3.1 Partial and Directional Derivatives

Let us first recall the notion of derivative for a function of one variable. Let $D \subseteq \mathbb{R}$ and let c be an interior point of D, that is, $(c-r, c+r) \subseteq D$ for some $r > 0$. A function $f : D \to \mathbb{R}$ is said to be differentiable at c if the limit

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

exists; in this case the value of the limit is denoted by $f'(c)$ and is called the derivative of f at c. Now suppose $D \subseteq \mathbb{R}^2$ and (x_0, y_0) is an interior point of D, that is, $\mathbb{S}_r(x_0, y_0) \subseteq D$ for some $r > 0$. For a function $f : D \to \mathbb{R}$, it might seem natural to consider a limit such as

$$\lim_{(h,k) \to (0,0)} \frac{f(x_0 + h, y_0 + k) - f(x_0, y_0)}{(h, k)}.$$

But this doesn't make sense for the simple reason that division of a real number by a point in $\mathbb{R}^2$ has not been defined. There are ways to get around this problem but they are not particularly easy, and we defer a discussion of the notion of differentiability for functions of two (or more) variables to a later section. For the moment, we shall see that choosing to become partial to one of the variables makes things easier and leads to a useful notion.

Partial Derivatives

Let $D \subseteq \mathbb{R}^2$ and let $f : D \to \mathbb{R}$ be any function. Fix $(x_0, y_0) \in D$ and define $D_1, D_2 \subseteq \mathbb{R}$ by $D_1 := \{x \in \mathbb{R} : (x, y_0) \in D\}$ and $D_2 := \{y \in \mathbb{R} : (x_0, y) \in D\}$. If x_0 is an interior point of D_1, we define the **partial derivative** of f with respect to x at (x_0, y_0) to be the limit

$$\lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},$$

provided this limit exists. It is denoted by $f_x(x_0, y_0)$. Likewise, if y_0 is an interior point of D_2, we define the **partial derivative** of f with respect to y at (x_0, y_0) to be the limit

$$\lim_{k \to 0} \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k},$$

provided this limit exists. It is denoted by $f_y(x_0, y_0)$. These partial derivatives are also called the **first-order partial derivatives** or simply the **first partials** of f at (x_0, y_0). They are sometimes denoted by

$$\frac{\partial f}{\partial x}(x_0, y_0) \quad \text{and} \quad \frac{\partial f}{\partial y}(x_0, y_0)$$

instead of $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$, respectively. If these partial derivatives exist, then the pair $(f_x(x_0, y_0), f_y(x_0, y_0))$ is called the **gradient** of f at (x_0, y_0) and is denoted by $\nabla f(x_0, y_0)$. Thus

$$\nabla f(x_0, y_0) = \left(\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0) \right).$$

The partial derivative $f_x(x_0, y_0)$ gives the rate of change in f at (x_0, y_0) along the x-axis, whereas $f_y(x_0, y_0)$ gives the rate of change in f at (x_0, y_0) along the y-axis. In practice, finding the partial derivative of f with respect to x amounts to taking the derivative of $f(x, y)$ as a function of x, treating y as a constant. Indeed, if $\phi : D_1 \to \mathbb{R}$ is the function of one variable defined by $\phi(x) := f(x, y_0)$, then ϕ is differentiable at x_0 if and only if the partial derivative of f with respect to x at (x_0, y_0) exists; in this case $f_x(x_0, y_0) = \phi'(x_0)$. Similarly, if $\psi : D_2 \to \mathbb{R}$ is defined by $\psi(y) = f(x_0, y)$ for $y \in D_2$, then ψ is differentiable at y_0 if and only if the partial derivative of f with respect to y at (x_0, y_0) exists; in this case $f_y(x_0, y_0) = \psi'(y_0)$. As a consequence, we see that partial derivatives of sums, scalar multiples, products, reciprocals, and radicals possess exactly the same properties as derivatives of functions of one variable. Moreover, since differentiability implies continuity for functions of one variable, we see that if the partial derivatives of f at (x_0, y_0) exist, then ϕ is continuous at x_0 and ψ is continuous at y_0. However, as Example 3.1 (iii) below shows, existence of both the partial derivatives at a point does not imply continuity at that point.

Analogous to the left(-hand) and the right(-hand) derivatives in one-variable calculus, we have the concepts of left(-hand) and right(-hand) partial derivatives at points that are akin to endpoints of an interval in $\mathbb{R}$. Let, as before, $D \subseteq \mathbb{R}^2$ and let $f : D \to \mathbb{R}$ be a function. Fix $(x_0, y_0) \in D$, and let $D_1 := \{x \in \mathbb{R} : (x, y_0) \in D\}$ and $D_2 := \{y \in \mathbb{R} : (x_0, y) \in D\}$. If there is $r > 0$ such that $(x_0 - r, x_0] \subseteq D_1$, then we define the **left(-hand) partial derivative** of f with respect to x at (x_0, y_0) to be the limit

$$\lim_{h \to 0^-} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},$$

provided this limit exists. It is denoted by $(f_x)_-(x_0, y_0)$. On the other hand, if there is $r > 0$ such that $[x_0, x_0 + r) \subseteq D$, then the **right(-hand) partial derivative** of f with respect to x at (x_0, y_0) is defined to be the above limit with $h \to 0^-$ replaced by $h \to 0^+$. It is denoted by $(f_x)_+(x_0, y_0)$. Likewise, we define the left(-hand) and right(-hand) partial derivatives of f with respect to y at (x_0, y_0). These are denoted by $(f_y)_-(x_0, y_0)$ and $(f_y)_-(x_0, y_0)$ respectively.

