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## UNIT II: Multivariable Calculus – Partial Differentiation

### Partial Differentiation:

Given a function of two variables,  $f(x, y)$  the derivative with respect to  $x$  only (treating  $y$  as a constant) is called the partial derivative of  $f$  with respect to  $x$

It is denoted by either  $\frac{\partial f}{\partial x}$  or  $f_x$

Similarly, the derivative of  $f$  with respect to  $y$  only (treating  $x$  as a constant) is called the partial derivative of  $f$  with respect to  $y$ . It is denoted by either  $\frac{\partial f}{\partial y}$  or  $f_y$

Differentiate  $f$  with respect to  $x$  twice. (i.e. differentiate  $f$  with respect to  $x$ ; then differentiate the result with respect to  $x$  again.)  $\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2}$  or  $f_{xx}$

Differentiate  $f$  with respect to  $y$  twice. (i.e. differentiate  $f$  with respect to  $y$ ; then differentiate the result with respect to  $y$  again.)  $\frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2}$  or  $f_{yy}$

Mixed partials derivative:

- First differentiate  $f$  with respect to  $x$ ; then differentiate the result with respect to  $y$ .

$$\frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x} \text{ or } f_{xy}$$

- First differentiate  $f$  with respect to  $y$ ; then differentiate the result with respect to  $x$ .

$$\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y} \text{ or } f_{yx}$$

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*"The Only things that will stop you from fulfilling your dreams is you"*



**Example 1:** If  $f(x, y) = 3x^2y + 5x - 2y^2 + 1$  find  $f_x$   $f_y$   $f_{xx}$   $f_{yy}$   $f_{xy}$   $f_{yx}$

**Solution:** Let  $f(x, y) = 3x^2y + 5x - 2y^2 + 1$

$f_x$  means differentiate  $f$  with respect to  $x$  (treating  $y$  as a constant)

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} (3x^2y + 5x - 2y^2 + 1)$$

$$f_x = \frac{\partial f}{\partial x} = 6xy + 5 - 0 + 0 = 6xy + 5$$

$f_y$  means differentiate  $f$  with respect to  $y$  (treating  $x$  as a constant)

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (3x^2y + 5x - 2y^2 + 1)$$

$$f_y = \frac{\partial f}{\partial y} = 3x^2 + 0 - 4y + 0 = 3x^2 - 4y$$

$\frac{\partial^2 f}{\partial x^2}$  or  $f_{xx}$  Differentiate  $f$  with respect to  $x$  twice.

$$\frac{\partial^2 f}{\partial x^2} \text{ or } f_{xx} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} (6xy + 5) = 6y$$

$\frac{\partial^2 f}{\partial y^2}$  or  $f_{yy}$  Differentiate  $f$  with respect to  $y$  twice.

$$\frac{\partial^2 f}{\partial y^2} \text{ or } f_{yy} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial y} (3x^2 - 4y) = -4$$

First differentiate  $f$  with respect to  $x$ ; then differentiate the result with respect to  $y$ .

$$\frac{\partial^2 f}{\partial y \partial x} \text{ or } f_{xy} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial y} (6xy + 5) = 6x$$

First differentiate  $f$  with respect to  $y$ ; then differentiate the result with respect to  $x$ .

$$\frac{\partial^2 f}{\partial x \partial y} \text{ or } f_{yx} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial x} (3x^2 - 4y) = 6x$$

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**Example 2:** If  $z = \tan(y + ax) + (y - ax)^{\frac{3}{2}}$  then find the value of  $\frac{\partial^2 z}{\partial x^2} - a^2 \frac{\partial^2 z}{\partial y^2}$

**Solution:** Let  $z = \tan(y + ax) + (y - ax)^{\frac{3}{2}}$

To find  $\frac{\partial^2 z}{\partial x^2}$  Differentiate  $z$  with respect to  $x$  twice

$$z = \tan(y + ax) + (y - ax)^{\frac{3}{2}}$$

Differentiating  $z$  with respect to  $x$  partially treating  $y$  as constant

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} \left[ \tan(y + ax) + (y - ax)^{\frac{3}{2}} \right] \quad \because \frac{d}{dx} \tan x = \sec^2 x \quad \frac{d}{dx} x^n = nx^{n-1}$$

$$\frac{\partial z}{\partial x} = \sec^2(y + ax) \frac{\partial}{\partial x} (y + ax) + \frac{3}{2} (y - ax)^{\frac{3}{2}-1} \frac{\partial}{\partial x} (y - ax)$$

$$\frac{\partial z}{\partial x} = \sec^2(y + ax) (0 + a) + \frac{3}{2} (y - ax)^{\frac{1}{2}} (0 - a)$$

$$\frac{\partial z}{\partial x} = a \sec^2(y + ax) - \frac{3a}{2} (y - ax)^{\frac{1}{2}} \dots \dots \dots (1)$$

Differentiating  $\frac{\partial z}{\partial x}$  with respect to  $x$  partially treating  $y$  as constant

$$\frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial x} \left[ a \sec^2(y + ax) - \frac{3a}{2} (y - ax)^{\frac{1}{2}} \right]$$

$$\frac{\partial^2 z}{\partial x^2} = 2 a \sec(y + ax) \frac{\partial}{\partial x} [\sec(y + ax)] - \frac{3a}{2} \frac{1}{2} (y - ax)^{\frac{1}{2}-1} \frac{\partial}{\partial x} (y - ax)$$

$$\frac{\partial^2 z}{\partial x^2} = 2 a \sec(y + ax) \sec(y + ax) \tan(y + ax) \frac{\partial}{\partial x} (y + ax)$$

$$- \frac{3a}{4} (y - ax)^{-\frac{1}{2}} (-a)$$

$$\frac{\partial^2 z}{\partial x^2} = 2 a \sec^2(y + ax) \tan(y + ax) (a) - \frac{3a}{4} (y - ax)^{-\frac{1}{2}} (-a)$$

$$\frac{\partial^2 z}{\partial x^2} = 2 a^2 \sec^2(y + ax) \tan(y + ax) + \frac{3a^2}{4} (y - ax)^{-\frac{1}{2}} \dots \dots \dots (A)$$

