INFINITE SERIES

2.1 Sequences: A sequence of real numbers is defined as a function $f: \boldsymbol{N} \rightarrow \boldsymbol{R}$,where $\boldsymbol{N}$ is a set of natural numbers and $\mathbf{R}$ is a set of real numbers. A sequence can be expressed as$\langle f_1, f_2, f_3, \dots\dots, f_n, \dots.\rangle$or $\langle f_n\rangle$. For example $\langle\frac{1}{n}\rangle = \langle\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \dots\dots.\rangle$ is a sequence.

Convergent sequence: A sequence $\langle u_n\rangle$ converges to a number l, if for given $\varepsilon > 0$, there exists a positive integer m depending on ε , such that $|u_n - l| < \varepsilon \; \forall \; n \geq m$.

Then l is called the limit of the given sequence and we can write

$$\lim_{n\to\infty} u_n = l \text{ or } u_n \rightarrow l$$

2.2 Definition of an Infinite Series

An expression of the form $u_1 + u_2 + u_3 + \cdots + u_n + \cdots$ is known as the infinite series of real numbers, where each $u_n$is a real number. It is denoted by$\sum_{n=1}^{\infty} u_n$.

For example $\sum_{n=1}^{\infty}\frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$is an infinite series.

Convergence of an infinite series

Consider an infinite series $\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \cdots$

Let us define$S_1 = u_1$, $S_2 = u_1 + u_2$, $S_3 = u_1 + u_2 + u_3$, ……..,

$S_n = u_1 + u_2 + u_3 + \cdots + u_n$ and so on .

Then the sequence $\langle S_n\rangle$ so formed is known as the sequence of partial sums (S.O.P.S.) of the given series.

Convergent series: A series $u_1 + u_2 + u_3 + \cdots + u_n + \cdots = \sum_{n=1}^{\infty} u_n$ converges if the sequence $\langle S_n \rangle$ of its partial sums converges i.e. if $\lim_{n\to\infty} S_n$ exists. Also if $\lim_{n\to\infty} S_n = S$ then S is called as the sum of the given series .

Divergent series: A series $u_1 + u_2 + u_3 + \cdots + u_n + \cdots = \sum_{n=1}^{\infty} u_n$ diverges if the sequence $\langle S_n \rangle$ of its partial sums diverges i.e. if $\lim_{n\to\infty} S_n = +\infty \; or - \infty$.

Example 1 Show that the Geometric series $\sum_{n=1}^{\infty} r^{n-1} = 1 + r + r^2 + r^3 + \ldots\ldots$, where $r > 0$, is convergent if $r < 1$ and diverges if $r \geq 1$.

Solution: Let us define $S_1 = 1$, $S_2 = 1 + r$, $S_3 = 1 + r + r^2$, ……..,

$$S_n = 1 + r + r^2 + \cdots + r^{n-1}$$

Case 1: $r < 1$

$$\text{Consider } \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1-r^n}{1-r} = \frac{1}{1-r} - \lim_{n\to\infty} \frac{r^n}{1-r}$$

$$= \frac{1}{1-r} \qquad (\text{As } \lim_{n\to\infty} r^n = 0 \text{ if } |r| < 1)$$

Since $\lim_{n\to\infty} S_n$ is finite $\therefore$ the sequence of partial sums i.e. $\langle S_n \rangle$ converges and hence the given series converges.

Case2: $r > 1$

$$\text{Consider } \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{r^n - 1}{r-1} = \lim_{n\to\infty} \frac{r^n}{1-r} - \frac{1}{r-1}$$

$\to \infty$ (As $r^n \to \infty$ if $r > 1$)

Since $\langle S_n \rangle$ diverges and hence the given series diverges.

Case2: $r = 1$

$$\text{Consider } S_n = 1 + r + r^2 + \cdots + r^{n-1}$$

$$= 1 + 1 + 1 + 1 + \cdots .. + 1 = n \Rightarrow \lim_{n\to\infty} S_n = \infty$$

Since $\langle S_n \rangle$ diverges and hence the given series diverges.

Positive term series

An infinite series whose all terms are positive is called a positive term series.

p-series: An infinite series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots (p > 0)$ is called p-series.

It converges if $p > 1$ and diverges if $p \leq 1$.

For example:

1. $\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots$ converges $\quad$ $(\text{As } p = 3 > 1)$

2. $\sum_{n=1}^{\infty} \frac{1}{n^{5/2}} = \frac{1}{1^{5/2}} + \frac{1}{2^{5/2}} + \frac{1}{3^{5/2}} + \cdots$ converges $\left(\text{As } p = \frac{5}{2} > 1\right)$

3. $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}} = \frac{1}{1^{1/2}} + \frac{1}{2^{1/2}} + \frac{1}{3^{1/2}} + \cdots$ converges $\left(\text{As } p = \frac{1}{2} < 1\right)$

Necessary condition for convergence:

If an infinite series $\sum_{n=1}^{\infty} u_n$ is convergent then $\lim_{n\to\infty} u_n = 0$. However, converse need not be true.

Proof: Consider the sequence $\langle S_n \rangle$ of partial sums of the series $\sum_{n=1}^{\infty} u_n$.

We know that $\quad S_n = u_1 + u_2 + u_3 + \cdots\cdots\cdots\cdots + u_n$

$$= u_1 + u_2 + u_3 + \cdots + u_{n-1} + u_n$$

$$\Rightarrow S_{n-1} = u_1 + u_2 + u_3 + \cdots\cdots\cdots + u_{n-1}$$

Now $\quad S_n - S_{n-1} = u_n$

Taking limit $n \to \infty$, we get

$$\lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} u_n$$

$\Rightarrow \lim\limits_{n\to\infty} S_n - \lim\limits_{n\to\infty} S_{n-1} = \lim\limits_{n\to\infty} u_n$ ………..(1)

As $\sum_{n=1}^{\infty} u_n$ is convergent $\therefore$ sequence $\langle S_n \rangle$ of its partial sums is also convergent.

Let $\lim\limits_{n\to\infty} S_n = l$, then $\lim\limits_{n\to\infty} S_{n-1} = l$

Substituting these values in equation (1), we get $\lim\limits_{n\to\infty} u_n = 0$.

To show that converse may not hold , let us consider the series$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{1}{n}$.

Here $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{1}{n} = 0$

But$\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent series (As $p = 1$)

Corollary: If $\lim\limits_{n\to\infty} u_n \neq 0$, then $\sum_{n=1}^{\infty} u_n$cannot converge.

Example 2 Test the convergence of the series $\sum_{n=1}^{\infty} cos\,\frac{1}{n}$

Solution: Here $u_n = cos\,\frac{1}{n} \Rightarrow \lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} cos\,\frac{1}{n} = 1 \neq 0$

Hence the given series is not convergent.

Example 3 Test the convergence of the series $\sum_{n=1}^{\infty} \sqrt{\frac{n}{n+1}}$

Solution: Here $u_n = \sqrt{\frac{n}{n+1}} \Rightarrow \lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \sqrt{\frac{n}{n+1}}$

$$\Rightarrow \lim_{n\to\infty} u_n = \lim_{n\to\infty} \sqrt{\frac{1}{1+\frac{1}{n}}} = 1 \neq 0$$

Hence the given series is not convergent.

2.3 Tests for the convergence of infinite series

1. Comparision Test:

Let $\sum_{n=1}^{\infty} u_n$and $\sum_{n=1}^{\infty} v_n$be two positive term series such that

$u_n \leq kv_n \ \ \forall \ \ n$ (where k is a positive number)

Then (i) If $\sum_{n=1}^{\infty} v_n$converges then $\sum_{n=1}^{\infty} u_n$also converges.

(ii) If $\sum_{n=1}^{\infty} u_n$diverges then $\sum_{n=1}^{\infty} v_{\,n}$also diverges.

Example 4 Test the convergence of the following series

$(i)\sum_{n=1}^{\infty} \frac{1}{n^n}$ $(ii)\sum_{n=2}^{\infty} \frac{1}{log\ n}$ $(iii)\sum_{n=1}^{\infty} \frac{1}{2^n+x} \forall x > 0$

Solution: (i) Here $u_n = \frac{1}{n^n}$ We know that $n^n > 2^n$ for $n > 2$

Hence $\frac{1}{n^n} < \frac{1}{2^n}$ for $n > 2$

Now $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series $(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots)$whose common ratio is $\frac{1}{2}$.

Since $\frac{1}{2} < 1 \therefore \sum_{n=1}^{\infty} \frac{1}{2^n}$ is a convergent series. Thus by comparision test $\sum_{n=1}^{\infty} \frac{1}{n^n}$ is also convergent.

(ii) Here $u_n = \frac{1}{\log n}$ We know that $\log n < n$ for $n \geq 2$

Hence $\frac{1}{\log n} > \frac{1}{n}$ for $n \geq 2$ $\Rightarrow \frac{1}{n} < \frac{1}{log\ n}$ for $n \geq 2$

Now $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent series (As $\mathrm{p} = 1$) . Thus by comparision test $\sum_{n=2}^{\infty} \frac{1}{\log n}$ is also divergent.

(iii) Here $u_n = \frac{1}{2^n+x}$. Clearly $2^n + x > 2^n\ (as\ x > 0)$

$$\therefore \ \frac{1}{2^n+x} < \frac{1}{2^n}$$

Now $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series $(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots)$whose common ratio is $\frac{1}{2}$.

Since $\frac{1}{2} < 1 \therefore \sum_{n=1}^{\infty} \frac{1}{2^n}$ is a convergent series. Thus by comparision test $\sum_{n=1}^{\infty} \frac{1}{2^n+x}$ is also convergent.

Example 4 Test the convergence of the series$\sum_{n=1}^{\infty}\left[\frac{1}{n^2}+\frac{1}{(n+1)^2}\right]$

Solution: Here $u_n = \left[\frac{1}{n^2}+\frac{1}{(n+1)^2}\right]$

Clearly $\frac{1}{n^2}+\frac{1}{(n+1)^2} < \frac{1}{n^2}+\frac{1}{n^2} < \frac{1}{n^2}$

Now $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent series (As p $= 2 > 1$) . Thus by comparision test $\sum_{n=1}^{\infty}\left[\frac{1}{n^2}+\frac{1}{(n+1)^2}\right]$ is also convergent.

2. Limit Form Test:

Let $\sum_{n=1}^{\infty} u_n$and $\sum_{n=1}^{\infty} v_n$be two positive term series such that

$\lim_{n\to\infty} \frac{u_n}{v_n} = l$ (where l is a finite and non zero number).

Then $\sum_{n=1}^{\infty} u_n$and $\sum_{n=1}^{\infty} v_n$behave in the same manner i.e. either both converge or both diverge.

Example 5 Test the convergence of the series $\frac{1}{3.7}+\frac{1}{4.9}+\frac{1}{5.11}+$…..

Solution: Here $u_n = \frac{1}{(n+2)(2n+5)}$

Let $v_n = \frac{1}{n^2}$. Now consider $\frac{u_n}{v_n} = \frac{1}{(n+2)(2n+5)}n^2 = \frac{n^2}{2n^2+9n+10}$

$$\Rightarrow \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n^2}{2n^2+9n+10}$$

$$= \lim_{n\to\infty} \frac{1}{2+\frac{9}{n}+\frac{10}{n^2}} = \frac{1}{2} \text{ (which is a finite and non zero number)}$$

Hence by Limit form test , $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ behave similarly.

Since $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (as $p = 2 > 1$)

$\therefore \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{1}{(n+2)(2n+5)}$ also converges.

Example 6 Test the convergence of the series

$(i) \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \frac{1}{\sqrt{4}+\sqrt{5}} + \ldots\ldots (ii) \sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n-1}}{n}$

Solution:(i) Here $u_n = \frac{1}{\sqrt{n+1}+\sqrt{n+2}}$

Let $v_n = \frac{1}{\sqrt{n}}$. Now consider $\frac{u_n}{v_n} = \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n+2}}$

$$\Rightarrow \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n+2}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+\sqrt{1+\frac{2}{n}}}$$

$$= \frac{1}{\sqrt{2}} \text{ (which is a finite and non zero number)}$$

Hence by Limit form test , $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ behave similarly.

Since $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges (as $p = \frac{1}{2} < 1$)

$\therefore \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n+2}}$ also diverges.

(ii) Here $u_n = \frac{\sqrt{n+1}-\sqrt{n-1}}{n}$

$$= \frac{\sqrt{n+1}-\sqrt{n-1}}{n} \cdot \frac{\sqrt{n+1}+\sqrt{n-1}}{\sqrt{n+1}+\sqrt{n-1}} = \frac{(n+1)-(n-1)}{n\sqrt{n+1}+\sqrt{n-1}} = \frac{2}{n\sqrt{n+1}+\sqrt{n-1}}$$

Let $v_n = \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$. Now consider $\frac{u_n}{v_n} = \frac{2\sqrt{n}}{\sqrt{n+1}+\sqrt{n-1}}$

$$\Rightarrow \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{2\sqrt{n}}{\sqrt{n+1}+\sqrt{n-1}}$$

$$= \lim_{n\to\infty}\frac{2}{\sqrt{1+\frac{1}{n}}+\sqrt{1-\frac{1}{n}}}$$

= 1 (which is a finite and non zero number)

Hence by Limit form test , $\sum_{n=1}^{\infty} u_n$and $\sum_{n=1}^{\infty} v_n$behave similarly.

Since$\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ converges (as $p = \frac{3}{2} > 1$)

$\therefore \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n-1}}{n}$ also converges.

Example 7 Test the convergence of the series

(i)$\sum_{n=1}^{\infty}\left[(n^3+1)^{1/3} - n\right]$ (ii) $\sum_{n=1}^{\infty} sin\frac{1}{n}$

Solution:(i) Here $u_n = (n^3+1)^{1/3} - n = n\left(1+\frac{1}{n^3}\right)^{1/3} - n$

$$= n\left[1 + \frac{1}{3n^3} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!} \ .\frac{1}{n^6} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!} \ .\frac{1}{n^9} + \cdots\right] - n$$

$$= \frac{1}{3n^2} - \frac{1}{9n^5}$$

Let $v_n = \frac{1}{n^2}$.

$\Rightarrow \lim_{n\to\infty}\frac{u_n}{v_n} = \frac{1}{3}$ (which is a finite and non zero number)

Since$\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty}\frac{1}{n^2}$converges (as $p = 2 > 1$)

$\therefore \sum_{n=1}^{\infty} u_n$ also converges (by Limit form test).

(ii) Here $u_n = sin\frac{1}{n}$. Let $v_n = \frac{1}{n}$.

Then $\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{sin\frac{1}{n}}{\frac{1}{n}}$

$=1$ (which is a finite and non zero number)

Since $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges (as $p = 1$)

$\therefore \sum_{n=1}^{\infty} u_n$ also diverges (by Limit form test).

Exercise 2A

Test the convergence of the following series:

1. $\sum_{n=1}^{\infty} e^{-n^2}$ Ans. Convergent
2. $\sum_{n=1}^{\infty} \frac{1}{n^2 \log n}$ Ans. Convergent
3. $\sum_{n=1}^{\infty} \left(\sqrt{n^3+1} - \sqrt{n^3}\right)$ Ans. Convergent
4. $\frac{1}{5} + \frac{\sqrt{2}}{7} + \frac{\sqrt{3}}{9} + \frac{\sqrt{4}}{11} \ldots\ldots$ Ans. Divergent
5. $\frac{1}{1.2^2} + \frac{1}{2.3^2} + \frac{1}{3.4^2} + \ldots\ldots$ Ans. Convergent
6. $\sum_{n=1}^{\infty} \left((n^3+1)^{1/3} - n\right)$ Ans. Divergent
7. $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$ Ans. Divergent
8. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} tan \frac{1}{n}$ Ans. Convergent
9. $\frac{1}{1+x} + \frac{1}{2+x} + \frac{1}{3+x} + \ldots\ldots$ Ans. Divergent
10. $\sum_{n=1}^{\infty} \frac{1}{n-1}$ Ans. Divergent

3. D' Alembert's Ratio Test

Let $\sum_{n=1}^{\infty} u_n$ be a positive term series such that

$$\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = l$$

Then (i) $\sum_{n=1}^{\infty} u_n$ converges if $l < 1$

(ii) $\sum_{n=1}^{\infty} u_n$ diverges if $l > 1$

(iii) Test fails if $l = 1$

Example 8 Test the convergence of the following series:

$(i)\frac{1}{3}+\frac{1}{2.3^2}+\frac{1}{3.3^3}+\frac{1}{4.3^4}\ldots..$ $(ii)\ \frac{1^2 2^2}{1}+\frac{2^2 3^2}{2!}+\frac{3^2 4^2}{3!}+\frac{4^2 5^2}{4!}\ldots..$ $(iii)\sum_{n=1}^{\infty}\frac{n!}{n^n}$

Solution: (i)Here $u_n=\frac{1}{n3^n}\quad\Rightarrow u_{n+1}=\frac{1}{(n+1)3^{n+1}}$

$$\text{Then }\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{n3^n}{(n+1)3^{n+1}}=\lim_{n\to\infty}\frac{n}{(n+1)3}$$

$$=\lim_{n\to\infty}\frac{1}{3\left(1+\frac{1}{n}\right)}=0<1$$

Hence by Ratio test ,the given series converges.

(ii) Here $u_n=\frac{n^2(n+1)^2}{n!}\Rightarrow u_{n+1}=\frac{(n+1)^2(n+2)^2}{(n+1)!}$

$$\text{Then }\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{(n+1)^2(n+2)^2}{(n+1)!}\cdot\frac{n!}{n^2(n+1)^2}$$

$$=\lim_{n\to\infty}\frac{(n+2)^2}{(n+1)}\cdot\frac{1}{n^2}=\lim_{n\to\infty}\frac{1}{(n+1)}\cdot\left(\frac{1+\frac{2}{n}}{1}\right)^2=0<1$$

Hence by Ratio test , the given series converges.

(iii) Here $u_n=\frac{n!}{n^n}\quad\Rightarrow u_{n+1}=\frac{(n+1)!}{(n+1)^{(n+1)}}$

$$\text{Then }\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{(n+1)}}\cdot\frac{n^n}{n!}=\lim_{n\to\infty}\frac{n^n}{(n+1)^n}$$

$$=\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n}=\frac{1}{e}=\frac{1}{2.718}<1$$

Hence by Ratio test , the given series converges.

Example 9 Test the convergence of the following series:

$(i)\ \frac{1}{7}+\frac{2!}{7^2}+\frac{3!}{7^3}+\frac{4!}{7^4}\ldots..$ $(ii)\ \left(\frac{1}{3}\right)^2+\left(\frac{1.2}{3.5}\right)^2+\left(\frac{1.2.3}{3.5.7}\right)^2+\left(\frac{1.2.3.4}{3.5.7.9}\right)^2+\ldots..$

Solution: (i) Here $u_n=\frac{n!}{7^n}\Rightarrow u_{n+1}=\frac{(n+1)!}{7^{n+1}}$

Then $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)!}{7^{(n+1)}} \cdot \frac{7^n}{n!}$

$$= \lim_{n\to\infty} \frac{n+1}{7} = \infty > 1$$

Hence by Ratio test , the given series diverges.

(ii) Here $u_n = \left[\frac{1.2.3.4.....n}{3.5.7.9....(2n+1)}\right]^2 \Rightarrow u_{n+1} = \left[\frac{1.2.3.4.....n(n+1)}{3.5.7.9....(2n+1)(2n+3)}\right]^2$

Then $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left(\frac{n+1}{2n+3}\right)^2 = \frac{1}{2^2} = \frac{1}{4} < 1$

Hence by Ratio test , the given series converges.

Example 10 Test the convergence of the following series:

$(i)\ \frac{x}{\sqrt{5}} + \frac{x^3}{\sqrt{7}} + \frac{x^5}{\sqrt{9}} + \frac{x^7}{\sqrt{11}} + \ldots.. \ (ii)\ \frac{x}{1.3} + \frac{x^2}{2.4} + \frac{x^3}{3.5} + \frac{x^4}{4.6} + \cdots (x > 0)$

Solution: (i) Here $u_n = \frac{x^{2n-1}}{\sqrt{2n+3}} \Rightarrow u_{n+1} = \frac{x^{2n+1}}{\sqrt{2n+5}}$

Then $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{x^{2n+1}}{\sqrt{2n+5}} \frac{\sqrt{2n+3}}{x^{2n-1}} = x^2$

Hence by Ratio test , the given series converges if $x^2 < 1$ and diverges if $x^2 > 1$.

Test fails if $x^2 = 1$. i.e. $x = 1$

When $x = 1$, $u_n = \frac{1}{\sqrt{2n+3}}$

Let $v_n = \frac{1}{\sqrt{n}}$. Now consider $\frac{u_n}{v_n} = \frac{\sqrt{n}}{\sqrt{2n+3}}$

$$\Rightarrow \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{2n+3}}$$

$= \frac{1}{2}$ (which is a finite and non zero number)

Since $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges (as $p = \frac{1}{2} < 1$) $\therefore \sum_{n=1}^{\infty} u_n$ also diverges for x=1 (by Limit form test).

$\therefore$ the given series converges for $x < 1$ and diverges for $x \geq 1$.

(ii) Here $u_n = \frac{x^n}{n(n+2)} \Rightarrow u_{n+1} = \frac{x^{n+1}}{(n+1)(n+3)}$

Then $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n(n+2)}{(n+1)(n+3)} x = x$

Hence by Ratio test , the given series converges if $x < 1$ and diverges if $x > 1$

Test fails if $x = 1$

When $x = 1$, $u_n = \frac{1}{n(n+2)}$

Let $v_n = \frac{1}{n^2}$.

$\Rightarrow \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n^2}{n(n+2)}$

$= 1$ (which is a finite and non zero number)

Since $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (as $p = 2 > 1$)

$\therefore \sum_{n=1}^{\infty} u_n$ also converges for $x=1$ (by Limit form test).

$\therefore$ the given series converges for $x \leq 1$ and diverges for $x > 1$.

3. Cauchy's n th Root Test

Let $\sum_{n=1}^{\infty} u_n$ be a positive term series such that

$$\lim_{n\to\infty} (u_n)^{1/n} = l$$

Then (i) $\sum_{n=1}^{\infty} u_n$ converges if $l < 1$

(ii) $\sum_{n=1}^{\infty} u_n$ diverges if $l > 1$

(iii) Test fails if $l = 1$

Example 11 Test the convergence of the following series:

$(i)\ 1+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{4^4}\ldots..$ $\quad (ii)\ \sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$ $\quad (iii)\ \sum_{n=1}^{\infty} 5^{-n-(-1)^n}$

Solution: (i) Here $u_n = \frac{1}{n^n}$

$$\Rightarrow \lim_{n\to\infty}(u_n)^{1/n} = \lim_{n\to\infty}\frac{1}{n} = 0 < 1$$

Hence by Cauchy's root test, the given series converges.

(ii) Here $u_n = \left(\frac{n}{n+1}\right)^{n^2} \Rightarrow \lim_{n\to\infty}(u_n)^{1/n} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{n}$

$$= \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^{n} = \frac{1}{e} < 1$$

Hence by Cauchy's root test, the given series converges.

(iii) Here $u_n = 5^{-n-(-1)^n} \Rightarrow \lim_{n\to\infty}(u_n)^{1/n} = \lim_{n\to\infty} 5^{-\{n+(-1)^n\}.1/n}$

$$= \lim_{n\to\infty} 5^{-\left\{1+\frac{(-1)^n}{n}\right\}} = 5^{-1}$$

$$= \frac{1}{5} < 1$$

Hence by Cauchy's root test, the given series converges.

Example 12 Test the convergence of the following series:

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\cdots$$

Solution: Here $\quad u_n = \left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right]^{-n}$

$$\Rightarrow \lim_{n\to\infty}(u_n)^{1/n} = \lim_{n\to\infty}\left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right]^{-1}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{-1} \left[\left(\frac{n+1}{n}\right)^{n} - 1\right]^{-1}$$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{-1} \left[\left(1 + \frac{1}{n}\right)^{n} - 1\right]^{-1}$$

$$= \frac{1}{e-1} < 1$$

Hence by Cauchy's root test, the given series converges.

Exercise 2B

Test the convergence of the following series:

1. $\sum_{n=1}^{\infty} \frac{2^n}{n^2+2}$ Ans. Convergent
2. $\sum_{n=1}^{\infty} \frac{n!}{2^{2n-1}}$ Ans. Divergent
3. $\sum_{n=1}^{\infty} \frac{1.2.3....n}{7.10....(3n+4)}$ Ans. Convergent
4. $1 + \frac{3}{2!} + \frac{5}{3!} + \frac{7}{4!}$ Ans. Convergent
5. $1 + \frac{2}{5}x + \frac{6}{9}x^2 + \frac{14}{17}x^3 + ...$ Ans. Convergent if $x < 1$, divergent if $x \geq 1$
6. $\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$ Ans. Convergent
7. $\sum_{n=1}^{\infty} \frac{n^{n^2}}{\left(n+\frac{1}{5}\right)^{n^2}}$ Ans. Convergent
8. $\sum_{n=1}^{\infty} \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$ Ans. Convergent
9. $1+\frac{2^p}{2!} + \frac{3^p}{3!} + \frac{4^p}{4!} + ...(p > 0)$ Ans. Convergent
10. $\sum_{n=1}^{\infty} \sqrt{\frac{n-1}{n^3+1}}\, x^n \ (x > 0)$ Ans.Convergent if $x < 1$, divergent if $x \geq 1$

4. Raabe's Test

Let $\sum_{n=1}^{\infty} u_n$ be a positive term series such that

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = l$$

Then (i) $\sum_{n=1}^{\infty} u_n$ converges if $l > 1$

(ii) $\sum_{n=1}^{\infty} u_n$ diverges if $l < 1$

(iii) Test fails if $l = 1$

Example 13 Test the convergence of the following series:

(i) $\frac{2}{3} + \frac{2.4}{3.5} + \frac{2.4.6}{3.5.7} + \frac{2.4.6.8}{3.5.7.9} + \ldots..$ (ii) $1 + \frac{3x}{7} + \frac{3.6\,x^2}{7.10} + \frac{3.6.9x^3}{7.10.13} + \cdots (x > 0)$

Solution:(i) Here $u_n = \frac{2.4.6....2n}{1.3.5....(2n+1)} \Rightarrow u_{n+1} = \frac{2.4.6....2n(2n+2)}{1.3.5....(2n+1)(2n+3)}$

Then $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{2n+2}{2n+3} = 1$

Hence Ratio test fails.

Now applying Raabe's test, we have

$$\operatorname{Lim}_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{2n+3}{2n+2} - 1\right)$$

$$= \lim_{n\to\infty} \left(\frac{n}{2n+1}\right) = \frac{1}{2} < 1$$

Hence by Raabe's test , the given series diverges.

(ii) Ignoring the first term, $u_n = \frac{3.6.9...3n}{7.10.13...(3n+4)} x^n$

$$\Rightarrow u_{n+1} = \frac{3.6.9...3n(3n+3)}{7.10.13...(3n+4)(3n+7)} x^{n+1}$$

Then $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{3n+3}{3n+7} x = x$

Hence by Ratio test , the given series converges if $x < 1$ and diverges if $x > 1$

Test fails if $x = 1$

When $x = 1$, $\frac{u_n}{u_{n+1}} = \frac{3n+7}{3n+3}$

$$\Rightarrow \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{3n+7}{3n+3} - 1\right)$$

$$= \lim_{n\to\infty} \frac{4n}{3n+3} = \frac{4}{3} > 1$$

Hence by Raabe's test, the given series converges if $x = 1$

∴ the given series converges if $x \leq 1$ and diverges if $x > 1$.

4. Logarithmic Test

Let $\sum_{n=1}^{\infty} u_n$ be a positive term series such that

$$\lim_{n\to\infty} n \frac{u_n}{u_{n+1}} = l$$

Then (i) $\sum_{n=1}^{\infty} u_n$ converges if $l > 1$

(ii) $\sum_{n=1}^{\infty} u_n$ diverges if $l < 1$

Example 14 Test the convergence of the series

$$x + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \frac{4^4x^4}{4!} + \cdots$$

Solution: Here $u_n = \frac{n^n x^n}{n!} \Rightarrow u_{n+1} = \frac{(n+1)^{n+1}x^{n+1}}{(n+1)!}$

$$\text{Then } \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)^{n+1}x}{(n+1)n^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)^n x}{n^n}$$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n x = e.x$$

Hence by Ratio test , the given series converges if $ex < 1$ $i.e. x < \frac{1}{e}$ and diverges if $ex > 1$ $i.e. x > \frac{1}{e}$

Test fails if $ex = 1$ $i.e. x = \frac{1}{e}$

Since $\frac{u_{n+1}}{u_n}$ involves e ∴ applying logarithmic test.

$$\frac{u_n}{u_{n+1}} = \frac{1}{\left(1+\frac{1}{n}\right)^n x}$$

∴ for $x = \frac{1}{e}$, $\frac{u_n}{u_{n+1}} = e.\frac{1}{\left(1+\frac{1}{n}\right)^n}$

$$log\left(\frac{u_n}{u_{n+1}}\right) = log\ e - log\left(1+\frac{1}{n}\right)^n = 1 - nlog\left(1+\frac{1}{n}\right)$$

$$=1 - n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + \cdots\right)$$

$$= \frac{1}{2n} - \frac{1}{3n^2} + \ldots$$

$$= \lim_{n\to\infty} nlog\left(\frac{u_n}{u_{n+1}}\right) = \lim_{n\to\infty} n\left(\frac{1}{2n} - \frac{1}{3n^2} + \cdots\right) = \frac{1}{2} < 1$$

∴ By logarithmic test , the series diverges for $x = \frac{1}{e}$.

Hence the given series converges for $x < \frac{1}{e}$ and diverges for $x \geq \frac{1}{e}$.

5. Cauchy's Integral Test

If $u(x)$ is non-negative , integrable and monotonically decreasing function such that $u(n)=u_n$, then if $\int_1^\infty u(x)\ d(x)$ converges then the series $\sum_{n=1}^\infty u_n$ also converges.

Example 15 Test the convergence of the following series

$(i)\sum_{n=1}^{\infty}\frac{1}{n^2+1}$ $\quad$ $(ii)\ \sum_{n=2}^{\infty}\frac{1}{n(\log n)}$

Solution:(i) Here $u_n = \frac{1}{n^2+1}$.

Let $u(x) = \frac{1}{x^2+1}$

Clearly $u(x)$ is non-negative , integrable and monotonically decreasing function.

Consider $\int_1^{\infty}\frac{1}{x^2+1}\ d(x) = [tan^{-1}x]_1^{\infty}$

$$= tan^{-1}\infty - tan^{-1}1$$

$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$ which is finite.

Hence $\int_1^{\infty}\frac{1}{x^2+1}\ d(x)$ converges so $\sum_{n=1}^{\infty}\frac{1}{n^2+1}$ also converges.

(ii) Here $u_n = \frac{1}{n(\log n)}$.

Let $u(x) = \frac{1}{x(\log x)}$

Clearly $u(x)$ is non-negative , integrable and monotonically decreasing function.

Consider $\int_2^{\infty}\frac{1}{x(\log x)}\ d(x) = \log(log\infty) - \log\ (log2) = \infty$

Hence $\sum_{n=2}^{\infty}\frac{1}{n(\log n)}$ diverges.

Exercise 2C

Test the convergence of the following series:

1. $\sum_{n=1}^{\infty} \frac{2.4.6....(2n+2)}{3.5.7......(2n+3)} x^{n-1} \;\; (x > 0)$

   Ans. Convergent if $x < 1$, divergent if $x \geq 1$

2. $\sum_{n=1}^{\infty} \frac{(2n!)}{(n!)^2} x^n \;\; (x > 0)$

   Ans. Convergent if $x < \frac{1}{4}$, divergent if $x \geq \frac{1}{4}$

3. $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$

   Ans. Convergent

4. $\sum_{n=1}^{\infty} \frac{1.3.5....(2n-1)}{2.4.6......2n} x^n (x > 0)$

   Ans. Convergent if $x < 1$, divergent if $x \geq 1$

5. $x^2 + \frac{2^2}{3.4} x^4 + \frac{2^2 4^2}{3.4.5.6} x^6 + \frac{2^2 4^2 6^2}{3.4.5.6.7.8} x^8 + \ldots..$

   Ans. Convergent if $|x| \leq 1$, divergent if $|x| > 1$

6. $1 + \frac{x}{2} + \frac{2! x^2}{3^2} + \frac{3! x^3}{4^3} + \frac{4! x^4}{5^4} + \cdots$

   Ans. Convergent if $x < e$, divergent if $x \geq e$.

7. $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$

   Ans. Convergent

2.4 Alternating Series

An infinite series of the form $u_1 - u_2 + u_3 - u_4 + \cdots \; (u_i > 0 \; \forall \; i)$

is called an infinite series.

We write $u_1 - u_2 + u_3 - u_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} u_n$

Leibnitz's Test

The alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}u_n$ converges if it satisfies the following conditions:

(i) $u_{n+1} \le u_n$

(ii) $\lim_{n\to\infty} u_n = 0$

Example 16 Test the convergence of the following series

$(i) 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ $\quad$ $(ii)\ 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

Solution: (i) The given series is $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$. Here $u_n = \frac{1}{n}$

Since $\frac{1}{n+1} < \frac{1}{n}$ $\quad \therefore\ u_{n+1} \le u_n$

Also $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{n} = 0$

Hence by Leibnitz's test , the given series converges.

(ii) The given series is $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2}$. Here $u_n = \frac{1}{n^2}$

Since $\frac{1}{(n+1)^2} < \frac{1}{n^2} \therefore\ u_{n+1} \le u_n$

Also $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{n^2} = 0$

Hence by Leibnitz's test , the given series converges.

Absolute Convergence

A series $\sum_{n=1}^{\infty} u_n$ is absolutely convergent if the series $\sum_{n=1}^{\infty}|u_n|$ is convergent.

For example $\sum_{n=1}^{\infty} u_n = 1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \cdots$ is absolutely convergent as $\sum_{n=1}^{\infty}|u_n| = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$ is a convergent series (Since it is a geometric series whose common ratio $\frac{1}{2} < 1$).

Result: Every absolutely convergent series is convergent. But the converse may not be true.

Conditional Convergence

A series which is convergent but not absolutely convergent is called conditionally convergent series.

Example 17 Test the convergence and absolute convergence of the following series:

$(i) 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ $\quad (ii)\ 1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots$

$(iii)\ \sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{log\ n}$

Solution: (i) The given series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ is convergent by Leibnitz's test.

Now, $\sum_{n=1}^{\infty}|u_n| = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots = \sum_{n=1}^{\infty}\frac{1}{n}$ is not convergent (As p=1)

Hence the given series is not absolutely convergent. This is an example of conditionally convergent series.

(ii) The given series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2}$ is convergent by Leibnitz's test.

Also, $\sum_{n=1}^{\infty}|u_n| = 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots = \sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent (As p = 2 > 1)

Hence the given series is absolutely convergent.

(iii)The given series $\sum_{n=2}^{\infty}(-1)^{n+1}u_n$

Here $u_n = \frac{1}{log\ n}$. Now $log\ x$ is an increasing function $\forall x > 0$

$$\therefore \log(n+2) > \log(n+1)$$

$$or\ \frac{1}{\log(n+2)} < \frac{1}{\log(n+1)}$$

$\therefore u_{n+1} \leq u_n$

Also $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\log n} = 0$

Hence by Leibnitz's test , the given series is convergent.

Now for absolute convergence , consider $\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \frac{1}{\log n}$

It is a divergent series (as discussed earlier).

Hence the given series is not absolutely convergent. This is an example of conditionally convergent series.

Example 18 Test the convergence of the series:

$(i)\sum_{n=1}^{\infty} (-1)^{n-1} \left[\frac{1}{n^2} + \frac{1}{(n+1)^2}\right]$ $(ii)\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n2^n}$

Solution:(i) The given series is $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$

Consider $\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \left[\frac{1}{n^2} + \frac{1}{(n+1)^2}\right]$

Now, $\frac{1}{n^2} + \frac{1}{(n+1)^2} < \frac{1}{n^2} + \frac{1}{n^2} = \frac{2}{n^2}$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$is convergent (As p = 2 > 1) $\therefore$ by Comparision test $\sum_{n=1}^{\infty} |u_n|$ is also convergent.

Hence the given series is absolutely convergent and so convergent.

(ii) The given series is $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$

Consider $\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \frac{1}{n2^n}$

Here $|u_n| = \frac{1}{n2^n}$

Now $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{n2^n}{(n+1)2^{n+1}}\right| = \frac{1}{2} < 1$

$\therefore$ by Ratio test $\sum_{n=1}^{\infty}|u_n|$ is convergent or the given series is absolutely convergent and hence convergent.

Example 19 Find the values of x for which the series

$x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots$ is absolutely convergent and conditionally convergent.

Solution: The given series is $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1} = \sum_{n=1}^{\infty}(-1)^{n-1}u_n$

Then $|u_n| = \left|\frac{x^{2n-1}}{2n-1}\right|$

Now, $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+1}}{2n+1}.\frac{2n-1}{x^{2n-1}}\right| = \frac{1}{x^2}$

Thus , by Ratio test $\sum_{n=1}^{\infty}|u_n|$ converges if $x^2 < 1$ $i.e. |x| < 1$, diverges if $x^2 > 1$ $i.e. |x| > 1$ and test fails if $|x| = 1$

When $|x| = 1$ $i.e. x = 1$ or $x = -1$, we have

For $x = 1$,

the given series is $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$, which is convergent by Leibnitz's test but not absolutely convergent.

For $x = -1$,

the given series is $-1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \cdots$, which is also convergent by Leibnitz's test but not absolutely convergent.

Hence the given series is absolutely convergent for $|x| < 1$ or

$-1 < x < 1$ and conditionally convergent for $|x| = 1$ $i.e.$

$x = 1$ $or -1$.

Exercise 2D

1. Show that the series $1-\frac{(2!)^2}{4!}+\frac{(3!)^2}{6!}-\cdots$is convergent.
2. Show that the series $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots$is absolutely convergent.
3. Test the convergence and absolute convergence of the series
   $1-\frac{1}{2.3}+\frac{1}{2^2.5}-\frac{1}{2^3.7}\ldots$
   Ans. Absolutely convergent
4. Show that the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{2n+3}$ is conditionally convergent.
5. Test the absolute convergence of the series
   $\sum_{n=1}^{\infty}(-1)^{n-1}\left(\sqrt{n^2+1}-n\right)$
   Ans. Not absolutely convergent
6. Show that the series $\frac{\sin x}{1^3}-\frac{\sin 2x}{2^3}+\frac{\sin 3x}{3^3}-\cdots$ converges absolutely.
7. Find the interval of convergence of the series $x-\frac{x^2}{\sqrt{2}}+\frac{x^3}{\sqrt{3}}-\frac{x^4}{\sqrt{4}}\ldots$
   Ans. $0<x\leq 1$

2.5 EXPANSION OF FUNCTIONS

Taylor Series:

If a function $f(x)$ is infinitely differentiable at the point *a* then $f(x)$ can be expanded about the point $'a'$ as

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f''(a)+\cdots+\frac{(x-a)^n}{n!}f^{(n)}(a)+\ldots\ldots$$

Also $f(a+h)$, where h is small, can be expanded as

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\frac{h^3}{3!}f'''(a)+\cdots+\frac{h^n}{n!}f^{(n)}(a)+\ldots\ldots$$

Malaurin Series:

It is the special case of Taylor series about the point 0 . Hence the Maclaurin series of $f(x)$ is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \ldots\ldots$$

Maclaurin series of standard functions:

1. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$
2. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$
3. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$
4. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, |x| < 1$
5. $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$
6. $(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots$

Example20 Expand $e^x cos\ x$ by Maclaurin series.

Solution : By Maclaurin's expansion , we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f''(0) + \cdots\ldots\ldots①$$

Here $f(x) = e^x cos\ x$

Now $f(0) = e^0 cos\ 0 = 1$

$$f'(x) = e^x cos\ x - e^x sin\ x$$

$$\Rightarrow f'(0) = e^0 cos\ 0 - e^0 sin\ 0 = 1$$

$$f''(x) = e^x cos\ x - e^x sin\ x - e^x sin\ x - e^x cos\ x$$

$$= -2e^x sin\ x$$

$$\Rightarrow f''(0) = -2e^0 sin\ 0 = 0$$

$$f'''(x) = -2e^x cos\ x - 2e^x sin\ x$$

$$\Rightarrow f'''(0) = -2e^0 cos\ 0 - 2e^0 sin\ 0 = -2$$

Similarly $f^{iv}(0) = -1$ and so on .

Putting these values in ①, we get

$$e^x \cos x = 1 + x - \frac{2x^3}{3!} - \frac{4x^4}{4!} - \cdots$$

Example21 Expand $tan\ x$ in powers of $(x - \frac{\pi}{4})$ upto forst four terms.

Solution: By Taylor's expansion , we have

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f''(a) + \cdots \ldots ①$$

Here $f(x) = tan\ x$ and $a = \frac{\pi}{4}$

Now $f\left(\frac{\pi}{4}\right) = tan\frac{\pi}{4} = 1$

$$f'(x) = sec^2 x$$

$$\Rightarrow f'\left(\frac{\pi}{4}\right) = sec^2\frac{\pi}{4} = 2$$

$$f''(x) = 2sec^{\ 2}x\ tan\ x$$

$$\Rightarrow f''\left(\frac{\pi}{4}\right) = 2sec^{\ 2}\left(\frac{\pi}{4}\right) tan\left(\frac{\pi}{4}\right) = 4$$

$$f'''(x) = 2sec^{\ 4}x + 4\ tan^2 x sec^2 x$$

$$\Rightarrow f'''\left(\frac{\pi}{4}\right) = 16$$

Putting these values in ①, we get

$$tan\ x = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3$$

Example22 Show that $log\ sec\ x = \frac{x^2}{2} + \frac{x^4}{12} + \cdots$

Solution: By Maclaurin's expansion , we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f''(0) + \cdots \ldots ①$$

Here $f(x) = log\ secx$

Now $f(0) = 0$

$f'(x) = tanx$

$\Rightarrow f'(0) = 0$

$f''(x) = sec^2x = 1 + tan^2x$

$\Rightarrow f''(0) = 1$

$f'''(x) = 2\sec x\ secx\ tanx$

$\Rightarrow f'''(0) = 0$

Similarly $f^{iv}(0) = 2$ and so on .

