SOLUTION(1)

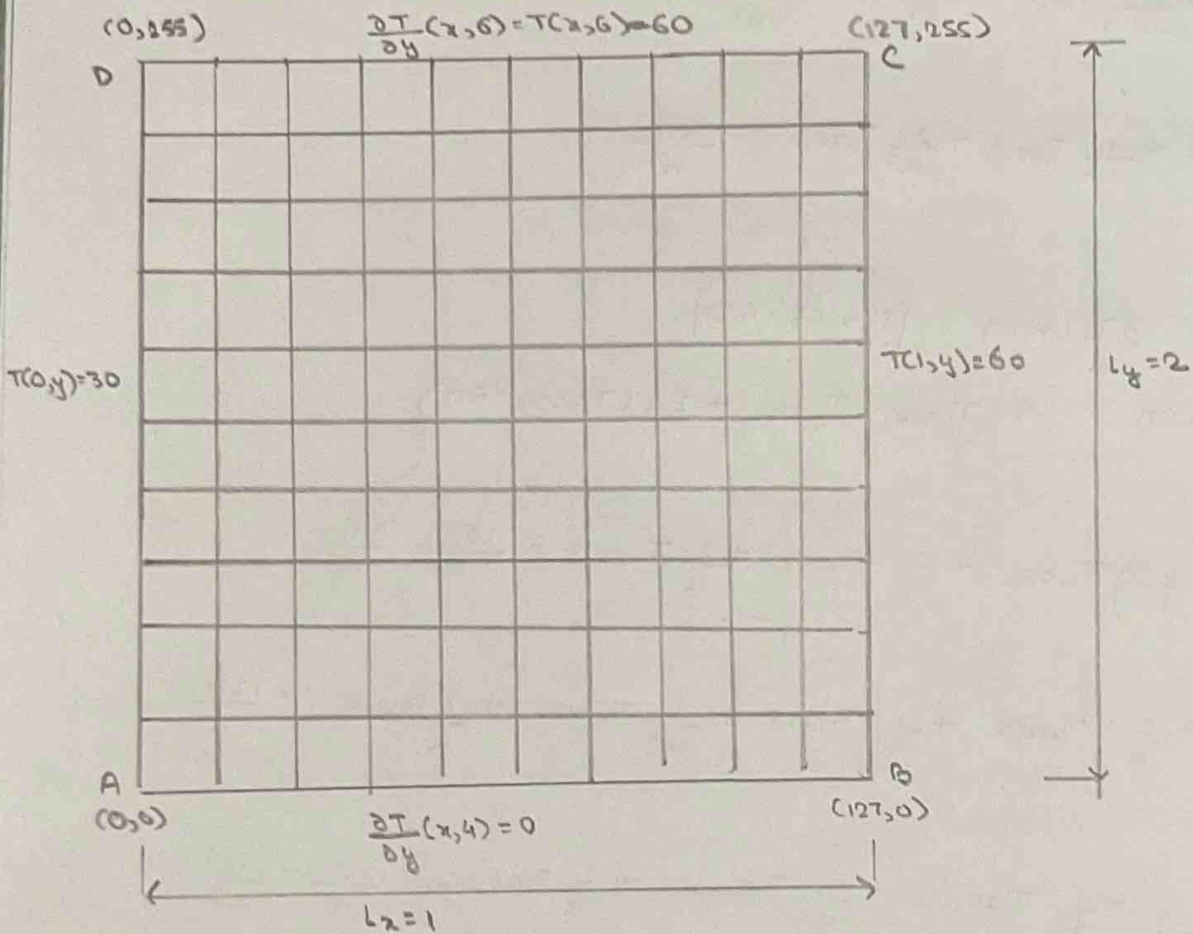
Heat conduction equation : $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$ .. (3)

Discreting the domain and approximating heat conduction equation using second order central difference Scheme

$$\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{h^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{k^2} = 0 \quad .. (2)$$

Boundary conditions :

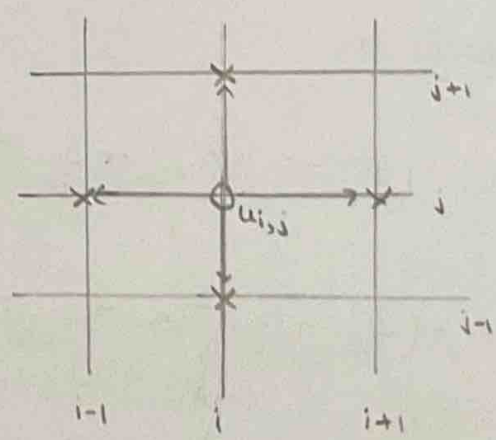
(i) $T(2,y) = 30$ (i) $T(3,y) = 60$ (i) $\frac{\partial T}{\partial y}(x,4) = 0$ (i) $\frac{\partial T}{\partial y}(x,6) = T(x,6) - 60$



Given : $N_x = 128$, $N_y = 256$

Then , $h = \Delta x = \frac{L_x}{N_x} = \frac{1}{128-1} = \frac{1}{127}$

$k = \Delta y = \frac{L_y}{N_y} = \frac{2}{256-1} = \frac{2}{255}$



b) Now, finding coefficient matrix

$$\frac{1}{h^2} T_{i+1,j} - \left(\frac{2}{h^2} + \frac{2}{k^2}\right) T_{i,j} + \frac{1}{h^2} T_{i-1,j} + \frac{1}{k^2} T_{i,j+1} + \frac{1}{k^2} T_{i,j-1} = 0 \quad \cdots (3)$$

Bottom surface AB:

BC: $\frac{\partial T}{\partial y}(x,4) = 0 \Rightarrow \left.\frac{\partial T}{\partial y}\right|_{(i,0)} = \frac{T_{i,1} - T_{i,-1}}{2k} = 0$

$\therefore T_{i,1} = T_{i,-1}$

Node (1,0):

$$\frac{1}{h^2} T_{2,0} - \left(\frac{2}{h^2} + \frac{2}{k^2}\right) T_{1,0} + \frac{2}{k^2} T_{1,1} = \frac{-30}{h^2}$$

Node (2,0):

$$\frac{1}{h^2} T_{3,0} - \left(\frac{2}{h^2} + \frac{2}{k^2}\right) T_{2,0} + \frac{1}{h^2} T_{1,0} + \frac{2}{k^2} T_{2,1} = 0$$

Node (126,0):

$$-\left(\frac{2}{h^2} + \frac{2}{k^2}\right) T_{126,0} + \frac{1}{h^2} T_{125,0} + \frac{2}{k^2} T_{126,1} = -\frac{60}{h^2}$$

Upper surface CD:

Boundary condition: $\frac{\partial T}{\partial y}(x,6) = [T(x,6) - 60]$

$\left.\frac{\partial T}{\partial y}\right|_{i,255} = T_{i,255} - 60 \Rightarrow T_{i,256} = T_{i,254} + 2k(T_{i,255} - 60)$

