A_4 x\sin x$

$\Rightarrow y_p'''(x) = 6A_2 - 2A_3\cos x - A_3\cos x - A_3 x\sin x - 2A_4\sin x - A_4\sin x - A_4 x\cos x$

$\therefore y'''_{(p)} + y'_{(p)} = 6A_2 - \overset{2}{\cancel{3}}A_3\cos x - A_3\sin x - 3A_4\sin x - A_4 x\cos x + A_0 + 2A_1 x + 3A_2 x^2 + A_3\cos x - A_3 x\sin x + A_4\sin x + A_4 x\cos x = 2x^2 + 4\sin x$

$\Rightarrow (A_0 + 6A_2) = 0 \Rightarrow A_0 = -6A_2 = -\overset{2}{\cancel{6}}\left(\frac{2}{\cancel{3}}\right) = -4$

$2A_1 = 0 \Rightarrow A_1 = 0$

$3A_2 = 2 \Rightarrow A_2 = 2/3$

$-2A_3 = 0 \Rightarrow A_3 = 0.$

$-3A_4 - \overset{0}{\cancel{A_3}} + A_4 = 4 \Rightarrow A_4 = -2$

Thus, $y_p(x) = -4x + \frac{2}{3}x^3 - 2x\sin x.$

$\therefore$ General soln: $C_1 + C_2\cos x + C_3\sin x - 4x + \frac{2}{3}x^3 - 2x\sin x.$

4 (b) $y^{(V)} + 2y''' + y' = 2x + \sin x + \cos x$.

Char. eqn: $\lambda^5 + 2\lambda^3 + \lambda = 0$

$\Rightarrow \lambda(\lambda^4 + 2\lambda^2 + 1) = 0$

$\Rightarrow \lambda(\lambda^2+1)^2 = 0$

$\Rightarrow \lambda = 0, \pm i, \pm i$

$y_h(x) = C_1 + C_2\cos x + xC_3\cos x + C_4\sin x + x\,C_4\sin x$.

Take $y_p(x) = A_0 + A_1 x + B_1\cos x + B_2\sin x$.

$\Rightarrow y_p'(x) = A_1 - B_1\sin x + B_2\cos x$

$y_p''(x) = -B_1\cos x - B_2\sin x$

$y_p'''(x) = +B_1\sin x - B_2\cos x$

$y^{(iv)}(x) = B_1\cos x + B_2\sin x$

$y^{(v)}(x) = -B_1\sin x + B_2\cos x$

$\therefore y_p^{(V)} + 2y_p''' + y_p' = (-\cancel{B_1} \cancel{+2B_1} - \cancel{B_1})\sin x + (\cancel{B_2} - \cancel{2B_2} + \cancel{B_2})\cos x + A_1$

$\longrightarrow$ Solution.

Try with $y_p(x) = x(A_0 + A_1 x + B_1\cos x + B_2\sin x) = xA_0 + A_1x^2 + B_1 x\cos x + B_2 x\sin x$

$\Rightarrow y_p'(x) = A_0 + 2A_1 x + B_1\cos x - B_1 x\sin x + B_2\sin x + B_2 x\cos x$

$y_p''(x) = 2A_1 - B_1\sin x - B_1\sin x - B_1 x\cos x + B_2\cos x + B_2\cos x - B_2 x\sin x$

$y_p'''(x) = -2B_1\cos x - B_1\cos x + B_1 x\sin x - 2B_2\sin x - B_2\sin x - B_2 x\cos x$

$y_p^{(iv)}(x) = +3B_1\sin x + B_1\sin x + B_1 x\cos x - 2B_2\cos x - B_2\cos x + B_2 x\sin x - B_2\cos x$

$y_p^{(V)}(x) = 4B_1\cos x + B_1\cos x - B_1 x\sin x + 4B_2\sin x + B_2\sin x + B_2 x\cos x$

$\therefore y_p^{(V)} + 2y_p''' + y_p' = \cos x(5B_1 - 6B_1 + B_1) + \sin x(5B_2 - 6B_2 + B_2)$
$+ x\sin x(-\cancel{B_1} + \cancel{2B_1} - \cancel{B_1}) + x\cos x(\cancel{B_2} - \cancel{2B_2} + \cancel{B_2})$
$+ A_0 + 2A_1 x$

$\longrightarrow$ solution.

Try with

$\Rightarrow y_p'$

Try with $y_p(x) = x^2(A_0 + A_1 x + B_1\cos x + B_2\sin x)$

$= A_0x^2 + A_1x^3 + B_1x^2\cos x + B_2x^2\sin x$

$\Rightarrow y_p'(x) = 2A_0x + 3A_1x^2 + 2B_1x\cos x - B_1x^2\sin x + 2B_2x\sin x + B_2x^2\cos x$

$y_p''(x) = 2A_0 + 6A_1x + 2B_1\cos x - 2B_1x\sin x - 2B_1x\sin x - B_1x^2\cos x + 2B_2\sin x + 2B_2x\cos x + 2B_2x\cos x - B_2x^2\sin x$

$y_p'''(x) = 6A_1 - 2B_1\sin x - 2B_1\sin x - 2B_1x\cos x - 2B_1\sin x - 2B_1x\cos x + B_1x^2\sin x - 2B_1x\cos x + 2B_2\cos x + 2B_2\cos x - 2B_2x\sin x + 2B_2\cos x - 2B_2x\sin x - 2B_2x\sin x - B_2x^2\cos x$

$= 6A_1 - 6B_1\sin x + 6B_2\cos x - 6B_1x\cos x + B_1x^2\sin x - B_2x^2\cos x - 6B_2x\sin x$

$y_p^{(iv)}(x) = -6B_1\cos x - 6B_2\sin x - 6B_1\cos x + 6B_1x\sin x + 2B_1x\sin x + B_1x^2\cos x - 2B_2x\cos x + B_2x^2\sin x - 6B_2\sin x - 6B_2x\cos x$

$= -12B_1\cos x - 12B_2\sin x + 8B_1x\sin x - 8B_2x\cos x + B_1x^2\cos x + B_2x^2\sin x$

