ME 282, Jan-April 2023
R. K. Shukla
Assignment, Due Thursday, April 20

1. Consider the FTCS, BTCS and a semi-implicit scheme for solution of the advection-diffusion equation $u_t + cu_x = bu_{xx}$:

$$\text{FTCS:} \quad \frac{u_j^{n+1} - u_j^n}{\Delta t} + c\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = b\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}$$

$$\text{BTCS:} \quad \frac{u_j^{n+1} - u_j^n}{\Delta t} + c\frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2\Delta x} = b\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2}$$

$$\text{Semi-Implicit CS:} \quad \frac{u_j^{n+1} - u_j^n}{\Delta t} + c\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = b\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2}$$

Derive a stability criteria for each of these methods using von Neumann stability analysis.

2. Show that the box scheme

$$\frac{1}{2\Delta t}\left[\left(u_i^{n+1} + u_{i+1}^{n+1}\right) - \left(u_i^n + u_{i+1}^n\right)\right] + \frac{c}{2\Delta x}\left[\left(u_{i+1}^{n+1} - u_i^{n+1}\right) - \left(u_{i+1}^n - u_i^n\right)\right] = f_i^n \tag{1}$$

is consistent with the one-way wave equation $u_t + cu_x = f$ and is unconditionally stable.

3. Consider the following one-way wave equation with periodic boundary conditions

$$\begin{aligned} &u_t + u_x = 0, \quad x \in (0,1),\ t > 0 \\ &u(x+1,t) = u(x,t), \quad u(x,0) = \cos(4\pi x) \end{aligned} \tag{2}$$

Write a computer code to solve this problem using the following four methods:

$$\text{FTBS}: \frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{u_i^n - u_{i-1}^n}{\Delta x} = 0 \tag{3}$$

$$\text{Leap Frog (CTCS)}: \frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} + \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = 0 \tag{4}$$

$$\text{BTBS}: \frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{u_i^{n+1} - u_{i-1}^{n+1}}{\Delta x} = 0 \tag{5}$$

$$\text{Crank Nicholson}: \frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{u_{i+1}^{n+1} - u_{i-1}^{n+1} + u_{i+1}^n - u_{i-1}^n}{4\Delta x} = 0 \tag{6}$$

Use your code to answer the following questions.

(a) Set CFL number $= \frac{\Delta t}{\Delta x} = 0.5$, and $\Delta x = 10^{-2}$. Plot the numerical solution computed using the four methods and the exact solution at $t = 5$. Compute and document the L_∞ error for all the four methods at time $t = 5$. Explain the key features (stability, dispersion/dissipation errors etc.) of each of the four schemes which you infer from the plots using amplification factor g and/or the modified equation.

(b) Repeat what you did in (a) for a CFL number of 1.

(c) Repeat what you did in (a) for a CFL number of 2.

Note: LeapFrog (CTCS) is a two step method which means you will need to devise a different method for the first time step. For simplicity, you can take exact solution for the first time step.

ME 282 – Computational Heat Transfer and Fluid Flow
Jan-April 2023 Semester
Assignment-4

Submitted By:
Saurabh Sharma
Sr. No: 21498

1.

For the given advection-diffusion equation

$$u_t + cu_x = bu_{xx} \tag{1.1}$$

Various schemes have been considered and the stability of these schemes are discussed as below

FTCS Scheme:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + c\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = b\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} \tag{1.2}$$

Rearranging

$$\left(u_j^{n+1} - u_j^n\right) + \frac{c\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right) = \frac{b\Delta t}{\Delta x^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

Let us define

$$\nu = \frac{c\Delta t}{\Delta x}, \qquad \mu = \frac{b\Delta t}{\Delta x^2}$$

$$\left(u_j^{n+1} - u_j^n\right) + \frac{\nu}{2}\left(u_{j+1}^n - u_{j-1}^n\right) = \mu\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right) \tag{1.3}$$

Now, from Von-Neuman Stability analysis, we know that we can write

$$u_j^n \sim g^n e^{ij\theta} \tag{1.4}$$

Where $g = amplification\ factor$

And for stability

$$\|g(\theta)\| \leq 1 \tag{1.6}$$

Substituting (1.4) in (1.3) we get

$$\left(g^{n+1}e^{ij\theta} - g^n e^{ij\theta}\right) + \frac{\nu}{2}\left(g^n e^{i(j+1)\theta} - g^n e^{i(j-1)\theta}\right) = \mu\left(g^n e^{i(j+1)\theta} - 2g^n e^{ij\theta} + g^n e^{i(j-1)\theta}\right)$$

Dividing by $g^n e^{ij\theta}$, we get

$$(g-1) + \frac{\nu}{2}\left(e^{i\theta} - e^{-i\theta}\right) = \mu\left(e^{i\theta} + e^{-i\theta} - 2\right)$$

Using $e^{\pm i\theta} = \cos\theta \pm i\sin\theta$ and rearranging we get

$$g = 1 - 2\mu(1 - \cos\theta) - i\nu\sin\theta \tag{1.7}$$

Using trigonometric identities

$$1 - \cos\theta = 2\sin^2\frac{\theta}{2}, \qquad \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$g = 1 - 4\mu \sin^2\frac{\theta}{2} - i\, 2\nu \sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

Now

$$|g|^2 = \left(1 - 4\mu \sin^2\frac{\theta}{2}\right)^2 + \left(2\nu \sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^2$$

For stability

$$|g|^2 = 1 - 8\mu \sin^2\frac{\theta}{2} + 16\mu^2 \sin^4\frac{\theta}{2} + 4\nu^2 \sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2} \le 1$$

$$4\sin^2\frac{\theta}{2}\left(-2\mu + 4\mu^2 \sin^2\frac{\theta}{2} + \nu^2\cos^2\frac{\theta}{2}\right) \le 0$$

$$-2\mu + 4\mu^2 \sin^2\frac{\theta}{2} + \nu^2\left(1 - \sin^2\frac{\theta}{2}\right) \le 0$$

$$(4\mu^2 - \nu^2)\sin^2\frac{\theta}{2} \le 2\mu - \nu^2$$

(1.8)

If $(4\mu^2 - \nu^2) \ge 0$ i.e., positive, for the above inequality to always hold, it should be true for the maxima of all the possible left bound i.e., maximum value of $\sin^2\frac{\theta}{2} = 1$. Then,

$$4\mu^2 - \nu^2 \le 2\mu - \nu^2$$

$$4\mu^2 \le 2\mu$$

$$\mu \le \frac{1}{2}$$

(1.9)

Similarly, if $(4\mu^2 - \nu^2) < 0$ i.e., negative, for the inequality in (1.8) to always hold, it should be true for the maxima of all the possible left bound i.e., minimum value of $\sin^2\frac{\theta}{2} = 0$. Then,

$$0 \le 2\mu - \nu^2$$

$$\therefore \nu^2 \le 2\mu$$

(1.10)

Substituting for ν & μ

$$\left(\frac{c\Delta t}{\Delta x}\right)^2 \le 2\frac{b\Delta t}{\Delta x^2}$$

$$\Delta t \le \frac{2b}{c^2}$$

(1.11)

<u>BTCS Scheme:</u>

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + c\frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2\Delta x} = b\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2}$$

(1.12)

Rearranging

$$(u_j^{n+1} - u_j^n) + \frac{c\Delta t}{2\Delta x}(u_{j+1}^{n+1} - u_{j-1}^{n+1}) = \frac{b\Delta t}{\Delta x^2}(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1})$$

$$(u_j^{n+1} - u_j^n) + \frac{\nu}{2}(u_{j+1}^{n+1} - u_{j-1}^{n+1}) = \mu(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1})$$

(1.13)

Now, from Von-Neuman Stability analysis, we know that we can write

$$u_j^n \sim g^n e^{ij\theta}$$

Substituting we get

$$\left(g^{n+1}e^{ij\theta} - g^n e^{ij\theta}\right) + \frac{\nu}{2}\left(g^{n+1}e^{i(j+1)\theta} - g^{n+1}e^{i(j-1)\theta}\right) = \mu\left(g^{n+1}e^{i(j+1)\theta} - 2g^{n+1}e^{ij\theta} + g^{n+1}e^{i(j-1)\theta}\right)$$

Dividing by $g^{n+1}e^{ij\theta}$, we get

$$\left(1 - \frac{1}{g}\right) + \frac{\nu}{2}\left(e^{i\theta} - e^{-i\theta}\right) = \mu\left(e^{i\theta} + e^{-i\theta} - 2\right)$$

Using $e^{\pm i\theta} = \cos\theta \pm i\sin\theta$ and rearranging we get

$$g = \frac{1}{1 + 2\mu(1 - \cos\theta) + i\nu\sin\theta} \tag{1.14}$$

$$|g|^2 = \frac{1}{\left(1 + 4\mu\sin^2\frac{\theta}{2}\right)^2 + (\nu\sin\theta)^2}$$

$$|g|^2 = \frac{1}{1 + 8\mu\sin^2\frac{\theta}{2} + 16\mu^2\sin^4\frac{\theta}{2} + \nu^2\sin^2\theta} \tag{1.15}$$

In RHS of (1.15) Denominator is always: $1 + (\textit{some positive value})$

$$\therefore |g|^2 \leq 1$$

Therefore, the BTCS scheme is unconditionally stable.

