Solution 1

a) Second order accurate finite difference approximation

To find: Second order accurate finite difference approximation for second order derivative $\frac{d^2u}{dx^2}$ using one sided stencil to the left of i on a non uniform grid

Let us ~~for~~ represent u as f.

Using method of undetermined coefficients

$$f''(x_i) = af(x_{i-1}) + bf(x_{i-2}) + cf(x_{i-3}) + df(x_i)$$

Expanding each terms in RHS using Taylor series expansion ~~do~~ :

$$f''(x_i) = a\left[f(x_i) - h_i f'(x_i) + \frac{h_i^2}{2!} f''(x_i) - \frac{h_i^3}{3!} f'''(x_i) + \cdots\right]$$

$$+ b\left[f(x_i) - (h_i + h_{i-1}) f'(x_i) + \frac{(h_i+h_{i-1})^2}{2!} f''(x_i) - \frac{(h_i+h_{i-1})^3}{3!} f'''(x_i) + \cdots\right]$$

$$+ c\left[f(x_i) - (h_i+h_{i-1}+h_{i-2}) f'(x_i) + \frac{(h_i+h_{i-1}+h_{i-2})}{2!} f''(x_i) - \frac{(h_i+h_{i-1}+h_{i-2})^3}{3!} f'''(x_i) + \cdots\right]$$

$$+ df(x_i)$$

$$= (a+b+c+d) f(x_i) - [ah_i + b(h_i+h_{i-1}) + c(h_i+h_{i-1}+h_{i-2})] f'(x_i)$$

$$+ \left[\frac{ah_i^2}{2!} + \frac{b(h_i+h_{i-1})^2}{2!} + \frac{c(h_i+h_{i-1}+h_{i-2})}{2!}\right] f''(x_i)$$

$$- \left[\frac{ah_i^3}{3!} + \frac{b(h_i+h_{i-1})^3}{3!} + \frac{c(h_i+h_{i-1}+h_{i-2})^3}{3!}\right] f'''(x_i)$$

Comparing coefficients of $f(x_i)$, $f'(x_i)$, $f''(x_i)$ & $f'''(x_i)$ on LHS & RHS :-

$$a + b + c + d = 0 \quad \ldots (1)$$

$$ah_i + b(h_i+h_{i-1}) + c(h_i+h_{i-1}+h_{i-2}) = 0 \quad \ldots (2)$$

$$ah_i^2 + b(h_i+h_{i-1})^2 + c(h_i+h_{i-1}+h_{i-2})^2 = 2 \quad \ldots (3)$$

$$ah_i^3 + b(h_i+h_{i-1})^3 + c(h_i+h_{i-1}+h_{i-2})^3 = 0 \quad \ldots (4)$$

Writing in matrix form :

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ h_i & h_i+h_{i-1} & h_i+h_{i-1}+h_{i-2} & 0 \\ h_i^2 & (h_i+h_{i-1})^2 & (h_i+h_{i-1}+h_{i-2})^2 & 0 \\ h_i^3 & (h_i+h_{i-1})^3 & (h_i+h_{i-1}+h_{i-2})^3 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

After solving matrix equation :

$$a = \frac{-2(h_{i-2}+2h_i+2h_{i-1})}{h_i(h_{i-1}+h_{i-2})h_{i-1}}$$

$$b = \frac{2(h_{i-2}+2h_i+h_{i-1})}{h_{i-1}(h_i+h_{i-1})h_{i-2}}$$

$$c = \frac{-2(2h_i+h_{i-1})}{(h_i+h_{i-2})h_{i-2}(h_i+h_{i-1}+h_{i-2})}$$

$$d = \frac{2(h_{i-2}+3h_i+2h_{i-1})}{h_i^2\left[h_i+2h_{i-1}h_i+h_i h_{i-2}+h_{i-1}^2+h_{i-1}h_{i-2}\right]}$$

$$\frac{d^2u}{dx^2} = \frac{-2(h_{i-2}+2h_i+2h_{i-1})}{h_i(h_{i-1}+h_{i-2})h_{i-1}}u(x_{i-1}) + \frac{2(h_{i-2}+2h_i+h_{i-1})}{h_{i-1}(h_i+h_{i-1})h_{i-2}}u(x_{i-2})$$

$$-\frac{2(2h_i+h_{i-1})}{(h_i+h_{i-2})h_{i-2}(h_i+h_{i-1}+h_{i-2})}u(x_{i-3}) + \frac{2(h_{i-2}+3h_i+2h_{i-1})}{h_i^2\left[h_i+2h_{i-1}h_i+h_ih_{i-2}+h_{i-1}^2+h_{i-1}h_{i-2}\right]}u(x_i)$$

c) To find : Second order accurate finite difference approximation for cross derivative $\frac{\partial^2 u}{\partial x \partial y}$ on a uniform grid.

$$\frac{\partial f}{\partial x} = \frac{f(x+\Delta x, y) - f(x-\Delta x, y)}{2\Delta x}, \quad \frac{\partial f}{\partial y} = \frac{f(x, y+\Delta y) - f(x, y-\Delta y)}{2\Delta y}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\frac{\partial f}{\partial y}(x+\Delta x, y) - \frac{\partial f}{\partial y}(x-\Delta x, y)}{2\Delta x}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\frac{f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y-\Delta y)}{2\Delta y} - \left[\frac{f(x-\Delta x, y+\Delta y) - f(x-\Delta x, y-\Delta y)}{2\Delta y}\right]}{2\Delta x}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y-\Delta y) - f(x-\Delta x, y+\Delta y) + f(x-\Delta x, y-\Delta y)}{4\Delta x \Delta y}$$

or $$\frac{\partial^2 u}{\partial x \partial y} = \frac{u(x_{i+1}, y_{j+1}) - u(x_{i+1}, y_{j-1}) - u(x_{i-1}, y_{j+1}) + u(x_{i-1}, y_{j-1})}{4\Delta x \Delta y} \quad \ldots (*)$$

Truncation error:

$$\frac{\partial^2 f}{\partial x \partial y} = a f(x_{i+1}, y_{j+1}) + b f(x_{i+1}, y_{j-1}) + c f(x_{i-1}, y_{j+1}) + d f(x_{i-1}, y_{j-1})$$

