Solution 1

a) Second order accurate finite difference approximation

To find: Second order accurate finite difference approximation for second order derivative $\frac{d^2u}{dx^2}$ using one sided stencil to the left of i on a non uniform grid

Let us ~~for~~ represent u as f.

Using method of undetermined coefficients

$$f''(x_i) = af(x_{i-1}) + bf(x_{i-2}) + cf(x_{i-3}) + df(x_i)$$

Expanding each terms in RHS using Taylor series expansion ~~as~~ :

$$f''(x_i) = a\left[f(x_i) - h_i f'(x_i) + \frac{h_i^2}{2!} f''(x_i) - \frac{h_i^3}{3!} f'''(x_i) + \dots\right]$$

$$+ b\left[f(x_i) - (h_i + h_{i-1}) f'(x_i) + \frac{(h_i+h_{i-1})^2}{2!} f''(x_i) - \frac{(h_i+h_{i-1})^3}{3!} f'''(x_i) + \dots\right]$$

$$+ c\left[f(x_i) - (h_i+h_{i-1}+h_{i-2}) f'(x_i) + \frac{(h_i+h_{i-1}+h_{i-2})}{2!} f''(x_i) - \frac{(h_i+h_{i-1}+h_{i-2})^3}{3!} f'''(x_i) + \dots\right]$$

$$+ df(x_i)$$

$$= (a+b+c+d) f(x_i) - \left[ah_i + b(h_i+h_{i-1}) + c(h_i+h_{i-1}+h_{i-2})\right] f'(x_i)$$

$$+ \left[\frac{ah_i^2}{2!} + \frac{b(h_i+h_{i-1})^2}{2!} + \frac{c(h_i+h_{i-1}+h_{i-2})}{2!}\right] f''(x_i)$$

$$- \left[\frac{ah_i^3}{3!} + \frac{b(h_i+h_{i-1})^3}{3!} + \frac{c(h_i+h_{i-1}+h_{i-2})^3}{3!}\right] f'''(x_i)$$

Comparing coefficients of $f(x_i)$, $f'(x_i)$, $f''(x_i)$ & $f'''(x_i)$ on LHS & RHS :-

$$a + b + c + d = 0 \quad \dots (1)$$

$$ah_i + b(h_i+h_{i-1}) + c(h_i+h_{i-1}+h_{i-2}) = 0 \quad \dots (2)$$

$$ah_i^2 + b(h_i+h_{i-1})^2 + c(h_i+h_{i-1}+h_{i-2})^2 = 2 \quad \dots (3)$$

$$ah_i^3 + b(h_i+h_{i-1})^3 + c(h_i+h_{i-1}+h_{i-2})^3 = 0 \quad \dots (4)$$

Writing in matrix form :

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ h_i & h_i+h_{i-1} & h_i+h_{i-1}+h_{i-2} & 0 \\ h_i^2 & (h_i+h_{i-1})^2 & (h_i+h_{i-1}+h_{i-2})^2 & 0 \\ h_i^3 & (h_i+h_{i-1})^3 & (h_i+h_{i-1}+h_{i-2})^3 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

After solving matrix equation:

$$a = \frac{-2(h_{i-2} + 2h_i + 2h_{i-1})}{h_i(h_{i-1} + h_{i-2})h_{i-1}}$$

$$b = \frac{2(h_{i-2} + 2h_i + h_{i-1})}{h_{i-1}(h_i + h_{i-1})h_{i-2}}$$

$$c = \frac{-2(2h_i + h_{i-1})}{(h_i + h_{i-2})h_{i-2}(h_i + h_{i-1} + h_{i-2})}$$

$$d = \frac{2(h_{i-2} + 3h_i + 2h_{i-1})}{h_i^2\left[h_i + 2h_{i-1}h_i + h_i h_{i-2} + h_{i-1}^2 + h_{i-1}h_{i-2}\right]}$$

$$\frac{d^2u}{dx^2} = \frac{-2(h_{i-2} + 2h_i + 2h_{i-1})}{h_i(h_{i-1} + h_{i-2})h_{i-1}}\,u(x_{i-1}) + \frac{2(h_{i-2} + 2h_i + h_{i-1})}{h_{i-1}(h_i + h_{i-1})h_{i-2}}\,u(x_{i-2})$$
$$- \frac{2(2h_i + h_{i-1})}{(h_i + h_{i-2})h_{i-2}(h_i + h_{i-1} + h_{i-2})}\,u(x_{i-3}) + \frac{2(h_{i-2} + 3h_i + 2h_{i-1})}{h_i^2\left[h_i + 2h_{i-1}h_i + h_i h_{i-2} + h_{i-1}^2 + h_{i-1}h_{i-2}\right]}\,u(x_i)$$

c) To find: Second order accurate finite difference approximation for cross derivative $\frac{\partial^2 u}{\partial x \partial y}$ on a uniform grid.

