ME 282, Jan-April 2023
R. K. Shukla
Assignment 2, Due Friday, Feb 24

1. Write a computer code to solve the following BVP using the standard second-order finite difference method on a uniform grid ($x_i = i/N$, $i = 0, 1, \dots, N$)

$$u'' = -\sin(\beta\pi x), \quad x \in (0,1) \tag{1}$$
$$u(0) = u(1) = 0. \tag{2}$$

Use Thomas algorithm to solve the resulting linear system of equations. Use your code to answer following questions.

(a) Assume $\beta = 10$. Plot the L_2 error, defined as

$$e_2 = \sqrt{\frac{\sum_{i=1}^{N}\left(u_i^{\text{calculated}} - u_i^{\text{exact}}\right)^2}{N}}, \tag{3}$$

as a function of N by choosing $N = 25, 50, 100, 200$ and 400 (use a log-log scale). What rate of convergence (slope of the graph) do you observe?

(b) Choose $N = 50$ and compute solutions for $\beta = 1, 10$ and 100. Plot u^{exact} and $u^{\text{calculated}}$ as a function of x for different values of β mentioned above (show all these on the same graph using different style/symbols, you may use Tecplot which is available at SERC or any other software of your choice). What do you observe ?

2. Let A be the matrix associated with the standard second-order finite difference approximation to the 2D BVP on a unit square

$$u_{xx} + u_{yy} = -f(x,y), \quad (x,y) \in (0,1)^2 \tag{4}$$
$$u(0,y) = u(1,y) = 0, \quad y \in [0,1] \tag{5}$$
$$u(x,0) = u(x,1) = 0, \quad x \in [0,1]. \tag{6}$$

Assume that $h = \Delta x = \Delta y = 1/N$ so that $x_i = ih, i = 0, 1, \dots, N$ and $y_j = jh, j = 0, 1, \dots N$.

(a) Verify analytically that the $(N-1)^2$ eigenvectors of A are given by the discrete functions $s^{p,q}$, whose (i,j)th entry is given by $\sin(p\pi x_i)\sin(q\pi y_j)$ for a value of p in $[1, N-1]$ and q in $[1, N-1]$.

(b) Give a formula for the eigenvalues and condition number of A.

(c) Give the analytical solution to $A\mathbf{u} = \mathbf{f}$, when $\mathbf{f}$ is the vector obtained by evaluating $-\sin(5\pi x)\sin(6\pi y)$ at the nodes $x_i = i/N, i = 1, 2, \dots, N-1$, $y_j = j/N, j = 1, 2, \dots, N-1$ (assuming $N > 6$). What do you observe when $N \to \infty$?

3. Consider the following 2D Helmholtz equation with Dirichlet boundary conditions:

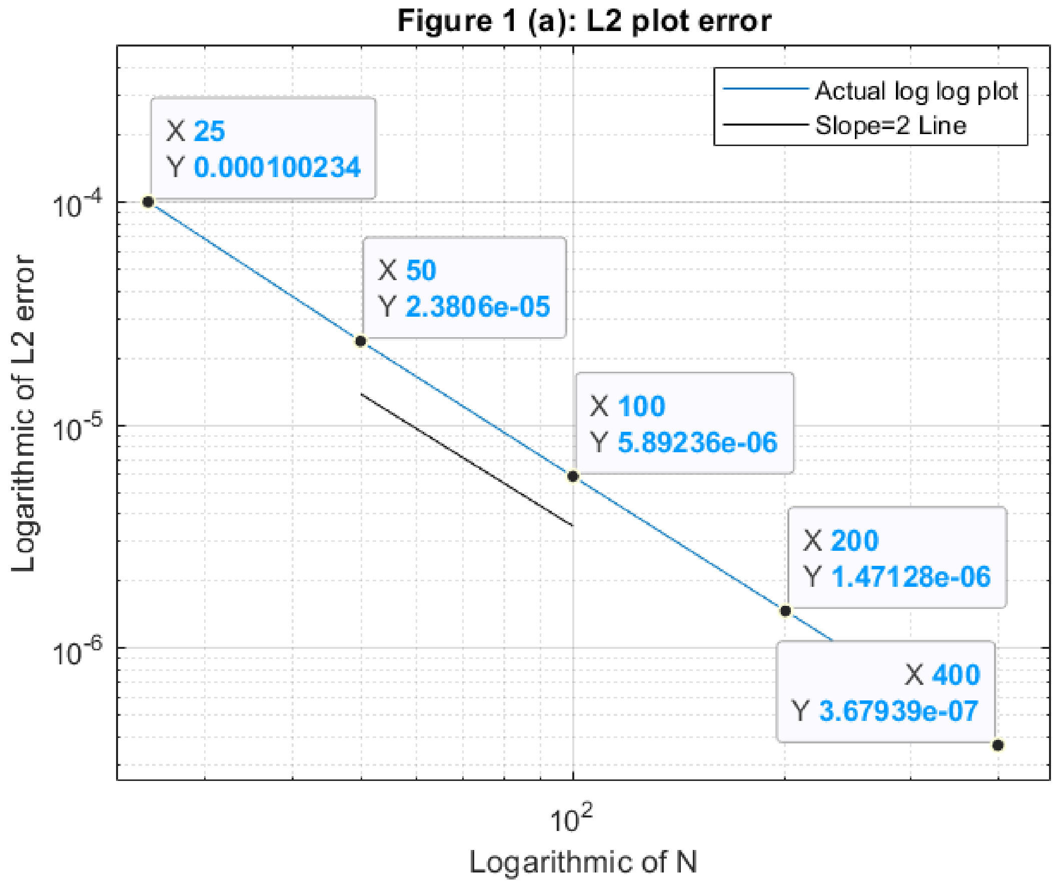
$$\begin{aligned} &\nabla^2 u + 1000u = (1000 - 200\pi^2)\sin(10\pi x)\cos(10\pi y), \quad (x,y) \in (0,1)^2 \\ &u(x,0) = u(x,1) = \sin(10\pi x) \\ &u(0,y) = u(1,y) = 0. \end{aligned} \tag{7}$$

