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SCAN ME



UNIT II: Multivariable Calculus – Partial Differentiation

Partial Differentiation:

Given a function of two variables, $f(x, y)$ the derivative with respect to x only (treating y as a constant) is called the partial derivative of f with respect to x

It is denoted by either $\frac{\partial f}{\partial x}$ or f_x

Similarly, the derivative of f with respect to y only (treating x as a constant) is called the partial derivative of f with respect to y . It is denoted by either $\frac{\partial f}{\partial y}$ or f_y

Differentiate f with respect to x twice. (i.e differentiate f with respect to x ; then differentiate the result with respect to x again.) $\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2}$ or f_{xx}

Differentiate f with respect to y twice. (i.e. differentiate f with respect to y ; then differentiate the result with respect to y again.) $\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2}$ or f_{yy}

Mixed partials derivative:

- First differentiate f with respect to x ; then differentiate the result with respect to y .

$$\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x} \text{ or } f_{xy}$$

- First differentiate f with respect to y ; then differentiate the result with respect to x .

$$\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y} \text{ or } f_{yx}$$

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Example 1: If $f(x, y) = 3x^2y + 5x - 2y^2 + 1$ find $f_x \ f_y \ f_{xx} \ f_{yy} \ f_{xy} \ f_{yx}$

Solution: Let $f(x, y) = 3x^2y + 5x - 2y^2 + 1$

f_x means differentiate f with respect to x (treating y as a constant)

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} (3x^2y + 5x - 2y^2 + 1)$$

$$f_x = \frac{\partial f}{\partial x} = 6xy + 5 - 0 + 0 = 6xy + 5$$

f_y means differentiate f with respect to y (treating x as a constant)

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (3x^2y + 5x - 2y^2 + 1)$$

$$f_y = \frac{\partial f}{\partial y} = 3x^2 + 0 - 4y + 0 = 3x^2 - 4y$$

$\frac{\partial^2 f}{\partial x^2}$ or f_{xx} Differentiate f with respect to x twice.

$$\frac{\partial^2 f}{\partial x^2} \text{ or } f_{xx} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} (6xy + 5) = 6y$$

$\frac{\partial^2 f}{\partial y^2}$ or f_{yy} Differentiate f with respect to y twice.

$$\frac{\partial^2 f}{\partial y^2} \text{ or } f_{yy} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial y} (3x^2 - 4y) = -4$$

First differentiate f with respect to x ; then differentiate the result with respect to y .

$$\frac{\partial^2 f}{\partial y \partial x} \text{ or } f_{xy} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial y} (6xy + 5) = 6x$$

First differentiate f with respect to y ; then differentiate the result with respect to x .

$$\frac{\partial^2 f}{\partial x \partial y} \text{ or } f_{yx} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial x} (3x^2 - 4y) = 6x$$

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Example 2: If $z = \tan(y + ax) + (y - ax)^{\frac{3}{2}}$ then find the value of $\frac{\partial^2 z}{\partial x^2} - a^2 \frac{\partial^2 z}{\partial y^2}$

Solution: Let $z = \tan(y + ax) + (y - ax)^{\frac{3}{2}}$

To find $\frac{\partial^2 z}{\partial x^2}$ Differentiate z with respect to x twice

$$z = \tan(y + ax) + (y - ax)^{\frac{3}{2}}$$

Differentiating z with respect to x partially treating y as constant

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} \left[\tan(y + ax) + (y - ax)^{\frac{3}{2}} \right] \because \frac{d}{dx} \tan x = \sec^2 x \frac{d}{dx} x^n = nx^{n-1}$$

$$\frac{\partial z}{\partial x} = \sec^2(y + ax) \frac{\partial}{\partial x}(y + ax) + \frac{3}{2}(y - ax)^{\frac{3}{2}-1} \frac{\partial}{\partial x}(y - ax)$$

$$\frac{\partial z}{\partial x} = \sec^2(y + ax)(0 + a) + \frac{3}{2}(y - ax)^{\frac{1}{2}}(0 - a)$$

$$\frac{\partial z}{\partial x} = a \sec^2(y + ax) - \frac{3a}{2}(y - ax)^{\frac{1}{2}} \dots \dots \dots (1)$$

Differentiating $\frac{\partial z}{\partial x}$ with respect to x partially treating y as constant

$$\frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial x} \left[a \sec^2(y + ax) - \frac{3a}{2}(y - ax)^{\frac{1}{2}} \right]$$

$$\frac{\partial^2 z}{\partial x^2} = 2 a \sec(y + ax) \frac{\partial}{\partial x} [\sec(y + ax)] - \frac{3a}{2} \frac{1}{2} (y - ax)^{\frac{1}{2}-1} \frac{\partial}{\partial x}(y - ax)$$

$$\begin{aligned} \frac{\partial^2 z}{\partial x^2} &= 2 a \sec(y + ax) \sec(y + ax) \tan(y + ax) \frac{\partial}{\partial x}(y + ax) \\ &\quad - \frac{3a}{4} (y - ax)^{-\frac{1}{2}}(-a) \end{aligned}$$

$$\frac{\partial^2 z}{\partial x^2} = 2 a \sec^2(y + ax) \tan(y + ax) (a) - \frac{3a}{4} (y - ax)^{-\frac{1}{2}}(-a)$$

$$\frac{\partial^2 z}{\partial x^2} = 2 a^2 \sec^2(y + ax) \tan(y + ax) + \frac{3a^2}{4} (y - ax)^{-\frac{1}{2}} \dots \dots \dots (A)$$

To find $\frac{\partial^2 z}{\partial y^2}$ Differentiate z with respect to y twice

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$$z = \tan(y + ax) + (y - ax)^{\frac{3}{2}}$$

