ME 282, Jan-April 2023
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Assignment 1, Due Friday, Feb 17

1. Read sections from the following references on classification of PDEs

   - Ian N. Sneddon, "Elements of partial differential equations" Dover Publications, 2006 (Chapter - 3, Section 5, pp. 105 - 115).
   - D. A. Anderson, J. C. Tannehill, and R. H. Pletcher, "Computational Fluid Mechanics and Heat Transfer," Hemisphere Publishing Corporation, 1984 (Chapter - 3).

2. Obtain the canonical form for the following partial differential equations

$$\frac{\partial^2 u}{\partial x^2} + 3\frac{\partial^2 u}{\partial x \partial y} + 2\frac{\partial^2 u}{\partial y^2} = 0, \tag{1}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{2}$$

$$\frac{\partial^2 u}{\partial x^2} - 6\frac{\partial^2 u}{\partial x \partial y} + 9\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} - e^{xy} = 1, \tag{3}$$

3. Let A be the $(N-1) \times (N-1)$ matrix associated with a finite difference solution to the BVP:

$$u''(x) = -f(x), u(0) = u(1) = 0, \tag{4}$$

   constructed using uniform grid points in [0,1] and the standard three-point, second order, discretization to approximate u_{xx} (let $x_i = i/N, i = 0, 1, \ldots, N$).

   (a) Verify analytically that the $N-1$ eigenvectors of A are given by the discrete sine functions: i.e. vectors whose elements consist of $\sin(k\pi x_i)$ for a value of k in $[1, N-1]$. Specifically the jth element of λ^k is given by $\lambda^k_j = \sin(k\pi x_j)$

   (b) Give a formula for the eigenvalues of A.

   (c) Give the analytical solution to $A\mathbf{u} = \mathbf{f}$, when $\mathbf{f}$ is the vector obtained by evaluating $-\sin(\pi x)$ at the nodes $x_i = i/N, i = 1, 2, \ldots, N-1$. What do you observe when $N \to \infty$?

4. Extend the second-order finite difference and finite volume methods that we developed in the class for (4) to approximate the BVP

$$\frac{d}{dx}\left(\alpha(x)\frac{du(x)}{dx}\right) + \beta(x)u(x) = -f(x), \quad x \in (0,1) \tag{5}$$

$$u(0) = a, u(1) = b. \tag{6}$$

   Do the two methods lead to the same linear system of equations for the unknowns $(u_i, i = 1, 2, \ldots, N-1)$?

5. Using Taylor series expansion determine the unknown coefficients ($a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, e_1$, and e_2) and the order of accuracy for the following finite difference approximations for the first and second derivatives on a uniform grid with spacing h:

   (a) $a_1 u'_{i-1} + u'_i + b_1 u'_{i+1} = c_1 u_{i-1} + d_1 u_i + e_1 u_{i+1}$

   (b) $a_2 u''_{i-1} + u''_i + b_2 u''_{i+1} = c_2 u_{i-1} + d_2 u_i + e_2 u_{i+1}$

6. Consider the following fourth-order accurate finite difference approximation on a uniform grid:

$$u'_i = \frac{1}{12h}\left(u_{i-2} - 8u_{i-1} + 8u_{i+1} - u_{i+2}\right). \tag{7}$$

   (a) Derive this formula using lagrange polynomial interpolation.

   (b) Use the above formula to give an expression for the approximate first derivative of $u(x) = x^n$, where n is an integer, at $x = x_i$. Give an expression for error in this approximation $e = u'_{exact}(x_i) - u'_i$. What is the value of e for $n \leq 4$?