<u>Semi-Implicit CS Scheme:</u>

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + c\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = b\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2} \tag{1.16}$$

Rearranging

$$\left(u_j^{n+1} - u_j^n\right) + \frac{c\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right) = \frac{b\Delta t}{\Delta x^2}\left(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}\right)$$

$$\left(u_j^{n+1} - u_j^n\right) + \frac{\nu}{2}\left(u_{j+1}^{n+1} - u_{j-1}^{n+1}\right) = \mu\left(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}\right) \tag{1.17}$$

Now, from Von-Neuman Stability analysis, we know that we can write

$$u_j^n \sim g^n e^{ij\theta}$$

Substituting we get

$$\left(g^{n+1}e^{ij\theta} - g^n e^{ij\theta}\right) + \frac{\nu}{2}\left(g^n e^{i(j+1)\theta} - g^n e^{i(j-1)\theta}\right) = \mu\left(g^{n+1}e^{i(j+1)\theta} - 2g^{n+1}e^{ij\theta} + g^{n+1}e^{i(j-1)\theta}\right)$$

Dividing by $g^n e^{ij\theta}$, we get

$$(g-1)+\frac{\nu}{2}\left(e^{i\theta}-e^{-i\theta}\right)=\mu\left(e^{i\theta}+e^{-i\theta}-2\right)g$$

Using $e^{\pm i\theta} = \cos\theta \ \pm i\sin\theta$ and rearranging we get

$$g=\frac{1-i\nu\sin\theta}{1+4\mu\sin^2\frac{\theta}{2}} \tag{1.18}$$

Now, for stability

$$|g|^2=\frac{1+(\nu\sin\theta)^2}{\left(1+4\mu\sin^2\frac{\theta}{2}\right)^2}\leq 1$$

$$1+\left(\nu.2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^2 \leq 1+8\mu\sin^2\frac{\theta}{2}+16\mu^2\sin^4\frac{\theta}{2}$$

$$4\nu^2\sin^2\frac{\theta}{2}\left(1-\sin^2\frac{\theta}{2}\right)\leq 8\mu\sin^2\frac{\theta}{2}+16\mu^2\sin^4\frac{\theta}{2}$$

$$\nu^2-2\mu\leq(\nu^2+4\mu^2)\sin^2\frac{\theta}{2}$$

As $(\nu^2+4\mu^2)\geq 0$ i.e., positive, for the above inequality to always hold, it should be true for the minima of all the possible right bounds i.e., minimum value of $\sin^2\frac{\theta}{2}=0$.

$$\nu^2-2\mu\leq 0$$

$$\nu^2\leq 2\mu \tag{1.19}$$

Similar to what we got in FTCS equation (1.10).

Substituting for ν & μ

$$\left(\frac{c\Delta t}{\Delta x}\right)^2\leq 2\frac{b\Delta t}{\Delta x^2}$$

$$\Delta t\leq\frac{2b}{c^2} \tag{1.20}$$

Therefore, for the given advection-diffusion equation using Von-Neuman Stability analysis, we found that

- FTCS conditionally stable in both space (Δx) and time (Δt)
- BTCS scheme is unconditionally stable
- Semi-Implicit scheme is conditionally stable in time and unconditionally stable in space.

2.

For the one-way wave equation

$$u_t + cu_x = f \tag{2.1}$$

Box scheme is as follows

$$\underbrace{\frac{1}{2\Delta t}\left[\left(u_j^{n+1} + u_{j+1}^{n+1}\right) - \left(u_j^n + u_{j+1}^n\right)\right]}_{(1)} + \underbrace{\frac{c}{2\Delta x}\left[\left(u_{j+1}^{n+1} - u_j^{n+1}\right) + \left(u_{j+1}^n - u_j^n\right)\right]}_{(2)} = f_j^n \tag{2.2}$$

Now, Taylor series expansion for various terms are

(Note: in all the equations below, derivates are at node x_j and time t_n i.e. $u_{x\dots t\dots} = u_{x\dots t\dots}(x_j, t_n)$)

$$u_j^{n+1} = u(x_j, t_n + \Delta t) = u_j^n + \Delta t\, u_t + \frac{\Delta t^2}{2!}u_{tt} + \frac{\Delta t^3}{3!}u_{ttt} \dots \tag{2.3}$$

$$u_{j+1}^n = u(x_j + \Delta x, t_n) = u_j^n + \Delta x\, u_x + \frac{\Delta x^2}{2!}u_{xx} + \frac{\Delta x^3}{3!}u_{xxx} \dots \tag{2.4}$$

$$\begin{aligned} u_{j+1}^{n+1} &= u(x_j + \Delta x, t_n + \Delta t) \\ &= u_j^n + \{\Delta x\, u_x + \Delta t\, u_t\} + \frac{1}{2!}\{\Delta x^2 u_{xx} + \Delta x \Delta t\, u_{xt} + \Delta t^2 u_{tt}\} \\ &\quad + \frac{1}{3!}\{\Delta x^3 u_{xxx} + \Delta x^2 \Delta t\, u_{xxt} + \Delta x \Delta t^2\, u_{xtt} + \Delta t^3 u_{ttt}\} \dots \end{aligned}$$

$$\begin{aligned} \therefore u_{j+1}^{n+1} &= u_j^n + \left(\Delta x\, u_x + \frac{\Delta x^2}{2!}u_{xx} + \frac{\Delta x^3}{3!}u_{xxx} \dots\right) + \left(\Delta t\, u_t + \frac{\Delta t^2}{2!}u_{tt} + \frac{\Delta t^3}{3!}u_{ttt} \dots\right) \\ &\quad + \left(\frac{\Delta x \Delta t}{2!}u_{xt} + \frac{\Delta x^2 \Delta t}{3!}u_{xxt} + \frac{\Delta x \Delta t^2}{3!}\, u_{xtt} \dots\right) \end{aligned} \tag{2.5}$$

Substituting in term (1) of equation (2.2)

$$\begin{aligned} &\left(u_j^{n+1} + u_{j+1}^{n+1}\right) - \left(u_j^n + u_{j+1}^n\right) \\ &= \left\{u_j^n + \left(\Delta t\, u_t + \frac{\Delta t^2}{2!}u_{tt} \dots\right)\right\} \\ &+ \left\{u_j^n + \left(\Delta x\, u_x + \frac{\Delta x^2}{2!}u_{xx} \dots\right) + \left(\Delta t\, u_t + \frac{\Delta t^2}{2!}u_{tt} \dots\right)\right. \\ &+ \left.\left(\frac{\Delta x \Delta t}{2!}u_{xt} + \frac{\Delta x^2 \Delta t}{3!}u_{xxt} + \frac{\Delta x \Delta t^2}{3!}\, u_{xtt} \dots\right)\right\} - u_j^n - \left\{u_j^n + \left(\Delta x\, u_x + \frac{\Delta x^2}{2!}u_{xx} \dots\right)\right\} \end{aligned}$$

$$\begin{aligned} &\left(u_j^{n+1} + u_{j+1}^{n+1}\right) - \left(u_j^n + u_{j+1}^n\right) \\ &= 2\left(\Delta t\, u_t + \frac{\Delta t^2}{2!}u_{tt} + \frac{\Delta t^3}{3!}u_{ttt} \dots\right) + \left(\frac{\Delta x \Delta t}{2!}u_{xt} + \frac{\Delta x^2 \Delta t}{3!}u_{xxt} + \frac{\Delta x \Delta t^2}{3!}\, u_{xtt} \dots\right) \end{aligned}$$

$$\frac{1}{2\Delta t}\left[\left(u_j^{n+1} + u_{j+1}^{n+1}\right) - \left(u_j^n + u_{j+1}^n\right)\right] = u_t + \left(\frac{\Delta t}{2!}u_{tt} + \frac{\Delta t^2}{3!}u_{ttt} \dots\right) + \frac{1}{2}\left(\frac{\Delta x}{2!}u_{xt} + \frac{\Delta x^2}{3!}u_{xxt} + \frac{\Delta x \Delta t}{3!}\, u_{xtt} \dots\right) \tag{2.6}$$

$$\frac{1}{2\Delta t}\left[\left(u_j^{n+1} + u_{j+1}^{n+1}\right) - \left(u_j^n + u_{j+1}^n\right)\right] = u_t + O(\Delta x) + O(\Delta t) + O(\Delta x \Delta t)$$