Differentiating z with respect to y partially treating x as constant

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} \left[\tan(y + ax) + (y - ax)^{\frac{3}{2}} \right]$$

$$\frac{\partial z}{\partial y} = \sec^2(y + ax) \frac{\partial}{\partial y}(y + ax) + \frac{3}{2}(y - ax)^{\frac{3}{2}-1} \frac{\partial}{\partial y}(y - ax)$$

$$\frac{\partial z}{\partial y} = \sec^2(y + ax)(1 + 0) + \frac{3}{2}(y - ax)^{\frac{1}{2}}(1 - 0)$$

$$\frac{\partial z}{\partial y} = \sec^2(y + ax) + \frac{3}{2}(y - ax)^{\frac{1}{2}} \dots \dots \dots (2)$$

Differentiating $\frac{\partial z}{\partial y}$ with respect to y partially treating x as constant

$$\frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial y} \left[\sec^2(y + ax) + \frac{3}{2}(y - ax)^{\frac{1}{2}} \right]$$

$$\frac{\partial^2 z}{\partial y^2} = 2 \sec(y + ax) \frac{\partial}{\partial x} [\sec(y + ax)] + \frac{3}{2} (y - ax)^{\frac{1}{2}-1} \frac{\partial}{\partial x}(y - ax)$$

$$\frac{\partial^2 z}{\partial y^2} = 2 \sec(y + ax) \sec(y + ax) \tan(y + ax) \frac{\partial}{\partial x}(y + ax) + \frac{3}{4}(y - ax)^{-\frac{1}{2}}(1)$$

$$\frac{\partial^2 z}{\partial y^2} = 2 \sec^2(y + ax) \tan(y + ax) (a) + \frac{3a}{4}(y - ax)^{-\frac{1}{2}} \dots \dots \dots (3)$$

Multiplying by $-a^2$

$$-a^2 \frac{\partial^2 z}{\partial y^2} = -2 a^2 \sec^2(y + ax) \tan(y + ax) - \frac{3a^2}{4}(y - ax)^{-\frac{1}{2}} \dots \dots \dots (B)$$

Equation (A) + (B)

$$\frac{\partial^2 z}{\partial x^2} - a^2 \frac{\partial^2 z}{\partial y^2} = 2 a^2 \sec^2(y + ax) \tan(y + ax) + \frac{3a^2}{4}(y - ax)^{-\frac{1}{2}}$$

$$= -2 a^2 \sec^2(y + ax) \tan(y + ax) - \frac{3a^2}{4}(y - ax)^{-\frac{1}{2}}$$

$$\therefore \frac{\partial^2 z}{\partial x^2} - a^2 \frac{\partial^2 z}{\partial y^2} = 0$$

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Example 3: If $u = \log(x^3 + y^3 - x^2y - xy^2)$ then prove that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 u = -\frac{4}{(x+y)^2}$$

Solution: Let $u = \log(x^3 + y^3 - x^2y - xy^2)$

$$u = \log(x^2(x-y) - y^2(x-y))$$

$$u = \log[(x-y)(x^2 - y^2)]$$

$$u = \log[(x-y)(x-y)(x+y)]$$

$$\because (a^2 - b^2) = (a+b)(a-b)$$

$$u = \log[(x-y)^2(x+y)]$$

$$\because \log ab = \log a + \log b$$

$$u = 2 \log(x-y) + \log(x+y)$$

$$\because \log a^n = n \log a$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 u = \left(\frac{\partial^2}{\partial x^2} + 2 \frac{\partial^2}{\partial x \partial y} + \frac{\partial^2}{\partial y^2}\right) u = \frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} \dots \dots \text{(A)}$$

To find $\frac{\partial^2 u}{\partial x^2}$ means Differentiate u with respect to x twice

$$u = 2 \log(x-y) + \log(x+y)$$

Differentiating u with respect to x partially y is constant

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} [2 \log(x-y) + \log(x+y)] \quad \because \frac{d}{dx} \log x = \frac{1}{x}$$

$$\frac{\partial u}{\partial x} = \frac{2}{(x-y)} \frac{\partial}{\partial x} (x-y) + \frac{1}{(x+y)} \frac{\partial}{\partial x} (x+y)$$

$$\frac{\partial u}{\partial x} = \frac{2}{(x-y)} (1-0) + \frac{1}{(x+y)} (1+0)$$

$$\frac{\partial u}{\partial x} = \frac{2}{(x-y)} + \frac{1}{(x+y)} \dots \dots \dots \text{(1)}$$

Differentiating $\frac{\partial u}{\partial x}$ with respect to x partially

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial x} \left[\frac{2}{(x-y)} + \frac{1}{(x+y)} \right]$$

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$$\frac{\partial^2 u}{\partial x^2} = \left[\frac{-2}{(x-y)^2} \frac{\partial}{\partial x} (x-y) + \frac{-1}{(x+y)^2} \frac{\partial}{\partial x} (x+y) \right] \quad \because \frac{d}{dx} \left(\frac{1}{x} \right) = \frac{-1}{x^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \left[\frac{-2}{(x-y)^2} (1-0) + \frac{-1}{(x+y)^2} (1+0) \right]$$

To find $\frac{\partial^2 u}{\partial y^2}$ Differentiate u with respect to y twice

$$u = 2 \log(x - y) + \log(x + y)$$

Differentiating u with respect to y partially

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y} [2 \log(x-y) + \log(x+y)]$$

$$\frac{\partial u}{\partial y} = -\frac{2}{(x-y)} \frac{\partial}{\partial y}(x-y) + \frac{1}{(x+y)} \frac{\partial}{\partial y}(x+y)$$

$$\frac{\partial u}{\partial y} = -\frac{2}{(x-y)} (0-1) + \frac{1}{(x+y)} (0+1)$$

$$\frac{\partial u}{\partial y} = \frac{-2}{(x-y)} + \frac{1}{(x+y)} \dots \dots \dots \quad (3)$$