From equation (*):

$$a = \frac{1}{4hk}, \; b = \frac{-1}{4hk}, \; c = \frac{-1}{4hk}, \; d = \frac{1}{4hk}$$

Now writing 4th order terms which will contribute to truncation error:

$$TE = \left\{ \begin{aligned} & a\left[h^4 \frac{\partial^4 f}{\partial x^4} + 6h^2k^2 \frac{\partial^4 f}{\partial x^2 \partial y^2} + 4hk^3 \frac{\partial^4 f}{\partial x \partial y^3} + 4h^3k \frac{\partial^4 f}{\partial x^3 \partial y} + k^4 \frac{\partial^4 f}{\partial y^4}\right] \\ & + b\left[h^4 \frac{\partial^4 f}{\partial x^4} + 6h^2k^2 \frac{\partial^4 f}{\partial x^2 \partial y^2} - 4hk^3 \frac{\partial^4 f}{\partial x \partial y^3} - 4h^3k \frac{\partial^4 f}{\partial x^3 \partial y} + k^4 \frac{\partial^4 f}{\partial y^4}\right] \\ & + c\left[h^4 \frac{\partial^4 f}{\partial x^4} + 6h^2k^2 \frac{\partial^4 f}{\partial x^2 \partial y^2} - 4hk^3 \frac{\partial^4 f}{\partial x \partial y^3} - 4h^3k \frac{\partial^4 f}{\partial x^3 \partial y} + k^4 \frac{\partial^4 f}{\partial y^4}\right] \\ & + d\left[h^4 \frac{\partial^4 f}{\partial x^4} + 6h^2k^2 \frac{\partial^4 f}{\partial x^2 \partial y^2} + 4hk^3 \frac{\partial^4 f}{\partial x \partial y^3} + 4h^3k \frac{\partial^4 f}{\partial x^2 \partial y} + k^4 \frac{\partial^4 f}{\partial y^4}\right] \end{aligned} \right\} \times \frac{1}{4!}$$

$$= \left[\begin{aligned} & (a+b+c+d) h^4 \frac{\partial^4 f}{\partial x^4} + (a+b+c+d) 6h^2k^2 \frac{\partial^4 f}{\partial x^2 \partial y^2} + (a-b-c+d) 4hk^3 \frac{\partial^4 f}{\partial x \partial y^3} \\ & + (a-b-c+d) 4h^3k \frac{\partial^4 f}{\partial x^3 \partial y} + (a+b+c+d) k^4 \frac{\partial^4 f}{\partial y^4} \end{aligned} \right] \times \frac{1}{4!}$$

$$= \frac{hk}{6}\left[\frac{\partial^4 f}{\partial x^3 \partial y} + \frac{\partial^4 f}{\partial x \partial y^3}\right]$$

$$= \frac{h^2}{6}\left[f_{xxxy} + f_{xyyy}\right]$$

SOLUTION 2

Solution 2:

a)

Given : $\frac{dy}{dt} = yt^2 - 1.1y$, $y(0) = 1$

$$\Rightarrow \frac{dy}{y} = (t^2 - 1.1)\,dt$$

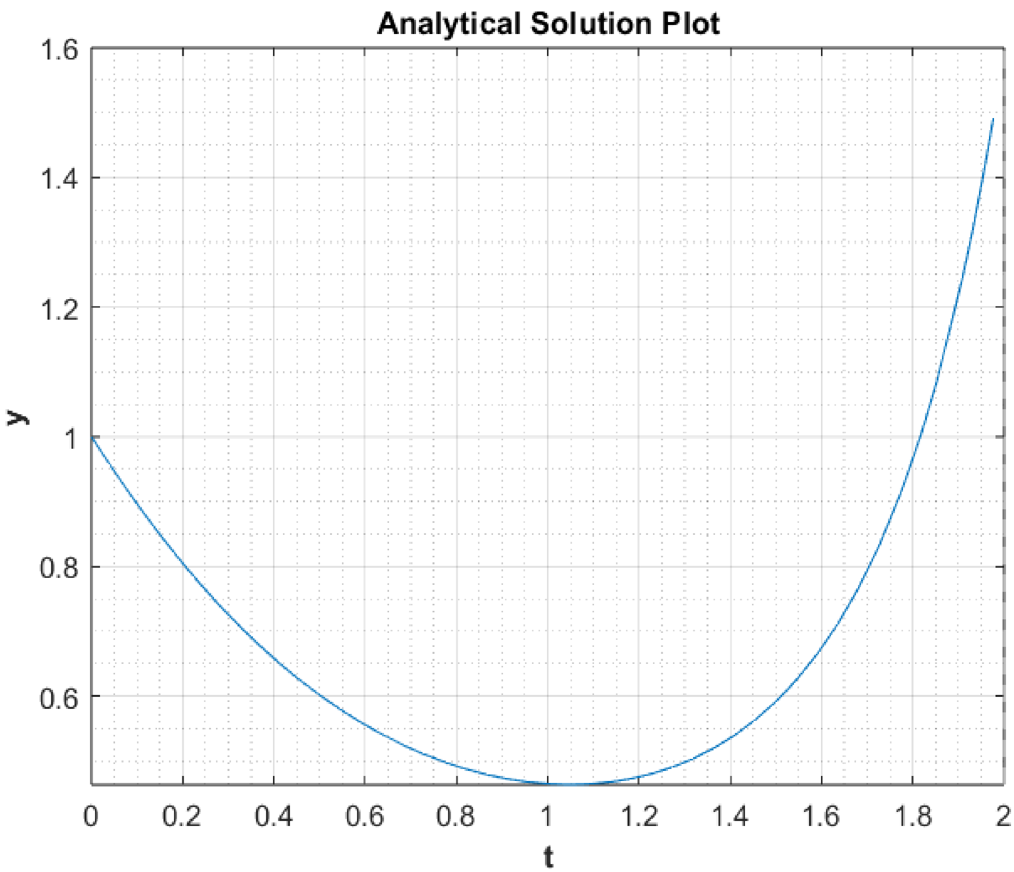
Integrating both the sides :