$$\frac{\partial f}{\partial x} = \frac{f(x+\Delta x, y) - f(x-\Delta x, y)}{2\Delta x}, \quad \frac{\partial f}{\partial y} = \frac{f(x, y+\Delta y) - f(x, y-\Delta y)}{2\Delta y}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\frac{\partial f}{\partial y}(x+\Delta x, y) - \frac{\partial f}{\partial y}(x-\Delta x, y)}{2\Delta x}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\frac{f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y-\Delta y)}{2\Delta y} - \left[\frac{f(x-\Delta x, y+\Delta y) - f(x-\Delta x, y-\Delta y)}{2\Delta y}\right]}{2\Delta x}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y-\Delta y) - f(x-\Delta x, y+\Delta y) + f(x-\Delta x, y-\Delta y)}{4\Delta x \Delta y}$$

or $$\frac{\partial^2 u}{\partial x \partial y} = \frac{u(x_{i+1}, y_{j+1}) - u(x_{i+1}, y_{j-1}) - u(x_{i-1}, y_{j+1}) + u(x_{i-1}, y_{j-1})}{4\Delta x \Delta y} \quad \ldots (*)$$

Truncation error:

$$\frac{\partial^2 f}{\partial x \partial y} = a f(x_{i+1}, y_{j+1}) + b f(x_{i+1}, y_{j-1}) + c f(x_{i-1}, y_{j+1}) + d f(x_{i-1}, y_{j-1})$$

From equation (*):

$$a = \frac{1}{4hk}, \quad b = \frac{-1}{4hk}, \quad c = \frac{-1}{4hk}, \quad d = \frac{1}{4hk}$$

Now writing 4th order terms which will contribute to truncation error:

$$TE = \left\{ \begin{aligned} & a \left[h^4 \frac{\partial^4 f}{\partial x^4} + 6h^2k^2 \frac{\partial^4 f}{\partial x^2 \partial y^2} + 4hk^3 \frac{\partial^4 f}{\partial x \partial y^3} + 4h^3k \frac{\partial^4 f}{\partial x^3 \partial y} + k^4 \frac{\partial^4 f}{\partial y^4} \right] \\ & + b \left[h^4 \frac{\partial^4 f}{\partial x^4} + 6h^2k^2 \frac{\partial^4 f}{\partial x^2 \partial y^2} - 4hk^3 \frac{\partial^4 f}{\partial x \partial y^3} - 4h^3k \frac{\partial^4 f}{\partial x^3 \partial y} + k^4 \frac{\partial^4 f}{\partial y^4} \right] \\ & + c \left[h^4 \frac{\partial^4 f}{\partial x^4} + 6h^2k^2 \frac{\partial^4 f}{\partial x^2 \partial y^2} - 4hk^3 \frac{\partial^4 f}{\partial x \partial y^3} - 4h^3k \frac{\partial^4 f}{\partial x^3 \partial y} + k^4 \frac{\partial^4 f}{\partial y^4} \right] \\ & + d \left[h^4 \frac{\partial^4 f}{\partial x^4} + 6h^2k^2 \frac{\partial^4 f}{\partial x^2 \partial y^2} + 4hk^3 \frac{\partial^4 f}{\partial x \partial y^3} + 4h^3k \frac{\partial^4 f}{\partial x^3 \partial y} + k^4 \frac{\partial^4 f}{\partial y^4} \right] \end{aligned} \right\} \times \frac{1}{4!}$$

$$= \left[\begin{aligned} & (a+b+c+d) h^4 \frac{\partial^4 f}{\partial x^4} + (a+b+c+d) 6h^2k^2 \frac{\partial^4 f}{\partial x^2 \partial y^2} + (a-b-c+d) 4hk^3 \frac{\partial^4 f}{\partial x \partial y^3} \\ & + (a-b-c+d) 4h^3k \frac{\partial^4 f}{\partial x^3 \partial y} + (a+b+c+d) k^4 \frac{\partial^4 f}{\partial y^4} \end{aligned} \right] \times \frac{1}{4!}$$

$$= \frac{hk}{6} \left[\frac{\partial^4 f}{\partial x^3 \partial y} + \frac{\partial^4 f}{\partial x \partial y^3} \right]$$

$$= \frac{h^2}{6} \left[4_{xxxy} + 4_{xyyy} \right]$$