To find  $\frac{\partial^2 z}{\partial y^2}$  Differentiate  $z$  with respect to  $y$  twice



$$z = \tan(y + ax) + (y - ax)^{\frac{3}{2}}$$

Differentiating  $z$  with respect to  $y$  partially treating  $x$  as constant

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} \left[ \tan(y + ax) + (y - ax)^{\frac{3}{2}} \right]$$

$$\frac{\partial z}{\partial y} = \sec^2(y + ax) \frac{\partial}{\partial y} (y + ax) + \frac{3}{2} (y - ax)^{\frac{3}{2}-1} \frac{\partial}{\partial y} (y - ax)$$

$$\frac{\partial z}{\partial y} = \sec^2(y + ax) (1 + 0) + \frac{3}{2} (y - ax)^{\frac{1}{2}} (1 - 0)$$

$$\frac{\partial z}{\partial y} = \sec^2(y + ax) + \frac{3}{2} (y - ax)^{\frac{1}{2}} \dots \dots \dots (2)$$

Differentiating  $\frac{\partial z}{\partial y}$  with respect to  $y$  partially treating  $x$  as constant

$$\frac{\partial}{\partial y} \left( \frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial y} \left[ \sec^2(y + ax) + \frac{3}{2} (y - ax)^{\frac{1}{2}} \right]$$

$$\frac{\partial^2 z}{\partial y^2} = 2 \sec(y + ax) \frac{\partial}{\partial x} [\sec(y + ax)] + \frac{3}{2} \frac{1}{2} (y - ax)^{\frac{1}{2}-1} \frac{\partial}{\partial x} (y - ax)$$

$$\frac{\partial^2 z}{\partial y^2} = 2 \sec(y + ax) \sec(y + ax) \tan(y + ax) \frac{\partial}{\partial x} (y + ax) + \frac{3}{4} (y - ax)^{-\frac{1}{2}} (1)$$

$$\frac{\partial^2 z}{\partial y^2} = 2 \sec^2(y + ax) \tan(y + ax) (a) + \frac{3a}{4} (y - ax)^{-\frac{1}{2}} \dots \dots \dots (3)$$

Multiplying by  $-a^2$

$$-a^2 \frac{\partial^2 z}{\partial y^2} = -2 a^2 \sec^2(y + ax) \tan(y + ax) - \frac{3a^2}{4} (y - ax)^{-\frac{1}{2}} \dots \dots \dots (B)$$

Equation (A) + (B)

$$\frac{\partial^2 z}{\partial x^2} - a^2 \frac{\partial^2 z}{\partial y^2} = 2 a^2 \sec^2(y + ax) \tan(y + ax) + \frac{3a^2}{4} (y - ax)^{-\frac{1}{2}}$$

$$= -2 a^2 \sec^2(y + ax) \tan(y + ax) - \frac{3a^2}{4} (y - ax)^{-\frac{1}{2}}$$

$$\therefore \frac{\partial^2 z}{\partial x^2} - a^2 \frac{\partial^2 z}{\partial y^2} = 0$$



**Example 3:** If  $u = \log(x^3 + y^3 - x^2y - xy^2)$  then prove that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 u = -\frac{4}{(x+y)^2}$$

**Solution:** Let  $u = \log(x^3 + y^3 - x^2y - xy^2)$

$$u = \log(x^2(x-y) - y^2(x-y))$$

$$u = \log[(x-y)(x^2 - y^2)]$$

$$u = \log[(x-y)(x-y)(x+y)] \quad \because (a^2 - b^2) = (a+b)(a-b)$$

$$u = \log[(x-y)^2(x+y)]$$

$$u = \log(x-y)^2 + \log(x+y) \quad \because \log ab = \log a + \log b$$

$$u = 2 \log(x-y) + \log(x+y) \quad \because \log a^n = n \log a$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 u = \left(\frac{\partial^2}{\partial x^2} + 2 \frac{\partial^2}{\partial x \partial y} + \frac{\partial^2}{\partial y^2}\right) u = \frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} \quad \dots \dots (A)$$

To find  $\frac{\partial^2 u}{\partial x^2}$  means Differentiate  $u$  with respect to  $x$  twice

$$u = 2 \log(x-y) + \log(x+y)$$

Differentiating  $u$  with respect to  $x$  partially  $y$  is constant

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} [2 \log(x-y) + \log(x+y)] \quad \because \frac{d}{dx} \log x = \frac{1}{x}$$

$$\frac{\partial u}{\partial x} = \frac{2}{(x-y)} \frac{\partial}{\partial x} (x-y) + \frac{1}{(x+y)} \frac{\partial}{\partial x} (x+y)$$

$$\frac{\partial u}{\partial x} = \frac{2}{(x-y)} (1-0) + \frac{1}{(x+y)} (1+0)$$

$$\frac{\partial u}{\partial x} = \frac{2}{(x-y)} + \frac{1}{(x+y)} \dots \dots \dots (1)$$

Differentiating  $\frac{\partial u}{\partial x}$  with respect to  $x$  partially

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial x} \left[ \frac{2}{(x-y)} + \frac{1}{(x+y)} \right]$$



$$\frac{\partial^2 u}{\partial x^2} = \left[ \frac{-2}{(x-y)^2} \frac{\partial}{\partial x} (x-y) + \frac{-1}{(x+y)^2} \frac{\partial}{\partial x} (x+y) \right] \quad \therefore \frac{d}{dx} \left( \frac{1}{x} \right) = \frac{-1}{x^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \left[ \frac{-2}{(x-y)^2} (1-0) + \frac{-1}{(x+y)^2} (1+0) \right]$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{2}{(x-y)^2} - \frac{1}{(x+y)^2} \dots \dots \dots (2)$$

To find  $\frac{\partial^2 u}{\partial y^2}$  Differentiate  $u$  with respect to  $y$  twice

$$u = 2 \log(x-y) + \log(x+y)$$

Differentiating  $u$  with respect to  $y$  partially

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y} [2 \log(x-y) + \log(x+y)]$$

$$\frac{\partial u}{\partial y} = \frac{2}{(x-y)} \frac{\partial}{\partial y} (x-y) + \frac{1}{(x+y)} \frac{\partial}{\partial y} (x+y)$$

$$\frac{\partial u}{\partial y} = \frac{2}{(x-y)} (0-1) + \frac{1}{(x+y)} (0+1)$$

$$\frac{\partial u}{\partial y} = \frac{-2}{(x-y)} + \frac{1}{(x+y)} \dots \dots \dots (3)$$

Differentiating  $\frac{\partial u}{\partial y}$  with respect to  $y$  partially