Substituting in term (2) of equation (2.2)

$$\begin{aligned}
\left(u_{j+1}^{n+1}-u_j^{n+1}\right)+\left(u_{j+1}^{n}-u_j^{n}\right)&\\
=\Big\{u_j^n+\left(\Delta x\, u_x+\frac{\Delta x^2}{2!}u_{xx}\dots\right)&+\left(\Delta t\, u_t+\frac{\Delta t^2}{2!}u_{tt}\dots\right)\\
+\left(\frac{\Delta x\Delta t}{2!}u_{xt}+\frac{\Delta t^2\Delta x}{3!}\,u_{xtt}+\frac{\Delta t\,\Delta x^2}{3!}u_{xxt}\dots\right)\Big\}&-\left\{u_j^n+\left(\Delta t\, u_t+\frac{\Delta t^2}{2!}u_{tt}\dots\right)\right\}\\
+\left\{u_j^n+\left(\Delta x\, u_x+\frac{\Delta x^2}{2!}u_{xx}\dots\right)\right\}&-u_j^n
\end{aligned}$$

$$\begin{aligned}
\left(u_{j+1}^{n+1}-u_j^{n+1}\right)+\left(u_{j+1}^{n}-u_j^{n}\right)&\\
=2\left(\Delta x\, u_x+\frac{\Delta x^2}{2!}u_{xx}+\frac{\Delta x^3}{3!}u_{xxx}\dots\right)&+\left(\frac{\Delta x\Delta t}{2!}u_{xt}+\frac{\Delta x^2\Delta t}{3!}u_{xxt}+\frac{\Delta x\Delta t^2}{3!}\,u_{xtt}\dots\right)
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2\Delta x}\left[\left(u_{j+1}^{n+1}-u_j^{n+1}\right)+\left(u_{j+1}^{n}-u_j^{n}\right)\right]&\\
=u_x+\left(\frac{\Delta x}{2!}u_{xx}+\frac{\Delta x^2}{3!}u_{xxx}\dots\right)&+\frac{1}{2}\left(\frac{\Delta t}{2!}u_{xt}+\frac{\Delta x\Delta t}{3!}u_{xxt}+\frac{\Delta t^2}{3!}\,u_{xtt}\dots\right)
\end{aligned}$$

(2.7)

$$\frac{1}{2\Delta x}\left[\left(u_{j+1}^{n+1}-u_j^{n+1}\right)+\left(u_{j+1}^{n}-u_j^{n}\right)\right]=u_x+O(\Delta x)+O(\Delta t)+O(\Delta x\Delta t)$$

Consistency Analysis

For the exact equation of the one-way wave (2.1)

$$u_t+cu_x=f$$

Let this operation be represented as P

$$Pu=u_t(x,t)+cu_x(x,t)-f(x,t)$$

(2.8)

Let the given box scheme it is represented as $P_{\Delta t,\Delta x}$

$$P_{\Delta t,\Delta x}u=\frac{1}{2\Delta t}\left[\left(u_j^{n+1}+u_{j+1}^{n+1}\right)-\left(u_j^n+u_{j+1}^n\right)\right]\;+\;\frac{c}{2\Delta x}\left[\left(u_{j+1}^{n+1}-u_j^{n+1}\right)+\left(u_{j+1}^{n}-u_j^{n}\right)\right]-f_j^n$$

(2.9)

From equations (2.6) and (2.7)

$$P_{\Delta t,\Delta x}u=u_t+\frac{\Delta t}{2!}u_{tt}+\frac{\Delta x}{4}u_{xt}+cu_x+\frac{c\Delta x}{2!}u_{xx}+\frac{c\Delta t}{4}u_{xt}-f_j^n+O(\Delta x^2)+O(\Delta t^2)+O(\Delta x\Delta t)$$

(2.10)

Truncation error is defined then as

$$T=Pu-P_{\Delta t,\Delta x}u$$

(2.11)

Substituting equation (2.8) and (2.10) gives

$$T\left(x_j,t_n\right)=\frac{\Delta t}{2!}u_{tt}+\frac{\Delta x}{4}u_{xt}+\frac{c\Delta x}{2!}u_{xx}+\frac{c\Delta t}{4}u_{xt}+O(\Delta x^2)+O(\Delta t^2)+O(\Delta x\Delta t)$$

$$T\left(x_j,t_n\right)=O(\Delta x)+O(\Delta t)+O(\Delta x\Delta t)$$

(2.12)

For scheme to be consistent, the truncation error should vanish as mesh is made finer and represent the exact solution i.e., $T\to 0$ as $\Delta t,h\to 0$.

As $\Delta t, h \to 0$

$$P_{\Delta t,\Delta x}u = u_t + cu_x - f_j^n$$

It approaches exact equation and hence $T \to 0$. Therefore, the box scheme is consistent with one-way wave equation.

Stability Analysis

From Von-Neuman Stability analysis, we know that we can write

$$u_j^n \sim g^n e^{ij\theta}$$

Where $g = amplification\ factor$

And for stability

$$\|g(\theta)\| \leq 1$$

Rearranging (2.2) and with $\nu = \frac{c\Delta t}{\Delta x}$

$$\left[\left(u_j^{n+1} + u_{j+1}^{n+1}\right) - \left(u_j^n + u_{j+1}^n\right)\right] + \nu\left[\left(u_{j+1}^{n+1} - u_j^{n+1}\right) + \left(u_{j+1}^n - u_j^n\right)\right] = 2\Delta t f_j^n$$

Substituting $u_j^n \sim g^n e^{ij\theta}$

$$\begin{aligned}&\left[\left(g^{n+1}e^{ij\theta} + g^{n+1}e^{i(j+1)\theta}\right) - \left(g^n e^{ij\theta} + g^n e^{i(j+1)\theta}\right)\right]\\ &\qquad + \nu\left[\left(g^{n+1}e^{i(j+1)\theta} - g^{n+1}e^{ij\theta}\right) + \left(g^n e^{i(j+1)\theta} - g^n e^{ij\theta}\right)\right] = 0\end{aligned}$$

Dividing by $g^n e^{ij\theta}$, we get

$$\left(g + ge^{i\theta} - 1 - e^{i\theta}\right) + \nu\left(ge^{i\theta} - g + e^{i\theta} - 1\right) = 0$$

$$g = \frac{(1+\nu) + (1-\nu)e^{i\theta}}{(1-\nu) + (1+\nu)e^{i\theta}}$$

Using $e^{i\theta} = \cos\theta + i\sin\theta$ and rearranging we get

$$g = \frac{\{(1+\nu) + (1-\nu)\cos\theta\} + i(1-\nu)\sin\theta}{\{(1-\nu) + (1+\nu)\cos\theta\} + i(1+\nu)\sin\theta} \tag{2.13}$$

$$|g|^2 = \frac{\{(1+\nu) + (1-\nu)\cos\theta\}^2 + \{(1-\nu)\sin\theta\}^2}{\{(1-\nu) + (1+\nu)\cos\theta\}^2 + \{(1+\nu)\sin\theta\}^2}$$

$$|g|^2 = \frac{(1+\nu)^2 + 2(1+\nu)(1-\nu)\cos\theta + (1-\nu)^2\cos^2\theta + (1-\nu)^2\sin^2\theta}{(1-\nu)^2 + 2(1-\nu)(1+\nu)\cos\theta + (1+\nu)^2\cos^2\theta + (1+\nu)^2\sin^2\theta}$$

As $\cos^2\theta + \sin^2\theta = 1$

$$|g|^2 = \frac{(1+\nu)^2 + 2(1+\nu)(1-\nu)\cos\theta + (1-\nu)^2}{(1-\nu)^2 + 2(1-\nu)(1+\nu)\cos\theta + (1+\nu)^2} = 1$$

$$|g| = 1 \tag{2.14}$$

Therefore, this box scheme is unconditionally stable.

Hence as it was seen that the box scheme is consistent with one-way wave equation and is unconditionally stable, therefore, this scheme is convergent.

3.

