Two-Dimensional Conduction:
Finite-Difference Equations
and
Solutions

Chapter 4

Sections 4.4 and 4.5

Numerical methods

- analytical solutions that allow for the determination of the exact temperature distribution are only available for limited ideal cases.
- graphical solutions have been used to gain an insight into complex heat transfer problems, where analytical solutions are not available, but they have limited accuracy and are primarily used for two-dimensional problems.
- advances in numerical computing now allow for complex heat transfer problems to be solved rapidly on computers, i.e., "numerical techniques".
- current numerical techniques include: finite-difference analysis; finite element analysis (FEA); and finite-volume analysis.
- in general, these techniques are routinely used to solve problems in heat transfer, fluid dynamics, stress analysis, electrostatics and magnetics, etc.
- We will show the use of finite-difference analysis to solve conduction heat transfer problems.

Finite-difference Analysis

- numerical techniques result in an **approximate solution**, however the error can be made very small.
- properties (e.g., temperature) are determined at discrete points in the region of interest-these are referred to as **nodal points** or **nodes**.

Consider the finite-difference technique for 2-D conduction heat transfer:

- in this case each node represents the temperature of a point on the surface being considered.
- the temperature at the node represents the average temperature of that region of the surface.
- algebraic expressions are used to define the relationship between adjacent nodes on the surface –usually the boundary conditions are specified.
- by increasing the number of nodes on the surface being considered it is possible to increase the spatial resolution of the solution and to potentially increase the accuracy of the numerical solution, however this increases the number of calculation is required to obtain a solution to the problem.

Finite-Difference Approximation

The Nodal Network and Finite-Difference Approximation

- The nodal network identifies discrete points at which the temperature is to be determined and uses an *m,n* notation to designate their location.
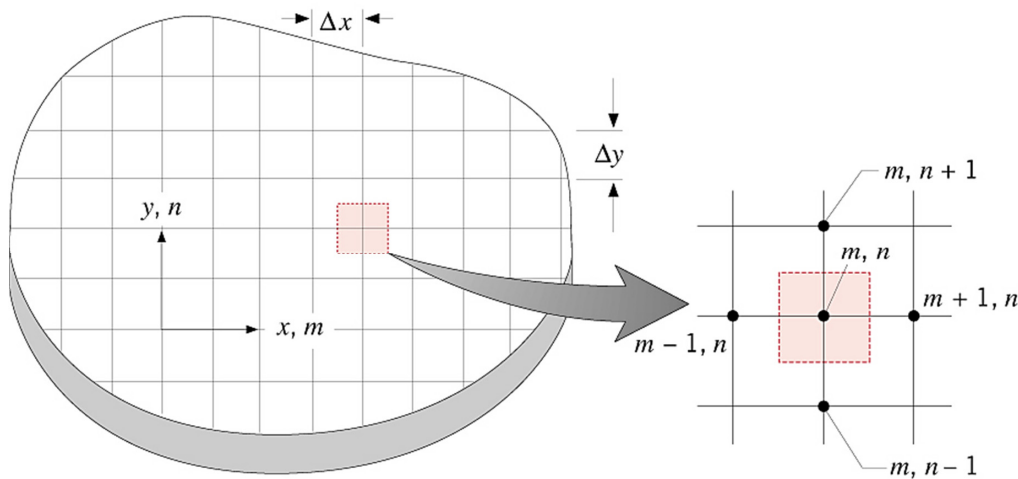
What is represented by the temperature determined at a nodal point, as for example, $T_{m,n}$?

How is the accuracy of the solution affected by construction of the nodal network?

What are the trade-offs between selection of a *fine* or a *coarse mesh*?
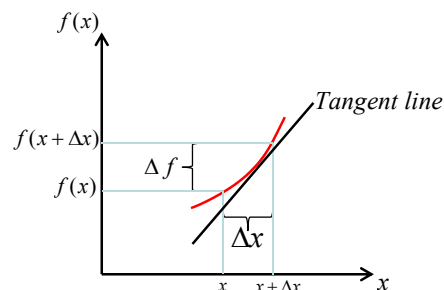
The Finite-Difference Method

Procedure:

- Represent the physical system by a *nodal network i.e., discretization* of problem.

- Use the energy balance method to obtain a finite-difference equation for each node of unknown temperature.

- Solve the resulting set of algebraic equations for the unknown nodal temperatures.

- Use the temperature field and Fourier's Law to determine the heat transfer in the medium

---

Finite difference formulation of the differential equation

- numerical methods are used for solving differential equations, i.e., the $\mathcal{DE}$ is replaced by algebraic equations

- in the finite difference method, derivatives are replaced by differences, i.e.,

$$\frac{d\,f(x)}{dx} \approx \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

- this is based on the premise that a reasonably accurate result can be obtained by replacing differential quantities by sufficiently small differences, letting

$$\frac{d\,f(x)}{dx} = \lim_{\Delta x \to 0}\frac{\Delta f(x)}{\Delta x}$$

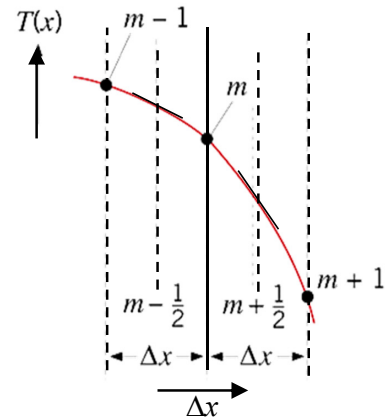# Finite-Difference Formulation of Differential Equation

For example: Consider the 1-D steady-state heat conduction equation with internal heat generation) i.e.,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{k} = 0$$

For a point m,n we approximate the first derivatives at points m-½Δx and m+ ½Δx as

$$\left.\frac{\partial T}{\partial x}\right|_{m-1/2,\,n} = \frac{T_{m,\,n} - T_{m-1,\,n}}{\Delta x}$$

$$\left.\frac{\partial T}{\partial x}\right|_{m+1/2,\,n} = \frac{T_{m+1,\,n} - T_{m,\,n}}{\Delta x}$$