Write a computer program to compute an approximate solution to this problem using standard second order central finite difference approximation on a uniform $N \times N$ computational grid. Use the matrix diagonalization method discussed in the class to solve the resulting linear system of equations (you could use the formulas for eigenvalues and eigenvectors from Problem 2 of this Assignment). Plot the L_2 and L_∞ errors as a function of N. What rate of convergence do you observe?

ME 282 ASSIGNMENT 2 – SAURABH SHARMA

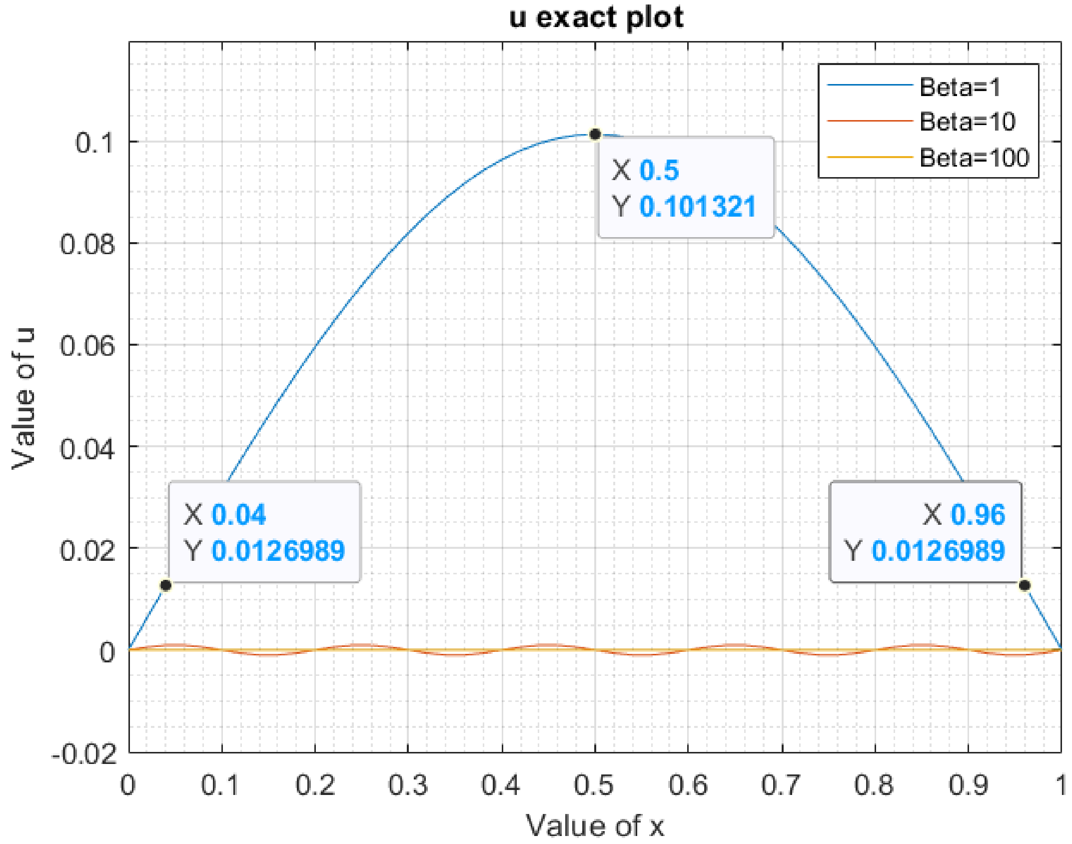
SOLUTION 1

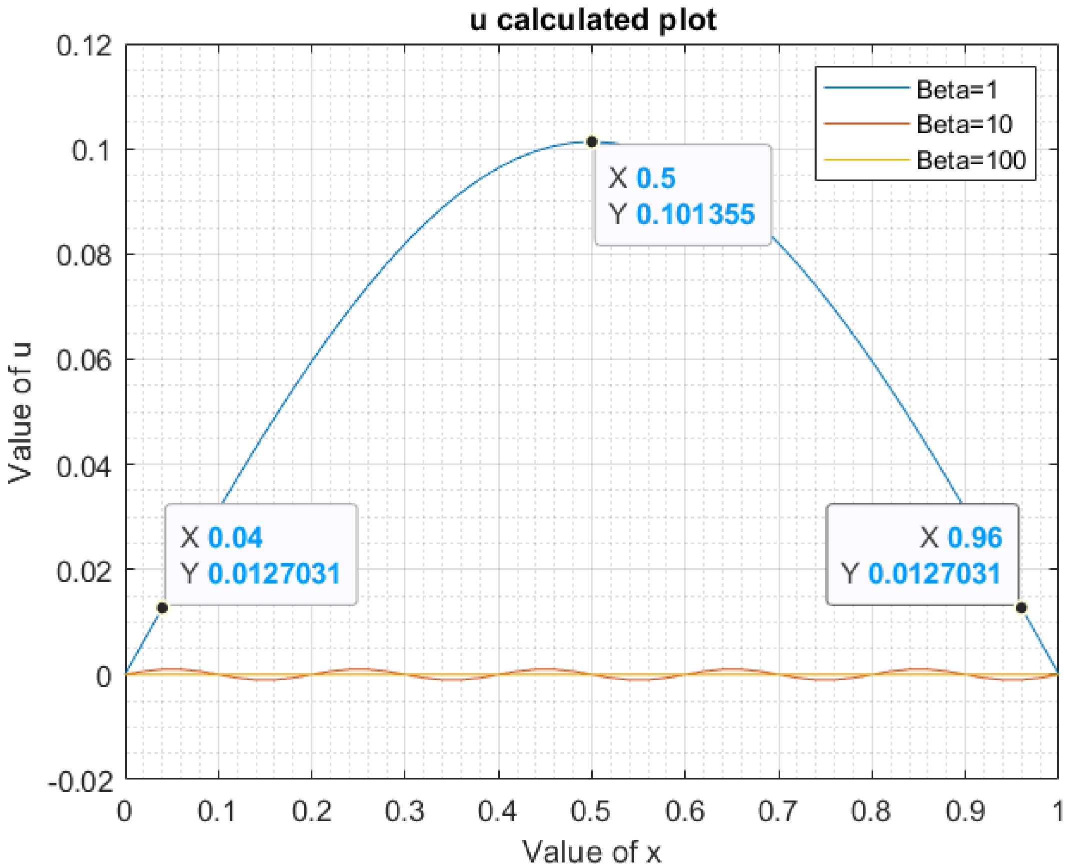
A.

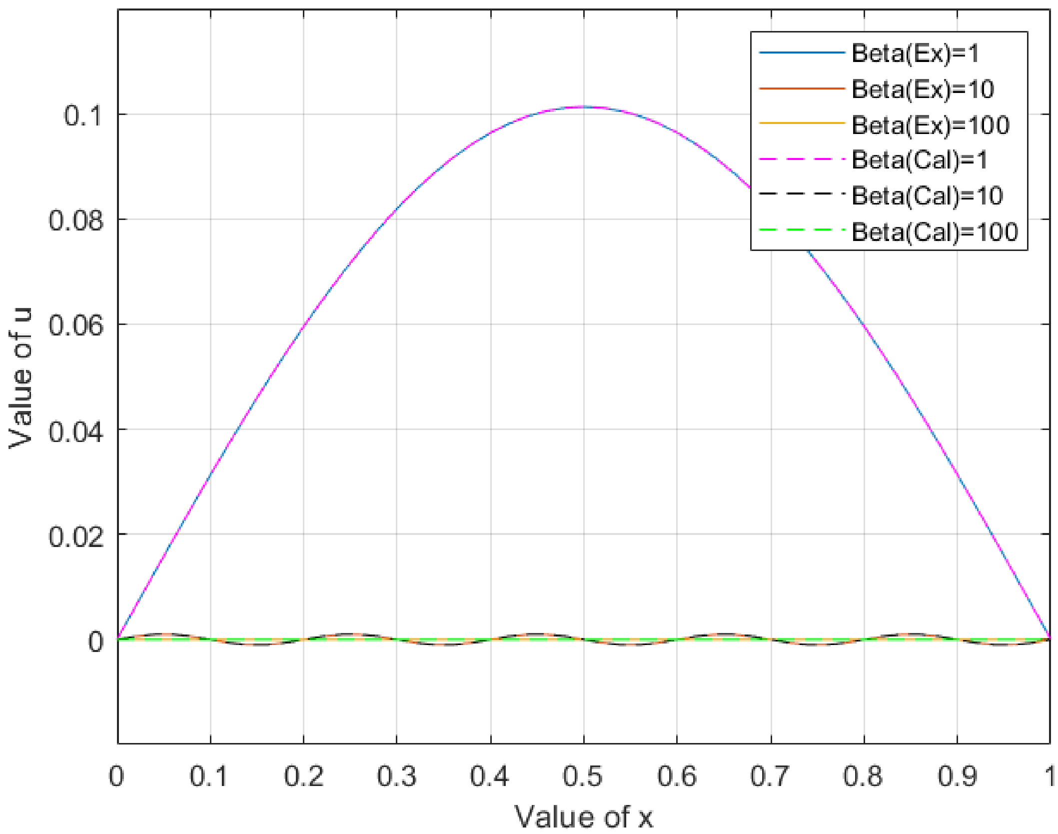


The rate of convergence is equal to 2.

B.


u exact plot
Beta=1
Beta=10
Beta=100
X 0.5
Y 0.101321
X 0.04
Y 0.0126989
X 0.96
Y 0.0126989
Value of u
Value of x
0.1
0.08
0.06
0.04
0.02
0
-0.02
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


u calculated plot
Beta=1
Beta=10
Beta=100
X 0.5
Y 0.101355
X 0.04
Y 0.0127031
X 0.96
Y 0.0127031
Value of u
Value of x
0.12
0.1
0.08
0.06
0.04
0.02
0
-0.02
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1



Beta(Ex)=1
Beta(Ex)=10
Beta(Ex)=100
Beta(Cal)=1
Beta(Cal)=10
Beta(Cal)=100
0.1
0.08
0.06
0.04
0.02
0
Value of u
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Value of x

SAURABH SHARMA
-21498
M.tech (ME)

