SOLUTION 1 Viscous Burgers equation:

- SAURABH SHARMA

$u_t + u u_x = \alpha u_{xx}$ where, $u(x,t)$ is velocity component along x direction, α is kinematic viscosity.

$t_{initial} = 0$, $t_{end} = 0.075$, $\Delta t = 0.0004$

$u(x,0) = \sin(4\pi x) + \sin(6\pi x) + \sin(10\pi x)$

Number of grid points : $N = \{64, 1024\}$

$$\Delta x = \frac{1}{64-1} \qquad \Delta x = \frac{1}{1024-1}$$

Physical gridpoints : $PN_x1 = 0$, $PN_x2 = N-1$

PART (a)

a) $\alpha = 0$

$u_t + u u_x = 0$

Using forward euler for time, and first order upwind in space:

i) Interior point : $i = PN_x1 + 1$ to $i = PN_x2 - 1$

When $u_i^n > 0$: $\dfrac{u_i^{n+1} - u_i^n}{\Delta t} + u_i^n \left(\dfrac{u_i^n - u_{i-1}^n}{\Delta x}\right) = 0$

$$\Rightarrow u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} u_i^n \left(u_i^n - u_{i-1}^n\right)$$

$$\Rightarrow u_i^{n+1} = u_i^n \left[1 - \frac{\Delta t}{\Delta x}\left(u_i^n - u_{i-1}^n\right)\right]$$

When $u_i^n < 0$: $\dfrac{u_i^{n+1} - u_i^n}{\Delta t} + u_i^n \left[\dfrac{u_{i+1}^n - u_i^n}{\Delta x}\right] = 0$

$$\Rightarrow u_i^{n+1} = u_i^n \left[1 - \frac{\Delta t}{\Delta x}\left(u_{i+1}^n - u_i^n\right)\right]$$

ii) End vertices : $i = PN_x1$ and $i = PN_x2$ (Periodic Boundary Condition)

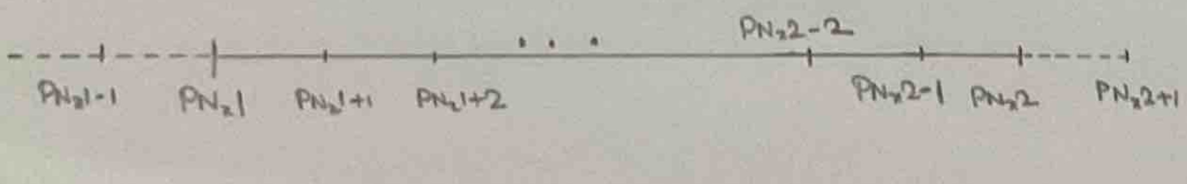
$i = PN_x1$: $u_{PN_x1}^{n+1} = u_{PN_x1}^n \left[1 - \dfrac{\Delta t}{\Delta x}\left(u_{PN_x1}^n - u_{PN_x1-1}^n\right)\right]$ $\{$ for $u_i^n > 0 \}$

$\left[u_{PN_x1-1}^n = u_{PN_x2-1}^n\right]$

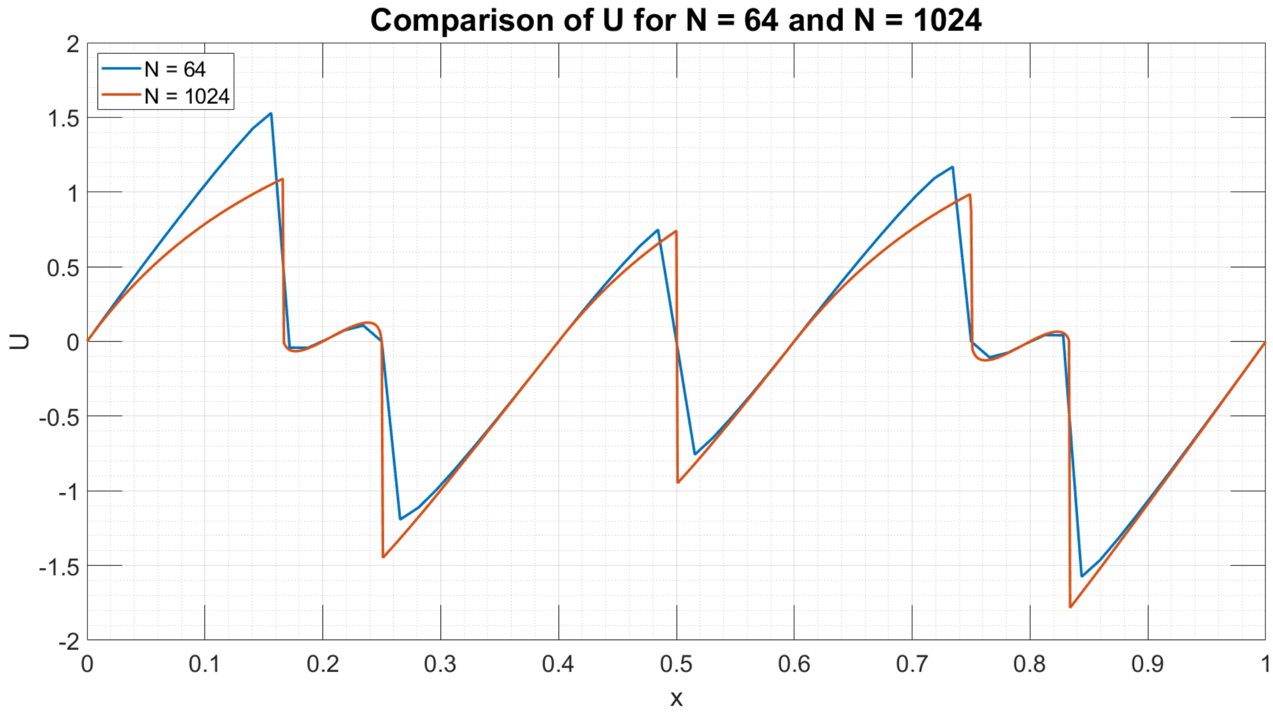
$u_{PN_x1}^{n+1} = u_{PN_x1}^n \left[1 - \dfrac{\Delta t}{\Delta x}\left(u_{PN_x1+1}^n - u_{PN_x1}^n\right)\right]$ $\{$ for $u_i^n < 0 \}$