Finite-Difference Formulation of Differential Equation

example: 1-D steady-state heat conduction equation with internal heat generation

For a point m we approximate the 2nd derivative as

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_m \approx \frac{\left.\frac{dT}{dx}\right|_{m+\frac{1}{2}} - \left.\frac{dT}{dx}\right|_{m-\frac{1}{2}}}{\Delta x} \approx \frac{\frac{T_{m+1} - T_m}{\Delta x} - \frac{T_m - T_{m-1}}{\Delta x}}{\Delta x}$$

$$\approx \frac{T_{m+1} - 2T_m + T_{m-1}}{\Delta x^2}$$

Now the finite-difference approximation of the heat conduction equation is

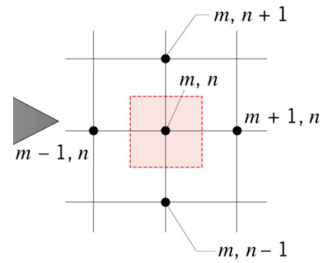$$\frac{T_{m+1} - 2T_m + T_{m-1}}{\Delta x^2} + \frac{\dot{q}_m}{k} = 0$$

This is repeated for all the modes in the region considered

Finite-Difference Formulation of Differential Equation

If this was a 2-D problem we could also construct a similar relationship in the both the x and Y-direction at a point *(m,n)* i.e.,

$$\frac{\partial^2 T}{\partial x^2}\bigg|_{m,n} \approx \frac{T_{m+1,n} - 2T_{m,n} + T_{m-1,n}}{(\Delta x)^2}$$

$$\frac{\partial^2 T}{\partial y^2}\bigg|_{m,n} \approx \frac{T_{m,n-1} - 2T_{m,n} + T_{m,n+1}}{(\Delta y)^2}$$

Now the finite-difference approximation of the 2-D heat conduction equation is

$$\frac{T_{m+1,n} - 2T_{m,n} + T_{m-1,n}}{(\Delta x)^2} + \frac{T_{m,n-1} - 2T_{m,n} + T_{m,n+1}}{(\Delta y)^2} + \frac{\dot{q}}{k} = 0$$

Once again this is repeated for all the modes in the region considered. We could also derive a similar equation for the 3-D case

Finite-Difference Formulation of Differential Equation

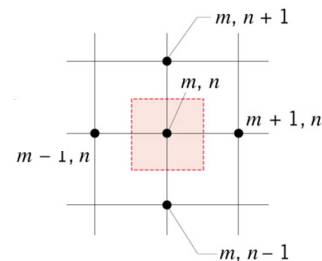If Δx =Δy, then the finite-difference approximation of the 2-D heat conduction equation is

$$T_{m-1,n} - 2T_{m,n} + T_{m+1,n} + T_{m,n-1} - 2T_{m,n} + T_{m,n+1} + \frac{\dot{q}(\Delta x)^2}{k} = 0$$

which can be reduced to

$$T_{m-1,n} + T_{m+1,n} + T_{m,n-1} + T_{m,n+1} - 4T_{m,n} + \frac{\dot{q}(\Delta x)^2}{k} = 0$$

and the relationship reduces to

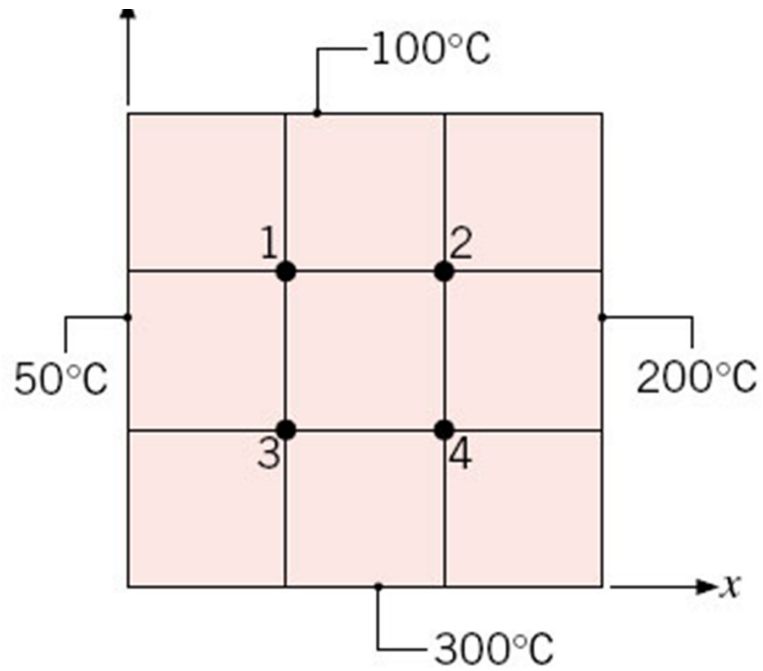$$T_{m,n} = \frac{1}{4}\left[T_{m-1,n} + T_{m+1,n} + T_{m,n-1} + T_{m,n+1} + \frac{\dot{q}(\Delta x)^2}{k}\right]$$
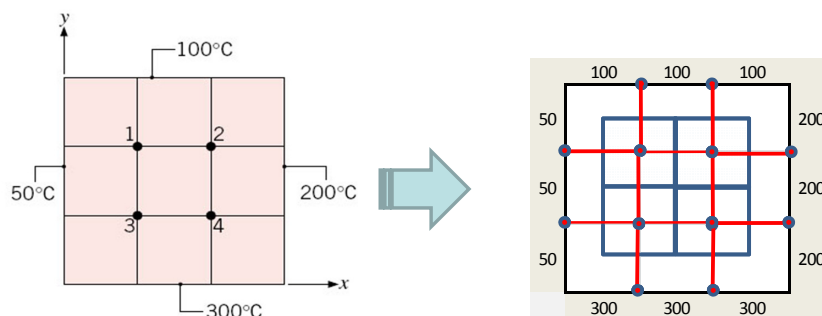
if there is no internal heat generation,

$$T_{m,n} = \frac{1}{4}\left[T_{m-1,n} + T_{m+1,n} + T_{m,n-1} + T_{m,n+1}\right]$$

Which is just the average of the surrounding node's temperatures!