$y_p^{(v)}(x) = 12B_1\sin x - 12B_2\cos x + 8B_1\sin x + 8B_1x\cos x - 8B_2\cos x + 8B_2x\sin x - B_1x^2\sin x + 2B_1x\cos x + 2B_2x\sin x + B_2x^2\cos x$

$= 20B_1\sin x - 20B_2\cos x + 10B_1x\cos x + 10B_2x\sin x - B_1x^2\sin x + B_2x^2\cos x.$

$\therefore y_p^{(v)} + 2y_p''' + y_p' = \sin x(20B_1 - 12B_1) + \cos(-20B_2 + 12B_2) + x\cos x(10B_1 - 12B_1 + 2B_1) + x\sin x(10B_2 - 12B_2 + 2B_2) + x^2\sin x(-B_1 + B_1 - B_1) + x^2\cos x(B_2 - B_2 + B_2) + 2A_0x + 3A_1x^2 = 2x + \sin x + \cos x$

$\Rightarrow 20B_1 - 12B_1 = 1 \Rightarrow 8B_1 = 1 \Rightarrow B_1 = 1/8$

$-20B_2 + 12B_2 = 1 \Rightarrow -8B_2 = 1 \Rightarrow B_2 = -1/8$

$2A_0 = 2,\ A_1 = 0 \Rightarrow A_0 = 1,\ A_1 = 0$

$\therefore y_p = x^2\left(1 + \frac{\cos x}{8} - \frac{\sin x}{8}\right)$

Hence, general soln: $c_1 + c_2\cos x + xc_3\cos x + c_4\sin x + xc_4\sin x + x^2 + x^2\frac{\cos x}{8} - x^2\frac{\sin x}{8}.$

(5) $y''' - 6y'' + 9y' - 4y = 8x^2 + 3 - 6e^{2x}$, $y(0) = 1, y'(0) = 7, y''(0) = 10$.

Char. eqn: $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

$\Rightarrow (\lambda - 1)(\lambda^2 - 5\lambda + 4) = 0$

$\Rightarrow (\lambda - 1)(\lambda - 4)(\lambda - 1) = 0$

$\Rightarrow \lambda = 1, 4, 1$

$\therefore y_h(x) = c_1 e^x + c_2 x e^x + c_3 e^{4x}$.

Take $y_p(x) = A_0 + A_1 x + A_2 x^2 + B e^{2x}$

$\Rightarrow y_p'(x) = A_1 + 2A_2 x + 2Be^{2x}$

$y_p''(x) = 2A_2 + 4Be^{2x}$

$y_p'''(x) = 8Be^{2x}$

$\therefore y_p''' - 6y'' + 9y' - 4y = 8Be^{2x} - 12A_2 - 24Be^{2x} + 9A_1 + 18A_2 x + 18Be^{2x} - 4A_0 - 4A_1 x - 4A_2 x^2 - 4Be^{2x} = 8x^2 + 3 - 6e^{2x}$

$\Rightarrow 8 - 24B + 18B - 4B = -6 \Rightarrow -2B = -6 \Rightarrow B = 3$

$-12A_2 + 9A_1 - 4A_0 = 3 \Rightarrow 4A_0 = +21 - 81 = -60 \Rightarrow A_0 = -15$

$18A_2 - 4A_1 = 0 \Rightarrow -36 = 4A_1 \Rightarrow A_1 = -9$

$-4A_2 = 8 \Rightarrow A_2 = -2$

$\therefore y_p(x) = -15 - 9x - 2x^2 + 3e^{2x}$.

Thus, general soln: $y = c_1 e^x + c_2 x e^x + c_3 e^{4x} - 15 - 9x - 2x^2 + 3e^{2x}$.

Given that

$y(0) = 1 \Rightarrow c_1 + 0 + c_3 - 15 + 3 = 1 \Rightarrow c_1 + c_3 = 13$ — (i)

$y'(0) = 7 \Rightarrow c_1 e^x + c_2 e^x + c_2 x e^x + 4c_3 e^{4x} - 9 - 4x + 6e^{2x} \Big|_0 = 7$

$\Rightarrow c_1 + c_2 + 4c_3 = 10$ — (ii)

$y''(0) = c_1 e^x + 2c_2 e^x + c_2 x e^x + 16 c_3 e^{4x} - 4 + 12e^{2x} \Big|_0 = 10$

$\Rightarrow c_1 + 2c_2 + 16c_3 + 12 - 4 = 10$

$\Rightarrow c_1 + 2c_2 + 16c_3 = 2$ — (iii)

(iii) $- 2$(ii) $\Rightarrow (c_1 - 2c_1) + (16c_3 - 8c_3) = (2 - 20)$

$\Rightarrow 8c_3 - c_1 = -18$

$\Rightarrow c_1 - 8c_3 = 18$ — (iv)

(6) $y''' + 3y'' + 3y' + y = 2e^{-x} - x^2e^{-x}$

Char. eqn: $\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$

$\Rightarrow (\lambda+1)(\lambda^2 + 2\lambda + 1) = 0$

$\Rightarrow \lambda = -1, -1, -1$

$y_h(x) = c_1e^{-x} + c_2xe^{-x} + c_3x^2e^{-x}$.

Now, $y_p(x) = e^{-x}\int \frac{W_1}{W} r(x)\,dx + xe^{-x}\int \frac{W_2}{W} r(x)\,dx + x^2e^{-x}\int \frac{W_3}{W} r(x)\,dx$.