For the given one-way wave equation

$$u_t + u_x = 0, \qquad x \in (0,1), \qquad t > 0$$

Subjected to boundary conditions

$$u(x+1,t) = u(x,t), \qquad u(x,0) = \cos(4\pi x)$$

This is discretised in space and time as

$$u_t(x_j,t_n) + u_x(x_j,t_n) = 0$$

$$\Delta x = \frac{1}{N}, \qquad x_j = \frac{j}{N} = j\Delta x, \qquad \Delta t = \frac{T}{N_t}, \qquad t_n = \frac{n}{N_t} = n\Delta t$$

Where, N is number of grid points in space and N_t is number of steps (or iterations) in time.

$$i = 0,1, \dots N, \qquad n = 0,1, \dots N_t$$

Also, defining CFL number as

$$\nu = \frac{\Delta t}{\Delta x}$$

Hence for all the problems we take

$$\Delta x = 10^{-2}$$

Hence for a given CFL number, Δt is evaluated as

$$\Delta t = \nu \Delta x$$

<u>Exact Solution:</u>

To determine the exact solution, we know for a one-way wave equation $u_t + cu_x = 0$, based on the characteristic, the solution is of the form

$$u(x,t) = f(x - ct)$$

In our case $c = 1$

$$\therefore u(x,t) = f(x - t)$$

using initial boundary condition at $t = 0$

$$u(x,0) = f(x) = \cos(4\pi x)$$

$$\therefore u(x,t) = f(x - t) = \cos(4\pi(x - t))$$

Now L_∞ error can be evaluated using below equation

$$e_\infty = Maximum\left(\left|u_j^{calculated} - u_j^{exact}\right|\right)_{j=0}^{N}$$

Various schemes have been considered and solved numerically as below for different CFL numbers to calculate the solution at time $T = 5$.

(a)

For CFL number $\boldsymbol{\nu} = \mathbf{0.5}$

$$\Delta t = \nu \Delta x = 5 \times 10^{-3}$$

We'll integrate from time $t = 0 \; to \; T = 5$ therefore, number of iterations $N_t = \frac{T}{\Delta t} = 1000$

<u>FTBS Scheme:</u>

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{u_j^n - u_{j-1}^n}{\Delta x} = 0$$

This can be rearranged as

$$u_j^{n+1} = (1-\nu)u_j^n + \nu u_{j-1}^n$$

This gives us the iterative update equation in time.

At $j = 0$

$$u_0^{n+1} = (1-\nu)u_0^n + \nu u_{-1}^n$$

Where using the periodic nature of the wave equation as given $u(x+1,t) = u(x,t)$

$$u_{-1}^n = u_{N-1}^n$$

$$u_0^{n+1} = (1-\nu)u_0^n + \nu u_{N-1}^n$$

Hence this can be written as linear system of equation as

$$\begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_i \\ \vdots \\ \vdots \\ u_{N-1} \end{bmatrix}^{n+1} = \begin{bmatrix} (1-\nu) & 0 & 0 & & 0 & \nu & 0 \\ \nu & (1-\nu) & 0 & & & 0 & 0 \\ 0 & & \ddots & & & & 0 \\ & \cdots & \nu & (1-\nu) & 0 & \cdots & \\ 0 & & & & \ddots & & 0 \\ 0 & 0 & & & \nu & (1-\nu) & 0 \\ 0 & 0 & 0 & & & \nu & (1-\nu) \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_i \\ \vdots \\ \vdots \\ u_{N-1} \end{bmatrix}^{n}$$

This can be solved iteratively from time $t = 0 \; to \; 5$, considering initial boundary conditions

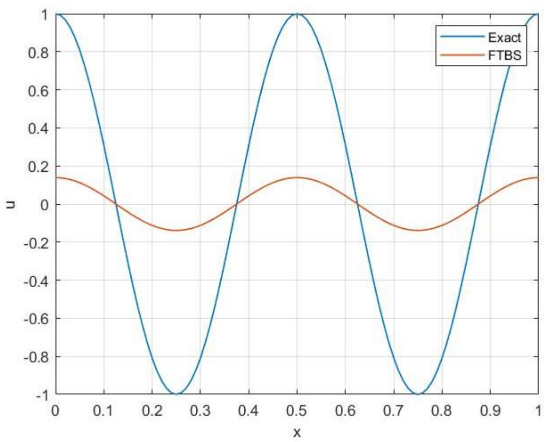
$$u_j^0 = \cos(4\pi x_j) = \cos(4\pi j \Delta x)$$

The computer code for solving this is given in the end, here we only discuss the solution.

Stability criteria: Using the Von-Neumann Stability analysis by taking $u_j^n \sim g^n e^{ij\theta}$ we get amplification factor as

$$g = 1 - \nu + \nu e^{-i\theta} = 1 - \nu(1 - \cos\theta) - i\nu \sin\theta$$

For stability $|g| \leq 1$, which gives stability criteria as $\nu \leq 1$



$$L_\infty = 0.8612$$

Here we observe that there is considerable dissipation observed in the solution. This is a first order accurate scheme $\sim O(\Delta x)$ & $O(\Delta t)$. If we consider these terms in the scheme and neglect instead even higher order terms (i.e., $\sim O(\Delta x^2)$ & $O(\Delta t^2)$), we get the modified equation of the form

$$u_t + cu_x = \frac{c\Delta x}{2}(1-\nu)u_{xx} = \mu_h u_{xx}$$

This is an equivalent advection diffusion equation which we are solving.

Here $\mu_h = 2.5 \times 10^{-3}$, however this apparent diffusion coefficient is very small but is considerably affecting the solution causing large dissipation errors.

<u>Leap Frog (CTCS) Scheme:</u>

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} + \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0$$

This can be rearranged as

$$u_j^{n+1} = -\nu u_{j+1}^n + \nu u_{j-1}^n + u_j^{n-1}$$

This gives us the iterative update equation in time.

At $j = 0$

$$u_0^{n+1} = -\nu u_1^n + \nu u_{-1}^n + u_0^{n-1}$$

Where using $u_{-1}^n = u_{N-1}^n$

$$u_0^{n+1} = -\nu u_1^n + \nu u_{N-1}^n + u_0^{n-1}$$

At $j = N$

$$u_N^{n+1} = -\nu u_{N+1}^n + \nu u_{N-1}^n + u_N^{n-1}$$

Where using $u_{N+1}^n = u_1^n$

$$u_N^{n+1} = -\nu u_1^n + \nu u_{N-1}^n + u_N^{n-1}$$

Hence this can be written as linear system of equation as

$$\begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_i \\ \vdots \\ \vdots \\ u_{N-1} \end{bmatrix}^{n+1} = \begin{bmatrix} 0 & -\nu & 0 & & 0 & \nu & 0 \\ \nu & 0 & -\nu & & & 0 & 0 \\ 0 & & \ddots & & & & 0 \\ & & \nu & 0 & -\nu & & \\ 0 & & & & \ddots & & 0 \\ 0 & 0 & & & \nu & 0 & -\nu \\ 0 & -\nu & 0 & & \nu & 0 & \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_i \\ \vdots \\ \vdots \\ u_{N-1} \end{bmatrix}^{n} + \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_i \\ \vdots \\ \vdots \\ u_{N-1} \end{bmatrix}^{n-1}$$

This can be solved iteratively from time $t = 0\ to\ 5$, considering initial boundary conditions

$$u_j^0 = \cos(4\pi x_j) = \cos(4\pi j\Delta x)$$

This is a two-step scheme, so we need to determine the first step using some other method. Using the exact solution to determine the first step at $t = \Delta t$

$$u_j^1 = u(x_j, t_1) = \cos(4\pi(x - \Delta t\,))$$

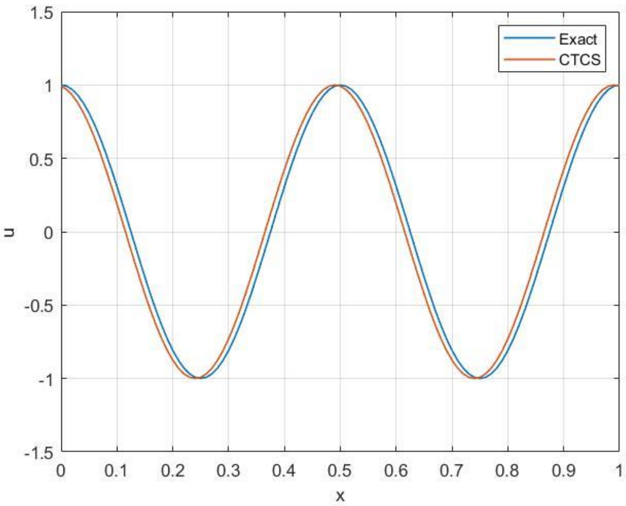
The computer code for solving this is given in the end, here we only discuss the solution.