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} \right) = \frac{\partial}{\partial y} \left[ \frac{-2}{(x-y)} + \frac{1}{(x+y)} \right]$$

$$\frac{\partial^2 u}{\partial y^2} = \left[ \frac{-(-2)}{(x-y)^2} \frac{\partial}{\partial y} (x-y) + \frac{-1}{(x+y)^2} \frac{\partial}{\partial y} (x+y) \right]$$

$$\frac{\partial^2 u}{\partial y^2} = \left[ \frac{2}{(x-y)^2} (0-1) + \frac{-1}{(x+y)^2} (0+1) \right]$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{2}{(x-y)^2} - \frac{1}{(x+y)^2} \dots \dots \dots (4)$$

$\frac{\partial^2 u}{\partial x \partial y}$  Differentiating  $u$  with respect to  $y$  partially then  $x$



$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial y} \right) = \frac{\partial}{\partial x} \left( \frac{-2}{(x-y)} + \frac{1}{(x+y)} \right) \quad \text{from (3)}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{-(-2)}{(x-y)^2} \frac{\partial}{\partial x} (x-y) + \frac{-1}{(x+y)^2} \frac{\partial}{\partial x} (x+y)$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{2}{(x-y)^2} (1-0) + \frac{-1}{(x+y)^2} (1+0)$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{2}{(x-y)^2} - \frac{1}{(x+y)^2} \dots \dots \dots (5)$$

$$\left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right)^2 u = \frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2}$$

From (2) (4) and (5)

$$\begin{aligned} \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right)^2 u &= -\frac{2}{(x-y)^2} - \frac{1}{(x+y)^2} + 2 \left[ \frac{2}{(x-y)^2} - \frac{1}{(x+y)^2} \right] - \frac{2}{(x-y)^2} - \frac{1}{(x+y)^2} \\ &= -\frac{2}{(x-y)^2} - \frac{1}{(x+y)^2} + \frac{4}{(x-y)^2} - \frac{2}{(x+y)^2} - \frac{2}{(x-y)^2} - \frac{1}{(x+y)^2} \\ &= -\frac{4}{(x+y)^2} \end{aligned}$$

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**Type II**  $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

If  $f(x, y)$  is Homogeneous Function then  $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

**Example 1:** If  $u = \log(x^2 + y^2)$  verify  $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

**Solution:** Let  $u = \log(x^2 + y^2)$

$\frac{\partial u}{\partial x}$  Differentiating  $u$  with respect to  $x$  partially  $y$  is constant

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} \log(x^2 + y^2)$$

$$\frac{\partial u}{\partial x} = \frac{1}{x^2 + y^2} \frac{\partial}{\partial x} (x^2 + y^2) \quad \because \frac{d}{dx} \log x = \frac{1}{x}$$

$$\frac{\partial u}{\partial x} = \frac{1}{x^2 + y^2} (2x + 0) = \frac{2x}{(x^2 + y^2)} \dots \dots \dots (1)$$

$\frac{\partial^2 u}{\partial y \partial x}$  Differentiating  $\frac{\partial u}{\partial x}$  with respect to  $y$  partially  $x$  is constant

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial y} \left( \frac{2x}{x^2 + y^2} \right)$$

$$\frac{\partial^2 u}{\partial y \partial x} = 2x \frac{\partial}{\partial y} \left( \frac{1}{x^2 + y^2} \right) \quad \because \frac{d}{dx} \left( \frac{1}{x} \right) = -\frac{1}{x^2}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{2x(-1)}{(x^2 + y^2)^2} \frac{\partial}{\partial y} (x^2 + y^2)$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{-2x}{(x^2 + y^2)^2} (0 + 2y)$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{-4xy}{(x^2 + y^2)^2} \dots \dots \dots (A)$$



$$u = \log(x^2 + y^2)$$

$\frac{\partial u}{\partial y}$  Differentiating  $u$  with respect to  $y$  partially

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \log(x^2 + y^2)$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^2 + y^2} \frac{\partial}{\partial y} (x^2 + y^2)$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^2 + y^2} (0 + 2y) = \frac{2y}{(x^2 + y^2)} \dots \dots \dots (2)$$

$\frac{\partial^2 u}{\partial x \partial y}$  Differentiating  $\frac{\partial u}{\partial y}$  with respect to  $x$  partially constant  $y$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial y} \right) = \frac{\partial}{\partial x} \left( \frac{2y}{x^2 + y^2} \right) = 2y \frac{\partial}{\partial x} \left( \frac{1}{x^2 + y^2} \right)$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{2y(-1)}{(x^2 + y^2)^2} \frac{\partial}{\partial x} (x^2 + y^2)$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{-2y}{(x^2 + y^2)^2} (2x + 0)$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{-4xy}{(x^2 + y^2)^2} \dots \dots \dots (B)$$

From (A) and (B)

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

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**Variable to be treated as Constant:**

Let  $f$  be function of  $x$  and  $y$  i.e.  $f(x, y)$

To find partial derivative with respect to  $x$  we have to treat  $y$  as constant

$$\left(\frac{\partial f}{\partial x}\right)_y$$

To find partial derivative with respect to  $y$  we have to treat  $x$  as constant

$$\left(\frac{\partial f}{\partial y}\right)_x$$

**For Example:**

$\left(\frac{\partial z}{\partial r}\right)_\theta$  Means  $z$  is function of  $r, \theta$   $z$  is differentiable w.r.t  $r, \theta$

here  $z$  is partially differentiated with respect to  $r$  and  $\theta$  as constant.

$\left(\frac{\partial y}{\partial \theta}\right)_r$  Means  $y$  is function of  $r, \theta$   $y$  is differentiable w.r.t  $r, \theta$

here  $y$  is partially differentiated with respect to  $\theta$  and  $r$  as constant.