$i = PN_x2$: $u_{PN_x2}^{n+1} = u_{PN_x2}^n \left[1 - \dfrac{\Delta t}{\Delta x}\left(u_{PN_x2}^n - u_{PN_x2-1}^n\right)\right]$ $\{$ for $u_i^n > 0 \}$

$u_{PN_x2}^{n+1} = u_{PN_x2}^n \left[1 - \dfrac{\Delta t}{\Delta x}\left(u_{PN_x1+1}^n - u_{PN_x2}^n\right)\right]$ $\{$ for $u_i^n < 0 \}$

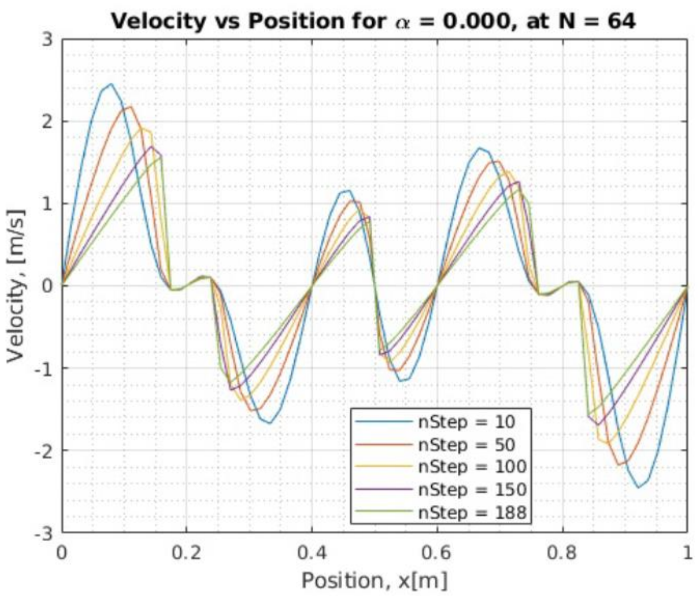


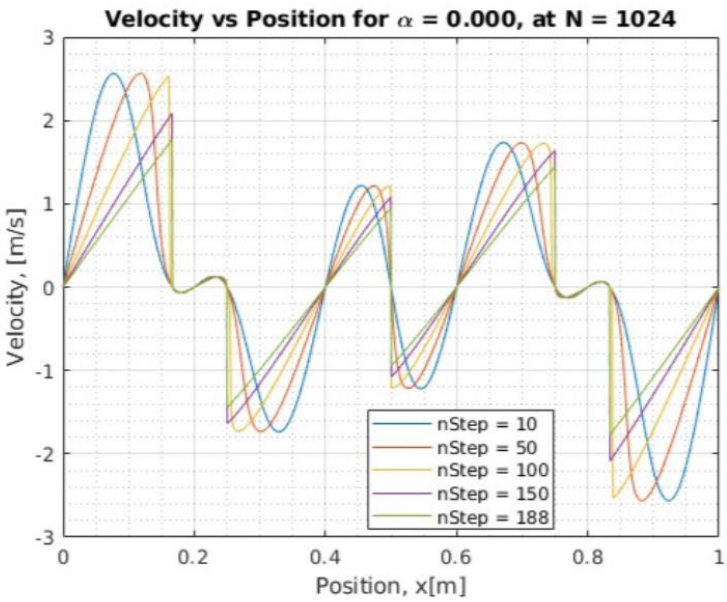
Solution 1(a)



Observations:

1. As the step size in numerical methods for solving differential equations decreases, the approximation of the solution approaches the true solution, resulting in a reduction of error.
2. Due to the nonlinearity of the given differential equation, the velocities may exhibit abrupt changes, which can manifest as discontinuities.
3. In order to accurately resolve these discontinuities, a higher number of grid points, such as 1024, may be necessary. This is because, with 64 grid points, the lines may become too smooth, resulting in inaccuracies and errors that do not accurately reflect the physical situation being modelled.