Putting these values in ①, we get

$$log\ sec\ x = \frac{x^2}{2} + \frac{x^4}{12} + \cdots$$

Example 23 Show that $sin\left(\frac{\pi}{4} + \theta\right) = \frac{1}{\sqrt{2}}\left(1 + \theta - \frac{\theta^2}{2!} - \frac{\theta^3}{3!} + \cdots\right)$

Solution: By Taylor's expansion , we have

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f''(a) + \cdots \ldots ①$$

Here $f(x) = sin\ x,\ a = \frac{\pi}{4}$ and $h = \theta$

So ① becomes

$$sin(a+h) = sin(a) + h\ cos\ a + \frac{h^2}{2!}(-sina) + \frac{h^3}{3!}(-cosa) + \cdots$$

or $sin\left(\frac{\pi}{4}+\theta\right)=sin\left(\frac{\pi}{4}\right)+\theta cos\ \frac{\pi}{4}+\frac{\theta^2}{2!}\left(-sin\frac{\pi}{4}\right)+\frac{\theta^3}{3!}\left(-cos\frac{\pi}{4}\right)+\cdots$

or $sin\left(\frac{\pi}{4}+\theta\right)=\frac{1}{\sqrt{2}}\left(1+\theta-\frac{\theta^2}{2!}-\frac{\theta^3}{3!}+\cdots\right)$

Example 24 Estimate the value of $\sqrt{10}$ correct to four places of decimal.

Solution: Let $f(x)=\sqrt{x}$

By Taylor's theorem, we have

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\frac{h^3}{3!}f''(a)+\cdots$$

$\Rightarrow\ (a+h)^{1/2}=a^{1/2}+h\left(\frac{d}{dx}x^{1/2}\right)_{at\ x=a}+\frac{h^2}{2!}\left(\frac{d^2}{dx^2}x^{1/2}\right)_{at\ x=a}+\frac{h^3}{3!}\left(\frac{d^3}{dx^3}x^{1/2}\right)_{at\ x=a}+\cdots....①$

Taking $a=9$ and $h=1$ in ①, we get

$10^{1/2}=9^{1/2}+\left(\frac{1}{2}x^{-1/2}\right)_{at\ x=9}+\frac{1}{2!}\left(\frac{1(-1)}{2.2}x^{-3/2}\right)_{at\ x=9}+\frac{1}{3!}\left(\frac{1(-1)(-3)}{2.2.2}x^{-5/2}\right)_{at\ x=9}+\cdots.$

$=3+\frac{1}{2.3}-\frac{1}{8.27}+\cdots$

=3.1623(approx.)

2.6 Approximate Error

Let y be a function of x i.e. y=f(x). If δx is a small change in x then the resulting change in y is denoted by δy and is given by

$\delta y=\frac{dy}{dx}\delta x$ approximately.

Example 25 Find the change in the total surface area of a right circular cone when

(i) the radius is constant but there is a small change in the altitude

(ii) the altitude is constant but there is a small change in the radius.

Solution: Let the radius of the base be r , altitude be h and the change in the altitude be the radius is constant but there is a small change in the altitude δh.

Let S be the total surface area of the cone , then

$$S = \pi r^2 + \pi r\sqrt{r^2 + h^2}$$

(i)If altitude changes then $\delta S = \frac{dS}{dh}\delta h$

$$\text{Now, } \frac{dS}{dh} = 0 + \frac{\pi r}{2}(r^2 + h^2)^{-1/2}.2h = \frac{\pi rh}{\sqrt{r^2+h^2}}$$

$$\therefore\ \delta S = \frac{dS}{dh}\delta h = \frac{\pi rh}{\sqrt{r^2+h^2}}\delta h \text{ approximately.}$$

(ii)If radius changes then $\delta S = \frac{dS}{dr}\delta r$

$$\text{Now, } \frac{dS}{dr} = 2\pi r + \pi\sqrt{r^2 + h^2} + \frac{2\pi r^2}{2\sqrt{r^2+h^2}} = 2\pi r + \frac{\pi(2r^2+h^2)}{\sqrt{r^2+h^2}}$$

$$\therefore\ \delta S = \frac{dS}{dr}\delta r = 2\pi r + \frac{\pi(2r^2+h^2)}{\sqrt{r^2+h^2}}\delta r \text{ approximately.}$$

Example 26 If a , b , c are the sides of the triangle ABC and S is the semi- perimeter , show that if there is a small error δcin the measurement of side c then the error $\delta\Delta$ in the area Δ of the triangle is given by

$$\delta\Delta = \frac{\Delta}{4}\left(\frac{1}{S} + \frac{1}{S-a} + \frac{1}{S-b} + \frac{1}{S-c}\right)\delta c$$

Solution: We know that $S = \frac{(a+b+c)}{2}$

$$\text{and } \Delta^2 = S(S-a)(S-b)(S-c)$$

$$or\ 2log\Delta = log\ S + \log(S-a) + \log(S-b) + \log\ (S-c)$$

On differentiating both the sides w.r.t. c , we get

$$\frac{2d\Delta}{\Delta dc} = \frac{1}{S}\frac{dS}{dc} + \frac{1}{S-a}\frac{d(S-a)}{dc} + \frac{1}{S-b}\frac{d(S-b)}{dc} + \frac{1}{S-c}\frac{d(S-c)}{dc}$$

$$= \frac{1}{S}\frac{1}{2} + \frac{1}{2(S-a)} + \frac{1}{2(S-b)} + \frac{1}{(S-c)}\left(\frac{1}{2} - 1\right)$$

$$\Rightarrow \frac{d\Delta}{dc} = \frac{\Delta}{4}\left(\frac{1}{S} + \frac{1}{(S-a)} + \frac{1}{(S-b)} + \frac{1}{(S-c)}\right)$$

$$\Rightarrow \delta\Delta = \frac{d\Delta}{dc}\delta c = \frac{\Delta}{4}\left(\frac{1}{S} + \frac{1}{(S-a)} + \frac{1}{(S-b)} + \frac{1}{(S-c)}\right)\delta c$$

Example 27 If $T = 2\pi\sqrt{\left({}^{l}/_{g}\right)}$ find the error in T corresponding to 2% error in l where g is constant.

Solution: Error in T is given by $\delta T = \frac{dT}{dl}\delta l$

$$\text{Now } \frac{dT}{dl} = \frac{2\pi}{\sqrt{g}}\frac{1}{2\sqrt{l}} \;\therefore\; \delta T = \frac{\pi}{\sqrt{g}}\frac{1}{\sqrt{l}}\delta l$$

$$\Rightarrow \frac{\delta T}{T} = \frac{\pi}{\sqrt{g}}\frac{\delta l}{\sqrt{l}}\frac{\sqrt{g}}{2\pi\sqrt{l}} = \frac{1}{2}\frac{\delta l}{l}$$

$$\Rightarrow \frac{\delta T}{T}.100 = \frac{1}{2}\frac{\delta l}{l}.100$$

As $\frac{\delta l}{l}.100 = 2 \;\therefore \frac{\delta T}{T}.100 = 1$ Hence error in T is 2%.

Exercise 2E

1. Expand $tan^{-1}x$ in powers of (x-1).

   Ans. $tan^{-1}x = \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \frac{1}{12}(x-1)^3 + \cdots$

2. Using Taylor's theorem find the approximate value of $f\left(\frac{11}{10}\right)$ where $f(x) = x^3 + 3^2 + 15x - 10$

   Ans. 11.461

3. Show that $\log(1 + sinx) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \cdots$

4. Show that $tan\left(\frac{\pi}{4}+x\right)=1+2x+2x^2+\frac{8}{3}x^3+\frac{10}{3}x^4+\cdots$ and hence find *tan* 46^o

   Ans. 1.0355

5. A soap bubble of radius 2cm shrinks to radius 1.9 cm. Finf the decrease in volume and surface area.

   Ans. -5.024 cm^3 and -.5.024cm^2

6. If $\log_{10}4$ = 0.6021 , find the approximate value of $\log_{10}404$.

   Ans. 2.61205

7. Let A , B and C be the angles of a triangle opposite to the sides a ,b and c respectively. If small errors $\delta a, \delta b$ and $\delta c$are made in the sides then show that $\delta A=\frac{a}{2\Delta}(\delta a-\delta b\ cosC-\delta c\ cosB)$ where Δ is the area of the triangle.

Limit of function of two variables ①

The function $f(x,y)$ is said to tend to the limit l as $x \to a$ and $y \to b$ if and only if the limit l is independent of the path followed by the point (x,y) as $x \to a$ and $y \to b$. Then

$$\lim_{\substack{x \to a \\ y \to b}} f(x,y) = l$$

Working Rule to find the limit :-

Step (I) Find $f(x,y)$ along $x \to a$ and $y \to b$

(II) Find $f(x,y)$ along ~~b~~ $y \to b$ and $x \to a$

If $\lim_{\substack{x \to a \\ y \to b}} f(x,y) = \lim_{\substack{y \to b \\ x \to a}} f(x,y)$, then Limit exist otherwise Not.

(III) If $a \to 0$ and $b \to 0$, find the limit along $y = mx$ or $y = mx^n$. If the value of limit does not contain m then limit exists. If it contains m, the limit does not exist.

Note. (i) Put $x = 0$, then $y = 0$ in $f(x,y)$, Find its value f_1

(ii) Put $y = 0$ and then $x = 0$ in $f(x,y)$, Find its value f_2

→ If $f_1 \neq f_2$ limit does not exist.

→ If $f_1 = f_2$ then

(II) Put $y = mx$ and find $f_3 = \lim_{x\to 0} f(x, mx)$ ②

If $f_1 = f_2 \neq f_3$, then limit does not exist

If $f_1 = f_2 = f_3$, then

(III) Put $y = mx^2$ and find $f_4 = \lim_{x\to 0} f(x, mx^2)$

If $f_1 = f_2 = f_3 \neq f_4$, then limit does not exist.

If $f_1 = f_2 = f_3 = f_4$, then limit exists.

ex: Evaluate $\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2 y}{x^4+y^2} = f(x,y)$

Soln: $\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2 y}{x^4+y^2} = \lim_{y\to 0} \frac{0}{0+y^2} = 0 = f_1$ (say)

$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2 y}{x^4+y^2} = \lim_{x\to 0} \frac{0}{x^4+0} = 0 = f_2$ (say)

Here $f_1 = f_2$, therefore

put $y = mx$

$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2 mx}{x^4+m^2x^2} = \lim_{x\to 0} \frac{mx}{x^2+m^2} = 0 = f_3$ (say)

Here $f_1 = f_2 = f_3$, therefore

put $y = mx^2$

$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2 . mx^2}{x^4+m^2x^4} = \frac{m}{1+m^2} = f_4$

Here $f_1 = f_2 = f_3 \neq f_4 \Rightarrow$ Limit does Not exists exp. Evaluate $\lim\limits_{\substack{x\to 1\\ y\to 2}} \dfrac{3x^2y}{x^2+y^2+5}$

Sol^n. $\lim\limits_{\substack{x\to 1\\ y\to 2}} \dfrac{3x^2y}{x^2+y^2+5} = \lim\limits_{y\to 2}\left\{\lim\limits_{x\to 1} \dfrac{3x^2y}{x^2+y^2+5}\right\} = \lim\limits_{y\to 2}\left[\dfrac{3y}{y^2+6}\right]$

$$= \frac{6}{10} = 3/5$$

Similarly $\lim\limits_{x\to 1}\left\{\lim\limits_{y\to 2} \dfrac{3x^2y}{x^2+y^2+5}\right\} = \lim\limits_{x\to 1}\left\{\dfrac{6x^2}{x^2+9}\right\}$

$$= \frac{6}{10} = 3/5$$

thus $\lim\limits_{x\to 1\, y\to 2} f(x,y) = \lim\limits_{y\to 2,\, x\to 1} f(x,y) \Rightarrow$ Limit exists.

Continuity

A function $f(x,y)$ is said to be continuous at a point (a,b) if $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

→ A function is said to be continuous in a domain if it is continuous at every point of the domain

Working Rule :- for continuity at a point (a,b)

(I) $f(a,b)$ should be well defined

(II) $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ should exists.

(III) $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

(4)

Exp. Test the continuity of $f(x,y)=\begin{cases}\frac{x^3-y^3}{x^2+y^2} & \text{if } x\neq 0, y\neq 0\\ 0 & \text{if } x=0, y=0\end{cases}$ at $(0,0)$.

Sol^n. $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = \lim_{x\to 0}\left\{\lim_{y\to mx} \frac{x^3-y^3}{x^2+y^2}\right\}$

$= \text{Limit}_{x\to 0} \frac{x^3-m^3x^2}{x^2+m^2x^2} = \lim_{x\to 0} \frac{x(1-m^3)}{1+m^2} = 0$

Thus Limit exists at origin with value 0.

also $f(0,0)=0$ (given)

Thus $f(0,0)= \lim_{(x,y)\to(0,0)} f(x,y)$

Hence $f(x,y)$ is continuous at $(0,0)$

Exp. $f(x,y)=\begin{cases}\frac{x}{\sqrt{x^2+y^2}}, & x\neq 0, y\neq 0\\ 2 & x=0, y=0\end{cases}$ at origin

Sol^n: $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0}\left[\lim_{y\to mx} \frac{x}{\sqrt{x^2+y^2}}\right] = \lim_{x\to 0} \frac{x}{\sqrt{x^2+m^2x^2}}$

$= \lim_{x\to 0} \frac{1}{\sqrt{1+m^2}}$ which gives different values for diff m.

So the $\lim_{\substack{x\to 0\\ y\to 0}} f(x,y)$ does not exists.

Partial Derivatives

Let $z = f(x,y)$ be function of two independent variables x and y. If we keep y constant and x varies then z becomes a function of x only. The derivative of z with respect to x, keeping y constant is called partial derivative of 'z' wrt 'x' and is denoted by $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$ or $f_x(x,y)$.

Then $$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x,y)}{\delta x}$$

Similarly, we can also find $\frac{\partial z}{\partial y}$ keeping x as constant.

$$\frac{\partial z}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x,y)}{\delta y}.$$

Note. $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$, $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$, $\frac{\partial^2 z}{\partial y^2} = t$.

exp' $z = f(x,y) = x^4 + 3xy + y^5$, find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, $\frac{\partial^2 z}{\partial x \partial y}$

Soln. We have $z = x^4 + 3xy + y^5$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^4 + 3xy + y^5) = 4x^3 + 3y + 0 = 4x^3 + 3y$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^4 + 3xy + y^5) = 0 + 3x + 5y^4 = 3x + 5y^4$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(3x + 5y^4) = 3 + 0 = 3$$

or

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(3y + 4x^3) = 3 + 0 = 3$$

exp If $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$, then find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$.

Sol^n We have $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$ — (1)

Differentiating (1) partially wrt x,

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \frac{1}{y} + \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(-\frac{y}{x^2}\right) = \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}$$

$$\Rightarrow x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2} \quad \text{— (2)}$$

Diff. partially (1) wrt y, we get

Now $$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \left(-\frac{x}{y^2}\right) + \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{-x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}$$

$$\Rightarrow y\frac{\partial u}{\partial y} = \frac{-x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2} \quad \text{— (3)}$$

$$(2)+(3) \Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \cancel{\frac{x}{\sqrt{y^2-x^2}}} - \cancel{\frac{xy}{x^2+y^2}} - \cancel{\frac{x}{\sqrt{y^2-x^2}}} + \cancel{\frac{xy}{x^2+y^2}}$$

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0}.$$

exp If $z = e^{ax+by} \cdot f(ax-by)$, prove that $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz$

Sol^n We have $z = e^{ax+by} \cdot f(ax-by)$ — (1)

Diff partially wrt x,

$$\frac{\partial z}{\partial x} = a \cdot e^{ax+by} \cdot f(ax-by) + e^{ax+by} \cdot a \cdot f'(ax-by)$$

$$b\frac{\partial z}{\partial x} = ab\, e^{ax+by}\left[f(ax-by) + f'(ax-by)\right] \quad \text{— (2)}$$

Diff ① partialy w.r.t y, we get

$$\frac{\partial z}{\partial y} = b\, e^{ax+by} f(ax-by) + e^{ax+by}.(b) f'(ax-by)$$

$$a\frac{\partial z}{\partial y} = ab\, e^{ax+by}\left[f(ax-by) - f'(ax-by)\right] \quad -③$$

Now taking ② + ③, we have

$$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2ab\, e^{ax+by}. f(ax-by)$$

$$= 2abz \quad \text{Proved}$$

Partial Derivatives of Higher Orders.

let $z = f(x,y)$, then $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ being the functions of x and y, Can be further differentiated partially with respect to x & y

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \text{ or } \frac{\partial^2 f}{\partial x^2} \text{ or } f_{xx}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) \text{ or } \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) \text{ or } \frac{\partial^2 f}{\partial x \partial y} = f_{xy}$$

Note. $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$ (In General)

exp If $u = \log(x^3+y^3+z^3-3xyz)$, Show that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$$

Solⁿ We have $u = \log(x^3+y^3+z^3-3xyz)$ — (1)

Diff (1) partially w.r.t x, we get

$$\frac{\partial u}{\partial x} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$$

Similarly $\frac{\partial u}{\partial y} = \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}$, $\frac{\partial u}{\partial z} = \frac{3z^2-3yx}{x^3+y^3+z^3-3xyz}$

Now taking

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{1}{[x^3+y^3+z^3-3xyz]}[3(x^2+y^2+z^2)-3(xy+yz+zx)]$$

$$= \frac{3[\cancel{x^2+y^2+z^2-xy-yz-zx}]}{(x+y+z)(\cancel{x^2+y^2+z^2-xy-yz-zx})} = \frac{3}{(x+y+z)}$$

$$\Rightarrow \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u = \frac{3}{(x+y+z)} \quad — (2)$$

Now we take $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u$

from (2) $= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\cdot\frac{3}{(x+y+z)}$

$$\Rightarrow \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = 3\left[\frac{\partial}{\partial x}\frac{1}{(x+y+z)}+\frac{\partial}{\partial y}\frac{1}{(x+y+z)}+\frac{\partial}{\partial z}\frac{1}{(x+y+z)}\right]$$

$$= -3\left[(x+y+z)^{-2}+(x+y+z)^{-2}+(x+y+z)^{-2}\right]$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = -9/(x+y+z)^2$$ Proved

$\rightarrow \left(\frac{\partial u}{\partial x}\right)_y$:= Diff partially u with respect to x, keeping y as constant. (9)

Homogeneous Function

A function $f(x,y)$ is a homogeneous function of degree n, if the degree of each of its terms in x and y is equal to n. Thus

$$f(x,y) = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_{n-1}xy^{n-1} + a_ny^n \quad -(1)$$

is a homogeneous function of degree n.

eq^n (1) can also be written as

$$f(x,y) = x^n\left[a_0 + a_1\frac{y}{x} + a_2\left(\frac{y}{x}\right)^2 + \cdots + a_{n-1}\left(\frac{y}{x}\right)^{n-1} + a_n\left(\frac{y}{x}\right)^n\right]$$

$$= x^n\phi(y/x) \quad -(2)$$

We can also rewrite (1) as follows

$$f(x,y) = y^n\psi(x/y) - (3)$$

exp $f(x,y) = x^3\left[1 + \frac{y}{x} + 3\left(\frac{y}{x}\right)^2 + 5\cdot\left(\frac{y}{x}\right)^3\right]$ is homogeneous of order 3.

exp. $\frac{\sqrt{x}+\sqrt{y}}{x^2+y^2} = \frac{\sqrt{x}\left[1+\sqrt{y/x}\right]}{x^2\left[1+(y/x)^2\right]} = x^{-3/2}\phi(y/x)$ is homogeneous of order 3.

exp. $\sin^{-1}\left[\frac{\sqrt{x}+\sqrt{y}}{x^2+y^2}\right]$ is Not Homogeneous fn as it can not be written as $x^n\phi(y/x)$

Euler's Theorem On Homogeneous fn

If z is a homogeneous function of x and y of order n, then

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz.$$

Proof. Since z is a homogeneous function of x and y of order n,

$\therefore z$ Can be written as follows

$$z = x^n \phi(y/x) \quad \text{— (1)}$$

Differentiating (1) partially with respect to x

$$\frac{\partial z}{\partial x} = n\cdot x^{n-1}\cdot\phi(y/x) + x^n\cdot\phi'(y/x)\cdot(-y/x^2)$$

$$= nx^{n-1}\cdot\phi(y/x) - x^{n-2}\cdot y\,\phi'(y/x)$$

$$x\frac{\partial z}{\partial x} = nx^n\phi(y/x) - x^{n-1}\cdot y\,\phi'(y/x) \quad \text{— (2)}$$

Differentiating (1) partially wrt y

$$\frac{\partial z}{\partial y} = x^n\phi'(y/x)\cdot\frac{1}{x} = x^{n-1}\phi'(y/x)$$

$$y\frac{\partial z}{\partial y} = x^{n-1}\cdot y\,\phi'(y/x) \quad \text{— (3)}$$

Now taking

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nx^n\phi(y/x)$$

$$\boxed{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz}$$ Proved

Note: $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$ (Check P12)

Note. If u is a homogeneous fn of x and y, z of degree n, then (11)

$$\boxed{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu}$$

Deductions From Euler's Theorem !—

(I) If z is a homogeneous function of x,y of degree n and $z=f(u)$

then $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+\cancel{z\frac{\partial u}{\partial z}}=n\frac{f(u)}{f'(u)}$

Proof. Since z is a homogeneous fn of x,y of degree n, we have

by Euler's theorem $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz.$ — (1)

Now $z=f(u)$, given $\frac{\partial z}{\partial x}=f'(u)\cdot\frac{\partial u}{\partial x}$ and $\frac{\partial z}{\partial y}=f'(u)\frac{\partial u}{\partial y}$

Substituting these values into (1)

$$x f'(u)\frac{\partial u}{\partial x}+y f'(u)\frac{\partial u}{\partial y}=n f(u)$$

$$\Rightarrow \boxed{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{f(u)}{f'(u)}}$$

Note!- If $z=f(u)$ is a homogeneous function in x,y,z of degree n, then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=n\frac{f(u)}{f'(u)}$$

(II) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)[g'(u)-1]$, where

$$g(u)=nf(u)/f'(u)$$

Proof. By Euler's I- deductions formula,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n.\frac{f(u)}{f'(u)}$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=g(u) \quad \text{(given)}. \quad \text{— (1)}$$

Differentiating (1) partially wrt x, we have

$$\left(x\frac{\partial^2 u}{\partial x^2} + 1\cdot\frac{\partial u}{\partial x}\right) + y\cdot\frac{\partial^2 u}{\partial x\partial y} = g'(u)\cdot\frac{\partial u}{\partial x}$$

$$\Rightarrow x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\partial y} = [g'(u)-1]\frac{\partial u}{\partial x} \quad \text{— (2)}$$

Similarly, on differentiating (1) partially wrt y, we have

$$y\frac{\partial^2 u}{\partial y^2} + x\cdot\frac{\partial^2 u}{\partial y\partial x} = [g'(u)-1]\frac{\partial u}{\partial y} \quad \text{— (3)}$$

Now taking (2) $\times x$ + (3) $\times y$, we get

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = [g'(u)-1]\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right]$$

$$= [g'(u)-1]\, g(u) \quad \text{from (1)}$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u)-1] \quad \text{proved.}$$

exp. If $z = \dfrac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}$, then show that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = -\dfrac{1}{6}z$.

Sol^n. we have $z = \dfrac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}} = x^{-1/6}\dfrac{[1+(y/x)^{1/3}]}{[1+(y/x)^{1/2}]}$

$z = x^{-1/6}\phi(y/x) \Rightarrow z$ is homogeneous fn of order $\boxed{-1/6 = n}$

Thus Euler's theorem, We have

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz = -\frac{1}{6}z \quad \text{Proved.}$$

exp: If $u = \cos^{-1}\left[\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right]$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + \dfrac{1}{2}\cot u = 0$.

Sol^n. we have $u = \cos^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ is not homogeneous.

if we put $z = \cos u$, then $z = \cos u = \dfrac{x+y}{\sqrt{x}+\sqrt{y}} = \dfrac{x^{1/2}[1+y/x]}{[1+\sqrt{y/x}]}$

$= x^{1/2}\phi(y/x)$

$\Rightarrow z$ is Homogeneous function of order $1/2 = n$.

By Euler's theorem for z, we have

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{1}{2}z$$

$$\Rightarrow x\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + y\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} = \frac{1}{2}z$$

$$\Rightarrow x\frac{\partial u}{\partial x}\cdot(-\sin u) + y\frac{\partial u}{\partial y}(-\sin u) = \frac{1}{2}\cos u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u$$

$$\Rightarrow \boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0}$$ proved.

exp. If $z = x^n f(y/x) + y^{-n}\phi(x/y)$, then prove that

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n^2 z.$$

Solⁿ: We have $z = x^n f(y/x) + y^{-n}\phi(x/y)$

$= u + v$ (Let us assume) — (1)

where $u = x^n f(y/x)$, $v = y^{-n}\phi(x/y)$

Since u is a homogeneous function of x and y of degree n,

then $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \quad — (2)$$

and $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u \quad — (3)$$

Also v is a homogeneous function of x and y of degree $-n$

therefore $$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = -nv \quad — (4)$$

and $$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\partial y} + y^2\frac{\partial^2 v}{\partial y^2} = -n(-n-1)v$$
$$= n(n+1)v \quad — (5)$$

taking (2) + (4), we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nu - nv$$

$$\Rightarrow x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nu - nv \quad \text{---} (6)$$

On adding (3) and (5), we have

$$x^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x^2}\right)+2xy\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial x\partial y}\right)+y^2\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

$$= n(n-1)u + n(n+1)v$$

$$\Rightarrow \boxed{x\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)u + n(n+1)v} \quad (7)$$

On adding (6) and (7), We get

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n(n-1)u + n(n+1)v + nu - nv$$

$$= nu[n\cancel{-1}\cancel{+1}] + nv[n\cancel{+1}\cancel{-1}]$$

$$= n^2(u+v) = n^2 z \quad \text{Proved}$$

exp: If $u = \tan^{-1}\left[\dfrac{x^3+y^3}{x-y}\right]$, prove that

(i) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sin 2u$ (ii) $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 2\cos 3u \sin u.$

Sol^n Since u is Not homogeneous, let us assume

$$Z = \tan u = \frac{x^3+y^3}{x-y} = x^2\phi(y/x)$$

So Z is homogeneous fn of order $\boxed{n=2}$ and $Z = f(u) = \tan u$

(i) By Euler's I-deduction, we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{n\,f(u)}{f'(u)} = 2\cdot\frac{\tan u}{\sec^2 u}$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\sin u\cdot\cos u = \sin(2u) \quad \text{proved.}$$

(ii) By Euler's II-deduction, we have

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u)-1]$$

where $g(u) = \dfrac{n f(u)}{f'(u)} = \sin(2u)$ (from (i))

$\Rightarrow g'(u) = 2\cos 2u.$

Thus $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \sin(4u)[2\cos 2u - 1]$

$= \sin(4u) - \sin 2u$

$= 2\cos 3u \sin u$ proved

— Total Differentiation —

let $z = f(x, y)$ — ①

If δx and δy be the increments in x and y respectively, let δz be the corresponding incremen in z.

then $z + \delta z = f(x+\delta x, y+\delta y)$ — ②

from ① and ②, we have

$$\delta z = f(x+\delta x, y+\delta y) - f(x, y) \quad — ③$$

$$\delta z = f(x+\delta x, y+\delta y) - f(x, y+\delta y) + f(x, y+\delta y) - f(x, y)$$

$$\delta z = \left[\frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta x}\right]\delta x + \left[\frac{f(x, y+\delta y) - f(x, y)}{\delta y}\right]\delta y$$

On taking $\delta x \to 0$ and $\delta y \to 0$ or $\frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy.$

$$\boxed{dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy} \quad — ④$$

is called total differential of z.

$\Rightarrow$ Change of two independent variable x and y by another variable t !

let $z = f(x, y)$, where $x = \phi(t)$ and $y = \psi(t)$

Here z is a composite function of t.

Dividing ④ by dt, we get

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \quad — ⑤$$

(II) change in the independent variables x and y by other two variables u and v.

Let $z = f(x,y)$, where $x = \phi(u,v)$, $y = \psi(u,v)$

Now from (5), we obtain

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} \quad — (6)$$

and $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} \quad — (7)$$

Exp If $u = x^3+y^3$, where $x = a\cos t$, $y = b\sin t$. Find $\frac{du}{dt}$

Sol^n We have $u = x^3+y^3$, $x = a\cos t$, $y = b\sin t$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} = (3x^2)(-a\sin t) + (3y^2)(b\cos t)$$

$$= -3a\cdot(a\cos t)^2\sin t + 3b\,(b\sin t)^2\cdot\cos t$$

$$\frac{du}{dt} = -3\,a^3\cos^2 t\cdot\sin t + 3b^3\sin^2 t\cdot\cos t$$

exp. If $z = f(x,y)$, where $x = e^u\cdot\cos v$ and $y = e^u\cdot\sin v$, Show

(i) $y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} = e^{2u}\cdot\frac{\partial z}{\partial y}$ (ii) $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = e^{-2u}\cdot\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$

Sol^n (i) We have $x = e^u\cos v$, $y = e^u\sin v$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}(e^u\cdot\cos v) + \frac{\partial z}{\partial y}[e^u\cdot\sin v]$$

$$y\frac{\partial z}{\partial u} = xy\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} \quad — (1)$$

and $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} = \frac{\partial z}{\partial x}[e^u(-\sin v)] + \frac{\partial z}{\partial y}[e^u\cos v]$$

$$= -y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y}$$

$$\Rightarrow \quad x\frac{\partial z}{\partial v} = -x\cdot y\frac{\partial z}{\partial x} + x^2\frac{\partial z}{\partial y} \quad - (2)$$

adding ① and ②, we get

$$y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} = (x^2+y^2)\frac{\partial z}{\partial y} = e^{2u}(\cos^2 u + \sin^2 u)\frac{\partial z}{\partial y}$$

$$= e^{2u}\cdot\frac{\partial z}{\partial y} \quad \text{proved.}$$

(ii) Similarly, we can calculate.

exp If $u = f(y-z, z-x, x-y)$, then show that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

Sol^n let $y-z = r$, $z-x = s$, $x-y = t$

thus $u = f(r, s, t)$

So $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}$$

$$= \frac{\partial u}{\partial r}\cdot 0 + \frac{\partial u}{\partial s}\cdot(-1) + \frac{\partial u}{\partial t}\cdot(1)$$

$$\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} \quad \text{— ①}$$

Now $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\overset{\to 0}{\frac{\partial s}{\partial y}} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$= \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t} \quad \text{— ②}$$

Similarly $$\frac{\partial u}{\partial z} = -\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} \quad \text{— ③}$$

Adding ①, ② and ③, We get

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0} \quad \text{proved.}$$

exp If $\phi(cx-az, cy-bz) = 0$, show that $ap+bq = c$.
where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$

Hint let $cx-az = r$, $cy-bz = s \Rightarrow \phi(r,s) = 0$

$\Rightarrow \frac{\partial \phi}{\partial x} = 0$, $\frac{\partial \phi}{\partial y} = 0$, $\frac{\partial \phi}{\partial z} = 0 \Rightarrow$ from these three eq^{ns} find required rel^n

Important Deductions: let $z = f(x,y)$, then

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

If $z=0 \Rightarrow dz=0 \Rightarrow 0 = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$

$$\Rightarrow \frac{\partial f}{\partial y}\frac{dy}{dx} = -\frac{\partial f}{\partial x}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}}$$

Similarly we can also find $\frac{d^2y}{dx^2} = -\frac{[q^2r - 2pqs + p^2t]}{q^3}$

where $p = \frac{\partial f}{\partial x}$, $q = \frac{\partial f}{\partial y}$, $r = \frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial x \partial y} = s$, $t = \frac{\partial^2 f}{\partial y^2}$

exp: If $x^3 + 3x^2y + 6xy^2 + y^3 = 1$, find $\frac{dy}{dx}$

Sol^n let $f(x,y) = x^3 + 3x^2y + 6xy^2 + y^3 - 1 = 0$

$\frac{\partial f}{\partial x} = 3x^2 + 6xy + 6y^2$, $\frac{\partial f}{\partial y} = 3x^2 + 12xy + 3y^2$

thus $\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{[3x^2+6xy+6y^2]}{[3x^2+12xy+3y^2]}$ Ans

exp. If $u = x\log xy$, where $x^3+y^3+3xy = 1$, Find $\frac{du}{dx}$

Sol^n. We have $u = x\log xy$, $\frac{\partial u}{\partial x} = x\left(\frac{1}{xy}\cdot y\right) + 1\cdot\log xy$

$$\frac{\partial u}{\partial x} = 1 + \log xy \quad —(1)$$

Similarly $\partial u/\partial y = x/y \quad —(2)$

Also we have $x^3+y^3+3xy = 1$,

On Differentiating wrt x, we get

$$3x^2 + 3y^2\frac{dy}{dx} + 3x\frac{dy}{dx} + 3y = 0$$

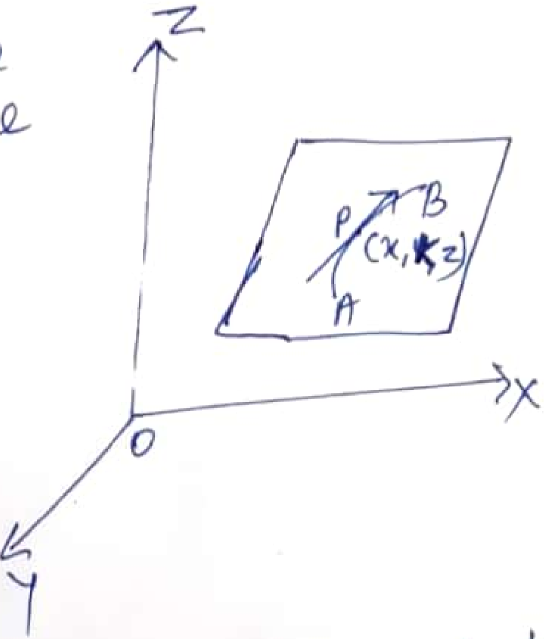
$$\frac{dy}{dx} = -\frac{(x^2+y)}{x+y^2} \quad —(3)$$

We know that
$$\frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$
$$= (1+\log xy)\cdot 1 + \left(\frac{x}{y}\right)\left(-\frac{x^2+y}{x+y^2}\right)$$
$$\Rightarrow \boxed{\frac{du}{dx} = 1+\log xy - \frac{x}{y}\,\frac{x^2+y}{x+y^2}}$$ Ans

Geometrical Interpretation of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

let $z = f(x,y)$ be a surface S
let $y = k$ be a plane parallel to X-Z plane, passing through $P(x,k,z)$ cutting the surface $z = f(x,y)$ along the curve APB this section APB is a plane curve whose equations are

$z = f(x,y)$
$y = k$



The slope of the tangent to this curve is given by $\partial z/\partial x$.

Similarly $\frac{\partial z}{\partial y}$ is the slope of the tangent to the curve of intersection of the surface $z = f(x,y)$ with a plane parallel to YZ-plane.

Jacobians

If u and v are functions of two independent variables x and y, then the determinant $\begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix}$ is called the Jacobian of u and v with respect to x and y and is denoted as $J\left(\frac{u,v}{x,y}\right)$ or $\frac{\partial(u,v)}{\partial(x,y)}$.

Similarly the Jacobian of u, v, w with respect to x, y and z, is

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{vmatrix}$$

exp. If $x = r\cos\theta$, $y = r\sin\theta$; Evaluate $\frac{\partial(x,y)}{\partial(r,\theta)}$; and $\frac{\partial(r,\theta)}{\partial(x,y)}$

Solⁿ We have $x = r\cos\theta \Rightarrow \frac{\partial x}{\partial r} = \cos\theta$ and $\frac{\partial x}{\partial \theta} = -r\sin\theta$

$y = r\sin\theta \Rightarrow \frac{\partial y}{\partial r} = \sin\theta$ and $\frac{\partial y}{\partial \theta} = r\cos\theta$

Thus
$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \partial x/\partial r & \partial x/\partial \theta \\ \partial y/\partial r & \partial y/\partial \theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta)$$

$$\boxed{J\left(\frac{x,y}{r,\theta}\right) = r}$$

Since $r^2 = x^2+y^2$ and $\theta = \tan^{-1}(y/x)$, we can calculate

$\boxed{\frac{\partial r}{\partial x} = \frac{x}{r},\ \frac{\partial r}{\partial y} = \frac{y}{r}}$; $\boxed{\frac{\partial \theta}{\partial x} = \frac{-y}{x^2+y^2} = -y/r^2}$; $\boxed{\frac{\partial \theta}{\partial x} = \frac{x}{r^2}}$

$$J\left(\frac{r,\theta}{x,y}\right) = \begin{vmatrix} \partial r/\partial x & \partial r/\partial y \\ \partial \theta/\partial x & \partial \theta/\partial y \end{vmatrix} = \begin{vmatrix} x/r & y/r \\ -y/r^2 & x/r^2 \end{vmatrix} = \frac{x^2}{r^3} + \frac{y^2}{r^3} = \frac{1}{r^3}(x^2+y^2)$$
$$= \frac{1}{r^3}\cdot r^2 = \frac{1}{r}.$$

exp. If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\cdot\sin\phi$, $z = r\cos\theta$ show that

$$J\left(\frac{x,y,z}{r,\theta,\phi}\right) = r^2\sin\theta.$$

Sol^n We have

$x = r\sin\theta\cos\phi$

$\partial x/\partial r = \sin\theta\cos\phi$

$\partial x/\partial\theta = r\cos\theta\cos\phi$

$\partial x/\partial\phi = -r\sin\theta\sin\phi$

$y = r\sin\theta\sin\phi$

$\partial y/\partial r = \sin\theta\cdot\sin\phi$

$\partial y/\partial\theta = r\cos\theta\sin\phi$

$\partial y/\partial\phi = r\sin\theta\cos\phi$

$z = r\cos\theta$

$\partial z/\partial r = \cos\theta$

$\partial z/\partial\theta = -r\sin\theta$

$\partial z/\partial\phi = 0$

Now Jacobian can be calculated as follows:

$$J\left(\frac{x,y,z}{r,\theta,\phi}\right) = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta & \partial x/\partial\phi \\ \partial y/\partial r & \partial y/\partial\theta & \partial y/\partial\phi \\ \partial z/\partial r & \partial z/\partial\theta & \partial z/\partial\phi \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{vmatrix}$$

$$= r^2\sin\theta \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta\sin\phi & \sin\theta\cos\phi & 0 \end{vmatrix}$$

$$\boxed{J\left(\frac{x,y,z}{r,\theta,\phi}\right) = r^2\sin\theta}$$

Proved (check plz)

Properties of Jacobians:—

(I) If u and v are the functions of x and y, then $\dfrac{\partial(u,v)}{\partial(x,y)} \times \dfrac{\partial(x,y)}{\partial(u,v)} = 1.$

(II) If u and v are the functions of r and s, where r and s are functions of x, y, then

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)}$$

(III) If the functions u, v and w of three independent variables x, y and z are not independent, or Dependent, then $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = 0.$

exp. If $u = xyz$, $v = x^2+y^2+z^2$, $w = x+y+z$, find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ (22)

Soln We have

$u = xyz$ $\quad$ $v = x^2+y^2+z^2$, $\quad$ $w = x+y+z$

$\partial u/\partial x = yz$ $\quad$ $\partial v/\partial x = 2x$ $\quad$ $\partial w/\partial x = 1$

$\partial u/\partial y = xz$ $\quad$ $\partial v/\partial y = 2y$ $\quad$ $\partial w/\partial y = 1$

$\partial u/\partial z = xy$ $\quad$ $\partial v/\partial z = 2z$ $\quad$ $\partial w/\partial z = 1$

$$J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{vmatrix} = \begin{vmatrix} yz & xz & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = -2(x-y)(y-z)(z-x)$$

Since $\frac{\partial(u,v,w)}{\partial(x,y,z)} \times \frac{\partial(x,y,z)}{\partial(u,v,w)} = 1 \Rightarrow \frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{-2(x-y)(y-z)(z-x)}$

Ans

exp. Find the value of the Jacobian $\frac{\partial(u,v)}{\partial(r,\theta)}$, where $u = x^2-y^2$, $v = 2xy$ and $x = r\cos\theta$, $y = r\sin\theta$

Solution:- ~~$u = r\cos\theta$~~ $u = x^2-y^2$, $v = 2xy$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2+y^2) = 4r^2$$

and $$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r \quad \text{(check it)}$$

Since $$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)} = 4r^2 \cdot r = 4r^3$$

Ans

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Exp. If $u = xy + yz + zx$, $v = x^2+y^2+z^2$ and $w = x+y+z$, determine wheather there is a functional relationship b/w u, v, w and if so find it.

Sol^n. We have $u = xy+yz+zx$, $v = x^2+y^2+z^2$, $w = x+y+z$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{vmatrix} = \begin{vmatrix} y+z & z+x & x+y \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} y+z & z+x & x+y \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} \xrightarrow[R_1+R_2]{R_1} = 2\begin{vmatrix} x+y+z & x+y+z & x+y+z \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = \underline{\underline{0}}$$

Hence, the functional relationship exists between u, v and w.