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**Example 1:** If  $u = 2x + 3y$ ,  $v = 3x - 2y$  Find  $\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v \left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial v}{\partial y}\right)_u$

**Solution:**

$$u = 2x + 3y \quad \dots \dots \dots (1)$$

$$v = 3x - 2y \quad \dots \dots \dots (2)$$

$u, v$  are functions of  $x, y$  variable i.e.  $u, v = f(x, y)$

$\left(\frac{\partial u}{\partial x}\right)_y$   $u$  is function of  $x, y$  variable i.e.  $u = 2x + 3y$

Diff  $u$  partially w.r.t.  $x$  keep  $y$  as constant

$$\left(\frac{\partial u}{\partial x}\right)_y = 2 \quad \dots \dots \dots (A)$$

$\left(\frac{\partial x}{\partial u}\right)_v$   $x$  is function of  $u, v$  variable

$$u = 2x + 3y \quad v = 3x - 2y \quad \text{Eliminate } y$$

$$u = 2x + 3y \quad 2y = 3x - v$$

$$u = 2x + 3y \quad y = \frac{3x - v}{2}$$

$$u = 2x + 3\left(\frac{3x - v}{2}\right)$$

*"The Only things that will stop you from fulfilling your dreams is you"*



$$2u = 4x + 3(3x - v)$$

$$2u = 4x + 9x - 3v$$

$$2u + 3v = 13x$$

$$x = \frac{2u + 3v}{13}$$

$\left(\frac{\partial x}{\partial u}\right)_v$  Diff partially w. r. t.  $u$  keep  $v$  as constant

$$\left(\frac{\partial x}{\partial u}\right)_v = \frac{2}{13} \quad \dots \dots \dots (B)$$

$\left(\frac{\partial y}{\partial v}\right)_x$   $y$  is function of  $v, x$  variable

$$\text{From (2)} \quad v = 3x - 2y \Rightarrow 2y = 3x - v \Rightarrow y = \frac{3x - v}{2}$$

Diff partially w. r. t.  $v$  keep  $x$  as constant

$$\left(\frac{\partial y}{\partial v}\right)_x = -\frac{1}{2} \quad \dots \dots \dots (C)$$

$\left(\frac{\partial v}{\partial y}\right)_u$   $v$  is function of  $y, u$  variable

$$u = 2x + 3y \quad v = 3x - 2y \quad \text{Eliminate } x$$

$$u = 2x + 3y \quad v + 2y = 3x$$

$$u = 2x + 3y \quad x = \frac{v + 2y}{3}$$

$$u = 2\left(\frac{v + 2y}{3}\right) + 3y$$

$$3u = 2v + 4y + 9y$$

$$3u = 2v + 13y$$

$$2v = 3u - 13y \quad v = \frac{3u - 13y}{2}$$

Diff partially w. r. t.  $y$  keep  $u$  as constant

$$\left(\frac{\partial v}{\partial y}\right)_u = -\frac{13}{2} \quad \dots \dots \dots (D)$$

From (A), (B), (C) and (D)

$$\therefore \left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v \left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial v}{\partial y}\right)_u = 2 \left(\frac{2}{13}\right) \left(-\frac{1}{2}\right) \left(-\frac{13}{2}\right) = 1$$



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**Example 2:** If  $x = \frac{\cos \theta}{r}$ ,  $y = \frac{\sin \theta}{r}$  Find  $\left(\frac{\partial x}{\partial r}\right)_\theta \left(\frac{\partial r}{\partial x}\right)_y + \left(\frac{\partial y}{\partial r}\right)_\theta \left(\frac{\partial r}{\partial y}\right)_x$

**Solution:**

$$x = \frac{\cos \theta}{r} \quad \dots \dots \dots (1)$$

$$y = \frac{\sin \theta}{r} \quad \dots \dots \dots (2)$$

$x, y$  are function of  $r, \theta$  variable i.e.  $x, y = f(r, \theta)$

$$\left(\frac{\partial x}{\partial r}\right)_\theta \quad x \text{ is function of } r, \theta \text{ variable i.e. } x = \frac{\cos \theta}{r}$$

Diff  $x$  partially w. r. t.  $r$  keep as  $\theta$  constant

$$\left(\frac{\partial x}{\partial r}\right)_\theta = -\frac{\cos \theta}{r^2} \quad \dots \dots \dots (A)$$

$$\frac{d}{dx} \left( \frac{1}{x} \right) = -\frac{1}{x^2}$$

$$\left(\frac{\partial r}{\partial x}\right)_y \quad r \text{ is function of } x, y \text{ variable}$$

$$x = \frac{\cos \theta}{r} \quad y = \frac{\sin \theta}{r} \quad \text{Eliminate } \theta$$

$$x^2 = \frac{\cos^2 \theta}{r^2} \quad y^2 = \frac{\sin^2 \theta}{r^2} \quad \text{squaring}$$

adding

$$x^2 + y^2 = \frac{\cos^2 \theta}{r^2} + \frac{\sin^2 \theta}{r^2}$$

$$x^2 + y^2 = \frac{\cos^2 \theta + \sin^2 \theta}{r^2}$$

$$x^2 + y^2 = \frac{1}{r^2}$$

$$r^2 = \frac{1}{x^2 + y^2}$$

$$r^2 = (x^2 + y^2)^{-1} \quad \text{taking square root on both side}$$

$$r = (x^2 + y^2)^{-1/2}$$

$$\left(\frac{\partial r}{\partial x}\right)_y = -\frac{1}{2} (x^2 + y^2)^{-\frac{3}{2}} \frac{\partial}{\partial x} (x^2 + y^2)$$

$$\frac{d}{dx} x^n = nx^{n-1} \frac{dx}{dx}$$

$$\left(\frac{\partial r}{\partial x}\right)_y = -\frac{1}{2} (x^2 + y^2)^{-\frac{3}{2}} (2x + 0)$$



$$= (-1) \left( \frac{1}{r^2} \right)^{\frac{-3}{2}} \left( \frac{\cos \theta}{r} \right)$$

$$x^2 + y^2 = \frac{1}{r^2} \text{ and } x = \frac{\cos \theta}{r}$$

$$= (-1) \frac{1}{r^{-3}} \left( \frac{\cos \theta}{r} \right)$$

$$= -\frac{\cos \theta}{r^{-2}}$$

$$= -r^2 \cos \theta \quad \dots \dots \dots (B)$$

$$\left( \frac{\partial y}{\partial r} \right)_{\theta} \quad y \text{ is function of } r, \theta \text{ variable} \quad i.e \quad y = \frac{\sin \theta}{r}$$