PART (b)

(b) $\alpha = 0.001$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \alpha \frac{\partial^2 u}{\partial x^2}$$

Forward Euler for time and second order central difference for space:

i.) Interior points : $i = PN_x1 + 1$ to $i = PN_x2 - 1$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + u_i^n\left[\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}\right] = \alpha\left[\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}\right]$$

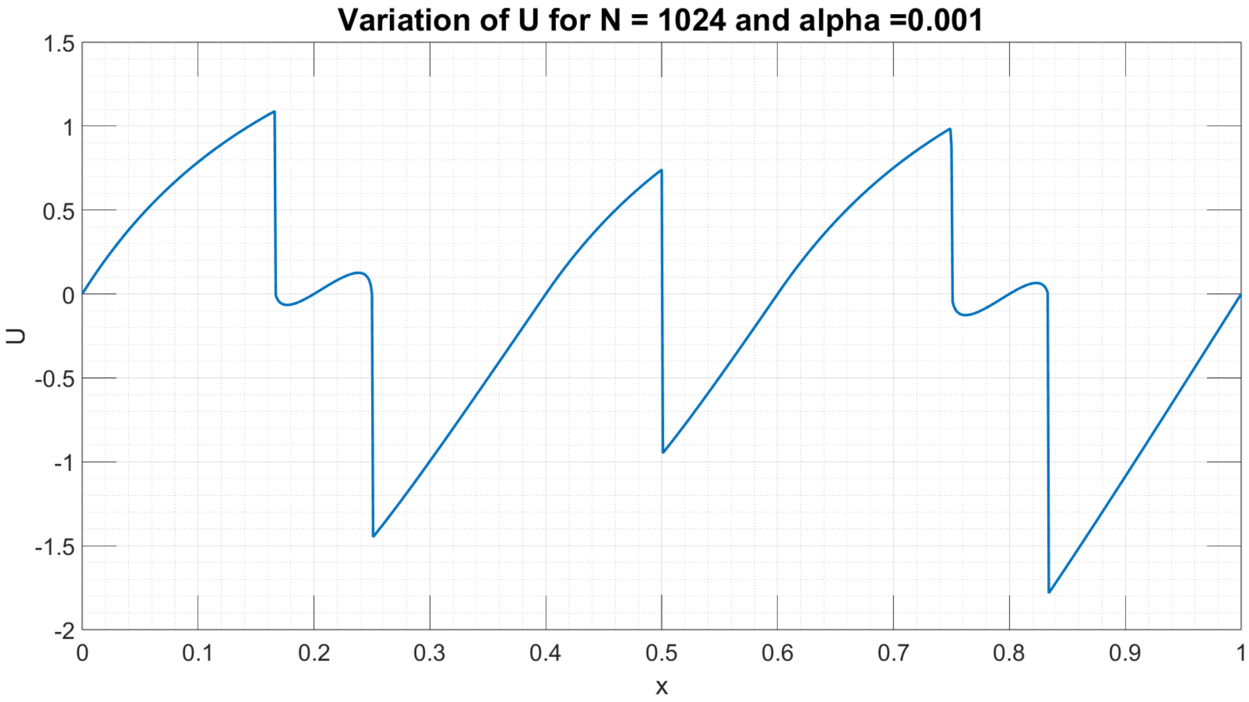
$$\Rightarrow u_i^{n+1} = u_i^n - u_i \frac{\Delta t}{2\Delta x}\left[u_{i+1}^n - u_{i-1}^n\right] + \alpha\frac{\Delta t}{\Delta x^2}\left[u_{i+1}^n - 2u_i^n + u_{i-1}^n\right]$$

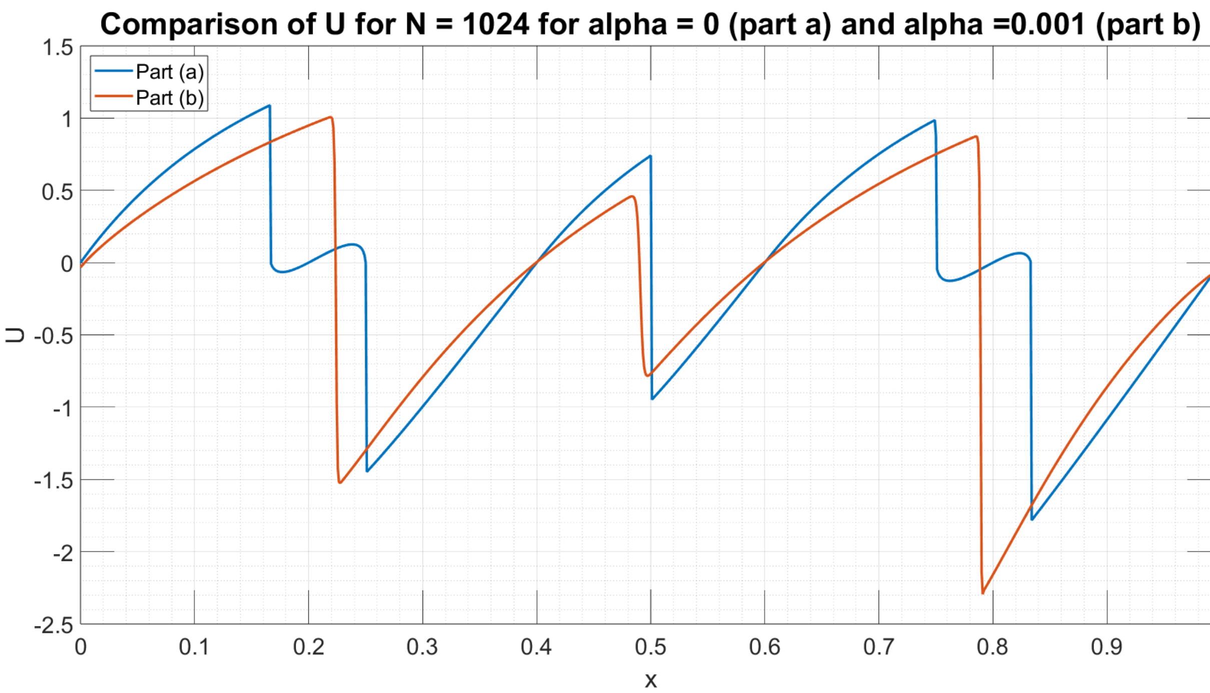
ii.) End vertices : $i = PN_x1$ and $i = PN_x2$ (using Periodic Boundary condition)