Stability criteria: Using the Von-Neumann Stability analysis by taking $u_j^n \sim g^n e^{ij\theta}$ we get amplification factor as

$$g = -i\nu \sin\theta \pm \sqrt{1 - \nu^2 \sin^2\theta}$$

For stability $|g| \leq 1$, which gives stability criteria as

$$\nu < 1, \qquad |g| = 1$$



$$L_\infty = 0.1241$$

Here we observe that there is no dissipation, which was also apparent from our amplification factor being 1 as $\nu < 1$; but a slight phase shift from the exact solution is observed, indicating dispersion error due to complex part of g being inexact, because of the difference approximation employed in this scheme.

BTBS Scheme:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{u_j^{n+1} - u_{j-1}^{n+1}}{\Delta x} = 0$$

This can be rearranged as

$$(1+\nu)u_j^{n+1} - \nu u_{j-1}^{n+1} = u_j^n$$

This gives us the iterative update equation in time.

At $j = 0$

$$(1+\nu)u_0^{n+1} - \nu u_{-1}^{n+1} = u_0^n$$

Where using $u_{-1}^n = u_{N-1}^n$

$$(1+\nu)u_0^{n+1} - \nu u_{N-1}^{n+1} = u_0^n$$

Hence this can be written as linear system of equation as

$$\begin{bmatrix} (1+\nu) & 0 & 0 & & 0 & -\nu & 0 \\ -\nu & (1+\nu) & 0 & & & 0 & 0 \\ 0 & & \ddots & & & & 0 \\ & \cdots & -\nu & (1+\nu) & 0 & \cdots & \\ 0 & & & & \ddots & & 0 \\ 0 & 0 & & & -\nu & (1+\nu) & 0 \\ 0 & 0 & 0 & & & -\nu & (1+\nu) \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_i \\ \vdots \\ \vdots \\ u_{N-1} \end{bmatrix}^{n+1} = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_i \\ \vdots \\ \vdots \\ u_{N-1} \end{bmatrix}^{n}$$

$$\mathbf{A}\mathbf{u}^{n+1} = \mathbf{u}^n$$

$$\therefore \mathbf{u}^{n+1} = \mathbf{A}^{-1}\mathbf{u}^n$$

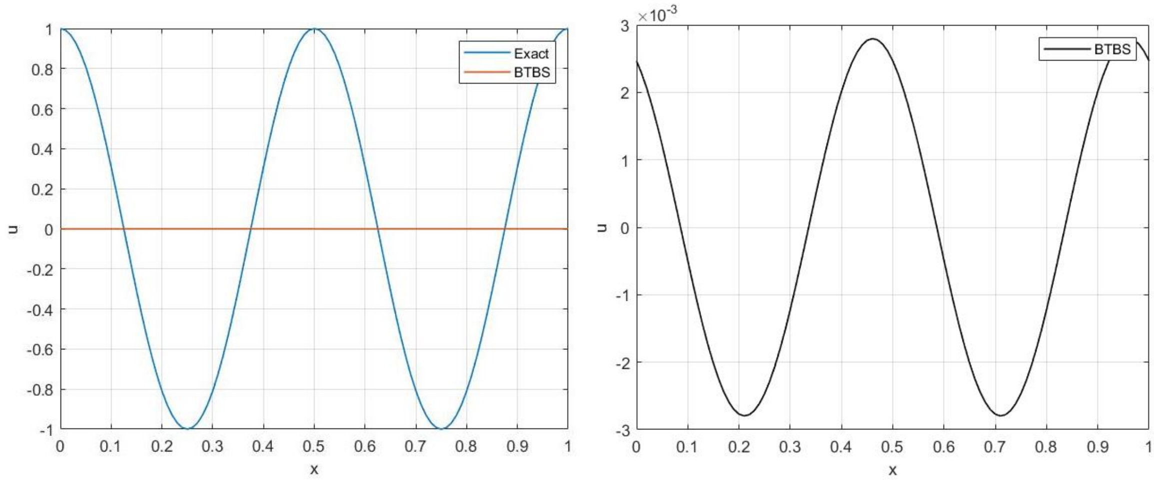
This can be solved iteratively from time $t = 0 \; to \; 5$, considering initial boundary conditions.

The computer code for solving this is given in the end, here we only discuss the solution.

Stability criteria: Using the Von-Neumann Stability analysis by taking $u_j^n \sim g^n e^{ij\theta}$ we get amplification factor as

$$g = \frac{1}{1 + \nu - \nu e^{-i\theta}} = \frac{1}{1 + \nu(1 - \cos\theta) + i\nu \sin\theta}$$

Here $|g| \leq 1$ always, which makes it unconditionally stable.



$$L_\infty = 0.9975 \equiv 1$$

Here we observe that there is a very large dissipation observed in the solution, even though the stability criteria is fulfilled. Possible reason being amplification factor $\ll 1$.

This is a first order accurate scheme $\sim O(\Delta x)$ & $O(\Delta t)$. If we consider these terms in the scheme and neglect instead even higher order terms as we did in FTBS,

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = u_t + \frac{\Delta t}{2!} u_{tt} + \frac{\Delta t^2}{3!} u_{ttt} \dots$$

$$\frac{u_j^{n+1} - u_{j-1}^{n+1}}{\Delta x} = u_x - \frac{\Delta x}{2!} u_{xx} + \frac{\Delta x^2}{3!} u_{xxx} \dots + \frac{2\Delta t}{2!} u_{xt} - \frac{3\Delta x \Delta t}{3!} u_{xxt} + \frac{3\Delta t^2}{3!} u_{xtt} \dots$$

Substituting in the BTBS scheme equation we get

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + c \frac{u_j^{n+1} - u_{j-1}^{n+1}}{\Delta x} = 0$$

$$u_t + u_x + \frac{\Delta t}{2} u_{tt} - \frac{\Delta x}{2} u_{xx} + \Delta t u_{xt} + O(\Delta x^2, \Delta t^2, \Delta x \Delta t) = 0$$

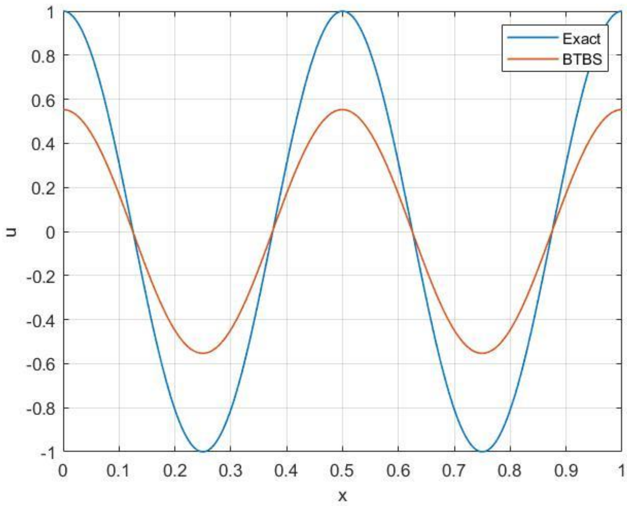
Differentiating with respect to x and t and eliminating u_{tt} and u_{xt} we get the modified equation of the form

$$u_t + c u_x = \frac{c\Delta x}{2}(1 + \nu) u_{xx} = \mu_h u_{xx}$$

This is an equivalent advection diffusion equation which we are solving.

Here $\mu_h = 7.5 \times 10^{-3}$, however this apparent diffusion coefficient is very small, but it is considerably affecting the solution, causing large dissipation errors.

It is expected that even finer meshing might reduce the dissipation error. For same CFL number $\boldsymbol{\nu = 0.5}$, taking $\Delta x = 10^{-3}$ and $\Delta t = 5 \times 10^{-4}$, we get the below solution; clearly dissipation error has been reduced. $L_\infty = 0.4468$



<u>Crank Nicholson Scheme:</u>

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{u_{j+1}^{n+1} - u_{j-1}^{n+1} + u_{j+1}^n - u_{j-1}^n}{4\Delta x} = 0$$

This can be rearranged as

$$-\frac{\nu}{4}u_{j-1}^{n+1} + u_j^{n+1} + \frac{\nu}{4}u_{j+1}^{n+1} = \frac{\nu}{4}u_{j-1}^n + u_j^n - \frac{\nu}{4}u_{j+1}^n$$

This gives us the iterative update equation in time.