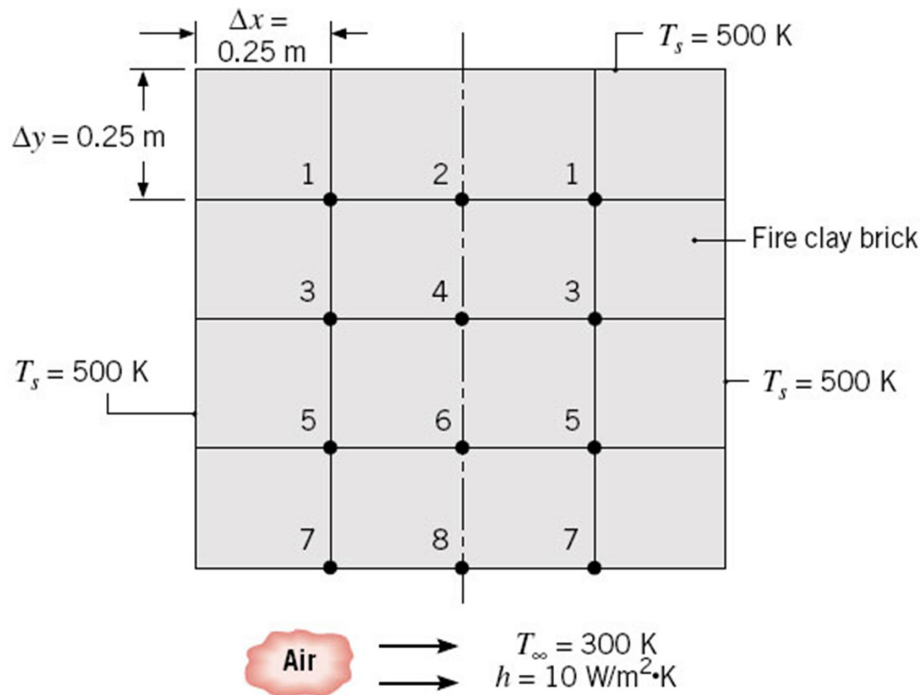**Consider this simple case**

Consider this simple case

$$T_1 = \frac{1}{4}[100 + T_3 + 50 + T_2]$$

$$T_2 = \frac{1}{4}[100 + T_1 + 200 + T_4]$$

$$T_3 = \frac{1}{4}[300 + T_1 + 50 + T_4]$$

$$T_4 = \frac{1}{4}[300 + T_3 + 200 + T_2]$$

What if the boundary conditions are different

Energy Balance Method

Derivation of the Finite-Difference Equations
- The Energy Balance Method -

- As a convenience that eliminates the need to predetermine the direction of heat flow, assume all heat flows are into the nodal region of interest, and express all heat rates accordingly.   Hence, the energy balance becomes:
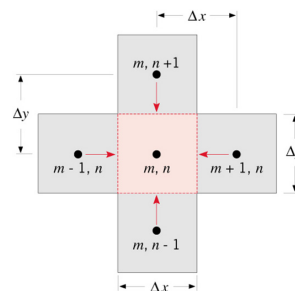
$$\dot{E}_{in} + \dot{E}_g = 0 \qquad (4.30)$$

- Consider application to an *interior nodal point* (one that exchanges heat by conduction with four, equidistant nodal points):

$$\sum_{i=1}^{4} q_{(i)\to(m,n)} + \dot{q} \ (\Delta x \cdot \Delta y \cdot \ell) = 0$$

  where, for example,

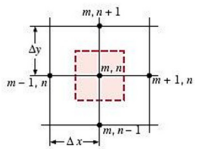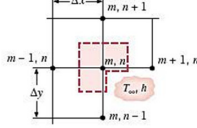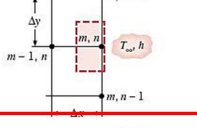$$q_{(m-1,n)\to(m,n)} = k(\Delta y \cdot \ell) \frac{T_{m-1,n} - T_{m,n}}{\Delta x} \qquad (4.31)$$
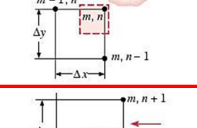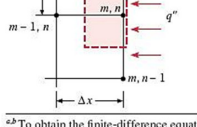
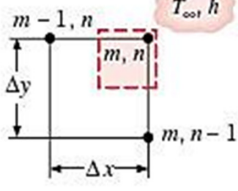 Is it possible for all heat flows to be into the *m,n* nodal region?

  What feature of the analysis insures a correct form of the energy balance equation despite the assumption of conditions that are not realizable?

- A summary of finite-difference equations for common nodal regions is provided in Table 4.2.

TABLE 4.2 Summary of nodal finite-difference equations

Configuration	Finite-Difference Equation for $\Delta x = \Delta y$	
	$T_{m,n+1} + T_{m,n-1} + T_{m+1,n} + T_{m-1,n} - 4T_{m,n} = 0$ **Case 1.** Interior node	(4.29)
	$2(T_{m-1,n} + T_{m,n+1}) + (T_{m+1,n} + T_{m,n-1})$ $+ 2\dfrac{h\Delta x}{k}T_\infty - 2\left(3 + \dfrac{h\Delta x}{k}\right)T_{m,n} = 0$ **Case 2.** Node at an internal corner with convection	(4.41)
	$(2T_{m-1,n} + T_{m,n+1} + T_{m,n-1}) + \dfrac{2h\Delta x}{k}T_\infty - 2\left(\dfrac{h\Delta x}{k} + 2\right)T_{m,n} = 0$ **Case 3.** Node at a plane surface with convection	(4.42)^a
	$(T_{m,n-1} + T_{m-1,n}) + 2\dfrac{h\Delta x}{k}T_\infty - 2\left(\dfrac{h\Delta x}{k} + 1\right)T_{m,n} = 0$ **Case 4.** Node at an external corner with convection	(4.43)
	$(2T_{m-1,n} + T_{m,n+1} + T_{m,n-1}) + \dfrac{2q''\Delta x}{k} - 4T_{m,n} = 0$ **Case 5.** Node at a plane surface with uniform heat flux	(4.44)^b

^{a,b} To obtain the finite-difference equation for an adiabatic surface (or surface of symmetry), simply set h or q'' equal to zero.
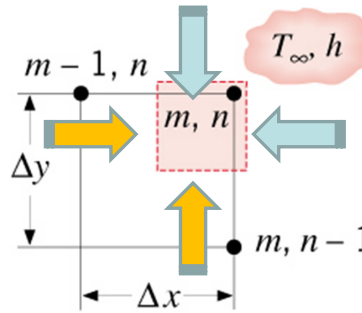

$(T_{m,n-1} + T_{m-1,n}) + 2\dfrac{h\Delta x}{k}T_\infty - 2\left(\dfrac{h\Delta x}{k} + 1\right)T_{m,n} = 0$