In case D is the rectangle $[a, b] \times [c, d]$, then for each $(x_0, y_0) \in D$, we have $D_1 = [a, b]$ and $D_2 = [c, d]$. If $a < x_0 < b$, then x_0 is an interior point of D_1 and it is clear that the partial derivative of f with respect to x at (x_0, y_0) exists if and only if both the left(-hand) and the right(-hand) partial derivatives of f with respect to x at (x_0, y_0) exist and are equal. Likewise for partial derivatives with respect to y when $c < y_0 < d$. If $x_0 = a$ or $x_0 = b$, then x_0 is not an interior point of D_1, but the right(-hand) partial derivative of f with respect to x at (x_0, y_0) can still be defined when $x_0 = a$, while the left(-hand) derivative of f with respect to x at (x_0, y_0) can be defined when $x_0 = b$. Likewise if $y_0 = c$ or $y_0 = d$. With this in view, we shall say that the **partial derivative** f_x of f exists on $[a, b] \times [c, d]$ if f_x exists at each point of $(a, b) \times [c, d]$, $(f_x)_+$ exists at each point of $\{a\} \times [c, d]$, and $(f_x)_-$ exists at each point of $\{b\} \times [c, d]$. In this case, we will simply write $f_x(a, y_0)$ to denote $(f_x)_+(a, y_0)$ and $f_x(b, y_0)$ to denote $(f_x)_-(b, y_0)$ for every $y_0 \in [c, d]$. In this way, we obtain a function $f_x : [a, b] \times [c, d] \to \mathbb{R}$. A similar convention holds for f_y.

Examples 3.1. (i) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) := x^2 + y^2$. Then both the partial derivatives of f exist at every point of $\mathbb{R}^2$; in fact, $f_x(x_0, y_0) = 2x_0$ and $f_y(x_0, y_0) = 2y_0$ for any $(x_0, y_0) \in \mathbb{R}^2$.

(ii) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the norm function given by $f(x, y) := \sqrt{x^2 + y^2}$. Then both the partial derivatives of f exist at every point of $\mathbb{R}^2$ except the origin; in fact, for any $(x_0, y_0) \in \mathbb{R}^2$ with $(x_0, y_0) \neq (0, 0)$,

$$f_x(x_0, y_0) = \frac{x_0}{\sqrt{x_0^2 + y_0^2}} \quad \text{and} \quad f_y(x_0, y_0) = \frac{y_0}{\sqrt{x_0^2 + y_0^2}}\,.$$

To examine whether any of the partial derivatives exist at $(0, 0)$, we have to resort to the definition. This leads to a limit of the quotient $h/|h|$ as h approaches 0. Clearly, such a limit does not exist. It follows that $f_x(0, 0)$ and $f_y(0, 0)$ do not exist.

(iii) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(0, 0) := 0$ and $f(x, y) := xy/(x^2 + y^2)$ for $(x, y) \neq (0, 0)$. Then for any $h, k \in \mathbb{R}$ with $h \neq 0$ and $k \neq 0$, we have

$$\frac{f(0 + h, 0) - f(0, 0)}{h} = 0 \quad \text{and} \quad \frac{f(0, 0 + k) - f(0, 0)}{k} = 0.$$

Hence $f_x(0, 0)$ and $f_y(0, 0)$ exist and are both equal to 0. However, as seen already in Example 2.16 (ii), f is not continuous at $(0, 0)$.

(iv) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = |x| + |y|$ for $(x, y) \in \mathbb{R}^2$. Clearly, f is continuous at $(0, 0)$. But for any $h, k \in \mathbb{R}$ with $h \neq 0$ and $k \neq 0$, we have

$$\frac{f(0 + h, 0) - f(0, 0)}{h} = \frac{|h|}{h} \quad \text{and} \quad \frac{f(0, 0 + k) - f(0, 0)}{k} = \frac{|k|}{k}.$$

Hence $f_x(0, 0)$ and $f_y(0, 0)$ do not exist. However, the left(-hand) and the right(-hand) partial derivatives of f at $(0, 0)$ do exist. Indeed, $(f_x)_+(0, 0) = 1 = (f_y)_+(0, 0)$, while $(f_x)_-(0, 0) = -1 = (f_y)_-(0, 0)$. On the other hand, if we let g and h denote the restrictions of f to the rectangles $[-1, 1] \times [0, 1]$ and $[0, 1] \times [-1, 1]$ respectively, then in accordance with our conventions, $g_y(0, 0)$ and $h_x(0, 0)$ do exist and are both equal to 1.

(ii) *If f_y exists on $[a,b] \times [c,d]$, then for each fixed $x_0 \in [a,b]$, the function from $[c,d]$ to $\mathbb{R}$ given by $y \longmapsto f(x_0, y)$ is continuous.*

(iii) *If both f_x and f_y exist, and if one of them is bounded on $[a,b] \times [c,d]$, then f is continuous on $[a,b] \times [c,d]$.*

Proof. (i) Fix $y_0 \in [c,d]$. The existence of $f_x(x_0, y_0)$ for every $x_0 \in [c,d]$ readily implies that the function of one variable given by $x \longmapsto f(x, y_0)$ is differentiable, and hence continuous, on $[a,b]$.

(ii) Proof of (ii) is similar to that of (i) above.