$$i = PN_x1 : u_{PN_x1}^{n+1} = u_{PN_x1}^n - u_{PN_x1}^n \frac{\Delta t}{2\Delta x}\left[u_{PN_x1+1}^n - \underbrace{u_{PN_x1-1}^n}_{PN_x2-2}\right] + \alpha\frac{\Delta t}{\Delta x^2}\left[u_{PN_x1+1}^n - 2u_{PN_x1}^n + \underbrace{u_{PN_x1-1}^n}_{PN_x2-1}\right]$$

$$i = PN_x2 : u_{PN_x2}^{n+1} = u_{PN_x2}^n - u_{PN_x2}^n \frac{\Delta t}{2\Delta x}\left[\underbrace{u_{PN_x2+1}^n}_{PN_x1+1} - u_{PN_x2-1}^n\right] + \frac{\alpha\Delta t}{\Delta x^2}\left[\underbrace{u_{PN_x2+1}^n}_{PN_x1+1} - 2u_{PN_x2}^n + u_{PN_x2-1}^n\right]$$

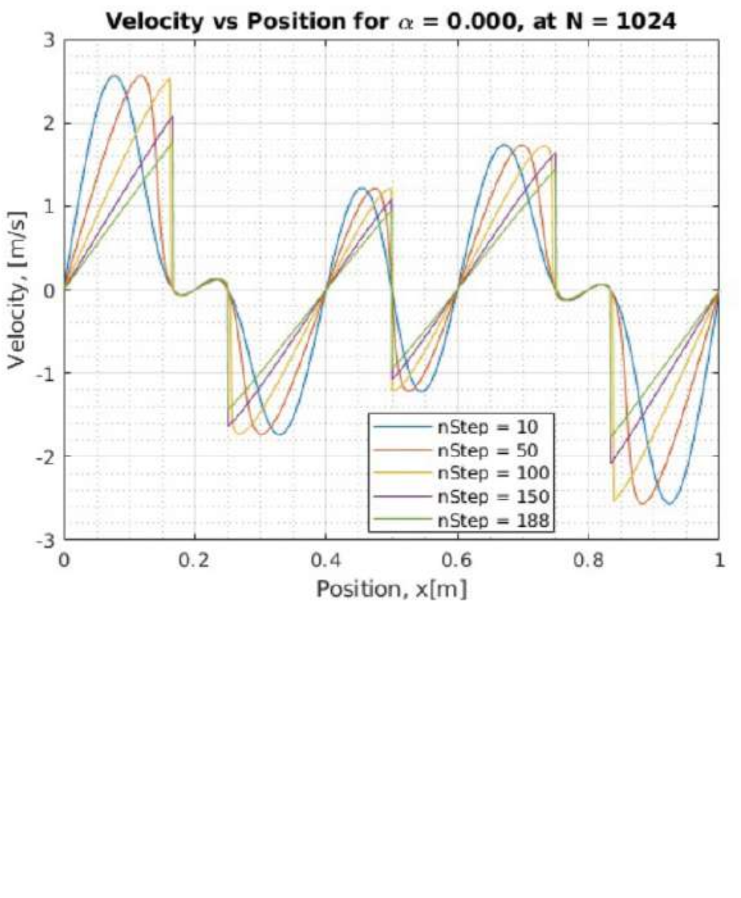
Solution 1 (b)



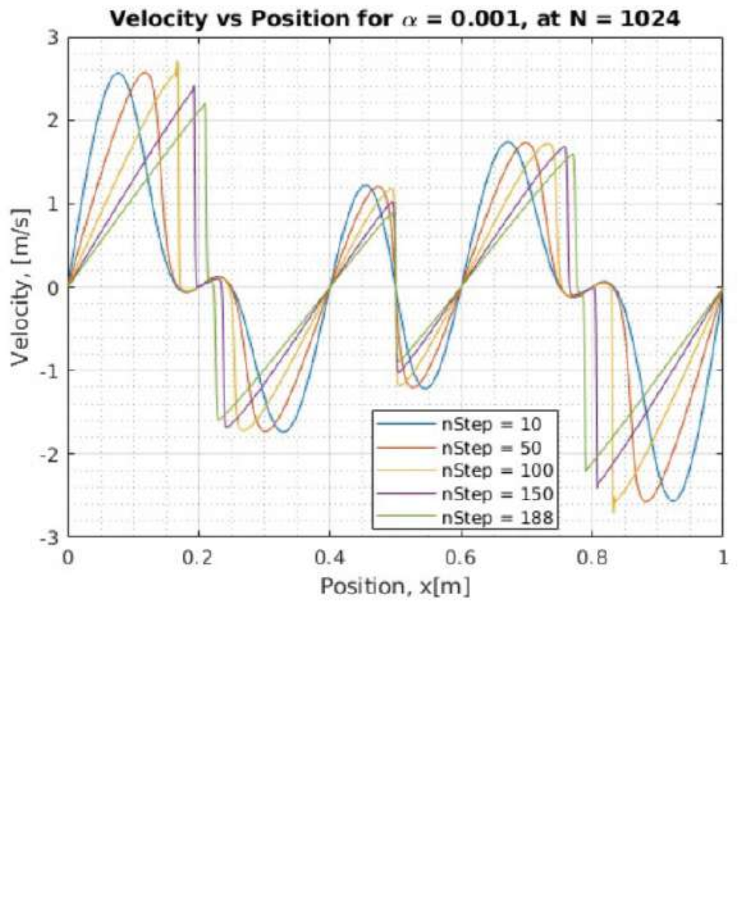


Observations:

1. A noticeable amount of dispersion is observed in the presence of discontinuities when alpha is set to 0.001.
2. Dispersion is only evident in the case of strong shocks, which occur within a specific range of x values (0 to 0.2), and is not observed in weaker shocks (between 0.4 to 0.5).
3. In the case where alpha equals zero, all discontinuities occur at sharp points. However, when alpha is set to 0.001, the discontinuities have smoother corners, which are more prevalent in this case.



Velocity vs Position for α = 0.000, at N = 1024
Velocity, [m/s]
Position, x[m]
nStep = 10
nStep = 50
nStep = 100
nStep = 150
nStep = 188



Velocity vs Position for α = 0.001, at N = 1024
Velocity, [m/s]
Position, x[m]
nStep = 10
nStep = 50
nStep = 100
nStep = 150
nStep = 188

SOLUTION 2 Linear wave equation -SAURABH SHARMA

$u_t + c u_x = 0$

Leap frog for time derivative and 2nd order central difference for space derivative :

Centered in time and space both :

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} + c\left[\frac{u_{j+1}^{n} - u_{j-1}^{n}}{2\Delta x}\right] = 0 \qquad \ldots(1)$$

Let $r = \frac{c\Delta t}{\Delta x}$ ⇒ r is always positive (if $c>0$ given)

Using Von Neumann's stability analysis : $u_j^n = V^n e^{i\beta x_j}$

where, β = wave number.

Substituting Ansatz in Equation (1) :

$$V^{n+1} e^{i\beta x_j} - V^{n-1} e^{i\beta x_j} + r\left[V^n e^{i\beta(x_j+h)} - V^n e^{i\beta(x_j-h)}\right] = 0$$

or $$V - \frac{1}{V} + r\left[e^{i\beta h} - e^{-i\beta h}\right] = 0$$

or $$V^2 - 1 + rV\left[2i\sin(\beta h)\right] = 0$$

or $$V^2 + 2irV\sin(\beta h) - 1 = 0$$

$$V = \frac{-2ir\sin(\beta h) \pm \sqrt{-4r^2\sin^2(\beta h) + 4}}{2}$$

$$V = -ir\sin(\beta h) \pm \sqrt{1 - r^2\sin^2(\beta h)}$$

CASE 1 If $1 - r^2\sin^2(\beta h) > 0$

$$V = -ir\sin(\beta h) \pm \sqrt{1 - r^2\sin^2(\beta h)}$$

$$|V|^2 = r^2\sin^2(\beta h) + 1 - r^2\sin^2(\beta h) = 1$$

CASE 2 If $1 - r^2\sin^2(\beta h) < 0$ → $1 < r^2\sin^2(\beta h)$ ⇒ $2r^2\sin^2(\beta h) > 2$ ...(2)

$$V = -ir\sin(\beta h) \pm i\sqrt{r^2\sin^2(\beta h) - 1}$$

$$|V|^2 = r^2\sin^2(\beta h) + r^2\sin^2(\beta h) - 1 = 2r^2\sin^2(\beta h) - 1$$

Then by equation (2), we can say that $|V|^2 > 1$

But we know that restricted stability condition : $|V| \leq 1$

∴ Leap frog method is unconditionally unstable scheme.

Expanding the terms of equation (1) by Taylor Series :

$$u_j^{n+1} = u_j^n + \Delta t \left.\frac{\partial u}{\partial t}\right|_j^n + \frac{(\Delta t)^2}{2!} \left.\frac{\partial^2 u}{\partial t^2}\right|_j^n + \frac{(\Delta t)^3}{3!} \left.\frac{\partial^3 u}{\partial t^3}\right|_j^n + \dots$$

$$u_j^{n-1} = u_j^n - \Delta t \left.\frac{\partial u}{\partial t}\right|_j^n + \frac{(\Delta t)^2}{2!} \left.\frac{\partial^2 u}{\partial t^2}\right|_j^n - \frac{(\Delta t)^3}{3!} \left.\frac{\partial^3 u}{\partial t^3}\right|_j^n + \dots$$

$$u_{j+1}^n = u_j^n + \Delta x \left.\frac{\partial u}{\partial x}\right|_j^n + \frac{(\Delta x)^2}{2!} \left.\frac{\partial^2 u}{\partial x^2}\right|_j^n + \frac{(\Delta x)^3}{3!} \left.\frac{\partial^3 u}{\partial x^3}\right|_j^n + \dots$$

$$u_{j-1}^n = u_j^n - \Delta x \left.\frac{\partial u}{\partial x}\right|_j^n + \frac{(\Delta x)^2}{2!} \left.\frac{\partial^2 u}{\partial x^2}\right|_j^n - \frac{(\Delta x)^3}{3!} \left.\frac{\partial^3 u}{\partial x^3}\right|_j^n + \dots$$