Now, $w^2 = (x+y+z)^2 = x^2+y^2+z^2 + 2(xy+yz+zx) = u + 2v$

$$\Rightarrow \boxed{w^2 - u - 2v = 0} \quad \underline{\underline{\text{Ans}}}$$

Jacobians of Implicit Functions

The variables x, y, u, v are connected by the implicit functions

$f_1(x,y,u,v) = 0$ — ①, $f_2(x,y,u,v) = 0$ — ②

where u and v are implicit functions of x and y.

then
$$\frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \frac{\partial(f_1,f_2)/\partial(x,y)}{\partial(f_1,f_2)/\partial(u,v)}$$

In general, the variables $x_1, x_2, \ldots x_n$ are connected with $u_1, u_2, \ldots u_n$ implicitly as $f_1(x_1, x_2, \ldots x_n; u_1, u_2, \ldots u_n) = 0$

$f_2(x_1, x_2, - x_n; u_1, u_2, u_3, - u_n) = 0$, —— $f_n(x_1, x_2, x_3, - x_n; u_1, u_2 - u_n) = 0$.

then we have

$$\frac{\partial(u_1, u_2, -, u_n)}{\partial(x_1, x_2 - - x_n)} = (-1)^n \frac{\partial(f_1, f_2, f_3 - f_n) / \partial(x_1, x_2 - x_n)}{\partial(f_1, f_2, - f_n) / \partial(u_1, u_2, - u_n)}$$

exp: If $u^3+v+w = x+y^2+z^2$; $u+v^3+w = x^2+y+z^2$, $u+v+w^3 = x^2+y^2+z$, prove that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{1-4(xy+yz+zx)+16xyz}{2-3(u^2+v^2+w^2)+27u^2v^2w^2}.$$

Soln: let $f_1 = u^3+v+w-x-y^2-z^2$, $f_2 = u+v^3+w-x^2-y-z^2$

$f_3 = u+v+w^3-x^2-y^2-z$

Now
$$\frac{\partial(f_1, f_2, f_3)}{\partial(x,y,z)} = \begin{vmatrix} -1 & -2y & -2z \\ -2x & -1 & -2z \\ -2x & -2y & -1 \end{vmatrix} = -1+4(yz+zx+xy)-16xyz$$

and
$$\frac{\partial(f_1, f_2, f_3)}{\partial(u,v,w)} = \begin{vmatrix} 3u^2 & 1 & 1 \\ 1 & 3v^2 & 1 \\ 1 & 1 & 3w^2 \end{vmatrix} = 2-3(u^2+v^2+w^2)+27u^2v^2w^2$$

Since
$$\frac{\partial(u, v, w)}{\partial(x,y,z)} = (-1)^3 \frac{\partial(f_1, f_2, f_3)}{\partial(x,y,z)} \bigg/ \frac{\partial(f_1, f_2, f_3)}{\partial(u,v,w)}$$

$$= \frac{1-4(yz+zx+xy)+16xyz}{2-3(u^2+v^2+w^2)+27u^2v^2w^2}$$ Proved

Taylor's Series of Two Variables

If $f(x,y)$ and all its partial derivatives upto nth order are finite and continuous for all points (x,y), where

$$a \le x \le a+h, \quad b \le y \le b+k,$$

then
$$f(a+h, b+k) = f(a,b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f + \frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f + \frac{1}{3!}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^3 f + \text{------}$$

exp: Suppose that $f(x+h, y+k)$ is a function of one variable only, say x where y is assumed as constant. Expanding by Taylor's Theorem for one variable, we have

$$f(x+\delta x, y+\delta y) = f(x, y+\delta y) + \delta x \cdot \frac{\partial}{\partial x} f(x, y+\delta y) + \frac{1}{2!}(\delta x)^2 \frac{\partial^2 f(x, y+\delta y)}{\partial x^2} + \text{---}$$

Now expanding for y, we get

$$= \left[f(x,y) + \delta y\frac{\partial}{\partial y}f(x,y) + \frac{(\delta y)^2}{2!}\frac{\partial^2}{\partial y^2}f(x,y) + \text{--}\right] + \delta x\cdot\frac{\partial}{\partial x}\left[f(x,y) + \delta y\frac{\partial}{\partial y}f(x,y) + \text{---}\right] + \frac{(\delta x)^2}{2!}\frac{\partial^2}{\partial x^2}\left[f(x,y) + \delta y\cdot\frac{\partial}{\partial y}f(x,y) + \text{--}\right] + \text{--}$$

$$= f(x,y) + \left[\delta x\frac{\partial}{\partial x}f(x,y) + \delta y\frac{\partial}{\partial y}f(x,y)\right] + \frac{1}{2!}\left[(\delta x)^2\frac{\partial^2}{\partial x^2}f(x,y) + 2\delta x\delta y\frac{\partial^2}{\partial x\partial y}f(x,y) + (\delta y)^2\frac{\partial^2}{\partial y^2}f(x,y)\right] + \text{--}$$

$$f(a+h, b+k) = f(a,b) + \left[h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right] + \frac{1}{2!}\left[h^2\frac{\partial^2}{\partial x^2}f + 2hk\frac{\partial^2 f}{\partial x\partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right] + \text{--}$$

$$= f(a,b) + \left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]f + \frac{1}{2!}\left[h^2\frac{\partial^2}{\partial x^2} + 2hk\frac{\partial^2}{\partial x\partial y} + k^2\frac{\partial^2}{\partial y^2}\right]f + \text{--}$$

On putting $a=0, b=0, h=x, k=y$, we have

$$f(x,y) = f(0,0) + \left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right) + \frac{1}{2!}\left(x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x\partial y} + y^2\frac{\partial^2 f}{\partial y^2}\right) + \text{--}$$

exp Expand $e^x \sin y$ in powers of x and y, $x=0$, $y=0$ as far as terms of third degree. (26)

Sol^n $f(x,y) = e^x.\sin y \Rightarrow$ at $(0,0)$ $f(0,0)=0$

$f_x(x,y) = e^x \sin y \Rightarrow$ at $(0,0)$ $f_x(0,0)=0$

$f_y(x,y) = e^x \cos y \Rightarrow$ at $(0,0)$ $f_y(0,0)=1$

$f_{xx}(x,y) = e^x \sin y \Rightarrow$ at $(0,0)$ $f_{xx}(0,0)=0$

$f_{yy}(x,y) = -e^x \sin y \Rightarrow$ at $(0,0)$ $f_{yy}(0,0)=0$

$f_{xy}(x,y) = e^x \cos y \Rightarrow$ at $(0,0)$ $f_{xy}(0,0)=1$

$f_{xxx}(x,y) = e^x.\sin y \Rightarrow$ at $(0,0)$ $f_{xxx}(0,0)=0$

$f_{xxy}(x,y) = e^x.\cos y \Rightarrow$ at $(0,0)$ $f_{xxy}(0,0)=1$

$f_{xyy}(x,y) = -e^x \sin y \Rightarrow$ at $(0,0)$ $f_{xyy}(0,0)=0$

$f_{yyy}(x,y) = -e^x \cos y \Rightarrow$ at $(0,0)$, $f_{yyy}(x,y) = -1$.

By Taylor's Series

$$f(x,y) = f(0,0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)f(0,0) + \frac{1}{2!}\cdot\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(0,0) + \frac{1}{3!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^3 f(0,0) + \text{---}$$

$$= f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{x^2}{2!}f_{xx}(0,0) + \frac{2.x.y}{2!}f_{xy}(0,0) + \frac{y^2}{2!}f_{yy}(0,0) + \frac{1}{3!}x^3 f_{xxx}(0,0) + \frac{3x^2y}{3!}f_{xxy}(0,0)$$

$$+ \frac{3}{3!}xy^2 f_{xyy}(0,0) + \frac{1}{3!}y^3 f_{yyy}(0,0) + \text{---}$$

$$= 0 + x.0 + y.1 + \frac{x^2}{2}.(0) + x.y.1 + \frac{y^2}{2}(0) + \frac{x^3}{6}(0) + \frac{3x^2y}{6}(1) + \frac{3xy^2}{6}.0 + \frac{y^3}{6}.(-1) + \text{---}$$

$$\boxed{f(x,y) = y + xy + \frac{x^2y}{2} - \frac{y^3}{6} + \text{---}}$$ Ans.

21

Gamma, Beta Functions, Differentiation Under the Integral Sign

21.1 GAMMA FUNCTION

$$\int_0^\infty e^{-x} x^{n-1}\, dx \qquad (n > 0)$$

is called gamma function of n. It is also written as $\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx$.

Example 1. *Prove that* $\Gamma 1 = 1$

Solution. $$\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx$$

Put $n = 1$, $$\Gamma 1 = \int_0^\infty e^{-x}\, dx = \left[\frac{e^{-x}}{-1}\right]_0^\infty = 1$$ **Proved**

Example 2. *Prove that*

(*i*) $\Gamma(n+1) = n\,\Gamma n$ (*ii*) $\Gamma(n+1) = \lfloor n$ (*Reduction formula*)

Solution.

(*i*) $$\Gamma n = \int_0^\infty x^{n-1} e^{-x}\, dx \qquad ...(1)$$

Integrating by parts, we have

$$= \left[x^{n-1}\frac{e^{-x}}{-1}\right]_0^\infty - (n-1)\int_0^\infty x^{n-2}\frac{e^{-x}}{-1}\, dx$$

$$= \left[\lim_{x \to 0} \frac{x^{n-1}}{e^x} = \lim_{x \to 0} 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \ldots + \frac{x^n}{\lfloor n} + \ldots + x^{n-1}\right] = 0$$

$$= (n-1)\int_0^\infty x^{n-2} e^{-x}\, dx$$

$$\therefore \qquad \Gamma n = (n-1)\,\Gamma(n-1) \qquad ...(2)$$

$$\Gamma(n+1) = n\,\Gamma n$$ Replacing n by $(n+1)$ **Proved**

(*ii*) Replace n by $n-1$ in (2), we get

$$\overline{|n-1} = (n-2)\ \overline{|n-2}$$

Putting the value $\Gamma n-1$ in (2), we get

$$\Gamma n = (n-1)\,(n-2)\,\overline{|n-2}$$

Similarly $$\Gamma n = (n-1)\,(n-2)\ldots 3.2.1\,\Gamma 1 \quad \ldots (3)$$

Putting the value of $\overline{|1}$ in (3), we have

$$\Gamma n = (n-1)\,(n-2)\ldots 3.2.1.1$$

$$\Gamma n = \underline{|n-1}$$

Replacing n by $n+1$, we have

$$\overline{|n+1} = \underline{|n}$$ **Proved**

Example 3. *Evaluate* $\int_0^\infty \sqrt[4]{x}\, e^{-\sqrt{x}}\, dx$

Solution. Let $I = \int_0^\infty x^{1/4}\, e^{-\sqrt{x}}\, dx$ ...(1)

Putting $\sqrt{x} = t$ or $x = t^2$ or $dx = 2\,t\,dt$ in (1), we get

$$I = \int_0^\infty t^{1/2}\, e^{-t}\, 2\,t\,dt = 2\int_0^\infty t^{3/2}\, e^{-t}\,dt$$

$$= 2\,\Gamma\frac{5}{2} \qquad \text{By definition}$$

$$= 2\,.\,\frac{3}{2}\,\Gamma\frac{3}{2} = 2\,.\,\frac{3}{2}\,.\,\frac{1}{2}\,\Gamma\frac{1}{2} = \frac{3}{2}\sqrt{\pi}$$ **Ans.**

Example 4. *Evaluate* $\int_0^\infty \sqrt{x}\, e^{-\sqrt[3]{x}}\, dx.$

Solution. Let $I = \int_0^\infty \sqrt{x}\, e^{-\sqrt[3]{x}}\, dx$ ...(1)

Putting $\sqrt[3]{x} = t$ or $x = t^3$ or $dx = 3\,t^2\,dt$ in (1) we get

$$I = \int_0^\infty t^{3/2}\, e^{-t}\, 3\,t^2\,dt = 3\int_0^\infty t^{7/2}\, e^{-t}\,dt = 3\,\Gamma\frac{9}{2} = 3\,.\,\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\frac{1}{2} = \frac{315}{16}\sqrt{\pi}$$ **Ans.**

Example 5. *Evaluate* $\int_0^\infty x^{n-1}\, e^{-h^2x^2}\, dx.$

Solution. Let $I = \int_0^\infty x^{n-1}\, e^{-h^2x^2}\, dx$ ...(1)

Putting $t = h^2x^2$ or $x = \dfrac{\sqrt{t}}{h}$ or $dx = \dfrac{dt}{2\,h\sqrt{t}}$,

(1) becomes $$I = \int_o^\infty \left(\frac{\sqrt{t}}{h}\right)^{n-1} e^{-t}\,\frac{dt}{2\,h\,\sqrt{t}}$$

$$= \frac{1}{2\,h^n}\int_0^\infty t^{\frac{n-1}{2}}\, e^{-t}\,\frac{dt}{\sqrt{t}} = \frac{1}{2\,h^n}\int_0^\infty t^{\frac{n-2}{2}}\, e^{-t}\,dt$$

$$= \frac{1}{2\,h^n}\,\Gamma\frac{n}{2}$$ **Ans.**

Example 6. *Evaluate* $\int_0^\infty \frac{x^a}{a^x}\,dx.$ $(a > 1)$

Solution: $I = \int_0^\infty \frac{x^a}{a^x}\,dx$ ...(1)

Putting $a^x = e^t$ or $x \log a = t$, $x = \frac{t}{\log a}$, $dx = \frac{dt}{\log a}$ in (1), we have

$$I = \int_0^\infty \left(\frac{t}{\log a}\right)^a e^{-t}\frac{dt}{\log a} = \frac{1}{(\log a)^{a+1}}\int_0^\infty e^{-t}\,t^a\,dt$$

$$= \frac{1}{(\log a)^{a+1}}\,\Gamma\overline{a+1}$$ **Ans.**

Example 7. *Evaluate* $\int_0^1 x^{n-1}.\left[\log_e.\left(\frac{1}{x}\right)\right]^{m-1}.\,dx$

Solution: Put $\log_e\frac{1}{x} = t$ or $x = e^{-t}$ $\therefore\ dx = -e^{-t}\,dt$

$$\int_0^1 x^{n-1}\left[\log_e\left(\frac{1}{x}\right)\right]^{m-1}dx = \int_\infty^0 (e^{-t})^{n-1}\,[t]^{m-1}\,(-e^{-t}\,dt) = \int_0^\infty e^{-nt}\,t^{m-1}\,dt$$

Put $nt = u$ or $t = \frac{u}{n}$ $\therefore$ $dt = \frac{du}{n}$

$$= \int_0^\infty e^{-u}\left(\frac{u}{n}\right)^{m-1}\frac{du}{n} = \frac{1}{n^m}\int_0^\infty e^{-u}\,u^{m-1}\,du = \frac{1}{n^m}\Gamma m$$ **Ans.**

21.2 TRANSFORMATION OF GAMA FUNCTION

Prove that (1) $\int_0^\infty e^{-ky}y^{n-1}\,dy = \frac{\Gamma n}{k^n}$ (2) $\Gamma\frac{1}{2} = \sqrt{\pi}$ (3) $\int_0^1\left(\log\frac{1}{y}\right)^{n-1}dy = \Gamma n$

Solution: We know that $\Gamma n = \int_0^\infty x^{n-1}\,e^{-x}\,dx$ ...(1)

(i) Replace x by $k\,y$, so that $dx = kdy$; then

(1) becomes $\Gamma n = \int_0^\infty (ky)^{n-1}\,e^{-ky}\,k\,dy.$

$$\Gamma n = k^n\int_0^\infty e^{-ky}\,y^{n-1}\,dy$$

$\therefore$ $$\boldsymbol{\int_0^\infty e^{-ky}\,y^{n-1}\,dy = \frac{\Gamma n}{k^n}}$$ ...(2) **Proved**

(*ii*) Replace x^n by y, $n\,x^{n-1}\,dx = dy$ in (1), then

$$\Gamma n = \int_0^\infty y^{\frac{n-1}{n}}\,e^{-y^{1/n}}\frac{dy}{nx^{n-1}}$$

$$= \int_0^\infty y^{\frac{n-1}{n}} e^{-y^{1/n}} \frac{dy}{ny^{\frac{n-1}{n}}} = \frac{1}{n}\int_0^\infty e^{-y^{1/n}}\, dy$$

When $n = \dfrac{1}{2},$
$$\Gamma\frac{1}{2} = \frac{1}{\frac{1}{2}}\int_0^\infty e^{-y^2}\, dy = 2\left[\frac{1}{2}\sqrt{\pi}\right]$$

Proved
$$\Gamma\frac{1}{2} = \sqrt{\pi}$$

(*iii*) Substitute e^{-x} by $y, -e^{-x}\, dx = dy$

$$-x = \log y,\ x = \log\frac{1}{y}, \quad \text{Then (1) becomes}$$

$$\Gamma n = -\int_1^0 \left(\log\frac{1}{y}\right)^{n-1} y \,.\, \frac{dy}{e^{-x}}$$

$$= \int_0^1 \left(\log\frac{1}{y}\right)^{n-1} y \,.\, \frac{dy}{y} = \int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy. \quad \textbf{Proved}$$

Exercise 21.1

Evaluate :

1. (*i*) $\Gamma -\frac{1}{2}$ (*ii*) $\Gamma\frac{-3}{2}$ (*iii*) $\Gamma\frac{-15}{2}$ (*iv*) $\Gamma\frac{7}{2}$ (*v*) $\Gamma 0$

Ans. (*i*) $-2\sqrt{\pi}$ (*ii*) $\frac{4}{3}\sqrt{\pi}$ (*iii*) $\dfrac{2^8\sqrt{\pi}}{15\times 13\times 11\times 9\times 7\times 5\times 3}$ (*iv*) $\dfrac{15\sqrt{\pi}}{8}$ (*v*) ∞

2. $\int_0^\infty \sqrt{x}\, e^{-x}\, dx$ **Ans.** $\Gamma\frac{3}{2}$

3. $\int_0^\infty x^4 e^{-x^2}\, dx$ **Ans.** $\dfrac{3\sqrt{\pi}}{8}$.

4. $\int_0^\infty e^{-h^2x^2}\, dx$ **Ans.** $\dfrac{\sqrt{\pi}}{2h}$

5. $\int_0^\infty\int_0^\infty e^{-(ax^2+by^2)} x^{2m-1} y^{2n-1}\, dx\, dy,\ a, b, m, n > 0$ **Ans.** $\dfrac{\Gamma m\,\Gamma n}{4a^m b^n}$

6. $\int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy,\quad n > 0$ **Ans.** Γn

7. $\int_0^1 \dfrac{dx}{\sqrt{-\log x}}$ **Ans.** $\sqrt{\pi}$

8. $\int_0^1 (x\log x)^3\, dx$ **Ans.** $-\dfrac{3}{128}$

9. $\int_0^1 \dfrac{dx}{\sqrt{x\log\frac{1}{x}}}$ **Ans.** $\sqrt{2\pi}$

10. Prove that $1.3.5.....(2n-1) = \dfrac{2^n\,\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}}$

11. $\int_0^\infty e^{-y^{1/m}}\, dy = m\Gamma m.$

21.3 BETA FUNCTION

$$\int_0^{\infty} x^{l-1}(1-x)^{m-1}\,dx \qquad (l>0,\ m>0)$$

is called the Beta function of *l, m.* It is also written as

$$\beta(l, m) = \int_0^1 x^{l-1}(1-x)^{m-1}\,dx.$$

21.4 EVALUATION OF BETA FUNCTION

$$\beta(l\,,m) = \frac{\Gamma l\,\Gamma m}{\Gamma l+m}$$

Solution. We have $\beta(l, m) = \int_0^1 x^{l-1}(1-x)^{m-1}\,dx = \int_0^1 (1-x)^{m-1}x^{l-1}\,dx$

Integrating by parts, we have

$$= \left[(1-x)^{m-1}\frac{x^l}{l}\right]_0^1 + (m-1)\int_0^1 (1-x)^{m-2}\left(\frac{x^l}{l}\right)dx$$

$$= \frac{(m-1)}{l}\int_0^1 (1-x)^{m-2}x^l\,dx$$

Again integrating by parts

$$= \frac{(m-1)(m-2)}{l(l+1)}\int_0^1 (1-x)^{m-3}x^{l+1}\,dx$$

$$= \frac{(m-1)(m-2)\ldots 2.1}{l(l+1)\ldots\ldots(l+m-2)}\int_0^1 x^{l+m-2}\,dx$$

$$= \frac{(m-1)(m-2)\ldots 2.1}{l(l+1)\ldots\ldots(l+m-2)}\left[\frac{x^{l+m-1}}{l+m-1}\right]_0^1$$

$$= \frac{(m-1)(m-2)\ldots 2.1}{l(l+1)\ldots(l+m-2)(l+m-1)}$$

$$= \frac{\underline{|m-1}}{l(l+1)\ldots(l+m-2)(l+m-1)} \times \frac{(l-1)(l-2)\ldots 1}{(l-1)(l-2\ldots 1}$$

$$= \frac{\underline{|m-1}\;\underline{|l-1}}{1.2\ldots(l-2)(l-1)\,.\,l(l+1)\ldots(l+m-2)(l+m-1)}$$

$$= \frac{\underline{|l-1}\;\;\underline{|m-1}}{\underline{|l+m-1}}$$

$$= \frac{\Gamma l\,\Gamma m}{\Gamma l+m}$$

And if only l is positive integer and not m then

$$\beta(l, m) = \frac{\underline{|l-1}}{m(m+1)\ldots(m+l-1)} \qquad \textbf{Ans.}$$

21.5 A PROPERTY OF BETA FUNCTION

$$\beta(l, m) = \beta(m, l)$$

Solution. We have

$$\beta(l, m) = \int_0^1 x^{l-1}(1-x)^{m-1}\,dx \qquad \left[\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^1 (1-x)^{l-1}\,[1-(1-x)]^{m-1}\,dx$$

$$= \int_0^1 (1-x)^{l-1}\,x^{m-1}\,dx$$

$$= \int_0^1 x^{m-1}(1-x)^{l-1}\,dx = \beta\,(m, l)$$ 1 and m are interchanged. **Proved**

Example 8. *Evaluate* $\int_0^1 x^4\,(1-\sqrt{x}\,)^5\,dx$

Solution. Let $\sqrt{x} = t$ or $x = t^2$ or $dx = 2\,tdt$

$$\int_0^1 x^4\,(1-\sqrt{x}\,)^5\,dx = \int_0^1 (t^2)^4\,(1-t)^5\,(2\,tdt)$$

$$= 2\int_0^1 t^9\,(1-t)^5\,dt = 2\,\beta\,(10, 6) = 2\,\frac{\Gamma 10\,\Gamma 6}{\Gamma 16} = 2\,\frac{\lfloor 9\,\lfloor 5}{\lfloor 15}$$

$$= 2\,.\,\frac{\lfloor 5}{10\times 11\times 12\times 13\times 14\times 15} = \frac{2\times 1\times 2\times 3\times 4\times 5}{10\times 11\times 12\times 13\times 14\times 15}$$

$$= \frac{1}{11\times 13\times 7\times 15} = \frac{1}{15015}$$ **Ans.**

Example 9. *Evaluate* $\int_0^1 (1-x^3)^{-\frac{1}{2}}\,dx$

Solution. Let $x^3 = y$ or $x = y^{1/3}$ or $dx = \frac{1}{3}\,y^{-\frac{2}{3}}\,dy$

$$\int_0^1 (1-x^3)^{-\frac{1}{2}}\,dx = \int_0^1 (1-y)^{-\frac{1}{2}}\left(\frac{1}{3}\,y^{-\frac{2}{3}}\,dy\right)$$

$$= \frac{1}{3}\int_0^1 y^{-\frac{2}{3}}\,(1-y)^{-\frac{1}{2}}\,dy = \frac{1}{3}\,\beta\left(\frac{1}{3}, \frac{1}{2}\right) = \frac{1}{3}\,\frac{\Gamma\frac{1}{3}\,\Gamma\frac{1}{2}}{\Gamma\frac{5}{6}}$$ **Ans.**

21.6 TRANSFORMATION OF BETA FUNCTION

We know that

$$\beta\,(l, m) = \int_0^1 x^{l-1}\,(1-x)^{m-1}\,dx$$

Putting $x = \dfrac{1}{1+y}$ so that $dx = -\dfrac{1}{(1+y)^2}\,dy$ and $1-x = \dfrac{y}{1+y}$.

$$\beta\,(l, m) = \int_\infty^0 \left(\frac{1}{1+y}\right)^{l-1}\left(\frac{y}{1+y}\right)^{m-1}\left[-\frac{1}{(1+y)^2}\,dy\right]$$

$$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{l+m}}\,dy$$

Since l, m can be interchanged in $\beta\,(l, m)$,

$$\beta(l, m) = \int_0^{\infty} \frac{y^{l-1}}{(1+y)^{m+l}}\, dy \quad \text{or} \quad \beta(l, m) = \int_0^{\infty} \frac{x^{l-1}}{(1+x)^{m+l}}\, dx$$

Example 10. *Evaluate* $\displaystyle\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\, dx$

Solution. We know that

$$\beta(m, n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\, dx \Rightarrow \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\, dx = \beta(m, n)$$

$$\Rightarrow \qquad \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\, dx + \int_1^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} = \beta(m, n) \qquad ...(1)$$

Consider $\displaystyle\int_1^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$ $\qquad \left(\text{Put } x = \dfrac{1}{t}\right)$

$$= \int_1^0 \frac{\left(\frac{1}{t}\right)^{m-1}}{\left(1+\frac{1}{t}\right)^{m+n}} \left(-\frac{1}{t^2}\, dt\right) = \int_0^1 \frac{\left(\frac{1}{t}\right)^{m-1} \frac{1}{t^2}}{\left(\frac{1}{t}\right)^{m+n} (t+1)^{m+n}}\, dt$$

$$= \int_0^1 \frac{t^{n-1}}{(1+t)^{m+n}}\, dt = \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$$

Putting the value of $\displaystyle\int_1^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$ in (1) we get

$$\int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\, dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\, dx = \beta(m, n)$$

$$\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\, dx = \beta(m, n) \qquad \textbf{Ans.}$$

21.7 RELATION BETWEEN BETA AND GAMMA FUNCTIONS

We know that

$$\Gamma l = \int_0^{\infty} e^{-x} x^{l-1}\, dx, \qquad \frac{\Gamma l}{z^l} = \int_0^{\infty} e^{-zx} x^{l-1}\, dx$$

$$\Gamma l = \int_0^{\infty} z^l e^{-zx} x^{l-1}\, dx$$

Multiplying both sides by $e^{-z} z^{m-1}$, we have

$$\Gamma l \,.\, e^{-z} \,.\, z^{m-1} = \int_0^{\infty} e^{-z} \,.\, z^{m-1} \,.\, z^l \,.\, e^{-zx} x^{l-1}\, dx$$

$$\Gamma l \,.\, e^{-z} \,.\, z^{m-1} = \int_0^{\infty} e^{-(1+x)z} z^{l+m-1} x^{l-1}\, dx$$

Integrating both sides w.r.t. 'x' we get

$$\int_0^{\infty} \Gamma l\, e^{-z} z^{m-1}\, dz = \int_0^{\infty} \int_0^{\infty} e^{-(1+x)z} z^{l+m-1} x^{l-1}\, dx\, dz$$

$$\Gamma l\, \Gamma m = \int_0^{\infty} x^{l-1}\, dx \int_0^{\infty} e^{-(1+x)z} z^{l+m-1}\, dz$$

$$= \int_0^\infty x^{l-1}\, dx \,.\, \frac{\overline{|l+m}}{(1+x)^{l+m}}$$

$$\Gamma l\, \Gamma m = \overline{|l+m} \int_0^\infty \frac{x^{l-1}}{(1+x)^{l+m}}\, dx = \overline{|l+m} \,.\, \beta\,(l, m)$$

$$\therefore \qquad \beta\,(l, m) = \frac{\Gamma l\, \Gamma m}{\overline{|l+m}}$$

This is the required relation.

Example 11. *Show that*

$$\int_0^{\frac{\pi}{2}} \sin^P \theta \cos^q \theta \, d\,\theta = \frac{\Gamma\left(\frac{P+1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{P+q+2}{2}\right)}$$

Solution. We know that

$$\beta\,(m, n) = \int_0^1 x^{m-1}\,(1-x)^{n-1}\, dx \qquad ...(1)$$

Putting $\quad x = \sin^2\theta, \quad dx = 2\sin\theta\cos\theta\, d\,\theta$

and $\quad 1 - x = 1 - \sin^2\theta = \cos^2\theta$

Then (1) becomes

$$\beta\,(m, n) = \int_0^{\frac{\pi}{2}} \sin^{2m-2}\theta \cos^{2n-2}\theta\; 2\sin\theta\cos\theta\, d\,\theta$$

or
$$\frac{\Gamma m\, \Gamma n}{\Gamma\, m+n} = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\,\theta$$

Putting $\quad 2m - 1 = p, \quad i.e. \quad m = \frac{p+1}{2}$

and $\quad 2n - 1 = q, \quad i.e. \quad n = \frac{q+1}{2}$

$$\frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{\Gamma\frac{P+q+2}{2}} = 2\int_0^{\frac{\pi}{2}} \sin^P\theta \,.\, \cos^q\theta\, d\,\theta$$

$$\int_0^{\frac{\pi}{2}} \sin^P\theta \,.\, \cos^q\theta\, d\,\theta = \frac{\overline{\left|\frac{P+1}{2}\right.}\;\overline{\left|\frac{q+1}{2}\right.}}{2\,\overline{\left|\frac{P+q+2}{2}\right.}}$$ **Proved**

Example 12. *Find the value of* $\overline{\left|\frac{1}{2}\right.}$.

Solution. We know that

$$\int_0^{\frac{\pi}{2}} \sin^P \theta \cos^q \theta \, d\theta = \frac{\Gamma\left(\frac{P+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{P+q+2}{2}\right)}$$

Putting $P = q = 0$
$$\int_0^{\frac{\pi}{2}} d\theta = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma(1)}$$

or
$$[\theta]_0^{\pi/2} = \frac{1}{2}\left(\Gamma\left(\frac{1}{2}\right)\right)^2 \quad \text{or} \quad \frac{\pi}{2} = \frac{1}{2}\left(\Gamma\left(\frac{1}{2}\right)\right)^2$$

or
$$\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = \pi \quad \text{or} \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$ **Ans.**

Example 13. *Show that*

$$\int_0^{\frac{\pi}{2}} \sqrt{(\cot \theta)}\, d\theta = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$$

Solution. We know that

$$\int_0^{\frac{\pi}{2}} \sin^P x \cos^q x \, dx = \frac{\Gamma\left(\frac{P+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{P+q+2}{2}\right)} \qquad \text{...(1)}$$

$$\int_0^{\frac{\pi}{2}} \sqrt{\cot \theta}\, d\theta = \int_0^{\frac{\pi}{2}} \frac{\cos^{1/2}\theta}{\sin^{1/2}\theta}\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^{-1/2}\theta \cos^{1/2}\theta \, d\theta$$

On applying formula (1), we have

$$= \frac{\Gamma\left(\frac{-1/2+1}{2}\right)\Gamma\left(\frac{1/2+1}{2}\right)}{2\,\Gamma\left(\frac{-1/2+1/2+2}{2}\right)} = \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{2\,\Gamma(1)} = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$$ **Proved**

Example 14. *Evaluate* $\int_{-1}^{+1} (1+x)^{P-1}(1-x)^{q-1}\, dx.$

Solution. Put $x = \cos 2\theta$, then $dx = -2\sin 2\theta \, d\theta$

$$\int_{-1}^{+1} (1+x)^{P-1}(1-x)^{q-1}\, dx = \int_{\frac{\pi}{2}}^{0} (1+\cos 2\theta)^{P-1}(1-\cos 2\theta)^{q-1}(-2\sin 2\theta\, d\theta)$$

$$= \int_{\frac{\pi}{2}}^{0} (1+2\cos^2\theta - 1)^{P-1}(1-1+2\sin^2\theta)^{q-1}(-4\sin\theta\cos\theta\, d\theta)$$

$$= 4\int_0^{\frac{\pi}{2}} 2^{P-1}\cos^{2P-2}\theta \,.\, 2^{q-1}\sin^{2q-2}\theta \,.\, \sin\theta\cos\theta\, d\theta$$

$$= 2^{P+q}\int_0^{\pi} \sin^{2q-1}\theta\cos^{2P-1}\theta\, d\theta$$

$$= 2^{P+q}\frac{\left|\frac{2q}{2}\right.\left|\frac{2P}{2}\right.}{2\left|\frac{2P+2q}{2}\right.} = 2^{P+q-1}\frac{|P\,|q}{|P+q} \qquad \textbf{Ans.}$$

Example 15. *Show that* $\lceil n \lceil 1-n = \dfrac{\pi}{\sin n\pi}$ *(0 < n < 1)*

Solution. We know that

$$\beta(m, n) = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$$

$$\frac{\lceil m \lceil n}{\lceil m+n} = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$$

Putting $m+n = 1$ or $m = 1-n$

$$\frac{\overline{|1-n}\lceil n}{\lceil 1} = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^1}\, dx$$

$$\overline{|1-n}\lceil n = \int_0^{\infty} \frac{x^{n-1}}{1+x}\, dx \qquad \left[\int_0^{\infty} \frac{x^{n-1}}{1+x}\, dx = \frac{\pi}{\sin n\pi}\right]$$

$$= \frac{\pi}{\sin n\pi} \qquad \textbf{Proved}$$

Example 16. *Evaluate* $\displaystyle\int_0^1 \frac{dx}{(1-x^n)^{1/n}}$.

Solution. Let $x^n = \sin^2\theta$ or $x = \sin^{2/n}\theta$

So that $\qquad dx = \dfrac{2}{n}\sin^{2/n-1}\theta\cos\theta\, d\theta$

$$\int_0^1 \frac{dx}{(1-x^n)^{1/n}} = \int_0^{\frac{\pi}{2}} \frac{\frac{2}{n}\sin^{2/n-1}\theta\cos\theta\, d\theta}{(1-\sin^2\theta)^{1/n}} = \frac{2}{n}\int_0^{\frac{\pi}{2}} \frac{\sin^{2/n-1}\theta\cos\theta\, d\theta}{(\cos^2\theta)^{1/n}}$$

$$= \frac{2}{n}\int_0^{\frac{\pi}{2}} \sin^{2/n-1}\theta\cos^{1-2/n}\theta\, d\theta$$

$$= \frac{2}{n}\,\frac{\left|\frac{\frac{2}{n}-1+1}{2}\right.\left|\frac{1-\frac{2}{n}+1}{2}\right.}{2\left|\frac{\frac{2}{n}-1+1+2-\frac{2}{n}}{2}\right.}$$

$$= \frac{1}{n} \frac{\Gamma\left(\frac{1}{n}\right) \Gamma\left(\frac{n-1}{n}\right)}{\Gamma 1} \qquad \left(\Gamma\left(\frac{1}{n}\right) \Gamma\left(1-\frac{1}{n}\right) = \frac{\pi}{\sin\frac{\pi}{n}} \right)$$

$$= \frac{\pi}{n \sin \frac{\pi}{n}}$$ **Ans.**

Example 17. Show that $\int_0^{\frac{\pi}{2}} tan^P\, \theta\, d\,\theta = \frac{\pi}{2} sec \frac{P\,\pi}{2}$ *and indicate the restriction on the values of P.*

Solution. $\int_0^{\frac{\pi}{2}} \tan^P \theta\, d\,\theta = \int_0^{\frac{\pi}{2}} \sin^P \theta \cos^{-P} \theta\, d\,\theta$

$$= \frac{\Gamma\left(\frac{P+1}{2}\right) \Gamma\left(\frac{-P+1}{2}\right)}{2\,\Gamma\left(\frac{P+1\;-P+1}{2}\right)} \qquad \left[\begin{array}{r} 1-P > 0 \\ 1 > P \end{array}\right]$$

$$= \frac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{-p+1}{2}\right)}{2\,\Gamma 1} \qquad \left[\begin{array}{r} 1+P>0 \\ P>-1 \end{array}\right]$$

$$= \frac{1}{2} \Gamma\left(\frac{1+p}{2}\right) \Gamma\left(\frac{-p+1}{2}\right) \qquad \therefore 1 > P > -1$$

$$= \frac{1}{2} \frac{\pi}{\sin \frac{p+1}{2}\pi} = \frac{1}{2} \frac{\pi}{\cos \frac{p\,\pi}{2}} = \frac{\pi}{2} \sec \frac{p\,\pi}{2}$$ **Proved**

Example 18. *Prove Duplication Formula*

$$\Gamma m\, \Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}} \Gamma 2\,m.$$

Hence show that $\beta(m, m) = 2^{1-2m} \beta\left(m, \frac{1}{2}\right)$ (*U.P., II Semeter, Summer 2001*)

Solution. We know that

$$\frac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)} = \int_0^{\frac{\pi}{2}} \sin^P \theta \cos^q \theta\, d\,\theta$$

Putting $q = p$ we get

$$\frac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{p+1}{2}\right)}{2\,\Gamma(p+1)} = \int_0^{\frac{\pi}{2}} \sin^P \theta \cos^P \theta\, d\,\theta = \int_0^{\frac{\pi}{2}} (\sin\theta\cos\theta)^P\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2^P}(2\sin\theta\cos\theta)^P \, d\theta = \frac{1}{2^P}\int_0^{\frac{\pi}{2}} (\sin 2\theta)^P \, d\theta$$

Putting $2\theta = t$, we have

$$= \frac{1}{2^p}\int_0^{\pi} \sin^P t \, \frac{dt}{2}$$

$$= \frac{1}{2^P}\cdot\frac{1}{2}\,.\,2\int_0^{\frac{\pi}{2}} \sin^P t \, dt = \frac{1}{2^P}\int_0^{\frac{\pi}{2}} \sin^P t\cos^{o} t \, dt$$

$$= \frac{1}{2^P}\,\frac{\Gamma\left(\frac{P+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\,\Gamma\left(\frac{P+2}{2}\right)}$$

or $$\frac{\Gamma\left(\frac{P+1}{2}\right)\Gamma\left(\frac{P+1}{2}\right)}{2\,\Gamma(P+1)} = \frac{1}{2^P}\,\frac{\Gamma\left(\frac{P+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(\frac{P+2}{2}\right)}$$

$\therefore$ or $$\frac{\Gamma\left(\frac{P+1}{2}\right)}{\Gamma(P+1)} = \frac{1}{2^P}\,\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{P+2}{2}\right)}$$

$\therefore$ or $$\frac{\Gamma\left(\frac{P+1}{2}\right)}{\Gamma(P+1)} = \frac{1}{2^P}\,\frac{\sqrt{\pi}}{\Gamma\left(\frac{P+2}{2}\right)}$$

Take $\frac{P+1}{2} = m$ or $P = 2m - 1$

or $$\frac{\Gamma m}{\Gamma 2m} = \frac{1}{2^{2m-1}}\,\frac{\sqrt{\pi}}{\Gamma\left(\frac{2m+1}{2}\right)} \qquad \text{...(1)}$$

$$\Gamma m\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma 2m$$ **Proved**

Multiplying both sides of (1) by Γm, we have

$$\frac{\Gamma m\,\Gamma m}{\Gamma 2m} = 2^{1-2m}\,\frac{\Gamma\left(\frac{1}{2}\right)\Gamma m}{\Gamma\left(m+\frac{1}{2}\right)}$$ **Proved**

$$\beta(m, m) = 2^{1-2m}\,\beta\left(m, \frac{1}{2}\right)$$

Example 19. *Evaluate* $\iint_A \dfrac{dx\,dy}{\sqrt{xy}}$, *using the substitutions*

$$x = \frac{u}{1+v^2}, \quad y = \frac{uv}{1+v^2}$$

where A is bounded by $x^2 + y^2 - x = 0,\ y = 0,\ y > 0.$

Solution. Here $\sqrt{xy} = \sqrt{\left(\dfrac{u}{1+v^2}\right)\left(\dfrac{uv}{(1+v^2)}\right)} = \dfrac{u\sqrt{v}}{1+v^2}$

$$dx\,dy = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\,dv$$

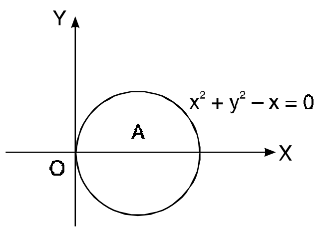
$$= \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} du\,dv = \begin{vmatrix} \dfrac{1}{1+v^2} & -\dfrac{2uv}{(1+v^2)^2} \\ \dfrac{v}{1+v^2} & \dfrac{u(1-v^2)}{(1+v^2)^2} \end{vmatrix} du\,dv$$

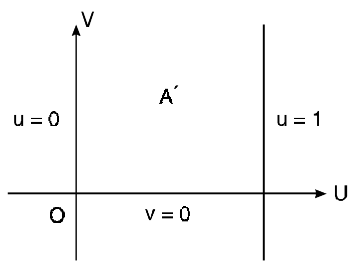
$$= \left[\frac{u(1-v^2)}{(1+v^2)^3} + \frac{2uv^2}{(1+v^2)^3}\right] du\,dv = \left[\frac{u - uv^2 + 2uv^2}{(1+v^2)^3}\right] du\,dv$$

$$= \frac{u(1+v^2)}{(1+v^2)^3}\,du\,dv = \frac{u}{(1+v^2)^2}\,du\,dv$$

Also the circle $x^2 + y^2 - x = 0$ is transformed into

$$\frac{u^2}{(1+v^2)^2} + \frac{u^2 v^2}{(1+v^2)^2} - \frac{u}{1+v^2} = 0 \quad \text{or} \quad \frac{u^2(1+v^2)}{(1+v^2)^2} - \frac{u}{1+v^2} = 0$$





$$\frac{u^2}{1+v^2} - \frac{u}{1+v^2} = 0 \quad \text{or} \quad u^2 - u = 0 \quad \text{or} \quad u(u-1) = 0 \Rightarrow u = 0, \quad u = 1$$

Further $\quad y = 0 \Rightarrow \dfrac{uv}{1+v^2} = 0 \Rightarrow u = 0, \quad v = 0$

and $y > 0 \Rightarrow uv > 0$ either both u and v are positive or both negative.