Diff partially w. r. t.  $r$  keep as  $\theta$  constant

$$\left( \frac{\partial x}{\partial r} \right)_{\theta} = -\frac{\sin \theta}{r^2} \quad \dots \dots \dots (C)$$

$$\frac{d}{dx} \left( \frac{1}{x} \right) = -\frac{1}{x^2}$$

$$\left( \frac{\partial r}{\partial y} \right)_x \quad r \text{ is function of } x, y \text{ variable} \quad r = (x^2 + y^2)^{-1/2}$$

$$\left( \frac{\partial r}{\partial y} \right)_x = -\frac{1}{2} (x^2 + y^2)^{\frac{-3}{2}} \frac{\partial}{\partial y} (x^2 + y^2)$$

$$\frac{d}{dx} x^n = nx^{n-1} \frac{dx}{dx}$$

$$\left( \frac{\partial r}{\partial y} \right)_x = -\frac{1}{2} (x^2 + y^2)^{\frac{-3}{2}} (0 + 2y)$$

$$= \left( -\frac{1}{2} \right) \left( \frac{1}{r^2} \right)^{\frac{-3}{2}} \left( \frac{2 \sin \theta}{r} \right)$$

$$x^2 + y^2 = \frac{1}{r^2} \text{ and } y = \frac{\sin \theta}{r}$$

$$= -\frac{\sin \theta}{r^{-2}}$$

$$\left( \frac{\partial r}{\partial y} \right)_x = -r^2 \sin \theta \quad \dots \dots \dots (D)$$

From (A), (B), (C) and (D)

$$\left( \frac{\partial x}{\partial r} \right)_{\theta} \left( \frac{\partial r}{\partial x} \right)_y + \left( \frac{\partial y}{\partial r} \right)_{\theta} \left( \frac{\partial r}{\partial y} \right)_x = \left( -\frac{\cos \theta}{r^2} \right) (-r^2 \cos \theta) + \left( -\frac{\sin \theta}{r^2} \right) (-r^2 \sin \theta)$$

$$\left( \frac{\partial x}{\partial r} \right)_{\theta} \left( \frac{\partial r}{\partial x} \right)_y + \left( \frac{\partial y}{\partial r} \right)_{\theta} \left( \frac{\partial r}{\partial y} \right)_x = \cos^2 \theta + \sin^2 \theta = 1$$

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## Composite Function:

A composite function is a function that depends on another function.

The chain rule exists for differentiating a function of another function.

The chain rule is a formula to compute the derivative of a composite function. If  $f(x, y)$  and  $x = h(r, \theta)$ ,  $y = g(r, \theta)$  are differentiable functions.

$f(x, y)$  is differentiable functions,

Diff partially w.r. t. to  $x$   $\frac{\partial f}{\partial x}$

Diff partially w.r. t. to  $y$   $\frac{\partial f}{\partial y}$

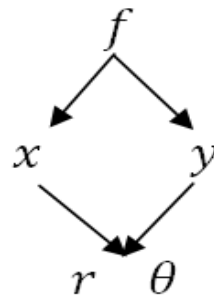
$x, y$  are differentiable function,

Diff partially w. r. t. to  $r, \theta$   $\frac{\partial x}{\partial r}, \frac{\partial x}{\partial \theta}$  and  $\frac{\partial y}{\partial r}, \frac{\partial y}{\partial \theta}$

By Chain

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r}$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \theta}$$

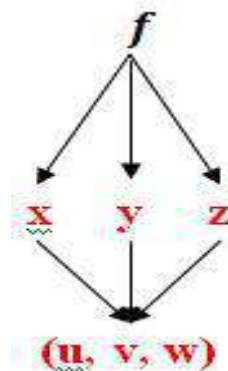


If  $f = f_1(x, y, z)$  and  $x, y, z = f_2(u, v, w)$  then

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial u}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial v}$$

$$\frac{\partial f}{\partial w} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial w} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial w} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial w}$$



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**Examples 1:** If  $z = f(u, v)$  and  $u = x \cos t - y \sin t$ ,  $v = x \sin t + y \cos t$ , where  $t$  is a constant, prove that :  $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = u \frac{\partial z}{\partial u} + v \frac{\partial z}{\partial v}$

**Solution:**  $z = f(u, v)$

$$u = x \cos t - y \sin t \quad \dots \dots \dots (1)$$

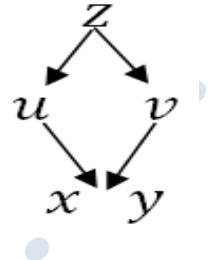
$$v = x \sin t + y \cos t \quad \dots \dots \dots (2)$$

$u$  and  $v$  is differentiable functions of  $x$  and  $y$

Diff  $u$  and  $v$  partially w. r. t. to  $x$  and  $y$  respectively

$$\frac{\partial u}{\partial x} = \cos t \quad \text{and} \quad \frac{\partial u}{\partial y} = -\sin t$$

$$\frac{\partial v}{\partial x} = \sin t \quad \text{and} \quad \frac{\partial v}{\partial y} = \cos t$$



By chain rule  $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x}$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cos t + \frac{\partial z}{\partial v} \sin t$$

Multiplying by  $x$   $x \frac{\partial z}{\partial x} = x \frac{\partial z}{\partial u} \cos t + x \frac{\partial z}{\partial v} \sin t$

$$x \frac{\partial z}{\partial x} = x \cos t \frac{\partial z}{\partial u} + x \sin t \frac{\partial z}{\partial v} \quad \dots \dots \dots (A)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} (-\sin t) + \frac{\partial z}{\partial v} \cos t$$

Multiplying by  $y$   $y \frac{\partial z}{\partial y} = y \frac{\partial z}{\partial u} (-\sin t) + y \frac{\partial z}{\partial v} \cos t$

$$y \frac{\partial z}{\partial y} = -y \sin t \frac{\partial z}{\partial u} + y \cos t \frac{\partial z}{\partial v} \quad \dots \dots \dots (B)$$

Adding equations (A) and (B)

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = x \cos t \frac{\partial z}{\partial u} + x \sin t \frac{\partial z}{\partial v} - y \sin t \frac{\partial z}{\partial u} + y \cos t \frac{\partial z}{\partial v}$$

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = (x \cos t - y \sin t) \frac{\partial z}{\partial u} + (x \sin t + y \cos t) \frac{\partial z}{\partial v}$$

From (1) and (2)

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = u \frac{\partial z}{\partial u} + v \frac{\partial z}{\partial v}$$