Differentiating $\frac{\partial u}{\partial y}$ with respect to y partially

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right) = \frac{\partial}{\partial} \left[\frac{-2}{(x-y)} + \frac{1}{(x+y)} \right]$$

$$\frac{\partial^2 u}{\partial y^2} = \left[\frac{-(-2)}{(x-y)^2} \frac{\partial}{\partial y}(x-y) + \frac{-1}{(x+y)^2} \frac{\partial}{\partial y}(x+y) \right]$$

$$\frac{\partial^2 u}{\partial y^2} = \left[\frac{2}{(x-y)^2} (0-1) + \frac{-1}{(x+y)^2} (0+1) \right]$$

$\frac{\partial^2 u}{\partial x \partial y}$ Differentiating u with respect to y partially then x

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$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial y} \right) = \frac{\partial}{\partial x} \left(\frac{-2}{(x-y)} + \frac{1}{(x+y)} \right) \quad from \ (3)$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{-(-2)}{(x-y)^2} \frac{\partial}{\partial x} (x-y) + \frac{-1}{(x+y)^2} \frac{\partial}{\partial x} (x+y)$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{2}{(x - y)^2} (1 - 0) + \frac{-1}{(x + y)^2} (1 + 0)$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{2}{(x-y)^2} - \frac{1}{(x+y)^2} \dots \dots \dots \quad (5)$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 u = \frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2}$$

From (2) (4) and (5)

$$\begin{aligned}
 \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 u &= -\frac{2}{(x-y)^2} - \frac{1}{(x+y)^2} + 2 \left[\frac{2}{(x-y)^2} - \frac{1}{(x+y)^2} \right] - \frac{2}{(x-y)^2} - \frac{1}{(x+y)^2} \\
 &= -\frac{2}{(x-y)^2} - \frac{1}{(x+y)^2} + \frac{4}{(x-y)^2} - \frac{2}{(x+y)^2} - \frac{2}{(x-y)^2} - \frac{1}{(x+y)^2} \\
 &= -\frac{4}{(x+y)^2}
 \end{aligned}$$



Type II $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

If $f(x, y)$ is Homogeneous Function then $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

Example 1: If $u = \log(x^2 + y^2)$ verify $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

Solution: Let $u = \log(x^2 + y^2)$

$\frac{\partial u}{\partial x}$ Differentiating u with respect to x partially y is constant

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} \log(x^2 + y^2)$$

$$\frac{\partial u}{\partial x} = \frac{1}{x^2 + y^2} \frac{\partial}{\partial x} (x^2 + y^2)$$

$$\therefore \frac{d}{dx} \log x = \frac{1}{x}$$

$$\frac{\partial u}{\partial x} = \frac{1}{x^2 + y^2} (2x + 0) = \frac{2x}{(x^2 + y^2)} \dots \dots \dots (1)$$

$\frac{\partial^2 u}{\partial y \partial x}$ Differentiating $\frac{\partial u}{\partial x}$ with respect to y partially x is constant

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial y} \left(\frac{2x}{x^2 + y^2} \right)$$

$$\frac{\partial^2 u}{\partial y \partial x} = 2x \frac{\partial}{\partial y} \left(\frac{1}{x^2 + y^2} \right) \quad \therefore \frac{d}{dx} \left(\frac{1}{x} \right) = -\frac{1}{x^2}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{2x (-1)}{(x^2 + y^2)^2} \frac{\partial}{\partial y} (x^2 + y^2)$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{-2x}{(x^2 + y^2)^2} (0 + 2y)$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{-4xy}{(x^2 + y^2)^2} \dots \dots \dots (A)$$

$$u = \log(x^2 + y^2)$$

$\frac{\partial u}{\partial y}$ Differentiating u with respect to y partially

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \log(x^2 + y^2)$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^2 + y^2} \frac{\partial}{\partial y} (x^2 + y^2)$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^2 + y^2} (0 + 2y) = \frac{2y}{(x^2 + y^2)} \dots \dots \dots \quad (2)$$

$\frac{\partial^2 u}{\partial x \partial y}$ Differentiating $\frac{\partial u}{\partial y}$ with respect to x partially constant y

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial y} \right) = \frac{\partial}{\partial x} \left(\frac{2y}{x^2 + y^2} \right) = 2y \frac{\partial}{\partial x} \left(\frac{1}{x^2 + y^2} \right)$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{2y(-1)}{(x^2 + y^2)^2} \frac{\partial}{\partial x} (x^2 + y^2)$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{-2y}{(x^2 + y^2)^2} (2x + 0)$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{-4xy}{(x^2 + y^2)^2} \dots \dots \dots (B)$$

From (A) and (B)

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

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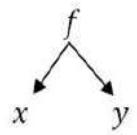


Variable to be treated as Constant:

Let f be function of x and y i.e. $f(x, y)$

To find partial derivative with respect to x we have to treat y as constant

$$\left(\frac{\partial f}{\partial x}\right)_y$$



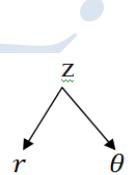
To find partial derivative with respect to y we have to treat x as constant

$$\left(\frac{\partial f}{\partial y}\right)_x$$

For Example:

$\left(\frac{\partial z}{\partial r}\right)_\theta$ Means z is function of r, θ z is differentiable w.r.t r, θ

here z is partially differentiated with respect to r and θ as constant.



$\left(\frac{\partial y}{\partial \theta}\right)_r$ Means y is function of r, θ y is differentiable w.r.t r, θ

here y is partially differentiated with respect to θ and r as constant.