Node (1,255):

$$\frac{1}{h^2} T_{2,255} - \left(\frac{2}{h^2} + \frac{2}{k^2}\right) T_{1,255} + \frac{2}{k^2} T_{1,254} + \frac{2}{k} T_{1,255} = \frac{120}{k} - \frac{30}{h^2}$$

Node (2,255):

$$\frac{1}{h^2} T_{3,255} - \left(\frac{2}{h^2} + \frac{2}{k^2}\right) T_{2,255} + \frac{1}{h^2} T_{1,255} + \frac{2}{k^2} T_{2,254} + \frac{2}{k} T_{2,255} = \frac{120}{k}$$

Node (126, 255):

$$-\left(\frac{2}{h^2} + \frac{2}{k^2}\right) T_{126,255} + \frac{1}{h^2} T_{125,255} + \frac{2}{k^2} T_{126,254} + \frac{2}{k} T_{126,255} = \frac{120}{k} - \frac{60}{h^2}$$

Interior points

Node (1,1)

$$\frac{1}{h^2} T_{2,1} - \left(\frac{2}{h^2} + \frac{2}{k^2}\right) T_{1,1} + \frac{1}{k^2} T_{1,2} + \frac{1}{k^2} T_{1,0} = -\frac{30}{h^2}$$

Node (1,2):

$$\frac{1}{h^2} T_{2,1} - \left(\frac{2}{h^2} + \frac{2}{k^2}\right) T_{1,2} + \frac{1}{k^2} T_{1,3} + \frac{1}{k^2} T_{1,1} = -\frac{30}{h^2}$$

Node (2,2):

$$\frac{1}{h^2} T_{3,2} - \left(\frac{2}{h^2} + \frac{2}{k^2}\right) T_{2,2} + \frac{1}{h^2} T_{1,2} + \frac{1}{k^2} T_{2,3} + \frac{1}{k^2} T_{2,1} = 0$$

Node (2,1):

$$\frac{1}{h^2}T_{3,1} - \left(\frac{2}{h^2}+\frac{2}{k^2}\right)T_{2,1} + \frac{1}{h^2}T_{1,1} + \frac{1}{k^2}T_{2,2} + \frac{1}{k^2}T_{2,0} = 0$$

Using all the equations we got and arranging them in coefficient matrix

$$\begin{bmatrix}
\begin{bmatrix} a & \frac{1}{h^2} & & 0 \\ \frac{1}{h^2} & a & \ddots & \\ & \ddots & a & \frac{1}{h^2} \\ 0 & & \frac{1}{h^2} & a \end{bmatrix} & \begin{bmatrix} \frac{2}{k^2} & & 0 \\ & \ddots & \\ 0 & & \frac{2}{k^2} \end{bmatrix} & & 0 \\
\begin{bmatrix} \frac{2}{k^2} & & 0 \\ & \ddots & \\ 0 & & \frac{2}{k^2} \end{bmatrix} & \begin{bmatrix} a & \frac{1}{h^2} & & 0 \\ \frac{1}{h^2} & \ddots & \ddots & \\ & \ddots & \ddots & \frac{1}{h^2} \\ 0 & & \frac{1}{h^2} & a \end{bmatrix} & \begin{bmatrix} \frac{2}{k^2} & & 0 \\ & \ddots & \\ 0 & & \frac{2}{k^2} \end{bmatrix} & \\
& \ddots & \ddots & \begin{bmatrix} \frac{2}{k^2} & & 0 \\ & \ddots & \\ 0 & & \frac{2}{k^2} \end{bmatrix} \\
0 & & \begin{bmatrix} \frac{2}{k^2} & & 0 \\ & \ddots & \\ 0 & & \frac{2}{k^2} \end{bmatrix} & \begin{bmatrix} b & \frac{1}{h^2} & & 0 \\ \frac{1}{h^2} & \ddots & \ddots & \\ & \ddots & \ddots & \frac{1}{h^2} \\ 0 & & \frac{1}{h^2} & b \end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
\begin{pmatrix} T_{1,0} \\ T_{2,0} \\ \vdots \\ T_{125,0} \\ T_{126,0} \end{pmatrix} \\
\begin{pmatrix} T_{1,1} \\ T_{2,1} \\ \vdots \\ T_{125,1} \\ T_{126,1} \end{pmatrix} \\
\vdots \\
\begin{pmatrix} T_{1,255} \\ T_{2,255} \\ \vdots \\ T_{125,255} \\ T_{126,255} \end{pmatrix}
\end{bmatrix}
=
\begin{bmatrix}
\begin{pmatrix} \frac{-30}{h^2} \\ 0 \\ \vdots \\ 0 \\ \frac{-60}{h^2} \end{pmatrix} \\
\begin{pmatrix} \frac{-30}{h^2} \\ 0 \\ \vdots \\ \frac{-60}{h^2} \end{pmatrix} \\
\vdots \\
\begin{pmatrix} \frac{120}{k}-\frac{30}{h^2} \\ \frac{120}{k} \\ \vdots \\ \frac{120}{k} \\ \frac{120}{k}-\frac{60}{h^2} \end{pmatrix}
\end{bmatrix}$$

where, $a = -\left(\frac{2}{h^2}+\frac{2}{k^2}\right)$, $b = -\left(\frac{2}{h^2}+\frac{2}{k^2}\right)+\frac{2}{k}$