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**Examples 2:** If  $z = f(x, y)$  where  $x = u + v$  and  $y = uv$ then prove that :  $u \frac{\partial z}{\partial u} + v \frac{\partial z}{\partial v} = x \frac{\partial z}{\partial x} + 2y \frac{\partial z}{\partial y}$ **Solution:**  $z = f(x, y)$  i.e.  $z$  is function of  $x, y$ 

$$x = u + v \quad \dots \dots \dots (1)$$

$$y = uv \quad \dots \dots \dots (2)$$

 $x$  and  $y$  is differentiable functions of  $u$  and  $v$ Diff partially w. r. t. to  $u$  and  $v$  respectively

$$\begin{aligned} \frac{\partial x}{\partial u} &= 1 & \text{and} & & \frac{\partial x}{\partial v} &= 1 \\ \frac{\partial y}{\partial u} &= v & \text{and} & & \frac{\partial y}{\partial v} &= u \end{aligned}$$

By chain rule  $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u}$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} 1 + \frac{\partial z}{\partial y} v$$

Multiplying by  $u$   $u \frac{\partial z}{\partial u} = u \frac{\partial z}{\partial x} 1 + u \frac{\partial z}{\partial y} v$

$$u \frac{\partial z}{\partial u} = u \frac{\partial z}{\partial x} + uv \frac{\partial z}{\partial y} \quad \dots \dots \dots (A)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} 1 + \frac{\partial z}{\partial y} u$$

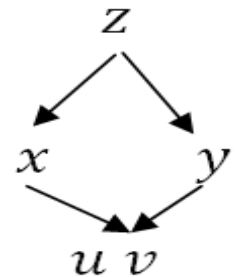
Multiplying by  $v$   $v \frac{\partial z}{\partial v} = v \frac{\partial z}{\partial x} 1 + v \frac{\partial z}{\partial y} u$

$$v \frac{\partial z}{\partial v} = v \frac{\partial z}{\partial x} + uv \frac{\partial z}{\partial y} \quad \dots \dots \dots (B)$$

Adding equations (A) and (B)

$$u \frac{\partial z}{\partial u} + v \frac{\partial z}{\partial v} = u \frac{\partial z}{\partial x} + uv \frac{\partial z}{\partial y} + v \frac{\partial z}{\partial x} + uv \frac{\partial z}{\partial y}$$

$$u \frac{\partial z}{\partial u} + v \frac{\partial z}{\partial v} = (u + v) \frac{\partial z}{\partial x} + (uv + uv) \frac{\partial z}{\partial y}$$



$$u \frac{\partial z}{\partial u} + v \frac{\partial z}{\partial v} = (u + v) \frac{\partial z}{\partial x} + 2uv \frac{\partial z}{\partial y}$$

From (1) and (2)

$$u \frac{\partial z}{\partial u} + v \frac{\partial z}{\partial v} = x \frac{\partial z}{\partial x} + 2y \frac{\partial z}{\partial y}$$

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## Homogeneous Function:

A function  $F(x, y)$  is said to be homogeneous function in which the power of each term is same.

$$F(x, y, z) = x^3 + y^3 + z^3 - 9x^1y^2 + 19x^2y^1 - \sqrt{3}y^2z^1$$

### Note:

- To verify homogeneous function put  $x = xt$ ,  $y = yt$  and  $z = zt$

$$F(x, y, z) = (xt)^3 + (yt)^3 + (zt)^3 - 9xt^1yt^2 + 19xt^2yt^1 - \sqrt{3}yt^2zt^1$$

$$F(x, y, z) = x^3t^3 + y^3t^3 + z^3t^3 - 9xy^2t^3 + 19x^2ty - \sqrt{3}y^2t^3z$$

$$F(x, y, z) = t^3F(x, y, z)$$

$F$  is homogeneous function of degree 3

- $f(x, y, z) = t^n f(x, y, z)$   $f$  is homogeneous function of degree  $n$

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## Euler's Theorem of Homogeneous Function:

Let  $f(x, y)$  be a homogeneous function of degree  $n$  in  $x$  and  $y$

For 2 variables and 1<sup>st</sup> order 
$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = n f(x, y)$$

Let  $f(x, y, z)$  be a homogeneous function of degree  $n$  so that  $f(x, y, z) = t^n f(x, y, z)$

For 3 variables and 1<sup>st</sup> order 
$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z} = n f(x, y, z)$$

Let  $f$  be a homogeneous function of degree  $n$  in  $x_1, x_2, x_3, \dots, x_n$

For  $n$  variables and 1<sup>st</sup> order

$$x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + x_3 \frac{\partial f}{\partial x_3} + \dots + x_n \frac{\partial f}{\partial x_n} = n f$$



**Euler's Theorem for second order**

Let  $f(x, y)$  be a homogenous function of two variables  $x, y$  of degree  $n$

For 2 variables and 2<sup>nd</sup> order 
$$x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = n(n-1)f$$

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**Example:** If  $f(x, y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{\log x - \log y}{x^2 + y^2}$  then prove that

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + 2f = 0 \quad \text{OR} \quad \text{Find } x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y}$$

**Solution:** Consider  $f(x, y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{\log x - \log y}{x^2 + y^2}$  ∴  $\log a - \log b = \log \left(\frac{a}{b}\right)$

$$f(x, y) = \frac{xy + x^2}{x^3y} + \frac{\log\left(\frac{x}{y}\right)}{x^2 + y^2}$$

$$f(x, y) = \frac{y + x}{x^2y} + \frac{\log\left(\frac{x}{y}\right)}{x^2 + y^2}$$

$$\text{Let } u = \frac{y + x}{x^2y} \quad v = \frac{\log\left(\frac{x}{y}\right)}{x^2 + y^2}$$

$$f(x, y) = u + v \dots \dots \dots (A)$$

$$\text{Put } x = xt, \quad y = yt$$

$$u = \frac{yt + xt}{(xt)^2 yt} \quad v = \frac{\log\left(\frac{xt}{yt}\right)}{(xt)^2 + (yt)^2}$$

$$u = \frac{t(y + x)}{t^3 x^2 y} \quad v = \frac{\log\left(\frac{x}{y}\right)}{t^2(x^2 + y^2)}$$

$$u = \frac{(y + x)}{t^2 x^2 y} \quad v = \frac{\log\left(\frac{x}{y}\right)}{t^2(x^2 + y^2)}$$

$$u = t^{-2} \frac{(y + x)}{x^2 y} \quad v = t^{-2} \frac{\log\left(\frac{x}{y}\right)}{(x^2 + y^2)}$$

$$u = t^{-2} u(x, y) \quad v = t^{-2} v(x, y)$$

∴  $u$  and  $v$  are homogeneous function of degree  $n = -2$



∴ by Eulers theorem

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu \Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = -2u \quad \dots \dots \dots (1)$$

$$x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = nv \Rightarrow x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = -2v \quad \dots \dots \dots (2)$$

$$f = u + v \quad \text{diff w.r.t } x, y$$

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \quad \& \quad \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$x \frac{\partial f}{\partial x} = x \frac{\partial u}{\partial x} + x \frac{\partial v}{\partial x} \quad y \frac{\partial f}{\partial y} = y \frac{\partial u}{\partial y} + y \frac{\partial v}{\partial y}$$

