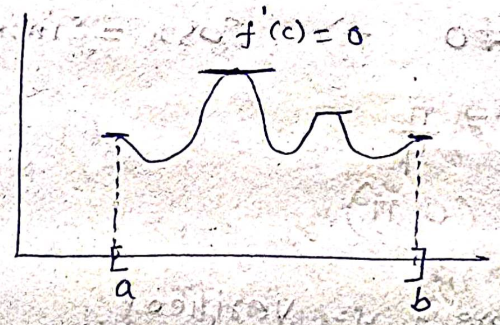
Mean Value Theorems

Rolle's Theorem :- $f(x)$ is continuous in the closed interval $[a,b]$ and differentiable in the open interval (a,b) $\{f'(x)$ exists for every value of x in the open interval $(a,b)\}$ and if $f(a)=f(b)$ then there exists at least one point $c \in (a,b)$ such that $f'(c)=0$

Geometrically :- Under the given conditions there is at least one point in (a,b) such that tangent at this point is parallel to x-axis.



① Verify the Rolle's theorem for the following function : $f(x) = 2x^3 + x^2 - 4x - 2$.

to find the interval

$f(x) = 0 \Rightarrow 2x^3 + x^2 - 4x - 2$

$(x^2-2)(2x+1) = 0 \Rightarrow x = \pm\sqrt{2}, \; x = -1/2$

$f(\sqrt{2}) = f(-\sqrt{2}) = 0$

Consider interval $(-\sqrt{2}, \sqrt{2})$, $f(x)$ is a polynomial in x

$f(x)$ is continuous in $[-\sqrt{2}, \sqrt{2}]$, differentiable for all values of x in $(-\sqrt{2}, \sqrt{2})$

$\Rightarrow$ all the conditions of Rolle's theorem are satisfied in this interval.

$\Rightarrow$ there is at least one value of x in $(-\sqrt{2}, \sqrt{2})$ for which $f'(x) = 0$

$\Rightarrow 6x^2 + 2x - 4 = 0 \Rightarrow 3x^2 + x - 2 = 0$

$\Rightarrow x = -1, 2/3$

Since both these points belong to $(-\sqrt{2}, \sqrt{2})$

$\Rightarrow$ Rolle's theorem is verified.

② **Verify Rolle's theorem:-**

$$f(x) = \frac{\sin x}{e^x} \text{ in } (0, \pi)$$

$$f(0) = \frac{\sin 0}{e^0} = 0 \quad , \quad f(\pi) = \frac{\sin \pi}{e^\pi} = 0$$

$f(x)$ is continuous in $[0, \pi]$
differentiable in $(0, \pi)$

$$\Rightarrow \quad f'(x) = e^{-x}(\cos x - \sin x) = 0$$

$$\Rightarrow \quad e^{-x} \neq 0 \quad \Rightarrow \quad \cos x = \sin x$$

$$\Rightarrow \quad x = \frac{\pi}{4}$$

$x = \frac{\pi}{4}$ lies in $(0, \pi)$

$\Rightarrow$ Rolle's theorem is verified.

Ex. (2) :- $f(x) = |x|$ in $[-1, 1]$

$$f(x) = |x| = -x \,, \quad -1 \le x \le 0$$
$$= x \,, \quad 0 \le x \le 1$$

(a) $f(x)$ is continuous in $[-1, 1]$

(b) for differentiability

$$R.H.D = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h - 0}{h} = 1$$

$$L.H.D = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \frac{-(-h) - 0}{-h} = -1$$

$\Rightarrow$ $f(x)$ is not differentiable at $x = 0$

$\Rightarrow$ Rolle's theorem is not applicable.

Ex.(3) :- $f(x) = \dfrac{\sin x}{e^x} = e^{-x}.\sin x$

in $[0, \pi]$

a). for continuity of $f(x)$ in $[0,\pi]$

$f(x)$ is a product of two continuous functions.

$\Rightarrow$ $f(x)$ is a continuous function,

(b). for differentiability in $[0,\pi]$

$f'(x) = -e^{-x}\sin x + e^{-x}\cos x$.

$f'(x) = e^{-x}(\cos x - \sin x)$

exists for every value of x in $(0,\pi)$.

$\Rightarrow$ $f(x)$ is differentiable in $(0,\pi)$

(c). $f(0) = f(\pi) = 0$

$\Rightarrow$ $f(x)$ satisfies all the conditions of Rolle's theorem.

$\Rightarrow$ $\exists$ $c \in (0,\pi)$ such that $f'(c) = 0$

$e^{-c}(\cos c - \sin c) = 0$

$\cos c - \sin c = 0$ $\quad \left\{ e^{-c} \neq 0 \right.$

$\tan c = 1$

$c = n\pi + \frac{\pi}{4}$, where n is an integer.