Case 4. Node at an external corner with convection

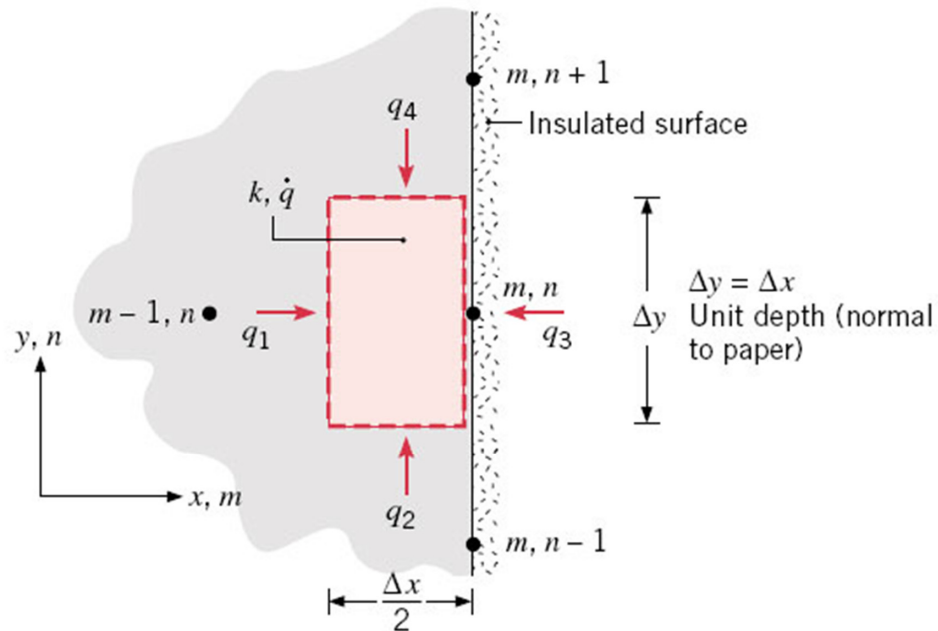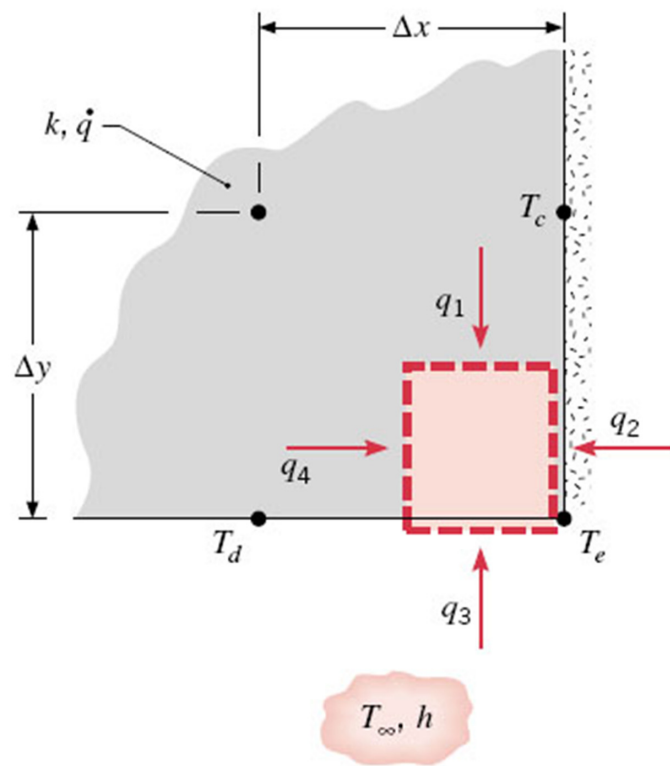Consider an *external corner with convection heat transfer*.

$$q_{(m-1,n)\to(m,n)} + q_{(m,n-1)\to(m,n)} + q_{(\infty)\to(m,n)} = 0$$

$$k\left(\frac{\Delta y}{2}\cdot\ell\right)\frac{T_{m-1,n}-T_{m,n}}{\Delta x} + k\left(\frac{\Delta x}{2}\cdot\ell\right)\frac{T_{m,n-1}-T_{m,n}}{\Delta y}$$

$$+ h\left(\frac{\Delta x}{2}\cdot\ell\right)\left(T_\infty - T_{m,n}\right) + h\left(\frac{\Delta y}{2}\cdot\ell\right)\left(T_\infty - T_{m,n}\right) = 0$$
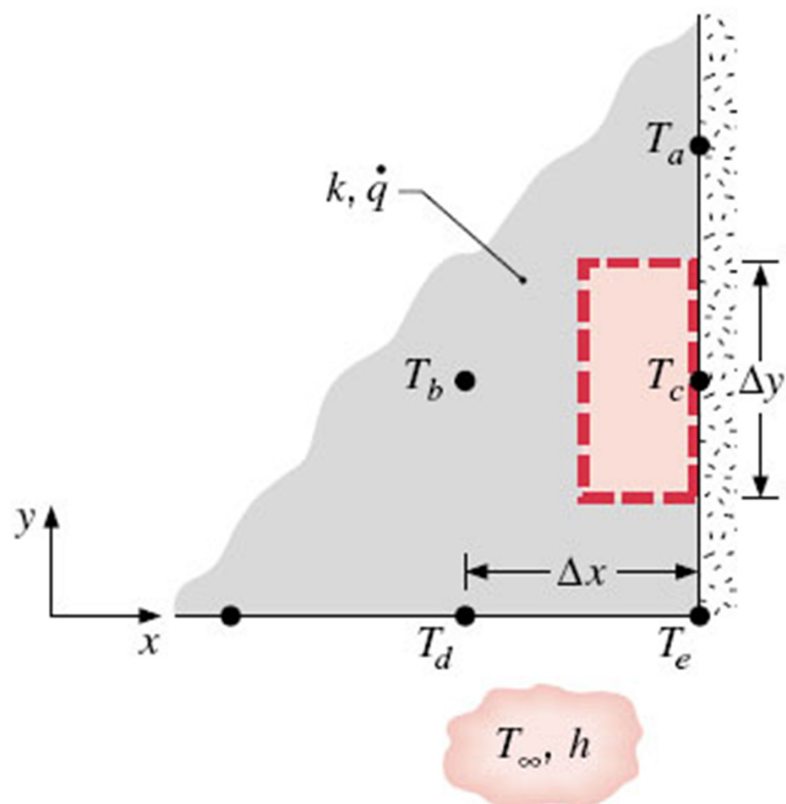
or, with $\Delta x = \Delta y$,