where,

$$W = \begin{vmatrix} e^{-x} & xe^{-x} & x^2e^{-x} \\ -e^{-x} & (e^{-x} - xe^{-x}) & (2xe^{-x} - x^2e^{-x}) \\ e^{-x} & (-e^{-x} - e^{-x} + xe^{-x}) & (2e^{-x} - 2xe^{-x} - 2xe^{-x} + x^2e^{-x}) \end{vmatrix}$$

$$= e^{-3x}\begin{vmatrix} 1 & x & x^2 \\ -1 & 1-x & 2x-x^2 \\ 1 & x-2 & 2-4x+x^2 \end{vmatrix}$$

$$= e^{-3x}\left[(2 - \cancel{4x} + x^2 - 2x + \cancel{4x^2} - \cancel{x^3}) - (2x^2 - \cancel{4x} - \cancel{x^3} + \cancel{2x^2}) - x(-2 + 4x - x^2 - 2x + x^2) + x^2(2 - \cancel{x} + \cancel{x} - 1)\right]$$

$$= e^{-3x}\left[\cancel{x^2} + \cancel{2x} + 2 + \cancel{2x^2} - 2x^2 + \cancel{x^2}\right]$$

$$= 2e^{-3x}$$

$$W_1(x) = \begin{vmatrix} 0 & xe^{-x} & x^2e^{-x} \\ 0 & e^{-x}(1-x) & e^{-x}(-x^2+2x) \\ 1 & e^{-x}(-2+x) & e^{-x}(x^2-4x+2) \end{vmatrix}$$

$$= +xe^{-x}(e^{-x}(-x^2+2x)) + x^2e^{-x}(-e^{-x}(1-x))$$

$$= e^{-2x}(-\cancel{x^3} + \cancel{2}x^2 - x^2 + \cancel{x^3})$$

$$= x^2e^{-2x}$$

$\Rightarrow c_1 = 18 + 8c_3 = 13 - c_3$

$\Rightarrow 9c_3 = -5 \Rightarrow c_3 = -\frac{5}{9}$

$\Rightarrow c_1 = 13 + \frac{5}{9} = \frac{117+5}{9} = \frac{122}{9}$

~~and, $c_3 = \frac{10 - c_1 - c_2}{4} = \frac{10 - \frac{112}{9} + \frac{5}{9}}{4} = \frac{95 - 112}{36} = \frac{17}{36}$~~

$c_2 = 10 - 4c_3 - c_1$

$= 10 + \frac{20}{9} - \frac{112}{9} = \frac{110 - 122}{9} = -\frac{12}{9} = -\frac{4}{3}$

$\therefore y(x) = \frac{122}{3}e^x - \frac{4}{3}xe^x - \frac{5}{9}e^{4x} + 3e^{2x} - 2x - 9x - 15.$

(6) (a) $y'' + a^2 y = \sec ax$

char. eqn : $\lambda^2 + a^2 = 0$

$\Rightarrow \lambda = \pm ai$

$y_h(x) = c_1 \cos ax + c_2 \sin ax.$

and, $y_p(x) = y_1 \int \frac{W_1}{W} r(x)\,dx + y_2 \int \frac{W_2}{W} r(x)\,dx,$

where $\left.\begin{array}{l} y_1 = \cos ax \\ y_2 = \sin ax \end{array}\right\}$ basis, $r(x) = \sec ax$

and, $W = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a\cos^2 ax + a\sin^2 ax = a$

$W_1 = \begin{vmatrix} 0 & \sin ax \\ 1 & a\cos ax \end{vmatrix} = -\sin ax$

$W_2 = \begin{vmatrix} \cos ax & 0 \\ -a\sin ax & 1 \end{vmatrix} = \cos ax$

$\Rightarrow y_p(x) = \cos ax \int -\frac{\sin ax}{a} \sec ax + \sin ax \int \frac{\cos ax}{a} \sec ax\,dx$

$= \frac{\cos ax}{a^2} \ln|\cos ax| + \frac{x\sin ax}{a}$

Hence, $y(x) = y_h(x) + y_p(x)$

$\Rightarrow y(x) = c_1 \cos ax + c_2 \sin ax + \frac{\cos ax}{a^2} \ln|\cos ax| + \frac{x \sin ax}{a}.$

(b)

$$W_2(x) = \begin{vmatrix} e^{-x} & 0 & e^{-x}x^2 \\ -e^{-x} & 0 & e^{-x}(-x^2+2x) \\ e^{-x} & 1 & e^{-x}(x^2-4x+2) \end{vmatrix}$$

$$\cancel{= e^{-x}\left(e^{-x}(-x^2+4x) - e^{-x}(-x^2+2x)\right) \; e^{-x}x^2(-e^{-x})}$$

$$= e^{-x}\left(-e^{-x}(-x^2+2x)\right) + e^{-x}x^2(-e^{-x})$$

$$= \cancel{e^{-2x}x^2} - 2x e^{-2x} - \cancel{x^2 e^{-2x}}$$

$$= -2x e^{-2x}$$

$$W_3(x) = \begin{vmatrix} e^{-x} & x e^{-x} & 0 \\ -e^{-x} & (1-x)e^{-x} & 0 \\ e^{-x} & (x-2)e^{-x} & 1 \end{vmatrix}$$

$$= e^{-x}(1-x)e^{-x} - x e^{-x}(-e^{-x})$$

$$= e^{-2x} - \cancel{x e^{-2x}} + \cancel{x e^{-2x}}$$

$$= e^{-2x}$$

$$\therefore y_p(x) = e^{-x}\int \frac{x^2 e^{-x}}{2e^{-3x}}(2e^{-x} - x^2 e^{-x})\,dx$$

$$+ x e^{-x}\int \frac{-2x e^{-2x}}{2e^{-3x}}(2e^{-x} - x^2 e^{-x})\,dx$$

$$+ x^2 e^{-x}\int \frac{e^{-2x}}{2e^{-3x}}(2e^{-x} - x^2 e^{-x})\,dx$$

$$= \frac{e^{-x}}{2}\int (2x^2 - x^4)\,dx - x e^{-x}\int (2x - x^3)\,dx + \frac{x^2 e^{-x}}{2}\int (2 - x^2)\,dx$$

$$\cancel{= \frac{e^{-x}}{2}\left[\int 2x^2 - x^4\,dx\right] - x e^{-x}\int (2x - x^3)\,dx}$$

$$= \frac{e^{-x}}{2}\left[\frac{2}{3}x^3 - \frac{x^5}{5}\right] - x e^{-x}\left[x^2 - \frac{x^4}{4}\right] + \frac{x^2 e^{-x}}{2}\left[2x - \frac{x^3}{3}\right]$$

$\therefore$ General soln: $y = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x}$
$+ \frac{e^{-x}}{2}\left(\frac{2}{3}x^3 - \frac{x^5}{5}\right) - x e^{-x}\left(x^2 - \frac{x^4}{4}\right) + \frac{x^2 e^{-x}}{2}\left(2x - \frac{x^3}{3}\right)$

y''