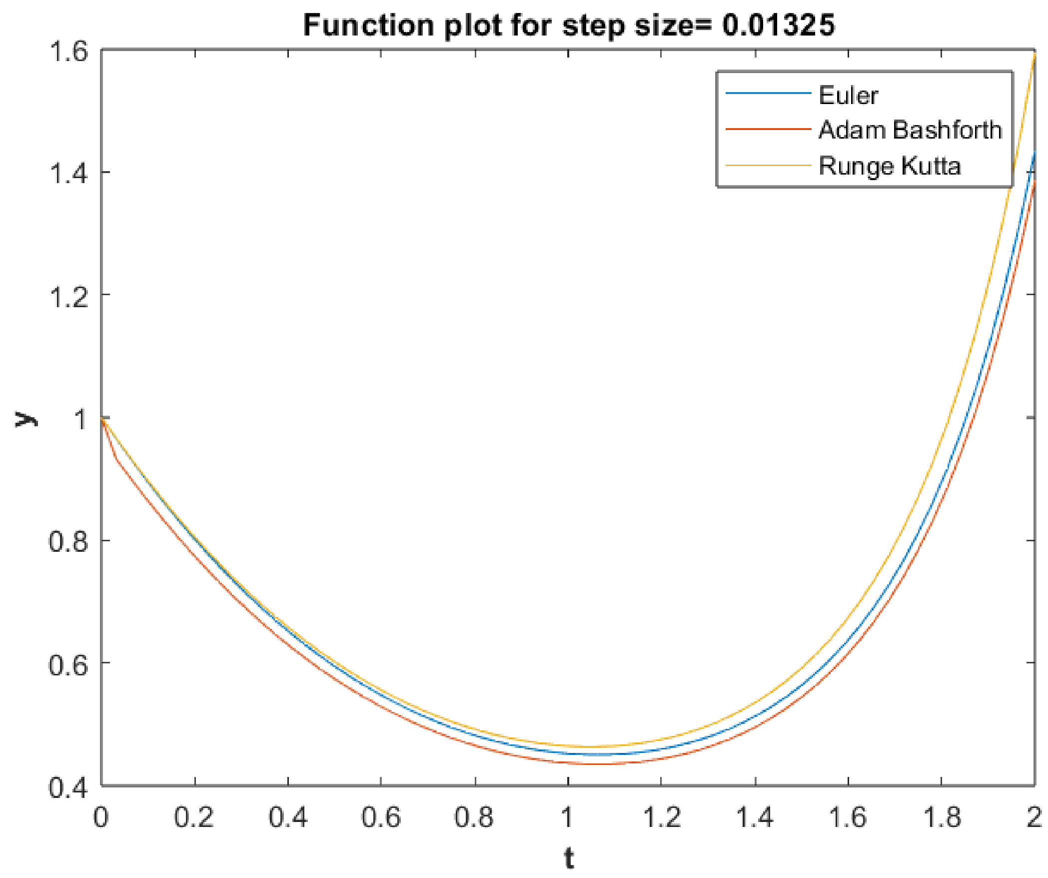
$$\int \frac{dy}{y} = \int (t^2 - 1.1)\,dt$$