$$T_{m-1,n} + T_{m,n-1} + 2\frac{h\Delta x}{k}T_\infty - 2\left(\frac{h\Delta x}{k}+1\right)T_{m,n} = 0 \qquad\text{(4.43)}$$
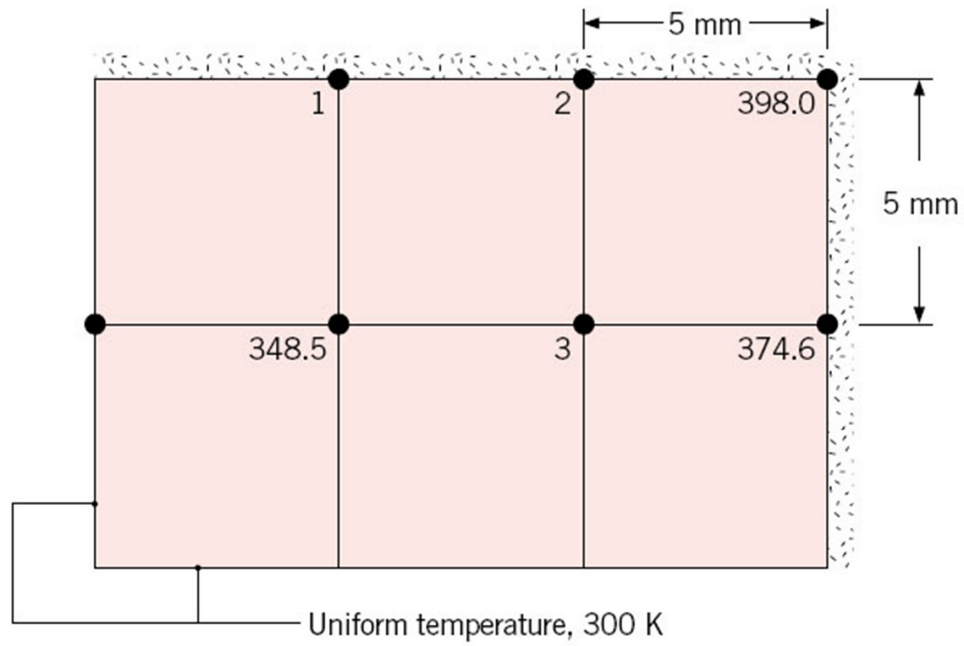

e_04_02

e 04 02 03

e 04 02 02

p_04_45
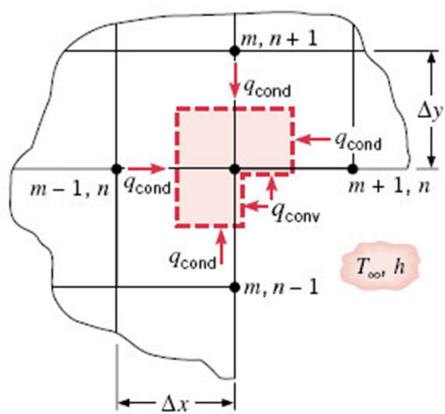
FIGURE 4.6
Formulation of the finite-difference equation for an
internal corner of a solid with surface convection.

fig_04_06

$T_\infty = 300$ K
$h = 50$ W/m²·K

$\Delta x = \Delta y = 0.01$ m

$T = 600$ K

$k = 1$ W/m·K
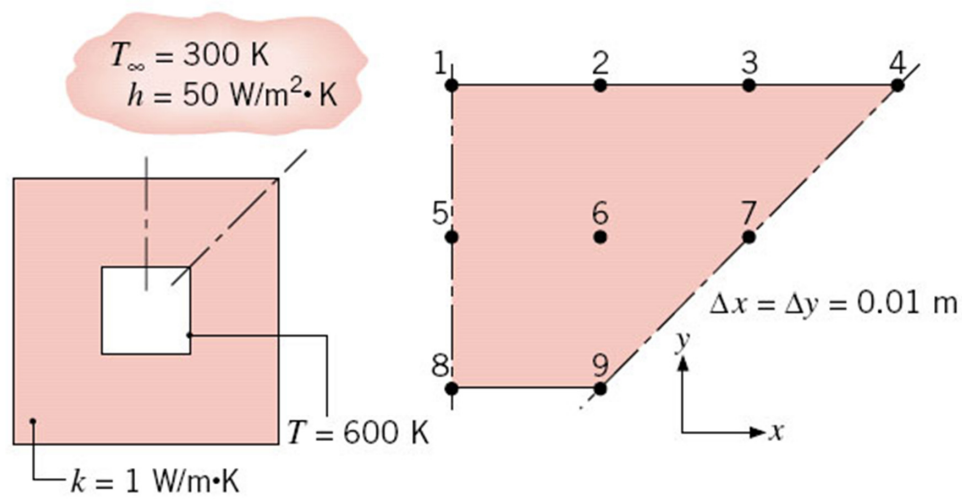
$T_1 = 430$ K $T_8 = T_9 = 600$ K
$T_3 = 394$ K $T_6 = 492$ K