$n = 0, 1, 2, \dots$

$$C = \frac{\pi}{4}, \frac{5\pi}{4}, \dots$$

~~Here~~ $C = \frac{\pi}{4}$ lies in the interval $(0, \pi)$

$\Rightarrow$ theorem is verified.

$\Rightarrow$

Example. 1:- Verify Rolle's theorem for the following functions:-

i) $f(x) = (x-a)^m (x-b)^n$ in $[a,b]$

where m, n are positive integers.

Solution:- (a) $f(x) = (x-a)^m (x-b)^n$ is a polynomial so $f(x)$ is continuous in $[a,b]$

(b). $f'(x) = m(x-a)^{m-1}(x-b)^n + n(x-a)^m (x-b)^{n-1}$

$= (x-a)^{m-1}(x-b)^{n-1} \cdot [m(x-b) + n(x-a)]$

$= (x-a)^{m-1}(x-b)^{n-1} [(m+n)x - (mb+na)]$

exists for every value of x in (a,b)

$\Rightarrow f(x)$ is differentiable in (a,b)

(c). $f(a) = f(b) = 0$

$\Rightarrow$ $f(x)$ satisfies all the conditions of Rolle's theorem. $\Rightarrow$ there exists at least one point c in (a,b) such that $f'(c) = 0$

$(c-a)^{m-1}(c-b)^{n-1} [(m+n)c - (mb+na)] = 0$

$$\Rightarrow \quad c = \frac{mb+na}{m+n}$$

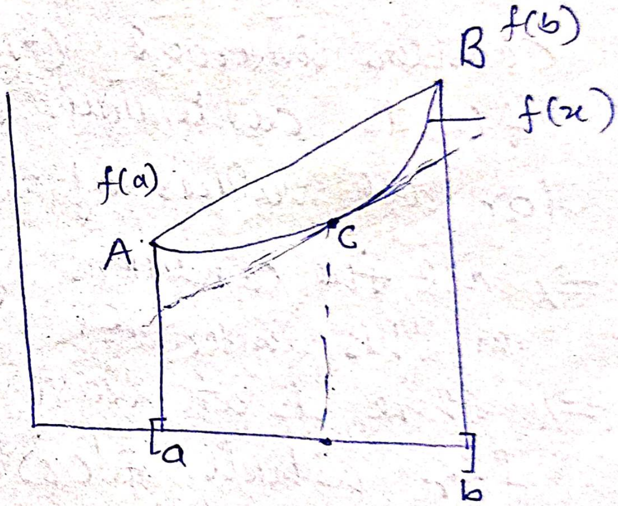
where c represents a point that divides the interval $[a, b]$ internally in the ratio of $m:n$. $\Rightarrow$ Thus c lies in (a, b)

hence, theorem is verified.

Lagranges Mean Value theorem :- If $f(x)$ is continuous in $[a,b]$ differentiable in (a,b) then there exist at least one point $c \in (a,b)$ such that

$$\boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}$$

Geometrically under the given conditions $\exists$ at least one point $c \in (a,b)$ such that tangent at this point is parallel to the chord AB



slope of chord AB is $\frac{f(b)-f(a)}{b-a}$

and slope of tangent at c is $f'(c)$

consider the function $\phi(x) = f(x) + kx$

$f(x)$ is continuous in $[a,b] \Rightarrow \phi(x)$ is continuous.

$f(x)$ is differentiable in $(a,b) \Rightarrow \phi(x)$ is differentiable

if $\phi(a) = \phi(b)$

$$\Rightarrow \quad f(a) + ka = f(b) + kb$$

$$\Rightarrow k = -\frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow \quad \phi(x) = f(x) - \frac{f(b)-f(a)}{b-a}\, x$$

$f(x)$ is differentiable $\Rightarrow \phi(x)$ is differentiable

$\Rightarrow \phi'(x)$ exists in (a,b)

$$\phi'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$$

$\Rightarrow \phi(x)$ satisfied all the conditions of Rolle's theorem $\Rightarrow \exists$ a point $c \in (a,b)$

such that $\phi'(c) = 0$

$$\Rightarrow \quad f'(c) - \frac{f(b)-f(a)}{b-a} = 0$$

$$\Rightarrow \boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}$$

if $\quad b = a+h \qquad a < c < b.$

$[a, a+h] \Rightarrow c \in (a, a+h)$

$\Rightarrow c = a + \theta h \quad , 0 < \theta < 1$

$$f'(a+\theta h) = \frac{f(a+h) - f(a)}{h}$$

$$\Rightarrow \boxed{f(a+h) = f(a) + h f'(a+\theta h)}$$

(1) Find c of Lagrange's M.V.T if

$f(x) = x(x-1)(x-2)$ in $[0, \frac{1}{2}]$

f(x) is a polynomial function is continuous and differentiable for each value of x.

by L.M.V.T $f'(c) = \frac{f(b)-f(a)}{b-a}$

$f(x) = x^3 - 3x^2 + 2x$

$f'(x) = 3x^2 - 6x + 2$

$3c^2 - 6c + 2 = \frac{f(\frac{1}{2}) - f(0)}{\frac{1}{2}}$

$3c^2 - 6c + 2 = \frac{3}{4}$

$3c^2 - 6c + \frac{5}{4} = 0 \Rightarrow c = 1 \pm \frac{\sqrt{21}}{6}$

Only $c = 1 - \frac{\sqrt{21}}{6}$ lies in the interval.