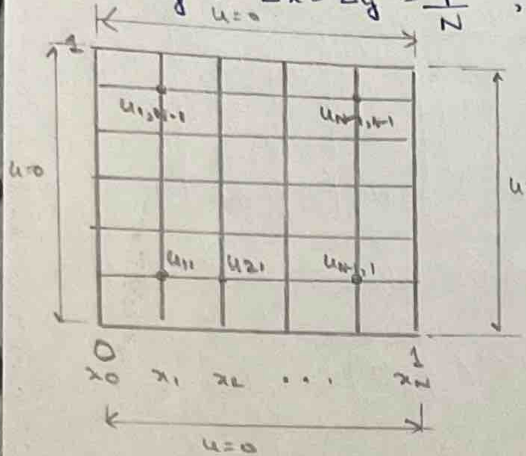
ME282 ASSIGNMENT-2

Problem 2: $u_{xx} + u_{yy} = -f(x,y) \quad , (x,y) \in (0,1)^2$

$u(0,y) = u(1,y) = 0 \quad , y \in [0,1]$

$u(x,0) = u(x,1) = 0 \quad , x \in [0,1]$

Assuming $h = \Delta x = \Delta y = \frac{1}{N}$; $x_i = ih$, $i = 0,1,\dots N$; $y_j = jh$, $j = 0,1,\dots,N$



Discretizing the given equation with second order difference :

$$\frac{u_{i+1,j} + u_{i-1,j} - 2u_{i,j}}{h^2} + \frac{u_{i,j+1} + u_{i,j-1} - 2u_{i,j}}{h^2} = -f_{i,j}$$

$$\Rightarrow u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = -h^2 f_{i,j}$$

Boundary condition : $u(x,0) = 0 \Rightarrow u_{i,0} = 0$

$u(x,1) = 0 \Rightarrow u_{i,1} = 0$

$u(0,y) = 0 \Rightarrow u_{0,j} = 0$

$u(1,y) = 0 \Rightarrow u_{1,j} = 0$

$$A = \begin{bmatrix} T & I & 0 & 0 & \cdots & 0 \\ I & T & I & 0 & \cdots & 0 \\ & \ddots & & \ddots & & \ddots \\ & \cdot & \cdot & 0 & I & T \end{bmatrix} , \quad T = \begin{bmatrix} -4 & -1 & 0 & 0 & \cdots & 0 \\ 1 & -4 & 1 & 0 & \cdots & 0 \\ & & \cdots & & & \\ & & & & 0 \; 1 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 4 & -1 & \cdots & \cdots & 0 \\ & & & & & \\ & & & & & -1 \\ & & & & -1 & 4 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \\ \vdots \\ u_{1,N-1} \\ \vdots \\ u_{ij} \\ \vdots \\ u_{N-1,N-1} \end{bmatrix} = \begin{bmatrix} F_{11} \\ F_{12} \\ \vdots \\ F_{ij} \\ \vdots \\ F_{N-1,N-1} \end{bmatrix}$$

Verifying eigenvector : $\sin(p\pi x_i)\sin(q\pi y_j)$

$[A]\vec{S}^{pq} = \lambda \vec{S}^{pq}$ where, $\vec{S}^{pq} = \begin{bmatrix} \sin\left(\frac{p\pi}{N}\right)\sin\left(\frac{q\pi}{N}\right) \\ \sin\left(\frac{2p\pi}{N}\right)\sin\left(\frac{2q\pi}{N}\right) \\ \vdots \\ \sin\left(\frac{ip\pi}{N}\right)\sin\left(\frac{jq\pi}{N}\right) \\ \vdots \\ \sin\left(\frac{(N-1)p\pi}{N}\right)\sin\left(\frac{(N-1)q\pi}{N}\right) \end{bmatrix}$

(i,j) : $-\sin\left[\frac{(i-1)p\pi}{N}\right]\sin\left[\frac{jq\pi}{N}\right] - \sin\left[\frac{ip\pi}{m}\right]\sin\left[\frac{(j-1)q\pi}{N}\right] + 4\sin\left(\frac{ip\pi}{N}\right)\cos\left(\frac{jq\pi}{N}\right) - \sin\left(\frac{ip\pi}{N}\right)\sin\left(\frac{(j+1)q\pi}{N}\right)$

$- \sin\left[\frac{(i+1)p\pi}{N}\right]\sin\left[\frac{jq\pi}{N}\right]$

$= -\sin\left[\frac{jq\pi}{N}\right]\left\{2\sin\left(\frac{ip\pi}{N}\right)\cos\left(\frac{p\pi}{N}\right)\right\} - \sin\frac{ip\pi}{N}\left[2\sin\left(\frac{jq\pi}{N}\right)\cos\left(\frac{q\pi}{N}\right)\right] + 4\sin\left(\frac{ip\pi}{N}\right)\sin\left(\frac{jq\pi}{N}\right)$

$= \left[4 - 2\cos\left(\frac{p\pi}{N}\right) - 2\cos\left(\frac{q\pi}{N}\right)\right]\left[\sin\left(\frac{ip\pi}{N}\right)\sin\left(\frac{jq\pi}{N}\right)\right] = \lambda\left[\vec{S}^{pq}\right]_{ij}$

This verified that eigenvectors of A are given by discrete function.

b.) $\lambda_{p,q} = 4 - 2\cos\left(\frac{p\pi}{N}\right) - 2\cos\left(\frac{q\pi}{N}\right)$

$= 2\left[1-\cos\left(\frac{p\pi}{N}\right)\right] + 2\left[1-\cos\left(\frac{q\pi}{N}\right)\right]$

$\Rightarrow \lambda_{p,q} = 4\left[\sin^2\left(\frac{p\pi}{2N}\right) + \sin^2\left(\frac{q\pi}{2N}\right)\right]$