Example 1: If $u = 2x + 3y$, $v = 3x - 2y$ Find $\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v \left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial v}{\partial y}\right)_u$

Solution:

$$u = 2x + 3y \dots \dots \dots (1)$$

$$v = 3x - 2y \dots \dots \dots (2)$$

u, v are functions of x, y variable i.e. $u, v = f(x, y)$

$\left(\frac{\partial u}{\partial x}\right)_y$ u is function of x, y variable i.e. $u = 2x + 3y$

Diff u partially w.r.t. x keep y as constant

$$\left(\frac{\partial u}{\partial x}\right)_y = 2 \dots \dots \dots \dots \dots (A)$$

$\left(\frac{\partial x}{\partial u}\right)_v$ x is function of u, v variable

$$u = 2x + 3y \quad v = 3x - 2y \quad \text{Eliminate } y$$

$$u = 2x + 3y \quad 2y = 3x - v$$

$$u = 2x + 3y \quad y = \frac{3x - v}{2}$$

$$u = 2x + 3\left(\frac{3x - v}{2}\right)$$

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$$2u = 4x + 3(3x - v)$$

$$2u = 4x + 9x - 3v$$

$$2u + 3v = 13x$$

$$x = \frac{2u + 3v}{13}$$

$\left(\frac{\partial x}{\partial u}\right)_v$ Diff partially w. r. t. u keep v as constant

$$\left(\frac{\partial x}{\partial u}\right)_v = \frac{2}{13} \quad \dots \dots \dots (B)$$

$\left(\frac{\partial y}{\partial v}\right)_x$ y is function of v, x variable

$$\text{From (2)} \quad v = 3x - 2y \quad \Rightarrow 2y = 3x - v \quad \Rightarrow y = \frac{3x - v}{2}$$

Diff partially w. r. t. v keep x as constant

$$\left(\frac{\partial y}{\partial v}\right)_x = \frac{-1}{2} \quad \dots \dots \dots (C)$$

$\left(\frac{\partial v}{\partial y}\right)_u$ v is function of y, u variable

$$u = 2x + 3y \quad v = 3x - 2y \quad \text{Eliminate } x$$

$$u = 2x + 3y \quad v + 2y = 3x$$

$$u = 2x + 3y \quad x = \frac{v + 2y}{3}$$

$$u = 2\left(\frac{v + 2y}{3}\right) + 3y$$

$$3u = 2v + 4y + 9y$$

$$3u = 2v + 13y$$

$$2v = 3u - 13y \quad v = \frac{3u - 13y}{2}$$

Diff partially w. r. t. y keep u as constant

$$\left(\frac{\partial v}{\partial y}\right)_u = -\frac{13}{2} \quad \dots \dots \dots (D)$$

From (A), (B), (C) and (D)

$$\therefore \left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v \left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial v}{\partial y}\right)_u = 2\left(\frac{2}{13}\right)\left(-\frac{1}{2}\right)\left(-\frac{13}{2}\right) = 1$$

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Example 2: If $x = \frac{\cos \theta}{r}$, $y = \frac{\sin \theta}{r}$ Find $\left(\frac{\partial x}{\partial r}\right)_\theta \left(\frac{\partial r}{\partial x}\right)_y + \left(\frac{\partial y}{\partial r}\right)_\theta \left(\frac{\partial r}{\partial y}\right)_x$

Solution:

$$x = \frac{\cos \theta}{r} \quad \dots \dots \dots (1)$$

$$y = \frac{\sin \theta}{r} \quad \dots \dots \dots (2)$$

x, y are function of r, θ variable i.e. $x, y = f(r, \theta)$

$\left(\frac{\partial x}{\partial r}\right)_\theta$ x is function of r, θ variable i.e. $x = \frac{\cos \theta}{r}$

Diff x partially w.r.t. r keep as θ constant

$$\left(\frac{\partial x}{\partial r}\right)_\theta = -\frac{\cos \theta}{r^2} \quad \dots \dots \dots \dots \dots (A)$$

$$\frac{d}{dx} \left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

$\left(\frac{\partial r}{\partial x}\right)_y$ r is function of x, y variable

$$x = \frac{\cos \theta}{r} \quad y = \frac{\sin \theta}{r} \quad \text{Eliminate } \theta$$

$$x^2 = \frac{\cos^2 \theta}{r^2} \quad y^2 = \frac{\sin^2 \theta}{r^2} \quad \text{squaring}$$

adding

$$x^2 + y^2 = \frac{\cos^2 \theta}{r^2} + \frac{\sin^2 \theta}{r^2}$$

$$x^2 + y^2 = \frac{\cos^2 \theta + \sin^2 \theta}{r^2}$$

$$x^2 + y^2 = \frac{1}{r^2}$$

$$r^2 = \frac{1}{x^2 + y^2}$$

$$r^2 = (x^2 + y^2)^{-1} \quad \text{taking square root on both side}$$

$$r = (x^2 + y^2)^{-1/2}$$

$$\left(\frac{\partial r}{\partial x}\right)_y = -\frac{1}{2}(x^2 + y^2)^{-\frac{3}{2}} \frac{\partial}{\partial x} (x^2 + y^2)$$

$$\frac{d}{dx} x^n = nx^{n-1} \frac{dx}{dx}$$

$$\left(\frac{\partial r}{\partial x}\right)_y = -\frac{1}{2}(x^2 + y^2)^{-\frac{3}{2}} (2x + 0)$$

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$$\begin{aligned}
 &= (-1) \left(\frac{1}{r^2} \right)^{\frac{-3}{2}} \left(\frac{\cos \theta}{r} \right) \\
 &= (-1) \frac{1}{r^{-3}} \left(\frac{\cos \theta}{r} \right) \\
 &= -\frac{\cos \theta}{r^{-2}} \\
 &= -r^2 \cos \theta \quad \dots \dots \dots \text{(B)}
 \end{aligned}$$

$$\left(\frac{\partial y}{\partial r} \right)_\theta \quad y \text{ is function of } r, \theta \text{ variable} \quad i.e \quad y = \frac{\sin \theta}{r}$$