6.(c) $y^{(iv)} - 16y = x^2 \sin 2x + x^4 e^{2x}$

Char. eqn: $\lambda^4 - 16 = 0 \Rightarrow \lambda^4 = 16$
$\Rightarrow \lambda^2 = 4, \lambda^2 = -4$
$\Rightarrow \lambda = \pm 2, \lambda = \pm 2i$

$y_h(x) = c_1 e^{2x} + c_2 e^{-2x} + c_3 \sin 2x + c_4 \cos 2x$

Now,
$y_p(x) = e^{2x}\int \frac{W_1}{W} r(x)\,dx + e^{-2x}\int \frac{W_2}{W} r(x)\,dx + \sin 2x \int \frac{W_3}{W} r(x)\,dx + \cos 2x \int \frac{W_4}{W} r(x)\,dx$
where,

$$W = \begin{vmatrix} e^{2x} & e^{-2x} & \sin 2x & \cos 2x \\ 2e^{2x} & -2e^{-2x} & 2\cos 2x & -2\sin 2x \\ 4e^{2x} & 4e^{-2x} & -4\sin 2x & -4\cos 2x \\ 8e^{2x} & -8e^{-2x} & -8\cos 2x & 8\sin 2x \end{vmatrix}$$

$$= e^{2x}\begin{vmatrix} -2e^{-2x} & 2\cos 2x & -2\sin 2x \\ 4e^{-2x} & -4\sin 2x & -4\cos 2x \\ -8e^{-2x} & -8\cos 2x & 8\sin 2x \end{vmatrix} - e^{-2x}\begin{vmatrix} 2e^{2x} & 2\cos 2x & -2\sin 2x \\ 4e^{2x} & -4\sin 2x & -4\cos 2x \\ -8e^{2x} & -8\cos 2x & 8\sin 2x \end{vmatrix}$$

$= -512$

$$W_1 = \begin{vmatrix} 0 & e^{-2x} & \cos 2x & \sin 2x \\ 0 & -2e^{-2x} & -2\sin 2x & 2\cos 2x \\ 0 & 4e^{-2x} & -4\cos 2x & -4\sin 2x \\ 1 & -8e^{-2x} & 8\sin 2x & -8\cos 2x \end{vmatrix} = -16e^{-2x}, \quad W_2 = \begin{vmatrix} e^{2x} & 0 & \cos 2x & \sin 2x \\ 2e^{2x} & 0 & -2\sin 2x & 2\cos 2x \\ 4e^{2x} & 0 & -4\cos 2x & -4\sin 2x \\ 8e^{2x} & 1 & 8\sin 2x & -8\cos 2x \end{vmatrix} = 16e^{-2x}$$

$$W_3 = \begin{vmatrix} e^{2x} & e^{-2x} & 0 & \sin 2x \\ 2e^{2x} & -2e^{-2x} & 0 & 2\cos 2x \\ 4e^{2x} & 4e^{-2x} & 0 & -4\sin 2x \\ 8e^{2x} & -8e^{-2x} & 1 & -8\cos 2x \end{vmatrix} = -32\sin 2x; \quad W_4 = \begin{vmatrix} e^{2x} & e^{-2x} & \cos 2x & 0 \\ 2e^{2x} & -2e^{-2x} & -2\sin 2x & 0 \\ 4e^{2x} & 4e^{-2x} & -4\cos 2x & 0 \\ 8e^{2x} & -8e^{-2x} & 8\sin 2x & 1 \end{vmatrix} = 32\cos 2x.$$

$y(x) = e^{2x}\int \frac{-16e^{-2x}}{-512}(x^2\sin 2x + x^4 e^{2x})\,dx + e^{-2x}\int \frac{16e^{-2x}}{-512}(x^2 \sin 2x + x^4 e^{2x})\,dx$
$+ \cos 2x \int \frac{-32\sin 2x}{-512}(x^2 \sin 2x + x^4 e^{2x})\,dx + \sin 2x \int \frac{32\cos 2x}{-512}(x^2\sin 2x + x^4 e^{2x})\,dx$

$y'' - 2y' + y = 2xe^{2x} + 6x^2$, $y(0) = 1$, $y'(0) = 0$.

Char. eqn: $\lambda^2 - 2\lambda + 1 = 0$

$\Rightarrow (\lambda - 1)^2 = 0 \Rightarrow \lambda = 1, 1$

$y_h(x) = c_1 e^x + c_2 x e^x$

Now,

$y_p(x) = e^x \int \frac{W_1}{W} r(x)\,dx + xe^x \int \frac{W_2}{W} r(x)\,dx$

where,

$W = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x} + \cancel{xe^{2x}} - xe^{+2x}$
$= e^{2x}$

$W_1 = \begin{vmatrix} 0 & xe^x \\ 1 & e^x + xe^x \end{vmatrix} = -xe^x$

$W_2 = \begin{vmatrix} e^x & 0 \\ e^x & 1 \end{vmatrix} = e^x$

$\therefore y_p(x) = e^x \int \frac{-xe^x}{e^{2x}} (2xe^{2x} + 6x^2)\,dx + xe^x \int \frac{e^x}{e^{2x}} (2xe^{2x} + 6x^2)\,dx$

$= -2e^x \int \left(x^2 e^x + 3\frac{x^3}{e^x}\right) dx + 2xe^x \int \left(e^x + \frac{3x}{e^x}\right) dx$

$= 2\left(-e^{-2x}(x^2 - 2x + 2)\right) + 6(x^3 + 3x^2 + 6x + 6)$

$+ e^x x (2(e^x x - e^x) - 6(x^2 + 2x + 2)$

$= 2e^{2x}(-x^2 + 2x - 2 + x^2) + 6(-x^2 + 4x + 6)$

$= 2e^{2x}(x - 2) + 6(-x^2 + 4x + 6)$

$\Rightarrow y_p(x) = 2e^{2x}(x-2) + 6(-x^2 + 4x + 6)$

Hence, $y(x) = c_1 e^x + c_2 x e^x + 2e^{2x}(x-2) + 6(-x^2 + 4x + 6)$.