or $\ln y = \frac{t^3}{3} - 1.1t + c$

At $t = 0$, $y = 1$: Putting values $\Rightarrow c = 0$

$$\boxed{\ln y = \frac{t^3}{3} - 1.1t}$$

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Euler
Adam Bashforth
Runge Kutta
y
t
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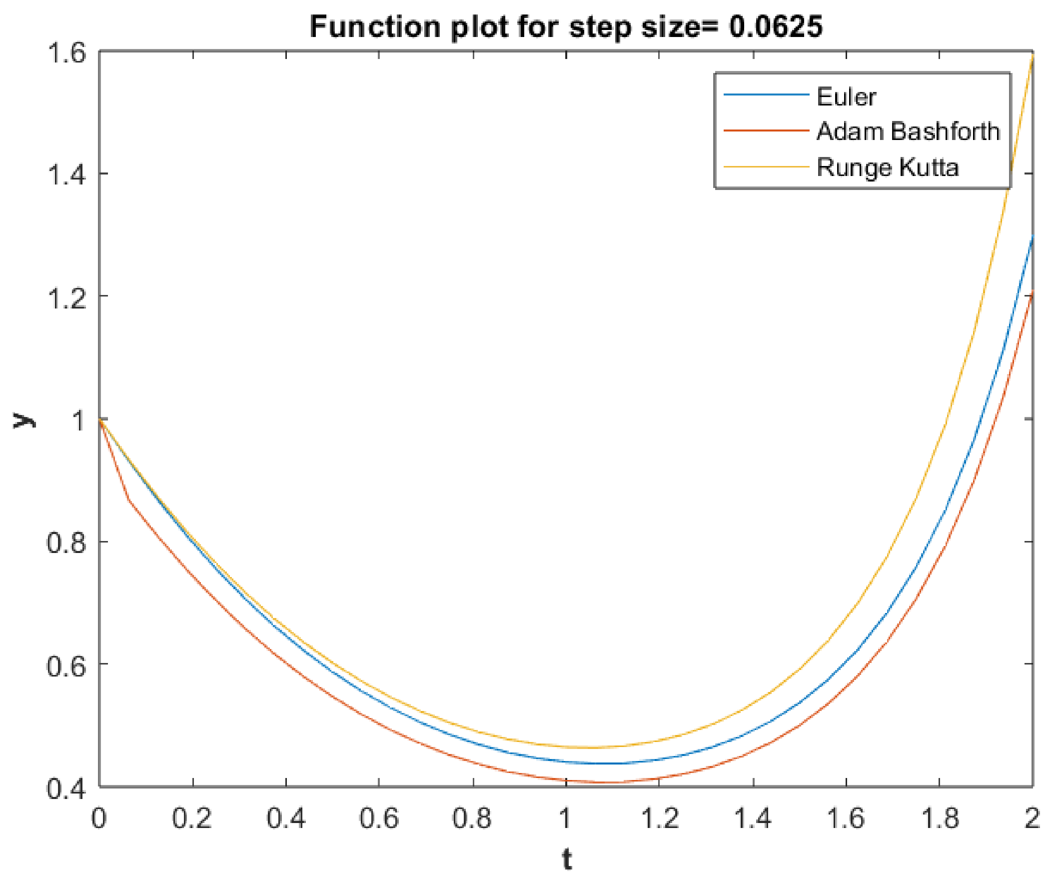
1.2
1
0.8
0.6
0.4
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2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Adam Bashforth
Runge Kutta
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t
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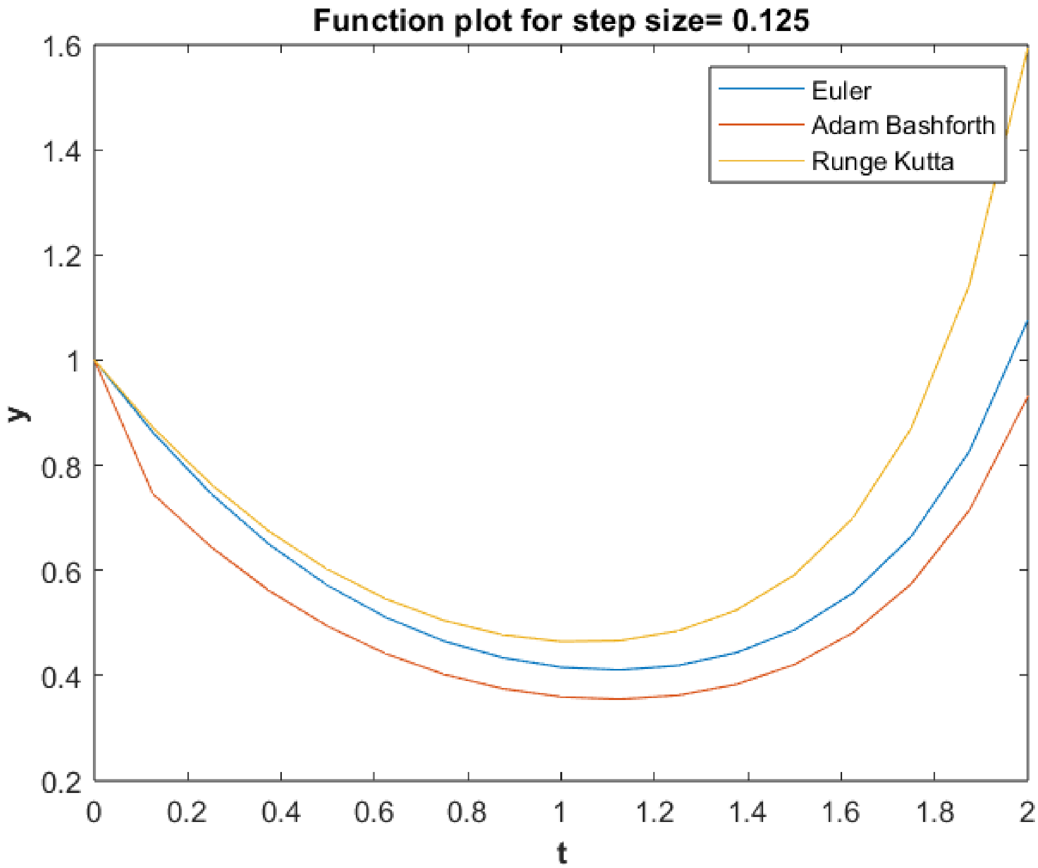
0.6
0.8
1
1.2
1.4
1.6
1.8
2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Adam Bashforth
Runge Kutta
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0
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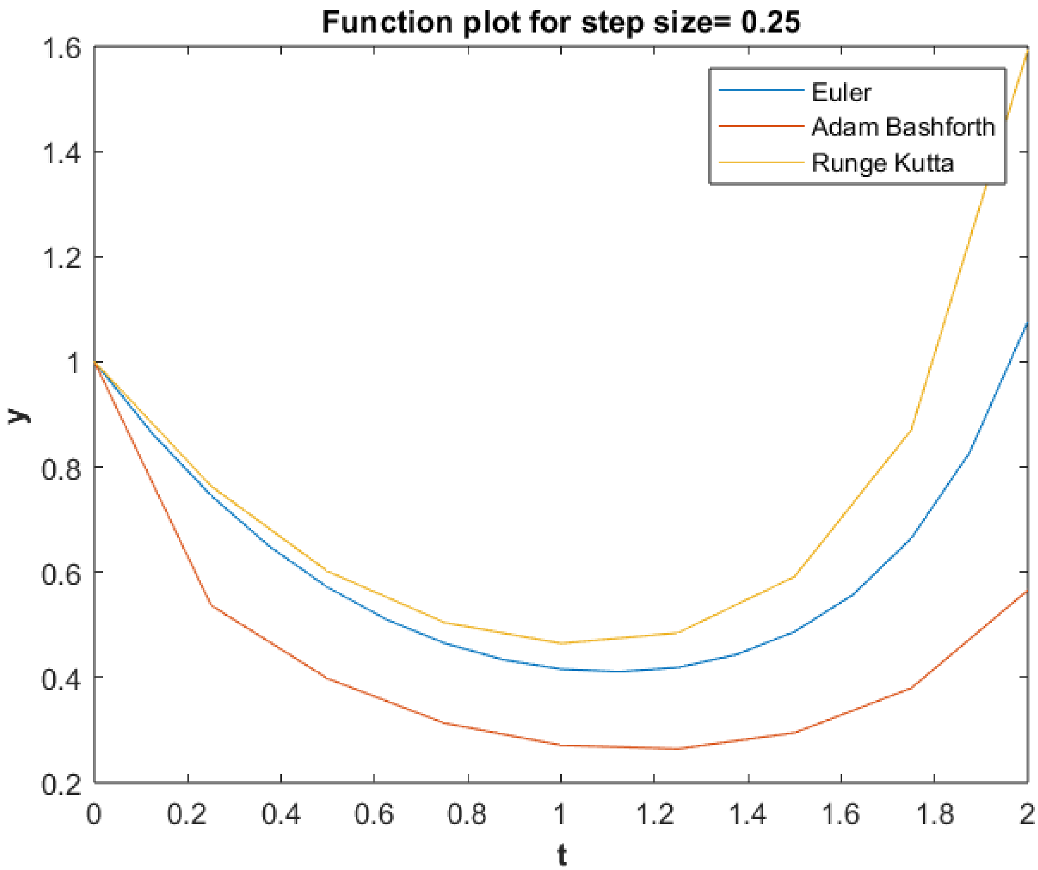
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2