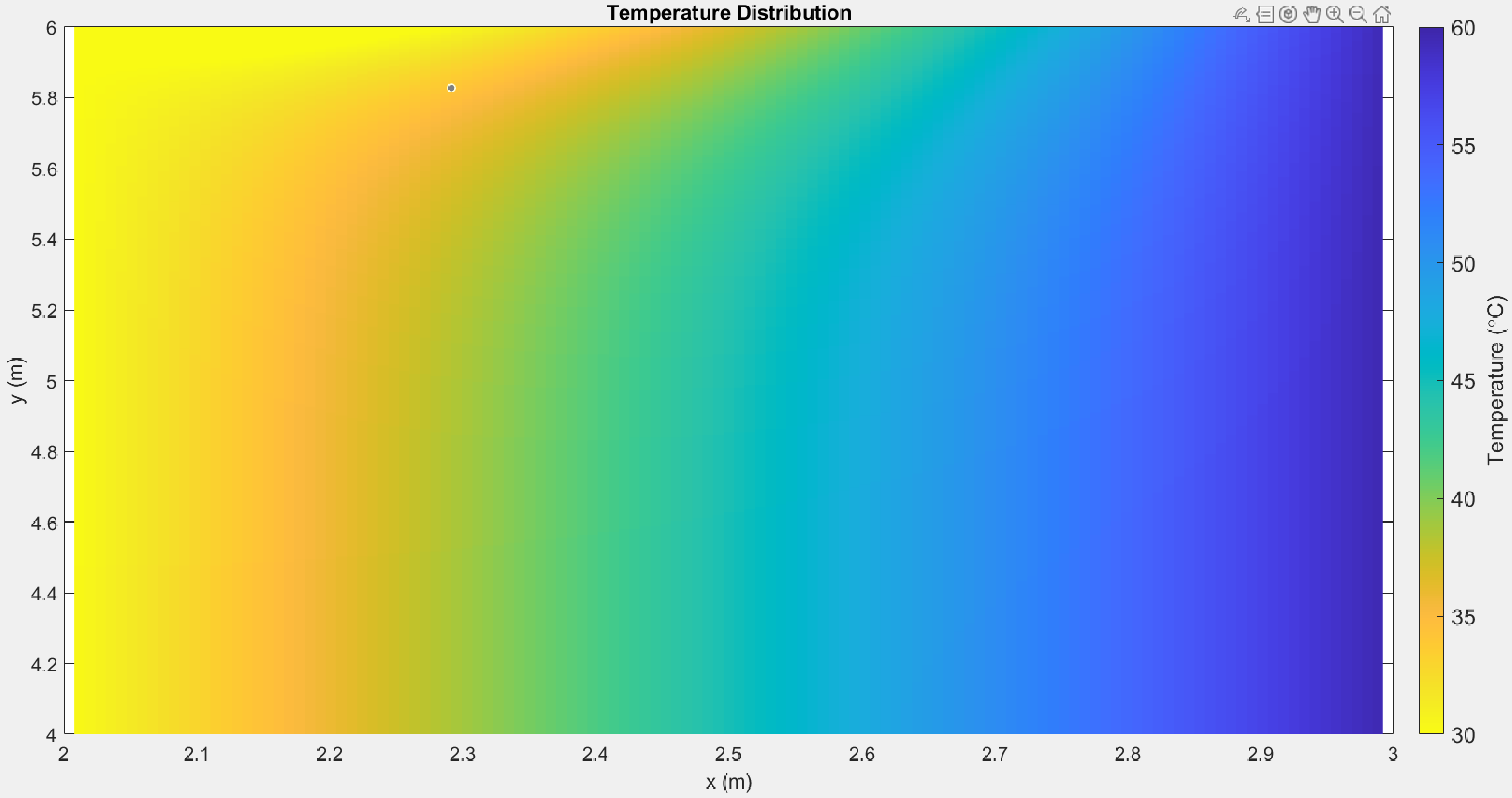
Above matrix can be written in terms of Block Tridiagonal matrix:

$$\begin{bmatrix}
[A] & [B] & F & & 0 \\
[B] & [A] & [B] & & \\
& [B] & [A] & \ddots & \\
& & \ddots & \ddots & \\
& & [B] & [A] & [B] \\
0 & & & [B] & [C]
\end{bmatrix}
\begin{bmatrix} \bar{T}_0 \\ \bar{T}_1 \\ \bar{T}_2 \\ \vdots \\ \bar{T}_{254} \\ \bar{T}_{255} \end{bmatrix}
=
\begin{bmatrix} \bar{F}_0 \\ \bar{F}_1 \\ \bar{F}_2 \\ \vdots \\ \bar{F}_{254} \\ \bar{F}_{255} \end{bmatrix}$$

where, $[A] = \begin{bmatrix} a & \frac{1}{h^2} & & 0 \\ \frac{1}{h^2} & a & \ddots & \\ & \ddots & \ddots & \frac{1}{h^2} \\ 0 & & \frac{1}{h^2} & a \end{bmatrix}$

$[B] = \begin{bmatrix} \frac{2}{k^2} & & 0 \\ & \frac{2}{k^2} & \\ 0 & & \ddots & \\ & & & \frac{2}{k^2} \end{bmatrix}$

$[C] = \begin{bmatrix} b & \frac{1}{h^2} & & 0 \\ \frac{1}{h^2} & b & \ddots & \\ & \ddots & \ddots & \frac{1}{h^2} \\ 0 & & \frac{1}{h^2} & b \end{bmatrix}$



Temperature Distribution
6
5.8
5.6
5.4
5.2
5
4.8
4.6
4.4
4.2
4
y (m)
2
2.1
2.2
2.3
2.4
2.5
2.6
2.7
2.8
2.9
3
x (m)
60
55
50
45
40
35
30
Temperature (°C)

- SAURABH

SOLUTION (2)

Crank Nicolson - θ scheme :

$u_t = \alpha u_{xx} \Rightarrow (u_t)_j^m = \alpha\left[(1-\theta)(u_{xx})_i^n + \theta (u_{xx})_i^n\right]$ where, $0 \leq \theta \leq 1$

$$\Rightarrow \frac{u_j^{m+1}-u_j^n}{\Delta t} = \alpha(1-\theta)\left[\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2}\right] + \alpha\theta\left[\frac{u_{j+1}^{n+1} - 2u_j^{m+1} + u_{j-1}^{n+1}}{(\Delta x)^2}\right] \quad \cdots (1)$$

Let $r = \frac{\alpha \Delta t}{(\Delta x)^2}$ $\Rightarrow$ r is always positive

Using Von Neumann's stability analysis : $u_{j,m} = V^m e^{i\beta p h}$

where, β = wave no.

$h = \Delta x$

Substituting Ansatz in Equation (1) :

$$V^{n+1} e^{i\beta p h} - V^n e^{i\beta p h} = r(1-\theta)\left[V^n e^{i\beta(p+1)h} - 2V^n e^{i\beta p h} + V^n e^{i\beta(p-1)h}\right]$$
$$+ r\theta\left[V^{n+1} e^{i\beta(p+1)h} - 2V^{n+1} e^{i\beta p h} + V^{n+1} e^{i\beta(p-1)h}\right]$$

$$\Rightarrow V - 1 = r(1-\theta)\left[e^{i\beta h} - 2 + e^{-i\beta h}\right] + r\theta\left[V e^{i\beta h} - 2V + V e^{-i\beta h}\right]$$

$$\Rightarrow V = 1 + r\left[V\theta\,(e^{i\beta h} + e^{-i\beta h} - 2) + (1-\theta)(e^{i\beta h} + e^{-i\beta h} - 2)\right]$$

$$\Rightarrow V = 1 + r\,(e^{i\beta h} + e^{-i\beta h} - 2)(V\theta + 1 - \theta)$$

$$\Rightarrow V = \frac{1 - 4r(1-\theta)\sin^2\left(\frac{\beta h}{2}\right)}{1 + 4r\theta \sin^2\left(\frac{\beta h}{2}\right)}$$

Stability condition : $|V| \leq 1$ $\Rightarrow -1 \leq V \leq 1$

Solving Right side inequality :

$$\frac{1 - 4r(1-\theta)\sin^2\left(\frac{\beta h}{2}\right)}{1 + 4r\theta \sin^2\left(\frac{\beta h}{2}\right)} \leq 1$$

The above inequality is always satisfied as terms are cancelling.
NOTE: j is denoted as p in the Von Neumann stability method.