p_04_44

$T_{\infty,o}, h_o$

$\Delta y = 1$ mm

Symmetry
adiabat

Symmetry
adiabat

$T_{\infty,i}, h_i$

$\Delta x = 1$ mm

Symmetry
adiabat
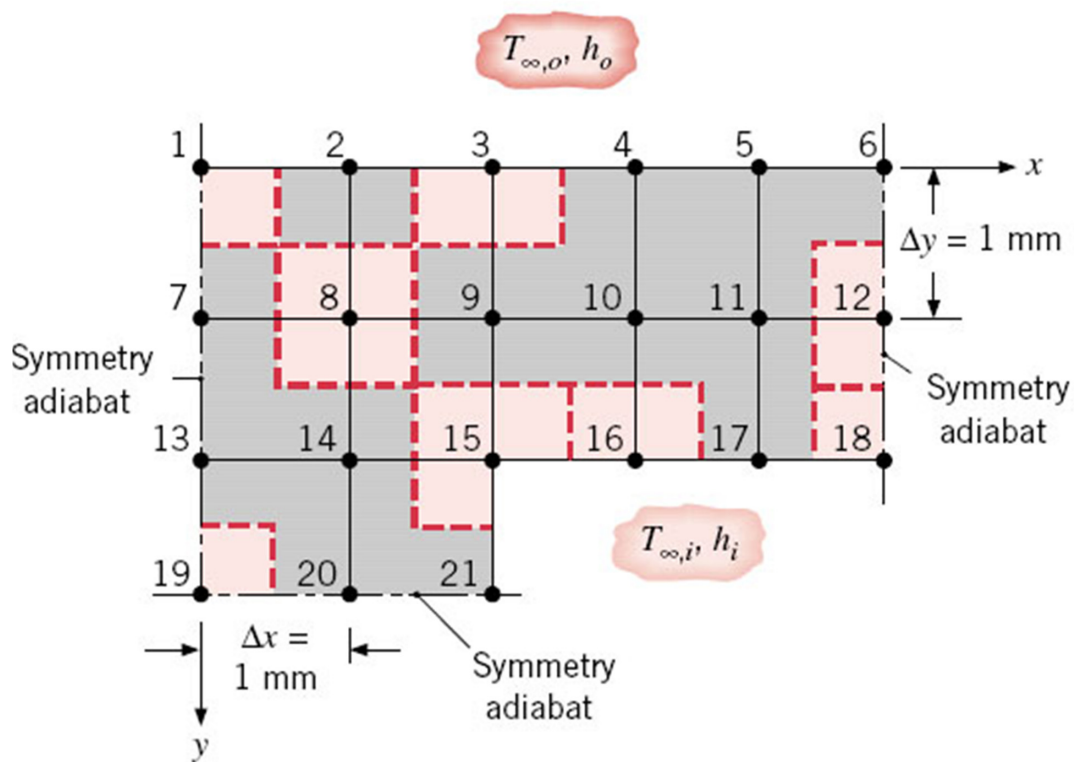
e_04_04_02

- Note potential utility of using thermal resistance concepts to express rate equations. E.g., conduction between adjoining dissimilar materials with an interfacial contact resistance.
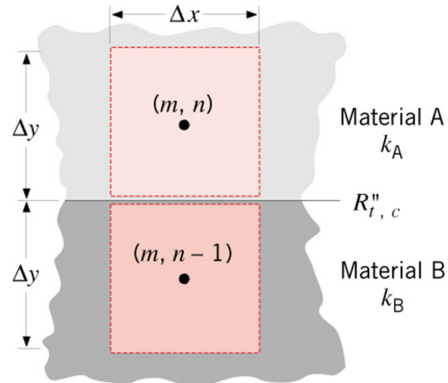
$$q_{(m,n-1)\to(m,n)} = \frac{T_{m,n-1} - T_{m,n}}{R_{tot}}$$

$$R_{tot} = \frac{\Delta y/2}{k_A(\Delta x \cdot \ell)} + \frac{R''_{t,c}}{\Delta x \cdot \ell} + \frac{\Delta y/2}{k_B(\Delta x \cdot \ell)} \qquad (4.46)$$

p_04_38

Solutions Methods

- **Matrix Inversion**: Expression of system of N finite-difference equations for N unknown nodal temperatures as:
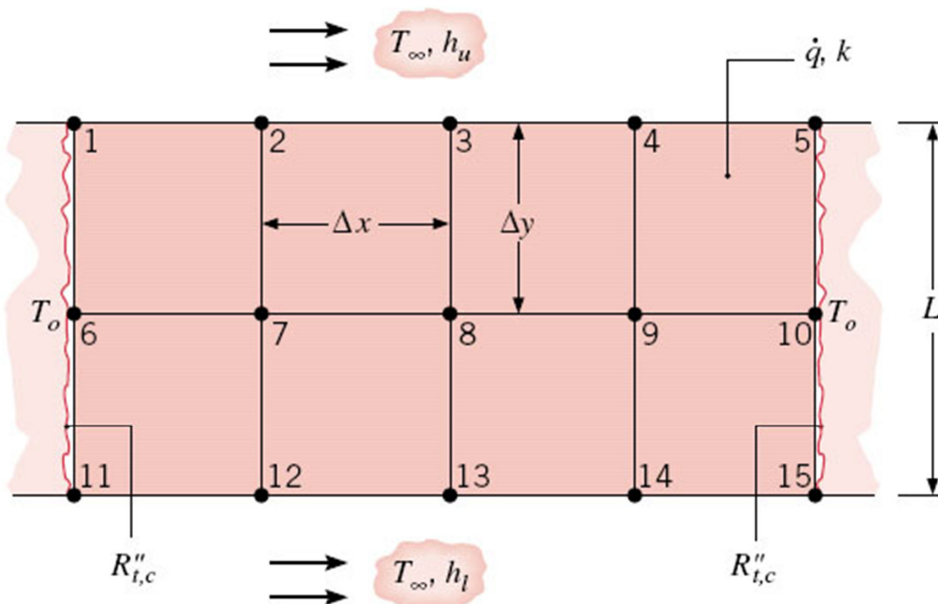
$$[A][T]=[C] \qquad (4.48)$$

Coefficient
Matrix (NxN) — Solution Vector $(T_1, T_2, \ldots T_N)$ — Right-hand Side Vector of Constants $(C_1, C_2 \ldots C_N)$

Solution →

$$[T]=[A]^{-1}[C]$$

Inverse of Coefficient Matrix   (4.49)

- **Gauss-Seidel Iteration**: Each finite-difference equation is written in explicit form, such that its unknown nodal temperature appears alone on the left-hand side:

$$T_i^{(k)} = \frac{C_i}{a_{ii}} - \sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} T_j^{(k)} - \sum_{j=i+1}^{N} \frac{a_{ij}}{a_{ii}} T_j^{(k-1)} \qquad (4.51)$$

where $i = 1, 2, \ldots, N$ and k is the level of iteration.

Iteration proceeds until satisfactory convergence is achieved for all nodes:

$$\left| T_i^{(k)} - T_i^{(k-1)} \right| \le \varepsilon$$

- What measures may be taken to insure that the results of a finite-difference solution provide an accurate prediction of the temperature field?

Problem: Finite-Difference Equations

Problem 4.39: Finite-difference equations for (a) nodal point on a diagonal surface and (b) tip of a cutting tool.
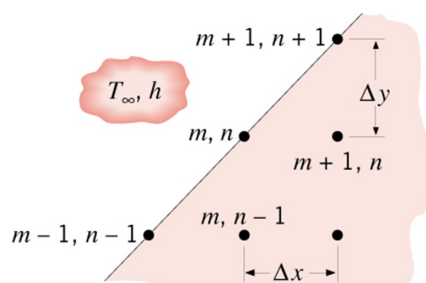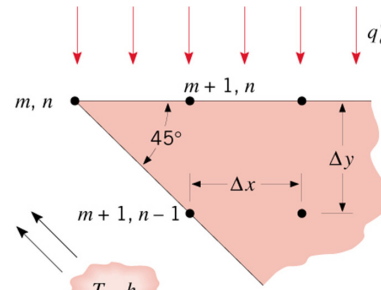
(a) Diagonal surface   (b) Cutting tool.