Diff partially w. r. t. r keep as θ constant

$$\left(\frac{\partial x}{\partial r} \right)_\theta = -\frac{\sin \theta}{r^2} \quad \dots \dots \dots \dots \dots \text{(C)}$$

$$\frac{d}{dx} \left(\frac{1}{x} \right) = -\frac{1}{x^2}$$

$$\left(\frac{\partial r}{\partial y} \right)_x \quad r \text{ is function of } x, y \text{ variable} \quad r = (x^2 + y^2)^{-1/2}$$

$$\left(\frac{\partial r}{\partial y} \right)_x = -\frac{1}{2} (x^2 + y^2)^{\frac{-3}{2}} \frac{\partial}{\partial y} (x^2 + y^2)$$

$$\frac{d}{dx} x^n = nx^{n-1} \frac{dx}{dx}$$

$$\left(\frac{\partial r}{\partial y} \right)_x = -\frac{1}{2} (x^2 + y^2)^{\frac{-3}{2}} (0 + 2y)$$

$$= \left(-\frac{1}{2} \right) \left(\frac{1}{r^2} \right)^{\frac{-3}{2}} \left(\frac{2 \sin \theta}{r} \right) \quad x^2 + y^2 = \frac{1}{r^2} \text{ and } y = \frac{\sin \theta}{r}$$

$$= -\frac{\sin \theta}{r^{-2}}$$

$$\left(\frac{\partial r}{\partial y} \right)_x = -r^2 \sin \theta \quad \dots \dots \dots \text{(D)}$$

From (A), (B), (C) and (D)

$$\left(\frac{\partial x}{\partial r} \right)_\theta \left(\frac{\partial r}{\partial x} \right)_y + \left(\frac{\partial y}{\partial r} \right)_\theta \left(\frac{\partial r}{\partial y} \right)_x = \left(-\frac{\cos \theta}{r^2} \right) (-r^2 \cos \theta) + \left(-\frac{\sin \theta}{r^2} \right) (-r^2 \sin \theta)$$

$$\left(\frac{\partial x}{\partial r} \right)_\theta \left(\frac{\partial r}{\partial x} \right)_y + \left(\frac{\partial y}{\partial r} \right)_\theta \left(\frac{\partial r}{\partial y} \right)_x = \cos^2 \theta + \sin^2 \theta = 1$$

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Composite Function:

A composite function is a function that depends on another function.

The chain rule exists for differentiating a function of another function.

The chain rule is a formula to compute the derivative of a composite function. If $f(x, y)$ and $x = h(r, \theta)$, $y = g(r, \theta)$ are differentiable functions.

$f(x, y)$ is differentiable functions,

Diff partially w. r. t. to x $\frac{\partial f}{\partial x}$

Diff partially w. r. t. to y $\frac{\partial f}{\partial y}$

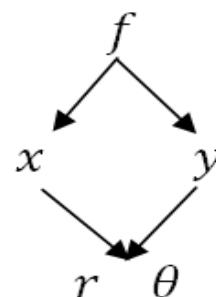
x, y are differentiable function,

Diff partially w. r. t. to r, θ $\frac{\partial x}{\partial r}, \frac{\partial x}{\partial \theta}$ and $\frac{\partial y}{\partial r}, \frac{\partial y}{\partial \theta}$

By Chain

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r}$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \theta}$$

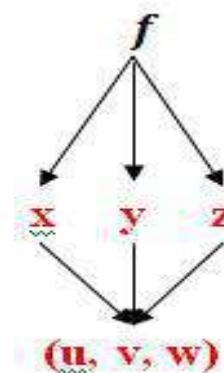


If $f = f_1(x, y, z)$ and $x, y, z = f_2(u, v, w)$ then

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial u}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial v}$$

$$\frac{\partial f}{\partial w} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial w} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial w} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial w}$$



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Examples 1: If $z = f(u, v)$ and $u = x \cos t - y \sin t$, $v = x \sin t + y \cos t$, where t is a constant, prove that: $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = u \frac{\partial z}{\partial u} + v \frac{\partial z}{\partial v}$

Solution: $z = f(u, v)$

$$u = x \cos t - y \sin t \quad \dots \dots \dots (1)$$

$$v = x \sin t + y \cos t \dots \dots \dots (2)$$

u and v are differentiable functions of x and y

Diff u and v partially w. r. t. to x and y respectively

$$\frac{\partial u}{\partial x} = \cos t \quad \text{and} \quad \frac{\partial u}{\partial y} = -\sin t$$

$$\frac{\partial v}{\partial x} = \sin t \quad \text{and} \quad \frac{\partial v}{\partial y} = \cos t$$

By chain rule

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cos t + \frac{\partial z}{\partial v} \sin t$$

Multiplying by x

$$x \frac{\partial z}{\partial x} = x \frac{\partial z}{\partial u} \cos t + x \frac{\partial z}{\partial v} \sin t$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} (-\sin t) + \frac{\partial z}{\partial v} \cos t$$

Multiplying by

$$y \frac{\partial z}{\partial v} = y \frac{\partial z}{\partial u} (-\sin t) + y \frac{\partial z}{\partial v} \cos t$$

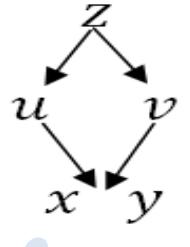
Adding equations (A) and (B)

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = x \cos t \frac{\partial z}{\partial u} + x \sin t \frac{\partial z}{\partial v} - y \sin t \frac{\partial z}{\partial u} + y \cos t \frac{\partial z}{\partial v}$$

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = (x \cos t - y \sin t) \frac{\partial z}{\partial u} + (x \sin t + y \cos t) \frac{\partial z}{\partial v}$$

From (1) and (2)

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = u \frac{\partial z}{\partial u} + v \frac{\partial z}{\partial v}$$



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* * * * *

Examples 2: If $z = f(x, y)$ where $x = u + v$ and $y = uv$

then prove that : $u \frac{\partial z}{\partial u} + v \frac{\partial z}{\partial v} = x \frac{\partial z}{\partial x} + 2y \frac{\partial z}{\partial y}$

Solution: $z = f(x, y)$ i.e. z is function of x, y

$$x = u + v \quad \dots \dots \dots (1)$$

$$y = uv \quad \dots\dots\dots (2)$$

x and *y* is differentiable functions of *u* and *v*

Diff partially w. r. t. to u and v respectively

$$\begin{aligned} \frac{\partial x}{\partial u} &= 1 & \text{and} & \quad \frac{\partial x}{\partial v} = 1 \\ \frac{\partial y}{\partial u} &= v & \text{and} & \quad \frac{\partial y}{\partial v} = u \end{aligned}$$

$$\text{By chain rule} \quad \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} 1 + \frac{\partial z}{\partial y} v$$