This shows right side inequality is always satisfied.

Left ~~[illegible]~~ side inequality :

$$-1 \leq \frac{1 - 4r(1-\theta)\sin^2\left(\frac{\beta h}{2}\right)}{1 + 4r\theta \sin^2\left(\frac{\beta h}{2}\right)}$$

$$\Rightarrow 2r\sin^2\left(\frac{\beta h}{2}\right)(1-2\theta) \leq 1$$

$$\Rightarrow r(1-2\theta) \leq \frac{1}{2\sin^2\left(\frac{\beta h}{2}\right)}$$

$r(1-2\theta) \leq \dfrac{1}{2 \times 1}$ {$\because r(1-2\theta)$ should be less than minimum of RHS also}

$\Rightarrow r(1-2\theta) \leq \dfrac{1}{2}$ .. (2)

CASE 1 for $0 \leq \theta \leq \dfrac{1}{2}$

Then stability condition : $r(1-2\theta) \leq \dfrac{1}{2}$ (Conditionally stable)

CASE 2 $\dfrac{1}{2} \leq \theta \leq 1$

Gives : $r \geq 0$ (which is always true)

This shows 2 level scheme with $\theta \geq 0.5$ is unconditionally stable for time derivative and second order central difference for space derivative.

- SAURABH

SOLUTION (3)

Diffusion equation : $u_t = \alpha u_{xx}$

Finite difference approximation of given PDE using Explicit Euler and central order central difference in space:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha\left[\frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{(\Delta x)^2}\right] \quad \cdot\cdot (1)$$

Let $r_d = \dfrac{\alpha \Delta t}{(\Delta x)^2}$

Then, $\dfrac{u_i^{n+1} - u_i^n}{\Delta t} = r_d\left[u_{i-1}^n - 2u_i^n + u_{i+1}^n\right] \Rightarrow u_i^{n+1} = u_i^n + r_d\left[u_{i+1}^n - 2u_i^n + u_{i-1}^n\right] \quad \cdot\cdot (2)$

Taylor expanding the terms about u_i^n :

$$u_i^{n+1} = u_i^n + k\frac{\partial u}{\partial t}\bigg|_{(x_i,t_n)} + \frac{k^2}{2}\frac{\partial^2 u}{\partial t^2}\bigg|_{(x_i,t_n)} + \frac{k^3}{6}\frac{\partial^3 u}{\partial t^3}\bigg|_{(x_i,t_n)} + \dots$$

$$u_{i+1}^n = u_i^n + h\frac{\partial u}{\partial x}\bigg|_{(x_i,t_n)} + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2}\bigg|_{(x_i,t_n)} + \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3}\bigg|_{(x_i,t_n)} + \frac{h^4}{24}\frac{\partial^4 u}{\partial x^4}\bigg|_{(x_i,t_n)} + \dots$$

$$u_{i-1}^n = u_i^n - h\frac{\partial u}{\partial x}\bigg|_{(x_i,t_n)} + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2}\bigg|_{(x_i,t_n)} - \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3}\bigg|_{(x_i,t_n)} + \frac{h^4}{24}\frac{\partial^4 u}{\partial x^4}\bigg|_{(x_i,t_n)} - \dots$$

Putting values of u_i^{n+1}, u_{i+1}^n & u_{i-1}^n in equation (2):

$$u_i^n + \left[k\frac{\partial u}{\partial t} + \frac{k^2}{2}\frac{\partial^2 u}{\partial t^2} + \frac{k^3}{6}\frac{\partial^3 u}{\partial t^3} + \frac{k^4}{24}\frac{\partial^4 u}{\partial t^4} + \dots\right]_{(x_i,t_n)}$$

$$= u_i^n + r_d\left[u_i^n + h\frac{\partial u}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2} + \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3} + \frac{h^4}{24}\frac{\partial^4 u}{\partial x^4} + \dots - 2u_i^n + u_i^n - h\frac{\partial u}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2} - \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3} + \frac{h^4}{24}\frac{\partial^4 u}{\partial x^4} - \dots\right]_{(x_i,t_n)}$$

Putting $r_d = \dfrac{\alpha \Delta t}{(\Delta x)^2} = \dfrac{\alpha k}{h^2}$

$$\left[\frac{\partial u}{\partial t} + \frac{k}{2}\frac{\partial^2 u}{\partial t^2} + \frac{k^2}{6}\frac{\partial^3 u}{\partial t^3} + \dots\right]_{(x_i,t_n)} = \frac{\alpha}{h^2}\left[h^2\frac{\partial^2 u}{\partial x^2} + \frac{h^4}{12}\frac{\partial^4 u}{\partial x^4} + \frac{h^6}{360}\frac{\partial^6 u}{\partial x^6} + \dots\right]_{(x_i,t_n)}$$

$$(u_t)_i^n - \alpha(u_{xx})_i^n = \alpha\left[\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4} + \frac{h^4}{360}\frac{\partial^6 u}{\partial x^6} + O(h^6)\right]_{(x_i,t_n)} - \left[\frac{k}{2}\left(\frac{\partial^2 u}{\partial t^2}\right) + \frac{k^2}{6}\left(\frac{\partial^3 u}{\partial t^3}\right) + \dots\right]_{(x_i,t_n)}$$

$$(u_t)_i^n - \alpha(u_{xx})_i^n = -\frac{k}{2}\left(\frac{\partial^2 u}{\partial t^2}\right)\bigg|_{(x_i,t_n)} + O(k^2) + \frac{\alpha h^2}{12}\left(\frac{\partial^4 u}{\partial x^4}\right)\bigg|_{(x_i,t_n)} + O(h^4)$$

We know that, $u_t = \alpha u_{xx} \Rightarrow u_{tt} = \alpha u_{txx} = \alpha^2 u_{xxxx}$

$$\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2} \Rightarrow \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \alpha\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial x^2}\right) = \alpha^2\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 u}{\partial x^2}\right) = \alpha^2\frac{\partial^4 u}{\partial x^4}$$

$$(u_t)_i^n - \alpha(u_{xx})_i^n = -\frac{\alpha^2 k}{2}\left(\frac{\partial^4 u}{\partial x^4}\right)\bigg|_{(x_i,t_n)} + \frac{\alpha h^2}{12}\left(\frac{\partial^4 u}{\partial x^4}\right)\bigg|_{(x_i,t_n)} + O(k^2 + h^4)$$

$$(u_t)_i^n - \alpha(u_{xx})_i^n = \left(-\frac{\alpha^2 k}{2} + \frac{\alpha h^2}{12}\right)(u_{xxxx})_i^n + O(k^2 + h^4)$$

(MODIFIED EQUATION)

We know that, if leading order term has even derivative in space, then the nature of error is DISSIPATIVE.