∴ Adding (1) and (2)

$$\Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = -2u - 2v$$

$$\Rightarrow x \frac{\partial u}{\partial x} + x \frac{\partial v}{\partial x} + y \frac{\partial u}{\partial y} + y \frac{\partial v}{\partial y} = -2(u + v)$$

$$\Rightarrow x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = -2f$$

$$\Rightarrow x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + 2f = 0$$

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**Example:** If  $u = \frac{x^3 + y^3}{x + y} + \frac{1}{x^5} \sin^{-1} \left( \frac{x^2 + y^2}{x^2 + 2xy} \right)$  then find the value of

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} \quad \text{at the point } (1, 2)$$

**Solution:** Consider  $f(x, y) = \frac{x^3 + y^3}{x + y} + \frac{1}{x^5} \sin^{-1} \left( \frac{x^2 + y^2}{x^2 + 2xy} \right)$

$$\text{Let } u = \frac{x^3 + y^3}{x + y} \quad v = \frac{1}{x^5} \sin^{-1} \left( \frac{x^2 + y^2}{x^2 + 2xy} \right)$$

$$f(x, y) = u + v \quad \dots \dots \dots (A)$$

$$\text{Put } x = xt, \quad y = yt$$

$$u = \frac{(xt)^3 + (yt)^3}{xt + yt} \quad v = \frac{1}{(xt)^5} \sin^{-1} \left( \frac{(xt)^2 + (yt)^2}{(xt)^2 + 2xtyt} \right)$$



$$u = \frac{t^3(x^3 + y^3)}{t(x + y)} \quad v = \frac{1}{x^5 t^5} \sin^{-1} \left( \frac{t^2(x^2 + y^2)}{t^2(x^2 + 2xy)} \right)$$

$$u = t^2 \frac{(x^3 + y^3)}{(x + y)} \quad v = t^{-5} \frac{1}{x^5} \sin^{-1} \left( \frac{x^2 + y^2}{x^2 + 2xy} \right)$$

$$u = t^2 u(x, y) \quad v = t^{-5} v(x, y)$$

$\therefore u$  is homogeneous function of degree  $n = 2$

$\therefore v$  is homogeneous function of degree  $n = -5$

$\therefore$  by Eulers theorem

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu$$

$$\Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2u \quad \dots \dots \dots (1)$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 2(2-1)u = 2u \quad \dots \dots \dots (2)$$

$$x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = nv$$

$$\Rightarrow x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = -5v \quad \dots \dots \dots (3)$$

$$x^2 \frac{\partial^2 v}{\partial x^2} + 2xy \frac{\partial^2 v}{\partial x \partial y} + y^2 \frac{\partial^2 v}{\partial y^2} = n(n-1)v$$

$$x^2 \frac{\partial^2 v}{\partial x^2} + 2xy \frac{\partial^2 v}{\partial x \partial y} + y^2 \frac{\partial^2 v}{\partial y^2} = -5(-5-1)v = 30v \quad \dots \dots \dots (4)$$

$$f = u + v \quad \text{diff w.r.t } x, y$$

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}$$

multiply by  $x$

$$x \frac{\partial f}{\partial x} = x \frac{\partial u}{\partial x} + x \frac{\partial v}{\partial x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

multiply by  $y$

$$y \frac{\partial f}{\partial y} = y \frac{\partial u}{\partial y} + y \frac{\partial v}{\partial y}$$



$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x^2} \quad \text{multiply by } x^2$$

$$x^2 \frac{\partial^2 f}{\partial x^2} = x^2 \frac{\partial^2 u}{\partial x^2} + x^2 \frac{\partial^2 v}{\partial x^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial y^2} \quad \text{multiply by } y^2$$

$$y^2 \frac{\partial^2 f}{\partial y^2} = y^2 \frac{\partial^2 u}{\partial y^2} + y^2 \frac{\partial^2 v}{\partial y^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x \partial y} \quad \text{multiply by } 2xy$$

$$2xy \frac{\partial^2 f}{\partial x \partial y} = 2xy \frac{\partial^2 u}{\partial x \partial y} + 2xy \frac{\partial^2 v}{\partial x \partial y}$$

∴ Adding (1), (2), (3) and (4)

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} + x^2 \frac{\partial^2 v}{\partial x^2} + 2xy \frac{\partial^2 v}{\partial x \partial y} + y^2 \frac{\partial^2 v}{\partial y^2} = 2u - 5v + 2u + 30v$$

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = 4u + 25v$$

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = 4 \frac{x^3 + y^3}{x + y} + 25 \frac{1}{x^5} \sin^{-1} \left( \frac{x^2 + y^2}{x^2 + 2xy} \right)$$

At point (1, 2) put  $x = 1$  and  $y = 2$

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = 4 \frac{1^3 + 2^3}{1 + 2} + 25 \frac{1}{1^5} \sin^{-1} \left( \frac{1^2 + 2^2}{1^2 + 2(1)(2)} \right)$$

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = 4 \left( \frac{9}{3} \right) + 25 \sin^{-1} \left( \frac{5}{5} \right)$$

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = 12 + 25 \frac{\pi}{2}$$

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**Example:** If  $u = x^n f\left(\frac{y}{x}\right) + y^{-n} g\left(\frac{x}{y}\right)$  then

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = n^2 u$$

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## Euler's Theorem of Non-Homogeneous Function:

Let  $u = f^{-1}(x, y)$  be a non-homogeneous function but  $f(u)$  is homogeneous function of order  $n$  so that  $f(u) = t^n f(u)$

For 2 variables and 1<sup>st</sup> order  $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = n \frac{f(u)}{f'(u)} = g(u)$

For 2 variables and 2<sup>nd</sup> order  $x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = g(u)[g'(u) - 1]$