At $j = 0$

$$-\frac{\nu}{4}u_{-1}^{n+1} + u_0^{n+1} + \frac{\nu}{4}u_1^{n+1} = \frac{\nu}{4}u_{-1}^n + u_0^n - \frac{\nu}{4}u_1^n$$

Where using $u_{-1}^n = u_{N-1}^n$ and $u_{-1}^{n+1} = u_{N-1}^{n+1}$

$$-\frac{\nu}{4}u_{N-1}^{n+1} + u_0^{n+1} + \frac{\nu}{4}u_1^{n+1} = \frac{\nu}{4}u_{N-1}^n + u_0^n - \frac{\nu}{4}u_1^n$$

At $j = N$

$$-\frac{\nu}{4}u_{N-1}^{n+1} + u_N^{n+1} + \frac{\nu}{4}u_{N+1}^{n+1} = \frac{\nu}{4}u_{N-1}^n + u_N^n - \frac{\nu}{4}u_{N+1}^n$$

Where using $u_{N+1}^n = u_1^n$ and $u_{N+1}^{n+1} = u_1^{n+1}$

$$-\frac{\nu}{4}u_{N-1}^{n+1} + u_N^{n+1} + \frac{\nu}{4}u_1^{n+1} = \frac{\nu}{4}u_{N-1}^n + u_N^n - \frac{\nu}{4}u_1^n$$

Hence this can be written as linear system of equation as

$$\begin{bmatrix} 1 & \nu/4 & 0 & & 0 & -\nu/4 & 0 \\ -\nu/4 & 1 & \nu/4 & & & 0 & 0 \\ 0 & & \ddots & & & & 0 \\ & & -\nu/4 & 1 & \nu/4 & & \\ 0 & & & & \ddots & & 0 \\ 0 & 0 & & & -\nu/4 & 1 & \nu/4 \\ 0 & \nu/4 & 0 & & & -\nu/4 & 1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_i \\ \vdots \\ \vdots \\ u_N \end{bmatrix}^{n+1} = \begin{bmatrix} 1 & -\nu/4 & 0 & & 0 & \nu/4 & 0 \\ \nu/4 & 1 & -\nu/4 & & & 0 & 0 \\ 0 & & \ddots & & & & 0 \\ & & \nu/4 & 1 & -\nu/4 & & \\ 0 & & & & \ddots & & 0 \\ 0 & 0 & & & \nu/4 & 1 & -\nu/4 \\ 0 & -\nu/4 & 0 & & & \nu/4 & 1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_i \\ \vdots \\ \vdots \\ u_N \end{bmatrix}^{n}$$

$$\mathbf{B}\mathbf{u}^{n+1} = \mathbf{C}\mathbf{u}^n$$

$$\therefore \mathbf{u}^{n+1} = \mathbf{B}^{-1}\mathbf{C}\mathbf{u}^n$$

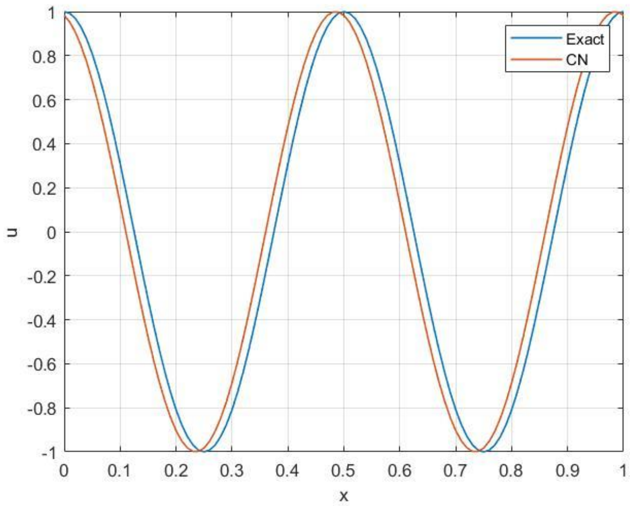
This can be solved iteratively from time $t = 0\ to\ 5$, considering initial boundary condition.

The computer code for solving this is given in the end, here we only discuss the solution.

Stability criteria: Using the Von-Neumann Stability analysis by taking $u_j^n \sim g^n e^{ij\theta}$ we get amplification factor as

$$g = \frac{1 - i\dfrac{\nu}{2}\sin\theta}{1 + i\dfrac{\nu}{2}\sin\theta}$$

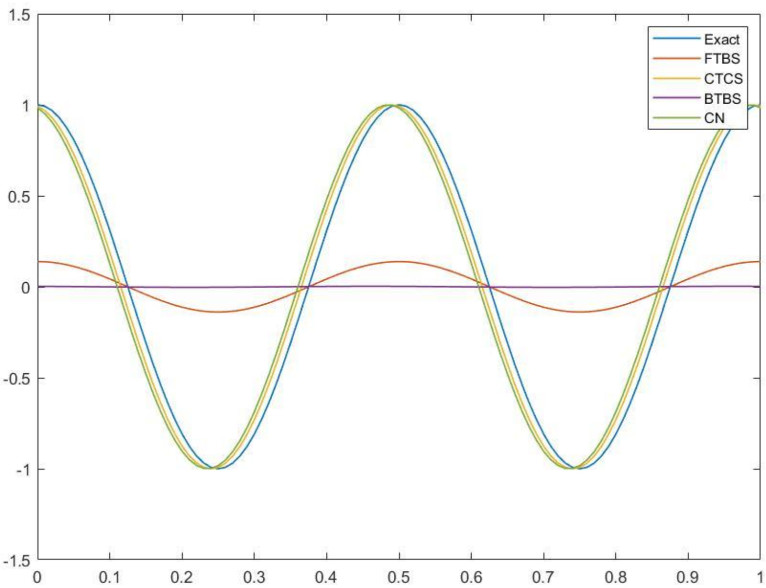
Here $|g| = 1$ always, which makes it unconditionally stable and non-dissipative scheme.



$$L_\infty = 0.1854$$

Here we observe that there is no dissipation, which was also apparent from our amplification factor being 1; but a slight phase shift from the exact solution is observed, indicating dispersion error due to complex part of g being inexact (when compared with the g_{exact} determined by Fourier analysis), because of the difference approximation employed in this scheme.

Below is the numerical solution computed using the four methods and the exact solution at t = 5 in a single plot for $CFL = 0.5$.



$CFL = 0.5$	L_∞
FTBS	0.8613
CTCS	0.1241
BTBS	0.9975
CN	0.1854

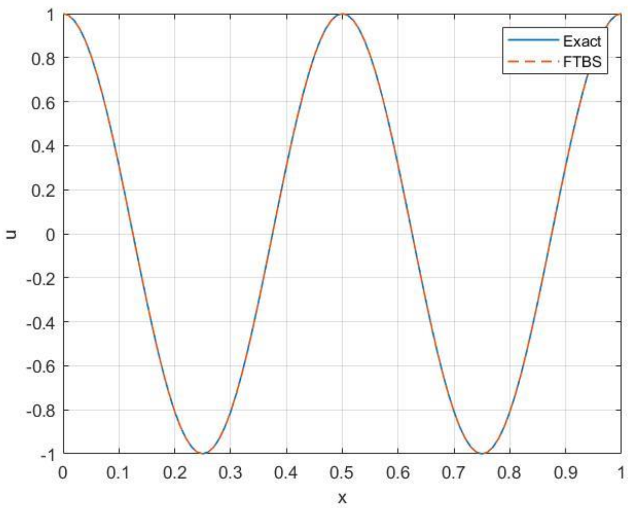
(b)

For CFL number $\boldsymbol{\nu} = \mathbf{1}$

$$\Delta t = \nu \Delta x = 10^{-2}$$

We'll integrate from time $t = 0 \; to \; T = \; 5$ therefore, number of iterations $N_t = \frac{T}{\Delta t} = 500$

<u>FTBS Scheme:</u>



$$L_\infty = 9.1 \times 10^{-15} \equiv 0$$

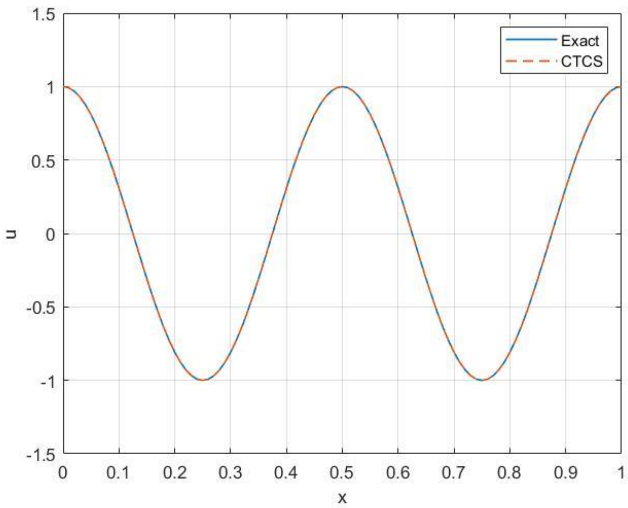
Here we observe that the solution agrees well with the exact solution.