SOLUTION (4)

Diffusion equation : $u_t = \alpha u_{xx}$, $\alpha = 1.5$

Initial condition : $u(x,0) = \sin(4x) + \sin(x)$

Boundary conditions (Periodic) :

$u(0,t) = u(2\pi,t)$ and $u_x(0,t) = u_x(2\pi,t)$

Using separation of variables : $u(x,t) = X(x)\,T(t)$

Plugging $u = XT$ into the heat equation :

$$X\frac{dT}{dt} = \alpha T\frac{d^2X}{dx^2} \Rightarrow \frac{dT}{\alpha T} = \frac{1}{X}\frac{d^2X}{dx^2} = -\lambda$$

$$\Rightarrow \frac{\dot{T}}{\alpha T} = \frac{\alpha X''}{X} = -\lambda$$

Solving for T : $\dfrac{\dot{T}}{\alpha T} = -\lambda \Rightarrow \int \frac{\dot{T}}{T}dt = \int -\lambda dt \Rightarrow \boxed{T(t) = Ae^{-\lambda t}}$

Solving second ODE :

$X'' + \dfrac{\lambda X}{\alpha} = 0 \Rightarrow X'' + \gamma^2 X = 0$ $\quad \{\text{Let } \gamma^2 = \frac{\lambda}{\alpha}\}$

Solution of ODE : $X(x) = C\cos(\gamma x) + D\sin(\gamma x)$

Using periodic boundary conditions :

$u(0,t) = u(2\pi,t) \Rightarrow X(0) = X(2\pi)$

$u_x(0,t) = u(2\pi,t) \Rightarrow X'(0) = X'(2\pi)$

$\rightarrow \gamma = n\pi$ where, $n = 0, \pm1, \pm2, \pm3, \ldots$

$X_n(x) = C_m\cos(n\pi x) + D_n\sin(n\pi x)$,

$u_n(x,t) = [C_n\cos(n\pi x) + D_n\sin(n\pi x)]Ae^{-\lambda_n t}$

Principal of superposition show that if $u_1, u_2, \ldots u_M$ are solution linear homogeneous problem, then any linear combination, $C_1u_2 + C_2u_2 + \ldots + C_Mu_M$ is also the solution.

$$u(x,t) = \sum_{n=0}^{\infty}[A_n\cos(\gamma_n x) + B_n\sin(\gamma_n x)]\,e^{-\lambda_n t}$$

Now using initial condition :

$$u(x,0) = \sin(4x) + \sin(x) = \sum_{n=-\infty}^{\infty} A_n\cos(\gamma_n x) + B_n\sin(\gamma_n x) \quad \ldots (2)$$

Now, we know the orthogonality of sines & cosines :

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 0 & , m \neq n \\ L/2 & , m = n \end{cases}$$

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = 0 \quad \text{always}$$

Multiplying equation (2) with $\sin(4x)$ and integrate :

$$\int_0^{2\pi} \sin^2(4x)\,dx + \int_0^{2\pi} \sin x \sin 4x \,dx = \sum_{n=-\infty}^{\infty} \int_0^{2\pi} A_n \cos(nx) \sin 4x \,dx + \sum_{n=-\infty}^{\infty} \int_0^{2\pi} B_n \sin nx \sin 4x \,dx$$

Using orgonality of sines & cosines : Only term remaining on RHS is B_4 :

$$\int_0^{2\pi} \sin^2(4x)\,dx = 0 + B_4(\pi)$$

$$\Rightarrow \quad B_4 = 1$$

Similarly, multiplying equation with $\sin x$ and integrate :

$$\int_0^{2\pi} \sin 4x \sin x \,dx + \int_0^{2\pi} \sin^2 x \,dx = \sum_{n=-\infty}^{\infty} \int_0^{2\pi} A_n \cos(nx) \sin x \,dx + \sum_{n=-\infty}^{\infty} \int_0^{2\pi} B_n \sin(nx) \sin x \,dx$$

$$0 + \int_0^{2\pi} \sin^2 x \,dx = B_1 \int_0^{2\pi} \sin^2 x \,dx$$

$$\Rightarrow \quad B_1 = 1$$

Multiplying equation (2) with $\cos(nx)$ gives $A_n = 0$

$$\therefore \quad u(x,t) = \sum_{n=1,4} e^{-n^2 \alpha t} \sin(nx) = e^{-\alpha t} \sin x + e^{-16\alpha t} \sin(4x)$$

$$\boxed{u(x,t) = e^{-1.5t} \sin x + e^{-24t} \sin(4x)}$$ ⇒ ANALYTICAL SOLUTION

And discretized equation used in the code is same as we obtained in ~~eqn~~ solution (3) :

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha \left[\frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{(\Delta x)^2} \right]$$

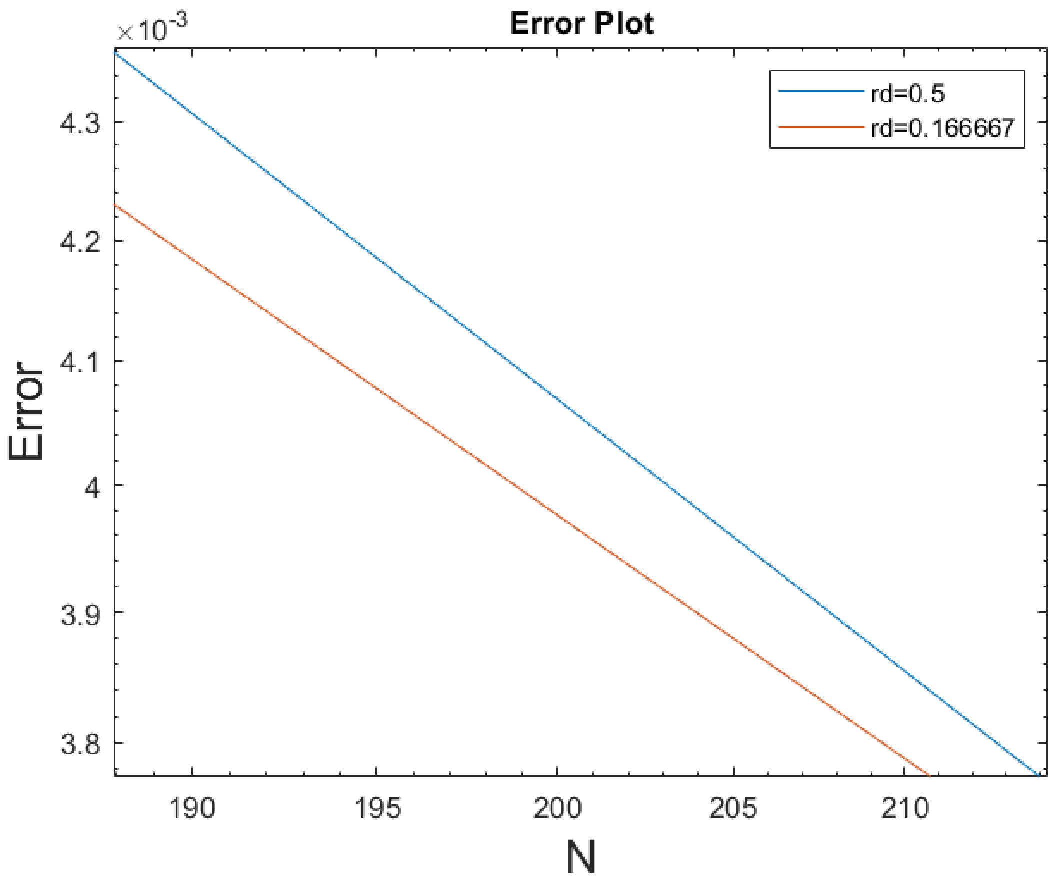
SOLUTION 4

(c)