Condition no. of A = cond (A) = $\dfrac{|\lambda_{p,q}|_{max}}{|\lambda_{p,q}|_{min}}$

$$= \frac{4\left[\sin^2\left(\frac{N-1}{N}\frac{\pi}{2}\right) + \sin^2\left(\frac{N-1}{N}\frac{\pi}{2}\right)\right]}{4\left[\sin^2\left(\frac{\pi}{2N}\right) + \sin^2\left(\frac{\pi}{2N}\right)\right]}$$

$N \to \infty$, $\sin\left(\frac{\pi}{2N}\right) \approx \frac{\pi}{2N}$, $\frac{N-1}{N} \to 1$

cond (A) $= \dfrac{8N^2}{2\pi^2} = \dfrac{4N^2}{\pi^2}$

c.) $f = -h^2 \begin{bmatrix} \sin\left(\frac{5\pi}{N}\right)\sin\left(\frac{6\pi}{N}\right) \\ \vdots \\ \sin\left(\frac{5\pi i}{N}\right)\sin\left(\frac{6\pi j}{N}\right) \\ \vdots \\ \sin\left[\frac{5\pi(N-1)}{N}\right]\sin\left[\frac{6\pi(N-1)}{N}\right] \end{bmatrix} = -h^2 \vec{s}_{5,6}$

$AU = F \Rightarrow [P][\lambda][P^{-1}]u = F$

as $\vec{F} = -h^2 \bar{s}_{5,6}$

let $P^{-1}F = F$ & $P^{-1}U = \bar{U}$

$\Rightarrow [\lambda]\vec{\bar{u}} = \vec{\bar{F}} \Rightarrow \overrightarrow{u_{i,j}} = \dfrac{\vec{F}_{i,j}}{\lambda_{i,j}}$

Matrix $\equiv$ $\begin{bmatrix} \uparrow & & \\ \sin\left(\frac{\pi i}{N}\right)\sin\left(\frac{\pi j}{N}\right) & \cdots & \sin\left(\frac{p\pi i}{N}\right)\sin\left(\frac{q\pi j}{N}\right) \\ \downarrow & & \end{bmatrix}\begin{bmatrix} 0 \\ \vdots \\ h^2 \\ \vdots \\ 0 \end{bmatrix} = h^2 \begin{bmatrix} -\sin\left(\frac{5\pi}{N}\right)\sin\left(\frac{6\pi}{N}\right) \\ \vdots \\ -\sin\left(\frac{5\pi i}{N}\right)\sin\left(\frac{6\pi j}{N}\right) \\ \vdots \\ -\sin\left(\frac{5\pi(N-1)}{N}\right)\sin\left(\frac{6\pi(N-1)}{N}\right) \end{bmatrix}$

$$\vec{F} = [0 \ldots -h^2 \ldots 0]^T$$

$$\bar{u} = \frac{\bar{F}}{\lambda} \Rightarrow \bar{u} = \left[0 \ldots \frac{-h^2}{4\left[\sin^2\left(\frac{5\pi}{2N}\right)+\sin^2\left(\frac{6\pi}{2N}\right)\right]} \ldots 0\right]^T$$

$$u = P\bar{u} \quad \Rightarrow u = -h^2 \begin{bmatrix} \dfrac{\sin\left(\frac{5\pi}{N}\right)\cdot\sin\left(\frac{6\pi}{N}\right)}{4\left[\sin^2\left(\frac{5\pi}{2N}\right)+\sin^2\left(\frac{6\pi}{2N}\right)\right]} \\ \vdots \\ \dfrac{\sin\left(\frac{5\pi i}{N}\right)\cdot\sin\left(\frac{6\pi i}{N}\right)}{4\left[\sin^2\left(\frac{5\pi}{2N}\right)+\sin^2\left(\frac{6\pi}{2N}\right)\right]} \end{bmatrix}$$

As $N \to \infty$: $\sin\left(\frac{5\pi}{2N}\right) \to \frac{5\pi}{2N}$, $\sin\left(\frac{6\pi}{2N}\right) \to \frac{6\pi}{2N}$

Then $(i,j) \Rightarrow u_{ij} = -\dfrac{\sin\left(\frac{5\pi i}{N}\right)\sin\left(\frac{6\pi i}{N}\right)}{(5\pi)^2+(6\pi)^2}$

Using $x_i = i/_N$, $y_j = \frac{j}{N}$

$\therefore u(x,y) = \dfrac{-\sin(5\pi x)\sin(6\pi y)}{(5\pi)^2+(6\pi)^2}$ ..(1) (Approximate solution)

In terms of Fourier series : $u(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm}\sin(n\pi x)\sin(m\pi y)$

$F(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} F_{nm}\sin(n\pi x)\sin(m\pi y)$

$u_{xx}+u_{yy} = -(n^2\pi^2+m^2\pi^2)\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm}\sin(n\pi x)\sin(m\pi y)$