Function plot for step size= 0.25
Euler
Adam Bashforth
Runge Kutta
y
t
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1.4
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1
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0.4
0.6
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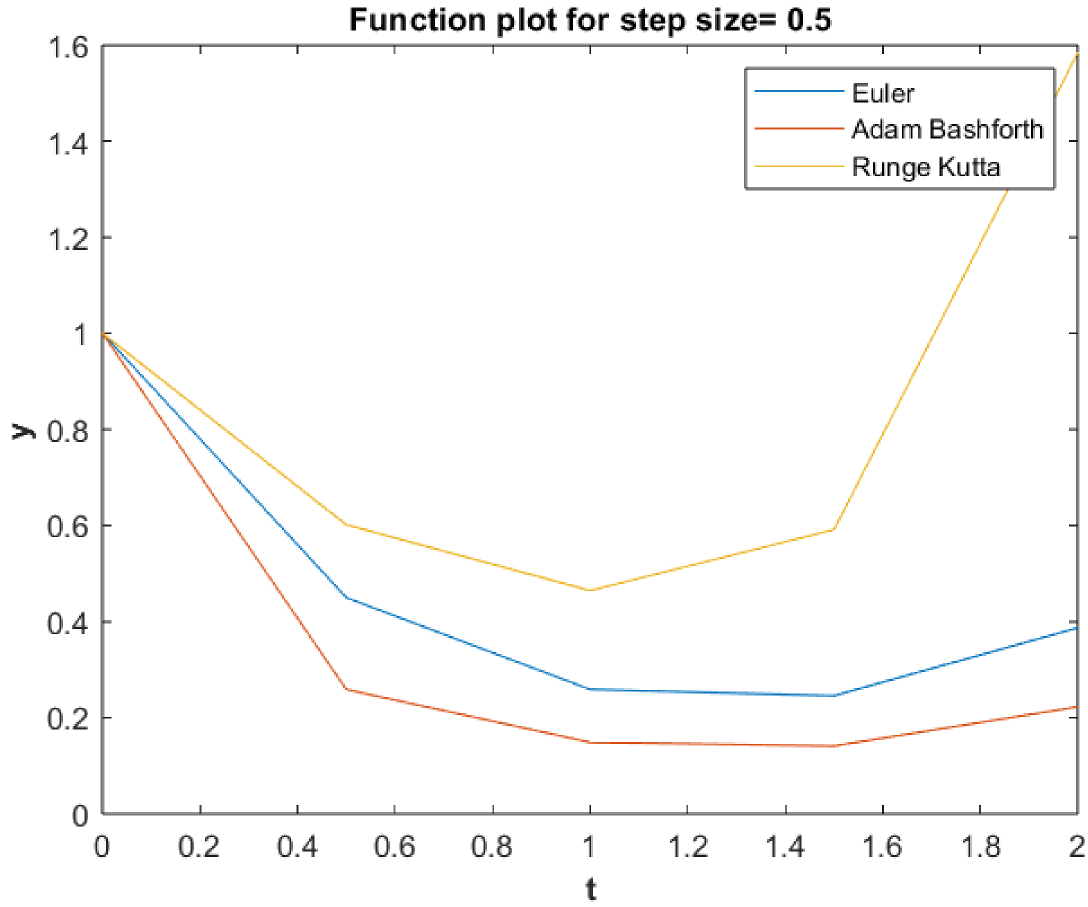
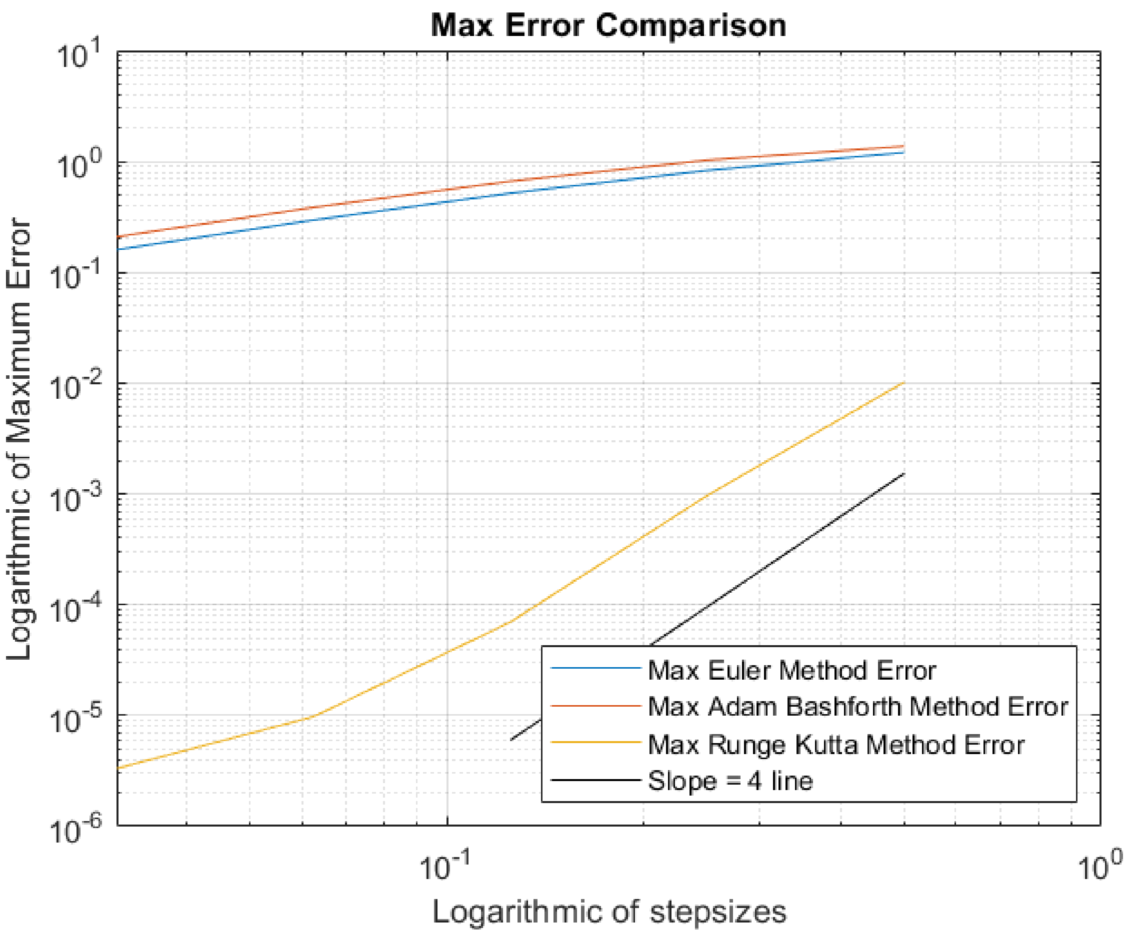
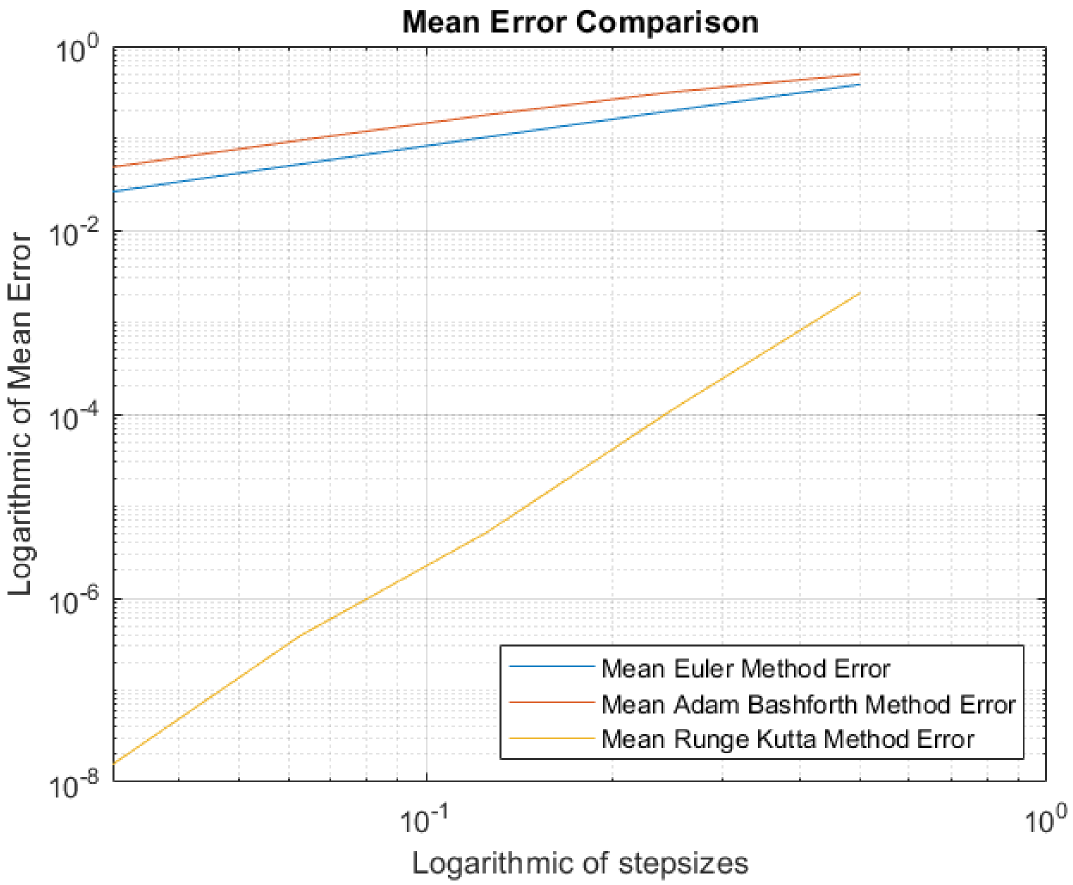
1
1.2
1.4
1.6
1.8
2