The area A, *i.e.*, $x^2 + y^2 - x = 0$ is transformed into A' bounded by $u = 0$, $v = 0$ and $u = 1$ *and* $v = \infty$.

$$\iint \frac{dx\,dy}{\sqrt{x}} = \int_0^1 \int_0^\infty \frac{\dfrac{u}{(1+v^2)^2}\,du\,dv}{\dfrac{u\sqrt{v}}{1+v^2}} = \int_0^1 \int_0^\infty \frac{1}{\sqrt{v}\,(1+v^2)}\,dv\,du$$

On putting $v = \tan\theta$, dv $= sec^2\,\theta\,d\theta$

$$= \int_0^1 \int_0^{\frac{\pi}{2}} \frac{sec^2 \theta \, d\theta \, du}{\sqrt{\tan\theta}\,(1+\tan^2\theta)} = \int_0^1 du \int_0^{\frac{\pi}{2}} \sqrt{\frac{\cos\theta}{\sin\theta}}\, d\theta = \int_0^1 du \int_0^{\frac{\pi}{2}} \sin\theta^{-\frac{1}{2}} \cos\theta^{\frac{1}{2}}\, d\theta$$

$$= \int_0^1 du \frac{\left|\overline{\frac{-\frac{1}{2}+1}{2}}\right.}{\lceil 2} \frac{\left|\overline{\frac{\frac{1}{2}+1}{2}}\right.}{\lceil 2} = \frac{1}{2}\int_0^1 du \left|\overline{\frac{1}{4}}\right. \left|\overline{\frac{3}{4}}\right. = \frac{1}{2}\int_0^1 du \left[\frac{\sqrt{\pi}}{2^{\frac{-1}{2}}} \left|\overline{\frac{1}{2}}\right.\right]$$

$$= \frac{1}{2}\int_0^1 du\ \sqrt{2}\sqrt{\pi}\cdot\sqrt{\pi} = \frac{\pi}{\sqrt{2}}[\,u\,]_0^1 = \frac{\pi}{\sqrt{2}}$$ **Ans.**

Example 20. *Prove that*

$$\iint_D x^{l-1} y^{m-1}\, dx\, dy = \frac{\lceil l \lceil m}{\lceil l+m+1}\, h^{l+m}$$

where D is the domain $x \ge 0,\quad y \ge 0$ *and* $x + y \le h.$

Solution. Putting $x = Xh$ and $y = Yh$, $dx\,dy = h^2\, dX\, dY$

$$\iint_D x^{l-1} y^{m-1}\, dx\, dy = \iint_{D'} (Xh)^{l-1} (Yh)^{m-1} h^2\, dX\, dY$$

where D' is the domain

$X \ge 0,\quad Y \ge 0,\quad X + Y \le 1$

$$= h^{l+m} \int_0^1 \int_0^{1-X} X^{l-1} Y^{m-1}\, dX\, dY = h^{l+m} \int_0^1 X^{l-1}\, dX \int_0^{1-X} y^{m-1}\, dY$$

$$= h^{l+m} \int_0^1 X^{l-1}\, dX \left[\frac{Y^m}{m}\right]_0^{1-X} = \frac{h^{l+m}}{m}\int_0^1 X^{l-1}(1-X)^m\, dX$$

$$= \frac{h^{l+m}}{m}\,\beta(l,\ m+1) = \frac{h^{l+m}}{m}\,\frac{\lceil l \left|\overline{m+1}\right.}{\left|\overline{l+m+1}\right.}$$

$$= \frac{h^{l+m}}{m}\,\frac{m\lceil l \lceil m}{\left|\overline{l+m+1}\right.} = h^{l+m}\,\frac{\lceil l \lceil m}{\left|\overline{l+m+1}\right.}.$$ **Proved.**

Example 21. *Establish* ***Dirichlet's integral***

$$\iiint_V x^{l-1} y^{m-l} z^{n-l}\, dx\, dy\, dz = \frac{\lceil l \lceil m \lceil n}{\left|\overline{l+m+n+1}\right.}$$

where V is the region $x \ge 0,\ y \ge 0,\ z \ge 0$ *and* $x + y + z \le 1.$

Solution. Putting $y + z \le 1 - x = h.$ Then $z \le h - y$

$$\iiint_V x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = \int_0^1 x^{l-1}\, dx \int_0^{1-x} y^{m-1}\, dy \int_0^{1-x-y} z^{n-1}\, dz$$

$$= \int_0^1 x^{l-1}\, dx \left[\int_0^h \int_0^{h-y} y^{m-1} z^{n-1}\, dy\, dz\right]$$

$$= \int_0^1 x^{l-1}\, dx \left[\frac{\lceil m \lceil n}{\left|\overline{m+n+1}\right.}\, h^{m+n}\right]$$

$$= \frac{\Gamma m \, \Gamma n}{\Gamma(m+n+1)} \int_0^1 x^{l-1} \, (1-x)^{m+n} \, dx$$

$$= \frac{\Gamma m \, \Gamma n}{\Gamma(m+n+1)} \, \beta\,(l, \; m+n+1)$$

$$= \frac{\Gamma m \, \Gamma n}{\Gamma(m+n+1)} \, \frac{\Gamma l \, \Gamma(m+n+1)}{\Gamma(l+m+n+1)}$$

$$= \frac{\Gamma l \, \Gamma m \, \Gamma n}{\Gamma(l+m+n+1)}$$ **Proved.**

Note. $\iiint_V x^{l-1} \, y^{m-1} \, z^{n-1} \, dx \, dy \, dz = \frac{\Gamma l \, \Gamma m \, \Gamma n}{\Gamma(l+m+n+1)} \, h^{l+m+n}$

where V is the domain, $x \geq 0, \; y \geq 0, \; z \geq 0$ and $x + y + z \leq h$.

Example 22. *Find the mass of an octant of the ellipsoid* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, *the density at any point being* $\rho = k \, x \, y \, z$.

Solution. Mass $= \iiint \rho \, dv = \iiint (k \, x \, y \, z) \, dx \, dy \, dz$

$$= k \iiint (xdx)\,(ydy)\,(zdz) \qquad \text{... (1)}$$

Putting $\frac{x^2}{a^2} = u, \; \frac{y^2}{b^2} = v, \; \frac{z^2}{c^2} = w$ and $u + v + w = 1$

so that $\frac{2xdx}{a^2} = du, \quad \frac{2\,y\,dy}{b^2} = dv, \quad \frac{2\,z\,dz}{c^2} = dw$

$$\text{Mass} = k \iiint \left(\frac{a^2\,du}{2}\right)\left(\frac{b^2\,dv}{2}\right)\left(\frac{c^2\,dw}{2}\right)$$

$$= \frac{k\,a^2\,b^2\,c^2}{8} \iiint du\,dv\,dw \qquad \text{where} \qquad u + v + w \leq 1$$

$$= \frac{k\,a^2\,b^2\,c^2}{8} \iiint u^{1-1}\,v^{1-1}\,w^{1-1}\,du\,dv\,dw$$

$$= \frac{k\,a^2\,b^2\,c^2}{8} \, \frac{\Gamma 1 \, \Gamma 1 \, \Gamma 1}{\Gamma(3+1)} = \frac{k\,a^2\,b^2\,c^2}{8 \times 6}$$

$$= \frac{k\,a^2\,b^2\,c^2}{48}$$ **Ans.**

Example 23. *Show that*

$$\int_0^1 \frac{x^{m-1}\,(1-x)^{n-1}}{(a+x)^{m+n}} \, dx = \frac{\beta\,(m, n)}{a^n\,(1+a)^m}$$

Solution: Put $\frac{x}{a+x} = \frac{t}{a+1}$

$$(a+1)\,x = t\,(a+x) \quad \text{or} \quad x = \frac{at}{a+1-t}$$

$$dx = \frac{(a+1-t)\,a\,dt - at\,(-dt)}{(a+1-t)^2}$$

$$= \frac{(a^2 + a - at + at)}{(a + 1 - t)^2}\, dt = \frac{a\,(a+1)}{(a+1-t)^2}\, dt$$

$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}}\, dx = \int_0^1 \frac{\left(\dfrac{at}{a+1-t}\right)^{m-1} \cdot \left(1 - \dfrac{at}{a+1-t}\right)^{n-1}}{\left(a + \dfrac{at}{a+1-t}\right)^{m+n}}\, \frac{a\,(a+1)}{(a+1-t)^2}\, dt$$

$$= \int_0^1 \frac{(at)^{m-1}\,(a+1-t-at)^{n-1}}{(a^2 + a - at + at)^{m+n}}\, a\,(a+1)\, dt$$

$$= \int_0^1 \frac{a^{m-1}\, t^{m-1}\, (a+1)^{n-1}\, (1-t)^{n-1}}{a^{m+n}\,(a+1)^{m+n}}\, a\,(a+1)\, dt$$

$$= \frac{1}{a^n\,(a+1)^m} \int_0^1 t^{m-1}\,(1-t)^{n-1}\, dt$$

$$= \frac{1}{a^n\,(a+1)^m}\, \beta\,(m, n)$$ **Proved**

Exercise 21.2

Prove that

1. (*a*) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2\theta \cos^4\theta \, d\theta = \frac{\pi}{32}$ (*b*) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^6\theta \, d\theta = \frac{5\pi}{32}$

2. (*a*) $\beta\,(m+1, n) = \dfrac{m}{m+n}\,\beta\,(m, n)$ (*b*) $\beta\,(m, n+1) = \dfrac{n}{m+n}\,\beta\,(m, n)$

(*c*) $\beta\,(m+1, n) + \beta\,(m, n+1) = \beta\,(m, n)$

3. $\displaystyle\int_0^1 \sqrt{x}\,\sqrt[3]{1-x^2}\, dx = \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{4}{3}\right)}{2\,\Gamma\left(\frac{7}{12}\right)}$

4. $\displaystyle\int_0^1 (1-x^n)^{-\frac{1}{2}}\, dx = \frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{1}{2}\right)}{n\,\Gamma\left(\frac{n+2}{2n}\right)}$

5. $\displaystyle\int_0^1 (1 - x^{1/n})^m\, dx = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

6. $\displaystyle\int_1^\infty \frac{dx}{x^{p+1}\,(x-1)^q} = \beta\,(p+q, 1-q)$ if $-p < q < 1$

7. $\displaystyle\int_0^1 x^m\,(1-x^n)^p\, dx = \frac{1}{n}\,\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma(p+1)}{\Gamma\left(\frac{m+1}{n} + p + 1\right)}$

8. $\displaystyle\int_0^b (x-a)^m\,(b-x)^n\, dx = (b-a)^{m+n+1}\,.\,\beta\,(m+1, n+1)$

9. $\displaystyle\int_3^7 \sqrt[4]{(x-3)\,(7-x)}\, dx = \frac{2\left(\Gamma\left(\frac{1}{4}\right)\right)^2}{3\sqrt{\pi}}$ Put $x = 4t + 3$

10. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - \frac{1}{2}\sin^2\theta}} = \frac{\left(\Gamma\left(\frac{1}{4}\right)\right)^2}{4\sqrt{\pi}}$

11. If $\int_0^\infty e^{-x} x^{n-1}\, dx = \text{In for } n > 0$ find $\frac{I_{n+1}}{I_n}$ (*A.M.I.E., Summer 2000*) **Ans.** n

21.8 LIOUVILLE'S EXTENSION OF DIRICHLET THEOREM

If the variables ***x,y,z* are all positive such that $h_1 < x + y + z < h_2$, then**

$$\iiint f(x+y+z)\, x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = \frac{\Gamma l\, \Gamma m\, \Gamma n}{\Gamma(l+m+n)} \int_{h_1}^{h_2} f(u)\, u^{l+m+n-1}\, du$$

Proof Let $$I = \iiint x^{l-1}\, y^{m-1}\, z^{n-1}\, dx\, dy\, dz$$

under the condition $x+y+z \le u$ then

$$I = u^{l+m+n} \frac{\Gamma l + \Gamma m + \Gamma n}{\Gamma(1+l+m+n)} \quad \text{... (1) (By Dirichlet Th.)}$$

If $x+y+z \le u + \delta u$, then

$$I = (u+\delta u)^{l+m+n} \frac{\Gamma l\, \Gamma m\, \Gamma n}{\Gamma(1+l+m+n)} \quad \text{... (2)}$$

If $u < x+y+z+ < u+\delta u$, then

$$\iiint x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = \frac{\Gamma l\, \Gamma m\, \Gamma n}{\Gamma(1+l+m+n)} \left[(u+\delta u)^{p+m+n} - u^{l+m+n}\right]$$

$$= \frac{\Gamma l\, \Gamma m\, \Gamma n}{\Gamma(1+l+m+n)} u^{l+m+n} \left[1 + \left(\frac{\delta u}{u}\right)^{l+m+n} - 1\right]$$

$$= \frac{\Gamma l\, \Gamma m\, \Gamma n}{\Gamma(1+l+m+n)} u^{l+m+n} \left[1 + (l+m+n)\frac{\delta u}{u} + \ldots - 1\right]$$

$$= \frac{\Gamma l\, \Gamma m\, \Gamma n}{\Gamma(1+l+m+n)} u^{l+m+n} (l+m+n) \frac{\delta u}{u} = \frac{\Gamma l\, \Gamma m\, \Gamma n}{\Gamma(l+m+n)} u^{l+m+n-1}\, \delta u$$

Let us consider $$\iiint f(x+y+z)\, x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz$$

under the condition $h_1 \le x+y+z \le h_2$

When $x + y + z$ lies between u and $u + \delta u$, the value of $f(x+y+z)$ can only differ from $f(u)$ by a small quantity of the same order as δu. Hence, neglecting square of δu, the part of the integral

$$\iiint f(x+y+z)\, x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz$$

$$= \frac{\Gamma l\, \Gamma m\, \Gamma n}{\Gamma(1+l+m+n)} f(u)\, u^{l+m+n-1}\, \delta u$$

(supposing the sum of variables to be betwee*n* u and $u + \delta u$)

So $$\iiint f(x+y+z)\, x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = \frac{\Gamma l\, \Gamma m\, \Gamma n}{(1+l+m+n)} \int_{h_1}^{h_2} f(u)\, u^{l+m+n-1}\, du$$

Example 24. Show that $\iiint \frac{dx\, dy\, dz}{(x+y+z)^3} = \frac{1}{2}\log 2 - \frac{5}{16}$, the integral being taken throughout the volume bounded by planes $x + 1 = 0$, $y = 0$, $z = 0$, $x + y + z = 1$.

Solution. By Liouville's theorem when $0 < x+y+z < 1$

$$\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3} = \iiint \frac{x^{1-1}\,y^{1-1}\,z^{1-1}\,dx\,dy\,dz}{(x+y+z+1)^3} \qquad (0 \le x+y+z \le 1)$$

$$= \frac{\Gamma(1)\,\Gamma(1)\,\Gamma(1)}{\Gamma(1+1+1)} \int_0^1 \frac{1}{(u+1)^3}\, u^{3-1}\; du$$

$$= \frac{1}{2} \int_0^1 \frac{u^2}{(u+1)^3}\; du$$

$$= \frac{1}{2} \int_0^1 \left[\frac{1}{u+1} - \frac{2}{(u+1)^2} - \frac{1}{(u+1)^3}\right] du \quad \text{(Partial fractions)}$$

$$= \frac{1}{2}\left[\log(u+1) + \frac{2}{u+1} + \frac{1}{2\,(u+1)^2}\right]_0^1$$

$$= \left[\log 2 + 2\left(\frac{1}{2} - 1\right) - \left(\frac{1}{8} - \frac{1}{2}\right)\right] = \frac{1}{2}\log 2 - \frac{5}{16} \quad \textbf{Proved}$$

Example 25. Find the value of $\iiint \log(x+y+z)\, dx\, dy\, dz$ the integral extending over all positive values of x, y, z subject to the condition $x + y + z < 1$.

Solution. By Liouville's theorem when $0 < x + y + z < 1$

$$\iiint \log(x+y+z)\, dx\, dy\, dz$$

$$= \iiint \log(x+y+z)\, x^{1-1}\, y^{1-1}\, z^{1-1}\; dx\, dy\, dz$$

$$= \frac{\Gamma(1)\,\Gamma(1)\,\Gamma(1)}{\Gamma(1+1+1-1)} \int_0^1 \log u\;\, u^{1+1+1}\, du$$

$$= \frac{1}{\Gamma(3)} \int u^2 \log u\; du = \frac{1}{2}\left[\log u\, \frac{u^3}{3} - \frac{1}{3}\frac{u^3}{3}\right]_0^1$$

$$= \frac{1}{2}\left(-\frac{1}{9}\right) = -\frac{1}{18} \qquad \textbf{Ans.}$$

Exercise 21.3.

Evaluate:

1. $\iiint e^{x+y+z}\; dx\, dy\, dz$ taken over the positive octant such that $x + y + z \le 1$. **Ans.** $\dfrac{e-2}{2}$

2. $\iiint \dfrac{dx\, dy\, dz}{(a^2 - x^2 - y^2 - z^2)}$ for all positive values of the variables for which the expression is real.

Hint. $a^2 - x^2 - y^2 - z^2 > 0 \Rightarrow 0 < x^2 + y^2 + z^2 < a^2$ **Ans.** $\dfrac{\pi^2 a^2}{8}$

3. $\iiint_R (x + y\,z + 1)^2\, dx\, dy\, dz$ where R is defined by $x \ge 0 \quad y \ge 0 \quad z \ge 0, \quad x + y + z \le 1$ **Ans.** $\dfrac{31}{60}$

4. $\iiint x^{-\frac{1}{2}}\; y^{-\frac{1}{2}}\; z^{-\frac{1}{2}}\; (1 - x - y - z)^{\frac{1}{2}}\; dx\, dy\, dz,\; x + y + z \le 1 \quad x > 0,\; y > 0,\; z > 0$ **Ans.** $\dfrac{\pi^2}{4}$

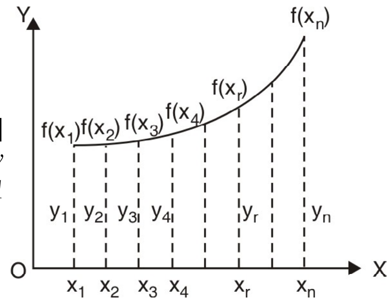
5. Evaluate $\int \int \int \dfrac{dx_1\, dx_2 \ldots\ldots\; dx_n}{\sqrt{1 - x_1^2 - x_2^2 \ldots\ldots - x_n^2}}$, integral being extended to all positive values of the variables for which the expression is real. (*U.P.., II Semester, Summer 2001*)

2

Multiple Integral

2.1 DOUBLE INTEGRATION

We know that



$$\int_a^b f(x)\,dx = \lim_{\substack{n\to\infty \\ \delta x\to 0}} [f(x_1)\,\delta x_1 + f(x_2)\,\delta x_2 + f(x_3)\,\delta x_3 + \ldots$$

$$+ f(x_n)\,\delta\, x_n]$$

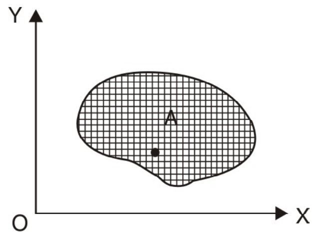
Let us consider a function $f(x, y)$ of two variable x and y defined in the finite region A of xy-plane. Divide the region A into elementary areas.

$$\delta A_1,\ \delta A_2,\ \delta A_3,\ \ldots\ldots\ \delta A_n$$

Then $$\iint_A f(x, y)\,dA$$

$$= \lim_{\substack{n\to\infty \\ \delta A\to 0}} \left[f(x_1, x_1)\,\delta A_1 + f(x_2, y_2)\,\delta A_2 + \ldots\ldots + f(x_n, y_n)\,\delta A_n \right]$$

2.2 EVALUATION OF DOUBLE INTEGRAL



Double integral over region A may be evaluated by two successive integrations.

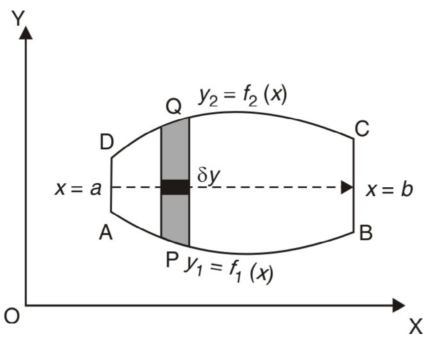
If A is described as $f_1\,(x) \le y \le f_2\,(x)$ $[y_1 \le y \le y_2]$

and $$a \le x \le b,$$

Then $$\int\int_A f(x, y)\,dA = \int_a^b \int_{y_1}^{y_2} f(x, y)\,dx\,dy$$

(1) First Method

$$\int\int_A f(x, y)\,dA = \int_a^b \boxed{\int_{y_1}^{y_2} f(x, y)\,dy}\,dx$$



$f(x, y)$ is first integratred with respect to y treating x as constant between the limits a and b.

In the region we take an elementary area $\delta x \delta y$. Then integration w.r.t y (x keeping constant). converts small rectangle $\delta x\delta y$ into a strip PQ ($y\ \delta x$). While the integration of the result w.r.t. x corresponding to the sliding to the strip PQ, from AD to BC covering the while region $ABCD$.

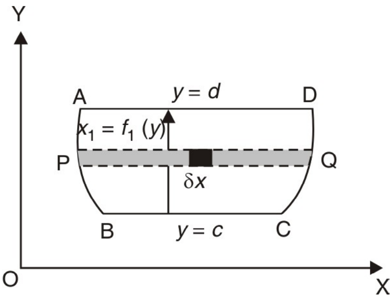
Second method

$$\int\int_A f(x, y)\,dxdy = \int_c^d \boxed{\int_{x_1}^{x_2} f\,(x, y)\,dx}\,dy$$

Here $f(x,y)$ is first integrated w.r.t x keeping y constant between the limits x_1 and x_2 and then the resulting expression is integrated with respect to y between the limits c and d

Take a small area $\delta x \delta y$. The integration w.r.t. x between the limits x_1, x_2 keeping y fixed indicates that integration is done, along PQ. Then the integration of result w.r.t y corresponds to sliding the strips PQ from BC to AD covering the whole region $ABCD$.

Note. For constant limits, it does not matter whether we first integrate w.r.t x and then w.r.t y or vice versa.



Example 1. *Evaluate* $\int_0^1 \int_0^x (x^2+y^2)\, dA$, *where dA indicates small area in xy-plane.*

(*Gujarat, I Semester, Jan. 2009*)

Solution. Let $$I = \int_0^1 \int_0^x (x^2+y^2)\, dy\, dx = \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_0^x dx$$

$$= \int_0^1 \left[x^2 (x-0) + \frac{1}{3}(x^3-0) \right] dx = \int_0^1 \left[x^3 + \frac{x^3}{3} \right] dx$$

$$= \int_0^1 \frac{4}{3} x^3\, dx = \frac{4}{3} \left[\frac{x^4}{4} \right]_0^1 = \frac{1}{3}[1-0] = \frac{1}{3} \text{ sq. units.}$$ **Ans.**

Example 2. *Evaluate* $\int_{-1}^{1} \int_0^{1-x} x^{1/3}\, y^{-1/2}\, (1-x-y)^{1/2}\, dy\, dx$. (*M.U., II Semester 2002*)

Solution. Here, we have

$$I = \int_{-1}^{1} \int_0^{1-x} x^{1/3}\, y^{-1/2}\, (1-x-y)^{1/2}\, dy\, dx \qquad ...(1)$$

Putting $(1-x) = c$ in (1), we get

$$I = \int_{-1}^{1} x^{1/3}\, dx \int_0^c y^{-1/2}\, (c-y)^{1/2}\, dy \qquad ...(2)$$

Again putting $y = ct \Rightarrow dy = c\, dt$ in (2), we get

$$I = \int_{-1}^{1} x^{\frac{1}{3}}\, dx \int_0^1 c^{-\frac{1}{2}}\, t^{-\frac{1}{2}}\, (c-ct)^{\frac{1}{2}}\, c\, dt$$

$$= \int_{-1}^{1} x^{1/3}\, dx \int_0^1 c^{-1/2}\, t^{-1/2}\, c^{1/2}\, (1-t)^{1/2}\, c\, dt$$

$$= \int_{-1}^{1} c\, x^{1/3}\, dx \int_0^c t^{-1/2}\, (1-t)^{1/2}\, dt = \int_{-1}^{1} c\, x^{1/3}\, dx \int_0^1 t^{1/2-1}\, (1-t)^{3/2-1}\, dt$$

$$= \int_{-1}^{1} c\, x^{\frac{1}{3}}\, dx\, \beta\left(\frac{1}{2}, \frac{3}{2}\right) \qquad \left[\int_0^1 x^{l-1}\, (1-x)^{m-1}\, dx = \beta(l, m) \right]$$

$$= \int_{-1}^{1} c\, x^{1/3}\, dx \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{3}{2}\right)} = \int_{-1}^{1} c\, x^{1/3}\, dx \frac{\Gamma\left(\frac{1}{2}\right) \cdot \frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)} = \int_{-1}^{1} c\, x^{1/3}\, dx \frac{\sqrt{\pi}\,\frac{1}{2}\sqrt{\pi}}{1}$$

$$= \int_{-1}^{1} c\, x^{1/3} \frac{\pi}{2}\, dx = \frac{\pi}{2} \int_{-1}^{1} x^{1/3} \cdot c\, dx$$

Putting the value of c, we get

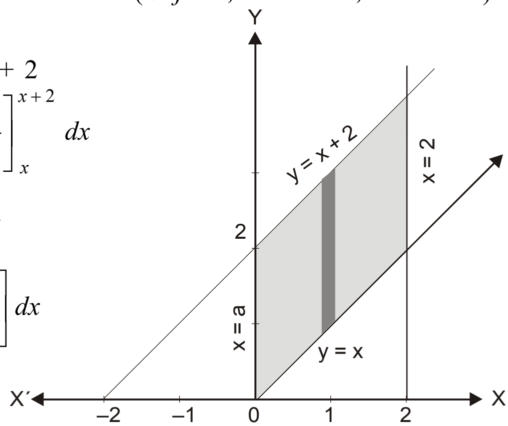
$$I = \frac{\pi}{2}\int_{-1}^{1} x^{1/3}(1-x)\,dx = \frac{\pi}{2}\int_{-1}^{1}(x^{1/3}-x^{4/3})\,dx = \frac{\pi}{2}\left[\frac{x^{4/3}}{\frac{4}{3}} - \frac{x^{7/3}}{\frac{7}{3}}\right]_{-1}^{1}$$

$$= \frac{\pi}{2}\left[\frac{3}{4}(1) - \frac{3}{7}(1) - \frac{3}{4}(-1) + \frac{3}{7}(-1)\right] = \frac{\pi}{2}\left[\frac{9}{14}\right] = \frac{9\pi}{28} \quad \textbf{Ans.}$$

Example 3. *Evaluate* $\iint_R (x+y)\,dy\,dx$, *R is the region bounded by $x = 0$, $x = 2$, $y = x$, $y = x + 2$.* (*Gujarat, I Semester, Jan. 2009*)

Solution. Let $I = \iint_R (x+y)\,dy\,dx$

The limits are $x = 0$, $x = 2$, $y = x$ and $y = x + 2$



$$I = \int_0^2 dx \int_x^{x+2}(x+y)\,dy = \int_0^2 \left[xy + \frac{y^2}{2}\right]_x^{x+2} dx$$

$$= \int_0^2 \left[x(x+2) + \frac{1}{2}(x+2)^2 - x^2 - \frac{x^2}{2}\right] dx$$

$$= \int_0^2 \left[x^2 + 2x + \frac{1}{2}(x^2+4x+4) - x^2 - \frac{x^2}{2}\right] dx$$

$$= \int_0^2 [2x + 2x + 2]\,dx$$

$$= 2\int_0^2 (2x+1)\,dx = 2[x^2 + x]_0^2 = 2\,[4 + 2] = 12 \quad \textbf{Ans.}$$

Example 4. *Evaluate* $\iint_R xy\,dx\,dy$

where R is the quadrant of the circle $x^2 + y^2 = a^2$ where $x \geq 0$ and $y \geq 0$.

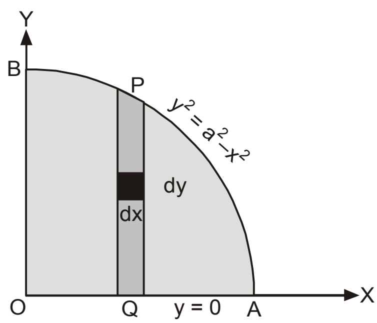
(*A.M.I.E.T.E, Summer 2004, 1999*)

Solution. Let the region of integration be the first quadrant of the circle OAB.

$$\iint_R xy\,dx\,dy \quad (x^2 + y^2 = a^2,\ y = \sqrt{a^2 - x^2})$$

First we integrate w.r.t. y and then w.r.t. x.

The limits for y are 0 and $\sqrt{a^2 - x^2}$ and for x, 0 to a.



$$= \int_0^a x\,dx \int_0^{\sqrt{a^2-x^2}} y\,dy = \int_0^a x\,dx \left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}}$$

$$= \frac{1}{2}\int_0^a x(a^2 - x^2)\,dx = \frac{1}{2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a = \frac{a^4}{8} \quad \textbf{Ans.}$$

Example 5. *Evaluate* $\iint_s \sqrt{xy - y^2}\,dy\,dx$,

where S is a triangle with vertices (0, 0), (10, 1) and (1, 1).

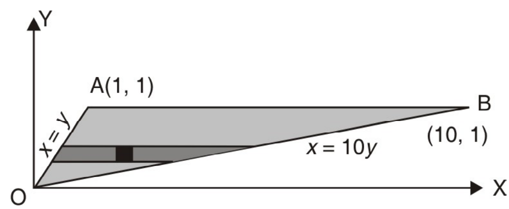
Solution. Let the verties of a triangle OBA be (0, 0) (10, 1) and (1, 1).

Equation of OA is $x = y$.

Equation of OB is $x = 10\,y$.

The region of ΔOBA, given by the limits

$y \leq x \leq 10\,y$ and $0 \leq y \leq 1$.



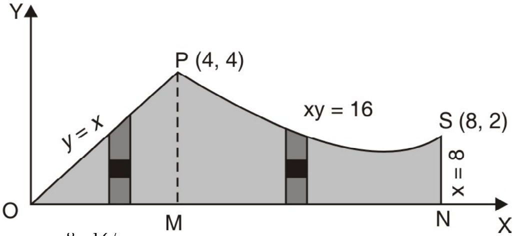
$$\iint_s \sqrt{xy - y^2}\,dy\,dx = \int_0^1 dy \int_y^{10y} (xy - y^2)^{½}\,dx$$

$$= \int_0^1 dy \left[\frac{2}{3}\frac{1}{y}(xy - y^2)^{3/2}\right]_y^{10y} = \int_0^1 \frac{2}{3}\frac{1}{y}(9y^2)^{3/2}\, dy = 18\int_0^1 y^2\, dy$$

$$= 18\left[\frac{y^3}{3}\right]_0^1 = \frac{18}{3} = 6$$ **Ans.**

Example 6. *Evaluate* $\iint_A x^2\, dx\, dy$, *where A is the region in the first quadrant bounded by the hyperbola xy = 16 and the lines y = x, y = 0 and x = 8.* (*A.M.I.E., Summer 2001*)

Solution. The line OP, $y = x$ and the curve PS, $xy = 16$ intersect at (4, 4).
The line SN, $x = 8$ intersects the hyperbola at S (8, 2). $y = 0$ is x-axis.
The area A is shown shaded.
Divide the area in to two part by PM perpendicular to OX.
For the area OMP, y varies from 0 to x, and then x varies from 0 to 4.
For the area PMNS, y -series from 0 to $16/x$ and then x varies from 4 to 8.



$$\therefore \quad \iint_A x^2\, dx\, dy = \int_0^4\int_0^x x^2\, dx\, dy + \int_4^8\int_0^{16/x} x^2\, dx\, dxy$$

$$= \int_0^4 x^2\, dx \int_0^x dy + \int_4^8 x^2\, dx \int_0^{16/x} dy = \int_0^4 x^2\, [y]_0^x\, dx + \int_4^8 x^2\, [y]_0^{16/x}\, dx$$

$$= \int_0^4 x^3\, dx + \int_4^8 16\, x\, dx = \left[\frac{x^4}{4}\right]_0^4 + 16\left[\frac{x^2}{2}\right]_4^8 = 64 + 8\,(8^2 - 4^2) = 64 + 384 = 448.$$ **Ans.**

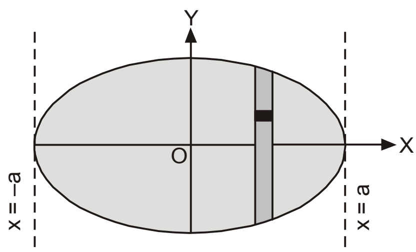
Example 7. *Evaluate* $\iint (x + y)^2\, dx\, dy$ *over the area bounded by the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
(*U.P. Ist Semester Compartment 2004*)

Solution. For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\Rightarrow \quad \frac{y}{b} = \pm\sqrt{1 - \frac{x^2}{a^2}} \quad \Rightarrow \quad y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$$

$\therefore$ The region of integration can be expressed as

$$-a \le x \le a \text{ and } -\frac{b}{a}\sqrt{a^2 - x^2} \le y \le \frac{b}{a}\sqrt{a^2 - x^2}$$



$$\therefore \quad \iint (x + y)^2\, dx\, dy = \iint (x^2 + y^2 + 2xy)\, dx\, dy$$

$$= \int_{-a}^{a}\int_{(-b/a)\sqrt{a^2 - x^2}}^{b/a\sqrt{a^2 - x^2}} (x^2 + y^2 + 2xy)\, dx\, dy$$

$$= \int_{-a}^{a}\int_{(-b/a)\sqrt{a^2 - x^2}}^{b/a\sqrt{a^2 - x^2}} (x^2 + y^2)\, dx\, dy \int_{-a}^{a}\int_{(-b/a)\sqrt{a^2 - x^2}}^{b/a\sqrt{a^2 - x^2}} 2xy\, dy\, dx$$

$$= \int_{-a}^{a}\int_{0}^{b/a\sqrt{a^2 - x^2}} 2\,(x^2 + y^2)\, dy\, dx + 0$$

[Since $(x^2 + y^2)$ is an even function of y and $2xy$ is an odd function of y]

$$= \int_{-a}^{a}\left[2\left(x^2 y + \frac{y^3}{3}\right)\right]_0^{\left(\frac{b}{a}\right)\sqrt{a^2 - x^2}} dx$$

$$= 2\int_{-a}^{a}\left[x^2 \times \frac{b}{a}\sqrt{a^2 - x^2} + \frac{1}{3}\frac{b^3}{a^3}(a^2 - x^2)^{3/2}\right]dx$$

$$= 4\int_{0}^{a}\left[\frac{b}{a}x^2\sqrt{a^2 - x^2} + \frac{b^3}{3a^3}(a^2 - x^2)^{3/2}\right]dx$$

[On putting $x = a \sin\theta$ and $dx = a\cos\theta\, d\theta$]

$$= 4\int_{0}^{\frac{\pi}{2}}\left(\frac{b}{a}.a^2\sin^2\theta.a\cos\theta + \frac{b^3}{3a^3}a^3\cos^3\theta\right)\times a\cos\theta\, d\theta$$

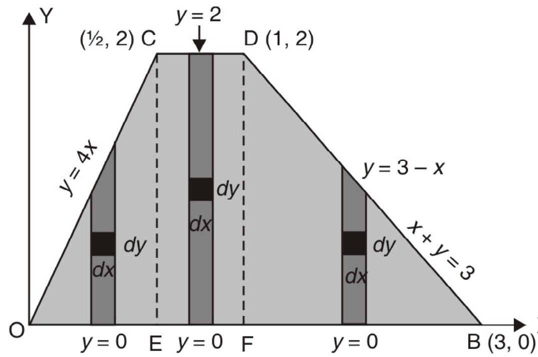
$$= 4\int_{0}^{\frac{\pi}{2}}\left(a^3 b\sin^2\theta\cos^2\theta + \frac{ab^3}{3}\cos^4\theta\right)d\theta = 4\left[a^3 b.\frac{1}{4}.\frac{1}{2}.\frac{\pi}{2} + \frac{ab^3}{3}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}\right]$$

$$= \frac{\pi}{4}(a^3 b + ab^3) = \frac{\pi}{4}ab(a^2 + b^2)$$ **Ans.**

Example 8. *Evaluate* $\iint (x^2 + y^2)\, dx\, dy$ *throughout the area enclosed by n the curves* $y = 4x$, $x + y = 3$, $y = 0$ *and* $y = 2$.

Solution. Let OC represent $y = 4x$; BD, $x + y = 3$; OB, $y = 0$, and CD, $y = 2$. The given integral is to be evaluated over the area A of the trapezium $OCDB$.

Area $OCDB$ consists of area OCE, area $ECDF$ and area FDB.



The co-ordinates of C, D and B are $\left(\frac{1}{2}, 2\right)$ (1, 2) and (3, 0) respectively.

$$\therefore \quad \iint_A (x^2 + y^2)\, dy\, dx$$

$$= \iint_{OCE}(x^2 + y^2)\,dy\,dx + \iint_{ECDE}(x^2 + y^2)\,dy\,dx + \iint_{FDB}(x^2 + y^2)\,dy\,dx$$

$$= \underbrace{\int_0^{½}dx\int_0^{4x}(x^2 + y^2)\,dy}_{I_1} + \underbrace{\int_{½}^{1}dx\int_0^{2}(x^2 + y^2)\,dy}_{I_2} + \underbrace{\int_1^{3}dx\int_0^{3-x}(x^2 + y^2)\,dy}_{I_3}$$

Now, $$I_1 = \int_0^{½}dx\int_0^{4x}(x^2 + y^2)\,dy = \int_0^{½}\left[x^2 y + \frac{y^3}{3}\right]_0^{4x}dx = \int_0^{½}\frac{76}{3}x^3\,dx$$

$$= \frac{76}{3}\int_0^{½}x^3\,dx = \frac{76}{3}\left[\frac{x^4}{4}\right]_0^{½} = \frac{76}{3}\left[\frac{1}{4}.\frac{1}{16}\right] = \frac{19}{48}$$

$$I_2 = \int_{½}^{1}dx\int_{½}^{1}(x^2 + y^2)\,dy = \int_{½}^{1}\left[x^2 y + \frac{y^3}{3}\right]_0^{2}dx = \int_{½}^{1}\left(2x^2 + \frac{8}{3}\right)dx$$

$$= \left[\frac{2x^3}{3} + \frac{8}{3}x\right]_{½}^{1} = \left[\left(\frac{2}{3} + \frac{8}{3}\right) - \left(\frac{2}{3}.\frac{1}{8} + \frac{8}{3}.\frac{1}{2}\right)\right] = \frac{23}{12}$$

$$I_3 = \int_1^{3}dx\int_0^{3-x}(x^2 + y^2)\,dy = \int_1^{3}\left[x^2 y + \frac{y^3}{3}\right]_0^{3-x}dx = \int_0^{3}\left[x^2(3 - x) + \frac{(3 - x)^3}{3}\right]dx$$

$$= \int_1^{3}\left[3x^2 - x^3 + \frac{(3 - x)^3}{3}\right]dx = \left[x^3 - \frac{x^4}{4} - \frac{(3 - x)^4}{3}\right]_1^{3}$$

$$= \left[27 - \frac{81}{4} - 0 - 1 + \frac{1}{4} + \frac{16}{12}\right] = \frac{22}{3}$$

$$\therefore \quad \int_A \int (x^2 + y^2)\, dy\, dx = I_1 + I_2 + I_3 = \frac{19}{48} + \frac{23}{12} + \frac{22}{3} = \frac{463}{48} = 9\frac{31}{48}. \qquad \textbf{Ans.}$$

EXERCISE 2.1

Evaluate

1. $\int_0^2 \int_0^{x^2} e^{\frac{y}{x}}\, dy\, dx$ **Ans.** $e^2 - 1$

2. $\int_0^a \int_0^{\sqrt{ay}} xy\, dx\, dy$ **Ans.** $\frac{a^4}{6}$

3. $\int_0^a \int_0^{\sqrt{a^2 - y^2}} dx\, dy$ **Ans.** $\frac{\pi a^2}{4}$

4. $\int_0^1 \int_{y^2}^{y} (1 + xy^2)\, dx\, dy$ **Ans.** $\frac{41}{210}$

5. $\int_0^{2a} \int_0^{\sqrt{2ax - x}} xy\, dy\, dx$ **Ans.** $\frac{2a^4}{3}$

6. $\int_0^{2a} \int_0^{\sqrt{2ax - x^2}} x^2\, dy\, dx$ **Ans.** $\frac{5\pi a^4}{8}$

7. $\int_0^a \int_0^{\sqrt{a^2 - x^2}} \sqrt{a^2 - x^2 - y^2}\, dy\, dx$ **Ans.** $\frac{\pi a^3}{4}$

8. $\int_0^1 \int_0^{\sqrt{\frac{1}{2}(1 - y^2)}} \frac{dx\, dy}{\sqrt{1 - x^2 - y^2}}$ **Ans.** $\frac{\pi}{4}$

9. $\int_0^a \int_0^{\sqrt{a^2 - x^2}} \frac{dx\, dy}{(1 + e^y)\sqrt{a^2 - x^2 - y^2}}$ **Ans.** $\frac{\pi}{2} \log \frac{2e^a}{1 + e^a}$

10. $\int_0^a \int_y^a \frac{x\, dx\, dy}{\sqrt{x^2 + y^2}}$ **Ans.** $\frac{a^2}{2} \log(\sqrt{2} + 1)$

11 $\int_{x=0}^{1} \int_{y=0}^{2} (x^2 + 3x\, y^2)\, dx\, dy$ (*A.M.I.E.T.E., June 2009*) **Ans.** $\frac{14}{3}$

12. $\iint_A (5 - 2x - y)\, dx\, dy$, where A is given by $y = 0$, $x + 2y = 3$, $x = y^2$. **Ans.** $\frac{217}{60}$

13. $\iint_A xy\, dx\, dy$, where A is given by $x^2 + y^2 - 2x = 0$, $y^2 = 2x$, $y = x$. **Ans.** $\frac{7}{12}$

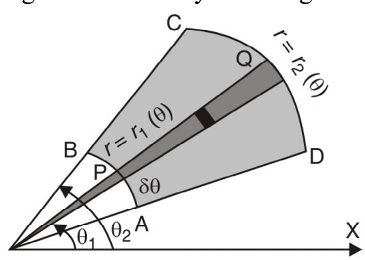
14. $\iint_A \sqrt{4x^2 - y^2}\, dx\, dy$, where A is the triangle given by $y = 0$, $y = x$ and $x = 1$. **Ans.** $\frac{1}{3}\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)$

15. $\iint_R x^2\, dx\, dy$, where R is the two-dimensional region bounded by the curves $y = x$ and $y = x^2$. **Ans.** $\frac{1}{20}$

16. $\iint_A \sqrt{xy(1 + x - y)}\, dx\, dy$ where A is the area bounded by $x = 0$, $y = 0$ and $x + y = 1$. **Ans.** $\frac{2\pi}{105}$

2.3 EVALUATION OF DOUBLE INTEGRALS IN POLAR CO-ORDINATES

We have to evalaute $\int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} f(r, \theta)\, dr\, d\theta$ over the region bounded by the staight lines $\theta = \theta_1$ and $\theta = \theta_2$ and the curves $r = r_1$ (q) and $r = r_2$ (θ). We first integrate with respect to r between the limits $r = r_1(\theta)$ and $r = r_2(\theta)$ and taking θ as constant. Then the resulting expression is integrated with respect to θ between the limits $\theta = \theta_1$ and $\theta = \theta_2$.

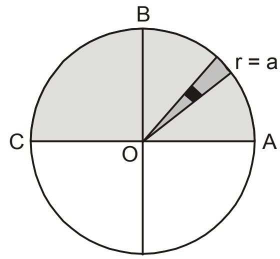


The area of integration is *ABCD*. On integrating first with respect to r, the strip extends from P to Q and the integration with respect to θ means the rotation ot this strip PQ from AD to BC.

Example 9. *Transform the integral to cartesian form and hence evaluate*

$\int_0^{\pi}\int_0^{a} r^3 \sin\theta \cos\theta \, dr \, d\theta.$ (*M.U., II Semester 2000*)

Solution. Here, we have



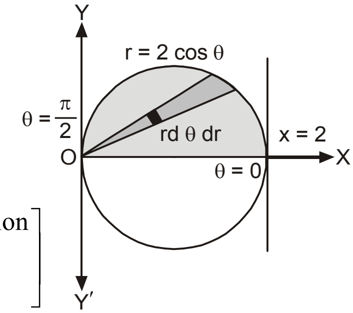
$$\int_0^{\pi}\int_0^{a} r^3 \sin\theta\cos\theta \, dr\, d\theta \quad ...(1)$$

Here the region *i.e.*, semicircle *ABC* of integration is bounded by $r = 0$, *i.e.*, x-axis.