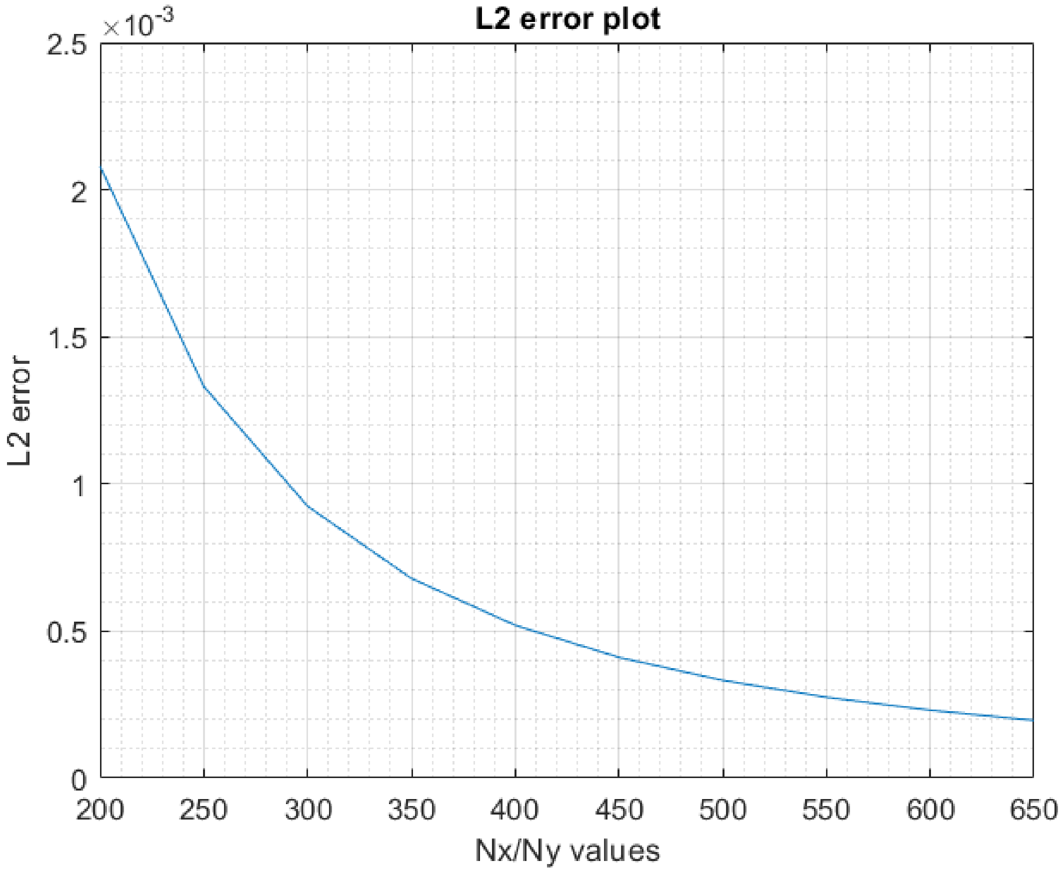
$\therefore a_{nm} = \dfrac{-F_{nm}}{(n\pi)^2+(m\pi)^2}$

$u(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm}\sin(n\pi x)\sin(m\pi y)$

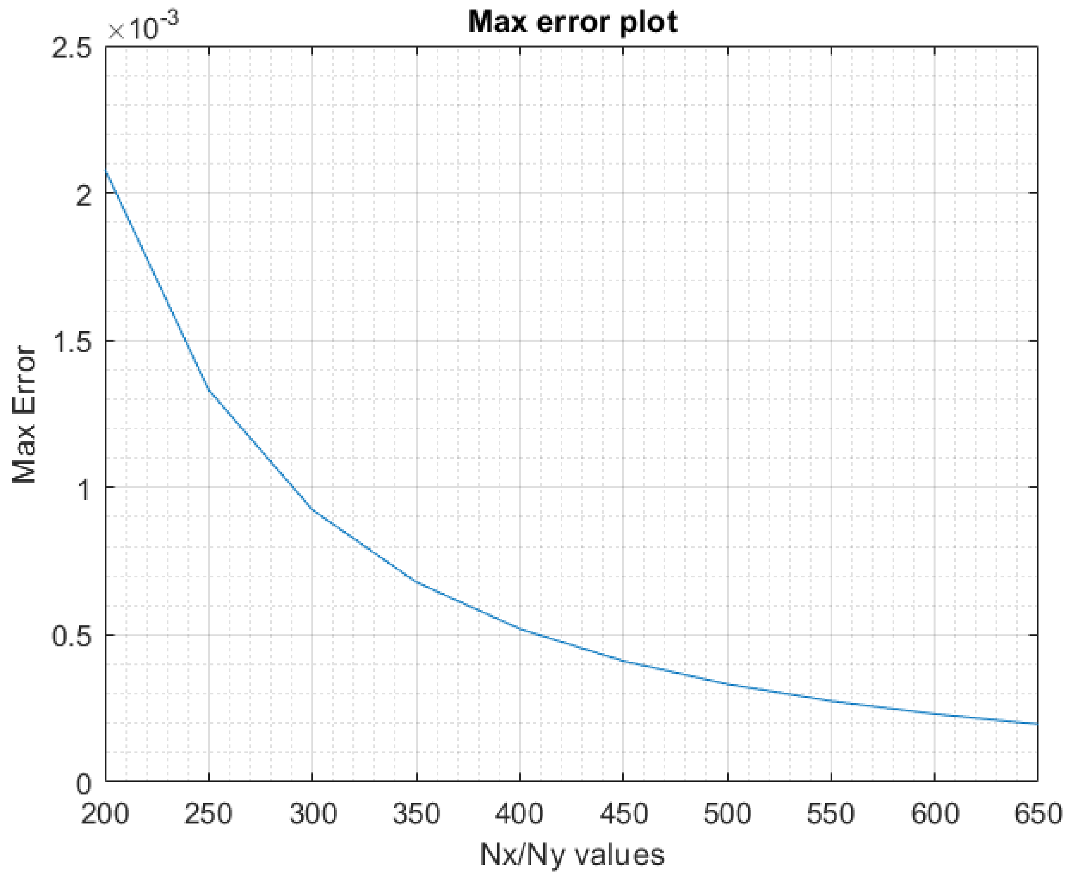
$= -\dfrac{\sin(5\pi x)\sin(6\pi y)}{(5\pi)^2+(6\pi)^2}$ ..(2) (Analytical solution)