(iii) Assume that both f_x and f_y exist, and f_x is bounded on $[a,b] \times [c,d]$. Then there is $\alpha \in \mathbb{R}$ such that $|f_x(u,v)| \le \alpha$ for all $(u,v) \in [a,b] \times [c,d]$. Fix $(x_0, y_0) \in [a,b] \times [c,d]$. Given any $(x,y) \in [a,b] \times [c,d]$ with $x \neq x_0$, by the MVT (Fact 3.2), we see that there is $u \in \mathbb{R}$ between x and x_0 such that

$$f(x,y) - f(x_0,y) = f_x(u,y)(x - x_0) \quad \text{and so} \quad |f(x,y) - f(x_0,y)| \le \alpha |x - x_0|.$$

Consequently,

$$\begin{aligned} |f(x,y) - f(x_0,y_0)| &\le |f(x,y) - f(x_0,y)| + |f(x_0,y) - f(x_0,y_0)| \\ &\le \alpha|x - x_0| + |f(x_0,y) - f(x_0,y_0)|\,. \end{aligned}$$

Moreover, these inequalities are clearly valid if $x = x_0$. Thus, in view of (ii), $f(x,y) \to f(x_0,y_0)$ as $(x,y) \to (x_0,y_0)$. So f is continuous at (x_0, y_0). □

Directional Derivatives

The notion of partial derivatives can be easily generalized to that of a *directional derivative*, which measures the rate of change of a function at a point along a given direction. We specify a direction by specifying a unit vector. Let $\mathbf{u} = (u_1, u_2)$ be a unit vector in $\mathbb{R}^2$, so that $u_1^2 + u_2^2 = 1$. Also let $D \subseteq \mathbb{R}^2$ and $f : D \to \mathbb{R}$ be any function. Let $(x_0, y_0) \in D$ be such that D contains a segment of the line passing through (x_0, y_0) in the direction of $\mathbf{u}$, that is, 0 is an interior point of $D_0 := \{t \in \mathbb{R} : (x_0 + tu_1, y_0 + tu_2) \in D\}$. We define the **directional derivative** of f at (x_0, y_0) along $\mathbf{u}$ to be the limit

$$\lim_{t \to 0} \frac{f(x_0 + tu_1, y_0 + tu_2) - f(x_0, y_0)}{t},$$

provided this limit exists. It is denoted by $\mathbf{D_u} f(x_0, y_0)$. Note that if $\mathbf{v} = -\mathbf{u}$, then $\mathbf{D_v} f(x_0, y_0) = -\mathbf{D_u} f(x_0, y_0)$. Note also that if $\mathbf{i} := (1,0)$ and $\mathbf{j} := (0,1)$, then $\mathbf{D_i} f(x_0, y_0) = f_x(x_0, y_0)$ and $\mathbf{D_j} f(x_0, y_0) = f_y(x_0, y_0)$.

Examples 3.4. (i) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) := x^2 + y^2$. Given any unit vector $\mathbf{u} = (u_1, u_2)$ in $\mathbb{R}^2$ and any $(x_0, y_0) \in \mathbb{R}^2$, for every $t \in \mathbb{R}$ with $t \neq 0$, the quotient $[f(x_0 + tu_1, y_0 + tu_2) - f(x_0, y_0)]/t$ is equal to

$$\frac{(x_0+tu_1)^2+(y_0+tu_2)^2-(x_0^2+y_0^2)}{t} = \frac{2tx_0u_1+t^2u_1^2+2ty_0u_2+t^2u_2^2}{t}$$
$$= 2x_0u_1+2y_0u_2+t.$$

It follows that $\mathbf{D_u}f(x_0,y_0)$ exists and is equal to $2x_0u_1+2y_0u_2$. Thus, in view of Example 3.1 (i), we see that $\mathbf{D_u}f(x_0,y_0) = f_x(x_0,y_0)u_1 + f_y(x_0,y_0)u_2$.

(ii) Let $f:\mathbb{R}^2\to\mathbb{R}$ be given by $f(x,y):=\sqrt{x^2+y^2}$. Given any unit vector $\mathbf{u}=(u_1,u_2)$ in $\mathbb{R}^2$ and any $(x_0,y_0)\in\mathbb{R}^2$, for every $t\in\mathbb{R}$ with $t\neq 0$, the quotient $[f(x_0+tu_1,y_0+tu_2)-f(x_0,y_0)]/t$ is equal to

$$\frac{\sqrt{(x_0+tu_1)^2+(y_0+tu_2)^2}-\sqrt{x_0^2+y_0^2}}{t}$$
$$= \frac{2tx_0u_1+2ty_0u_2+t^2}{t\left(\sqrt{(x_0+tu_1)^2+(y_0+tu_2)^2}+\sqrt{x_0^2+y_0^2}\right)}.$$

It follows that if $(x_0,y_0)\neq(0,0)$, then $\mathbf{D_u}f(x_0,y_0)$ exists and

$$\mathbf{D_u}f(x_0,y_0) = \frac{x_0u_1+y_0u_2}{\sqrt{x_0^2+y_0^2}}.$$

Thus, in view of Example 3.1 (ii), we see once again that $\mathbf{D_u}f(x_0,y_0) = f_x(x_0,y_0)u_1 + f_y(x_0,y_0)u_2$ for $(x_0,y_0)\neq(0,0)$. On the other hand, $\mathbf{D_u}f(0,0)$ does not exist, since the quotient $t/|t|$ does not have a limit as $t\to 0$.

(iii) Let $f:\mathbb{R}^2\to\mathbb{R}$ be given by $f(0,0):=0$ and $f(x,y):=x^2y/(x^4+y^2)$ for $(x,y)\neq(0,0)$. Given any unit vector $\mathbf{u}=(u_1,u_2)$ in $\mathbb{R}^2$ and any $t\in\mathbb{R}$ with $t\neq 0$, the quotient $[f(0+tu_1,0+tu_2)-f(0,0)]/t$ is equal to $u_1^2u_2/(u_1^4t^2+u_2^2)$. It follows that $\mathbf{D_u}f(0,0)$ exists and

$$\mathbf{D_u}f(0,0) = \begin{cases} \dfrac{u_1^2}{u_2} & \text{if } u_2\neq 0,\\[2ex] 0 & \text{if } u_2 = 0.\end{cases}$$

In particular, $f_x(0,0) = 0 = f_y(0,0)$. Thus, we see this time that $\mathbf{D_u}f(0,0)\neq f_x(0,0)u_1+f_y(0,0)u_2$, unless $u_1=0$ or $u_2=0$. Notice that in view of Example 2.16 (iv), f is not continuous at $(0,0)$ even though all the directional derivatives of f at $(0,0)$ exist.