$r = a$ *i.e.*, circle, $\theta = 0$ and $\theta = \pi$ *i.e.*, x-axis in the second quadrant.

$$\iint (r\sin\theta)(r\cos\theta)(r\, d\theta\, dr)$$

Putting $x = r\cos\theta$, $y = r\sin\theta$, $dx\, dy = r\, d\theta\, dr$ in (1), we get



$$\int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} xy\, dy\, dx = \int_{-a}^{a} x\, dx \int_0^{\sqrt{a^2-x^2}} y\, dy$$

$$= \int_{-a}^{a} x\, dx \left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} = \int_{-a}^{a} x\, dx\, \frac{(a^2-x^2)}{2}$$

$$= \frac{1}{2}\int_{-a}^{a} (a^2 x - x^3)\, dx = 0 \textbf{ Ans.} \quad \left[\begin{array}{l}\text{Since } f(x) \text{ is odd function} \\ \int_{-a}^{a} f(x)\, dx = 0\end{array}\right]$$

Example 10. *Evaluate* $\int_0^2\int_0^{\sqrt{2x-x^2}} (x^2+y^2)\, dy\, dx$

Solution. $\int_0^2\int_0^{\sqrt{2x-x^2}} (x^2+y^2)\, dy\, dx$

Limits of $\quad y = \sqrt{2x-x^2} \Rightarrow y^2 = 2x - x^2 \Rightarrow x^2 + y^2 - 2x = 0 \quad ...(1)$

(1) represents a circle whose centre is (1, 0) and radius = 1.

Lower limit of y is 0 *i.e., x*-axis.

Region of integration is upper half circle.

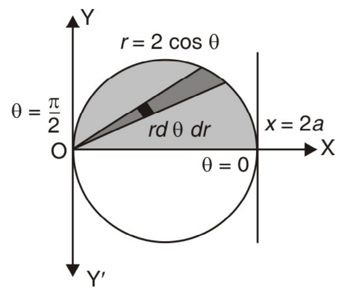
Let us convert (1) into polar co-ordinate by putting

$x = r\cos\theta$, $y = r\sin\theta$

$r^2 - 2r\cos\theta = 0 \Rightarrow r = 2\cos\theta$

Limits of r are 0 to $2\cos\theta$

Limits of θ are 0 to $\frac{\pi}{2}$



$$\int_0^2\int_0^{\sqrt{2x-x^2}} (x^2+y^2)\, dy\, dx = \int_0^{\frac{\pi}{2}}\int_0^{2\cos\theta} r^2\,(r\, d\theta\, dr) = \int_0^{\frac{\pi}{2}} d\theta \int_0^{2\cos\theta} r^3\, dr = \int_0^{\frac{\pi}{2}} d\theta \left[\frac{r^4}{4}\right]_0^{2\cos\theta}$$

$$= 4\int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta = 4\times\frac{3\times 1\times\pi}{4\times 2\times 2} = \frac{3\pi}{4} \quad \textbf{Ans.}$$

Example 11. *Evaluate* $\int_0^2\int_0^{\sqrt{2x-x^2}} \frac{x\, dy\, dx}{\sqrt{x^2+y^2}}$ *by changing to polar coordinates.*

Solution. In the given integral, y varies from 0 to $\sqrt{2x-x^2}$ and x varies from 0 to 2.

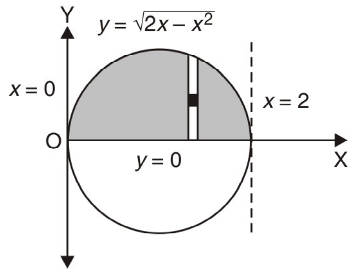
$$y = \sqrt{2x-x^2}$$

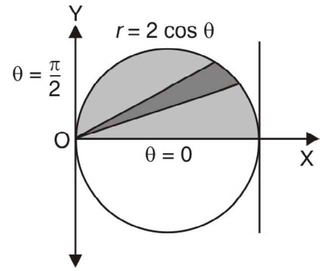
$$\Rightarrow \qquad y^2 = 2x - x^2$$
$$\Rightarrow \qquad x^2 + y^2 = 2x$$

In polar co-ordinates, we have $r^2 = 2r\cos\theta \quad \Rightarrow \quad r = 2\cos\theta$.

$\therefore$ For the region of integarion, r varies from 0 to $2\cos\theta$ and θ varies from 0 to $\frac{\pi}{2}$.

In the given integral, replacing x by $r\cos\theta$, y by $r\sin\theta$, $dy\,dx$ by $r\,dr\,d\theta$, we have





$$I = \int_0^{\pi/2}\int_0^{2\cos\theta} \frac{r\cos\theta .\, r\,dr\,d\theta}{r} = \int_0^{\pi/2}\int_0^{2\cos\theta} r\cos\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2}\cos\theta\left[\frac{r^2}{2}\right]_0^{2\cos\theta} d\theta = \int_0^{\pi/2} 2\cos^3\theta\,d\theta = 2.\frac{2}{3} = \frac{4}{3}. \qquad \textbf{Ans.}$$

EXERCISE 2.2

Evaluate the folloing:

1. $\int_0^{\pi}\int_0^{a(1-\cos\theta)} 2\pi r^2 \sin\theta\,d\theta\,dr$ **Ans.** $\frac{8}{3}\pi a^3$

2. $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2\cos\theta\,dr\,d\theta$ **Ans.** $\frac{5}{8}\pi a^3$

3. $\iint_A \frac{r\,dr\,d\theta}{\sqrt{r^2+a^2}}$ where A is a loop of $r^2 = a^2\cos 2\theta$ **Ans.** $2a - \frac{\pi a}{2}$

4. $\iint_A r^2\sin\theta\,d\theta\,dr$ where A is $r = 2a\cos\theta$ above initial line. (*A.M.I.E. Winter 2001*) **Ans.** $\frac{2a^3}{3}$

5. Calculate the integral $\iint \frac{(x-y)^2}{x^2+y^2}\,dx\,dy$ over the circle $x^2 + y^2 \le 1$. **Ans.** $\pi - 2$

6. $\iint (x^2 + y^2)\,x\,dx\,dy$ over the positive quadrant of the circle $x^2 + y^2 = a^2$ by changing to polar coordinates.

Ans. $\frac{a^2}{5}$

7. $\iint_R \sqrt{x^2+y^2}\,dx\,dy$ by changing to polar coordinates, R is the region in the xy-plane bounded by the circles $x^2 + y^2 = 4$ (*AMIETE, Dec. 2009*) **Ans.** $\frac{38\pi}{3}$

8. Convert into polar coordinates

$$\int_0^{2a}\int_0^{2ax-x^2} dx\,dy$$ **Ans.** $\int_0^{\pi/2}\int_0^{2a\cos\theta} r\,d\theta\,dr$

9. $\iint r^3\,dr\,d\theta$, over the area bounded between the circles $r = 2b\cos\theta$ and $r = 2b\cos\theta$. **Ans.** $\frac{3\pi}{2}(a^4 - b^4)$

10. $\iint r\sin\theta\,dr\,d\theta$ over the area of the cardiod $r = a(1 + \cos\theta)$ above the initial line. **Ans.** $\frac{5}{8}\pi a^3$

11. $\int\int_A x^2\, dr\, d\theta$, where A is the area between the circles $r = a\cos\theta$ and $r = 2a\cos\theta$. **Ans.** $\dfrac{28a^3}{9}$

12. Transform the integral $\int_0^1 \int_0^x f(x, y)\, dy\, dx$ to the integral in polar co-ordinates.

Ans. $\int_0^{\pi/4} \int_0^{\sec\theta} f(r, \theta)\, r\, d\theta\, dr$

2.4 CHANGE OF ORDER OF INTEGRATION

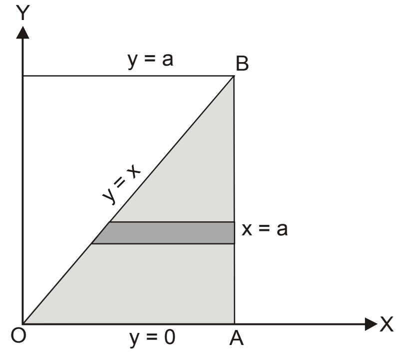
On changing the order of integration, the limits of integration change. To find the new limits, we draw the rough sketch of the region of integration.

Some of the problems connected with double integrals, which seem to be complicated, can be made easy to handle by a change in the order of integration.

Example 12. *Evaluate* $\int_0^a \int_y^a \dfrac{x}{x^2 + y^2}\, dx\, dy$ *by changing the order of integration.*

(*AMIETE, June 2010, Nagpur University, Summer 2008*)

Solution. Here we have



$$I = \int_0^a \int_y^a \frac{x}{x^2 + y^2}\, dx\, dy$$

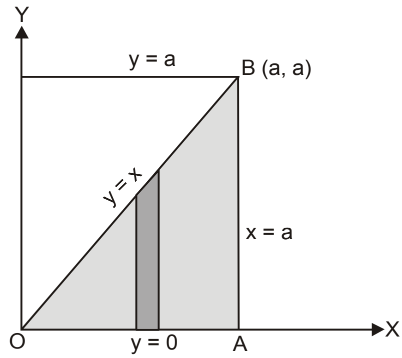
Here $\quad x = a, x = y, y = 0$ and $y = a$

The area of integration is OAB.

On changing the order of integration Lower limit of $y = 0$ and

upper limit is $y = x$.

Lower limit of $x = 0$ and upper limit is $x = a$.



$$I = \int_0^a x\,dx \int_0^{y=x} \frac{1}{x^2 + y^2}\, dy$$

$$= \int_0^a x\,dx \left[\frac{1}{x}\tan^{-1}\frac{y}{x}\right]_0^{y=x}$$

$$= \int_0^a \frac{x}{x}\,dx \left(\tan^{-1}\frac{x}{x} - \tan^{-1} 0\right)$$

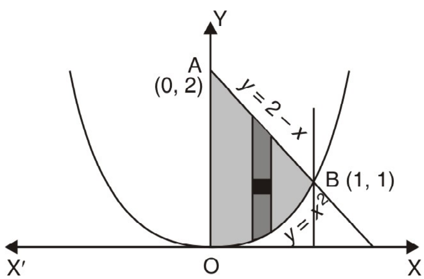
$$= \int_0^a dx \left(\frac{\pi}{4}\right) \quad = \frac{\pi}{4}[x]_0^a = \frac{a\pi}{4} \quad \textbf{Ans.}$$

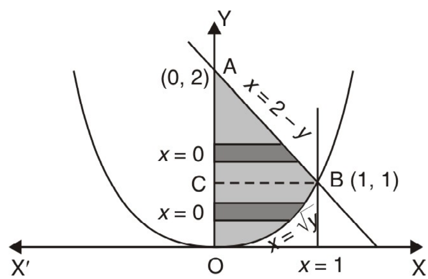
Example 13. *Change the order of integration in*

$$I = \int_0^1 \int_{x^2}^{2-x} xy\, dx\, dy \text{ and hence evaluate the same.}$$

(*A.M.I.E.T.E., June 2010, 2009, U.P. I Sem., Dec., 2004*)

Solution. $I = \int_0^1 \int_{x^2}^{2-x} xy\, dx\, dy$





The region of integration is shown by shaded portion in the figure bounded by parabola $y = x^2$ and the line $y = 2 - x$.
The point of intersection of the parabola $y = x^2$ and the line $y = 2 - x$ is B (1, 1).
In the figure below (left) we have taken a strip parallel to y-axis and the order of integration is

$$\int_0^1 x\,dx \int_{x^2}^{2-x} y\,dy$$

In the second figure above we have taken a strip parallel to x-axis in the area OBC and second strip in the area ABC. The limits of x in the area OBC are 0 and $\sqrt{y}$ and the limits of x in the area ABC are 0 and $2 - y$.

$$= \int_0^1 y\,dy \int_0^{\sqrt{y}} x\,dx + \int_1^2 y\,dx \int_0^{2-y} x\,dx = \int_0^1 y\,dy \left[\frac{x^2}{2}\right]_0^{\sqrt{y}} + \int_0^{\sqrt{y}} y\,dy \left[\frac{x^2}{2}\right]_0^{2-y}$$

$$= \frac{1}{2}\int_0^1 y^2\,dy + \frac{1}{2}\int_1^2 y\,(2-y)^2\,dy = \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 + \frac{1}{2}\int_1^2 (4y - 4y^2 + y^3)\,dy$$

$$= \frac{1}{6} + \frac{1}{2}\left[2y^2 - \frac{4}{3}y^3 + \frac{y^4}{4}\right]_1^2 = \frac{1}{6} + \frac{1}{2}\left[8 - \frac{32}{3} + 4 - 2 + \frac{4}{3} - \frac{1}{4}\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left[\frac{96 - 128 + 48 - 24 + 16 - 3}{12}\right] = \frac{1}{6} + \frac{5}{24} = \frac{9}{24} = \frac{3}{8}$$ **Ans.**

Example 14. *Evaluate the integral* $\int_0^\infty \int_0^x x \exp\left(-\frac{x^2}{y}\right) dx\,dy$ *by changing the order of integration* (*U.P. I Semester Dec., 2005*)

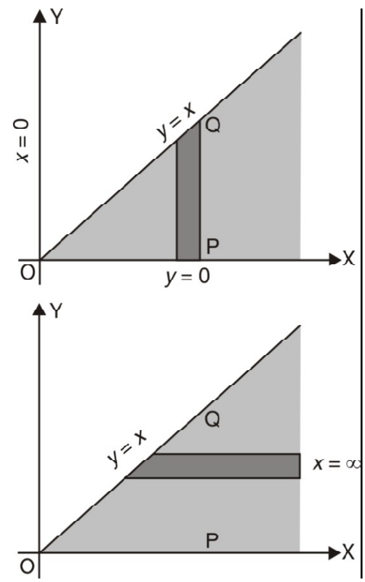
Solution. Limits are given

$y = 0$ and $y = x$
$x = 0$ and $x = \infty$

Here, the elementary strip PQ extends from $y = 0$ to $y = x$ and this vertical strip slides from $x = 0$ to $x = \infty$.

The region of integration is shown by shaded portion in the figure bounded by $y = 0$, $y = x$, $x = 0$ and $x = \infty$.

On changing the order of integration, we first integrate with respect to x along a horizontal strip RS which extends from $x = y$ to $x = \infty$ and this horizontal strip slides from $y = 0$ to $y = \infty$ to cover the given region of integration.



New limits :

$x = y$ and $x = \infty$
$y = 0$ and $y = \infty$

We first integrate with respect to x.
Thus,

$$\int_0^\infty dy \int_y^\infty x e^{-\frac{x^2}{y}}\,dx = \int_0^\infty dy \int_y^\infty -\frac{y}{2}\left(-\frac{2x}{y} e^{-\frac{x^2}{y}}\right) dx$$

$$= \int_0^\infty dy \left[-\frac{y}{2} e^{-\frac{x^2}{y}}\right]_y^\infty = \int_0^\infty dy \left[0 + \frac{y}{2} e^{-\frac{y^2}{2}}\right] = \int_0^\infty \frac{y}{2} e^{-y}\,dy$$

$$= \left[\frac{y}{2}(-e^{-y}) - \left(\frac{1}{2}\right)(e^{-y})\right]_0^{\infty}$$ (Integrating by parts)

$$= \left[(0-0) - \left(0 - \frac{1}{2}\right)\right] = \frac{1}{2}$$ **Ans.**

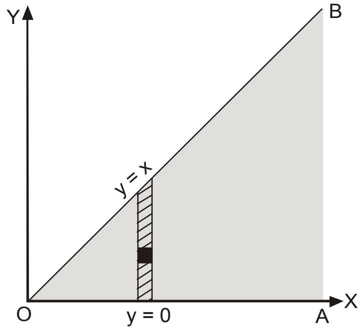
Example 15. *Change the order of the integration*

$$\int_0^{\infty}\int_0^{x} e^{-xy}\ y\,dy\,dx$$ (*B.P.U.T.; I Semester 2008*)

Solution. Here, we have

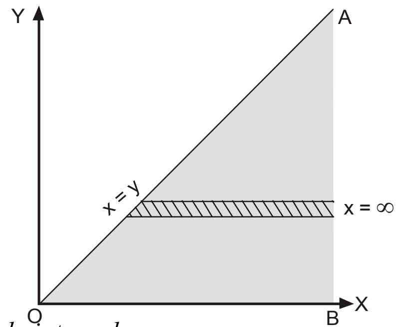
$$\int_0^{\infty}\int_0^{x} e^{-xy}\ y\,dy\,dx$$

Here the region OAB of integration is bounded by $y = 0$ (x-axis), $y = x$ (a straight line), $x = 0$, *i.e.*, y axis. A strip is drawn parallel to y-axis, y varies 0 to x and x varies 0 to ∞.



On changing the order of integration, first we integrate w.r.t. x and then w.r.t. y.

A strip is drawn parallel to x-axis. On this strip x varies from y to ∞ and y varies from 0 to ∞.



Hence $\displaystyle\int_0^{\infty}\int_0^{x} e^{-xy}\ y\,dy\,dx = \int_0^{\infty} y\,dy\int_y^{\infty} e^{-xy}dx$

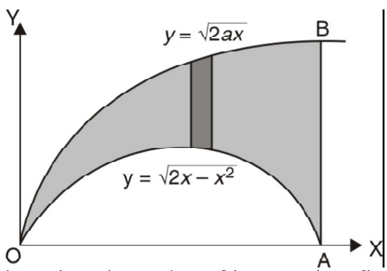
$$= \int_0^{\infty} y\,dy\left(\frac{e^{-xy}}{-y}\right)_y^{\infty}$$

$$= \int_0^{\infty}\frac{y\,dy}{-y}\left[0 - e^{y^2}\right]$$

$$= \int_0^{\infty} e^{-y^2}\,dy = \frac{1}{2}\sqrt{\pi}$$ **Ans.**

Example 16. *Change the order of integration in the double integral*

$$\int_0^{2a}\int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} V\,dx\,dy$$



Solution. Limits are given as

$$x = 0,\ x = 2a$$

$$y = \sqrt{2ax}$$

and $\quad y = \sqrt{2ax - x^2} \quad \Rightarrow \quad y^2 = 2ax$

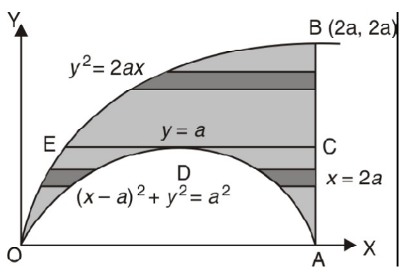
and $\quad (x-a)^2 + y^2 = a^2$

The area of integration is the shaded portion OAB. On changing the order of integration first we have to integrate w.r.t. x, The area of integration has three portions BCE, ODE and ACD.

$$\int_0^{2a} dx\int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} V\,dy$$

$$= \int_0^{2a} dy\int_{y^2/2a}^{2a} V\,dx + \int_0^{a} dy\int_{y^2/2a}^{a+\sqrt{a^2+y^2}} V\,dx$$

$$+ \int_0^{a} dy\int_{a+\sqrt{a^2-y^2}}^{2a} V\,dx$$ **Ans.**



EXERCISE 2.3

Change the order of integration and hence evaluate the following:

1. $\int_0^a \int_0^x \frac{\cos y\, dy}{\sqrt{(a-x)(a-y)}}\, dx$ **Ans.** $(a) \int_0^a dy \int_y^a \frac{\cos y\, dx}{\sqrt{(a-x)(a-y)}}$ $(b)\ 2 \sin a$.

2. $\int_0^{2a} \int_{\frac{x^2}{4a}}^{3a-x} (x^2+y^2)\, dy\, dx$ **Ans.** $(a) \int_0^a dy \int_0^{2\sqrt{ay}} (x^2+y^2)\, dx + \int_a^{3a} dy \int_0^{3a-y} (x^2+y^2)\, dx$ $(b)\ \frac{314\, a^4}{35}$.

3. $\int_0^1 \int_{x^2}^{x} (x^2+y^2)^{-1/2}\, dy\, dx$ **Ans.** $\int_0^1 dy \int_y^{\sqrt{y}} (x^2+y^2)^{-1/2}\, dx.$

4. $\int_0^a \int_{\sqrt{a^2-y^2}}^{y+a} f(x,y)\, dx\, dy$ **Ans.** $\int_0^a dx \int_{\sqrt{a^2-x^2}}^{a} f(x,y)\, dy + \int_a^{2a} dx \int_{x-a}^{a} f(x,y)\, dy$

5. $\int_{-a}^{a} \int_0^{\sqrt{a^2-y^2}} f(x,y)\, dx\, dy$ **Ans.** $\int_0^a dx \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x,y)\, dy$

6. $\int_0^1 \int_x^{2-x} \frac{x}{y}\, dy\, dx$ **Ans.** $\int_0^a \frac{dy}{y} \int_0^y x dx + \int_1^2 \frac{dy}{y} \int_0^{2-y} x dx;\ \log \frac{4}{e}$

7. $\int_0^b \int_y^a \frac{x\, dy\, dx}{x^2+y^2}$ (*M.P. 2003*)

8. $\int_0^a \int_0^{bx/a} x\, dy\, dx$ **Ans.** $(a) \int_0^b dy \int_{ay/b}^{a} x\, dx$ $(b)\ \frac{1}{3} a^2 b$

9. $\int_0^5 \int_{2-x}^{2+x} f(x,y)\, dx\, dy$ **Ans.** $\int_0^2 dy \int_{2-y}^{5} f(x,y)\, dx + \int_2^7 dy \int_{y-2}^{5} f(x,y)\, dx$

10. $\int_0^\infty \int_{-y}^{y} (y^2-x^2)\, e^{-y}\, dx\, dy$ **Ans.** $\int_{-\infty}^{\infty} dx \int_{-x}^{x} (y^2-x^2)\, e^{-y}\, dy$ (*A.M.I.E., Summer 2000*)

11. $\int_{y=0}^{1} \int_{x=\sqrt{y}}^{2-y} \times yx\, dy$ (*A.M.I.E.T.E., June 2009*)

12. $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy\, dx\, dy$ (*U.P. I Semester, Dec., 2007*) **Ans.** $\int_0^a \int_0^{\sqrt{ay}} xy\, dx\, dy + \int_0^{2a-y} xy\, dx\, dy,\ \frac{3a^2}{8}$

13. $\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} xy\, dx\, dy$ **Ans.** $\int_0^{2a} x\, dx \int_0^{\sqrt{a^2-(x-a)^2}} y\, dy,\ \frac{2}{3} a^4$

[**Hint:** Put $x = a \sin^2 \theta \Rightarrow dx = 2\, a \sin\theta \cos\theta\, d\, \theta$]

14. $\int_0^1 \int_{-1}^{1-y} x^{1/3}\, y^{-1/2}\, (1-x-y)^{1/2}\, dx\, dy$ **Ans.** $\int_{-1}^{1} x^{\frac{1}{3}}\, dx \int_0^{1-x} y^{-\frac{1}{2}}\, (1-x-y)^{\frac{1}{2}}\, dy,\ -\frac{3\pi}{7}$

15. $\int_0^{2a} dx \int_0^{\frac{x^2}{4a}} (x+y)^3\, dy$ **Ans.** $\int_0^a dy \int_{\sqrt{4a\, y}}^{2a} (x+y)^3\, dx$

16. $\int_0^1 \int_0^y (x^2+y^2)\, dx\, dy + \int_1^2 \int_0^{2-y} (x^2+y^2)\, dx\, dy$ **Ans.** $\int_0^1 dx \int_x^{2-x} (x^2+y^2)\, dy,\ \frac{5}{3}$

17. $\int_0^a \int_0^{\sqrt{a^2-x^2}} y^2 \sqrt{x^2+y^2}\, dx\, dy$ by changing into polar coordinates. **Ans.** $\frac{\pi\, a^5}{20}$

(*U.P., I Semester, Dec. 2007, A.M.I.E., Summer 2001*)

18. $\int_0^1 \int_1^2 \frac{1}{x^2+y^2}\, dx\, dy + \int_0^2 \int_y^2 \frac{1}{x^2+y^2}\, dx\, dy = \int_R \int \frac{1}{x^2+y^2}\, dy\, dx$

Recognise the region R of integration on the R.H.S. and then evaluate the integral on the right in the order indicated. (*AMIETE, Dec. 2004*)

Ans. Region R is $x = 0$, $x = y$, $y = 1$ and $y = 2$, $\frac{\pi}{4} \log 2$.

19. Express as single integral and evaluate :

$$\int_0^{\frac{a}{\sqrt{2}}} \int_0^x x\, dx\, dy + \int_{\frac{a}{\sqrt{2}}}^{a} \int_0^{\sqrt{a^2-x^2}} x\, dx\, dy$$

Ans. $\int_0^{\frac{a}{\sqrt{2}}} dy \int_y^{\sqrt{a^2-y^2}} x\, dx,\ \frac{5a^3}{6\sqrt{2}}$

20. Express as single integral and evaluate :

$$\int_0^1 \int_0^y (x^2 + y^2)\, dx\, dy + \int_1^2 \int_0^{2-y} (x^2 + y^2)\, dx\, dy \qquad \textbf{Ans. } \int_0^1 dx \int_x^{2-x} (x^2 + y^2)\, dy, \frac{5}{3}$$

21. If $f(x, y)\, dx\, dy$, where R is the circle $x^2 + y^2 = a^2$, is R equivalent to the repeated integral.

(*AMIE winter 2001*) [**Ans.** $\int_0^{2\pi} \int_0^1 (r, \theta)\, r\, dr\, d\theta.$]

2.5 CHANGE OF VARIABLES

Sometimes the problems of double integration can be solved easily by change of independent variables. Let the double integral as be $\iint_R f(x, y)\, dx\, dy$. It is to be changed by the new variables u, v.

The relation of x, y with u, v are given as $x = f(u, v)$, $y = \Psi(u, v)$. Then the double integration is converted into.

$\iint_{R'} f\, \{\phi(u, v), \Psi(u, v)\}\, |J|\, du\, dv$, where

$$dx\, dy = |J|\, du\, dv = \frac{\partial(x, y)}{\partial(u, v)}\, du\, dv = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} du\, dv$$

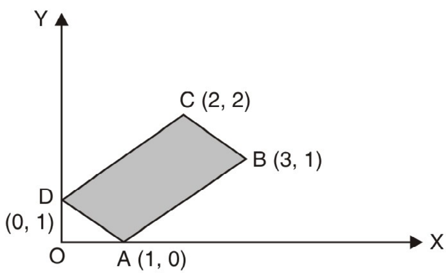
Example 17. *Evaluate* $\iint_R (x + y)^2\, dx\, dy$, *where R is the parallelogram in the xy-plane with vertices (1, 0), (3, 1), (2, 2), (0, 1), using the transformation u = x + y and v = x – 2y.*

(*U.P., I Semester, 2003*)

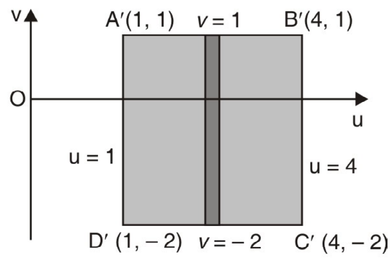
Solution. The region of integration is a parallelogram $ABCD$, where A (1, 0), B (3, 1), C (2, 2) and D (0, 1) in xy-plane.

The new region of integration is a rectangle $A'B'C'D'$ in uv-plane

xy-plane	$A \equiv (x, y)$ $A \equiv (1, 0)$	$B \equiv (x, y)$ $B \equiv (3, 1)$	$C \equiv (x, y)$ $C \equiv (2, 2)$	$D \equiv (x, y)$ $D \equiv (0, 1)$
uv-plane	$A' \equiv (u, v)$ $A' \equiv (x + y, x - 2y)$ $A' \equiv (1 + 0, 1 - 2 \times 0)$ $A' \equiv (1, 1)$	$B' \equiv (u, v)$ $B' \equiv (x + y, x - 2y)$ $B' \equiv (3 + 1, 3 - 2 \times 1)$ $B' \equiv (4, 1)$	$C' \equiv (u, v)$ $C' \equiv (u, v)$ $C' \equiv (2 + 2, 2 - 2 \times 2)$ $C' \equiv (4, -2)$	$D' \equiv (u, v)$ $D' \equiv (0 + 1, 0 - 2 \times 1)$ $D' \equiv (1, -2)$



(xy-plane)



(uv-plane)

and $\left\{\begin{array}{l} u = x + y \\ v = x - 2y \end{array}\right\} \Rightarrow x = \frac{1}{3}(2u + v)$ and $y = \frac{1}{3}(u - v)$

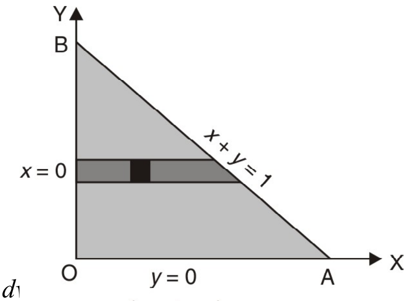
$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix} = -\frac{1}{3}$$

$$dx\,dy = |J|\,du\,dv = \frac{1}{3}\,du\,dv$$

$$\iint_R (x+y)^2\,dx\,dy = \int_{-2}^{1}\int_{1}^{4} u^2.\frac{1}{3}\,du\,dv = \int_{-2}^{1}\frac{1}{3}\left[\frac{u^3}{3}\right]_1^4 dv = \int_{-2}^{1} 7\,dv = 7\,[v]_2^1 = 7\times 3 = 21 \textbf{ Ans.}$$

Example 18. *Using the transformation $x + y = u$, $y = uv$, show that*

$$\iint [xy\,(1-x-y)]^{1/2}\,dx\,dy = \frac{2\pi}{105},$$ *integration being taken over the area of the tringle bounded by the lines $x = 0$, $y = 0$, $x + y = 1$.*



(xy-plane)

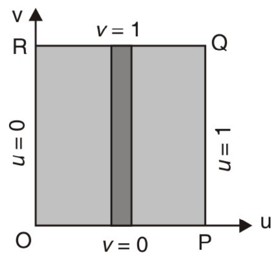
Solution. $\iint [xy\,(1-x-y)]^{1/2}\,dx\,dy$

$$x + y = u \text{ or } x = u - y = u - uv,$$

$$dx\,dy = \frac{\partial(x, y)}{\partial(u, v)}\,du\,dv = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} du\,dv$$

$$dx\,dy = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} du\,dv = u\,du\,dv$$

$$\begin{aligned} x = 0 \quad &\Rightarrow \quad u(1-v) = 0 \\ &\Rightarrow \quad u = 0,\ v = 1 \\ y = 0 \quad &\Rightarrow \quad uv = 0 \\ &\Rightarrow \quad u = 0,\ v = 0 \\ x + y = 1 \quad &\Rightarrow \quad u = 1 \end{aligned}$$



(uv-plane)

Hence, the limits of u are from 0 to 1 and the limits of v are from 0 to 1.

The area of integration is a square $OPQR$ in uv-plane.

On putting $x = u - uv$, $y = uv$, $dx\,dy = u\,du\,dv$ in (1), we get

$$\iint (u - uv)^{1/2}\,(uv)^{1/2}\,(1-v)^{1/2}\,u\,du\,dv$$

$$= \int_0^1 u^2\,(1-u)^{1/2}\,du \int_0^1 v^{1/2}\,(1-v)^{1/2}\,dv = \frac{\Gamma 3}{\Gamma\frac{9}{2}} \times \frac{\Gamma\frac{3}{2}\,\Gamma\frac{3}{2}\,\Gamma\frac{3}{2}}{\Gamma 5}$$

$$= \frac{2.\Gamma\frac{3}{2}}{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\Gamma\frac{3}{2}} \times \frac{\frac{1}{2}\Gamma\frac{1}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{2} = \frac{1}{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}} \times \frac{\frac{1}{2}\sqrt{\pi}.\frac{1}{2}\sqrt{\pi}}{1} = \frac{2\pi}{105} \qquad \textbf{Ans.}$$

EXERCISE 2.4

1. Evaluate $\int_0^\infty\int_0^\infty e^{-(x+y)}\sin\left(\frac{\pi y}{x+y}\right)dx\,dy$ by means of the transformation $u = x + y$, $v = y$ from (x, y) to (u, v) **Ans.** $\frac{1}{\pi}$

2. Using the transformation $x + y = u$, $y = uv$, show that $\int_0^1\int_0^{1-x}\frac{y}{e^{x+y}}\,dy\,dx = \frac{1}{2}(e-1)$ (*A.M.I.E. Winter 2001*)

3. Using the transformation $u = x - y$, $v = x + y$, prove that $\iint_R \cos\frac{x-y}{x+y}\,dx\,dy = \frac{1}{2}\sin 1$ where R is bounded by $x = 0$, $y = 0$, $x + y = 1$

$$\left[\textbf{Hint: } x = \frac{1}{2}(u+v),\ y = \frac{1}{2}(v-u) \text{ so that } |J| = \frac{1}{2}\right]$$

2.6 AREA IN CARTESIAN CO-ORDINATES

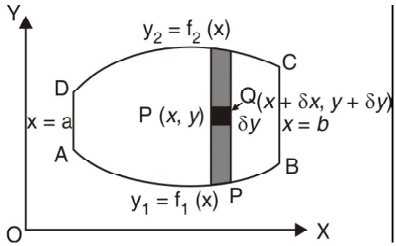
Let the curves AB and CD be $y_1 = f_1(x)$ and $y_2 = f_2(x)$.

Let the ordinates AD and BC be $x = a$ and $x = b$.

So the area enclosed by the two curves $y_1 = f_1(x)$ and $y_2 = f_2(x)$ and $x = a$ and $x = b$ is $ABCD$.

Let $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ be two neighbouring points, thent the area of the small rectangle $PQ = \delta x . \delta y$.

Area of the vertical strip = $\lim\limits_{\partial y \to 0} \sum\limits_{y_1}^{y_2} \delta x\, \delta y = \delta x \int_{y_1}^{y_2} dy\, \delta x$ the width of the strip is constant throughout.

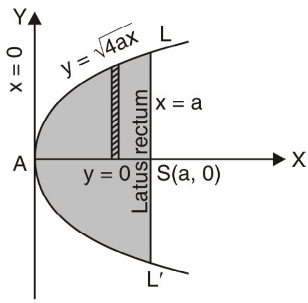


If we add all the strips from $x = a$ to $x = b$, we get

The area $ABCD = \lim\limits_{\partial x \to 0} \sum\limits_{a}^{b} \delta x \int_{y_1}^{y_2} dy = \int_a^b dx \int_{y_1}^{y_2} dy$

$$\boxed{\textbf{Area} = \int_a^b \int_{y_1}^{y_2} dx\, dy}$$

Example 19. *Find the area bounded by the parabola $y^2 = 4ax$ and its latus rectum.*



Solution. Required area = 2 (area (ASL)

$$= 2\int_0^a \int_0^{2\sqrt{ax}} dy\, dx$$

$$= 2\int_0^a 2\sqrt{ax}\, dx$$

$$= 4\sqrt{a}\left(\frac{x^{3/2}}{3/2}\right)_0^a = \frac{8a^2}{3}$$

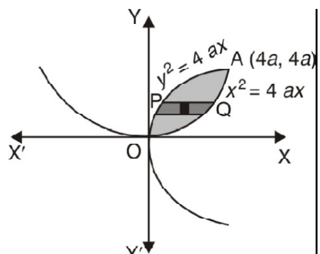
Example 20. *Find the area between the parabolas $y^2 = 4\,ax$ and $x^2 = 4\,ay$.*

Solution. $y^2 = 4ax$ ...(1)

$x^2 = 4ay$ ...(2)

On solving the equations (1) and (2) we get the point of intersection $(4a, 4a)$.

Divide the area into horizontal strips of width δy, x varies from $P, \dfrac{y^2}{4a}$ to $Q, \sqrt{4\,ay}$ and then y varies from $O(y = 0)$ to $A\,(y = 4a)$.



$\therefore$ The required area = $\int_0^{4a} dy \int_{y^2/4a}^{\sqrt{4ay}} dx$

$$= \int_0^{4a} dy\,[x]_{y^2/4a}^{\sqrt{4ay}} = \int_0^{4a} dy \left[\sqrt{4ay} - \frac{y^2}{4a}\right] = \left[\sqrt{4a}\,\frac{y^{3/2}}{\frac{3}{2}} - \frac{y^3}{12\,a}\right]_0$$

$$= \left[\frac{4\sqrt{a}}{3}(4a)^{3/2} - \frac{(4a)^3}{12a}\right] = \left[\frac{32}{3}a^2 - \frac{16}{3}a^2\right] = \frac{16}{3}a^2 \qquad \textbf{Ans.}$$

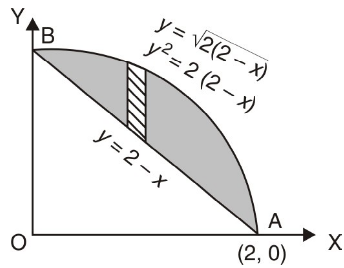
Example 21. *Find by double integration the area enclosed by the pair of curves*

$y = 2 - x$ and $y^2 = 2(2 - x)$

Solution. $y = 2 - x$

$y^2 = 2\,(2 - x)$

On solving the equations (1) and (2), we get the points of intersection (2, 0) and (0, 2).



$$A = \int \int dx\, dy$$

$$\text{The required area} = \int_0^2 dx \int_{2-x}^{\sqrt{2(2-x)}} dy$$

$$= \int_0^2 dx\, [y]_{2-x}^{\sqrt{2(2-x)}} = \int_0^2 dx\, [\sqrt{4-2x} - 2 + x]$$

$$= \left[\frac{2}{3 \times -2}(4-2x)^{3/2} - 2x + \frac{x^2}{2}\right]_0^2$$

$$= \left[-\frac{1}{3}(4-2x)^{3/2} - 2x + \frac{x^2}{2}\right]_0^2 = \left(-4 + \frac{4}{2}\right) + \frac{8}{3} = \frac{2}{3}$$ **Ans.**

EXERCISE 2.5

Use double integration in the following questions:

1. Find the area bounded by $y = x - 2$ and $y^2 = 2x + 4$. **Ans.** 18.

2. Find the area between the circle $x^2 + y^2 = a^2$ and the line $x + y = a$ in the first quadrant. **Ans.** $(\pi - 2)a^2/4$

3. Find the area of a plate in the form of quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. **Ans.** $\frac{\pi ab}{4}$

4. Find the area included between the curves $y^2 = 4a(x + a)$ and $y^2 = 4b(b - x)$. **Ans.** $\frac{8\sqrt{ab}}{3}$

(*A.M.I.E.T.E., Summer 2001*)

5. Find the area bounded by (*a*) $y^2 = 4 - x$ and $y^2 = x$. **Ans.** $\frac{16\sqrt{2}}{3}$

(*b*) $x - 2y + 4 = 0$, $x + y - 5 = 0$, $y = 0$ (*A.M.I.E., Winter 2001*) **Ans.** $\frac{27}{2}$

6. Find the area enclosed by the leminscate $r^2 = a^2 \cos 2\theta$. **Ans.** a^2

7. Find the area common to the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = 2ax$. **Ans.** $\left[\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right]a^2$

8. Find the area included between the curves $y = x^2 - 6x + 3$ and $y = 2x + 9$.

(*A.M.I.E., Summer 2001*) **Ans.** $\frac{88\sqrt{22}}{3}$

9. Determine the area of region bounded by the curves $xy = 2$, $4y = x^2$, $y = 4$. **Ans.** $\frac{28}{3} - 4\log 2$

(*U.P. I Semester 2003*)

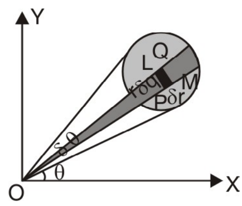
2.7 AREA IN POLAR CO-ORDINATES

$$\text{Area} = \iint r\, d\theta\, dr$$

Let us consider the area enclosed by the curve $r = f(\theta)$.

Let $P(r, \theta)$, $Q(r + \delta r, \theta + \delta\theta)$ be two neighbouring points.

Draw ares PL and QM, radii r and $r + \delta r$.



$$PL = r\delta\theta,\ PM = \delta r$$

Area of rectangle $PLQM = PL \times PM$

$$= (r\delta\theta)(d\delta r) = r\,\delta\theta\,\delta r.$$

The whole area A is composed of such small rectangles.

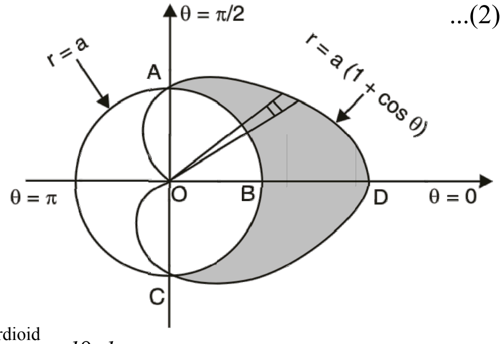
Hence,

$$A = \lim_{\substack{\partial r \to 0 \\ \delta\theta \to 0}} \sum \sum r\, \delta\theta . \delta r = \iint r\, d\theta\, dr$$

Example 22. *Find by double integration, the area lying inside the cardioid r = a (1 + cos θ) and outside the circle r = a.* (*Nagpur University, Winter 2000*)

Solution. $r = a\,(1 + \cos\theta)$ ...(1)

$r = a$ ...(2)



Solving (1) and (2), by eliminating r, we get

$a(1 + \cos\theta) = a \quad \Rightarrow \quad 1 + \cos\theta = 1$

$\cos\theta = 0 \quad \Rightarrow \quad \theta = -\dfrac{\pi}{2} \text{ or } \dfrac{\pi}{2}$

limits of r are a and $a(1 + \cos\theta)$

limits of θ are $-\dfrac{\pi}{2}$ to $\dfrac{\pi}{2}$

Required area = Area ABCDA

$$= \int_{-\pi/2}^{\pi/2} \int_{r \text{ for circle}}^{\text{for cardioid}} r\, d\theta\, dr$$

$$= \int_{-\pi/2}^{\pi/2} \int_{a}^{a(1+\cos\theta)} r\, d\theta\, dr \qquad = \int_{-\pi/2}^{\pi/2} \left(\frac{r^2}{2}\right)_{a}^{a(1+\cos\theta)} d\theta$$

$$= \frac{a^2}{2} \int_{-\pi/2}^{\pi/2} [(1+\cos\theta)^2 - 1]\, d\theta \qquad = \frac{a^2}{2} \int_{-\pi/2}^{\pi/2} (\cos^2\theta + 2\cos\theta)\, d\theta$$

$$= a^2 \int_0^{\pi/2} (\cos^2\theta + 2\cos\theta)\, d\theta \qquad = a^2 \left[\int_0^{\pi/2} \cos^2\theta\, d\theta + 2\int_0^{\pi/2} \cos\theta\, d\theta\right]$$

$$= a^2 \left[\frac{\pi}{4} + 2\,(\sin\theta)_0^{\pi/2}\right] = a^2\left[\frac{\pi}{4} + 2\right] \qquad = \frac{a^2}{4}(\pi + 8)$$ **Ans.**

Example 23. *Find by double integration, the area lying inside the circle r = a sin θ and outside the cardioid r = a (1 – cos θ).*

Solution. We have,

$r = a\sin\theta$ ...(1)

$r = a\,(1 - \cos\theta)$ ...(2)

Solving (1) and (2) by eliminating r, we have

$$\sin\theta = 1 - \cos\theta \Rightarrow \sin\theta + \cos\theta = 1$$

Squaring above, we get

$\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = 1$

$$\Rightarrow \qquad 1 + \sin 2\theta = 1 \Rightarrow \sin 2\theta = 0 \Rightarrow 2\theta = 0 \text{ or } \pi = \theta = 0 \text{ or } \frac{\pi}{2}$$

The required area is shaded portion in the fig.