From the modified equation which we discussed earlier

$$u_t + cu_x = \frac{c\Delta x}{2}(1 - \nu)u_{xx} = \mu_h u_{xx}$$

Here $\mu_h = 0$, as $\nu = 1$, and hence no dissipation error observed, which as was seen earlier.

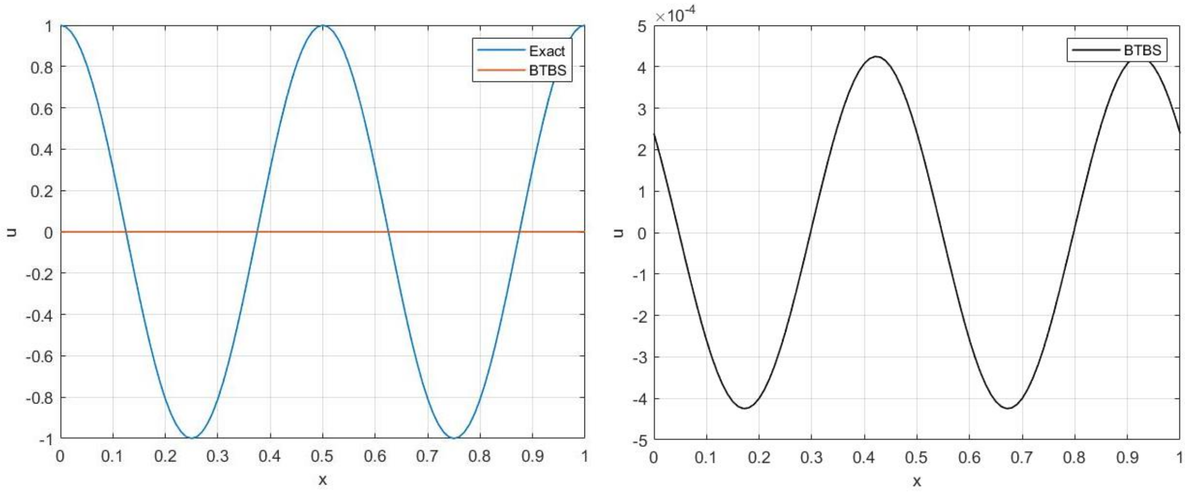
<u>Leap Frog (CTCS) Scheme:</u>



$$L_\infty = 3.77 \times 10^{-14} \equiv 0$$

Here we observe that the solution agrees well with the exact solution.

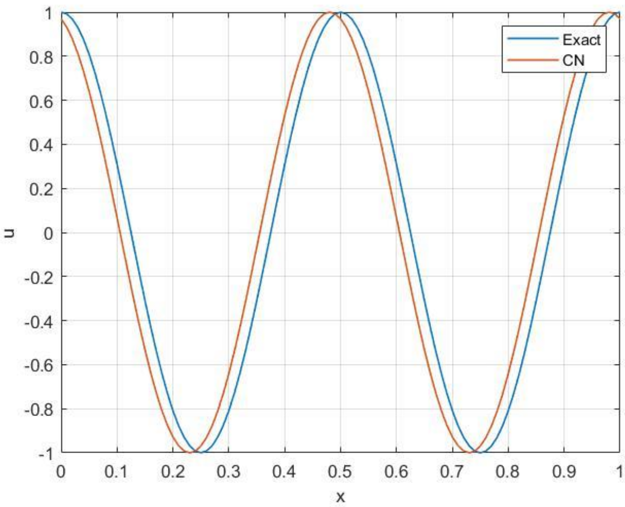
BTBS Scheme:



$$L_\infty = 0.9998 \equiv 1$$

Here we observe that there is a very large dissipation observed in the solution, even though the stability criteria are fulfilled. Possible reason being same as discussed earlier $\mu_h = \frac{c\Delta x}{2}(1+\nu) \neq 0$.

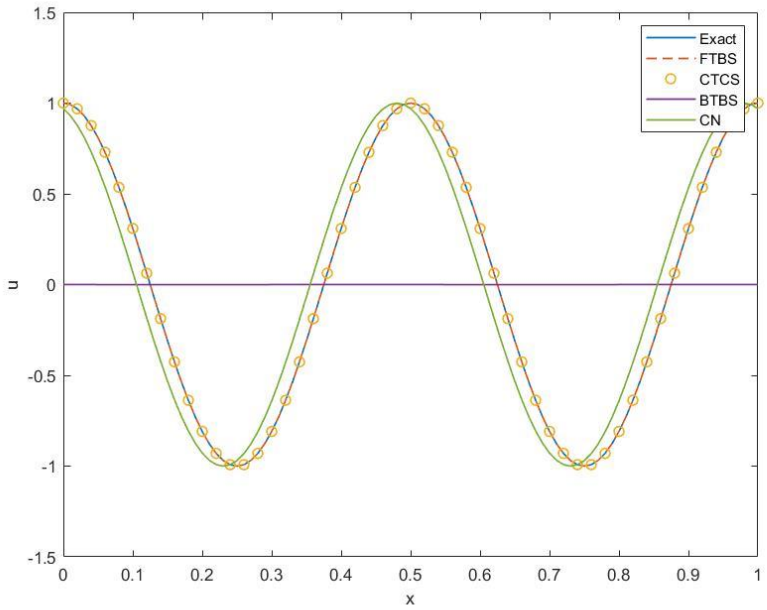
Crank Nicholson Scheme:



$$L_\infty = 0.2460$$

Here we observe that there is no dissipation, which was also apparent from our amplification factor being 1; but a slight phase shift from the exact solution is observed, indicating dispersion due to complex part of g being inexact (when compared with the g_{exact} determined by Fourier analysis) because of the difference approximation employed in this scheme.

Below is the numerical solution computed using the four methods and the exact solution at t = 5 in a single plot for $CFL = 1$.



$CFL = 1$	L_∞
FTBS	9.1×10^{-15}
CTCS	3.8×10^{-14}
BTBS	0.9998
CN	0.2460

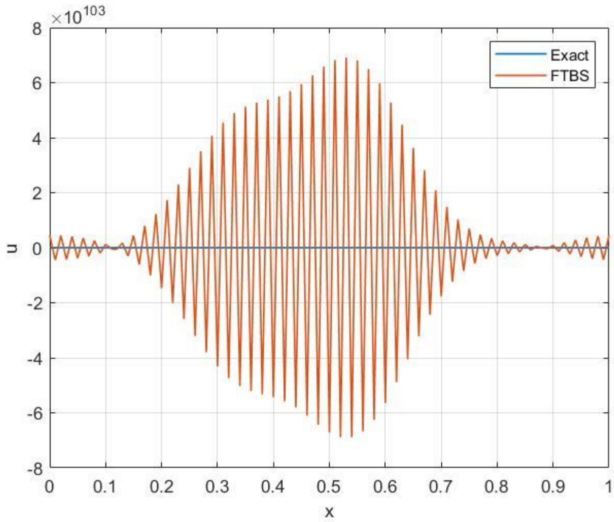
(c)

For CFL number $\boldsymbol{\nu} = \mathbf{2}$

$$\Delta t = \nu \Delta x = 2 \times 10^{-2}$$

We'll integrate from time $t = 0 \; to \; T = 5$ therefore, number of iterations $N_t = \frac{T}{\Delta t} = 250$

<u>FTBS Scheme:</u>



$$L_\infty = 6.91 \times 10^{103}$$

Here we observe that the solution blows up and clearly high frequency modes are highly amplified. Here, $\nu = 2 > 1$, which violates the stability criteria and hence the scheme becomes unstable.

Also, from the modified equation which we discussed earlier

$$u_t + cu_x = \frac{c\Delta x}{2}(1-\nu)u_{xx} = \mu_h u_{xx}$$

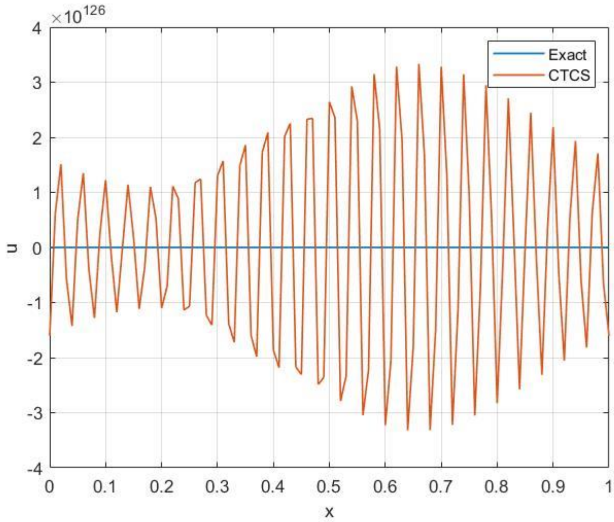
Here $\mu_h < 0$, as $\nu = 2$, and hence the problem becomes ill posed. If we take the Fourier transform of the modified equation, we get

$$\frac{d\hat{u}}{dt} + c(ik)\hat{u} = \mu_h (ik)^2 \hat{u}$$

$$\hat{u}(k,t) = \hat{u}(k,0)e^{-ikct}e^{-\mu_h k^2 t}$$

For the well posedness of the problem, clearly $\mu_h \geq 0$, which is violated for $\nu = 2$.