Multiplying by u $u \frac{\partial z}{\partial u} = u \frac{\partial z}{\partial x} 1 + u \frac{\partial z}{\partial y} v$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v}$$

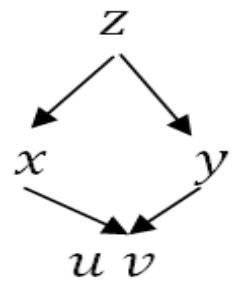
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} 1 + \frac{\partial z}{\partial y} u$$

Multiplying by v $v \frac{\partial z}{\partial v} = v \frac{\partial z}{\partial x} 1 + v \frac{\partial z}{\partial y} u$

Adding equations (A) and (B)

$$u \frac{\partial z}{\partial u} + v \frac{\partial z}{\partial v} = u \frac{\partial z}{\partial x} + uv \frac{\partial z}{\partial y} + v \frac{\partial z}{\partial x} + uv \frac{\partial z}{\partial y}$$

$$u \frac{\partial z}{\partial u} + v \frac{\partial z}{\partial v} = (u + v) \frac{\partial z}{\partial x} + (uv + uv) \frac{\partial z}{\partial y}$$



$$u \frac{\partial z}{\partial u} + v \frac{\partial z}{\partial v} = (u + v) \frac{\partial z}{\partial x} + 2uv \frac{\partial z}{\partial y}$$

From (1) and (2)

$$u \frac{\partial z}{\partial u} + v \frac{\partial z}{\partial v} = x \frac{\partial z}{\partial x} + 2y \frac{\partial z}{\partial y}$$



Homogeneous Function:

A function $F(x, y)$ is said to be homogeneous function in which the power of each term is same.

$$F(x, y, z) = x^3 + y^3 + z^3 - 9x^1y^2 + 19x^2y^1 - \sqrt{3}y^2z^1$$

Note:

- To verify homogeneous function put $x = xt$, $y = yt$ and $z = zt$

$$F(x, y, z) = (xt)^3 + (yt)^3 + (zt)^3 - 9xt^1yt^2 + 19xt^2yt^1 - \sqrt{3}yt^2zt^1$$

$$F(x, y, z) = x^3t^3 + y^3t^3 + t^3z^3 - 9xy^2t^3 + 19x^2t^3y - \sqrt{3}y^2t^3z$$

$$F(x, y, z) = t^3F(x, y, z)$$

F is homogeneous function of degree 3

- $f(x, y, z) = t^n f(x, y, z)$ f is homogeneous function of degree n

Euler's Theorem of Homogeneous Function:

Let $f(x, y)$ be a homogenous function of degree n in x and y

$$\text{For 2 variables and 1}^{\text{st}} \text{ order} \quad x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = n f(x, y)$$

Let $f(x, y, z)$ be a homogenous function of degree n so that $f(x, y, z) = t^n f(x, y, z)$

$$\text{For 3 variables and 1}^{\text{st}} \text{ order} \quad x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z} = n f(x, y, z)$$

Let f be a homogenous function of degree n in $x_1, x_2, x_3, \dots, x_n$

For n variables and 1st order

$$x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + x_3 \frac{\partial f}{\partial x_3} + \dots + x_n \frac{\partial f}{\partial x_n} = n f$$

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Euler's Theorem for second order

Let $f(x, y)$ be a homogenous function of two variables x, y of degree n

$$\text{For 2 variables and 2nd order} \quad x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = n(n - 1)f$$

Example: If $f(x, y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{\log x - \log y}{x^2 + y^2}$ then prove that

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + 2f = 0 \quad \text{OR} \quad \text{Find } x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y}$$

Solution: Consider $f(x, y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{\log x - \log y}{x^2 + y^2} \quad \because \log a - \log b = \log \left(\frac{a}{b}\right)$

$$f(x, y) = \frac{xy + x^2}{x^3 y} + \frac{\log \left(\frac{x}{y}\right)}{x^2 + y^2}$$

$$f(x, y) = \frac{y + x}{x^2 y} + \frac{\log \left(\frac{x}{y}\right)}{x^2 + y^2}$$

$$\text{Let } u = \frac{y + x}{x^2 y} \quad v = \frac{\log \left(\frac{x}{y}\right)}{x^2 + y^2}$$

$$f(x, y) = u + v \dots \dots \dots (A)$$

$$\text{Put } x = xt, \quad y = yt$$

$$u = \frac{yt + xt}{(xt)^2 yt} \quad v = \frac{\log \left(\frac{xt}{yt}\right)}{(xt)^2 + (yt)^2}$$

$$u = \frac{t(y + x)}{t^3 x^2 y} \quad v = \frac{\log \left(\frac{x}{y}\right)}{t^2 (x^2 + y^2)}$$

$$u = \frac{(y + x)}{t^2 x^2 y} \quad v = \frac{\log \left(\frac{x}{y}\right)}{t^2 (x^2 + y^2)}$$

$$u = t^{-2} \frac{(y + x)}{x^2 y} \quad v = t^{-2} \frac{\log \left(\frac{x}{y}\right)}{(x^2 + y^2)}$$

$$u = t^{-2} u(x, y) \quad v = t^{-2} v(x, y)$$

$\therefore u$ and v are homogeneous function of degree $n = -2$

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∴ by Eulers theorem

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu \Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = -2u \dots \dots \dots (1)$$

$$x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = nv \Rightarrow x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = -2v \dots \dots \dots (2)$$

$$f = u + v \quad \text{diff w.r.t } x, y$$

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \quad \& \quad \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$x \frac{\partial f}{\partial x} = x \frac{\partial u}{\partial x} + x \frac{\partial v}{\partial x} \quad y \frac{\partial f}{\partial y} = y \frac{\partial u}{\partial y} + y \frac{\partial v}{\partial y}$$