Limits of r are $a(1 - \cos\theta)$ and $a\sin\theta$, limits of θ are 0 and $\dfrac{\pi}{2}$.

$$\text{Required area} = \int_0^{\frac{\pi}{2}} \int_{a(1-\cos\theta)}^{a\sin\theta} r\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{r^2}{2}\right]_{a(1-\cos\theta)}^{a\sin\theta} d\theta = \frac{1}{2}\int_0^{\pi/2} a^2\,[\sin^2\theta - (1-\cos\theta)^2]\, d\theta$$

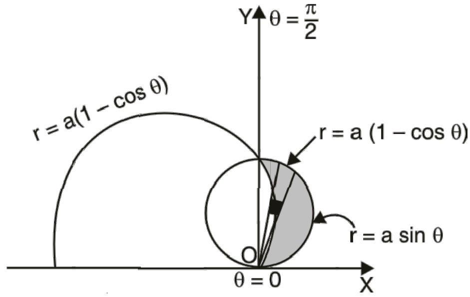
$$= \frac{a^2}{2}\int_0^{\pi/2} (\sin^2\theta - 1 - \cos^2\theta + 2\cos\theta]\,d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2} (-2\cos^2\theta + 2\cos\theta)\,d\theta$$

$$= \frac{a^2}{2}\left[\int_0^{\pi/2} -2\cos^2\theta\,d\theta + \int_0^{\pi/2} 2\cos\theta\,d\theta\right]$$

$$= \frac{a^2}{2}\left[\left(-2.\frac{\pi}{4}\right) + 2\,(\sin\theta)_0^{\pi/2}\right]$$

$$= \frac{a^2}{2}\left[-\frac{\pi}{2} + 2\left(\sin\frac{\pi}{2} - \sin 0\right)\right] = \frac{a^2}{2}\left[-\frac{\pi}{2} + 2\right] = a^2\left(1 - \frac{\pi}{4}\right)$$ **Ans.**

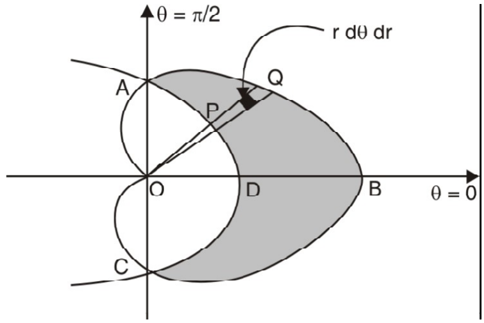


Example 24. *Find by double integration, the area lying inside a cardioid r = 1 + cos θ and outside the parabola r (1 + cos θ) = 1.*

Solutio. We have,

$$r = 1 + \cos\theta \quad ...(1)$$

$$r\,(1 + \cos\theta) = 1 \quad ...(2)$$



Solving (1) and (2), we get

$$(1 + \cos\theta)\,(1 + \cos\theta) = 1$$

$$(1 + \cos\theta)^2 = 1$$

$$1 + \cos\theta = 1$$

$$\cos\theta = 0 \Rightarrow \theta = \pm\frac{\pi}{2}$$

limits of r are $1 + \cos\theta$ and $\dfrac{1}{1+\cos\theta}$ limits of θ are $-\dfrac{\pi}{2}$ to $\dfrac{\pi}{2}$.

Required area = Area *ADCBA* (Shaded portion)

$$= \int_{-\pi/2}^{\pi/2}\int_{\frac{1}{1+\cos\theta}}^{1+\cos\theta} r\,d\theta\,dr = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\left(\frac{r^2}{2}\right)_{\frac{1}{1+\cos\theta}}^{1+\cos\theta} d\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2}\left[(1+\cos\theta)^2 - \frac{1}{(1+\cos\theta)^2}\right] d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2}\left[(1 + \cos^2\theta + 2\cos\theta) - \frac{1}{\left(2\cos^2\frac{\theta}{2}\right)^2}\right] d\theta$$

$$= 2\times\frac{1}{2}\int_0^{\pi/2}\left[(1 + \cos^2\theta + 2\cos\theta) - \frac{1}{4}\sec^4\frac{\pi}{2}\right] d\theta$$

$$= \int_0^{\pi/2}\left[(1 + \cos^2\theta + 2\cos\theta) - \frac{1}{4}\left(1 + \tan^2\frac{\theta}{2}\right)\sec^2\frac{\theta}{2}\right] d\theta$$

$$= \int_0^{\pi/2}\left[\left(1 + \frac{1+\cos 2\theta}{2} + 2\cos\theta\right) - \frac{1}{4}\left(1 + \tan^2\frac{\pi}{2}\right)\sec^2\frac{\pi}{2}\right] d\theta$$

$$= \int_0^{\pi/2}\left[1 + \frac{1}{2} + \frac{\cos 2\theta}{2} + 2\cos\theta - \frac{1}{4}\left(\sec^2\frac{\theta}{2} + \tan^2\frac{\theta}{2}\times\sec^2\frac{\theta}{2}\right)\right] d\theta$$

$$= \left[\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4} + 2\sin\theta - \frac{1}{4}\left(2\tan\frac{\theta}{2} + \frac{2}{3}\tan^3\frac{\theta}{2}\right)\right]_0^{\frac{\pi}{2}}$$

$$= \left[\frac{\pi}{2} + \frac{\pi}{4} + 0 + 2\sin\frac{\pi}{2} - \frac{1}{2}\tan\frac{\pi}{4} - \frac{1}{6}\tan^3\frac{\pi}{4}\right] = \left[\frac{3\pi}{4} + 2 - \frac{1}{2} - \frac{1}{6}\right] = \left[\frac{3\pi}{4} + \frac{4}{3}\right]$$ **Ans.**

EXERCISE 2.6

1. Find the area of cardioid $r = a(1 + \cos\theta)$. **Ans.** $\dfrac{3\pi a^2}{2}$
2. Find the area of the curve $r^2 = a^2 \cos 2\theta$. **Ans.** a^2
3. Find the area enclosed by the curve $r = 2\, a \cos\theta$ **Ans.** πa^2
4. Find the area enclosed by the curve $r = 3 + 2 \cos\theta$. **Ans.** $11\,\pi$
5. Find the area enclosed by the curve

   $r^3 = a^2 \cos^2\theta + b^2 \sin^2\theta$. **Ans.** $\dfrac{\pi}{2}(a^2 + b^2)$
6. Show that the area of the region included between the cardioides $r = a(1 + \cos\theta)$ and $r = a\,(1 - \cos\theta)$ is $\dfrac{a^2}{2}(3\pi - 8)$.
7. Find the area outside the circle $r = 2$ and inside the cardioid $r = 2(1 + \cos\theta)$. **Ans.** $(\pi + 8)$
8. Find the area inside the circle $r = 2a \cos\theta$ and outside the circle $r = a$. **Ans.** $2a^2\left(\dfrac{\pi}{3} + \dfrac{\sqrt{3}}{4}\right)$
9. Find the area inside the circle $r = 4 \sin\theta$ and outside the lemniscate $r^2 = 8 \cos 2\,\theta$.

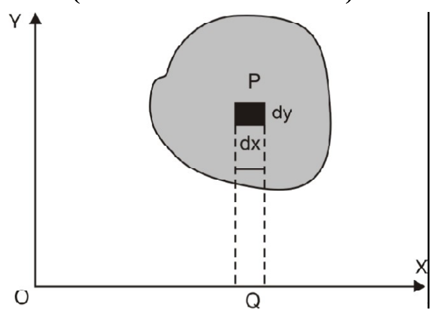
   Ans. $\left(\dfrac{8}{3}\pi + 4\sqrt{3} - 4\right)$

2.8 VOLUME OF SOLID BY ROTATION OF AN AREA (DOUBLE INTEGRAL)

When the area enclosed by a curve $y = f(x)$ is revolved about an axis, a solid is generated, we have to find out the volume of solid generated.

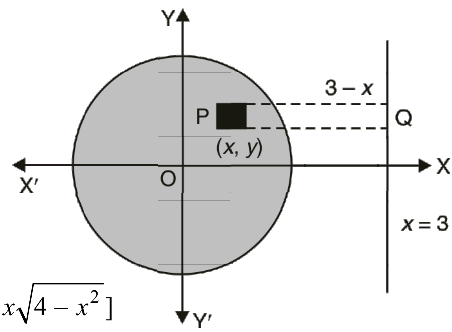
Volume of the solid generated about x-axis

$$= \int_a^b \int_{y_1(x)}^{y_2(x)} 2\pi\, PQ\, dx\, dy$$



Example 25. *Find the volume of the torus generated by revolving the circle $x^2 + y^2 = 4$ about the line $x = 3$.*

Solution. $x^2 + y^2 = 4$



$$V = \int\int (2\pi\, PQ)\, dx\, dy = 2\pi \int\int (3 - x)\, dx\, dy$$

$$= 2\pi \int_{-2}^{+2} dx \int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}} (3 - x)\, dy$$

$$= 2\pi \int_{-2}^{+2} dx\, \left(3y - x\,y\right)_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}}$$

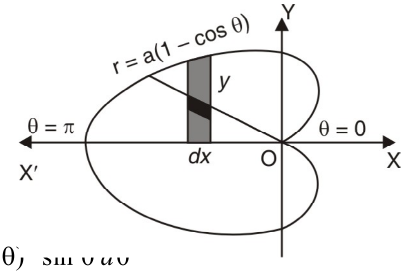
$$= 2\pi \int_{-2}^{+2} dx\, [3\sqrt{4-x^2} - x\sqrt{4-x^2} + 3\sqrt{4-x^2} - x\sqrt{4-x^2}]$$

$$= 4\pi [3\sqrt{4-x^2} - x\sqrt{4-x^2}]\, dx = 4\pi \left[3\frac{x}{2}\sqrt{4-x^2} + 3\times\frac{4}{2}\sin^{-1}\frac{x}{2} + \frac{1}{3}(4-x^2)^{3/2}\right]_{-2}^{2}$$

$$= 4\pi\left[6\times\frac{\pi}{2} + 6\times\frac{\pi}{2}\right] = 24\,\pi^2$$ **Ans.**

Example 26. *Calculate by double integration the volume generated by the revolution of the cardioid $r = a\,(1 - \cos\theta)$ about its axis.* (*AMIETE, June 2010*)

Solution. $r = a\,(1 - \cos\theta)$



$$V = 2\pi \int \int y\,dx\,dy \Rightarrow V = 2\pi \int \int (r\,d\theta\,dr)\,y$$

$$= 2\pi \int d\theta \int r\,dr\,(r\sin\theta)$$

$$= 2\pi \int_0^{\pi} \sin\theta\,d\theta \int_0^{a(1-\cos\theta)} r^2\,dr$$

$$= 2\pi \int_0^{\pi} \sin\theta\,d\theta \left[\frac{r^3}{3}\right]_0^{a\,(1-\cos\theta)} = \frac{2\pi}{3}\int_0^{\pi} a^3\,(1-\cos\theta)^3 \sin\theta\,d\theta$$

$$= \frac{2\pi\,a^3}{3}\left[\frac{(1-\cos\theta)^4}{4}\right]_0^{\pi} = \frac{2\pi\,a^3}{12}[16] = \frac{8}{3}\pi\,a^3$$ **Ans.**

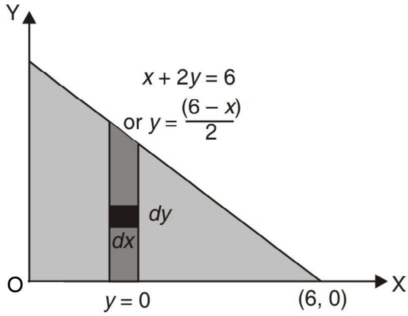
Example 27. *A pyramid is bounded by the three co-ordinate planes and the plane $x + 2y + 3z = 6$. Compute this volume by double integration.*

Solution. $x + 2y + 3z = 6$ ...(1)

$x = 0,\ y = 0,\ z = 0$ are co-ordinate planes.

The line of intersection of plane (1) and xy plane ($z = 0$) is

$$x + 2y = 6 \qquad ...(2)$$



The base of the pyramid may be taken to be the triangle bounded by x-axis, y-axis and the line (2).

An elementary area on the base is $dx\,dy$.

Consider the elementary rod standing on this area and having height z, where

$$3z = 6 - x - 2y \text{ or } z = \frac{6 - x - 2y}{3}$$

Volume of the rod = $dx\,dy$, z, Limits for z are 0 and $\dfrac{6 - x - 2y}{3}$.

Limits of y are 0 and $\dfrac{6 - x}{2}$ and limits of x are 0 and 6.

Required volume = $\displaystyle\int_0^6 \int_0^{\frac{6-x}{2}} z\,dx\,dy = \int_0^6 dx \int_0^{\frac{6-x}{2}} \frac{6 - x - 2y}{3}\,dy$

$$= \frac{1}{3}\int_0^6 dx\left(6x - xy - y^2\right)_0^{\frac{6-x}{2}} = \frac{1}{3}\int_0^6 \left(\frac{6(6-x)}{2} - \frac{x(6-x)}{2} - \left(\frac{6-x}{2}\right)^2\right) dx$$

$$= \frac{1}{3}\int_0^6 \left(\frac{36 - 6x}{2} - \frac{6x - x^2}{2} - \frac{36 + x^2 - 12x}{4}\right) dx$$

$$= \frac{1}{12}\int_0^6 (72 - 12x - 12x + 2x^2 - 36 - x^2 + 12x)\,dx$$

$$= \frac{1}{12}\int_0^6 (x^2 - 12x + 36)\,dx = \frac{1}{12}\left[\frac{x^3}{3} - \frac{12x^2}{2} + 36x\right]_0^6 = \frac{1}{12}[72 - 216 + 216] = 6$$ **Ans.**

EXERCISE 2.7

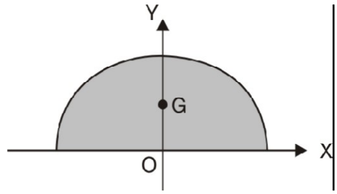
1. Find the volume of the sphere $x^2 + y^2 + z^2 = a^2$ by revolving area of the circle $x^2 + y^2 = a^2$. **Ans.** $\frac{4}{3}\pi a^3$

2.9 CENTRE OF GRAVITY

$$\bar{x} = \frac{\int\int \rho\,x\,dx\,dy}{\int\int \rho\,dx\,dy},\ \bar{y} = \frac{\int\int \rho\,y\,dx\,dy}{\int\int \rho\,dx\,dy}$$

Example 28. *Find the position of the C.G. of a semi-circular lamina of radius a if its density varies as the square of the distance from the diameter.* (*AMIETE, Dec. 2010*)

Solution. Let the bounding diameter be as the x-axis and a line perpendicular to the diameter and passing through the centre is y-axis. Equation of the circle is $x^2 + y^2 = a^2$. By symmetry $\bar{x} = 0$.



$$\bar{y} = \frac{\int\int y\,\rho\,dx\,dy}{\int\int \rho\,dx\,dy} = \frac{\int\int(\lambda\,y^2)\,y\,dx\,dy}{\int\int(\lambda\,y^2)\,dx\,dy} = \frac{\int_{-a}^{a} dx \int_0^{\sqrt{a^2-x^2}} y^3\,dy}{\int_{-a}^{a} dx \int_0^{\sqrt{a^2-x^2}} y^2\,dy}$$

$$= \frac{\int_{-a}^{a} dx \left[\frac{y^4}{4}\right]_0^{\sqrt{a^2-x^2}}}{\int_{-a}^{a} dx \left(\frac{y^3}{3}\right)_0^{\sqrt{a^2-x^2}}} = \frac{3\int_{-a}^{a}(a^2-x^2)^2\,dx}{4\int_{-a}^{a}(a^2-x^2)^{3/2}\,dx}$$

Put $x = a\sin\theta$

$$= \frac{3\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(a^2-a^2\sin^2\theta)^2\,a\cos\theta\,d\theta}{4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(a^2-a^2\sin^2\theta)^{3/2}\,a\cos\theta\,d\theta} = \frac{3\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a^5\cos^5\theta\,d\theta}{4\int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} a^4\cos^4\theta\,d\theta}$$

$$= \frac{3a}{4}\,\frac{\frac{4\times 2}{5\times 3}}{\frac{3\times 1}{4\times 2}\,\frac{\pi}{2}} = \left(\frac{3a}{4}\right)\left(\frac{8}{15}\right)\left(\frac{16}{3\pi}\right) = \frac{32a}{15\pi}$$

Hence C.G. is $\left(0, \frac{32\,a}{15\,\pi}\right)$ **Ans.**

Example 29. *Find C.G. of the area in the positive quadrant of the curve*
$$x^{2/3} + y^{2/3} = a^{2/3}.$$

Solution. For C.G. of area; $\bar{x} = \frac{\int\int x\,dx\,dy}{\int\int dx\,dy}$, $\bar{y} = \frac{\int\int y\,dx\,dy}{\int\int dx\,dy}$

$$\bar{x} = \frac{\int_0^a x\,dx \int_0^{(a^{2/3}-x^{2/3})^{3/2}} dy}{\int_0^a dx \int_0^{(a^{2/3}-x^{2/3})^{3/2}} dy} = \frac{\int_0^a x\,dx\,[y]_0^{(a^{2/3}-x^{2/3})^{3/2}}}{\int_0^a dx\,[y]_0^{(a^{2.3}-x^{2/3})^{3/2}}}$$

[Put $x = a\cos^3\theta$]

$$= \frac{\int_0^a x\,dx\,(a^{2/3}-x^{2/3})^{3/2}}{\int_0^a dx\,(a^{2/3}-x^{2/3})^{3/2}} = \frac{\int_{\frac{\pi}{2}}^{0} a\cos^3\theta\,(a^{2/3}-a^{2/3}\cos^2\theta)^{3/2}\,(-3a\cos^2\theta\sin\theta\,d\theta)}{\int_{\frac{\pi}{2}}^{0}(a^{2/3}-a^{2/3}\cos^2\theta)^{3/2}\,(-3a\cos^2\theta\sin\theta\,d\,\theta)}$$

$$= \frac{\int_0^{\frac{\pi}{2}} 3a^3\cos^3\theta\sin^3\theta\cos^2\theta\sin\theta\,d\,\theta}{\int_0^{\frac{\pi}{2}} 3a^2\sin^3\theta\cos^2\theta\sin\theta\,d\,\theta} = \frac{a\int_0^{\frac{\pi}{2}}\sin^4\theta\cos^5\theta\,d\,\theta}{\int_0^{\frac{\pi}{2}}\sin^4\theta\cos^2\theta\,d\,\theta} = \frac{\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{6}{2}\right)}{2\,\Gamma\left(\frac{4}{2}\right)}\,a}{\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{2\,\Gamma 4}}$$

$$= \frac{\lceil 3 \lceil 4\, a}{\left\lceil \frac{3}{2} \right. \left\lceil \frac{11}{2} \right.} = \frac{(2)\,(6)\,a}{\frac{1}{2}\cdot\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\pi} = \frac{256\,a}{315\,\pi}, \quad \text{Similarly,} \quad \bar{y} = \frac{256\,a}{315\,\pi}$$

Hence, C.G. of the area is $\left(\frac{256\,a}{315\,\pi}, \frac{256\,a}{315\,\pi}\right)$.

Example 30. *Find by double integration, the centre of gravity of the area of the cardioid $r = a\,(1 + \cos\theta)$.*

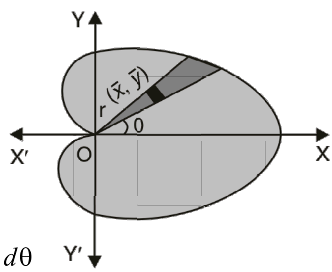
Solution. Let $(\bar{x}, \bar{y})$ be the C.G. the cardioid

By Symmetry, $\bar{y} = 0$.

$$\bar{x} = \frac{\int\int_A x\,dx\,dy}{\int\int_A dx\,dy} = \frac{\int\int_A x\,dx\,dy}{\int\int_A dx\,dy}$$

$$= \frac{\int_{-\pi}^{\pi}\int_0^{a(1+\cos\theta)} (r\cos\theta)\,(r\,d\theta\,dr)}{\int_{-\pi}^{\pi}\int_0^{a(1+\cos\theta)} r\,d\theta\,dr} = \frac{\int_{-\pi}^{\pi}\cos\theta\,d\theta\int_0^{a\,(1+\cos\theta)} r^2\,dr}{\int_{-\pi}^{\pi}d\theta\int_0^{a(1+\cos\theta)} r\,dr}$$

$$= \frac{\int_{-\pi}^{\pi}\cos\theta\,d\theta\left[\frac{r^3}{3}\right]_0^{a\,(a+\cos\theta)}}{\int_{-\pi}^{\pi}d\theta\left(\frac{r^2}{2}\right)_0^{a\,(1+\cos\theta)}} = \frac{\int_{-\pi}^{\pi}\cos\theta\,d\theta.\,\frac{a^3}{3}\,(1+\cos\theta)^3}{\int_{-\pi}^{\pi}d\theta\,\frac{a^2}{2}\,(1+\cos\theta)^2}$$



$$= \frac{\frac{a^3}{3}\int_{-\pi}^{\pi}\left(2\cos^2\frac{\theta}{2}-1\right)\left(1+2\cos^2\frac{\theta}{2}-1\right)^3 d\theta}{\frac{a^2}{2}\int_{-\pi}^{\pi}\left(1+2\cos^2\frac{\theta}{2}-1\right)d\theta}$$

$$= \frac{a^3}{3}\int_{-\pi}^{\pi}\left(2\cos^2\frac{\theta}{2}-1\right)\left(8\cos^6\frac{\theta}{2}\right)d\theta \div \frac{a^2}{2}\int_{-\pi}^{\pi}4\cos^4\frac{\theta}{2}\,d\theta$$

$$= \frac{8a^3}{3}\int_{-\pi}^{\pi}\left(2\cos^8\frac{\theta}{2}-\cos^6\frac{\theta}{2}\right)d\theta \div 2a^2\int_{-\pi}^{\pi}\cos^4\frac{\theta}{2}\,d\theta$$

$$= \frac{2\times 8a^3}{3}\int_0^{\pi}\left(2\cos^8\frac{\theta}{2}-\cos^6\frac{\theta}{2}\right)d\theta \div 4a^2\int_0^{\pi}\cos^4\frac{\theta}{2}\,d\theta$$

$$= \frac{16a^3}{3}\int_0^{\pi/2}(2\cos^8 t - \cos^6 t)\,(2\,dt) \div 4a^2\int_0^{\pi/2}\cos^4 t\,(2\,dt)$$

$$= \frac{32\,a^3}{3}\left[\frac{2\times 7\times 5\times 3\times 1}{8\times 6\times 4\times 2}\,\frac{\pi}{2} - \frac{5\times 3\times 1}{6\times 4\times 2}\,\frac{\pi}{2}\right] \div 8a^2\left(\frac{3\times 1}{4\times 2}\,\frac{\pi}{2}\right)$$

$$= \frac{32a^3}{3}\left(\frac{35\,\pi}{128} - \frac{5\,\pi}{32}\right) \div 8a^2\left(\frac{3\pi}{16}\right) = \frac{8a^3}{3}\times\frac{15\pi}{128}\times\frac{16}{8a^2\times 3\pi} = \frac{5a}{24}$$ **Ans.**

2.10 CENTRE OF GRAVITY OF AN ARC

Example 31. *Find the C.G. of the arc of the curve*

$x = a\,(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ in the positive quadrant.

Solution. We know that, $\bar{x} = \frac{\int x\,ds}{\int ds}, \bar{y} = \frac{\int y\,ds}{\int ds}$

$$\text{Now, } ds = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \sqrt{\{a^2 (1+\cos\theta)^2 + a^2 \sin^2\theta\}}\, d\theta = a\sqrt{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta}\, d\theta$$

$$= a\sqrt{1 + 2\cos\theta + 1}\, d\theta = a\sqrt{2(1+\cos\theta)}\, d\theta = a\sqrt{4\cos^2\frac{\theta}{2}}\, d\theta = 2a\cos\frac{\theta}{2}\, d\theta$$

$$\bar{x} = \frac{\int x\, dx}{\int ds} = \frac{\int_0^\pi a(\theta + \sin\theta)\, 2a\cos\frac{\theta}{2}\, d\theta}{\int_0^\pi 2a\cos\frac{\theta}{2}\, d\theta} = \frac{a\int_0^\pi\left(\theta + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\, d\theta\right)}{\left[2\sin\frac{\pi}{2}\right]_0^\pi}$$

$$= \frac{a}{2}\int_0^\pi\left[\theta\cos\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}\right] d\theta = \frac{a}{2}\int_0^{\frac{\pi}{2}}(2t\cos t + 2\sin t\cos^2 t)\, 2\, dt$$

$$= 2a\left[t\sin t + \cos t - \frac{\cos^3 t}{3}\right]_0^{\frac{\pi}{2}} = 2a\left[\frac{\pi}{2} - 1 + \frac{1}{3}\right] = a\left[\pi - \frac{4}{3}\right]$$

$$\bar{y} = \frac{\int y\, ds}{\int ds} = \frac{\int_0^\pi a(1-\cos\theta)\, 2a\cos\frac{\theta}{2}\, d\theta}{\int_0^\pi 2a\cos\frac{\theta}{2}\, d\theta} = \frac{a\int_0^\pi 2\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}\, d\theta}{\int_0^\pi \cos\frac{\theta}{2}\, d\theta}$$

$$= \frac{a\, r\left[\sin^3\frac{\theta}{2}\right]_0^\pi}{3\left[2\sin\frac{\theta}{2}\right]_0^\pi} = \frac{4a}{3\times 2} = \frac{2a}{3} \quad \text{Hence, C.G. of the arc is} \left[a\left(\pi - \frac{4}{3}\right), \frac{2a}{3}\right]$$ **Ans.**

EXERCISE 2.8

1. Find the centre of gravity of the area bounded by the parabola $y^2 = x$ and the line $x + y = 2$.
Ans. $\left(\frac{8}{5}, -\frac{1}{2}\right)$

2. Find the centroid of the tetrahedron bounded by the coordinate planes and the plane $x + y + z = 1$, the density at any point varying as its distance from the face $z = 0$. **Ans.** $\left(\frac{1}{5}, \frac{1}{5}, \frac{2}{5}\right)$

3. Find the centroid of the area enclosed by the parabola $y^2 = 4\,ax$, the axis of x and latus rectum.
Ans. $\left(\frac{3a}{20}, \frac{3a}{16}\right)$

4. Find the centroid of the loop of curve $r^2 = a^2 \cos 2\,\theta$. **Ans.** $\left(\frac{\pi a\sqrt{2}}{8}, 0\right)$

5. Find the centroid of solid formed by revolving about the x-axis that part of the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which lies in the first quadrant. **Ans.** $\left(\frac{3a}{8}, 0\right)$

6. Find the average density of the sphere of radius a whose density at a distance r from the centre of the sphere is $\rho = \rho_0\left[1 + k\frac{r^3}{a^3}\right]$. $\rho_0\left(1 + \frac{k}{2}\right)$

7. The density at a point on a circular lamina varies as the distance from a point O on the circumference. Show that the C.G. divides the diameter through O in the ratio 3 : 2.

1.11 TRIPLE INTEGRATION

Let a function $f(x, y, z)$ be a continuous at every point of a finite region S of three dimensional space. Consider n sub-spaces $\delta S_1, \delta S_2, \delta S_3, \ldots \delta S_n$ of the space S.

If (x_r, y_r, z_r) be a point in the rth subspace.

The limit of the sum $\sum_{r=1}^{n} f(x_r, y_r, z_r)\, \delta S_r$, as $n \to \infty$, $\delta S_r \to 0$ is known as the triple integral of $f(x, y, z)$ over the space S.

Symbolically, it is denoted by

$$\iiint_S f(x, y, z)\, dS$$

It can be calculated as $\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x, y, z)\; dx\; dy\; dz$. First we integrate with respect to z treating x, y as constant between the limits z_1 and z_2. The resulting expression (function of x, y) is integrated with respect to y keeping x as constant between the limits y_1 and y_2. At the end we integrate the resulting expression (function of x only) within the limits x_1 and x_2.

$$\boxed{\int_{x_1=a}^{x_2=b} \Psi(x)\, dx \;\boxed{\int_{y_1=\phi_1(x)}^{y_2=\phi_2(x)} \phi(x, y)\, dy \;\boxed{\int_{z_1=f_1(x,y)}^{z_2=f_2(x,y)} f(x, y, z)\, dz}}}$$

First we integrate from inner most integral w.r.t. z, then we integrate with respect to y and finally the outer most with respect to x.

But the above order of integration is immaterial provided the limits change accordingly.

Example 32. *Evaluate* $\iiint_R (x + y + z)\; dx\; dy\; dz$, *where* $R : 0 \le x \le 1,\ 1 \le y \le 2,\ 2 \le z \le 3$.

Solution.
$$\int_0^1 dx\int_1^2 dy\int_2^3 (x+y+z)\, dz = \int_0^1 dx\int_1^2 dy\left[\frac{(x+y+z)^2}{2}\right]_2^3$$
$$= \frac{1}{2}\int_0^1 dx\int_1^2 dy\,[(x+y+3)^2-(x+y+2)^2] = \frac{1}{2}\int_0^1 dx\int_1^2 (2x+2y+5)\,.\,1.dy$$
$$= \frac{1}{2}\int_0^1 dx\left[\frac{(2x+2y+5)^2}{4}\right]_1^2 = \frac{1}{8}\int_0^1 dx\,[(2x+4+5)^2-(2x+2+5)^2]$$
$$= \frac{1}{8}\int_0^1 (4x+16)\,.\,2\,dx = \int_0^1 (x+4)\,dx = \left[\frac{x^2}{2}+4x\right]_0^1 = \frac{1}{2}+4 = \frac{9}{2}$$ **Ans.**

Example 33. *Evaluate the integral :* $\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\; dz\; dy\; dx$.

Solution. $\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\; dz\; dy\; dx.$

$$= \int_0^{\log 2} e^x dx\int_0^x e^y dy\int_0^{x+\log y} e^z dz = \int_0^{\log 2} e^x\; dx\int_0^x e^y dy (e^z)_0^{x+\log y}$$
$$= \int_0^{\log 2} e^x dx\int_0^x e^y dy\,(e^{x+\log y}-1) = \int_0^{\log 2} e^x\; dx\int_0^x e^y dy\,(e^{\log y}\,.\,e^x-1)$$
$$= \int_0^{\log 2} e^x dx\int_0^x e^y\,(y e^x-1)\,dy = \int_0^{\log 2} e^x dx\left[(ye^x-1)\,e^y-\int e^x\,.\,e^y\,dy\right]_0^x$$
$$= \int_0^{\log 2} e^x dx\left[(ye^x-1)\,e^y-e^{x+y}\right]_0^x = \int_0^{\log 2} e^x dx\,[(xe^x-1)\,e^x-e^{2x}+1+e^x]$$
$$= \int_0^{\log 2} e^x dx\,[xe^{2x}-e^x-e^{2x}+1+e^x] = \int_0^{\log 2}(xe^{3x}-e^{3x}+e^x)\,dx$$

$$=\left[x\,\frac{e^{3x}}{3}-\int 1\,.\,\frac{e^{3x}}{3}\,dx-\frac{e^{3x}}{3}+e^{x}\right]_0^{\log 2}=\left[\frac{x}{3}\,e^{3x}-\frac{e^{3x}}{9}-\frac{e^{3x}}{3}+e^{x}\right]_0^{\log 2}$$

$$=\frac{\log 2}{3}\,e^{3\log 2}-\frac{e^{3\log 2}}{9}-\frac{e^{3\log 2}}{3}+e^{\log 2}+\frac{1}{9}+\frac{1}{3}-1$$

$$=\frac{\log 2}{3}\,e^{\log 2^3}-\frac{e^{\log 2^3}}{9}-\frac{e^{\log 2^3}}{3}+e^{\log 2}+\frac{1}{9}+\frac{1}{3}-1$$

$$=\frac{8}{3}\log 2-\frac{8}{9}-\frac{8}{3}+2+\frac{1}{9}+\frac{1}{3}-1=\frac{8}{3}\log 2-\frac{19}{9}$$ **Ans.**

Example 34. *Evaluate* $\int_0^{\log 2}\int_0^{x}\int_0^{x+y} e^{x+y+z}\,dx\,dy\,dz.$

(*M.U. II Semester, 2005, 2003, 2002*)

Solution. $I=\int_0^{\log 2}\int_0^{x} e^{x+y}\left[e^{z}\right]_0^{x+y}dx\,dy$

$$=\int_0^{\log 2}\int_0^{x} e^{x+y}(e^{x+y}-1)\,dx\,dy=\int_0^{\log 2}\int_0^{x}\left[e^{2(x+y)}-e^{(x+y)}\right]dx\,dy$$

$$=\int_0^{\log 2}\left[e^{2x}\,.\,\frac{e^{2y}}{2}-e^{x}\,.\,e^{y}\right]_0^{x}dx=\int_0^{\log 2}\left(\frac{e^{4x}}{2}-e^{2x}-\frac{e^{2x}}{2}+e^{x}\right)dx$$

$$=\left[\frac{e^{4x}}{8}-\frac{e^{2x}}{2}-\frac{e^{2x}}{4}+e^{x}\right]_0^{\log 2}=\left[\frac{e^{4\log 2}}{8}-\frac{e^{2\log 2}}{2}-\frac{e^{2\log 2}}{4}+e^{\log 2}\right]-\left(\frac{1}{8}-\frac{1}{2}-\frac{1}{4}+1\right)$$

$$=\left(\frac{e^{\log 16}}{8}-\frac{e^{\log 4}}{2}-\frac{e^{\log 4}}{4}+e^{\log 2}\right)-\left(\frac{1}{8}-\frac{1}{2}-\frac{1}{4}+1\right)$$

$$=\left(\frac{16}{8}-\frac{4}{2}-\frac{4}{4}+2\right)-\left(\frac{1}{8}-\frac{1}{2}-\frac{1}{4}+1\right)=\frac{5}{8}$$ **Ans.**

Example 35. *Evaluate* $\iiint_R (x^2+y^2+z^2)\,dx\,dy\,dz$

where R denotes the region bounded by $x=0,\ y=0,\ z=0$ *and* $x+y+z=a,\ (a>0)$

Solution. $\iiint_R (x^2+y^2+z^2)\,dx\,dy\,dz$

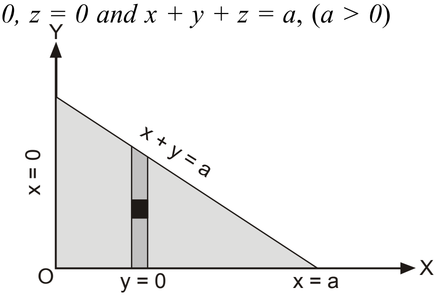
$x+y+z=a$ or $z=a-x-y$

Upper limit of $z=a-x-y$

On x-y plane, $x+y+z=a$ becomes $x+y=a$ as shown in the figure.

Upper limit of $y=a-x$

Upper limit of $x=a$



$$=\int_{x=0}^{a}dx\int_{y=0}^{a-x}dy\int_{z=0}^{a-x-y}(x^2+y^2+z^2)\,dz=\int_0^{a}dx\int_0^{a-x}dy\left(x^2z+y^2z+\frac{z^3}{3}\right)_0^{a-x-y}$$

$$=\int_0^{a}dx\int_0^{a-x}dy\left[x^2(a-x-y)+y^2(a-x-y)+\frac{(a-x-y)^3}{3}\right]$$

$$=\int_0^{a}dx\int_0^{a-x}\left[x^2(a-x)-x^2y+(a-x)\,y^2-y^3+\frac{(a-x-y)^3}{3}\right]dy$$

$$=\int_0^{a}dx\left[x^2(a-x)\,y-\frac{x^2y^2}{2}+(a-x)\,\frac{y^3}{3}-\frac{y^4}{4}-\frac{(a-x-y)^4}{12}\right]_0^{a-x}$$

$$= \int_0^a dx\left[x^2(a-x)^2 - \frac{x^2}{2}(a-x)^2 + (a-x)\frac{(a-x)^3}{3} - \frac{(a-x)^4}{4} + \frac{(a-x)^4}{12}\right]$$

$$= \int_0^a \left[\frac{x^2}{2}(a-x)^2 + \frac{(a-x)^4}{6}\right]dx = \int_0^a \left[\frac{1}{2}(a^2x^2 - 2ax^3 + x^4) + \frac{(a-x)^4}{6}\right]dx$$

$$= \left[\frac{1}{2}a^2\frac{x^3}{3} - \frac{ax^4}{4} + \frac{x^5}{10} - \frac{(a-x)^5}{30}\right]_0^a = \frac{a^5}{6} - \frac{a^5}{4} + \frac{a^5}{10} + \frac{a^5}{30} = \frac{a^5}{20}$$ **Ans.**

Example 36. *Compute* $\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3}$ *if the region of integration is bounded by the coordinate planes and the plane $x + y + z = 1$.* (*M.U., II Semester 2007, 2006*)

Solution. Let the given region be R, then R is expressed as

$$0 \le z \le 1 - x - y, \quad 0 \le y \le 1 - x, \quad 0 \le x \le 1.$$

$$\iiint_R \frac{dx\,dy\,dz}{(x+y+z+1)^3} = \int_0^1 dx\int_0^{1-x} dy\int_0^{1-x-y} \frac{dz}{(x+y+z+1)^3}$$

$$= \int_0^1 dx\int_0^{1-x} dy\left[\frac{1}{-2(x+y+z+1)^2}\right]_0^{1-x-y}$$

$$= -\frac{1}{2}\int_0^1 dx\int_0^{1-x} dy\left[\frac{1}{(x+y+1-x-y+1)^2} - \frac{1}{(x+y+1)^2}\right]$$

$$= -\frac{1}{2}\int_0^1 dx\int_0^{1-x}\left[\frac{1}{4} - \frac{1}{(x+y+1)^2}\right]dy = -\frac{1}{2}\int_0^1 dx\left[\frac{y}{4} + \frac{1}{x+y+1}\right]_0^{1-x}$$

$$= -\frac{1}{2}\int_0^1 dx\left[\frac{1-x}{4} + \frac{1}{x+1+1-x} - \frac{1}{x+1}\right] = -\frac{1}{2}\int_0^1 \left[\frac{1-x}{4} + \frac{1}{2} - \frac{1}{x+1}\right]dx$$

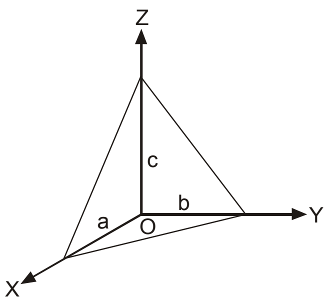
$$= -\frac{1}{2}\left[-\frac{(1-x)^2}{8} + \frac{x}{2} - \log(x+1)\right]_0^1 = -\frac{1}{2}\left[\frac{1}{2} - \log 2 + \frac{1}{8}\right] = -\frac{1}{2}\left[\frac{5}{8} - \log 2\right]$$

$$= \frac{1}{2}\log 2 - \frac{5}{16}$$ **Ans.**

Example 37. *Evaluate* $\iiint x^2yz\ dx\,dy\,dz$ *throughout the volume bounded by the planes $x = 0$, $y = 0$, $z = 0$, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.* (*M.U. II Semester* 2003, 2002, 2001)

Solution. Here, we have

$$I = \iiint x^2yz\ dx\,dy\,dz \quad ...(1)$$



Putting $x = au, \quad y = bv, \quad z = cw$

$dx = a\,du, \quad dy = b\,dv, \quad dz = c\,dw$ in (1), we get

$$I = \iiint a^2bc\,u^2vw\,a\,bc\,du\,dv\,dw$$

Limits are for $u = 0, 1$ for $v = 0, 1 - u$ and for $w = 0, 1 - u - v$

$$u + v + w = 1$$

$$I = \int_{u=o}^1 \int_{v=0}^{1-u} \int_{w=0}^{1-u-v} a^3b^2c^2\,u^2vw\,du\,dv\,dw = \int_0^1 \int_0^{1-u} a^3b^2c^2\,u^2v\left[\frac{w^2}{2}\right]_0^{1-u-v} du\,dv$$

$$= \frac{a^3b^2c^2}{2}\int_0^1 \int_0^{1-u} u^2v(1-u-v)^2\,du\,dv$$

$$= \frac{a^3b^2c^2}{2}\int_0^1 \int_0^{1-u} u^2v\left[(1-u)^2 - 2(1-u)v + v^2\right]du\,dv$$

$$= \frac{a^3b^2c^2}{2}\int_0^1\int_0^{1-u} u^2[(1-u)^2 v - 2(1-u)v^2 + v^3]\, du\, dv$$

$$= \frac{a^3b^2c^2}{2}\int_0^1 u^2\left[(1-u)^2\frac{v^2}{2} - 2(1-u)\frac{v^3}{3} + \frac{v^4}{4}\right]_0^{1-u} du$$

$$= \frac{a^3b^2c^2}{2}\int_0^1 u^2\left[\frac{(1-u)^4}{2} - \frac{2(1-u)^4}{3} + \frac{(1-u)^4}{4}\right] du$$

$$= \frac{a^3b^2c^2}{2}\int_0^1 \frac{u^2(1-u)^4}{12}\, du = \frac{a^3b^2c^2}{24}\int_0^1 u^{3-1}(1-u)^{5-1}\, du$$

$$= \frac{a^3b^2c^2}{24}\beta(3,5) = \frac{a^3b^2c^2}{24}\cdot\frac{\lceil 3\, \lceil 5}{\lceil 8} = \frac{a^3b^2c^2}{24}\cdot\left(\frac{2!\,4!}{7!}\right) = \frac{a^3\,b^2\,c^2}{2520}. \qquad \textbf{Ans.}$$

2.12 INTEGRATION BY CHANGE OF CARTESIAN COORDINATES INTO SPHERICAL COORDINATES

Sometime it becomes easy to integrate by changing the cartesian coordinates into spherical coordinates.

The relations between the cartesian and spherical polar co-ordinates of a point are given by the relations

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \\ dx\,dy\,dz &= |J|\,dr\,d\theta\,d\phi \\ &= r^2\sin\theta\,dr\,d\theta\,d\phi \end{aligned}$$

Note. 1. Spherical coordinates are very useful if the expression $x^2 + y^2 + z^2$ is involved in the problem.

2. In a sphere $x^2 + y^2 + z^2 = a^2$ the limits of r are 0 and a and limits of θ are $0, \pi$ and that of ϕ are 0 and 2π.

Example 38. *Evaluate the integral* $\iiint (x^2 + y^2 + z^2)\, dx\, dy\, dz$ *taken over the volume enclosed by the sphere* $x^2 + y^2 + z^2 = 1$.