	rd	N	E(N)
	Number	Number	Number
1	0.5	32	0.0309577
2	0.5	64	0.0118341
3	0.5	128	0.00667739
4	0.5	256	0.00309388
5	0.166667	32	0.0265206
6	0.166667	64	0.0122321
7	0.166667	128	0.00619624
8	0.166667	256	0.00311103

(d)

Logarithmic Scale: Plot N ns E(N)



The order of accuracy for rd =1/2 is 2 in space and 1 in time

The order of accuracy for rd=1/6 is 4 in space and 2 in time

(e)

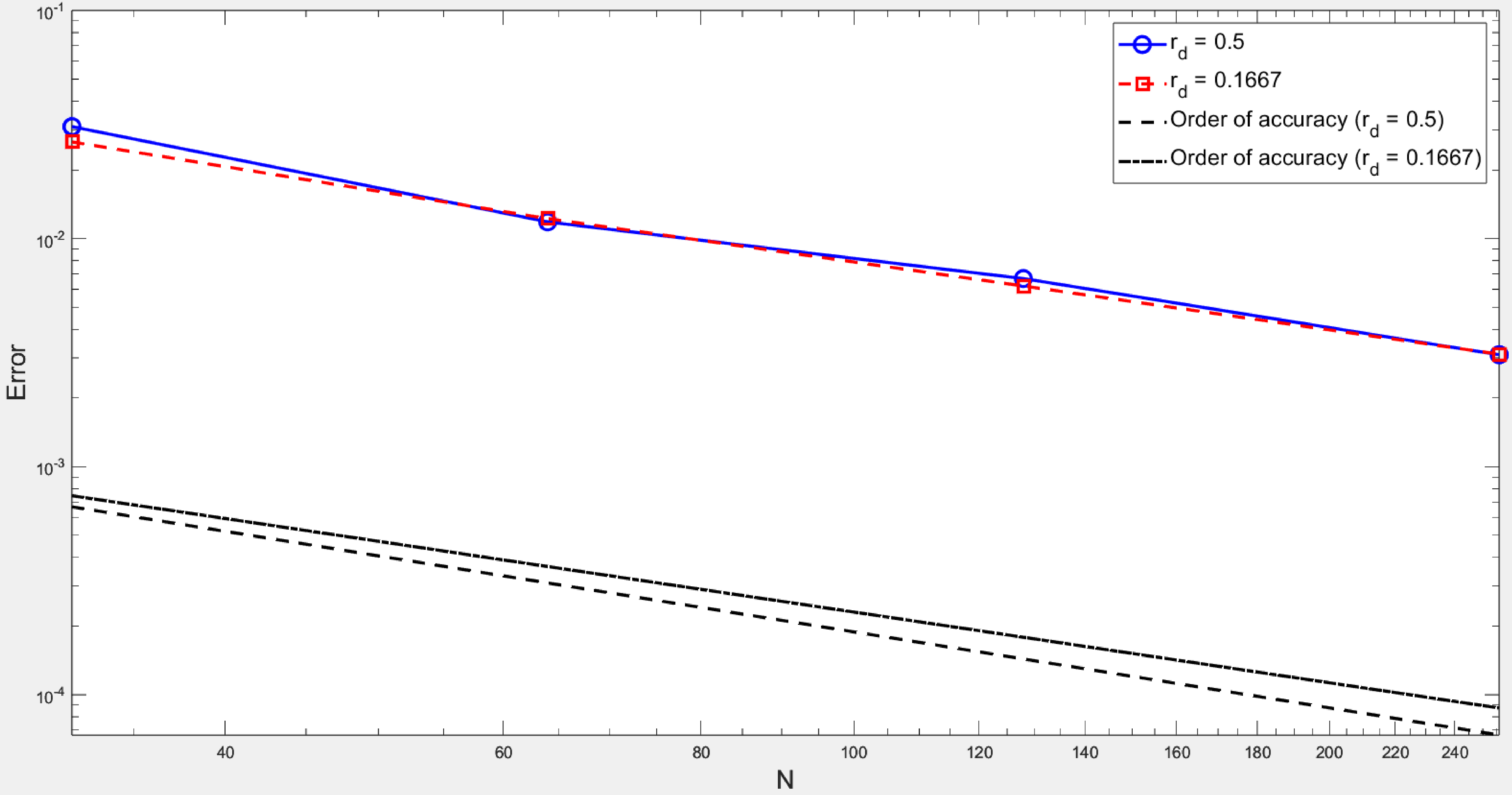
e) As we derived ~~order of accura~~ modified equation in solution (3):

$$(u_t)_i^n - \alpha (u_{xx})_i^n = \left(-\alpha^2 \frac{k}{2} + \frac{\alpha h^2}{12}\right) (u_{xxxx})_i^n \quad + O(k^2+h^4)$$

If $-\frac{\alpha^2 k}{2} + \frac{\alpha h^2}{12} = 0 \Rightarrow \frac{\alpha h^2}{12} = \frac{\alpha^2 k}{2} \Rightarrow \alpha k = \frac{h^2}{6} \Rightarrow \frac{\alpha k}{h^2} = \frac{1}{6}$

, i.e., $r_d = \frac{1}{6}$

Then, Order of accuracy is $O(k^2+h^4)$ $\Rightarrow$ Justifies abnormal order of accuracy observed in $r_d = 1/6$ case.



r_d = 0.5
r_d = 0.1667
Order of accuracy (r_d = 0.5)
Order of accuracy (r_d = 0.1667)
Error
N
10^-1
10^-2
10^-3
10^-4
40
60
80
100
120
140
160
180
200
220
240

SOLUTION (5)

Diffusion equation : $u_t = \alpha u_{xx}$

Using implicit Euler and second order central difference :