Solving : $u_j^{n+1} - u_j^{n-1} = 2\left[\Delta t \left(\frac{\partial u}{\partial t}\right) + \frac{(\Delta t)^3}{3!} \frac{\partial^3 u}{\partial t^3} + \frac{(\Delta t)^5}{5!} \frac{\partial^5 u}{\partial t^2} + \dots\right]_j^n$

$$u_{j+1}^n - u_{j-1}^n = 2\left[\Delta x \left(\frac{\partial u}{\partial x}\right) + \frac{(\Delta x)^3}{3!} \frac{\partial^3 u}{\partial x^3} + \frac{(\Delta x)^5}{5!} \frac{\partial^5 u}{\partial x^5} + \dots\right]_j^n$$

Now, solving $u_t + cu_x$:

$$u_t + cu_x = \frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} + c\left[\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}\right]$$

$$= \frac{1}{\Delta t}\left[\Delta t\left(\frac{\partial u}{\partial t}\right) + \frac{(\Delta t)^3}{3!} \frac{\partial^3 u}{\partial t^3} + \frac{(\Delta t)^5}{5!} \frac{\partial^5 u}{\partial t^5} + \dots\right]_j^n$$

$$+ \frac{c}{\Delta x}\left[\Delta x\left(\frac{\partial u}{\partial x}\right) + \frac{(\Delta x)^3}{3!} \frac{\partial^3 u}{\partial x^3} + \frac{(\Delta x)^5}{5!} \frac{\partial^5 u}{\partial x^5} + \dots\right]_j^n$$

$$= \left(\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}\right)_j^n + \frac{(\Delta t)^2}{3!} \left.\frac{\partial^3 u}{\partial t^3}\right|_j^n + O(\Delta t)^4 + \frac{c(\Delta x)^2}{3!} \left.\frac{\partial^3 u}{\partial x^3}\right|_j^n + O(\Delta x)^4$$

$$= \frac{\Delta t^2}{3!} u_{ttt} + O(\Delta t)^4 + c\frac{(\Delta x)^2}{3!} u_{xxx} + O(\Delta x)^4 \qquad \dots (3)$$

We have $u_t + cu_x = 0$ (Approximate without truncation error)

Differentiating with respect to t:

$$u_{tt} = -cu_{xt} = +c^2 u_{xx}$$

Again differentiating with respect to t:

$$u_{ttt} = -c^3 u_{xxt} = -c^3 u_{xxx}$$

Putting value of u_{ttt} in equation (3):

$$u_t + cu_x = \left[-c^3 \frac{(\Delta t)^2}{6} + c\frac{(\Delta x)^2}{6}\right] u_{xxx} + O(\Delta t)^4 + O(\Delta x)^4$$ (MODIFIED EQUATION)

Highest order derivative term in truncation error is odd order term.

Hence, errors will be dispersive error.

SOLUTION 3 Weak form

a) Beam on elastic foundation :

$$\frac{d^2}{dx^2}\left(b\frac{d^2w}{dx^2}\right) + kw = f \quad \text{for} \quad 0<x<L$$

$$w = b\frac{d^2w}{dx^2} = 0 \quad \text{at } x = 0, L$$

where, $b = EI = b(x)$
$f = f(x)$
k is constant

Method of weighted residual : $R(x) = \frac{d^2}{dx^2}\left(b\frac{d^2w_n}{dx^2}\right) + kw_n - f$

where, w_n is finite basis

R is required to vanish in weighted integral sense:

$$\int (R\,w_i)\,dx = 0 \qquad \text{where, } i = 1, 2, \ldots N$$

w_i = weighting functions such that $w_i(0) = 0$, $w_i(L) = 0$

$$\int \left[\frac{d^2}{dx^2}\left(b\frac{d^2w_n}{dx^2}\right) + kw_n - f\right] w_i\,dx = 0$$

$$\Rightarrow \int w_i\frac{d^2}{dx^2}\left(b\frac{d^2w_n}{dx^2}\right)dx + k\int w_n w_i\,dx = \int f w_i\,dx$$

Solving term 1 : $\int w_i \frac{d^2}{dx^2}\left(b\frac{d^2w_n}{dx^2}\right)dx$

$$= \left[\frac{d}{dx}\left(b\frac{d^2w_n}{dx^2}\right)w_i\right]_0^L - \int_0^L \frac{dw_i}{dx}\frac{d}{dx}\left(b\frac{d^2w_n}{dx^2}\right)dx \qquad \{\text{Appling by parts}\}$$

$$= \left[\frac{d}{dx}\left(b\frac{d^2w_n}{dx^2}\right)w_i\right]_0^L - \left[\frac{dw_i}{dx}\,b\frac{d^2w_n}{dx^2}\right]_0^L + \int_0^L b\frac{d^2w_i}{dx^2}\frac{d^2w_n}{dx^2}dx$$

$$= 0 - 0 + \int_0^L b\frac{d^2w_i}{dx^2}\frac{d^2w_n}{dx^2}dx \qquad \left\{\because w_i(0) = w_i(L) = 0 = b\frac{d^2w}{dx^2}(0) = b\frac{d^2w}{dx^2}(L) = 0\right\}$$

Putting the value back :

$$\Rightarrow \boxed{\int_0^L b\frac{d^2w_i}{dx^2}\frac{d^2w_n}{dx^2}dx + k\int w_n w_i\,dx = \int f w_i\,dx}$$