(iv) Let $f:\mathbb{R}^2\to\mathbb{R}$ be given by $f(x,y)=|x|+|y|$ for $(x,y)\in\mathbb{R}^2$. Let $\mathbf{u}=(u_1,u_2)$ be any unit vector. Then $|u_1|+|u_2|\neq 0$ and for any $t\in\mathbb{R}$ with $t\neq 0$, we have

$$\frac{f(0+tu_1,0+tu_2)-f(0,0)}{t} = \frac{|t|}{t}\left(|u_1|+|u_2|\right).$$

Hence $\mathbf{D_u}f(0,0)$ does not exist. Notice that here, f is clearly continuous at $(0,0)$, but none of the directional derivatives of f at $(0,0)$ exist. ◇

In case the identity $\mathbf{D_u} f = \nabla f \cdot \mathbf{u}$ is satisfied by f as well as its higher-order directional derivatives, we obtain the following alternative version of the Bivariate Taylor Theorem that is analogous to Corollary 3.6.

Corollary 3.22. *Let $D \subseteq \mathbb{R}^2$ be an open set and n a nonnegative integer. Let (x_0, y_0) and (x_1, y_1) be distinct points in D and let L be the line segment joining them. Assume that $L \subseteq D$. Let $f : D \to \mathbb{R}$ be such that f has continuous partial derivatives of order $\leq n+1$ at every point of D and moreover, the higher-order directional derivatives $\mathbf{D}^i_{\mathbf{u}} f$ exist at every point of D for all unit vectors $\mathbf{u}$ in $\mathbb{R}^2$ and $i = 0, 1, \ldots, n+1$. Assume further that for any unit vector $\mathbf{u}$ in $\mathbb{R}^2$, the functions $f_i := \mathbf{D}^i_{\mathbf{u}} f$ satisfy*

$$(\mathbf{D_u} f_i)\,(x, y) = \nabla f_i(x, y) \cdot \mathbf{u} \quad \textit{for all } (x, y) \in D \textit{ and } i = 0, \ldots, n,$$

where $f_0 := f$. Then there is $(c, d) \in L \setminus \{(x_0, y_0), (x_1, y_1)\}$ such that

$$f(x_1, y_1) = \sum_{i=0}^{n} \frac{1}{i!} \left(\mathcal{D}^i_{h,k} f\right)(x_0, y_0) + \frac{1}{(n+1)!} \left(\mathcal{D}^{n+1}_{h,k} f\right)(c, d),$$

where $h := x_1 - x_0$ and $k := y_1 - y_0$.

Proof. With h and k as above, let $r := \sqrt{h^2 + k^2}$ and $\mathbf{u} := (h/r,\ k/r)$. Then $\mathbf{u}$ is a unit vector in $\mathbb{R}^2$ with $r\mathbf{u} = (h, k)$. As in the proof of Corollary 3.6, for any $(x, y) \in D$ and $i = 0, 1, \ldots, n$, we have

$$r f_{i+1}(x, y) = r \mathbf{D_u} f_i(x, y) = \left(h \frac{\partial f_i}{\partial x} + k \frac{\partial f_i}{\partial y}\right)(x, y) = (\mathcal{D}_{h,k} f_i)\,(x, y).$$

Successively using the above identity, we see that $r^i \left(\mathbf{D}^i_{\mathbf{u}} f\right) = r^i f_i = (\mathcal{D}_{h,k} f_i)^i$ for $i = 0, \ldots, n+1$. So the desired result follows from Proposition 3.21. $\square$

3.2 Differentiability

The difficulties involved in generalizing the notion of differentiability from functions of one variable to functions of two (or more) variables were discussed at the beginning of Section 3.1. We shall show in this section how to overcome these difficulties. The key idea here is twofold: (i) a realization that the derivative of a real-valued function of two variables may not be a single number but possibly a pair of real numbers, and (ii) an observation that the problem of division by a point (h, k) in $\mathbb{R}^2$ can be solved by replacing (h, k) with its norm $|(h, k)| := \sqrt{h^2 + k^2}$. To understand this better, let us first note that if $D \subseteq \mathbb{R}$ and c is an interior point of D, then a function $f : D \to \mathbb{R}$ is differentiable at c if and only if there is $\alpha \in \mathbb{R}$ such that

$$\lim_{h \to 0} \frac{f(c+h) - f(c) - \alpha h}{|h|} = 0.$$

In this case, α is the derivative of f at c. Now suppose $D \subseteq \mathbb{R}^2$ and (x_0, y_0) is an interior point of D. A function $f : D \to \mathbb{R}$ is said to be **differentiable** at (x_0, y_0) if there is $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ such that

$$\lim_{(h,k)\to(0,0)} \frac{f(x_0 + h, y_0 + k) - f(x_0, y_0) - \alpha_1 h - \alpha_2 k}{\sqrt{h^2 + k^2}} = 0.$$

In this case, we call the pair (α_1, α_2) the **total derivative**[1] of f at (x_0, y_0).

Let us note that if f is differentiable at (x_0, y_0) and if (α_1, α_2) is the total derivative of f at (x_0, y_0), then letting (h, k) approach $(0, 0)$ along the x-axis or the y-axis, we see that

$$\lim_{h\to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0) - \alpha_1 h}{|h|} = 0 = \lim_{k\to 0} \frac{f(x_0, y_0 + k) - f(x_0, y_0) - \alpha_2 k}{|k|}.$$

Consequently, both $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist and are equal to α_1 and α_2, respectively. In other words, if f is differentiable, then the gradient of f at (x_0, y_0) exists and

$$\text{the total derivative of } f \text{ at } (x_0, y_0) = \nabla f(x_0, y_0).$$

Thus in checking the differentiability of f at (x_0, y_0), it is clear which values of α_1 and α_2 can possibly work, and the task reduces to checking whether the corresponding two-variable limit exists and is equal to zero. Also, if either of the partial derivatives does not exist at a point, then we can be sure that f is not differentiable at that point. This is illustrated by Example 3.23 (iii). On the other hand, existence of both the partial derivatives at (x_0, y_0) is not sufficient for f to be differentiable at (x_0, y_0), and this will be seen later in Example 3.29 (i).

