INDIAN INSTITUTE OF SPACE SCIENCE AND TECHNOLOGY
THIRUVANANTHAPURAM 695 547

End Semester Examination - November 2022

B.Tech - III Semester

MA211 - Linear Algebra, Complex Analysis and Fourier Series

Date: 28/11/2022 Time: 01.30 pm - 04.30 pm Max. Marks: 50

Note: Use of Scientific Calculator is not allowed

PART A (Answer ALL questions - 10x2.5= 25 marks.)

1. For what values of a and b the following system of equations

$$\begin{aligned} x + 2y - 4z &= 3 \\ x + y - 3z &= a \\ 2x + 3y + (b-8)z &= 2a + 6 \end{aligned}$$

has (i) unique solution
(ii) infinitely many solutions
(iii) no solution.

2. Write down two different subspaces of the vector space $V = \{(x_1, x_2, x_3, x_4)/x_4 = 0 \text{ and } x_2 - x_3 = 0\}$ and give a basis for each subspace.

3. The trace of the matrix $\begin{bmatrix} 0 & 1 & 3 & 0 \\ -2 & 3 & 0 & 4 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 1 & 6 \end{bmatrix}$ is 15 and the determinant is 70. If two eigenvalues of the matrix are 1, 5 find the remaining eigenvalues.

4. Find $T(2,5)$, where $T : R^2 \to R^2$ has the matrix representation $\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ with respect to the ordered bases $B_1 = \{(1,1), (2,3)\}$ and $B_2 = \{(1,2), (0,1)\}$ for the domain and codomain respectively.

5. Evaluate $\int_C \frac{\tanh z}{z^2}\, dz$, where C is positively oriented circle $|z| = 2$.

6. Find the residue of $f(z) = \dfrac{1}{z - \sin z}$ at its pole.

7. Using residues, find the Cauchy principal value of the following integral (write all steps clearly).

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 2x + 2}\, dx.$$

8. Evaluate $\int_C \frac{z+1}{z^2 + 2z + 4}\, dz$, where C is $|z + 1 + i| = 2$. By using
   (i) Cauchy's Integral formula.
   (ii) Residue Theorem.

9. Consider a function f defined by

$$f(x) = \begin{cases} x, & 0 \leq x < \pi/2, \\ \pi - x, & \pi/2 < x \leq \pi, \\ 2022, & x = \pi/2 \end{cases}$$

(a) Determine Fourier coefficients of odd extension of f, say, f_{odd}.

(b) Using Fourier series of f_{odd}, evaluate sum of the series: $\dfrac{1}{3^2} + \dfrac{1}{5^2} + \dfrac{1}{7^2} + \cdots$.

10. Establish the following identity using Fourier integral and determine α:

$$\alpha \int_0^\infty \frac{\sin \pi t}{1 - t^2} \sin tx \, dt = \begin{cases} \sin x, & \text{for } |x| \leq \pi, \\ 0, & \text{otherwise} \end{cases}$$

PART B (Answer any FIVE questions - 5x5= 25 marks.)

11. (a) Find all values of a and b so that

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & b+1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ a+2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\}$$

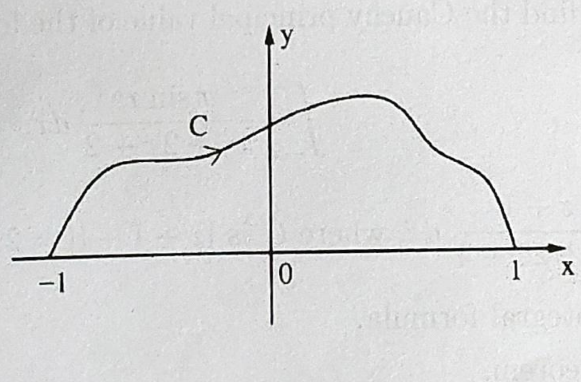
is a basis for $M_{2\times 2}(R)$. [2.5]

(b) Using Gram-Schimdt process obtain an orthonormal basis for R^3 from the ordered basis $\{(1, 2, 3), (2, 1, 1), (2, 3, 2)\}$. [2.5]

12. (a) Let $T : P_3 \to P_2$ be the linear map $T(a_0 + a_1x + a_2x^2 + a_3x^3) = a_3x^2 + a_1x$. Find the matrix representation of T with respect to the ordered basis $\{2, 1 + x, x^2, 2 + x^3\}$ of P_3 and $\{1, 1 + x, 1 + x^2\}$ of P_2. [2.5]

(b) Let $T : P_3 \to R^3$ be the linear map defined by $T(a_0 + a_1x + a_2x^2 + a_3x^3) = (a_0 + a_3, a_2, a_1 - a_3)$. Find Ker$(T)$ and Range(T) and give a basis for each of them. [2.5]

13. (a) Evaluate $\int_C z^i \, dz$, where z^i is the principal branch and C is the contour shown in the following figure. Write all steps clearly with proper justification.



[3]

(b) Does there exist a non-constant function $f(z) = u + iv$ on a domain $D \subseteq \mathbb{C}$ such that u and v are harmonic conjugate of each other on D. If yes, give an example. Justify your answer. [2]

14. (a) Discuss continuity and differentiability of the following functions at the origin

(i) $f(z) = \begin{cases} 0 & \text{if } z = 0 \\ e^{\left(\frac{-1}{z^4}\right)} & \text{if } z \neq 0 \end{cases}$

(ii) $f(z) = \begin{cases} 0 & \text{if } z = 0 \\ \dfrac{\operatorname{Im}(z^2)}{\bar{z}} & \text{if } z \neq 0 \end{cases}$

[4]

(b) Find the residue of $f(z) = \dfrac{z}{z^4 + 4}$ at any one of its singular points. [1]

15. Consider a function f defined by

$$f(x) = \begin{cases} 1 - |x|, & \text{if } |x| \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find Fourier integral of f. [4]

(b) Using (a), evaluate $\displaystyle\int_0^\infty \frac{\sin^2 t}{t^2}\, dt$. [1]

16. Consider a function f defined by

$$f(x) = \begin{cases} \frac{1}{4}\pi x, & 0 \leq x \leq \pi/2, \\ \frac{1}{4}\pi(\pi - x), & \pi/2 < x \leq \pi. \end{cases}$$

(i) Determine Fourier coefficients of even extension of f, say, f_{ev}. [4]

(ii) Write Fourier series of f_{ev} (upto first three non-zero terms). [1]

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