Now,

$y(0) = 1 \Rightarrow c_1 + 0 - 4 + 36 = 1 \Rightarrow c_1 = -31$

$y'(0) = 0 \Rightarrow \left(c_1 e^x + e^x (c_2)(x+1) + 2e^x + 2e^{2x} 2 - 4e^{2x} 2 + 6(-2x + 4)\right)\Big|_0 = 0$

$\Rightarrow c_1 + c_2 + 2 - 8 + 24 \Rightarrow c_2 = 13$

$\therefore y(x) = -31e^x + 13xe^x + 2e^{2x}(x-2) + 6(-x^2 + 4x + 6)$

8. (a) $x^3y''' + x^2y'' + 2xy' - 2y = x^3$

$\Rightarrow y''' + \frac{y''}{x} + \frac{2y'}{x^2} - \frac{2y}{x^3} = 1.$

Take $y = x^m$ in homogeneous eqn $\left(y''' + \frac{y''}{x} + \frac{2y'}{x^2} - \frac{2y}{x^3} = 0\right)$.

$y' = m x^{m-1}$, $y'' = m(m-1)x^{m-2}$

$y''' = m(m-1)(m-2)x^{m-3}$

$\therefore m(m-1)(m-2)x^{m-3} + \frac{m(m-1)x^{m-2}}{x} + \frac{2mx^{m-1}}{x^2} - \frac{2x^m}{x^3} = 0.$

$\Rightarrow m(m-1)(m-2)x^{m-3} + m(m-1)x^{m-3} + 2mx^{m-3} - 2x^{m-3} = 0$

$\Rightarrow x^{m-3}\left[m(m-1)(m-2) + m(m-1) + 2m - 2\right] = 0$

$\Rightarrow x^{m-3}\left[(m-1)\left[m(m-2) + m + 2\right]\right] = 0$

$\Rightarrow (x^{m-3})(m-1)(m^2 - m + 2) = 0$

$m = 1$, $m = \frac{1 \pm \sqrt{1-8}}{2} = \frac{1 \pm 7i}{2}$

$\therefore y_h(x) = \cancel{c_1x + x^{1/2}\left(c_2\cos\left(\frac{7}{2}\ln|x|\right) + c_3\sin\left(\frac{7}{2}\ln|x|\right)\right)}$

$= c_1 x + x^{1/2}\left(c_2 x^{7i/2} + c_3 x^{-7i/2}\right)$

8. (a) $x^3y''' - x^2y'' + 2xy' - 2y = x^3$

$\Rightarrow y''' - \frac{y''}{x^2} + \frac{2y'}{x^2} - \frac{2y}{x^3} = 1 \quad -①$

Take $y = x^m$ in homogeneous part.

$y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$

$y''' = m(m-1)(m-2)x^{m-3}$

$\therefore x^m\left(m(m-1)(m-2) - m(m-1) + 2m - 2\right) = 0$

$\Rightarrow (m-1)\left[m(m-2) - m + 2\right] = 0$

$\Rightarrow (m-1)(m^2 - 3m + 2) = 0$

$\Rightarrow (m-1)(m-2)(m-1) = 0$

$\Rightarrow m = 1, 2, 1.$

$\therefore y_h(x) = c_1x + c_2x^2 + c_3x\ln x$

mt. (i) $y_p(x) =$

cont.

Now,

$$y_p(x) = x\int \frac{W_1}{W} r(x)\,dx + x^2\int \frac{W_2}{W} r(x)\,dx + x\ln x \int \frac{W_3}{W} r(x)\,dx$$

where,

$$W = \begin{vmatrix} x & x^2 & x\ln x \\ 1 & 2x & \ln x + 1 \\ 0 & 2 & \frac{1}{x} \end{vmatrix} = -2x\ln x - x + 2x\ln x$$
$$= -x$$

$$W_1 = \begin{vmatrix} 0 & x^2 & x\ln x \\ 0 & 2x & \ln x + 1 \\ 1 & 2 & \frac{1}{x} \end{vmatrix} = x^2 - x^2\ln x$$

$$W_2 = \begin{vmatrix} x & 0 & x\ln x \\ 1 & 0 & \ln x + 1 \\ 0 & 1 & 1/x \end{vmatrix} = -x$$

$$W_3 = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix} = x^2$$

$$\therefore y_p(x) = x\int \frac{x^2(1-\ln x)}{-x}\,dx + x^2\int \frac{-x}{-x}\,dx + x\ln x \int \frac{x^2}{-x}\,dx$$

$$= x\int (x\ln x - x)\,dx + x^3 - \frac{x^3\ln x}{2}$$

$$= x\left[\frac{x^2}{2}\ln x - \frac{3x^2}{4}\right] + x^3 - \frac{x^3\ln x}{2}$$

$$= x^3\left[\frac{\ln x}{2} - \frac{3}{4} + 1 - \frac{\ln x}{2}\right] = \frac{x^3}{4}.$$

$$\therefore y(x) = c_1 x + c_2 x^2 + c_3 x\ln x + \frac{x^3}{4}.$$

⑧ ⓑ $x^3y''' - 3x^2y'' + 6xy' - 6y = 0$.

$\Rightarrow y''' - \frac{3}{x}y'' + \frac{6}{x^2}y' - \frac{6}{x^3} = 0$

Take $y = x^m$

$\Rightarrow y' = m\,x^{m-1},\ y'' = m(m-1)x^{m-2},\ y''' = m(m-1)(m-2)x^{m-3}$.

$\therefore m(m-1)(m-2)x^{m-3} - \frac{3m(m-1)x^{m-2}}{x} + \frac{6mx^{m-1}}{x^2} - \frac{6}{x^3}x^m = 0$

$\Rightarrow m(m-1)(m-2)x^{m-3} - 3m(m-1)x^{m-3} + 6mx^{m-3} + 6x^{m-3} = 0$

$\Rightarrow x^{m-3}\left[m(m-1)(m-2) - 3m(m-1) + 6m - 6\right] = 0$

$\Rightarrow x^{m-3}\left[(m-1)(m(m-2) - 3m + 6)\right] = 0$

$\Rightarrow x^{m-3}(m-1)(m^2 - 5m + 6) = 0$

$\Rightarrow x^{m-3}(m-1)(m-3)(m-2) = 0$..

$\Rightarrow m = 1, 3, 2$

$\therefore y = c_1x + c_2x^3 + c_3x^2$.