Function plot for step size= 0.5
Euler
Adam Bashforth
Runge Kutta
y
t



Max Error Comparison
Logarithmic of Maximum Error
Logarithmic of stepsizes
Max Euler Method Error
Max Adam Bashforth Method Error
Max Runge Kutta Method Error
Slope = 4 line



Solution 2:

(i) Slope of errorgraphs:

Runge Kutta's slope $= \dfrac{\ln(0.00207148) - \ln(5.15462 \times 10^{-6})}{\ln(0.5) - \ln(0.125)}$ (Mean line)

$\simeq 4.32$

Runge Kutta's slope (for max line) $= \dfrac{\ln(0.0102198) - \ln(6.97582 \times 10^{-5})}{\ln(0.5) - \ln(0.125)}$

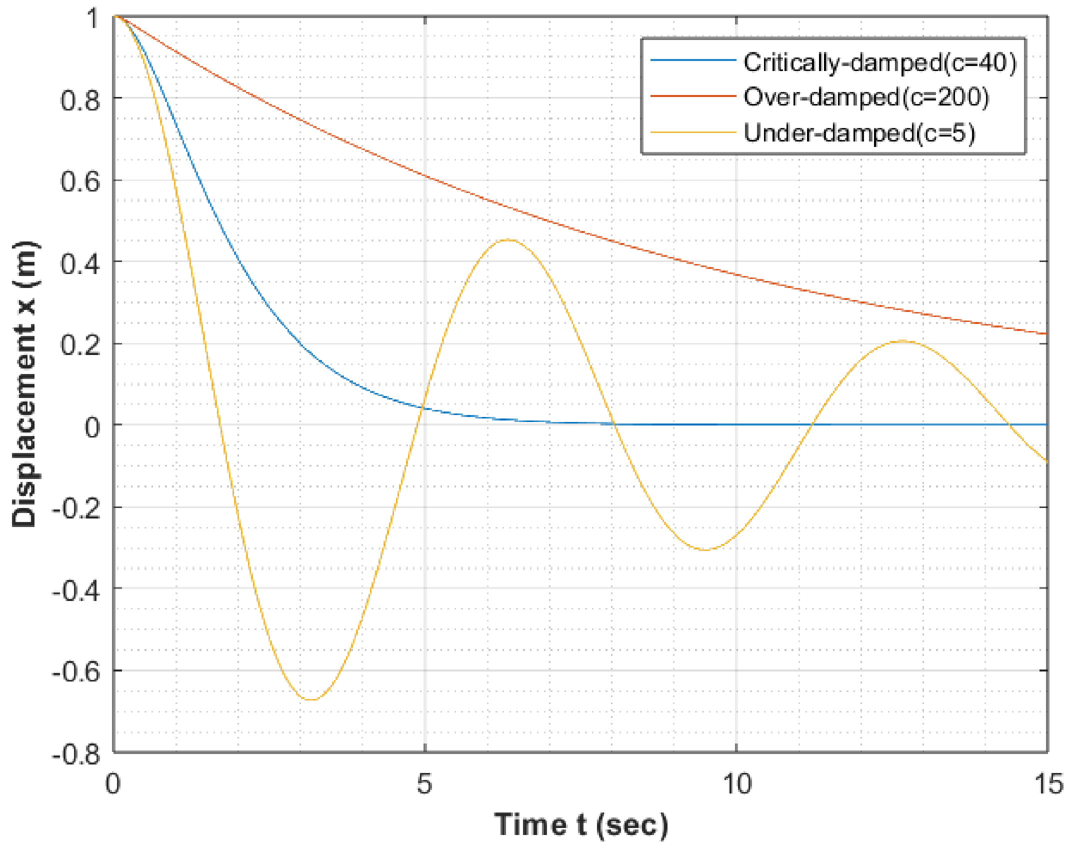
$\simeq 3.69$

Euler Method's slope (Mean line) $= \dfrac{\ln(1.20751) - \ln(0.51745)}{\ln(0.5) - \ln(0.125)} \approx 0.611$

Euler Method's slope (Max line) $= \dfrac{\ln(0.382113) - \ln(0.102322)}{\ln(0.5) - \ln(0.125)} \simeq 0.95$

It is shown that Runge Kutta is the fourth-order accurate scheme, and Euler's Explicit Method is first-order accurate. Similarly, it can be found in Adam Bashforth's method. Error is minimum in the case of the Fourth-order RK Method. Finer, the step size results will be close to the Analytical plot.

Solution 3:



Critical Damping (c=40)

The condition in which the damping of an oscillator causes it to return as quickly as possible to its equilibrium position without oscillating back and forth about this position.

Overdamping (c=200)

The condition in which the damping of an oscillator causes it to return to equilibrium without oscillating. Oscillator moves more slowly toward equilibrium than in the critically damped system.

Under Damping (c=5)

The condition in which the damping of an oscillator causes it to return to equilibrium with the amplitude gradually decreasing to zero; the system returns to equilibrium faster but overshoots and crosses the equilibrium position one or more times.

Solution 4:

Solution 4: $\frac{dy}{dx} = -2,00,000y + 2,00,000e^{-x} - e^{-x}$

a) Explicit Euler Method: $y_{i+1} = y_i + h \left.\frac{dy}{dx}\right|_{(x_i, y_i)}$

$$y_{i+1} = y_i + h\left[-200000y_i + 200000e^{-x_i} - e^{-x_i}\right]$$

or $y_{i+1} = y_i\left[1 - 200000h\right] + 199999\, e^{-x_i}$

At large value of x: $e^{-x} \rightarrow 0$

So, $x_i \rightarrow \infty$, $e^{-x_i} \rightarrow 0$

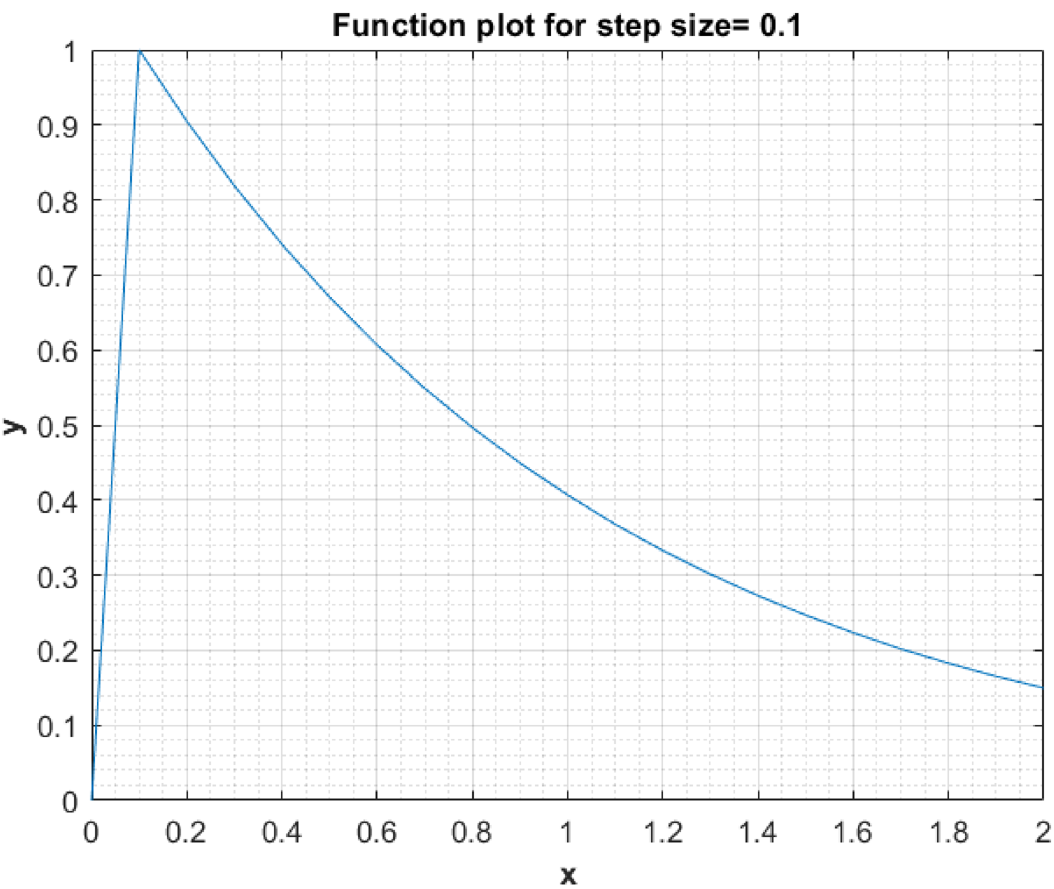
$$\Rightarrow \quad \frac{y_{i+1}}{y_i} = 1 - 200000h$$