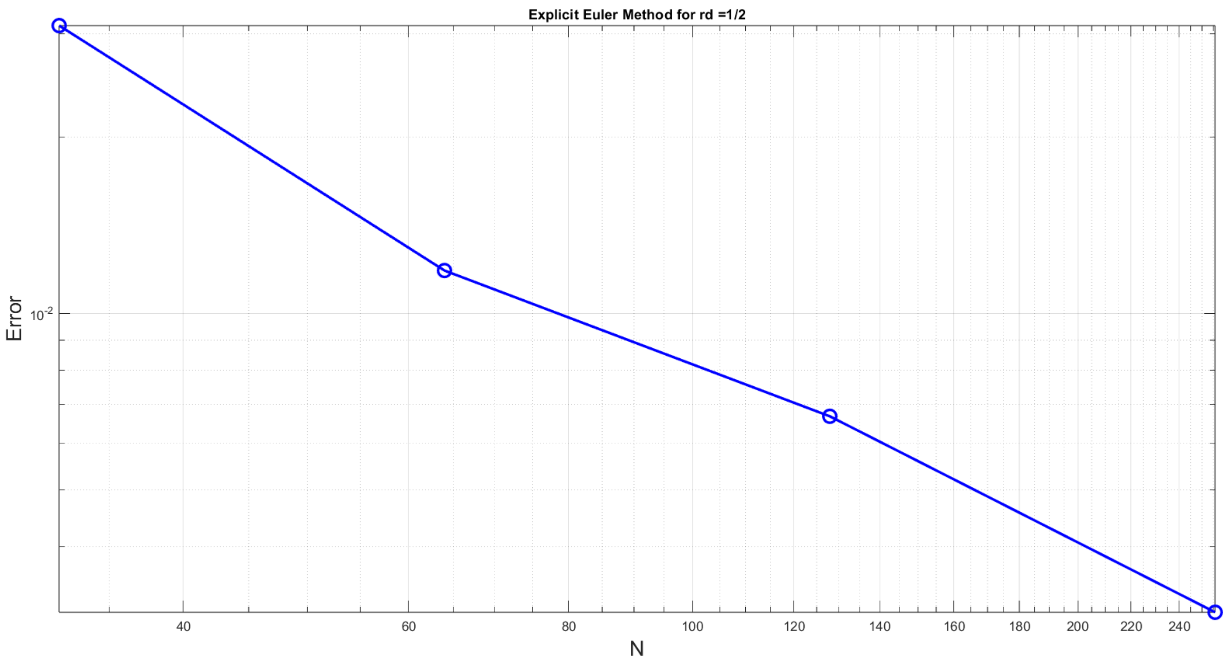
$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha \left[\frac{u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}}{(\Delta x)^2} \right]$$

Let $r_d = \frac{\alpha \Delta t}{(\Delta x)^2}$

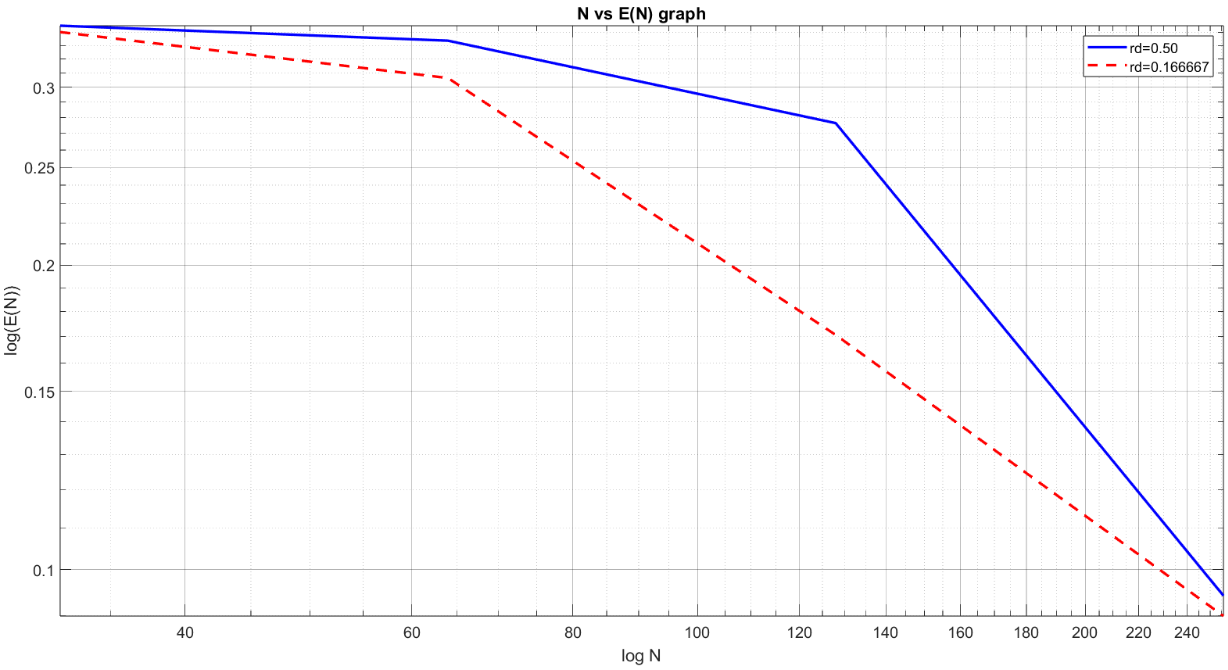
$$u_i^{n+1} - u_i^n = r_d \left[u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1} \right]$$ → Discretized equation