⑨ ⓐ $y'' - (1+x)y' + x^2y = x \quad -①,\ y(0) = 1,\ y'(0) = 1$.

Take $y = \sum_{n=0}^{\infty} a_nx^n$

$\Rightarrow y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$

$y'' = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}$.

Substituting in ①,

$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - (1+x)\sum_{n=1}^{\infty} a_n n x^{n-1} + x^2\sum_{n=0}^{\infty} a_nx^n = x$.

$\Rightarrow \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=1}^{\infty} a_n n x^{n-1} - \sum_{n=1}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} = x$.

$\Rightarrow (a_2 2x^0 + a_3\cdot 6x + a_4\cdot 12\cdot x^2 + a_5\cdot 20x^3 + \dots)$

$(-a_1 - 2a_2x - 3a_3x^2 - 4a_4x^3 - \dots)$

$(-a_1x - a_2 2x^2 - a_3 3x^3 - a_4 4x^4 \dots)$

$(+a_0x^2 + a_1x^3 + a_2x^4 + a_3x^5 + \dots) = x$.

$2a_2 - a_1 = 0$, $6a_3 - 2a_2 - a_1 = 1$

$\Rightarrow a_2 = \frac{a_1}{2}$, $6a_3 - 2a_1 = 1 \Rightarrow a_3 = \frac{1+2a_1}{6}$

$12a_4 - 3a_3 - 2a_2 + a_0 = 0$, $y(0) = 1 \Rightarrow a_0 = 1$

$\Rightarrow 12a_4 - \frac{3}{6}(1+2) - 1 + 1 = 0 \Rightarrow a_4 = \frac{9}{6\times12} = \frac{1}{8}$

$y'(0) = 1 \Rightarrow a_1 = 1$

$a_0 = 1, a_1 = 1, a = \frac{1}{2}, a_3 = \frac{1}{2}$

$a_4 = \frac{1}{8}, a_5 = \frac{1}{20}$

Hence, $y = 1 + x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{8} + \frac{x^5}{20} + \dots$

(9) (b) $y'' - xy' + y = 0$ — (1)

Take $y = \sum_{n=0}^{\infty} a_n x^n$

$\Rightarrow y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$

$y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$

Substitituting in (1),

$\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} - \sum_{n=1}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$

$\Rightarrow (2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \dots)$

$-(a_1 x + 2a_2 x^2 + 3a_3 x^3 + 4a_4 x^4 + \dots)$

$+a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots = 0$

$\Rightarrow a_0 + 2a_2 = 0 \Rightarrow 2a_2 = -a_0 \Rightarrow a_2 = -a_0/2$

$6a_3 - a_1 + a_1 = 0 \Rightarrow a_3 = 0$

$12a_4 - 2a_2 + a_2 = 0 \Rightarrow 12a_4 = a_2 \Rightarrow a_4 = \frac{-a_0}{2\times12} = \frac{-a_0}{24}$

$20a_5 - 3a_3 + a_3 = 0 \Rightarrow 20a_5 = 2a_3 = 0 \Rightarrow a_5 = 0$

Thus, $a_0, a_1, a_0 = -a_0/2$, $a_3 = 0$, $a_4 = -a_0/24$, $a_5 = 0$.

$\therefore y = a_0 + a_1 x - \frac{a}{2} x^2 + 0 - \frac{a_0}{24} x^4 + \dots$

Let $a_1 = 0$. (as any value of a_1 satisfies)

Hence, $y = a_0 \left(1 - \frac{x^2}{2} - \frac{x^4}{24} + \dots\right)$.

(10) (a) $9x(1+x)y'' - 6y' + 2y = 0$ — (1), $x_0 = 0$.

Take $y = (x-x_0)^r \sum_{n=0}^{\infty} a_n (x-x_0)^n$

As $x_0 = 0$,

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}.$$

$$\Rightarrow y = x^r (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots)$$

$$= a_0 x^r + a_1 x^{r+1} + a_2 x^{r+2} + \dots$$

$$\Rightarrow y' = r a_0 x^{r-1} + a_1 (r+1) x^r + a_2 (r+2) x^{r+1} + \dots$$

$$y'' = r(r-1) a_0 x^{r-2} + a_1 r(r+1) x^{r-1} + \dots$$

Eq^n (1) becomes

$$9x(1+x)\left(r(r-1)a_0 x^{r-2} + a_1 r(r+1) x^{r-1} + \dots\right)$$
$$-6\left(r a_0 x^{r-1} + a_1 (r+1) x^r + a_2 (r+2) x^{r+1} + \dots\right)$$
$$+2\left(a_0 x^r + a_1 x^{r+1} + a_2 x^{r+2} + \dots\right) = 0$$

$$\Rightarrow 9r(r-1) a_0 x^r + 9r(r-1) a_0 x^{r-1} + 9a_1 r(r+1) x^{r+1} +$$
$$9a_1 r(r+1) x^r - 6 r a_0 x^{r-1} - 6a_1 (r+1) x^r + \dots$$
$$+ 2a_0 x^r + 2a_1 x^{r+1} + \dots = 0$$

Equating coeff. of lowest degree,

$$9r(r-1)a_0 - 6ra_0 = 0 \rightarrow \text{Indicial eq}^n$$

$$\Rightarrow 9r(r-1) - 6r = 0$$
$$\Rightarrow 3r(3(r-1) - 2) = 0$$
$$\Rightarrow 3r(3r - 3 - 2) = 0 \quad \Rightarrow r = 0, \frac{5}{3}$$
$$\Rightarrow 3r(3r-5) = 0$$

So, $y_1 = x^0 \sum a_m x^m$, $y_2 = x^{5/3} \sum A_m x^m = \sum A_m x^{m+5/3}$

For $y_1 = \sum a_m x^m$

$$y_1' = \sum_{m=1} a_m (m) x^{m-1}, \quad y_1'' = \sum_{m=2} m(m-1) a_m x^{m-2}$$