Examples 3.23. (i) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the constant function given by $f(x, y) := 1$ for all $(x, y) \in \mathbb{R}^2$. It is clear that f is differentiable at any $(x_0, y_0) \in \mathbb{R}^2$ and the total derivative at (x_0, y_0) is $(0, 0)$.

(ii) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) := x^2 + y^2$. Given any $(x_0, y_0) \in \mathbb{R}^2$, we have $f_x(x_0, y_0) = 2x_0$ and $f_y(x_0, y_0) = 2y_0$, and moreover,

$$\begin{aligned}
&\lim_{(h,k)\to(0,0)} \frac{f(x_0 + h, y_0 + k) - f(x_0, y_0) - 2x_0 h - 2y_0 k}{\sqrt{h^2 + k^2}} \\
&= \lim_{(h,k)\to(0,0)} \frac{h^2 + k^2}{\sqrt{h^2 + k^2}} = \lim_{(h,k)\to(0,0)} \sqrt{h^2 + k^2} = 0.
\end{aligned}$$

It follows that f is differentiable at (x_0, y_0) and $\nabla f(x_0, y_0) = (2x_0, 2y_0)$.

[1] In modern treatments of multivariable calculus, instead of the pair (α_1, α_2), the linear map from $\mathbb{R}^2$ to $\mathbb{R}$ given by $(h, k) \mapsto \alpha_1 h + \alpha_2 k$ is called the (total) derivative of f at (x_0, y_0). However, the pair (α_1, α_2) and the corresponding linear map determine each other uniquely. For this reason and for the sake of simplicity, we have chosen to call the pair (α_1, α_2) the (total) derivative of f at (x_0, y_0).

(iii) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the norm function given by $f(x, y) := \sqrt{x^2 + y^2}$. We have seen in Example 3.1 (ii) that both $f_x(0, 0)$ and $f_y(0, 0)$ do not exist. Hence f is not differentiable at $(0, 0)$.

(iv) Let $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = |xy|$. It is easily seen that $f_x(0, 0) = 0 = f_y(0, 0)$. Moreover, for any $(h, k) \in \mathbb{R}^2$, we have $|h| \le \sqrt{h^2 + k^2}$ and thus if $(h, k) \ne (0, 0)$, then

$$\frac{f(h,k) - f(0,0) - 0 \cdot h - 0 \cdot k}{\sqrt{h^2 + k^2}} = \frac{|hk|}{\sqrt{h^2 + k^2}} \le |k| \to 0 \text{ as } (h, k) \to (0, 0).$$

Hence, f is differentiable at $(0, 0)$ and $\nabla f(0, 0) = (0, 0)$. ◇

For functions of one variable, one has a useful characterization of differentiability given by Carathéodory's lemma. Let us recall its statement; for a proof we refer to page 107 of ACICARA.

Fact 3.24 (Carathéodory's Lemma). *Let $D \subseteq \mathbb{R}$ and let c be an interior point of D. Then $f : D \to \mathbb{R}$ is differentiable at c if and only if there exists a function $f_1 : D \to \mathbb{R}$ such that $f(x) - f(c) = (x - c)f_1(x)$ for all $x \in D$, and f_1 is continuous at c. Moreover, if these conditions hold, then $f'(c) = f_1(c)$.*

If the conditions of Fact 3.24 hold, then the function f_1 is uniquely determined by f and c, and f_1 is called the **increment function** associated with f and c. For functions of two variables, there is an analogous characterization of differentiability, and it will play an important role in the sequel.

Proposition 3.25 (Increment Lemma). *Let $D \subseteq \mathbb{R}^2$ and let (x_0, y_0) be an interior point of D. Then $f : D \to \mathbb{R}$ is differentiable at (x_0, y_0) if and only if there exist functions $f_1, f_2 : D \to \mathbb{R}$ such that f_1 and f_2 are continuous at (x_0, y_0) and*

$$f(x, y) - f(x_0, y_0) = (x - x_0)f_1(x, y) + (y - y_0)f_2(x, y) \quad \text{for all } (x, y) \in D.$$

Moreover, if these conditions hold, then $\nabla f(x_0, y_0) = (f_1(x_0, y_0), f_2(x_0, y_0))$.

Proof. Assume that $f : D \to \mathbb{R}$ is differentiable at (x_0, y_0). Then there is $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ such that

$$\lim_{(x,y)\to(x_0,y_0)} \frac{F(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0,$$

where

$$F(x, y) := f(x, y) - f(x_0, y_0) - \alpha_1(x - x_0) - \alpha_2(y - y_0) \quad \text{for } (x, y) \in D.$$

Define $f_1, f_2 : D \to \mathbb{R}$ by $f_i(x_0, y_0) := \alpha_i$ for $i = 1, 2$, and for $(x, y) \ne (x_0, y_0)$,

$$f_1(x, y) := \alpha_1 + \frac{(x - x_0)F(x, y)}{(x - x_0)^2 + (y - y_0)^2}, \quad f_2(x, y) := \alpha_2 + \frac{(y - y_0)F(x, y)}{(x - x_0)^2 + (y - y_0)^2}.$$

a function is not differentiable at a point. Likewise, if either of the partial derivatives does not exist at a point, then the function cannot be differentiable at that point. We have also seen in Examples 3.29 (i) and (ii) that neither continuity nor the existence of partial derivatives is sufficient to ascertain differentiability. But it turns out that the existence and continuity of partial derivatives imply differentiability. This result, proved below, gives a very useful set of sufficient conditions for differentiability.