Where  $g(u) = n \frac{f(u)}{f'(u)}$

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**Example:** If  $u = \sin^{-1}(x^2 + y^2)^{1/5}$  then Prove that

$$x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = \frac{2 \tan u}{5} \left[ \frac{2 \tan^2 u}{5} - \frac{3}{5} \right]$$

**Solution:** Consider  $u(x, y) = \sin^{-1}(x^2 + y^2)^{1/5}$

Put  $x = xt, \quad y = yt$

$$u = \sin^{-1}[(xt)^2 + (yt)^2]^{1/5}$$

$$u = \sin^{-1}[t^2(x^2 + y^2)]^{1/5}$$

$$(a^n)^m = a^{nm}$$

$$u = \sin^{-1}[t^{2/5}(x^2 + y^2)^{1/5}]$$

$$\sin^{-1}(ax) \neq a \sin^{-1}(x)$$

$$u(x, y) \neq t^n u(x, y) \quad \therefore u \text{ is not homogeneous function}$$

$$\sin u = t^{2/5}(x^2 + y^2)^{1/5}$$

$$\therefore f(u) = \sin u \text{ is homogeneous function of degree } n = \frac{2}{5}$$

$$x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = g(u)[g'(u) - 1] \quad \dots \dots \dots (A)$$

where  $g(u) = n \frac{f(u)}{f'(u)}$

here  $f(u) = \sin u \Rightarrow f'(u) = \cos u$

$$\Rightarrow g(u) = n \frac{f(u)}{f'(u)} = \frac{2 \sin u}{5 \cos u} = \frac{2}{5} \tan u$$

$$g'(u) = \frac{2}{5} \sec^2 u$$





Equation (A)

$$x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = \frac{2}{5} \tan u \left[ \frac{2 \sec^2 u}{5} - 1 \right]$$

$$x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = \frac{2}{5} \tan u \left[ \frac{2(\tan^2 u + 1)}{5} - 1 \right]$$

$$x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = \frac{2}{5} \tan u \left[ \frac{2 \tan^2 u + 2}{5} - 1 \right]$$

$$x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = \frac{2}{5} \tan u \left[ \frac{2 \tan^2 u + 2 - 5}{5} \right]$$

$$x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = \frac{2}{5} \tan u \left[ \frac{2 \tan^2 u}{5} - \frac{3}{5} \right]$$

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**Example:** If  $u = \operatorname{cosec}^{-1} \sqrt{\frac{x^{1/2} + y^{1/2}}{x^{1/3} + y^{1/3}}}$  then Prove that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{12} \left[ \frac{\tan^2 u}{12} + \frac{13}{12} \right]$$

**Solution:** Consider  $u = \operatorname{cosec}^{-1} \sqrt{\frac{x^{1/2} + y^{1/2}}{x^{1/3} + y^{1/3}}}$

Put  $x = xt$ ,  $y = yt$

$$u = \operatorname{cosec}^{-1} \sqrt{\frac{(xt)^{1/2} + (yt)^{1/2}}{(xt)^{1/3} + (yt)^{1/3}}}$$

$$u = \operatorname{cosec}^{-1} \sqrt{\frac{t^{1/2}(x^{1/2} + y^{1/2})}{t^{1/3}(x^{1/3} + y^{1/3})}}$$

$$u = \operatorname{cosec}^{-1} \left[ \frac{t^{1/2}(x^{1/2} + y^{1/2})}{t^{1/3}(x^{1/3} + y^{1/3})} \right]^{1/2}$$

$$u = \operatorname{cosec}^{-1} \left[ \frac{t^{1/2} t^{-1/3} (x^{1/2} + y^{1/2})}{(x^{1/3} + y^{1/3})} \right]^{1/2}$$

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$$u = \operatorname{cosec}^{-1} \left[ \frac{t^{1/6}(x^{1/2} + y^{1/2})}{(x^{1/3} + y^{1/3})} \right]^{1/2}$$

$$u = \operatorname{cosec}^{-1} \left[ t^{1/12} \sqrt{\frac{x^{1/2} + y^{1/2}}{x^{1/3} + y^{1/3}}} \right]$$

$u(x, y) \neq t^n u(x, y) \therefore u$  is not homogeneous function

$$\operatorname{cosec} u = t^{1/12} \sqrt{\frac{x^{1/2} + y^{1/2}}{x^{1/3} + y^{1/3}}}$$

$\therefore f(u) = \operatorname{cosec} u$  is homogeneous function of degree  $n = \frac{1}{12}$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$$

$$\text{where } g(u) = n \frac{f(u)}{f'(u)}$$

$$\text{here } f(u) = \operatorname{cosec} u \Rightarrow f'(u) = -\operatorname{cosec} u \cot u$$

$$\Rightarrow g(u) = n \frac{f(u)}{f'(u)} = -\frac{1}{12} \frac{\operatorname{cosec} u}{\operatorname{cosec} u \cot u}$$

$$g(u) = -\frac{1}{12} \frac{1}{\cot u} = -\frac{1}{12} \tan u$$

$$g'(u) = -\frac{1}{12} \sec^2 u$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = -\frac{1}{12} \tan u \left[ \frac{-\sec^2 u}{12} - 1 \right]$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{1}{12} \tan u \left[ \frac{\sec^2 u}{12} + 1 \right]$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{1}{12} \tan u \left[ \frac{\tan^2 u + 1}{12} + 1 \right]$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{1}{12} \tan u \left[ \frac{\tan^2 u + 1 + 12}{12} \right]$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{1}{12} \tan u \left[ \frac{\tan^2 u + 13}{12} \right]$$

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**Example:** If  $u = \sec^{-1} \left[ \frac{x+y}{x^{1/2} + y^{1/2}} \right]$  then Prove that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = -\frac{\cot u}{4} [3 + \cot^2 u]$$

\*\*\*\*\*

**Example:** If  $u = \tan^{-1} \left[ \frac{x^3 + y^3}{x + y} \right]$  then Prove that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \sin 2u [1 - 4 \sin^2 u]$$

\*\*\*\*\*

**Example:** If  $u = \sin^{-1} \left[ \frac{x+y}{x^{1/2} + y^{1/2}} \right]$  then Prove that

$$x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = \frac{-\sin u \cos 2u}{4 \cos^3 u}$$

\*\*\*\*\*

**Example:** If  $u = \sin^{-1} \left[ \frac{x^2 + y^2}{x+y} \right]^{\frac{1}{2}}$  then Prove that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{4} [\tan^2 u - 1]$$

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