Putting in (1),

$$(9x^2 + 9x) y'' - 6y' + 2y = 0$$
$$\Rightarrow 9x^2 y'' + 9xy'' - 6y' + 2y = 0$$
$$\Rightarrow 9x^2 \sum_2 m(m-1) a_m x^{m-2} + 9x\left(\sum_2 m(m-1) a_m x^{m-2}\right)$$
$$-6 \sum_1 m a_m x^{m-1} + 2 \sum_0 a_m x^m = 0.$$

(3) contd.

$$9\sum_{2} m(m-1)a_m x^m + 9\sum_{2} m(m-1)a_m x^{m-1} - 6\sum_{1} m a_m x^{m-1} + 2\sum_{0} a_m x^m = 0$$

Equating coefficients,

$$2a_0 - 6a_1 = 0 \Rightarrow 2a_0 = 6a_1 \Rightarrow a_1 = a_0/3$$

x^1: $9.2.a_2 - 6.2.a_2 + 2a_1 = 0$

$\Rightarrow 18a_2 - 12a_2 + 2a_1 = 0$

$\Rightarrow 6a_2 + 2a_1 = 0 \Rightarrow a_2 = -a_1/3$

x^2: $9.2.a_2 + 9.3.2a_3 + 6.3.a_3 + 2.a_2 = 0$

$\Rightarrow 18a_2 + 54a_3 - 18a_3 + 2a_2 = 0$

$\Rightarrow 36a_3 + 20a_2 = 0 \Rightarrow a_3 = \frac{-20a_3}{36} = \frac{-5a_2}{9}$.

x^3: $9.3.2.a_3 + 9.4.3.a_4 - 6.4.a_4 + 2a_3 = 0$

$\Rightarrow 54a_3 + 108a_4 - 24a_4 - 2a_3 = 0$

$\Rightarrow 56a_3 + 84a_4 = 0$

$\Rightarrow a_4 = -\frac{56}{84}a_3 = -\frac{8}{12}a_3 = \frac{-2}{3}a_3$

$a_1 = \frac{a_0}{3}$, $a_2 = \frac{-a_1}{3} = \frac{-a_0}{9}$

$a_3 = -\frac{5}{9}a_2 = -\frac{5}{9}\frac{a_0}{9} = \frac{5}{81}a_0$.

$a_4 = -\frac{2}{3}a_3 = -\frac{2}{3}.\frac{5}{81} = \frac{-10}{243}a_0$.

$y_1 = a_0 + \frac{a_1}{3}x - \frac{a_0}{9}x^2 + \frac{5}{81}a_0x^3 - \frac{10}{243}a_0x^4 + \dots$

$= a_0\left(1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5}{81}x^3 - \frac{10}{243}x^4 + \dots\right)$

For $y_2 = \sum_{m=0} a_m x^{m+5/3} = a_0x^{5/3} + a_1x^{8/3} + a_2x^{11/3} + a_3x^{14/3}$.

$y_2' = \sum\left(m+\frac{5}{3}\right)a_m x^{m+5/3-1} = \sum\left(m+\frac{5}{3}\right)a_m x^{m+2/3}$

$y_2'' = \sum\left(m+\frac{5}{3}\right)\left(m+\frac{2}{3}\right)a_m x^{(m+\frac{2}{3}-1)}$

$= \sum\left(m+\frac{5}{3}\right)\left(m+\frac{2}{3}\right)a_m x^{m+1/3}$

Putting in (1),

$(9x + 9x^2)y'' - 6y' + 2y = 0$

$\Rightarrow 9xy'' + 9x^2y'' - 6y' + 2y = 0$.

$$\Rightarrow 9x\left(\sum(m+\tfrac{5}{3})(m+\tfrac{2}{3})a_m x^{m-1/3}\right)+9x^2\left(\sum(m+\tfrac{5}{3})(m+\tfrac{2}{3})a_m x^{m-1/3}\right)$$

$$-6\sum(m+\tfrac{5}{3})a_m x^{m+2/3}+2\sum a_m x^{m+5/3}=0$$

$$\Rightarrow \sum 9(m+\tfrac{5}{3})(m+\tfrac{2}{3})a_m x^{m+2/3}+\sum 9(m+5/3)(m+2/3)a_m x^{5/3+m}$$

$$-\sum 6(m+\tfrac{5}{3})a_m x^{m+2/3}+\sum 2a_m x^{m+5/3}=0$$

$$\Rightarrow \sum 9(m+\tfrac{5}{3})(m+\tfrac{2}{3})a_m x^m+\sum 9(m+\tfrac{5}{3})(m+\tfrac{2}{3})(a_m x^{m+1})$$

$$-\sum 6(m+\tfrac{5}{3})a_m x^m+\sum 2a_m x^{m+1}=0.$$

Coeff. of x^0: $9\times\frac{5}{3}\cdot\frac{2}{3}a_0-6\cdot\frac{5}{3}a_0 \Rightarrow 10a_0-10a_0=0$

Coeff. of x^1: $9(1+\frac{5}{3})(1+\frac{2}{3})a_1+10a_0-6\cdot\frac{8}{3}a_1+2a_0=0$

$\Rightarrow 40a_1+10a_0-16a_1+2a_0=0$

$\Rightarrow 24a_1+12a_0=0$

$\Rightarrow a_1=-a_0/2$

x^2: $11\cdot 8a_2-40a_1-66a_2+2a_1=0$

$\Rightarrow 88a_2-66a_2+42a_1=0$

$\Rightarrow 22a_2+42a_1=0$

$\Rightarrow a_2=\frac{-42a_1}{22}=-\frac{42}{22}\cdot-\frac{a_0}{2}=\frac{21}{22}a_0$

x^3: $14\cdot 11a_3+11\cdot 8a_2-84a_3+2a_2=0$

$\Rightarrow 154a_3-84a_3+88a_2+2a_2=0$

$\Rightarrow 70a_3+90a_2=0$

$\Rightarrow a_3=-\frac{90}{70}a_2=-\frac{90}{70}\cdot\frac{21}{22}=\frac{-27}{22}a_0$

x^4: $17\cdot 14a_4+154a_3-102a_4+2a_3=0$

$\Rightarrow 136a_4-156a_3=0$

$\Rightarrow a_4=\frac{-156}{136}a_3=\frac{-156}{136}\left(-\frac{27}{22}\right)a_0=\frac{39}{34}\cdot\frac{27}{22}a_0$