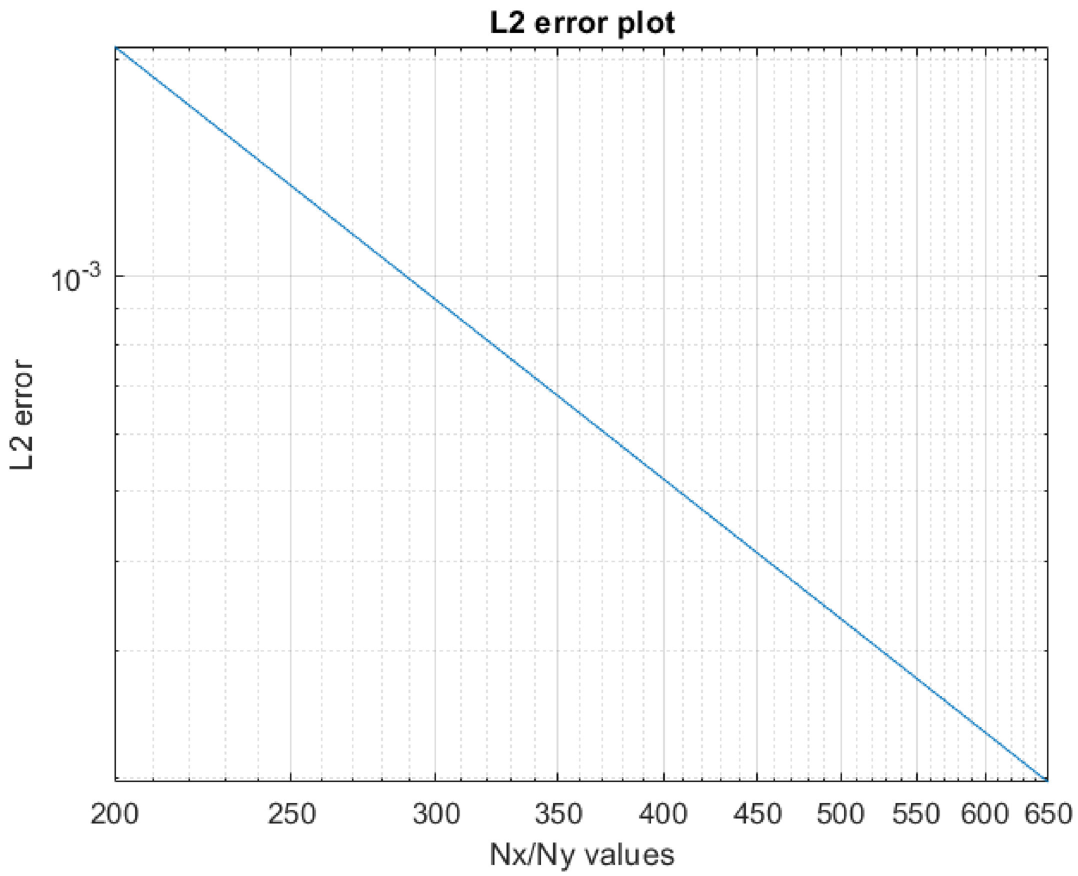
As $N\to\infty$; Eq^n (1) is equivalent to equation (2).