(WEAK FORM)

b) A non linear equation:

$$-\frac{d}{dx}\left(u\frac{du}{dx}\right) + f = 0 \quad \text{for} \quad 0 < x < 1$$

$$\left.\frac{du}{dx}\right|_{x=0} = 0 \quad , u(1) = \sqrt{2}$$

Similarly using method of weighted residual : $\int R w_i dx = 0$, $i = 1$ to N

where , $w_i(x)$ is weighting function

$$R(x) = -\frac{d}{dx}\left(u_n \frac{du_n}{dx}\right) + f$$

$$\int R w_i dx = 0 \Rightarrow \int_0^1 -w_i \frac{d}{dx}\left(u_n \frac{du_n}{dx}\right) dx + \int_0^1 w_i f dx = 0$$

$$\Rightarrow \int_0^1 w_i \frac{d}{dx}\left(u_n \frac{du_n}{dx}\right) dx = \int_0^1 w_i f dx$$

$$LHS = \int_0^1 w_i \frac{d}{dx}\left(u_n \frac{du_n}{dx}\right) dx = \left[w_i u_n \frac{du_n}{dx}\right]_0^1 - \int_0^1 u_n \frac{dw_i}{dx}\frac{du_n}{dx} dx$$

$$= w_i u_n(1) \left.\frac{du_n}{dx}\right|_{x=1} - w_i u_n(0) \left.\frac{du_n}{dx}\right|_{x=0} - \int_0^1 u_n \frac{dw_i}{dx}\frac{du_n}{dx} dx$$

$$= w_i \sqrt{2} \left.\frac{du_n}{dx}\right|_{x=1} - 0 - \int_0^1 u_n \frac{dw_i}{dx}\frac{du_n}{dx} dx$$

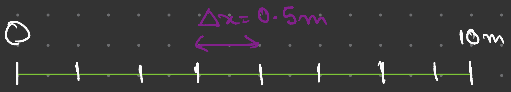
LHS = RHS

$$w_i \sqrt{2} \left.\frac{du_n}{dx}\right|_{x=1} - \int_0^1 u_n \frac{dw_i}{dx}\frac{du_n}{dx} dx = \int_0^1 w_i f dx$$

$$\Rightarrow \boxed{\int_0^1 u_n \frac{dw_i}{dx}\frac{du_n}{dx} dx + \int_0^1 w_i f dx = w_i \sqrt{2} \left.\frac{du_n}{dx}\right|_{x=1}} \quad \text{(WEAK FORM)}$$

SOLUTION 4

Given equation: $A_c E \frac{d^2u}{dx^2} = P(x)$; Here, $A_c = 0.1\,m^2$, $E = 200\times10^9\,N/m^2$, $L = 10\,m$, $P(x) = 100\,N/m$



No of elements $= \frac{10}{0.5} = 20$

Considering an element 'e' in the domain:

Let us approximate function 'u' on this element using an approximate function u_h which is linear combination of some basis function.

$$u_h^e(x) = c_1^e \phi_1^e(x) + c_2^e \phi_2^e(x) + \dots + c_n^e \phi_n^e(x) = \sum_{i=1}^{n} c_i^e \phi_i^e(x)$$

Residue, $R^e = A_c E \frac{d^2u_h^e}{dx^2} - P$

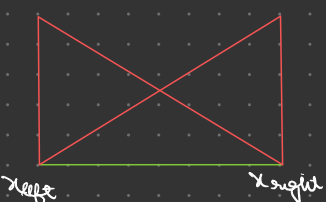
Using method of weighted residual : $\int_{x_{left}}^{x_{right}} R^e w_i^e \, dx = 0$

$$\Rightarrow \int_{x_{left}}^{x_{right}} \left(A_c E \frac{d^2u_h^e}{dx^2} - P\right) w_i^e \, dx = 0$$

$$\Rightarrow \left[\frac{du_h^e}{dx} w_i^e\right]_{x_{left}}^{x_{right}} - \int_{x_{left}}^{x_{right}} \frac{du_h^e}{dx}\frac{dw_i^e}{dx} dx - \int_{x_{left}}^{x_{right}} \frac{P}{A_c E} w_i^e \, dx = 0$$

$$\Rightarrow \int_{x_{left}}^{x_{right}} \frac{du_h^e}{dx}\frac{dw_i^e}{dx} dx + \int_{x_{left}}^{x_{right}} \frac{P}{A_c E} w_i^e \, dx = \left[\left(\frac{du_h^e}{dx}\right)_{x_{right}} w_i^e(x_{right}) - \left(\frac{du_h^e}{dx}\right)_{x_{left}} w_i^e(x_{left})\right]$$

(WEAK FORM)



Linear basis for element e:

$$u_h^e = L_{left}^e u_{left}^e + L_{right}^e u_{right}^e$$

where, $L_{left}^e = \frac{x - x_{right}}{x_{left} - x_{right}}$, $L_{right}^e = \frac{x - x_{left}}{x_{right} - x_{left}}$

Using Galerkin approach, weighting functions are: $w_i^e(x) = L_i^e(x)$

$$\text{So, } \int_{x_{left}}^{x_{right}} \frac{du_h^e}{dx}\frac{dw_i^e}{dx} dx = \int_{x_{left}}^{x_{right}} \frac{d}{dx}\left[L_j^e u_j^e\right] \frac{dL_i^e}{dx} dx = \int_{x_{left}}^{x_{right}} \frac{dL_j^e}{dx}\frac{dL_i^e}{dx} u_j^e \, dx$$

$$= \underbrace{\left[\int_{x_{left}}^{x_{right}} \frac{dL_j^e}{dx}\frac{dL_i^e}{dx} dx\right]}_{K_{ij}^e} u_j^e$$