Schematic:

ASSUMPTIONS: (1) Steady-state, 2-D conduction, (2) Constant properties

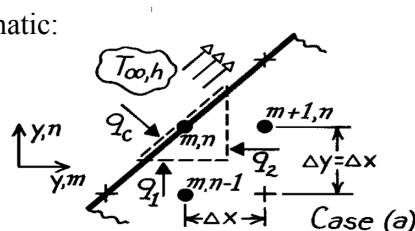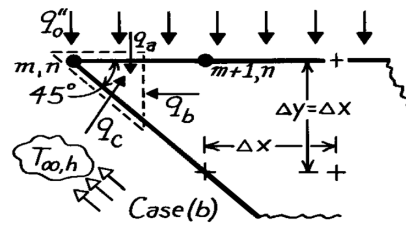**ANALYSIS:** (a) The control volume about node m,n is triangular with sides Δx and Δy and diagonal (surface) of length $\sqrt{2}\ \Delta x$.

The heat rates associated with the control volume are due to conduction, q_1 and q_2, and to convection, q_c. An energy balance for a unit depth normal to the page yields

$$\dot{E}_{in} = 0$$

$$q_1 + q_2 + q_c = 0$$

$$k(\Delta x \cdot 1)\frac{T_{m,n-1} - T_{m,n}}{\Delta y} + k(\Delta y \cdot 1)\frac{T_{m+1,n} - T_{m,n}}{\Delta x} + h\left(\sqrt{2}\ \Delta x \cdot 1\right)\left(T_\infty - T_{m,n}\right) = 0.$$

With $\Delta x = \Delta y$, it follows that

$$T_{m,n-1} + T_{m+1,n} + \sqrt{2}\ \cdot \frac{h\Delta x}{k}T_\infty - \left[2 + \sqrt{2}\ \cdot \frac{h\Delta x}{k}\right]T_{m,n} = 0.$$

(b) The control volume about node m,n is triangular with sides $\Delta x/2$ and $\Delta y/2$ and a lower diagonal surface of length $\sqrt{2}\left(\Delta x/2\right)$.

The heat rates associated with the control volume are due to the uniform heat flux, q_a, conduction, q_b, and convection q_c. An energy balance for a unit depth yields

$$\dot{E}_{in} = 0$$

$$q_a + q_b + q_c = 0$$

$$q_o'' \cdot \left[\frac{\Delta x}{2} \cdot 1\right] + k \cdot \left[\frac{\Delta y}{2} \cdot 1\right]\frac{T_{m+1,n} - T_{m,n}}{\Delta x} + h \cdot \left[\sqrt{2} \cdot \frac{\Delta x}{2}\right]\left(T_\infty - T_{m,n}\right) = 0.$$

or, with $\Delta x = \Delta y$,

$$T_{m+1,n} + \sqrt{2} \cdot \frac{h\Delta x}{k} \cdot T_\infty + q_o'' \cdot \frac{\Delta x}{k} - \left(1 + \sqrt{2} \cdot \frac{h\Delta x}{k}\right)T_{m,n} = 0.$$


Problem 4.76: Analysis of cold plate used to thermally control IBM multi-chip, thermal conduction module.

Features:

- Heat dissipated in the chips is transferred by conduction through spring-loaded aluminum pistons to an aluminum cold plate.

- Nominal operating conditions may be assumed to provide a uniformly distributed heat flux of $q_o'' = 10^5\ \text{W/m}^2$ at the base of the cold plate.

- Heat is transferred from the cold plate by water flowing through channels in the cold plate.

Find: (a) Cold plate temperature distribution for the prescribed conditions. (b) Options for operating at larger power levels while remaining within a maximum cold plate temperature of 40°C.
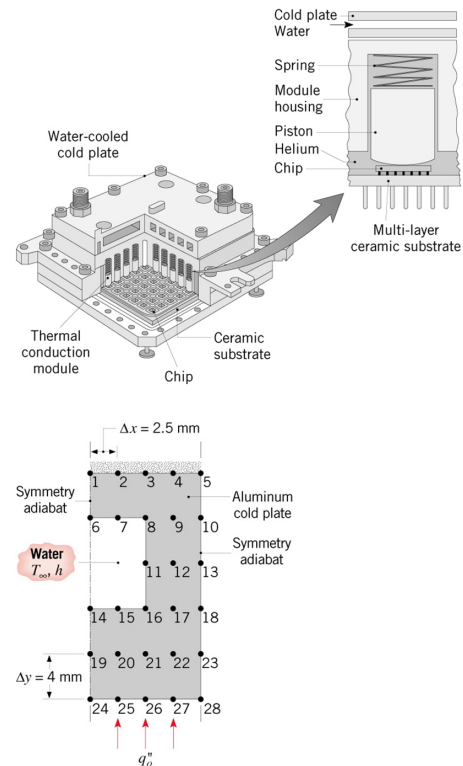
Schematic:
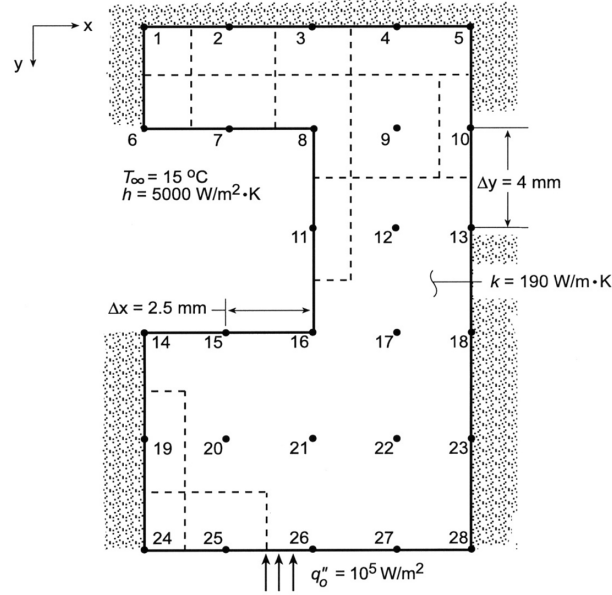
ASSUMPTIONS: (1) Steady-state conditions, (2) Two-dimensional conduction, (3) Constant properties.