Proposition 3.33. *Let $D \subseteq \mathbb{R}^2$ and let (x_0, y_0) be an interior point of D. Let $f : D \to \mathbb{R}$ be such that both f_x and f_y exist on $D \cap \mathbb{S}_\delta(x_0, y_0)$ for some $\delta > 0$. If one of them is continuous at (x_0, y_0), then f is differentiable at (x_0, y_0).*

Proof. Suppose f_x is continuous at (x_0, y_0). Since (x_0, y_0) is an interior point of D, we may assume without loss of generality that $\mathbb{S}_\delta(x_0, y_0) \subseteq D$. In view of Remark 3.26 (i), it suffices to find a pair of increment functions associated with $f_{|\mathbb{S}_\delta(x_0,y_0)}$ and (x_0, y_0). To this end, let us first observe that for any $(x, y) \in \mathbb{S}_\delta(x_0, y_0)$, we have $(x_0, y) \in \mathbb{S}_\delta(x_0, y_0)$, and moreover, we can decompose $f(x, y) - f(x_0, y_0)$ along the "hook" (Figure 3.1) linking (x, y) and (x_0, y_0), that is, we can write $f(x, y) - f(x_0, y_0) = A(x, y) + B(y)$, where

$$A(x, y) := f(x, y) - f(x_0, y) \quad \text{and} \quad B(y) := f(x_0, y) - f(x_0, y_0).$$

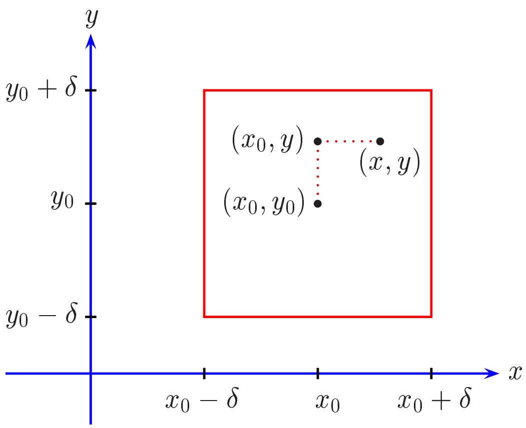

Fig. 3.1. The "hook" linking (x, y) and (x_0, y_0).

Let us define functions $f_1, f_2 : \mathbb{S}_\delta(x_0, y_0) \to \mathbb{R}$ by

$$f_1(x, y) := \begin{cases} \dfrac{A(x, y)}{x - x_0} & \text{if } x \neq x_0, \\[2ex] f_x(x_0, y) & \text{if } x = x_0, \end{cases} \quad \text{and} \quad f_2(x, y) := \begin{cases} \dfrac{B(y)}{y - y_0} & \text{if } y \neq y_0, \\[2ex] f_y(x_0, y_0) & \text{if } y = y_0. \end{cases}$$

Then it is easily seen that

$$f(x,y)-f(x_0,y_0)=(x-x_0)f_1(x,y)+(y-y_0)f_2(x,y) \text{ for all } (x,y)\in \mathbb{S}_\delta(x_0,y_0).$$

Moreover, since f_x exists on $\mathbb{S}_\delta(x_0,y_0)$, we see that if $x\in(x_0-\delta,x_0+\delta)$ with $x\neq x_0$ and $y\in(y_0-\delta,y_0+\delta)$, then by the MVT (Fact 3.2), there is $c\in\mathbb{R}$ between x_0 and x such that $A(x,y)=f(x,y)-f(x_0,y)=(x-x_0)f_x(c,y)$, and hence $f_1(x,y)=f_x(c,y)$. Now, since f_x continuous at (x_0,y_0), we see that f_1 is continuous at (x_0,y_0). Also, since $f_y(x_0,y_0)$ exists, we see that f_2 is continuous at (x_0,y_0). Thus it follows from the Increment Lemma that f is differentiable at (x_0,y_0). The case in which f_y is continuous at (x_0,y_0) is proved similarly. □

The first example below illustrates how Proposition 3.33 can be used to determine differentiability, while the second example shows that the converse of Proposition 3.33 is not true. Another example of a differentiable function whose partial derivatives exist but are not continuous is given in Exercise 29.

Examples 3.34. (i) Consider $f:\mathbb{R}^2\to\mathbb{R}$ defined by $f(0,0):=0$ and $f(x,y):=x^2y^2/(x^4+y^2)$ for $(x,y)\neq(0,0)$. It is easily seen that $f_x(0,0)=0=f_y(0,0)$. Also, for $(x_0,y_0)\neq(0,0)$,

$$f_x(x_0,y_0)=\frac{2x_0y_0^2(y_0^2-x_0^4)}{(x_0^4+y_0^2)^2} \quad\text{and}\quad f_y(x_0,y_0)=\frac{2x_0^6y_0}{(x_0^4+y_0^2)^2}.$$

Moreover, since $(x_0^4+y_0^2)^2\geq y_0^4$ and $(x_0^4+y_0^2)^2\geq 2x_0^4y_0^2$, we see that $|f_x(x_0,y_0)|\leq 2|x_0|+|x_0|=3|x_0|$ for $(x_0,y_0)\neq(0,0)$, and therefore f_x is continuous at $(0,0)$. Hence by Proposition 3.33, f is differentiable at $(0,0)$. Note, however, that $f_y(x_0,x_0^2)=\frac{1}{2}$ for all $x_0\neq 0$, and hence f_y is not continuous at $(0,0)$.