Applying stability condition: $\left|\frac{y_{i+1}}{y_i}\right| < 1$

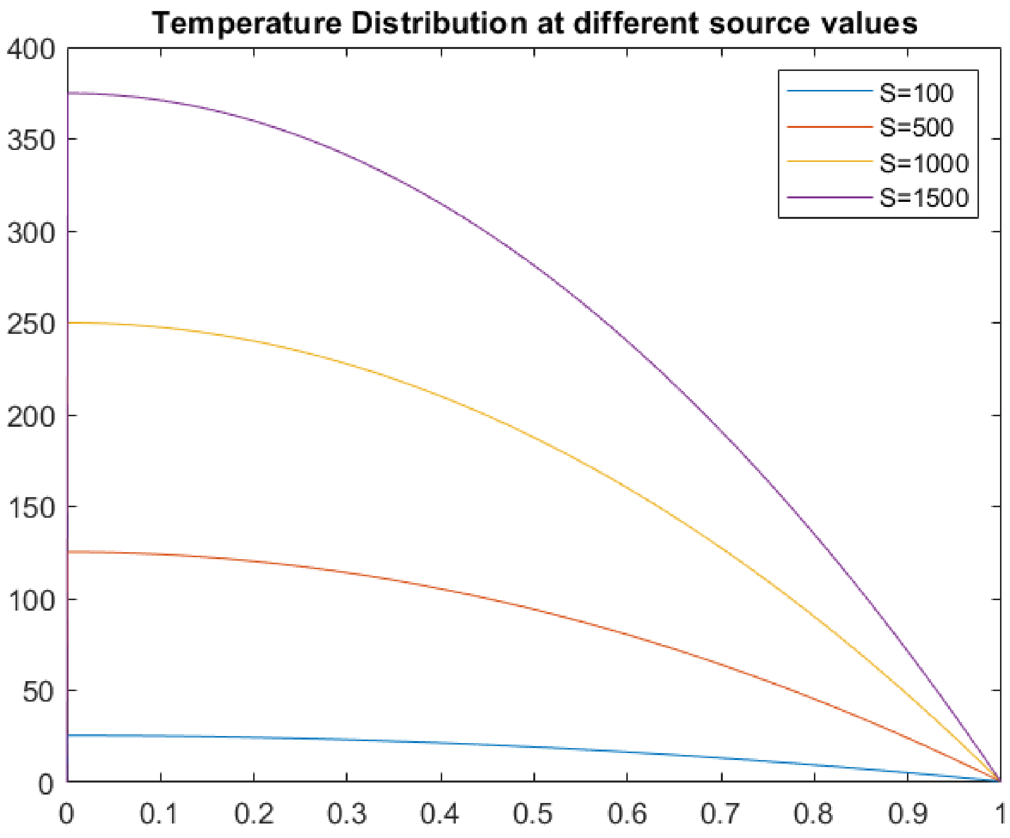
$$\Rightarrow \quad -1 < \frac{y_{i+1}}{y_i} \quad \Rightarrow \quad 1 - 200000h > -1$$

$$\Rightarrow \quad h < \frac{2}{200000}$$

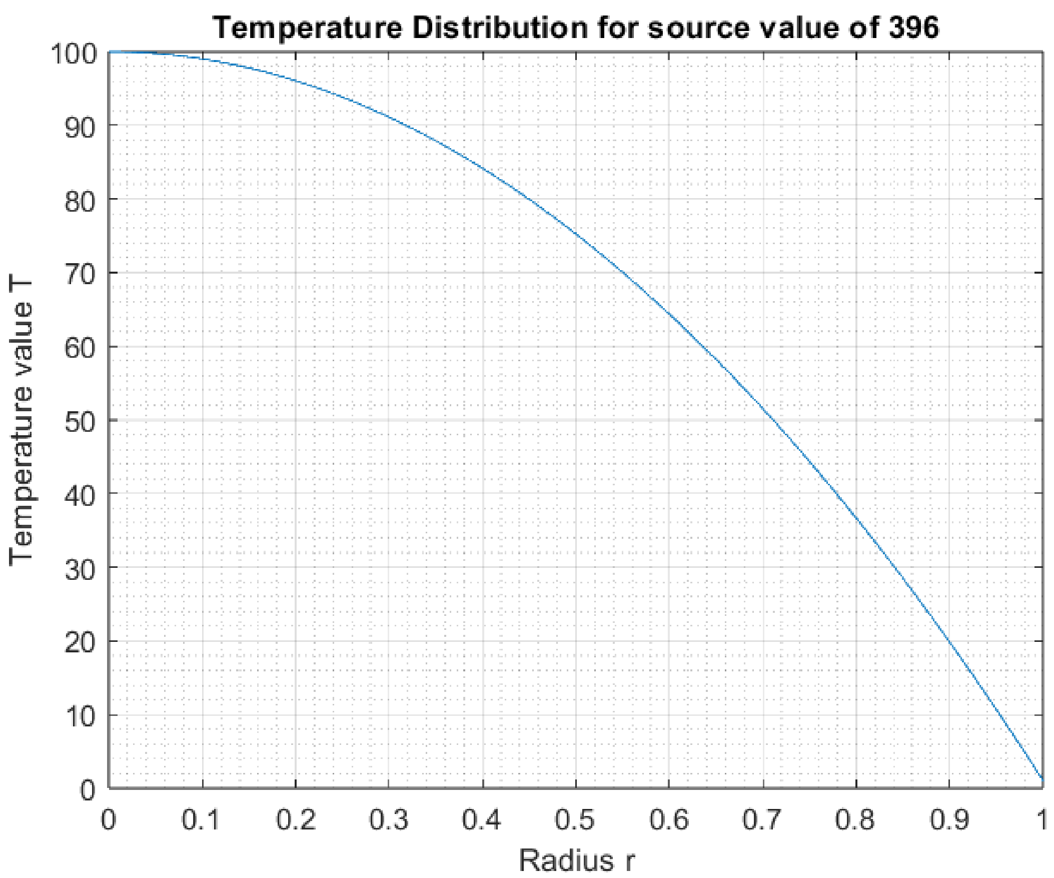
$$\Rightarrow \quad \boxed{h < 10^{-5}}$$



SOLUTION 5



b. Maximum temperature is at r=0 where insulation is provided. The largest source has the largest temperature at r=0.



By interpolation and checking values, at source value=396 the peak temperature in the domain does not exceed 100K. (Can be checked by changing values in the input file)