General equation: $\underset{\sim}{K}\,\underset{\sim}{u} = \underset{\sim}{F}$

ISO PARAMETRIC ELEMENT:

$z_e = 0$ —— $z_e = 1$ $\Rightarrow L_{eft} = 1 - z_e$, $L_{right} = z_e$

Steps:

Compute 2x2 local stiffness matrix given by: $k^e = \frac{1}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Element 0: $$\begin{bmatrix} K_{00}^0 & K_{01}^0 \\ K_{10}^0 & K_{11}^0 \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = -\begin{bmatrix} f_0 \\ f_1 \end{bmatrix} + [J]^{-1}\begin{bmatrix} -q_0 \\ q_1 \end{bmatrix}$$

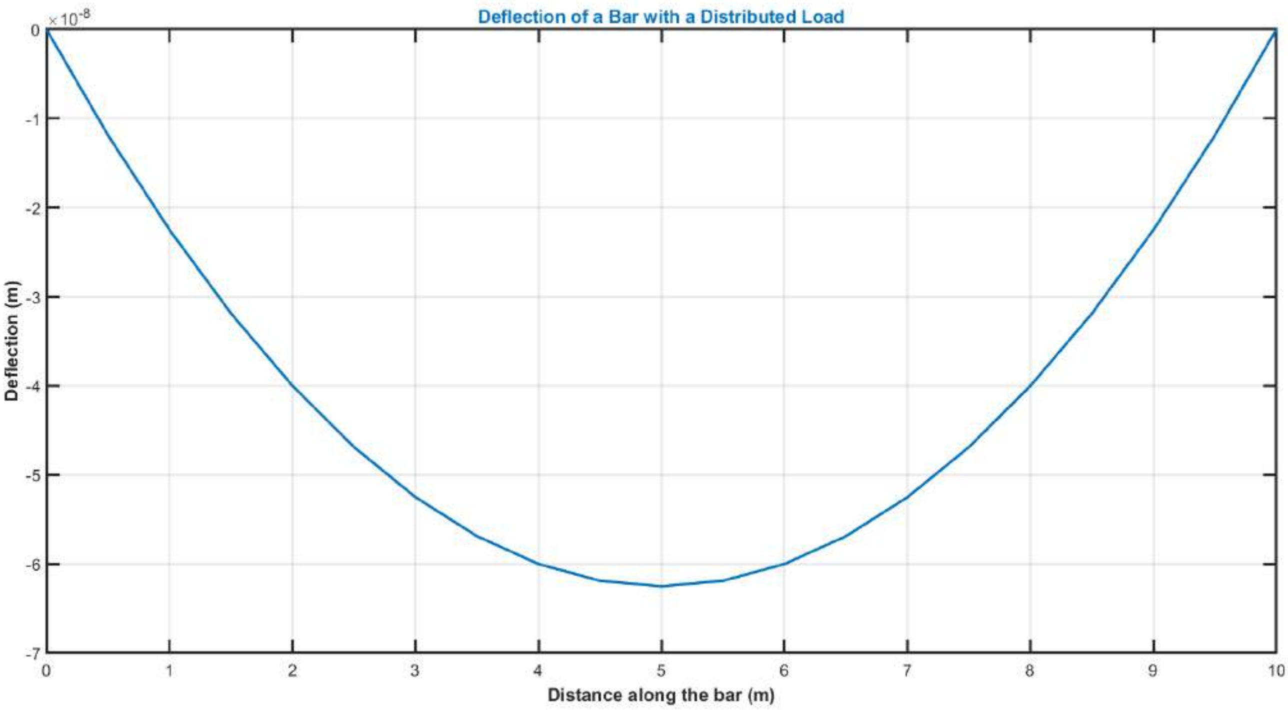
Element 1: $$\begin{bmatrix} K_{00}^1 & K_{01}^1 \\ K_{10}^1 & K_{11}^1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = -\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + [J]^{-1}\begin{bmatrix} -q_1 \\ q_2 \end{bmatrix}$$

$\vdots$

Element N-1: $$\begin{bmatrix} K_{00}^{N-1} & K_{01}^{N-1} \\ K_{10}^{N-1} & K_{11}^{N-1} \end{bmatrix}\begin{bmatrix} u_{N-1} \\ u_N \end{bmatrix} = -\begin{bmatrix} f_{N-1} \\ f_N \end{bmatrix} + [J]^{-1}\begin{bmatrix} -q_{N-1} \\ q_N \end{bmatrix}$$

Combining all matrices:

$$\begin{bmatrix} K_{00}^0 & K_{01}^0 & & & \\ K_{10}^0 & K_{11}^0 + K_{00}^1 & K_{01}^1 & & \\ & K_{10}^1 & K_{11}^1 + K_{00}^2 & \ddots & \\ & & \ddots & \ddots & \ddots \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{bmatrix} = -\begin{bmatrix} f_0 \\ 2f_1 \\ 2f_2 \\ \vdots \\ 2f_{N-1} \\ f_N \end{bmatrix} + [J]^{-1}\begin{bmatrix} -q_0 \\ 0 \\ \vdots \\ 0 \\ q_N \end{bmatrix}$$



Deflection of a Bar with a Distributed Load
×10^-8
0
-1
-2
-3
-4
-5
-6
-7
Deflection (m)
0
1
2
3
4
5
6
7
8
9
10
Distance along the bar (m)