(ii) Consider $f:\mathbb{R}^2\to\mathbb{R}$ defined by $f(x,y):=|xy|$. We have seen in Example 3.23 (iv) that f is differentiable at $(0,0)$. Let $(x_0,y_0)\in\mathbb{R}^2$. For any $h\in\mathbb{R}$ with $h\neq 0$,

$$\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}=\frac{|y_0|\left(|x_0+h|-|x_0|\right)}{h}.$$

Consequently, $f_x(x_0,0)=0$, whereas $f_x(0,y_0)$ does not exist if $y_0\neq 0$. Similarly, it can be seen that $f_y(0,y_0)=0$, whereas $f_y(x_0,0)$ does not exist if $x_0\neq 0$. Thus neither f_x nor f_y exists on $\mathbb{S}_\delta(0,0)$ for any $\delta>0$. ◇

Differentiability and Directional Derivatives

We shall now show that if a function f of two variables is differentiable, then all its directional derivatives exist and they can be computed by the simple formula $\mathbf{D_u}f=\nabla f\cdot\mathbf{u}$. More precisely, we have the following.

Proposition 3.35. *Let $D \subseteq \mathbb{R}^2$ and let (x_0, y_0) be an interior point of D. If $f : D \to \mathbb{R}$ is differentiable at (x_0, y_0), then for every unit vector $\mathbf{u} = (u_1, u_2)$ in $\mathbb{R}^2$, the directional derivative $\mathbf{D_u} f(x_0, y_0)$ exists and moreover,*

$$\mathbf{D_u} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \mathbf{u} = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2.$$

Proof. Let (f_1, f_2) be a pair of increment functions associated with f and (x_0, y_0). Then for any $t \in \mathbb{R}$ such that $(x_0 + tu_1, y_0 + tu_2) \in D$,

$$f(x_0+tu_1, y_0+tu_2)-f(x_0, y_0) = tu_1 f_1(x_0+tu_1, y_0+tu_2)+tu_2 f_2(x_0+tu_1, y_0+tu_2).$$

Thus, using Proposition 2.15 and the continuity of f_1 and f_2 at (x_0, y_0), we see that $\mathbf{D_u} f(x_0, y_0)$ exists and moreover,

$$\mathbf{D_u} f(x_0, y_0) = f_1(x_0, y_0)u_1 + f_2(x_0, y_0)u_2 = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2,$$

as desired. □

The above result suggests the following geometric interpretation of the gradient. Let $D \subseteq \mathbb{R}^2$ and let (x_0, y_0) be an interior point of D. Let $f : D \to \mathbb{R}$ be differentiable at (x_0, y_0) and suppose $\nabla f(x_0, y_0) \neq (0, 0)$. Given any unit vector $\mathbf{u} = (u_1, u_2)$,

$$\mathbf{D_u} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \mathbf{u} = |\nabla f(x_0, y_0)| \cos\theta\,,$$

where $\theta \in [0, \pi]$ is the angle between $\nabla f(x_0, y_0)$ and $\mathbf{u}$. Thus, if we keep in mind the fact that $\mathbf{D_u}(x_0, y_0)$ measures the rate of change in f in the direction of $\mathbf{u}$, then we can make the following observations.

1. $\mathbf{D_u} f(x_0, y_0)$ is maximum when $\cos\theta = 1$, that is, when $\theta = 0$. Thus near (x_0, y_0), $\mathbf{u} = \nabla f(x_0, y_0)/|\nabla f(x_0, y_0)|$ is the direction in which f increases most rapidly.
2. $\mathbf{D_u} f(x_0, y_0)$ is minimum when $\cos\theta = -1$, that is, when $\theta = \pi$. Thus near (x_0, y_0), $\mathbf{u} = -\nabla f(x_0, y_0)/|\nabla f(x_0, y_0)|$ is the direction in which f decreases most rapidly.
3. $\mathbf{D_u} f(x_0, y_0) = 0$ when $\cos\theta = 0$, that is, when $\theta = \pi/2$. Thus near (x_0, y_0), $\mathbf{u} = \pm\left(f_y(x_0, y_0), -f_x(x_0, y_0)\right)/|\nabla f(x_0, y_0)|$ are the directions of no change in f.

For example, consider $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = 4 - x^2 - y^2$. We have $f_x = -2x$ and $f_y = -2y$. So at $(x_0, y_0) = (1, 1)$, the gradient is given by $\nabla f(1, 1) = (-2, -2)$. Thus, near $(1, 1)$, the steepest ascent on the surface $z = f(x, y)$ is in the direction of $\nabla f(1, 1)/|\nabla f(1, 1)| = \left(-1/\sqrt{2}\,, -1/\sqrt{2}\right)$, while the steepest descent is in the reverse direction, namely, $\left(1/\sqrt{2}\,, 1/\sqrt{2}\right)$. The directions of no change are $\pm\left(1/\sqrt{2}\,, -1/\sqrt{2}\right)$.

Proposition 3.35 is also useful in showing that certain functions are not differentiable even though the gradient may exist. Indeed, it suffices to find a single unit vector $\mathbf{u}$ such that the identity $\mathbf{D_u} f = \nabla f \cdot \mathbf{u}$ fails to hold. On the other hand, even when this identity holds for all unit vectors $\mathbf{u}$, the function f may not be differentiable. These remarks are illustrated below.

Examples 3.36. (i) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) := \sqrt{|xy|}$. It is easy to see that f is continuous at $(0, 0)$ and $f_x(0, 0) = 0 = f_y(0, 0)$. On the other hand, given a unit vector $\mathbf{u} = (u_1, u_2)$ in $\mathbb{R}^2$ and any $t \in \mathbb{R}$ with $t \neq 0$, we have

$$\frac{f(0 + tu_1, 0 + tu_2) - f(0, 0)}{t} = \frac{|t|\sqrt{|u_1u_2|}}{t}.$$

It follows that the directional derivative $\mathbf{D_u}f(0, 0)$ does not exist whenever u_1 and u_2 are nonzero, for example, if $\mathbf{u} = (1/\sqrt{2}, 1/\sqrt{2})$. Hence, by Proposition 3.35, we conclude that f is not differentiable at $(0, 0)$.