∴ Adding (1) and(2)

$$\Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = -2u - 2v$$

$$\Rightarrow x \frac{\partial u}{\partial x} + x \frac{\partial v}{\partial x} + y \frac{\partial u}{\partial y} + y \frac{\partial v}{\partial y} = -2(u + v)$$

$$\Rightarrow x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = -2f$$

$$\Rightarrow x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + 2f = 0$$

Example: If $u = \frac{x^3+y^3}{x+y} + \frac{1}{x^5} \sin^{-1} \left(\frac{x^2+y^2}{x^2+2xy} \right)$ then find the value of

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} \quad \text{at the point (1, 2)}$$

Solution: Consider $f(x, y) = \frac{x^3+y^3}{x+y} + \frac{1}{x^5} \sin^{-1} \left(\frac{x^2+y^2}{x^2+2xy} \right)$

$$\text{Let } u = \frac{x^3+y^3}{x+y} \quad v = \frac{1}{x^5} \sin^{-1} \left(\frac{x^2+y^2}{x^2+2xy} \right)$$

$$f(x, y) = u + v \dots \dots \dots (A)$$

$$\text{Put } x = xt, \quad y = yt$$

$$u = \frac{(xt)^3+(yt)^3}{xt+yt} \quad v = \frac{1}{(xt)^5} \sin^{-1} \left(\frac{(xt)^2+(yt)^2}{(xt)^2+2xtyt} \right)$$

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$$u = \frac{t^3(x^3 + y^3)}{t(x+y)} \quad v = \frac{1}{x^5 t^5} \sin^{-1} \left(\frac{t^2(x^2 + y^2)}{t^2(x^2 + 2xy)} \right)$$

$$u = t^2 \frac{(x^3 + y^3)}{(x+y)} \quad v = t^{-5} \frac{1}{x^5} \sin^{-1} \left(\frac{x^2 + y^2}{x^2 + 2xy} \right)$$

$$u = t^2 u(x, y) \quad v = t^{-5} v(x, y)$$

$\therefore u$ is homogeneous function of degree $n = 2$

$\therefore v$ is homogeneous function of degree $n = -5$

\therefore by Eulers theorem

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu$$

$$\Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2u \quad \dots \dots \dots (1)$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 2(2-1)u = 2u \quad \dots \dots \dots (2)$$

$$x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = nv$$

$$\Rightarrow x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = -5v \quad \dots \dots \dots (3)$$

$$x^2 \frac{\partial^2 v}{\partial x^2} + 2xy \frac{\partial^2 v}{\partial x \partial y} + y^2 \frac{\partial^2 v}{\partial y^2} = n(n-1)v$$

$$x^2 \frac{\partial^2 v}{\partial x^2} + 2xy \frac{\partial^2 v}{\partial x \partial y} + y^2 \frac{\partial^2 v}{\partial y^2} = -5(-5-1)v = 30v \quad \dots \dots \dots (4)$$

$$f = u + v \quad \text{diff w.r.t } x, y$$

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \quad \text{multiply by } x \quad x \frac{\partial f}{\partial x} = x \frac{\partial u}{\partial x} + x \frac{\partial v}{\partial x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \text{multiply by } y \quad y \frac{\partial f}{\partial y} = y \frac{\partial u}{\partial y} + y \frac{\partial v}{\partial y}$$



$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x^2} \quad \text{multiply by } x^2$$

$$x^2 \frac{\partial^2 f}{\partial x^2} = x^2 \frac{\partial^2 u}{\partial x^2} + x^2 \frac{\partial^2 v}{\partial x^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial y^2} \quad \text{multiply by } y^2$$

$$y^2 \frac{\partial^2 f}{\partial y^2} = y^2 \frac{\partial^2 u}{\partial y^2} + y^2 \frac{\partial^2 v}{\partial y^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x \partial y} \quad \text{multiply by } 2xy$$

$$2xy \frac{\partial^2 f}{\partial x \partial y} = 2xy \frac{\partial^2 u}{\partial x \partial y} + 2xy \frac{\partial^2 v}{\partial x \partial y}$$

\therefore Adding (1), (2), (3)and (4)

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} + x^2 \frac{\partial^2 v}{\partial x^2} + 2xy \frac{\partial^2 v}{\partial x \partial y} + y^2 \frac{\partial^2 v}{\partial y^2}$$

$$= 2u - 5v + 2u + 30v$$

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = 4u + 25v$$

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = 4 \frac{x^3 + y^3}{x+y} + 25 \frac{1}{x^5} \sin^{-1} \left(\frac{x^2 + y^2}{x^2 + 2xy} \right)$$

At point (1, 2) put $x = 1$ and $y = 2$

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = 4 \frac{1^3 + 2^3}{1+2} + 25 \frac{1}{1^5} \sin^{-1} \left(\frac{1^2 + 2^2}{1^2 + 2(1)(2)} \right)$$

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = 4 \left(\frac{9}{3} \right) + 25 \sin^{-1} \left(\frac{5}{5} \right)$$

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = 12 + 25 \frac{\pi}{2}$$

Example: If $u = x^n f \left(\frac{y}{x} \right) + y^{-n} g \left(\frac{x}{y} \right)$ then

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = n^2 u$$

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Euler's Theorem of Non-Homogeneous Function:

Let $u = f^{-1}(x, y)$ be a non-homogenous function but $f(u)$ is homogenous function of order n so that $f(u) = t^n f(u)$

$$\text{For 2 variables and 1st order} \quad x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = n \frac{f(u)}{f'(u)} = g(u)$$

$$\text{For 2 variables and 2nd order} \quad x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = g(u)[g'(u) - 1]$$

$$\text{Where } g(u) = n \frac{f(u)}{f'(u)}$$

Example: If $u = \sin^{-1}(x^2 + y^2)^{1/5}$ then Prove that

$$x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = \frac{2 \tan u}{5} \left[\frac{2 \tan^2 u}{5} - \frac{3}{5} \right]$$