ANALYSIS: Finite-difference equations must be obtained for each of the 28 nodes. Applying the energy balance method to regions 1 and 5, which are similar, it follows that

Node 1: $\quad (\Delta y/\Delta x)T_2 + (\Delta x/\Delta y)T_6 - [(\Delta y/\Delta x)+(\Delta x/\Delta y)]T_1 = 0$

Node 5: $(\Delta y/\Delta x)T_4 + (\Delta x/\Delta y)T_{10} - [(\Delta y/\Delta x)+(\Delta x/\Delta y)]T_5 = 0$

Nodal regions 2, 3 and 4 are similar, and the energy balance method yields a finite-difference equation of the form

Nodes 2,3,4:

$$(\Delta y/\Delta x)(T_{m-1,n} + T_{m+1,n}) + 2(\Delta x/\Delta y)T_{m,n-1} - 2[(\Delta y/\Delta x)+(\Delta x/\Delta y)]T_{m,n} = 0$$

Energy balances applied to the remaining combinations of similar nodes yield the following finite-difference equations.

Nodes 6, 14: $\quad (\Delta x/\Delta y)T_1 + (\Delta y/\Delta x)T_7 - [(\Delta x/\Delta y)+(\Delta y/\Delta x)+(h\Delta x/k)]T_6 = -(h\Delta x/k)T_\infty$

$\qquad\qquad\quad (\Delta x/\Delta y)T_{19} + (\Delta y/\Delta x)T_{15} - [(\Delta x/\Delta y)+(\Delta y/\Delta x)+(h\Delta x/k)]T_{14} = -(h\Delta x/k)T_\infty$

Nodes 7, 15: $\quad (\Delta y/\Delta x)(T_6 + T_8) + 2(\Delta x/\Delta y)T_2 - 2[(\Delta y/\Delta x)+(\Delta x/\Delta y)+(h\Delta x/k)]T_7 = -(2h\Delta x/k)T_\infty$

$\qquad\qquad\quad (\Delta y/\Delta x)(T_{14} + T_{16}) + 2(\Delta x/\Delta y)T_{20} - 2[(\Delta y/\Delta x)+(\Delta x/\Delta y)+(h\Delta x/k)]T_{15} = -(2h\Delta x/k)T_\infty$

Nodes 8, 16: $\quad (\Delta y/\Delta x)T_7 + 2(\Delta y/\Delta x)T_9 + (\Delta x/\Delta y)T_{11} + 2(\Delta x/\Delta y)T_3 - [3(\Delta y/\Delta x) + 3(\Delta x/\Delta y)$
$$+ (h/k)(\Delta x + \Delta y)]T_8 = -(h/k)(\Delta x + \Delta y)T_\infty$$
$$(\Delta y/\Delta x)T_{15} + 2(\Delta y/\Delta x)T_{17} + (\Delta x/\Delta y)T_{11} + 2(\Delta x/\Delta y)T_{21} - [3(\Delta y/\Delta x) + 3(\Delta x/\Delta y)$$
$$+ (h/k)(\Delta x + \Delta y)]T_{16} = -(h/k)(\Delta x + \Delta y)T_\infty$$

Node 11: $\quad (\Delta x/\Delta y)T_8 + (\Delta x/\Delta y)T_{16} + 2(\Delta y/\Delta x)T_{12} - 2[(\Delta x/\Delta y) + (\Delta y/\Delta x) + (h\Delta y/k)]T_{11} = -(2h\Delta y/k)T_\infty$

Nodes 9, 12, 17, 20, 21, 22:
$$(\Delta y/\Delta x)T_{m-1,n} + (\Delta y/\Delta x)T_{m+1,n} + (\Delta x/\Delta y)T_{m,n+1} + (\Delta x/\Delta y)T_{m,n-1} - 2[(\Delta x/\Delta y) + (\Delta y/\Delta x)]T_{m,n} = 0$$

Nodes 10, 13, 18, 23:
$$(\Delta x/\Delta y)T_{n+1,m} + (\Delta x/\Delta y)T_{n-1,m} + 2(\Delta y/\Delta x)T_{m-1,n} - 2[(\Delta x/\Delta y) + (\Delta y/\Delta x)]T_{m,n} = 0$$

Node 19: $\qquad\qquad (\Delta x/\Delta y)T_{14} + (\Delta x/\Delta y)T_{24} + 2(\Delta y/\Delta x)T_{20} - 2[(\Delta x/\Delta y) + (\Delta y/\Delta x)]T_{19} = 0$

Nodes 24, 28: $\quad (\Delta x/\Delta y)T_{19} + (\Delta y/\Delta x)T_{25} - [(\Delta x/\Delta y) + (\Delta y/\Delta x)]T_{24} = -(q_o''\Delta x/k)$
$$(\Delta x/\Delta y)T_{23} + (\Delta y/\Delta x)T_{27} - [(\Delta x/\Delta y) + (\Delta y/\Delta x)]T_{28} = -(q_o''\Delta x/k)$$

Nodes 25, 26, 27:
$$(\Delta y/\Delta x)T_{m-1,n} + (\Delta y/\Delta x)T_{m+1,n} + 2(\Delta x/\Delta y)T_{m,n+1} - 2[(\Delta x/\Delta y) + (\Delta y/\Delta x)]T_{m,n} = -(2q_o''\Delta x/k)$$

Evaluating the coefficients and solving the equations simultaneously, the steady-state temperature distribution (°C), tabulated according to the node locations, is:

23.77	23.91	24.27	24.61	24.74
23.41	23.62	24.31	24.89	25.07
		25.70	26.18	26.33
28.90	28.76	28.26	28.32	28.35
30.72	30.67	30.57	30.53	30.52
32.77	32.74	32.69	32.66	32.65

(b) For the prescribed conditions, the maximum allowable temperature ($T_{24} = 40°C$) is reached when

$$q_o'' = 1.407 \times 10^5 \text{ W/m}^2 \ (14.07 \text{ W/cm}^2).$$

Options for extending this limit could include use of a copper cold plate ($k \approx 400$ W/m·K) and/or increasing the convection coefficient associated with the coolant.

With k = 400 W/m·K, a value of $q_o'' = 17.37$ W/cm² may be maintained.

. With k = 400 W/m·K and h = 10,000 W/m²·K (a practical upper limit), $q_o'' = 28.65$ W/cm².

Additional, albeit small, improvements may be realized by relocating the coolant channels closer to the base of the cold plate.