(ii) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(0, 0) := 0$ and $f(x, y) := x^2y/(x^2 + y^2)$ for $(x, y) \neq (0, 0)$. We have seen in Example 2.16 (iii) that f is continuous at $(0, 0)$. Also, given a unit vector $\mathbf{u} = (u_1, u_2)$ in $\mathbb{R}^2$ and any $t \in \mathbb{R}$ with $t \neq 0$, we have

$$\frac{f(0 + tu_1, 0 + tu_2) - f(0, 0)}{t} = \frac{t^3u_1^2u_2}{t(t^2u_1^2 + t^2u_2^2)} = u_1^2u_2.$$

It follows that the directional derivative $\mathbf{D_u}f(0, 0)$ exists and is equal to $u_1^2u_2$. In particular, $f_x(0, 0) = 0 = f_y(0, 0)$. Consequently, $\mathbf{D_u}f(0, 0) \neq \nabla f(0, 0) \cdot \mathbf{u}$ whenever u_1 and u_2 are nonzero. Hence, by Proposition 3.35, we conclude that f is not differentiable at $(0, 0)$.

(iii) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(0, 0) := 0$ and $f(x, y) := x^3y/(x^4 + y^2)$ for $(x, y) \neq (0, 0)$. We have seen in Example 2.16 (v) that f is continuous at $(0, 0)$. Moreover, for any unit vector $\mathbf{u} = (u_1, u_2)$ in $\mathbb{R}^2$ and any $t \in \mathbb{R}$ with $t \neq 0$, we have

$$\frac{f(0 + tu_1, 0 + tu_2) - f(0, 0)}{t} = \frac{t^4u_1^3u_2}{t(t^4u_1^4 + t^2u_2^2)} = \frac{tu_1^3u_2}{t^2u_1^4 + u_2^2},$$

and so, considering separately the cases $u_2 = 0$ and $u_2 \neq 0$, we see that $\mathbf{D_u}f(0, 0)$ exists and is equal to 0. In particular, $f_x(0, 0) = 0 = f_y(0, 0)$. Consequently, $\mathbf{D_u}f(0, 0) = \nabla f(0, 0) \cdot \mathbf{u}$ for all unit vectors $\mathbf{u}$. On the other hand, if we consider

$$Q(h, k) := \frac{f(0 + h, 0 + k) - f(0, 0) - 0 \cdot h - 0 \cdot k}{\sqrt{h^2 + k^2}} = \frac{h^3k}{(h^4 + k^2)\sqrt{h^2 + k^2}},$$

then $Q(h, k) \not\to 0$ as $(h, k) \to (0, 0)$. To see this, consider a sequence in $\mathbb{R}^2 \setminus \{(0, 0)\}$ approaching $(0, 0)$ along the parabola $k = h^2$. For example, if $(a_n, b_n) := (1/n, 1/n^2)$ for $n \in \mathbb{N}$, then $Q(a_n, b_n) \to 1/2$. It follows that f is not differentiable at $(0, 0)$. This shows that the converse of Proposition 3.35 is not true. In fact, it shows that a function can satisfy all the necessary conditions for differentiability given in Propositions 3.28 and 3.35, but still it may fail to be differentiable. $\diamond$

It may be worthwhile to record the following consequence of the sufficient and the necessary conditions for differentiability proved in this section.

Corollary 3.37. *Let $D \subseteq \mathbb{R}^2$ and let (x_0, y_0) be an interior point of D. If $f : D \to \mathbb{R}$ is such that both f_x and f_y exist on $D \cap \mathbb{S}_\delta(x_0, y_0)$ for some $\delta > 0$, and one of them is continuous at (x_0, y_0), then*

(i) *f is continuous at (x_0, y_0),*
(ii) *for every unit vector $\mathbf{u} = (u_1, u_2)$, the directional derivative $\mathbf{D_u} f(x_0, y_0)$ exists and moreover,*

$$\mathbf{D_u} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \mathbf{u} = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2.$$

Proof. By Proposition 3.33, f is differentiable at (x_0, y_0). Hence (i) follows from Proposition 3.28, while (ii) follows from Proposition 3.35. □

Implicit Differentiation

In calculus of functions of a single variable, one encounters the process of implicit differentiation. Typically, this is applied to equations of the form $f(x, y) = 0$, which are "implicitly differentiated," treating y as a function of x, so as to obtain an equation such as

$$P(x, y) + Q(x, y)\frac{dy}{dx} = 0.$$

Using this, the derivative of y with respect to x is computed at points where $Q(x, y)$ does not vanish. To gain a proper perspective on this process and to put it on a firm footing, one has to take recourse to functions of two variables and an important result known as the Implicit Function Theorem. To begin with, note that $P(x, y)$ and $Q(x, y)$ are, in fact, the partial derivatives $f_x(x, y)$ and $f_y(x, y)$, and the process of differentiation at a point can be justified if the chain rule is applicable and if the equation $f(x, y) = 0$ does indeed define y as a function of x, at least around the point at which derivatives are taken. The Implicit Function Theorem, in the form given below, enables us to justify the latter. It may be recalled that we had already proved a version of the Implicit Function Theorem in the context of continuous functions. The following is the classical version and the one that is most often used in practice.

Proposition 3.38 (Classical Version of Implicit Function Theorem). *Let $D \subseteq \mathbb{R}^2$, $f : D \to \mathbb{R}$, and $(x_0, y_0) \in D$ be such that $f(x_0, y_0) = 0$. Assume that there is $r > 0$ with $\mathbb{S}_r(x_0, y_0) \subseteq D$ and the following conditions hold:*

(a) *f_x and f_y exist at every point of $\mathbb{S}_r(x_0, y_0)$,*
(b) *f_y is continuous at (x_0, y_0) and $f_y(x_0, y_0) \neq 0$.*