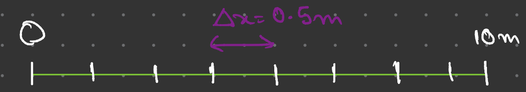
SOLUTION 4

Given equation: $A_c E \dfrac{d^2u}{dx^2} = P(x)$; Here, $A_c = 0.1\,m^2$, $E = 200 \times 10^9\ N/m^2$, $L = 10\,m$, $P(x) = 100\,N/m$



No of elements $= \dfrac{10}{0.5} = 20$

Considering an element 'e' in the domain:

Let us approximate function 'u' on this element using an approximate function u_h which is linear combination of some basis function.

$$u_h^e(x) = c_1^e \phi_1^e(x) + c_2^e \phi_2^e(x) + \ldots + c_n^e \phi_n^e(x) = \sum_{j=1}^{n} c_j^e \phi_j^e(x)$$

Residue, $R^e = A_c E \dfrac{d^2 u_h^e}{dx^2} - P$

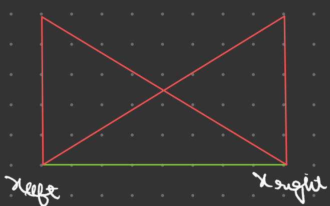
Using method of weighted residual : $\displaystyle\int_{x_{left}}^{x_{right}} R^e w_i^e \, dx = 0$

$$\Rightarrow \int_{x_{left}}^{x_{right}} \left(A_c E \frac{d^2 u_h^e}{dx^2} - P \right) w_i^e \, dx = 0$$

$$\Rightarrow \left[\frac{du_h^e}{dx} w_i^e \right]_{x_{left}}^{x_{right}} - \int_{x_{left}}^{x_{right}} \frac{du_h^e}{dx} \frac{dw_i^e}{dx} \, dx - \int_{x_{left}}^{x_{right}} \frac{P}{A_c E} w_i^e \, dx = 0$$

$$\Rightarrow \int_{x_{left}}^{x_{right}} \frac{du_h^e}{dx} \frac{dw_i^e}{dx} \, dx + \int_{x_{left}}^{x_{right}} \frac{P}{A_c E} w_i^e \, dx = \left[\left(\frac{du_h^e}{dx}\right)_{x_{right}} w_i^e(x_{right}) - \left(\frac{du_h^e}{dx}\right)_{x_{left}} w_i^e(x_{left}) \right]$$

(WEAK FORM)



Linear basis for element e:

$$u_h^e = L_{left}^e u_{left}^e + L_{right}^e u_{right}^e$$

where, $L_{left}^e = \dfrac{x - x_{right}}{x_{left} - x_{right}}$, $L_{right}^e = \dfrac{x - x_{left}}{x_{right} - x_{left}}$

Using Galerkin approach, weighting function are: $w_i^e(x) = L_i^e(x)$

So, $$\int_{x_{left}}^{x_{right}} \frac{du_h^e}{dx} \frac{dw_i^e}{dx} \, dx = \int_{x_{left}}^{x_{right}} \frac{d}{dx}\left[L_j^e u_j^e \right] \frac{dL_i^e}{dx} \, dx = \int_{x_{left}}^{x_{right}} \frac{dL_j^e}{dx} \frac{dL_i^e}{dx} u_j^e \, dx$$

$$= \underbrace{\left[\int_{x_{left}}^{x_{right}} \frac{dL_j^e}{dx} \frac{dL_i^e}{dx} \, dx \right]}_{K_{ij}^e} u_j^e$$

General equation: $\underset{\sim}{K}\,\underset{\sim}{u} = \underset{\sim}{F}$

ISO PARAMETRIC ELEMENT:

$z_e = 0$ —— $z_e = 1$ $\Rightarrow L_{eft} = 1 - z_e$, $L_{right} = z_e$

Steps:

Compute 2×2 local stiffness matrix given by: $k^e = \frac{1}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Element 0: $\begin{bmatrix} K_{00}^0 & K_{01}^0 \\ K_{10}^0 & K_{11}^0 \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = -\begin{bmatrix} f_0 \\ f_1 \end{bmatrix} + [J]^{-1}\begin{bmatrix} -q_0 \\ q_1 \end{bmatrix}$

Element 1: $\begin{bmatrix} K_{00}^1 & K_{01}^1 \\ K_{10}^1 & K_{11}^1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = -\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + [J]^{-1}\begin{bmatrix} -q_1 \\ q_2 \end{bmatrix}$

$\vdots$

Element N-1: $\begin{bmatrix} K_{00}^{N-1} & K_{01}^{N-1} \\ K_{10}^{N-1} & K_{11}^{N-1} \end{bmatrix}\begin{bmatrix} u_{N-1} \\ u_N \end{bmatrix} = -\begin{bmatrix} f_{N-1} \\ f_N \end{bmatrix} + [J]^{-1}\begin{bmatrix} -q_{N-1} \\ q_N \end{bmatrix}$

Combining all matrices:

$$\begin{bmatrix} K_{00}^0 & K_{01}^0 & & \\ K_{10}^0 & K_{11}^0 + K_{00}^1 & K_{01}^1 & \\ & K_{10}^1 & K_{11}^1 + K_{00}^2 & \ddots \\ & \ddots & \ddots & \ddots \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{bmatrix} = \begin{bmatrix} f_0 \\ 2f_1 \\ 2f_2 \\ \vdots \\ 2f_{N-1} \\ f_N \end{bmatrix} + [J]^{-1}\begin{bmatrix} -q_0 \\ 0 \\ \vdots \\ 0 \\ q_N \end{bmatrix}$$