So, $y_2=a_0-\frac{a_0}{2}x+\frac{21}{22}a_0x^2-\frac{27}{22}a_0x^3+\frac{39}{34}\cdot\frac{27}{22}a_0x^4+\dots$

$=a_0\left(1-\frac{x}{2}+\frac{21}{22}x^2-\frac{27}{22}x^3+\dots\right)$

$\therefore$ Soln: $y=c_1\left(1+\frac{x}{3}-\frac{x^2}{9}+\frac{5}{81}x^3-\frac{10}{243}x^4+\dots\right)+c_2\left(1-\frac{x}{2}+\frac{21}{22}x^2-\frac{27}{22}x^3+\dots\right)$

3) $y''-y=0$ —(1)

$x=0$ is a regular singular point

so, take $y=(x-x_0)^r \sum a_m (x-x_0)^m$

Put $x_0=0$

$\Rightarrow y = x^r \sum a_m x^m = \sum_{m=0}^{\infty} a_m x^{m+r}$

$y' = \sum (m+r)\, a_m x^{m+r-1}$

$y'' = \sum (m+r)(m+r-1)\, a_m x^{m+r-2}$

Substituting in (1),

$x \sum (m+r)(m+r-1)\, a_m x^{m+r-2} - \sum a_m x^{m+r} = 0$

$\Rightarrow \sum (m+r)(m+r-1)\, a_m x^{m+r-1} - \sum a_m x^{m+r} = 0$

$\Rightarrow \sum (m+r)(m+r-1)\, a_m x^{m+r-1} - \sum a_m x^{m+r} = 0$

$\Rightarrow \left(r(r-1)a_0 x^{r-1} + (r+1)(r) a_1 x^{r} + \dots\right) - \left(a_0 x^r + a_1 x^{r+1} + \dots\right)$

comparing coff. of lowest power,

$r(r-1)a_0 = 0 \Rightarrow r = 0, 1$

$r_1 = 1,\ r_2 = 0.$

differ by integer

$y_1 = \sum a_m x^{m+1}, \quad y_2 = k_1 y_1 \ln x + x^{r_2} \sum a_m x^m$

$= k_1 \ln x \sum a_m x^{m+1} + \sum b_m x^m.$

For y_1,

$y_1 = \sum a_m x^{m+1} \Rightarrow y_1' = \sum a_m (m+1) x^m$

$\Rightarrow y_1'' = \sum a_m m(m+1) x^{m-1}$

Put into (1),

$x \sum a_m m(m+1) x^{m-1} - \sum a_m x^{m+1} = 0$

$\Rightarrow \sum a_m m(m+1) x^m - \sum a_m x^{m+1} = 0$

Coeff. of x^0: 0

x^1: $2a_1 - a_0 = 0 \Rightarrow a_1 = a_0/2$

x^2: $6a_2 - a_1 = 0 \Rightarrow a_2 = \frac{a_1}{6} = \frac{a_0}{12}$

x^3: $12a_3 - a_2 = 0 \Rightarrow a_3 = \frac{a_2}{12} = \frac{a_0}{144}$

x^4: $20a_4 - a_3 = 0 \Rightarrow a_4 = \frac{a_3}{20} = \frac{a_0}{2880}$

$$\Rightarrow a_m = \frac{a_{m-1}}{m(m+1)}$$

$$y_1 = a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + a_4 x^5 + \cdots$$

$$= a_0 x + \frac{a_0}{2} x^2 + \frac{a_0}{12} x^3 + \frac{a_0}{144} x^4 + \frac{a_0}{2880} x^5 + \cdots$$

$$= a_0 \left[x + \frac{x^2}{2} + \frac{x^3}{12} + \frac{x^4}{144} + \frac{x^5}{2880} + \cdots \right]$$

For y_2,

$$y_2 = k \ln x \sum a_m x^{m+1} + \sum a_m x^m$$

$$\Rightarrow y_2' = \frac{k}{x} \sum a_m x^{m+1} + k \ln x \sum a_m (m+1) x^m + \sum b_m m \cdot x^{m-1}$$

$$y_2'' = \frac{-k}{x^2} \sum a_m x^{m+1} + \frac{k}{x} \sum a_m (m+1) x^m + \frac{k}{x} \sum a_m (m+1) x^m + k \ln x + \sum a_m m(m+1) x^{m-1} + \sum b_m m(m-1) x^{m-2}$$

$$= -k \sum a_m x^{m-1} + 2k \sum a_m (m+1) x^{m-1} + k \ln x \sum a_m m(m+1) x^{m-1} + \sum b_m m(m-1) x^{m-2}$$

Put into ①,

$$x\left[-k \sum a_m x^{m-1} + 2k \sum a_m (m+1) x^{m-1} + k \ln x \sum m(m+1) a_m x^{m-1} + \sum b_m m(m-1) x^{m-2}\right] - k \ln x \sum a_m x^{m+1} - k \sum b_m x^m = 0.$$

$$\Rightarrow -k \sum a_m x^m + 2k \sum a_m (m+1) x^m + k \ln x \sum m(m+1) a_m x^m + \sum b_m m(m-1) x^{m-1} - k \ln x \sum a_m x^{m+1} - k \sum b_m x^m = 0.$$

Equating x^0 coeff.: $-ka_0 + 2ka_0 - kb_0 = 0 \Rightarrow ka_0 = kb_0 \Rightarrow a_0 = b_0$

x^1: $-ka_1 + 4ka_1 + 2b_2 - kb_1 = 0 \Rightarrow 3ka_1 - kb_1 + 2b_2 = 0$

x^2: $-ka_2 + 6ka_2 + 6b_3 - kb_2 = 0 \Rightarrow 5ka_2 - kb_2 + 6b_3 = 0.$

Equating coeff., $k = 0$, $b_1 = \frac{b_0}{2}$, $b_2 = \frac{b_0}{12}$, $b_3 = \frac{b_0}{144}$.

$$\therefore y = a_0 \left[x + \frac{x^2}{2} + \frac{x^3}{12} + \frac{x^4}{144} + \cdots \right] + b_0 \left[1 + \frac{x}{2} + \frac{x^2}{12} + \frac{x^3}{144} + \cdots \right]$$