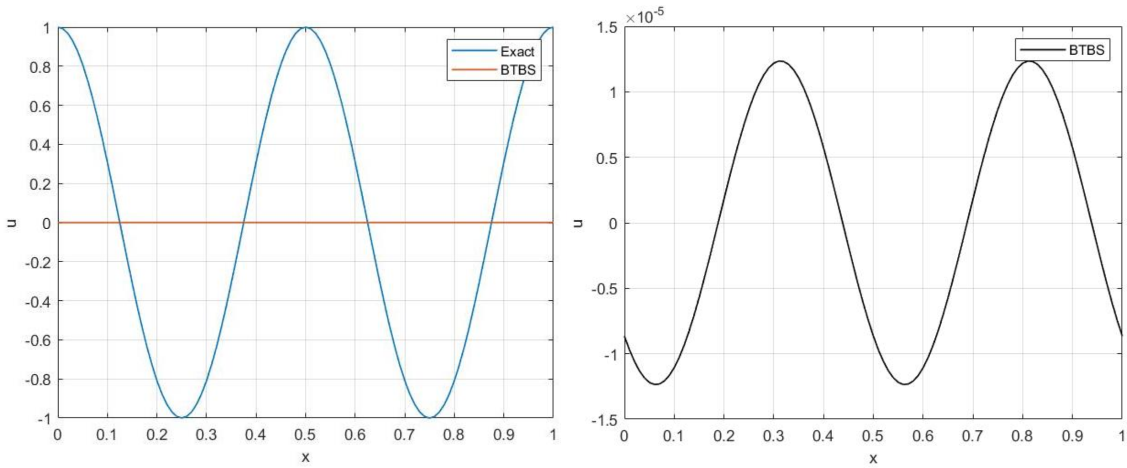
Leap Frog (CTCS) Scheme:



$$L_\infty = 3.3 \times 10^{126}$$

Here we observe that the solution blows up and clearly high frequency modes are highly amplified. Here, $\nu = 2 > 1$, which violates the stability criteria and hence the scheme becomes unstable.

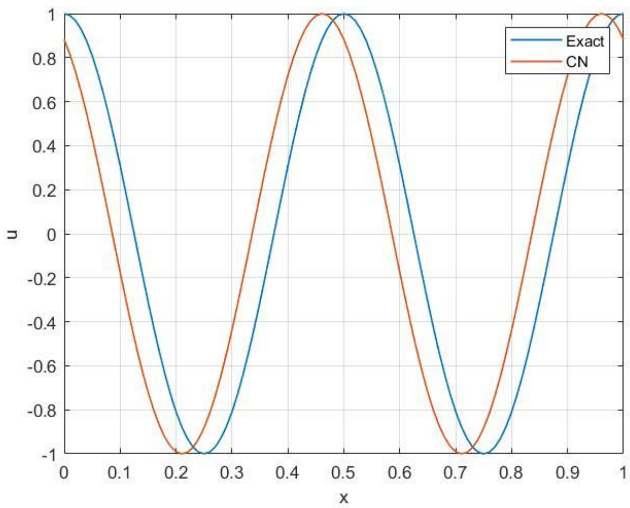
BTBS Scheme:



$$L_\infty \equiv 1$$

Here we observe that there is a very large dissipation observed in the solution, even though the stability criteria are fulfilled. Possible reason being same as discussed earlier $\mu_h = \frac{c\Delta x}{2}(1+\nu) \neq 0$.

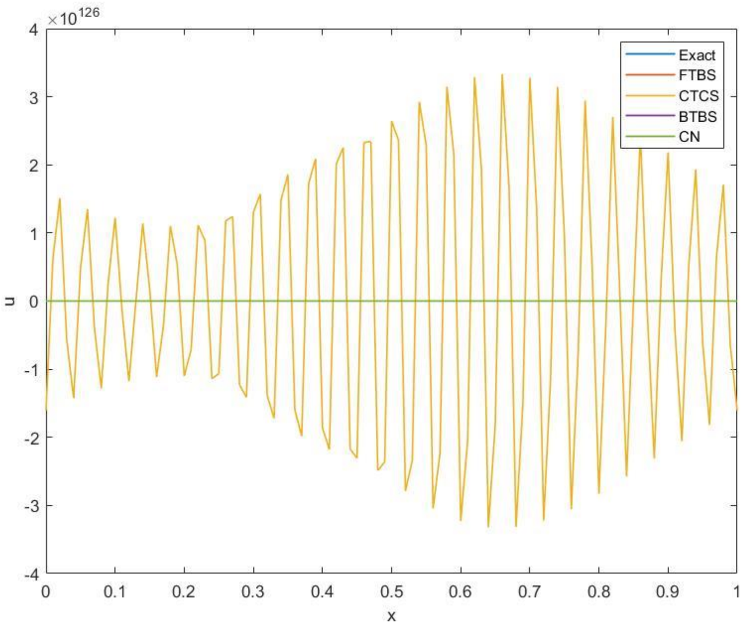
Crank Nicholson Scheme:



$$L_{\infty} = 0.4846$$

Here we observe that there is no dissipation, which was also apparent from our amplification factor being 1; but a slight phase shift from the exact solution is observed, indicating dispersion due to complex part of g being inexact (when compared with the g_{exact} determined by Fourier analysis) because of the difference approximation employed in this scheme.

Below is the numerical solution computed using the four methods and the exact solution at t = 5 in a single plot for $CFL = 2$.



$CFL = 2$	L_{∞}
FTBS	6.91×10^{103}
CTCS	3.33×10^{126}
BTBS	1
CN	0.4846

Computer Code:

```
nu = 0.5; %CFL
dx = 0.01;
dt= dx*nu;

X=1;
T=5;
N=X/dx;
n=T/dt;

%Exact solution
x=transpose(linspace(0,X,N+1));
ut0=cos(4*pi*x);
ut1=cos(4*pi*(x-dt));
uT=cos(4*pi*(x-T));

%FTBS
A1=zeros([N+1 N+1]);
for i=1:(N+1)
    A1(i,i)=1-nu;
    if (i+1)<=(N+1)
        A1(i+1,i)=nu;
    end
end
A1(1,N)=nu;

u1=ut0;
for i=1:n
    u1=A1*u1;
end
Uerr1=(u1-uT);
Li_FTBS = max(max(abs(Uerr1)));

%CTCS
A2=zeros([N+1 N+1]);
for i=1:(N+1)
    if (i-1)>0
        A2(i-1,i)=-nu;
    end
    if (i+1)<=(N+1)
        A2(i+1,i)=nu;
    end
end
A2(1,N)=nu;
A2(N+1,2)=-nu;

u2=ut1;
ui=ut0;
for i=2:n
    u2p=A2*u2+ui;
    ui=u2;
    u2=u2p;
end
Uerr2=(u2-uT);
Li_CTCS = max(max(abs(Uerr2)));

%BTBS
A3=zeros([N+1 N+1]);
for i=1:(N+1)
    A3(i,i)=1+nu;
    if (i+1)<=(N+1)
        A3(i+1,i)=-nu;
    end
end
A3(1,N)=-nu;

u3=ut0;
ubc =zeros([N+1 1]);
for i=1:n
    u3=A3\u3;
end
Uerr3=(u3-uT);
Li_BTBS = max(max(abs(Uerr3)));
```

```
%CN
A4=eye(N+1);
for i=1:(N+1)
    if (i-1)>0
        A4(i-1,i)=nu/4;
    end
    if (i+1)<=(N+1)
        A4(i+1,i)=-nu/4;
    end
end
A4L=A4;
A4L(1,N)=-nu/4;
A4L(N+1,2)=nu/4;

A4R=transpose(A4);
A4R(1,N)=nu/4;
A4R(N+1,2)=-nu/4;

u4=ut0;
for i=1:n
    u4=A4L\A4R*u4;
end
Uerr4=(u4-uT);
Li_CN = max(max(abs(Uerr4)));
```