(2). prove that if $(0 < a < b < 1)$

$\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}$

if $f(x) = \tan^{-1} x$

$f'(x) = \frac{1}{1+x^2}$

~~f(x)~~ by L-MVT. $c \in (a, b)$

$f'(c) = \frac{f(b)-f(a)}{b-a}$

$\frac{1}{1+c^2} = \frac{\tan^{-1} b - \tan^{-1} a}{b-a}$

$a < c < b$

$1+a^2 < 1+c^2 < 1+b^2 \Rightarrow \frac{1}{1+b^2} < \frac{1}{1+c^2} < \frac{1}{1+a^2}$

$$\frac{1}{1+b^2} < \frac{\tan^{-1} b - \tan^{-1} a}{b-a} < \frac{1}{1+a^2}$$

$$\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}$$

(2). prove that $\frac{x}{1+x} < \log(1+x) < x$, $\underline{\underline{x>0}}$

$f(x) = \log(1+x)$ $\quad f'(x) = \frac{1}{1+x}$

Consider an interval $[0, x]$

$$0 < c < x$$

$$1 < 1+c < 1+x$$

$$\frac{1}{1+x} < \frac{1}{1+c} < 1$$

by L.M.V.T. $\quad f'(c) = \frac{f(b)-f(a)}{b-a}$.

$$\frac{1}{1+c} = \frac{f(x) - f(0)}{x-0}$$

$$\frac{1}{1+c} = \frac{\log(1+x)}{x}$$

$$\Rightarrow \frac{1}{1+x} < \frac{\log(1+x)}{x} < 1$$

$$\Rightarrow \boxed{\frac{x}{1+x} < \log(1+x) < x.}$$

Cauchy's mean value theorem →

(i). If $f(x)$ and $g(x)$ be two continuous functions in the closed interval $[a,b]$

(ii). and $f'(x)$ and $g'(x)$ exist in (a,b) {differentiable in the open interval (a,b)}

(iii). $g'(x) \neq 0$ for any value of x in (a,b);

then there exists at least one point $c \in (a,b)$ such that

$$\boxed{\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}}$$

①. if in cauchy's mean value theorem, we write $f(x) = 1/\sqrt{x}$ $g(x) = \sqrt{x}$ then c is the geometric mean between a and b.

by cauchy's mean value theorem.

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

$f(x) = 1/\sqrt{x}$ $f'(x) = -\frac{1}{2} \cdot x^{-3/2}$ $g(x) = \sqrt{x}$ $g'(x) = \frac{1}{2}x^{-1/2}$.

$$\frac{-1/2 \cdot c^{-3/2}}{1/2\, c^{-1/2}} = \frac{1/\sqrt{b} - \frac{1}{\sqrt{a}}}{\sqrt{b} - \sqrt{a}.} \Rightarrow -\frac{1}{c} = -\frac{1}{\sqrt{ab}}$$

$$\Rightarrow \boxed{c = \sqrt{ab}}$$

(ii). if $f(x) = 1/x$ $g(x) = 1/x^2$. show c is the harmonic mean between a and b.

$f'(x) = -\frac{1}{x^2}$ $g'(x) = -\frac{2}{x^3}$

$$\frac{-1/c^2}{-2/c^3} = \frac{1/b - \frac{1}{a}}{1/b^2 - 1/a^2}$$

$$\frac{c}{2} = \frac{(a-b)\,ab}{(a-b)(a+b)} = \frac{ab}{a+b}.$$

$$\boxed{c = \frac{2ab}{a+b}}$$

(2) : if $f(x) = e^{x}$, $g(x) = e^{-x}$ show that by using C.M.V.T c is the arithmetic mean between a and b

$f(x) = e^{x}$ $\quad f'(x) = e^{x}$ $\quad g(x) = e^{-x}$ $\quad g'(x) = -e^{-x}$

By C.M.V.T. $\quad \dfrac{f'(c)}{g'(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$

$$\frac{e^{c}}{-e^{-c}} = \frac{e^{b} - e^{a}}{e^{-b} - e^{-a}} = \frac{e^{b} - e^{a}}{\frac{1}{e^{b}} - \frac{1}{e^{a}}} = -e^{a} \cdot e^{b}$$

$$-e^{2c} = -e^{a+b} \Rightarrow 2c = a+b \Rightarrow \boxed{c = \frac{a+b}{2}}$$