Solution: Consider $u(x, y) = \sin^{-1}(x^2 + y^2)^{1/5}$

$$\text{Put } x = xt, \quad y = yt$$

$$u = \sin^{-1}[(xt)^2 + (yt)^2]^{1/5}$$

$$u = \sin^{-1}[t^2(x^2 + y^2)]^{1/5} \quad (\mathbf{a^n m = a^{nm}})$$

$$u = \sin^{-1}[t^{2/5}(x^2 + y^2)^{1/5}] \quad \mathbf{\sin^{-1}(ax) \neq a \sin^{-1}(x)}$$

$u(x, y) \neq t^n u(x, y) \therefore u$ is not homogeneous function

$$\sin u = t^{2/5}(x^2 + y^2)^{1/5}$$

$\therefore f(u) = \sin u$ is homogeneous function of degree $n = \frac{2}{5}$

$$x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = g(u)[g'(u) - 1] \quad \dots \dots \dots (A)$$

$$\text{where } g(u) = n \frac{f(u)}{f'(u)}$$

$$\text{here } f(u) = \sin u \Rightarrow f'(u) = \cos u$$

$$\Rightarrow g(u) = n \frac{f(u)}{f'(u)} = \frac{2 \sin u}{5 \cos u} = \frac{2}{5} \tan u$$

$$g'(u) = \frac{2}{5} \sec^2 u$$

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Equation (A)

$$x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = \frac{2}{5} \tan u \left[\frac{2 \sec^2 u}{5} - 1 \right]$$

$$x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = \frac{2}{5} \tan u \left[\frac{2(\tan^2 u + 1)}{5} - 1 \right]$$

$$x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = \frac{2}{5} \tan u \left[\frac{2 \tan^2 u + 2}{5} - 1 \right]$$

$$x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = \frac{2}{5} \tan u \left[\frac{2 \tan^2 u + 2 - 5}{5} \right]$$

$$x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = \frac{2}{5} \tan u \left[\frac{2 \tan^2 u}{5} - \frac{3}{5} \right]$$

Example: If $u = \operatorname{cosec}^{-1} \sqrt{\frac{x^{1/2} + y^{1/2}}{x^{1/3} + y^{1/3}}}$ then Prove that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{12} \left[\frac{\tan^2 u}{12} + \frac{13}{12} \right]$$

Solution: Consider $u = \operatorname{cosec}^{-1} \sqrt{\frac{x^{1/2} + y^{1/2}}{x^{1/3} + y^{1/3}}}$

Put $x = xt, y = yt$

$$u = \operatorname{cosec}^{-1} \sqrt{\frac{(xt)^{1/2} + (yt)^{1/2}}{(xt)^{1/3} + (yt)^{1/3}}}$$

$$u = \operatorname{cosec}^{-1} \sqrt{\frac{t^{1/2}(x^{1/2} + y^{1/2})}{t^{1/3}(x^{1/3} + y^{1/3})}}$$

$$u = \operatorname{cosec}^{-1} \left[\frac{t^{1/2}(x^{1/2} + y^{1/2})}{t^{1/3}(x^{1/3} + y^{1/3})} \right]^{1/2}$$

$$u = \operatorname{cosec}^{-1} \left[\frac{t^{1/2}t^{-1/3}(x^{1/2} + y^{1/2})}{(x^{1/3} + y^{1/3})} \right]^{1/2}$$

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$$u = \operatorname{cosec}^{-1} \left[\frac{t^{1/6}(x^{1/2} + y^{1/2})}{(x^{1/3} + y^{1/3})} \right]^{1/2}$$

$$u = \operatorname{cosec}^{-1} \left[t^{1/12} \sqrt{\frac{x^{1/2} + y^{1/2}}{x^{1/3} + y^{1/3}}} \right]$$

$u(x, y) \neq t^n u(x, y) \therefore u$ is not homogeneous function

$$\operatorname{cosec} u = t^{1/12} \sqrt{\frac{x^{1/2} + y^{1/2}}{x^{1/3} + y^{1/3}}}$$

$\therefore f(u) = \operatorname{cosec} u$ is homogeneous function of degree $n = \frac{1}{12}$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$$

$$\text{where } g(u) = n \frac{f(u)}{f'(u)}$$

$$\text{here } f(u) = \operatorname{cosec} u \Rightarrow f'(u) = -\operatorname{cosec} u \cot u$$

$$\Rightarrow g(u) = n \frac{f(u)}{f'(u)} = -\frac{1}{12} \frac{\operatorname{cosec} u}{\operatorname{cosec} u \cot u}$$

$$g(u) = -\frac{1}{12} \frac{1}{\cot u} = -\frac{1}{12} \tan u$$

$$g'(u) = -\frac{1}{12} \sec^2 u$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = -\frac{1}{12} \tan u \left[\frac{-\sec^2 u}{12} - 1 \right]$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{1}{12} \tan u \left[\frac{\sec^2 u}{12} + 1 \right]$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{1}{12} \tan u \left[\frac{\tan^2 u + 1}{12} + 1 \right]$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{1}{12} \tan u \left[\frac{\tan^2 u + 1 + 12}{12} \right]$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{1}{12} \tan u \left[\frac{\tan^2 u + 13}{12} \right]$$

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Example: If $u = \sec^{-1} \left[\frac{x+y}{x^{1/2} + y^{1/2}} \right]$ then Prove that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = -\frac{\cot u}{4} [3 + \cot^2 u]$$

Example: If $u = \tan^{-1} \left[\frac{x^3 + y^3}{x + y} \right]$ then Prove that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \sin 2u [1 - 4 \sin^2 u]$$

Example: If $u = \sin^{-1} \left[\frac{x+y}{x^{1/2} + y^{1/2}} \right]$ then Prove that

$$x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = \frac{-\sin u \cos 2u}{4 \cos^3 u}$$

Example: If $u = \sin^{-1} \left[\frac{x^2 + y^2}{x+y} \right]^{\frac{1}{2}}$ then Prove that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{4} [\tan^2 u - 1]$$