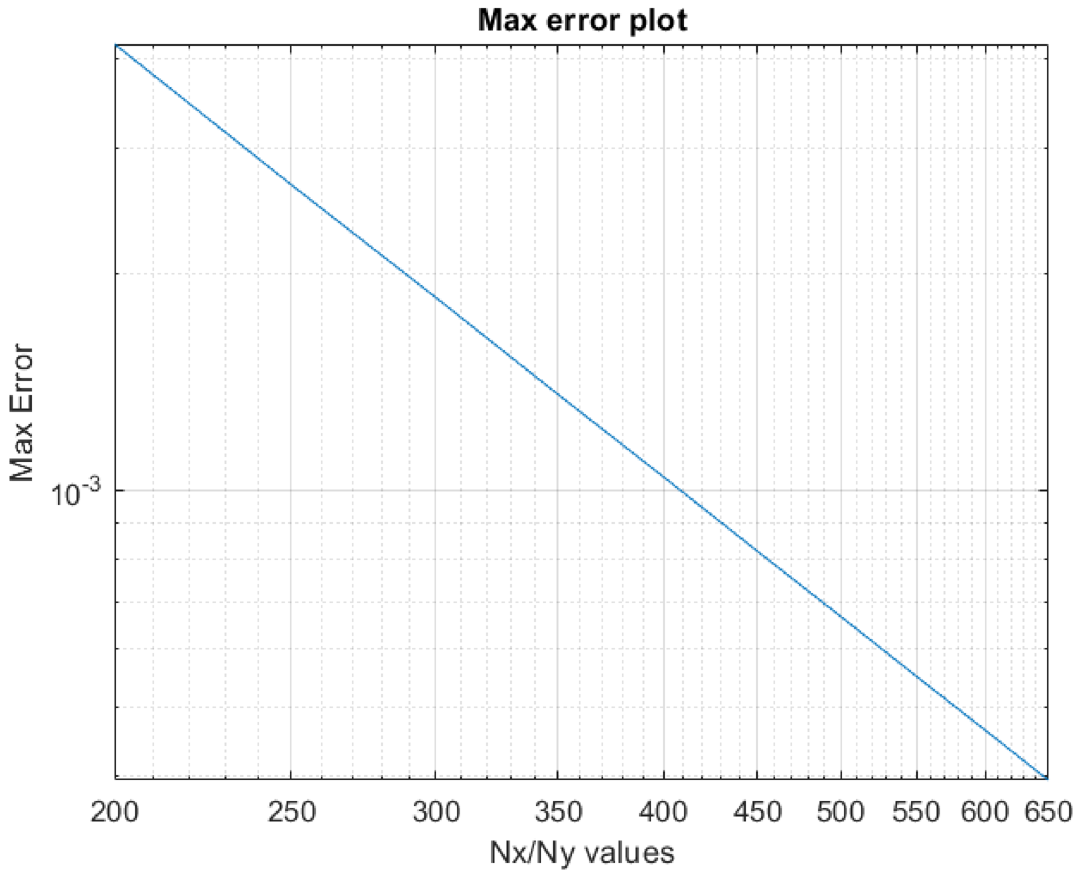
SOLUTION 3



The rate of convergence is equal to one for the L2 error plot.


Max error plot
×10^-3
2.5
2
1.5
1
0.5
0
Max Error
200
250
300
350
400
450
500
550
600
650
Nx/Ny values


L2 error plot
10^-3
L2 error
200
250
300
350
400
450
500
550
600
650
Nx/Ny values


Max error plot
Max Error
10^-3
200
250
300
350
400
450
500
550
600
650
Nx/Ny values

Problem 3 $\nabla^2 u + 1000u = (1000-200\pi^2)\sin(10\pi x)\cos(10\pi y)$, $(x,y) \in (0,1)^2$

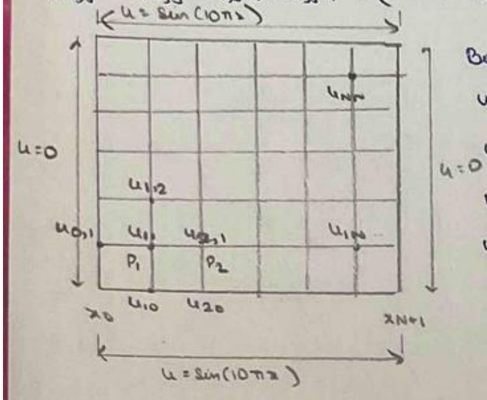
$u(x,0) = u(x,1) = \sin(10\pi x)$

$u(0,y) = u(1,y) = 0$

Ans: Discretizing given equation with second order difference

$$\frac{u_{i+1,j} + u_{i-1,j} - 2u_{i,j}}{h^2} + \frac{u_{i,j+1} + u_{i,j-1} - 2u_{i,j}}{h^2} + 1000u_{i,j} = (1000-200\pi^2)\sin(10\pi x_i)\cos(10\pi y_j)$$

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} + (1000h^2-4)u_{i,j} = (1000-200\pi^2)\sin(10\pi x_i)\cos(10\pi y_j)$$



Boundary conditions:

$u(0,y) = 0 \Rightarrow u_{0,j} = 0$

$u(1,y) = 0 \Rightarrow u_{1,j} = 0$

$u(x,0) = u_{i,0} = \sin(10\pi x_i)$

$u(x,1) = u_{i,1} = \sin(10\pi x_i)$

For point P_1: $i=1, j=1$; $u_{2,1} + u_{0,1} + u_{1,2} + u_{1,0} + (1000h^2-4)u_{1,1} = (1000-200\pi^2)\sin(10\pi x_1)\cos(10\pi y_1)$

$u_{2,1} + 0 + u_{1,2} + \sin(10\pi x_1) + (1000h^2-4)u_{1,1} = (1000-200\pi^2)\sin(10\pi x_1)\cos(10\pi y_1)$

$\Rightarrow u_{1,2} + (1000h^2-4)u_{1,1} = (1000-200\pi^2)\sin(10\pi x_1)\cos(10\pi y_1) - \sin(10\pi x_1) - \sin(10\pi x_1)$

For point P_2: $i=2, j=1$; $u_{3,1} + u_{1,1} + u_{2,2} + u_{2,2} + (1000h^2-4)u_{2,1}$
$= (1000-200\pi^2)\sin(10\pi x_2)\cos(10\pi y_1)$

$$\begin{bmatrix} (1000h^2-4) & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & \ddots & \ddots & \ddots & & & & \\ & \ddots & \ddots & \ddots & \ddots & & & \\ & & & & & & & 1 \\ & & & & 1 & & & (1000h^2-4) \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{1,2} \\ \vdots \\ u_{1N} \\ \vdots \\ u_{ij} \\ \vdots \\ u_{N,N} \end{bmatrix}$$

For P_N: $i=N, j=N$

$u_{N+1,N} + u_{N-1,N} + u_{N,N+1} + u_{N,N-1} + (1000h^2-4)u_{N,N} = (1000-200\pi^2)\sin(10\pi x_N)\cos(10\pi y_N)$

$0 + u_{N-1,N} + \sin(10\pi x_N) + u_{N,N-1} + (1000h^2-4)u_{N,N} = (1000-200\pi^2)\sin(10\pi x_N)\cos(10\pi y_N)$

Diagonalization method: $A = [P]\,[\Lambda][P^{-1}]$

$Au = F$

$[P]\,[\Lambda]\,[P^{-1}]\,u = F$