Solution. Let us convert the given integral into spherical polar co-ordinates. By putting

$x = r\sin\theta\cos\phi$; $y = r\sin\theta\sin\phi$; $z = r\cos\theta$

$$\iiint (x^2 + y^2 + z^2)\, dx\, dy\, dz = \int_0^{2\pi}\int_0^{\pi}\int_0^1 r^2(r^2\sin\theta\, d\theta\, d\phi\, dr)$$

$$= \int_0^{2\pi} d\phi\int_0^{\pi}\sin\theta\, d\theta\int_0^1 r^4 dr = \int_0^{2\pi} d\phi\int_0^{\pi}\sin\theta\, d\theta\left(\frac{r^5}{5}\right)_0^1 = \frac{1}{5}\int_0^{2\pi} d\phi\,[-\cos\theta]_0^{\pi} = \frac{2}{5}\int_0^{2\pi} d\phi$$

$$= \frac{2}{5}(\phi)_0^{2\pi} = \frac{4\pi}{5} \qquad \textbf{Ans.}$$

Example 39. *Evaluate* $\iiint (x^2 + y^2 + z^2)\, dx\, dy\, dz$ *over the first octant of the sphere* $x^2 + y^2 + z^2 = a^2$. (*M.U. II Semester 2007*)

Solution. Here, we have

$$I = \iiint (x^2 + y^2 + z^2)\, dx\, dy\, dz \qquad \ldots(1)$$

Putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ and $dx\,dy\,dz = r^2\sin\theta\, dr\, d\theta\, d\phi$ in (1), we get

Limits of r are $0, a$ for θ are $0, \frac{\pi}{2}$ for ϕ are $0, \frac{\pi}{2}$.

$$I = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^a r^2 . r^2 \sin\theta \, dr \, d\theta \, d\phi = \int_0^{\frac{\pi}{2}} d\phi \int_0^{\frac{\pi}{2}} \sin\theta \, d\theta \int_0^a r^4 \, dr$$

$$\left(\begin{aligned} x^2 + y^2 + z^2 &= r^2 \sin^2\theta \cos^2\phi + r^2 \sin^2\theta \sin^2\phi + r^2 \cos^2\theta \\ &= r^2 \sin^2\theta + r^2 \cos^2\theta = r^2 \end{aligned} \right)$$

$$= [\phi]_0^{\pi/2} [-\cos\theta]_0^{\pi/2} \left[\frac{r^5}{5}\right]_0^a = \frac{\pi}{2} . (1) . \frac{a^5}{5} = \pi . \frac{a^5}{10}.$$ **Ans.**

Example 40. *Evaluate* $\iiint \frac{dx\, dy\, dz}{x^2 + y^2 + z^2}$ *throughout the volume of the sphere* $x^2 + y^2 + z^2 = a^2$.

(*M.U. II Semester 2002, 2001*)

Solution. Here, we have

$$I = \iiint \frac{dx\, dy\, dz}{x^2 + y^2 + z^2} \qquad ...(1)$$

Putting $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r\cos\theta$ and $dx\, dy\, dz = r^2 \sin\theta \, dr\, d\theta\, d\phi$ in (1), we get

The limits of r are 0 and a, for θ are 0 and $\frac{\pi}{2}$ for ϕ are 0 and $\frac{\pi}{2}$ in first octant.

$$I = 8 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^a \frac{r^2 \sin\theta \, dr\, d\theta\, d\phi}{r^2} \qquad [\text{Sphere } x^2 + y^2 + z^2 \text{ lies in 8 quadrants}]$$

$$I = 8 \int_0^{\frac{\pi}{2}} d\phi \int_0^{\frac{\pi}{2}} \sin\theta \, d\theta \int_0^a dr = 8 [\phi]_0^{\pi/2} [-\cos\theta]_0^{\pi/2} [r]_0^a = 8\left(\frac{\pi}{2} - 0\right)(0+1)(a+0)$$

$$= 8\frac{\pi}{2} . 1 . a = 4\pi a$$ **Ans.**

EXERCISE 2.9

Evaluate the following :

1. $\int_{-1}^{1} \int_{-2}^{2} \int_{-3}^{3} dx\, dy\, dz$ (*M.U., II Semester 2002*) **Ans.** 48

2. $\int_0^4 \int_0^x \int_0^{x+y} z\, dz\, dy\, dx$ (*R.G.P.V. Bhopal I Sem. 2003*) **Ans.** 70

3. $\int_1^2 \int_0^1 \int_{-1}^1 (x^2 + y^2 + z^2)\, dx\, dy\, dz$ **Ans.** 6

4. $\int_0^1 \int_0^1 \int_0^1 (x^2 + y^2 + z^2)\, dz\, dy\, dx$ (*AMIETE, June 2006*) **Ans.** 1

5. $\int_{-1}^1 \int_0^z \int_{x-z}^{x+z} (x - y + z)\, dx\, dy\, dz$ (*AMIETE, Summer 2004*) **Ans.** 0

6. $\iiint_R (x - y - z)\, dx\, dy\, dz$, where $R : 1 \le x \le 2;\ 2 \le y \le 3;\ 1 \le z \le 3$ **Ans.** 2

7. $\int_0^2 \int_1^3 \int_1^2 xy^2 z\, dx\, dy\, dz$ (*AMIETE, Dec. 2007*) **Ans.** 26

8. $\int_0^1 dx \int_0^2 dy \int_1^2 x^2\, yz\, dz$ **Ans.** 1

9. $\iiint x^2 yz\, dx\, dy\, dz$ throughout the volume bounded by $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$.

(*M.U. II Semester, 2003*) **Ans.** $\frac{1}{2520}$

10. $\int_0^1 \int_0^{1-x} \int_0^{1-x^2-y^2} dz\, dy\, dx$ **Ans.** $\frac{1}{3}$

11. $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z\, dz\, dx\, dy$ **Ans.** $\frac{1}{2}(e^2 - 8e + 13)$

12. $\iiint_T y\,dx\,dy\,dz$, where T is the region bounded by the surfaces $x = y^2$, $x = y + 2$, $4z = x^2 + y^2$ and $z = y + 3$. (*AMIETE Dec. 2008*)

13. $\int_0^2 \int_0^x \int_0^{2x+2y} e^{x+y+z}\,dz\,dy\,dx$ **Ans.** $\frac{1}{3}\left[\frac{e^{12}}{6} - \frac{e^6}{3} - \frac{1}{6} + \frac{1}{3}\right] - \frac{1}{2}[e^4 - 1] + [e^2 - 1]$ (*M.U. II Sem., 2003*)

14. $\iiint (x + y + z)\,dx\,dy\,dz$ over the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$. **Ans.** $\frac{1}{8}$

15. $\int_0^a \int_0^{a-x} \int_0^{a-x-y} x^2\,dx\,dy\,dz$ **Ans.** $\frac{a^5}{60}$

16. $\int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx$ **Ans.** $8\sqrt{2}\pi$

17. $\int_{-1}^{1} \int_0^z \int_{x-z}^{x+z} (x + y + z)\,dz\,dx\,dy$ (*M.U. II Semester, 2000, 02*) **Ans.** 0

18. $\int_0^2 \int_0^y \int_{x-y}^{x+y} (x + y + z)\,dx\,dy\,dz$ (*M.U. II Semester 2004*) **Ans.** 16

19. $\iiint \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}}\,dx\,dy\,dz$ throughout the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Ans. $\frac{\pi^2}{4} abc$

20. $\iiint \sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}}\,dx\,dy\,dz$ over the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. **Ans.** $\frac{4\pi}{3} abc$

21. $\iiint x^{l-1} y^{m-1} z^{n-1}\,dx\,dy\,dz$ throughout the volume of the tetrahedron

$x \ge 0$, $y \ge 0$, $z \ge 0$, $x + y + z \le 1$. **Ans.** $\frac{1}{(l+m+n)} \cdot \frac{\overline{|l}\,\overline{|m}\,\overline{|n}}{\overline{|l+m+n}}$

22. $\iiint \frac{dx\,dy\,dz}{\sqrt{1 - x^2 - y^2 - z^2}}$ taken throughout the volume of the sphere $x^2 + y^2 + z^2 = 1$, lying in the first octant. **Ans.** $\frac{\pi^2}{8}$

23. $\int_0^{\pi} 2d\theta \int_0^{a(1+\cos\theta)} r\,dr \int_0^h \left[1 - \frac{r}{a(1+\cos\theta)}\right] dz$ **Ans.** $\frac{\pi a^2}{2} h$

24. $\int_0^{\pi/2} \int_0^{a\sin\theta} \int_0^{(a^2-r^2)/a} r\,d\theta\,dr\,dz$ **Ans.** $\frac{5a^3}{64}$

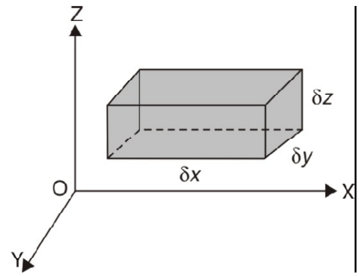
25. $\iiint z^2\,dx\,dy\,dz$ over the volume common to the sphere $x^2 + y^2 + z^2 = a^2$ and the cylinder $x^2 + y^2 + z^2 = ax$. **Ans.** $\frac{2a^5\pi}{15}$

26. $\iiint_V \frac{dx\,dy\,dz}{(1 + x^2 + y^2 + z^2)^2}$ where V is the volume in the first octant. **Ans.** $\frac{\pi^2}{8}$

27. $\iiint \frac{dx\,dy\,dz}{(x^2 + y^2 + z^2)^{3/2}}$ over the volume bounded by the spheres $x^2 + y^2 + z^2 = 16$ and $x^2 + y^2 + z^2 = 25$. (*M.U. II Semester, 2001, 03*) **Ans.** $4\pi \log (5/4)$

28. $\iiint_T z^2\,dx\,dy\,dz$ over the volume bounded by the cylinder $x^2 + y^2 = a^2$ and the paraboloid $x^2 + y^2 = z$ and the plane $z = 0$. **Ans.** $\frac{\pi a^8}{12}$

2.13 VOLUME = $\iiint dx\, dy\, dz$.



The elementary volume δv is $\delta x \,.\, \delta y \,.\, \delta z$ and therefore the volume of the whole solid is obtained by evaluating the triple integral.

$$\delta V = \delta x\, \delta y\, \delta z$$

$$\boxed{V = \iiint dx\, dy\, dz.}$$

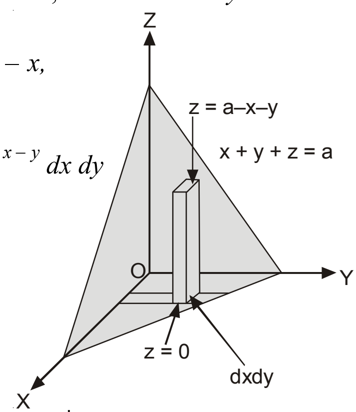
Note : (*i*) Mass = volume × density = $\iiint \rho\, dx\, dy\, dz$ if ρ is the density.

(*ii*) In cylindrical co-ordinates, we have $\boxed{V = \iiint_V r\, dr\, d\varphi\, dz}$

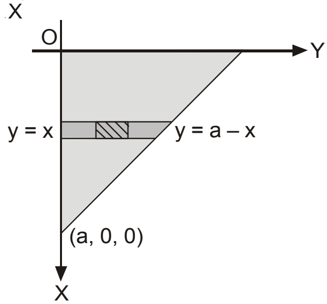
(*iii*) In spherical polar co-ordinates, we have $\boxed{V = \iiint_V r^2 \sin\theta dr\, d\theta\, d\varphi}$

Example 41. *Find the volume of the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = a$.* (*M.U. II Semester, 2005, 2000*)

Solution. Here, we have a solid which is bounded by $x = 0$, $y = 0$, $z = 0$ and $x + y + z = a$ planes.



The limits of z are 0 and $a - x - y$, the limits of y are 0 and $1 - x$, the limits of x are 0 and a.



$$V = \int_{x=0}^{a} \int_{y=0}^{a-x} \int_{z=0}^{a-x-y} dx\, dy\, dz = \int_{x=0}^{a} \int_{y=0}^{a-x} [z]_0^{a-x-y}\, dx\, dy$$

$$= \int_{x=0}^{a} \int_{y=0}^{a-x} (a - x - y)\, dx\, dy$$

$$= \int_{x=0}^{a} \left[ay - xy - \frac{y^2}{2}\right]_0^{a-x} dx$$

$$= \int_0^a \left[a(a-x) - x(a-x) - \frac{(a-x)^2}{2}\right] dx$$

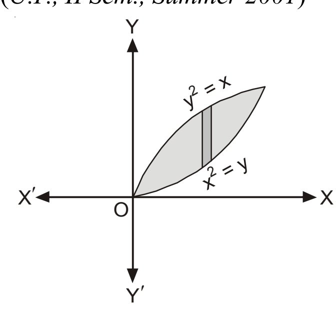
$$= \int_0^a \left[a^2 - ax - ax + x^2 - \frac{a^2}{2} + ax - \frac{x^2}{2}\right] dx$$

$$= \int_0^a \left(\frac{a^2}{2} - ax + \frac{x^2}{2}\right) dx$$

$$= \left[\frac{a^2}{2}\,.\,x - \frac{ax^2}{2} + \frac{x^3}{6}\right]_0^a = a^3\left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right) = \frac{a^3}{6}. \quad \textbf{Ans.}$$

Example 42. *Find the volume of the cylindrical column standing on the area common to the parabolas $y^2 = x$, $x^2 = y$ and cut off by the surface $z = 12 + y - x^2$.* (*U.P., II Sem., Summer 2001*)

Solution. We have,



$$y^2 = x$$
$$x^2 = y$$
$$z = 12 + y - x^2$$

$$V = \int_0^1 dx \int_{x^2}^{\sqrt{x}} dy \int_0^{12+y-x^2} dz = \int_0^1 dx \int_{x^2}^{\sqrt{x}} (12 + y - x^2)\, dy$$

$$= \int_0^1 dx \left(12y + \frac{y^2}{2} - x^2 y\right)_{x^2}^{\sqrt{x}}$$

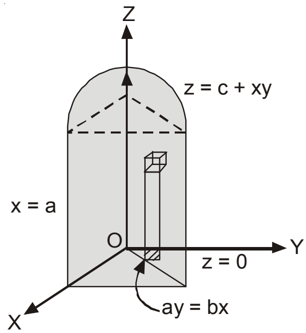
$$= \int_0^1 \left(12\sqrt{x} + \frac{x}{2} - x^{5/2} - 12x^2 - \frac{x^4}{2} + x^4\right) dx$$

$$= \left[\frac{2}{3} \times 12x^{3/2} + \frac{x^2}{4} - \frac{2}{7}x^{7/2} - 4x^3 - \frac{x^5}{10} + \frac{x^5}{5}\right]_0^1$$

$$= 8 + \frac{1}{4} - \frac{2}{7} - 4 - \frac{1}{10} + \frac{1}{5} = 4 + \frac{1}{4} - \frac{2}{7} - \frac{1}{10} + \frac{1}{5} = \frac{560 + 35 - 40 - 14 + 28}{140} = \frac{569}{140}$$ **Ans.**

Example 43. *A triangular prism is formed by planes whose equations are ay = bx, y = 0 and x = a. Find the volume of the prism between the planes z = 0 and surface z = c + xy.*

(*M.U. II Semester 2000; U.P., Ist Semester, 2009 (C.O) 2003*)



Solution. Required volume $= \int_0^a \int_0^{\frac{bx}{a}} \int_0^{c+xy} dz\, dy\, dx$

$$= \int_0^a \int_0^{\frac{bx}{a}} (c + xy)\, dy\, dx$$

$$= \int_0^a \left(cy + \frac{xy^2}{2}\right)_0^{\frac{bx}{a}} dx$$

$$= \int_0^a \left(\frac{cbx}{a} + \frac{b^2}{2a^2}x^3\right) dx = \frac{bc}{a}\left(\frac{x^2}{2}\right)_0^a + \frac{b^2}{2a^2}\left(\frac{x^4}{4}\right)_0^a$$

$$= \frac{abc}{2} + \frac{b^2a^2}{8} = \frac{ab}{8}(4c + ab)$$ **Ans.**

2.14 VOLUME OF SOLID BOUNDED BY SPHERE OR BY CYLINDER

We use spherical coordinates (r, θ, ϕ) and the cylindrical coordinates are (ρ, ϕ, z) and the relations are $x = \rho \cos \phi$, $y = \rho \sin \phi$.

Example 44. *Find the volume of a solid bounded by the spherical surface $x^2 + y^2 + z^2 = 4a^2$ and the cylinder $x^2 + y^2 - 2ay = 0$.*

Solution. $x^2 + y^2 + z^2 = 4a^2$ ...(1)

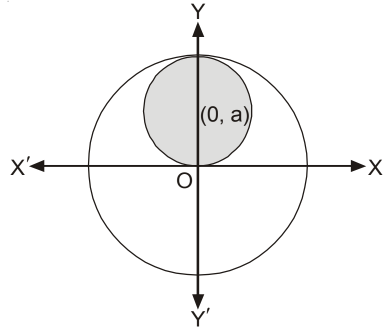
$x^2 + y^2 - 2ay = 0$ ...(2)

Considering the section in the positive quadrant of the *xy*-plane and taking *z* to be positive (that is volume above the *xy*-plane) and changing to polar co-ordinates, (1) becomes

$$r^2 + z^2 = 4a^2 \quad \Rightarrow \quad z^2 = 4a^2 - r^2$$

$$\therefore \quad z = \sqrt{4a^2 - r^2}$$

(2) becomes $r^2 - 2ar \sin\theta = 0 \Rightarrow r = 2a \sin\theta$



$$\text{Volume} = \iiint dx\, dy\, dz$$

$$= 4\int_0^{\pi/2} d\theta \int_0^{2a\sin\theta} r\, dr \int_0^{\sqrt{4a^2 - r^2}} dz$$ (Cylindrical coordinates)

$$= 4\int_0^{\pi/2} d\theta \int_0^{2a\sin\theta} r\,dr\,[z]_0^{\sqrt{4a^2-r^2}} = 4\int_0^{\pi/2} d\theta \int_0^{2a\sin\theta} r\,dr\,.\sqrt{4a^2-r^2}$$

$$= 4\int_0^{\pi/2} d\theta \left[-\frac{1}{3}(4a^2-r^2)^{3/2}\right]_0^{2a\sin\theta} = \frac{4}{3}\int_0^{\pi/2}\left[-(4a^2-4a^2\sin^2\theta)^{3/2}+8a^3\right]d\theta$$

$$= \frac{4}{3}\int_0^{\pi/2}(-8a^3\cos^3\theta+8a^3)\,d\theta = \frac{8\times 4a^3}{3}\int_0^{\pi/2}(1-\cos^3\theta)\,d\theta$$

$$= \frac{32a^3}{3}\int_0^{\pi/2}\left(1-\frac{1}{4}\cos 3\theta-\frac{3}{4}\cos\theta\right)d\theta$$

$$= \frac{32a^3}{3}\left[\theta-\frac{1}{12}\sin 3\theta-\frac{3}{4}\sin\theta\right]_0^{\pi/2} = \frac{32a^3}{3}\left(\frac{\pi}{2}+\frac{1}{12}-\frac{3}{4}\right) = \frac{32a^3}{3}\left[\frac{\pi}{2}-\frac{2}{3}\right] \textbf{ Ans.}$$

Example 45. *Find the volume enclosed by the solid*

$$\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}+\left(\frac{z}{c}\right)^{2/3} = 1$$

Solution. The equation of the solid is

$$\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}+\left(\frac{z}{c}\right)^{2/3} = 1$$

Putting $\left(\frac{x}{a}\right)^{1/3} = u \Rightarrow x = a\,u^3 \Rightarrow dx = 3\,au^2\,du$

$\left(\frac{y}{b}\right)^{1/3} = v \Rightarrow y = b\,v^3 \Rightarrow dy = 3\,bv^2\,dv$

$\left(\frac{z}{c}\right)^{1/3} = w \Rightarrow z = c\,w^3 \Rightarrow dz = 3\,c\,w^2\,dw$

The equation of the solid becomes

$$u^2+v^2+w^2 = 1 \qquad \text{...(1)}$$

$$V = \iiint dx\,dy\,dz \qquad \text{...(2)}$$

On putting the values of dx, dy and dz in (2), we get

$$V = \iiint 27abc\,u^2v^2w^2du\,dv\,dw \qquad \text{...(3)}$$

(1) represents a sphere.
Let us use spherical coordinates.

$$u = r\sin\theta\cos\phi, \qquad v = r\sin\theta\sin\phi,$$
$$w = r\cos\theta, \qquad du\,dv\,dw = r^2\sin\theta\,dr\,d\theta\,d\phi$$

On substituting spherical coordinates in (3), we have

$$V = 27abc\,.\,8\int_{r=0}^{1}\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2} r^2\sin^2\theta\cos^2\phi\,.\,r^2\sin^2\theta\sin^2\phi\,.\,r^2\cos^2\theta\,.\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= 216\,abc\int_{r=0}^{1} r^8\,dr\int_{\phi=0}^{\pi/2}\sin^2\phi\cos^2\phi\,d\phi\int_{\theta=0}^{\pi/2}\sin^5\theta\cos^2\theta\,d\theta$$

$$= 216\,abc\left[\frac{r^9}{9}\right]_0^1.\left(\frac{\Gamma\frac{3}{2}\,\Gamma\frac{3}{2}}{2\,\Gamma 3}\right)\left(\frac{\Gamma 3\,\Gamma\frac{3}{2}}{2\,\Gamma\frac{9}{2}}\right) = 24\,abc\,.\,\frac{1}{2}\,.\,\frac{\Gamma\frac{3}{2}\,\Gamma\frac{3}{2}}{\Gamma 3}\,.\,\frac{1}{2}\,.\,\frac{\Gamma 3\,\Gamma\frac{3}{2}}{\Gamma\frac{9}{2}}$$

$$= 6abc\,.\frac{\left[\left(\frac{1}{2}\right)\Gamma\frac{1}{2}\right]^2}{2!}\,.\,\frac{2!\,\Gamma\frac{3}{2}}{\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\frac{3}{2}\,\Gamma\frac{3}{2}} = 6abc\,.\frac{1}{4}\,.\,\pi\,\frac{1}{\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)} = \frac{4}{35}\,abc\,\pi$$

Ans.

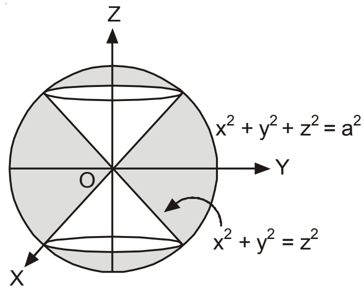
Example 46. *Find the volume bounded above by the sphere $x^2 + y^2 + z^2 = a^2$ and below by the cone $x^2 + y^2 = z^2$.* (*U.P. II Semester 2002*)

Solution. The equation of the sphere is $x^2 + y^2 + z^2 = a^2$ ...(1)

and that of the cone is $x^2 + y^2 = z^2$ ...(2)

In polar coordinates $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r\cos\theta$

The equation (1) in polar co-ordinates is



$$(r \sin\theta \cos\phi)^2 + (r \sin\theta \sin\phi)^2 + (r\cos\theta)^2 = a^2$$

$$\Rightarrow \quad r^2 \sin^2\theta \cos^2\phi + r^2 \sin^2\theta \sin^2\phi + r^2\cos^2\theta = a^2$$

$$\Rightarrow \quad r^2 \sin^2\theta\,(\cos^2\phi + \sin^2\phi) + r^2\cos^2\theta = a^2$$

$$\Rightarrow \quad r^2 \sin^2\theta + r^2\cos^2\theta = a^2$$

$$\Rightarrow \quad r^2 (\sin^2\theta + \cos^2\theta) = a^2$$

$$\Rightarrow \quad r^2 = a^2 \quad \Rightarrow \quad r = a$$

The equation (2) in polar co-ordinates is

$$(r \sin\theta \cos\phi)^2 + (r \sin\theta \sin\phi)^2 = (r\cos\theta)^2$$

$$\Rightarrow \quad r^2 \sin^2\theta\,(\cos^2\phi + \sin^2\phi) = r^2\cos^2\theta \;\Rightarrow\; r^2\sin^2\theta = r^2\cos^2\theta$$

$$\Rightarrow \quad \tan^2\theta = 1 \;\Rightarrow\; \tan\theta = 1 \Rightarrow \quad \theta = \pm\frac{\pi}{4}$$

Thus equations (1) and (2) in polar coordinates are respectively,

$$r = a \quad \text{and} \quad \theta = \pm\frac{\pi}{4}$$

The volume in the first octant is one fourth only.

Limits in the first octant : r varies 0 to a, θ from 0 to $\frac{\pi}{4}$ and ϕ from 0 to $\frac{\pi}{2}$.

The required volume lies between $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 = z^2$.

$$V = 4\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}}\int_0^{a} r^2 \sin\theta\, dr\, d\theta\, d\phi = 4\int_0^{\frac{\pi}{2}} d\phi \int_0^{\frac{\pi}{4}} \sin\theta\, d\theta \left[\frac{r^3}{3}\right]_0^a$$

$$= 4\int_0^{\frac{\pi}{2}} d\phi \int_0^{\frac{\pi}{4}} \sin\theta\, d\theta\,.\,\frac{a^3}{3} = \frac{4a^3}{3}\int_0^{\frac{\pi}{2}} d\phi\,[-\cos\theta]_0^{\frac{\pi}{4}} = \frac{4a^3}{3}(\phi)_0^{\frac{\pi}{2}}\left[-\frac{1}{\sqrt{2}} + 1\right]$$

$$= \frac{2}{3}\pi a^3\left(1 - \frac{1}{\sqrt{2}}\right)$$

Ans.

2.15 VOLUME OF SOLID BOUNDED BY CYLINDER OR CONE

We use cylindrical coordinates (r, θ, z).

Example 47. *Find the volume of the solid bounded by the parabolic $y^2 + z^2 = 4x$ and the plane $x = 5$.*

Solution. $y^2 + z^2 = 4x$, $x = 5$

$$V = \int_0^5 dx \int_{-2\sqrt{x}}^{2\sqrt{x}} dy \int_{-\sqrt{4x - y^2}}^{\sqrt{4x - y^2}} dz = 4\int_0^5 dx \int_0^{2\sqrt{x}} dy \int_0^{\sqrt{4x - y^2}} dz$$

$$= 4\int_0^5 dx \int_0^{2\sqrt{x}} dy\,[z]_0^{\sqrt{4x-y^2}} = 4\int_0^5 dx \int_0^{2\sqrt{x}} dy\,\sqrt{4x-y^2}$$

$$= 4\int_0^5 dx\left[\frac{y}{2}\sqrt{4x-y^2} + \frac{4x}{2}\sin^{-1}\frac{y}{2\sqrt{x}}\right]_0^{2\sqrt{x}} = 4\int_0^5\left[0 + 2x\left(\frac{\pi}{2}\right)\right]dx = 4\pi\int_0^5 x\,dx$$

$$= 4\pi\left[\frac{x^2}{2}\right]_0^5 = 50\pi \qquad \textbf{Ans.}$$

Example 48. *Calculate the volume of the solid bounded by the following surfaces :*

$$z = 0, \quad x^2 + y^2 = 1, \quad x + y + z = 3$$

Solution.
$$x^2 + y^2 = 1 \qquad ...(1)$$
$$x + y + z = 3 \qquad ...(2)$$
$$z = 0 \qquad ...(3)$$

$$\text{Required Volume} = \iiint dx\,dy\,dz = \iint dx\,dy\,[z]_0^{3-x-y} = \iint (3 - x - y)\,dx\,dy$$

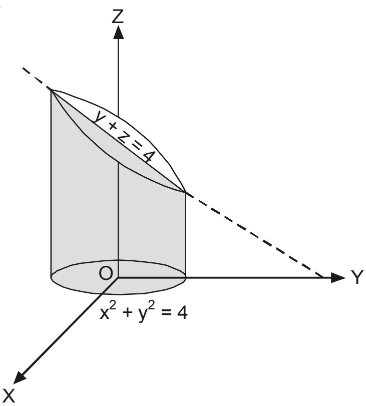
On putting $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,d\theta\,dr$, we get

$$= \iint (3 - r\cos\theta - r\sin\theta)\,r\,d\theta\,dr = \int_0^{2\pi} d\theta \int_0^1 (3r - r^2\cos\theta - r^2\sin\theta)\,dr$$

$$= \int_0^{2\pi} d\theta\left(\frac{3r^2}{2} - \frac{r^3}{3}\cos\theta - \frac{r^3}{3}\sin\theta\right)_0^1 = \int_0^{2\pi}\left(\frac{3}{2} - \frac{1}{3}\cos\theta - \frac{1}{3}\sin\theta\right)d\theta$$

$$= \left[\frac{3}{2}\theta - \frac{1}{3}\sin\theta + \frac{1}{3}\cos\theta\right]_0^{2\pi} = 3\pi - \frac{1}{3}\sin 2\pi + \frac{1}{3}\cos 2\pi - \frac{1}{3} = 3\pi \qquad \textbf{Ans.}$$

Example 49. *Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.*



Solution. $x^2 + y^2 = 4 \Rightarrow y = \pm\sqrt{4 - x^2}$

$y + z = 4 \;\Rightarrow\; z = 4 - y$ and $z = 0$

x varies from -2 to $+2$.

$$V = \iiint dx\,dy\,dz = \int_{-2}^{2} dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy \int_0^{4-y} dz$$

$$= \int_{-2}^{2} dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy\,[z]_0^{4-y}$$

$$= \int_{-2}^{2} dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy\,(4 - y) = \int_{-2}^{2} dx\left[4y - \frac{y^2}{2}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}$$

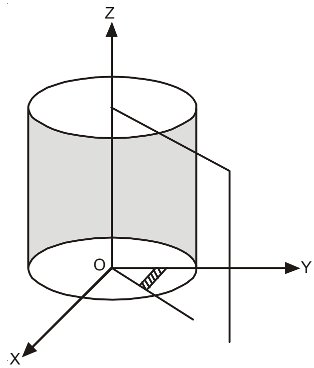
$$= \int_{-2}^{2} dx\left[4\sqrt{4-x^2} - \frac{1}{2}(4 - x^2) + 4\sqrt{4-x^2} + \frac{1}{2}(4 - x^2)\right]$$

$$= 8\int_{-2}^{2}\sqrt{4-x^2}\,dx = 8\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_{-2}^{2} = 16\pi \qquad \textbf{Ans.}$$

Example 50. *Find the volume in the first octant bounded by the cylinder $x^2 + y^2 = 2$ and the planes $z = x + y$, $y = x$, $z = 0$ and $x = 0$.* (*M.U. II Semester 2005*)

Solution. Here, we have the solid bounded by

$$x^2 + y^2 = 2 \text{ (cylinder)}$$
$$(\text{or} \quad r^2 = 2)$$
$$z = x + y \quad \Rightarrow \quad z = r(\cos\theta + \sin\theta) \quad \text{(plane)}$$
$$y = x \quad \Rightarrow \quad r\sin\theta = r\cos\theta \quad \text{(plane)}$$



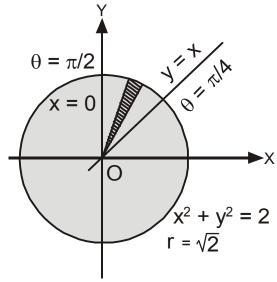
$$\Rightarrow \quad \tan\theta = 1 \quad \Rightarrow \quad q = \frac{\pi}{4}$$

$$x = 0 \Rightarrow r\cos\theta = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$$

z varies from 0 to $r(\cos\theta + \sin\theta)$

r varies from 0 to $\sqrt{2}$

θ varies from $\frac{\pi}{4}$ to $\frac{\pi}{2}$



$$\therefore \quad V = \int_{\theta=\pi/4}^{\pi/2} \int_{r=0}^{\sqrt{2}} \int_{z=0}^{r(\cos\theta + \sin\theta)} r\, dr\, d\theta\, dz$$

$$= \int_{\theta=\pi/4}^{\pi/2} \int_{r=0}^{\sqrt{2}} r[z]_0^{r(\cos\theta + \sin\theta)}\, dr\, d\theta$$

$$= \int_{\theta=\pi/4}^{\pi/2} \int_{r=0}^{\sqrt{2}} r^2 (\cos\theta + \sin\theta)\, dr\, d\theta$$

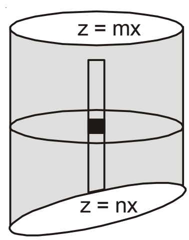
$$= \int_{\theta=\pi/4}^{\pi/2} (\cos\theta + \sin\theta)\left[\frac{r^3}{3}\right]_0^{\sqrt{2}} d\theta = \frac{2\sqrt{2}}{3}\int_{\theta=\pi/4}^{\pi/2} (\cos\theta + \sin\theta)\, d\theta$$

$$= \frac{2\sqrt{2}}{3}\left[\sin\theta - \cos\theta\right]_{\pi/4}^{\pi/2} = \frac{2\sqrt{2}}{3}\left[(1-0) - \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)\right] = \frac{2\sqrt{2}}{3}$$ **Ans.**

Example 51. *Show that the volume of the wedge intercepted between the cylinder $x^2 + y^2 = 2ax$ and planes $z = mx$, $z = nx$ is $\pi(m - n)\, a^3$.* (*M.U. II Semester, 2000*)

Solution. The equation of the cylinder is $x^2 + y^2 = 2\,a\,x$

we convert the cartesian coordinates into cylindrical coordinates.



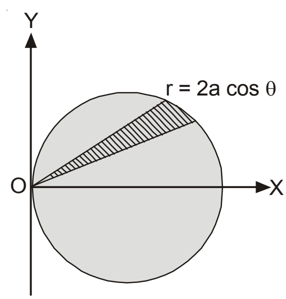
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$x^2 + y^2 = 2ax \quad \Rightarrow \quad r^2 = 2ar\cos\theta$$
$$\Rightarrow \quad r = 2a\cos\theta$$

r varies from 0 to $2a\cos\theta$

θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

and z varies from $z = nx$ ($z = nr\cos\theta$) to $z = m\,x$ ($z = m\,r\cos\theta$)

$$V = 2\int_{\theta=0}^{\pi/2} \int_{r=0}^{2a\cos\theta} \int_{z=nr\cos\theta}^{mr\cos\theta} r\, dr\, d\theta\, dz$$



$$= 2\int_{\theta=0}^{\pi/2} \int_{r=0}^{2a\cos\theta} r[z]_{nr\cos\theta}^{mr\cos\theta}\, dr\, d\theta$$

$$= 2\int_{\theta=0}^{\pi/2} \int_{r=0}^{2a\cos\theta} r.(m-n)\, r\cos\theta\, dr\, d\theta$$

$$= 2(m-n)\int_{\theta=0}^{\pi/2} \int_{r=0}^{2a\cos\theta} r^2 \cos\theta\, dr\, d\theta$$

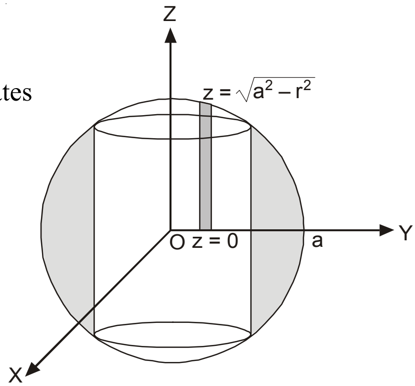
$$= 2(m-n)\int_{\theta=0}^{\pi/2}\left[\frac{r^3}{3}\right]_0^{2a\cos\theta}\cos\theta\, d\theta = 2(m-n)\int_{\theta=0}^{\pi/2}\frac{8a^3}{3}\cos^3\theta\cos\theta\, d\theta$$

$$= \frac{16(m-n)}{3}a^3\int_{\theta=0}^{\pi/2}\cos^4\theta\, d\theta = \frac{16(m-n)}{3}\,.\,a^3\,.\,\frac{3}{4}\,.\,\frac{1}{2}\,.\,\frac{\pi}{2} = (m-n)\pi a^3 \quad \textbf{Ans.}$$

Example 52. *A cylindrical hole of radius b is bored through a sphere of radius a. Find the volume of the remaining solid.* (*M.U. II Semester 2004*)

Solution. Let the equation of the sphere be

$$x^2 + y^2 + z^2 = a^2$$



Now, we will solve this problem using cylindrical coordinates

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

Limits of z are 0 and $\sqrt{a^2-(x^2+y^2)}$ *i.e.*, $\sqrt{a^2-r^2}$

Limits of r are a and b.

and the limits of θ are 0 and $\frac{\pi}{2}$

$$V = 8\int_{\theta=0}^{\pi/2}\int_{r=b}^{a}\int_{z=0}^{\sqrt{a^2-r^2}} r\,dr\,d\theta\,dz = 8\int_{\theta=0}^{\pi/2}\int_{r=b}^{a}[z]_0^{\sqrt{a^2-r^2}}\, r\,dr\,d\theta$$

$$= 8\int_{\theta=0}^{\pi/2}\int_{r=b}^{a}(a^2-r^2)^{1/2}\,.\,r\,dr\,d\theta$$

$$= 8\int_{\theta=0}^{\pi/2}\left[\frac{(a^2-r^2)^{3/2}}{3/2}\cdot\left(-\frac{1}{2}\right)\right]_b^a d\theta = -\frac{8}{3}\int_0^{\pi/2}-(a^2-b^2)^{\frac{3}{2}}\,d\theta$$

$$= \frac{8}{3}(a^2-b^2)^{\frac{3}{2}}[\theta]_0^{\pi/2} = \frac{4\pi}{3}(a^2-b^2)^{\frac{3}{2}} \quad \textbf{Ans.}$$

Example 53. *Find the volume cut off from the paraboloid*

$x^2 + \frac{y^2}{4} + z = 1$ by the plane $z = 0$. (*M.U. II Semester 2005*)

Solution. We have

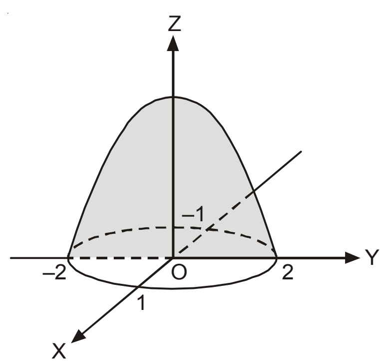
$$x^2 + \frac{y^2}{4} + z = 1 \quad \text{(Paraboloid)} \qquad ...(1)$$

$$z = 0 \quad (x\text{-}y \text{ plane}) \qquad ...(2)$$

z varies from 0 to $1 - x^2 - \frac{y^2}{4}$

y varies from $-2\sqrt{1-x^2}$ to $2\sqrt{1-x^2}$

x varies from –1 to 1.



$$V = \iiint dx\,dy\,dz = \int_{-1}^{1}dx\int_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}}dy\int_0^{\left(1-x^2-\frac{y^2}{4}\right)}dz$$

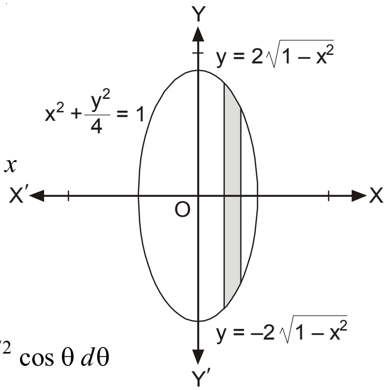
$$= \int_{-1}^{1}\int_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}}\left(1-x^2-\frac{y^2}{4}\right)dx\,dy$$

$$= 4\int_0^1\int_0^{2\sqrt{1-x^2}}\left(1-x^2-\frac{y^2}{4}\right)dx\,dy$$

$$= 4\int_0^1 \left[(1-x^2)\,y - \frac{y^3}{12}\right]_0^{2\sqrt{1-x^2}} . dx$$

$$= 4\int_0^1 \left[(1-x^2)\,.\,2\sqrt{1-x^2} - \frac{8}{12}(1-x^2)^{3/2}\right] d\,x$$

$$= 4\int_0^1 \left[2\,(1-x^2)^{3/2} - \frac{2}{3}(1-x^2)^{3/2}\right] d\,x$$



On putting $x = \sin\theta$, we get

$$V = 4\int_0^1 \frac{4}{3}(1-x^2)^{3/2}\,dx = \frac{16}{3}\int_0^{\pi/2} (-\sin^2\theta)^{3/2}\cos\theta\,d\theta$$

$$= \frac{16}{3}\int_0^{\pi/2} \cos^4\theta\,d\theta = \frac{16}{3}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \pi$$ **Ans.**

Example 54. *Find the volume enclosed between the cylinders $x^2 + y^2 = a\,x$, and $z^2 = a\,x$.*

Solution. Here, we have $x^2 + y^2 = ax$ ...(1)

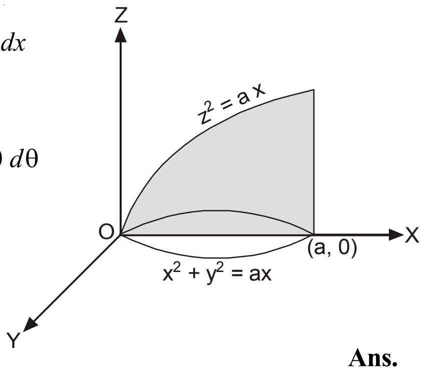
$z^2 = ax$ ...(2)

$$V = \iiint dx\,dy\,dz$$

$$= \int_0^a dx \int_{-\sqrt{ax-x^2}}^{\sqrt{ax-x^2}} dy \int_{-\sqrt{ax}}^{\sqrt{ax}} dz = 2\int_0^a dx \int_{-\sqrt{a^2-x^2}}^{\sqrt{ax-x^2}} dy \int_0^{\sqrt{ax}} dz$$

$$= 2\int_0^a dx \int_{-\sqrt{ax-x^2}}^{\sqrt{ax-x^2}} dy\,(z)_0^{\sqrt{ax}} = 2\int_0^a dx \int_{-\sqrt{ax-x^2}}^{\sqrt{ax-x^2}} dy\,\sqrt{ax} = 2\int_0^a \sqrt{ax}\,dx\,[y]_{-\sqrt{ax-x^2}}^{\sqrt{ax-x^2}}$$

$$= 2\int_0^a \sqrt{ax}\,dx\left(2\sqrt{ax-x^2}\right) = 4\sqrt{a}\int_0^a x\sqrt{a-x}\,dx$$



Putting $x = a\sin^2\theta$ so that $dx = 2a\sin\theta\cos\theta\,d\theta$, we get

$$V = 4\sqrt{a}\int_0^{\pi/2} a\sin^2\theta\,\sqrt{a - a\sin^2\theta}\,.\,2a\sin\theta\cos\theta\,d\theta$$

$$= 8a^3\int_0^{\pi/2} \sin^3\theta\cos^2\theta\,d\theta$$

$$= 8a^3\,\frac{\Gamma 2\,\Gamma\frac{3}{2}}{2\,\Gamma\frac{7}{2}} = 4a^3\,\frac{\Gamma\frac{3}{2}}{\frac{5}{2}\cdot\frac{3}{2}\,\Gamma\frac{3}{2}} = \frac{16a^3}{15}$$ **Ans.**

EXERCISE 2.10

1. Find the volume bounded by the coordinate planes and the plane. $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$ **Ans.** $\dfrac{abc}{6}$

2. Find the volume bounded by the cylinders $y^2 = x$ and $x^2 = y$ between the planes $z = 0$ and $x + y + z = 2$. **Ans.** $\dfrac{11}{30}$

3. Find the volume bounded by the co-ordinate planes and the plane.
$l\,x + m\,y + n\,z = 1$ (*A.M.I.E.T.E. Winter 2001*) **Ans.** $\dfrac{1}{6\,l\,m\,n}$

4. Find the volume of the sphere $x^2 + y^2 + z^2 = a^2$ by triple integration. (*AMIETE, June 2009*) **Ans.** $\dfrac{4}{3}\pi a^3$

5. Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ **Ans.** $\frac{4\pi a b c}{3}$

6. Find the volume bounded by the cylinder $x^2 + y^2 = a^2$ and the planes $y + z = 2a$ and $z = 0$. (*M.U. II Semester 2000, 02, 06*) **Ans.** $2\pi a^3$

7. Find the volume bounded by the cylinder $x^2 + y^2 = a^2$ and the planes $z = 0$ and $y + z = b$. **Ans.** $\pi a^2 b$

8. Find the volume of the region bounded by $z = x^2 + y^2$, $z = 0$, $x = -a$, $x = a$ and $y = -a$, $y = a$. **Ans.** $\frac{8}{3}a^4$

9. Find the volume enclosed by the cylinder $x^2 + y^2 = 9$ and the planes $x + z = 5$ and $z = 0$. **Ans.** $45\pi - 36$

10. Compute the volume of the solid bounded by $x^2 + y^2 = z$, $z = 2x$.(*A.M.I.E., Summer 2000*) **Ans.** 2π

11. Find the volume cut from the paraboloid $4z = x^2 + y^2$ by plane $z = 4$. (*U.P. I Semester, Dec. 2005*) **Ans.** 32π

12. By using triple integration find the volume cut off from the sphere $x^2 + y^2 + z^2 = 16$ by the plane $z = 0$ and the cylinder $x^2 + y^2 = 4x$. **Ans.** $\frac{64}{9}(3\pi - 4)$

13. The sphere $x^2 + y^2 + z^2 = a^2$ is pierced by the cylinder $x^2 + y^2 = a^2(x^2 - y^2)$.
Prove that the volume of the sphere that lies inside the cylinder is $\frac{8}{3}\left[\frac{\pi}{4}+\frac{5}{3}-\frac{4\sqrt{2}}{3}\right]a^3$.