SOLUTION 5

Explicit Euler

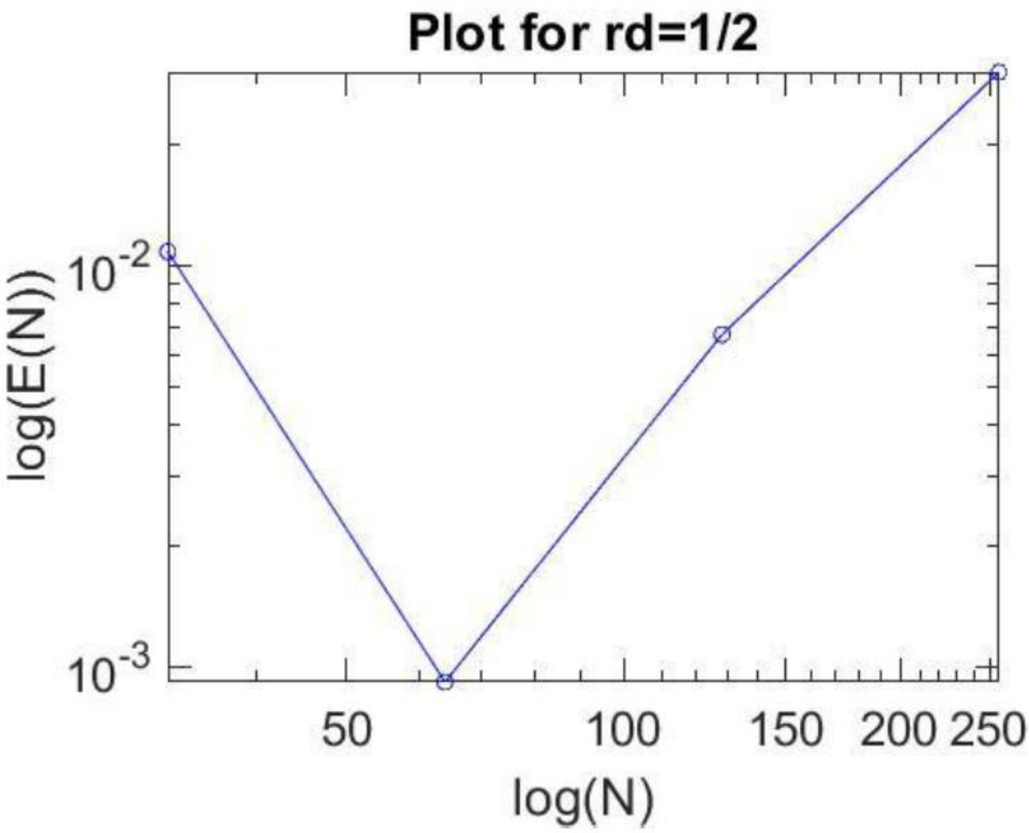


Implicit Euler



(b) ***Observations***

Plot for rd = 1/2



1) As we can see, the implicit method is unconditionally stable.

2) Explicit method is stable only when rd<0.25

3) Implicit method is better than the explicit one, as it requires fewer steps.

4) Implicit method is more accurate and stable

SOLUTION (6)

Transient 1D Heat Conduction : $\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$ .. (1)

Given discretized equation : $\frac{T_i^{n+1} - T_i^n}{\Delta t} = \alpha \left[\frac{T_{i+1}^{n-k} - 2T_i^n + T_{i-1}^n}{(\Delta x)^2} \right]$ .. (2)

Expanding terms by Taylor series :

Truncation error , $\tau_i^n = \cancel{\frac{\partial T}{\partial t}} - \cancel{\alpha \frac{\partial^2 T}{\partial x^2}} = \frac{T_i^{n+1} - T_i^n}{\Delta t}$ ~~malgr~~ $- \alpha \left[\frac{T_{i+1}^{n-k} - 2T_i^n + T_{i-1}^n}{(\Delta x)^2} \right]$ .. (3)

Solving LHS of equation (2):

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \frac{1}{\Delta t} \left[T_i^n + \Delta t \frac{\partial T}{\partial t} + \frac{(\Delta t)^2}{2!} \frac{\partial^2 T}{\partial t^2} + \frac{(\Delta t)^3}{3!} \frac{\partial^3 T}{\partial t^3} + \dots - T_i^n \right]_i^n$$

$$= \frac{\partial T}{\partial t} + \left[\frac{(\Delta t)}{2!} \frac{\partial^2 T}{\partial t^2} + \frac{(\Delta t)^2}{3!} \frac{\partial^3 T}{\partial t^3} \right]_i^n + O(\Delta t)^3$$

Solving: $T_{i+1}^{n-k} - 2T_i^n + T_{i-1}^n = \Big[T + (\Delta x) \frac{\partial T}{\partial x} - k\Delta t \frac{\partial T}{\partial t} + \frac{(\Delta x)^2}{2} \frac{\partial^2 T}{\partial x^2} + (\Delta x)(-k\Delta t) \frac{\partial^2 T}{\partial x \partial t}$

$+ \frac{(-k\Delta t)^2}{2} \frac{\partial^2 T}{\partial t^2} + \frac{(\Delta x)^3}{6} \frac{\partial^3 T}{\partial x^3} + \frac{(\Delta x)^2 (-k\Delta t)}{2} \frac{\partial^3 T}{\partial x^2 \partial t}$

$+ \frac{(\Delta x)(-k\Delta t)^2}{3} \frac{\partial^3 T}{\partial x \partial t^2} + \frac{(-k\Delta t)^3}{6} \frac{\partial^3 T}{\partial t^3} + \frac{(\Delta x)^4}{24} \frac{\partial^4 T}{\partial x^4} - 2T$

$+ T - \Delta x \frac{\partial T}{\partial x} + \frac{(\Delta x)^2}{2} \frac{\partial^2 T}{\partial x^2} - \frac{(\Delta x)^3}{6} \frac{\partial^3 T}{\partial x^3} + \frac{(\Delta x)^4}{24} \frac{\partial^4 T}{\partial x^4} + \dots \Big]_i^n$

$= \Big[-k\Delta t \frac{\partial T}{\partial t} + (\Delta x)^2 \frac{\partial^2 T}{\partial x^2} - k\Delta t \Delta x \frac{\partial^2 T}{\partial x \partial t} - \frac{(k\Delta t)^3}{2} \frac{\partial^2 T}{\partial t^2} - \frac{k\Delta t (\Delta x)^2}{2} \frac{\partial^3 T}{\partial x^2 \partial t} + \frac{k^2 \Delta x \Delta t^2}{2} \frac{\partial^3 T}{\partial x \partial t^2}$

$- \frac{k^3 \Delta t^3}{6} \frac{\partial^3 T}{\partial t^3} + \frac{\Delta x^4}{12} \frac{\partial^4 T}{\partial x^4} \Big]_i^n$

$$\tau_i^n = \frac{T_i^{n+1} - T_i^n}{\Delta t} = \alpha \left[\frac{T_{i+1}^{n-k} - 2T_i^n + T_{i-1}^n}{(\Delta x)^2} \right]$$

$= \underbrace{\frac{\partial T}{\partial t} - \alpha \frac{\partial^2 T}{\partial x^2}}_{=0} + \Big[\frac{\Delta t}{2} \frac{\partial^2 T}{\partial t^2} + \frac{(\Delta t)^2}{6} \frac{\partial^3 T}{\partial t^3} + \frac{\alpha k \Delta t}{(\Delta x)^2} \frac{\partial T}{\partial t} + \frac{\alpha k \Delta t}{\Delta x} \frac{\partial^2 T}{\partial x \partial t} + \frac{\alpha k^2 (\Delta t)^2}{2(\Delta x)^2} \frac{\partial^2 T}{\partial t^2}$

$+ \frac{\alpha k \Delta t}{2} \frac{\partial^3 T}{\partial x^2 \partial t} - \frac{\alpha k^2 (\Delta t)^2}{2\Delta x} \frac{\partial^3 T}{\partial x \partial t^2} + \frac{\alpha k^3 (\Delta t)^3}{6(\Delta x)^2} \frac{\partial^3 T}{\partial t^3} + \dots \Big]_i^n$

$$\tau_i^n = O\left(\Delta t, (\Delta x)^2, \frac{k\Delta t}{\Delta x}, \frac{k\Delta t}{(\Delta x)^2} \right) = \tau(x,t)$$

A method is said to be consistent if $\tau(x,t) \to 0$ as $k, h \to 0$

But as the denominator contains , Δx, Δt. Then the expression will tend to infinity.

∴ Given finite difference approximation is NOT CONSISTENT.