14. Find the volume of the solid bounded by the surfaces $z = 0$, $3z = x^2 + y^2$ and $x^2 + y^2 = 9$. (*A.M.I.E.T.E., Summer 2005*) **Ans.** $\frac{27\pi}{2}$

15. Obtain the volume bounded by the surface $z = c\left(1-\frac{x}{a}\right)\left(1-\frac{y}{b}\right)$ and a quadrant of the elliptic cylinder $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $z > 0$ and where $a, b > 0$. (*A.M.I.E.T.E., Dec. 2005*)

16. Find the volume of the paraboloid $x^2 + y^2 = 4z$ cut off by the plane $z = 4$. **Ans.** 32π

17. Find the volume bounded by the cone $z^2 = x^2 + y^2$ and the paraboloid $z = x^2 + y^2$. **Ans.** $\frac{\pi}{6}$

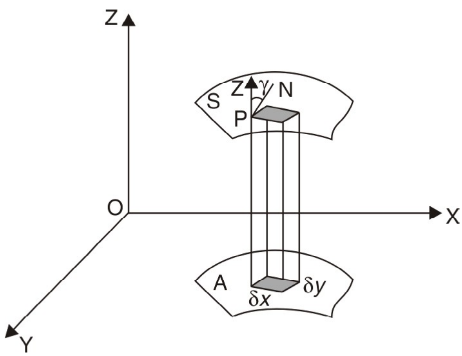
18. Find the volume enclosed by the cylinders $x^2 + y^2 = 2ax$ and $z^2 = 2ax$. **Ans.** $\frac{128a^3}{15}$

19. Find the volume of the solid bounded by the plane $z = 0$, the paraboloid $z = x^2 + y^2 + 2$ and the cylinder $x^2 + y^2 = 4$. **Ans.** 16π

20. The triple integral $\iiint dx\,dy\,dz$ gives
(*a*) Volume of region (*b*) Surface area of region T
(*c*) Area of region *T* (*d*) Density of region *T*. (*A.M.I.E.T.E., 2002*) **Ans.** (*a*)

2.16 SURFACE AREA

Let $z = f(x,y)$ be the surface *S*. Let its projection on the *x-y* plane be the region *A*. Consider an element $8x.\ \delta y$ in the region *A*. Erect a cylinder on the element $\delta x.\ \delta y$ having its generator parallel to *OZ* and meeting the surface *S* in an element of area δs.



$\therefore \quad \delta x\ \delta y = \delta s \cos\gamma,$

Where γ is the angle between the *xy*-plane and the tangent plane to *S* at *P, i.e.,* it is the angle between the *Z*-axis and the normal to *S* at *P*.

The direction cosines of the normal to the surface $F(x, y, z) = 0$ are proportional to

$$\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$$

$\therefore$ The direction of the normal to S $[F = f(x, y) - z]$ are proportional to $-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1$ and those of the Z-axis are 0, 0 , 1.

$$\text{Direction cosines} = \frac{-\frac{\partial z}{\partial x}}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}}, \frac{-\frac{\partial z}{\partial y}}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}}, \frac{1}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}},$$

Hence $$\cos\gamma = \frac{1}{\sqrt{\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]}} \quad (\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2)$$

$$\delta S = \frac{\delta x\, \delta y}{\cos\gamma} = \sqrt{\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]}\, \delta x\, \delta y\,; \quad S = \iint_A \sqrt{\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]}\, dx\, dy$$

Example 55. *Find the surface area of the cylinder* $x^2 + z^2 = 4$ *inside the cylinder* $x^2 + y^2 = 4$.

Solution. $x^2 + y^2 = 4$

$$2x + 2z\frac{\partial z}{\partial x} = 0 \text{ or } \frac{\partial z}{\partial x} = -\frac{x}{z}, \quad \frac{\partial z}{\partial y} = 0$$

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1 = \frac{x^2}{z^2} + 1 = \frac{x^2 + z^2}{z^2} = \frac{4}{4 - x^2}$$

Hence, the required surface area $= 8\int_0^2 \int_0^{\sqrt{4-x^2}} \sqrt{\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]}\, dx\, dy$

$$= 8\int_0^2 \int_0^{\sqrt{4-x^2}} \frac{2}{\sqrt{4 - x^2}}\, dx\, dy = 16\int_0^2 \frac{1}{\sqrt{4 - x^2}}\, [y]_0^{\sqrt{4-x^2}}\, dx = 16\int_0^2 \frac{1}{\sqrt{4 - x^2}}\, [\sqrt{4 - x^2}\,]\, dx$$

$$= 16\int_0^2 dx = 16\,(x)_0^2 = 32$$ **Ans.**

Example 56. *Find the surface area of the sphere* $x^2 + y^2 + z^2 = 9$ *lying inside the cylinder* $x^2 + y^2 = 3y$.

Solution. $x^2 + y^2 + z^2 = 9$

$$2x + 2z\frac{\partial z}{\partial x} = 0, \quad \frac{\partial z}{\partial x} = -\frac{x}{z}$$

$$2x + 2z\frac{\partial z}{\partial y} = 0, \quad \frac{\partial z}{\partial y} = -\frac{y}{z}$$

$$\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right] = \frac{x^2}{z^2} + \frac{y^2}{z^2} + 1 = \frac{x^2 + y^2 + z^2}{z^2} = \frac{9}{9 - x^2 - y^2} = \frac{9}{9 - r^2} \quad \begin{bmatrix} x = r\cos\theta \\ y = r\sin\theta \end{bmatrix}$$

$$x^2 + y^2 = 3y \text{ or } r^2 = 3r\sin\theta \text{ or } r = 3\sin\theta.$$

Hence, the required surface area

$$= \iint \sqrt{\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]}\, dx\, dy = 4\int_0^{\pi/2}\int_0^{3\sin\theta} \frac{3}{\sqrt{9 - r^2}}\, r\, d\theta\, dr = 12\int_0^{\pi/2} d\theta \int_0^{3\sin\theta} \frac{r\, dr}{\sqrt{9 - r^2}}$$

$$= 12\int_0^{\pi/2} d\theta\, [-\sqrt{9 - r^2}\,]_0^{3\sin\theta} = 12\int_0^{\pi/2} [-\sqrt{9 - 9\sin^2\theta} + 3]\, d\theta$$

$$= 36\int_0^{\pi/2}(-\cos\theta+1)\,d\theta = 36\,(-\sin\theta+\theta)_0^{\pi/2} = 36\left(-1+\frac{\pi}{2}\right) = 18\,(\pi-2)$$ **Ans.**

Example 57. *Find the surface area of the section of the cylinder $x^2 + y^2 = a^2$ made by the plane $x + y + z = a$.*

Solution.
$$x^2+y^2=a^2 \qquad \text{... (1)}$$
$$x+y+z=a \qquad \text{... (2)}$$

The projection of the surface area on xy-plane is a circle
$x^2 + y^2 = a^2$

$$1+\frac{\partial z}{\partial x}=0 \quad \text{or} \quad \frac{\partial z}{\partial x}=-1$$
$$1+\frac{\partial z}{\partial y}=0 \quad \text{or} \quad \frac{\partial z}{\partial y}=-1$$
$$\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1} = \sqrt{(-1)^2+(-1)^2+1} = \sqrt{3}$$

Hence the required surface area

$$= 4\int_0^a\int_0^{\sqrt{a^2-x^2}}\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}\;dx\,dy = 4\int_0^a\int_0^{\sqrt{a^2-x^2}}\sqrt{3}\,dx\cdot dy$$
$$= 4\sqrt{3}\int_0^a[y]_0^{\sqrt{a^2-x^2}}\,dx = 4\sqrt{3}\int_0^a\sqrt{a^2-x^2}\,dx$$
$$= 4\sqrt{3}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a = 4\sqrt{3}\left[0+\frac{a^2}{2}\,\frac{\pi}{2}\right] = 4\sqrt{3}\left(\frac{a^2\pi}{4}\right) = \sqrt{3}\,\pi\,a^2$$ **Ans.**

Example 58. *Find the area of that part of the surface of the paraboloid of the paraboloid $y^2 + z^2 = 2\,ax$, which lies between the cylinder, $y^2 = ax$ and the plane $x = a$.*

Solution.
$$y^2+z^2=2\,ax \qquad \text{... (1)}$$
$$y^2=ax \qquad \text{... (2)}$$
$$x=a \qquad \text{... (3)}$$

Differentiating (1), we get

$$2\,z\frac{\partial z}{\partial x} = 2\,a,\ \frac{\partial z}{\partial x}=\frac{a}{z}$$
$$2y+2\,z\frac{\partial z}{\partial y} = 0,\ \frac{\partial z}{\partial y}=-\frac{y}{z}$$

$$\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1 = \frac{a^2}{z^2}+\frac{y^2}{z^2}+1 = \frac{a^2+y^2}{z^2}+1 \qquad \left[\begin{aligned} y^2+z^2&=2\,ax\\ z^2&=2\,ax-y^2\end{aligned}\right]$$
$$= \frac{a^2+y^2}{2\,a\,x-y^2}+1 = \frac{a^2+y^2+2\,a\,x-y^2}{2\,a\,x-y^2} = \frac{a^2+2\,a\,x}{2\,a\,x-y^2}$$

$$S = \int_0^a\int_{-\sqrt{ax}}^{\sqrt{ax}}\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}\;dx\,dy = \int_0^a\int_{-\sqrt{ax}}^{\sqrt{ax}}\sqrt{\frac{a^2+2\,ax}{2\,ax-y^2}}\,dx\,dy \qquad \left[\begin{aligned} y^2&=ax\\ y&=\pm\sqrt{ax}\end{aligned}\right]$$
$$= \sqrt{a}\int_0^a\int_{-\sqrt{ax}}^{\sqrt{ax}}\sqrt{\frac{a+2\,x}{2\,ax-y^2}}\,dx\,dy = \sqrt{a}\int_0^a\sqrt{a+2\,x}\,dx\int_{-\sqrt{ax}}^{\sqrt{ax}}\frac{1}{\sqrt{2\,ax+y^2}}\,dy$$

$$= \sqrt{a}\int_0^a \sqrt{a+2x}\,dx\left[\sin^{-1}\frac{y}{\sqrt{2ax}}\right]_{-\sqrt{ax}}^{\sqrt{ax}}$$

$$= \sqrt{a}\int_0^a \sqrt{a+2x}\,dx\left[\sin^{-1}\frac{1}{\sqrt{2}} - \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)\right] = \sqrt{a}\int_0^a \sqrt{a+2x}\,dx\left[\frac{\pi}{4}-\left(\frac{\pi}{4}\right)\right]$$

$$= \sqrt{a}\,\frac{\pi}{2}\int_0^a \sqrt{a+2x}\,dx = \frac{\pi}{2}\cdot\frac{\sqrt{a}}{2}\cdot\frac{2}{3}[(a+2x)^{3/2}]_0^a$$

$$= \frac{\pi\sqrt{a}}{6}[(3a)^{3/2} - a^{3/2}] = \frac{\pi a^2}{6}[3\sqrt{3}-1]$$ **Ans.**

EXERCISE 2.11

1. Find the surface area of sphere $x^2 + y^2 + z^2 = 16$. **Ans.** 64π
2. Find the surface area of the portion of the cylinder $x^2 + y^2 = 4y$ lying inside the sphere $x^2 + y^2 + z^2 = 16$. **Ans.** 64.
3. Show that the area of surfaces cz = xy intercepted by the cylinder $x^2 + y^2 = b^2$ is $\iint_A \frac{\sqrt{c^2+x^2+y^2}}{c}\,dx\,dy$, where A is the area of the circle $x^2 + y^2 = b^2$, $z = 0$

   Ans. $\frac{2}{3}\frac{\pi}{c}\left[(c^2+b^2)^{\frac{1}{2}} - c^2\right]$
4. Find the area of the portion of the sphere $x^2 + y^2 + z^2 = a^2$ lying inside the cylinder $x^2 + y^2 = ax$. **Ans.** $2(\pi - 2)a^2$
5. Find the area of the surface of the cone $z^2 = 3(x^2 + y^2)$ cut out by the paraboloid $z = x^2 + y^2$ using surface integral. **Ans.** 6π

2.17 CALCULATION OF MASS

We have,

Volume = $\iint_V dx\,dy\,dz$ Density = $\rho = f(x, y, z)$

[Density = Mass per unit volume] Mass = Volume × Density

Mass = $\iint_V dx\,dy\,dz$ $\boxed{Mass = \iint_V f(x, y, z)\,dx\,dy\,dz}$

Example 59. Find the mass of a plate which is formed by the co-ordinate planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, the density is given by $\rho = k\,x\,y\,z$. (*U.P., I Semester*, *Dec.*, 2003)

Solution. The plate is bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

$$\text{Mass} = \iiint dx\,dy\,dz\,\rho = \int_0^c\int_0^{b\left(1-\frac{z}{c}\right)}\int_0^{a\left(1-\frac{y}{b}-\frac{z}{c}\right)} dx\,dy\,dz\,(k\,xyz)$$

$$= k\int_0^c z\,dz\int_0^{b\left(1-\frac{z}{c}\right)} y\,dy\int_0^{a\left(1-\frac{y}{b}-\frac{z}{c}\right)} x\,dx = k\int_0^c z\,dz\int_0^{b\left(1-\frac{z}{c}\right)} y\,dy\left(\frac{x^2}{2}\right)_0^{a\left(1-\frac{y}{b}-\frac{z}{c}\right)}$$

$$= k\int_0^c z\,dz\int_0^{b\left(1-\frac{z}{c}\right)} y\,dy\,\frac{a^2}{2}\left(1-\frac{y}{b}-\frac{z}{c}\right)^2 = \frac{k\,a^2}{2}\int_0^c z\,dz\int_0^{b\left(1-\frac{z}{c}\right)} y\left[\left(1-\frac{z}{c}\right)-\frac{y}{b}\right]^2 dy$$

$$= \frac{k\,a^2}{2}\int_0^c z\,dz\int_0^{b\left(1-\frac{z}{c}\right)}\left[y\left(1-\frac{z}{c}\right)^2 + \frac{y^3}{b^2} - \frac{2\,y^2}{b}\left(1-\frac{z}{c}\right)\right]dy$$

$$= \frac{k\,a^2}{2}\int_0^c z\,dz\left[\frac{y^2}{2}\left(1-\frac{z}{c}\right)^2+\frac{y^4}{4\,b^2}-\frac{2\,y^3}{3\,b}\left(1-\frac{z}{c}\right)\right]_0^{b\left(1-\frac{z}{c}\right)}$$

$$= \frac{k\,a^2}{2}\int_0^c z\,dz\left[\frac{b^2}{2}\left(1-\frac{z}{c}\right)^4+\frac{b^4}{4\,b^2}\left(1-\frac{z}{c}\right)^4-\frac{2}{3}\cdot\frac{b^3}{b}\left(1-\frac{z}{c}\right)^4\right]$$

$$= \frac{k\,a^2}{2}\int_0^c z\left[\frac{b^2}{2}+\frac{b^2}{4}-\frac{2b^2}{3}\right]\left(1-\frac{z}{c}\right)^4 dz = \frac{k\,a^2}{2}\,\frac{b^2}{12}\int_0^c\left(1-\frac{z}{c}\right)^4 dz \qquad [\text{Put } z = c\sin^2\theta]$$

$$= \frac{k\,a^2\,b^2\,c^2}{12}\int_0^{\frac{\pi}{2}} c\sin^2\theta\,(1-\sin^2\theta)^4\,(2\,c\sin\theta\cos\theta\,d\theta)$$

$$= \frac{k^2\,a^2\,b^2\,c^2}{12}\int_0^{\pi/2}\sin^2\theta\,(\cos^8\theta)\sin\theta\cos\theta\,d\theta = \frac{k^2\,a^2\,b^2\,c^2}{12}\int_0^{\pi/2}\sin^3\theta\cos^9\theta\,d\theta$$

$$= \frac{k^2\,a^2\,b^2\,c^2}{12}\;\frac{\left\lceil\frac{3+1}{2}\right.\left\lceil\frac{9+1}{2}\right.}{2\left\lceil\frac{3+9+2}{2}\right.} = \frac{k\,a^2\,b^2\,c^2}{12}\cdot\frac{\lceil 2\,\lceil 5}{2\,\lceil 7} = \frac{k\,a^2\,b^2\,c^2}{12}\;\frac{(1)\,(\lceil 5)}{2\times 6\times 5\,\lceil 5} = \frac{k\,a^2\,b^2\,c^2}{720} \quad \textbf{Ans.}$$

2.18 CENTRE OF GRAVITY

$$\bar{x} = \frac{\iiint x\,\rho\,dx\,dy\,dz}{\iiint \rho\,dx\,dy\,dz},\ \bar{y} = \frac{\iiint y\,\rho\,dx\,dy\,dz}{\iiint \rho\,dx\,dy\,dz},\ \bar{z} = \frac{\iiint z\,\rho\,dx\,dy\,dz}{\iiint \rho\,dx\,dy\,dz}$$

Example 60. *Find the co-ordinates of the centre of gravity of the positive octant of the sphere $x^2 + y^2 + z^2 = a^2$, density being given = k xyz.*

Solution. $$\bar{x} = \frac{\iiint_V x\,\rho\,dx\,dy\,dz}{\iiint_V \rho\,dx\,dy\,dz} = \frac{\iiint z\,\rho\,dx\,dy\,dz}{\iiint \rho\,dx\,dy\,dz} = \frac{\iiint_V x^2yz\,dx\,dy\,dz}{\iiint_V xyz\,dx\,dy\,dz}$$

Converting into polar co-ordinates, $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$,
$dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$

$$\bar{x} = \frac{\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a (r\sin\theta\cos\phi)^2\,(r\sin\theta\sin\phi)\,(r\cos\theta)\,(r^2\sin\theta\,dr\,d\theta\,d\phi)}{\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a (r\sin\theta\cos\phi)\,(r\sin\theta\sin\phi)\,(r\cos\theta)\,(r^2\sin\theta\,dr\,d\theta\,d\phi)}$$

$$= \frac{\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a r^6\sin^4\theta\cos\theta\sin\phi\cos^2\phi\,dr\,d\theta\,d\phi}{\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a r^5\sin^3\theta\cos\theta\sin\phi\cos\phi\,dr\,d\theta\,d\phi}$$

$$= \frac{\int_0^{\pi/2}\sin\phi\cos^2\phi\,d\phi\int_0^{\pi/2}\sin^4\theta\cos\theta\,d\theta\int_0^a r^6\,dr}{\int_0^{\pi/2}\sin\phi\cos\phi\,d\phi\int_0^{\pi/2}\sin^3\theta\cos\theta\,d\theta\int_0^a r^5\,dr}$$

$$= \frac{\left[-\frac{\cos^3\phi}{3}\right]_0^{\pi/2}\left[\frac{\sin^5\theta}{5}\right]_0^{\pi/2}\left[\frac{r^7}{7}\right]_0^a}{-\left[\frac{\cos^2\phi}{2}\right]_0^{\pi/2}\left[\frac{\sin^4\theta}{4}\right]_0^{\pi/2}\left[\frac{r^6}{6}\right]_0^a} = \frac{\left(\frac{1}{3}\right)\left(\frac{1}{5}\right)\left(\frac{a^7}{7}\right)}{\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(\frac{a^6}{6}\right)} = \frac{16\,a}{35}$$

Similarly, $\bar{y} = \bar{z} = \dfrac{16\,a}{35}$; Hence, C.G. is $\left(\dfrac{16\,a}{35}, \dfrac{16\,a}{35}, \dfrac{16\,a}{35}\right)$ **Ans.**

2.19 MOMENT OF INERTIA OF A SOLID

Let the mass of an element of a solid of volume V be $\rho\ \delta x\ \delta y\ \delta z$.

Perpendicular distance of this element from the x-axis $= \sqrt{y^2 + z^2}$

$M.I.$ of this element about the x-axis $= \rho\ \delta x\ \delta y\ \delta z\ \sqrt{y^2 + z^2}$

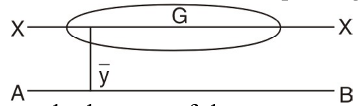
$M.I.$ of the solid about x-axis $= \iiint_V \rho\,(y^2 + z^2)\,dx\,dy\,dz$

$M.I.$ of the solid about y-axis $= \iiint_V \rho\,(x^2 + z^2)\,dx\,dy\,dz$

$M.I.$ of the solid about z-axis $= \iiint_V \rho\,(x^2 + y^2)\,dx\,dy\,dz$

The Perpendicular Axes Theorem

If I_{ox} and I_{oy} be the moments of inertia of a lamina about x-axis and y-axis respectively and I_{oz} be the moment of inertia of the lamina about an axis perpendicular to the lamina and passing through the point of intersection of the axes OX and OY.



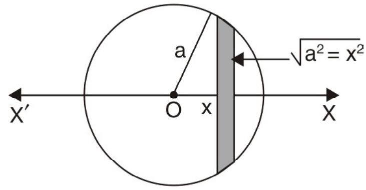
$$I_{OZ} = I_{OX} + I_{OY}$$

The Parallel Axes Theorem

M.I. of a lamina about an axis in the plane of the lamina equals the sum of the moment of inertia about a parallel centroidal axis in the plane of lamina together with the product of the mass of the lamin a and square of the distance between the two axes.

$$I_{AB} = I_{XX} + M\bar{y}^2$$

Example 61. *Find M.I. of a sphere about diameter.*



Solution. Let a circular disc of $\delta\ x$ thickness be perpendicular to the given diameter XX' at a distance x from it.

The radius of the disc $= \sqrt{a^2 - x^2}$

Mass of the disc $= \rho\ \pi\ (a^2 - x^2)$

Moment of inertia of the disc about a diameter perpendicular on it

$$= \frac{1}{2} MR^2 = \frac{1}{2}[\rho\,\pi\,(a^2 - x^2)]\,(a^2 - x^2) = \frac{1}{2}\rho\,\pi\,(a^2 - x^2)^2$$

$$\text{M.I. of the sphere} = \int_{-a}^{a} \frac{1}{2}\rho\,\pi\,(a^2 - x^2)^2\,dx = 2\left(\frac{1}{2}\rho\,\pi\right)\int_0^a [a^4 - 2\,a^2\,x^2 + x^4]\,dx$$

$$= \rho\,\pi\left[a^4 x - \frac{2\,a^2\,x^3}{3} + \frac{x^5}{5}\right]_0^a = \rho\,\pi\left[a^5 - \frac{2\,a^5}{3} + \frac{a^5}{5}\right]$$

$$= \frac{8}{15}\pi\,\rho\,a^5 = \frac{2}{5}\left(\frac{4\,\pi}{3}a^3\,\rho\right)a^2 = \frac{2}{5}M\,a^2 \qquad \textbf{Ans.}$$

Example 62. *The mass of a solid right circular cylinder of radius a and height h* is *M. Find the moment of inertia of the cylinder about* (*i*) *its axis* (*ii*) *a line through its centre of gravity perpendicular to its axis* (*iii*) *any diameter through its base.*

Solution. *To find M.I. about OX. Consider a disc at a distance x from O at the base.*

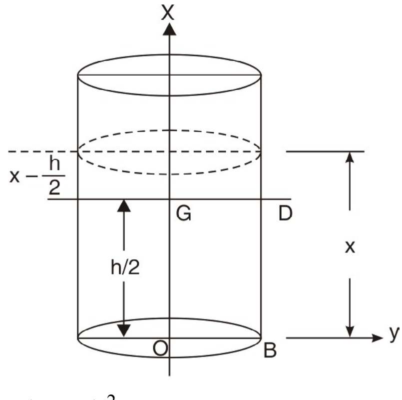
$$\text{M.I. of the about } OX, = \frac{(\pi\,a^2\,\rho\,dx)\,a^2}{2} = \frac{\pi\,\rho\,a^4\,dx}{2}$$

(*i*) M.I. of the cylinder about OX

$$\int_0^h \frac{\pi\,\rho\,a^4 dx}{2} = \frac{\pi\,\rho\,a^4}{2}(x)_0^h = \frac{\pi\,\rho\,a^4\,h}{2} = (\pi\,a^2\,h)\,\rho\cdot\frac{a^2}{2} = \frac{M\,a^2}{2}$$

(*ii*) *M.I.* of the disc about a line through *C.G.* and perpendicular to *OX*.

$$I_{OX} + I_{OY} = I_{OZ}$$
$$I_{OX} + I_{OX} = I_{OZ}$$
$$I_{OX} = \frac{1}{2} I_{OZ}$$



M.I. of the disc about a line through

$$C.G. = \frac{1}{2}\left(\frac{M\,a^2}{2}\right) = \frac{M\,a^2}{4}$$

M.I. of the disc about the diameter $= \left(\dfrac{\pi\, a^2\, \rho\, dx}{4}\right) a^2$

M.I. of the disc about line $GD = \dfrac{\pi\, a^2\, \rho\, dx}{4} + (\pi\, a^2\, \rho\, dx)\left(x - \dfrac{h}{2}\right)^2$

Hence, *M.I.* of cylinder about $GD = \displaystyle\int_0^h \frac{\pi\, a^2\, \rho}{4}\, dx + \int_0^h (\pi\, a^2\, \rho\, dx)\left(x - \frac{h}{2}\right)^2$

$$= \frac{\pi\, a^2\, \rho}{4}(x)_0^h + \left[\frac{\pi\, a^2\, \rho}{4}\left(x - \frac{h}{2}\right)^3\right]_0^h = \frac{\pi\, a^2\, \rho\, h}{4} + \left[\frac{\pi\, a^2\, \rho}{3}\left(\frac{h}{2}\right)^3 + \frac{\pi\, a^2\, \rho}{3}\left(\frac{h}{2}\right)^3\right]$$

$$= \frac{\pi\, a^2\, \rho\, h}{4} + \frac{\pi\, a^2\, \rho\, h^3}{12} = \frac{M\, a^2}{4} + \frac{M\, h^2}{12}$$

(*iii*) *M.I.* of cylinder about line *OB* (through) base

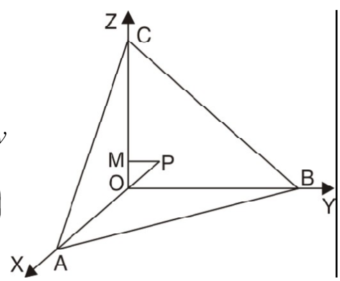
$$I_{OB} = I_G + M\left(\frac{h}{2}\right)^2 = \frac{M\, a^2}{4} + \frac{M\, h^2}{12} + \frac{M\, h^2}{4} = \frac{M\, a^2}{4} + \frac{M\, h^2}{3}$$ **Ans.**

Example 63. *Find the moment of inertia and radius of gyration about z-axis of the region in the first octant bounded by* $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$.

Solution. Let *r* be the density. M.I. of tetrahedron about *z*-axis

$$= \iiint (\rho\, dx\, dy\, dz)\,(x^2 + y^2)$$

$$= \rho \int_0^a dx \int_0^{b\left(1-\frac{x}{a}\right)} (x^2 + y^2)\, dy \int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dz = \rho \int_0^a dx \int_0^{b\left(1-\frac{x}{a}\right)} (x^2 + y^2)\, dy\, (z)_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)}$$



$$= \rho \int_0^a dx \int_0^{b\left(1-\frac{x}{a}\right)} (x^2 + y^2)\, dy\, c\, c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$$

$$= \rho c \int_0^a dx \int_0^{b\left(1-\frac{x}{a}\right)} \left[x^2\left(1 - \frac{x}{a}\right) - \frac{x^2 y}{b} + y^2\left(1 - \frac{x}{a}\right) - \frac{y^3}{b}\right] dy$$

$$= \rho c \int_0^a dx \left[x^2\left(1 - \frac{x}{a}\right) y - \frac{x^2 y^2}{2\,b} + \frac{y^3}{3}\left(1 - \frac{x}{a}\right) - \frac{y^4}{4\,b}\right]_0^{b\left(1-\frac{x}{a}\right)}$$

$$= \rho c \int_0^a dx \left[x^2\left(1 - \frac{x}{a}\right) b\left(1 - \frac{x}{a}\right) - \frac{x^2}{2\,b} b^2\left(1 - \frac{x}{a}\right)^2 + \frac{b^3}{3}\left(1 - \frac{x}{a}\right)^3\left(1 - \frac{x}{a}\right) - \frac{b^4}{4\,b}\left(1 - \frac{x}{a}\right)^4\right]$$

$$= b\rho c \int_0^a \left[x^2\left(1 - \frac{x}{a}\right)^2 - \frac{x^2}{2}\left(1 - \frac{x}{a}\right)^2 - \frac{b^2}{3}\left(1 - \frac{x}{a}\right)^4 - \frac{b^2}{4}\left(1 - \frac{x}{a}\right)^4\right] dx$$

$$= \rho bc \int_0^a \left[\frac{x^2}{2}\left(1-\frac{x}{a}\right)^2 + \frac{b^2}{12}\left(1-\frac{x}{a}\right)^4 \right] dx$$

$$= \rho bc \int_0^a \left[\frac{1}{2}\left(x^2 - \frac{2x^3}{a} + \frac{x^4}{a^2}\right) + \frac{b^2}{12}\left(1 - \frac{4x}{a} + \frac{6x^2}{a^2} - \frac{4x^3}{a^3} + \frac{x^4}{a^4}\right) \right] dx$$

$$= \rho bc \int_0^a \left[\frac{1}{2}\left(\frac{x^3}{3} - \frac{x^4}{2a} + \frac{x^5}{5a^2}\right) + \frac{b^2}{12}\left(x - \frac{2x^2}{a} + \frac{6x^2}{a^2} - \frac{4x^3}{a^3} + \frac{x^4}{a^4}\right) \right]_0^a dx$$

$$= \rho bc \int_0^a \left[\frac{1}{2}\left(\frac{a^3}{3} - \frac{a^3}{2} + \frac{a^3}{5}\right) + \frac{b^2}{12}\left(a - 2a + 2a - a + \frac{a}{5}\right) \right]$$

$$= \rho bc \left[\frac{a^3}{60} + \frac{ab^2}{60} \right] = \rho \frac{abc}{60}(a^2 + b^2)$$

Radius of gyration $= \sqrt{\dfrac{M.I.}{Mass}} = \dfrac{\sqrt{\dfrac{\rho abc}{60}(a^2+b^2)}}{\dfrac{\rho abc}{6}} = \sqrt{\dfrac{1}{10}(a^2+b^2)}$ **Ans.**

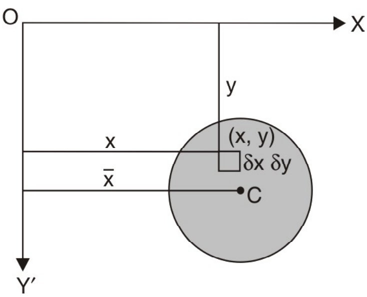
2.20 CENTRE OF PRESSURE

The centre of pressure of a plane area immersed in a fluid is the point at which the resultant force acts on the area.

Consider a plane area A immersed vertically in a homogeneous liquid. Let x-axis be the line of intersection of the plane with the free surface. Any line in this plane and perpendicular to x-axis is the *y-axis.*

Let P be the pressure at the point (x, y). Then the pressure on elementary area $\delta x\ \delta y$ is $P\ \delta x\ \delta y$.

Let $(\bar{x}, \bar{y})$ be the centre of pressure. Taking moment about y-axis.



$$\bar{x} \cdot \iint_A P\,dx\,dy = \iint_A Px\,dx\,dy$$

$$\bar{x} = \frac{\iint_A Px\,dx\,dy}{\iint_A P\,dx\,dy}$$

Similarly, $$\bar{y} = \frac{\iint_A Py\,dx\,dy}{\iint_A P\,dx\,dy}$$

Example 64. *A uniform semi-circular lamina is immersed in a fluid with its plane vertical and its bounding diameter on the free surface. If the density at any point of the fluid varies as the depth of the point below the free surface, find the position of the centre of pressure of the lamina.*

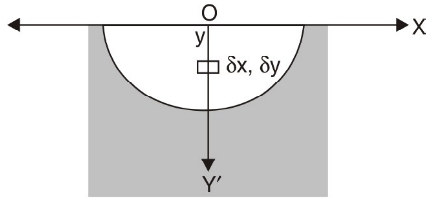
Solution. Let the semi-circular lamina be

$$x^2 + y^2 = a^2$$

By symmetry its centre of pressure lies on OY. Let ky be the density of the fluid.

$$\bar{y} = \frac{\iint_A Py\,dx\,dy}{\iint_A P\,dx\,dy} = \frac{\iint_A (\rho y)\,y\,dx\,dy}{\iint_A (\rho y)\,dx\,dy} \qquad (\because \rho = ky)$$

$$= \frac{\iint_A (ky\,.\,y)\,y\,dx\,dy}{\iint_A (ky\,.\,y)\,dx\,dy} = \frac{\iint_A y^3\,dx\,dy}{\iint_A y^2\,dx\,dy} = \frac{\int_{-a}^{a} dx \int_0^{\sqrt{a^2-x^2}} y^3\,dy}{\int_{-a}^{a} dx \int_0^{\sqrt{a^2-x^2}} y^2\,dy}$$



$$= \frac{\int_{-a}^{a} dx \left[\frac{y^4}{4}\right]_0^{\sqrt{a^2-x^2}}}{\int_{-a}^{a} dx \left[\frac{y^3}{3}\right]_0^{\sqrt{a^2-x^2}}} = \frac{3}{4}\,\frac{\int_{-a}^{a} dx\,(a^2-x^2)^2}{\int_{-a}^{a} dx\,(a^2-x^2)^{3/2}}$$

$$= \frac{3}{4}\,\frac{\int_{-\pi/2}^{\pi/2}(a\cos\theta\, d\theta)\,(a^2-a^2\sin^2\theta)^2}{\int_{-\pi/2}^{\pi/2}(a\cos\theta\, d\theta)\,(a^2-a^2\sin^2\theta)^{3/2}} \qquad (\text{Put } x = a\sin\theta)$$

$$= \frac{3\,a}{4}\,\frac{\int_{-\pi/2}^{\pi/2}\cos^5\theta\, d\theta}{\int_{-\pi/2}^{\pi/2}\cos^4\theta\, d\theta} = \frac{3\,a}{4}\,\frac{2\int_0^{\pi/2}\cos^5\theta\, d\theta}{2\int_0^{\pi/2}\cos^4\theta\, d\theta} = \frac{3\,a}{4}\,\frac{\frac{4\times 2}{5\times 3}}{\frac{3\times 1}{4\times 2}\,\frac{\pi}{2}} = \frac{32\,a}{15\,\pi} \qquad \textbf{Ans.}$$

EXERCISE 2.12

1. Find the mass of the solid bounded by the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and the co-ordinate planes, where the density at any point $P\ (x, y, z)$ is $k\ xyz$. **Ans.** P

2. If the density at a point varies as the square of the distance of the point from XOY plane, find the mass of the volume common to the sphere $x^2+y^2+z^2=a^2$ and cylinder $x^2+y^2=ax$.

Ans. $\frac{4\,k}{15}\,a^5\left(\frac{\pi}{2}-\frac{8}{15}\right)$

3. Find the mass of the plate in the form of one loop of leminscate $r^2 = a^2 \sin 2\,\theta$, where $\rho = k\,r^2$.

Ans. $\frac{k\,\pi\,a^4}{16}$

4. Find the mass of the plate which is inside the circle $r = 2a\cos\theta$ and outside the circle $r = a$, if the density varies as the distance from the pole.

5. Find the mass of a lamina in the form of the cardioid $r = a\,(1+\cos\theta)$ whose density at any point varies as the square of its distance from the initial line. **Ans.** $\frac{21\,\pi\,\mathrm{k}\,a^4}{32}$

6. Find the centroid of the region in the first octant bounded by $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. **Ans.** $\left(\frac{a}{4},\frac{b}{4},\frac{c}{4}\right)$

7. Find the centroid of the region bounded by $z = 4 - x^2 - y^2$ and xy-plane. **Ans.** $\left(0, 0, \frac{4}{3}\right)$

8. Find the position of $C.G.$ of the volume intercepted between the parallelepiped $x^2+y^2 = a(a-z)$ and the plane $z = 0$. **Ans.** $\left(0, 0, \frac{a}{3}\right)$

9. A solid is cut off the cylinder $x^2+y^2=a^2$ by the plane $z = 0$ and that part of the olane $z = mx$ for which z is positive. The density of the solid cut off at any point varies as the height of the point above plane $z = 0$. Find $C.G.$ of the solid. **Ans.** $\bar{z} = \frac{64\,ma}{45\,\pi}$

10. If an area is bounded by two concentric semi-circles with their common bounding diameter in a free surface, prove that the depth of the centre of pressure is $\frac{3\,\pi}{16}\,\frac{(a+b)\,(a^2+b^2)}{a^2+ab+b^2}$

11. An ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is immersed vertically in a fluid with its major axis horizontal. If its centre be at depth h, find the depth of its centre of pressure. **Ans.** $h+\frac{b^2}{4\,h}$

12. A horizontal boiler has a flat bottom and its ends are plane and semi-circular. If it is just full of water, show that the depth of centre of pressure of either end is 0.7 × total depth approximately.

13. A quadrant of a circle of radius a is just immersed vertically in a homogeneous liquid with one edge in the surface. Determine the co-ordinates of the centre of pressure. **Ans.** $\left(\frac{3a}{8}, \frac{3\pi a}{16}\right)$

14. Find the product of inertia of an equilateral triangle about two perpendicular axes in its plane at a vertex, one of the axes being along a side.

15. Find the *M.I.* of a right circular cylinder of radius a and height h about axis if density varies as distance from the axis. **Ans.** $\frac{2}{5} k \pi a^5 h$

16. Compute the moment of inertia of a right circular cone whose altitude is h and base radius r, about (*i*) the axis of symmetry (*ii*) the diameter of the base. **Ans.** (*i*) $\frac{\pi h r^4}{10}$ (*ii*) $\frac{\pi h r^2}{60}(2h^2 + 3r^2)$

17. Find the moment of inertia for the area of the cardioid $r = a(1 - \cos\theta)$ relative to the pole.

Ans. $\frac{35\pi a^4}{16}$

18. Find the M.I. about the line $\theta = \frac{\pi}{2}$ of the area enclosed by $r = a(1 + \cos\theta)$.

19. Find the moment of inertia of the uniform solid in the form of octant of the ellipsoid

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ about OX **Ans.** $\frac{M}{5}(b^2 + c^2)$

20. Prove that the moment of inertia of the area included between the curves $y^2 = 4ax$ and $x^2 = 4ay$ about the x-axis is $\frac{144}{35} M a^2$, where M is the mass of area included between the curves.

21. A solid body of density p is the shape of solid formed by revolution of the cardioid $r = a(1 + \cos\theta)$ about the initial line. Show that its moment of inertia about a straight line through the pole perpendicular to the initial line is $\left(\frac{352}{105}\right)\pi l a^5$. (*U. P. II Semester, Summer 2001*)

22. Find the product of inertia of a disc in the form of a quadrant of a circle of radius 'a' about bounding radii. (*U. P. II Semester, Summer 2002*) **Ans.** $\rho \frac{a^4}{4}$

23. Show that the principal axes at the origin of the triangle enclosed by $x = 0, y = 0, \frac{x}{a} + \frac{y}{b} = 1$ are inclined at angles α and $\alpha + \frac{\pi}{2}$ to the x-axis, where $a = \frac{1}{2}\tan^{-1}\left(\frac{ab}{a^2 - b^2}\right)$ (*U.P. II Semester Summer* 2001)

Choose the correct answer:

24. The triple integral $\iiint_T dx\, dy\, dz$ gives

(*i*) Volume of region T (*ii*) Surface area of region T
(*ii*) Area of region T (*iv*) Density of region T. (*A.M.I.E.T.E. 2002*)

Ans. (*i*)

25. The volume of the solid under the surface $az = x^2 + y^2$ and whose base R is the circle $x^2 + y^2 = a^2$ is given as

(*i*) $\frac{\pi}{2a}$ (*ii*) $\frac{\pi a^3}{2}$ **Ans.** (*ii*)

(*iii*) $\frac{4}{3}\pi a^3$ (*iv*) None of the above. [*U.P., I. Sem. Dec. 2008*]