The rank of Matrix:-

The matrix is said to be of rank r if there is,

i) at least one minor of order r which is non-zero.

ii) Every minor of order $(r+1)$ is equals to zero.

OR The rank of matrix is order of its non-vanishing minor and it is denoted as.

$$\boxed{\rho(A) = r}$$

Note: Elementary transformations of a matrix do not changes rank and order of matrix.

(*) Rank of matrix by Reducing the matrix to Echelon form:-

Echelon or canonical form:-

let A be a matrix of order $m \times n$. then the canonical or Echelon form of A is a matrix in which,

i) one or more elements in each of the first r rows are non-zero and elements in remaining rows are zero.

ii) In the first r rows, the first non-zero elements in each row appears in a column to the right of first non-zero element of the preceding row e.g. $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

above matrix is in Echelon form.

Note: In Echelon form no. of non-zero rows is equal to rank of matrix.

∴ In above e.g. $\boxed{\rho(A) = 3}$

Some Examples :-

① find rank of A where $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 4 & 0 & -1 \\ -1 & 0 & -2 & 7 \end{bmatrix}$

Soln: $R_2 - 3R_1$, $R_3 + R_1$

$$A \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -2 & 3 & -10 \\ 0 & 2 & -3 & 10 \end{bmatrix}$$

$R_3 + R_2$

$$A \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -2 & 3 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ which is in Echelon form.

∴ Here no. of non-zero rows = 2

$\boxed{\therefore \rho(A) = 2}$

2) Find the rank of $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

Sol^n $R_1 \leftrightarrow R_2$

$R_2 - 2R_1,\ R_3 - 3R_1,\ R_4 - 6R_1$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$R_2 - R_3$

$R_3 - 4R_2,\ R_4 - 9R_2$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 66 & 44 \end{bmatrix}$$

$R_4 - 2R_3$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ which is in echelon form.

here no. of non-zero rows = 3

$$\boxed{\therefore\ \rho(A) = 3}$$

(*) Homework:

Q. Find rank of following matrices :-

① $\begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix}$ Ans $\rho(A) = 4$

② $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ Ans $\rho(A) = 2$

③ $\begin{bmatrix} 4 & 2 & -1 & 2 \\ 1 & -1 & 2 & 1 \\ 2 & 2 & -2 & 0 \end{bmatrix}$ Ans $\rho(A) = 3$

"Techno-Social Excellence"

Marthwada Mitramandal's Institute of Technology (MMIT)

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LECTURE NOTES

Faculty :- Mr. Rahul Mali

Subject :- Eng. maths-I

Reference for Topic:-

Name of the Book/paper/Website/Other	Author Name	Page No/Link

Topic Name :- System of Linear Equations.

System of linear Equations:-

consider a system of m linear equations in n unknowns, $x_1, x_2, x_3 \dots x_n$.

$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$

$\vdots$

$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$.

The above system of eqn can be written in matrix form as.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

i.e. $AX = B$

where A is called ~~matr~~ coefficient matrix

Augmented matrix:-

It is denoted by $[A|B]$ and is given by

$$[A|B] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ & & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$$

* Non-Homogeneous Equations :-

For a system $AX = B$ if matrix B is not zero matrix then system is called non-homogeneous system of equations.

If matrix B is zero matrix.

i.e. $AX = 0$

then system is called Homogeneous system of equations.

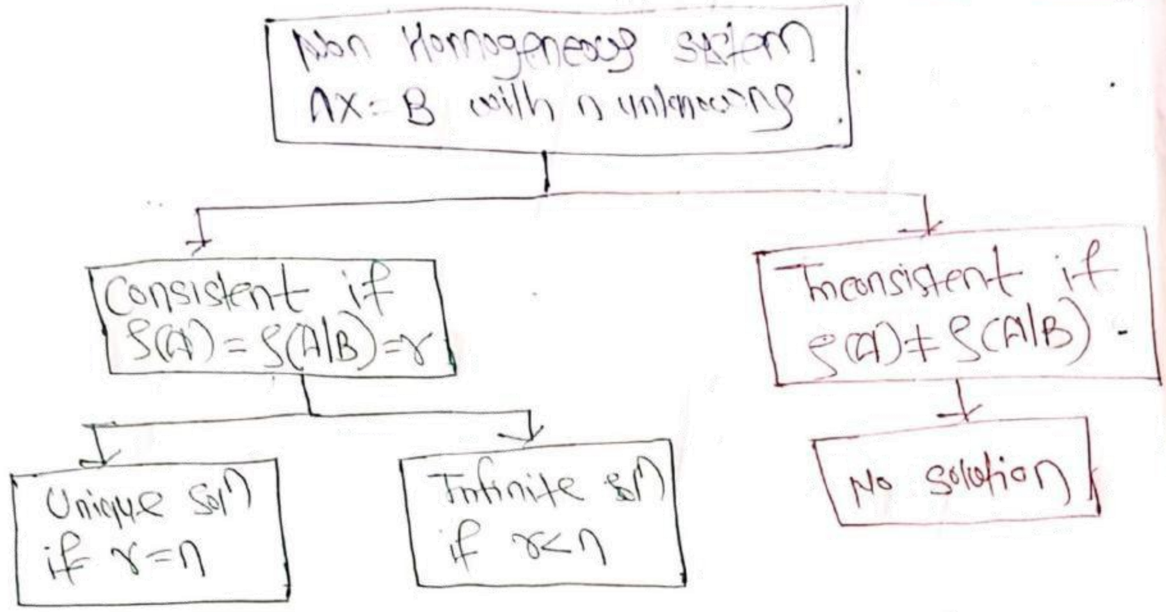
Note: If system of equation has solution then that system is called consistent system otherwise it is called inconsistent.

* Condition for consistency of non-homogeneous eq^n's :-

Consider a system of m eq^n's in n unknowns

$AX = B$

(*) Charts for system of non-homogeneous Equation:



*Note: while solving system of non-homogeneous or homogeneous equations we will perform only elementary row transformation.

(*) Examples on Non-homogeneous system of eqn:-

① By considering ranks of relevant matrices, examine for consistency of system of equations and solve them if found consistent.

$$2x - y - z = 2$$
$$x + 2y + z = 2$$
$$4x - 7y - 5z = 2$$

Soln: The matrix form of given system can be written as.

$$\begin{bmatrix} 2 & -1 & -1 \\ 1 & 2 & 1 \\ 4 & -7 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

i.e. $AX = B$

Consider an augmented matrix,

$$(A|B) = \left[\begin{array}{ccc|c} 2 & -1 & -1 & 2 \\ 1 & 2 & 1 & 2 \\ 4 & -7 & -5 & 2 \end{array}\right]$$

R_{12}

$$= \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & -1 & -1 & 2 \\ 4 & -7 & -5 & 2 \end{array}\right] \underset{R_2-2R_1,\ R_3-4R_1}{=} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -5 & -3 & -2 \\ 0 & -15 & -9 & -6 \end{array}\right]$$

now $R_3 - 3R_2$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -5 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

here $\rho(A) = 2$, $\rho(A|B) = 2$

i.e. $\rho(A) = \rho(A|B)$

$\therefore$ System is consistent.

but $\rho(A) = \rho(A|B) = 2 < 3$

$\therefore$ System has infinte sol^{n}s.

Let $z = t$,

from R_2 $\quad -5y - 3z = -2 \Rightarrow -5y - 3t = -2$

$\Rightarrow -5y = 3t - 2$

$\Rightarrow y = \dfrac{3t-2}{-5} = \dfrac{2-3t}{5}$

from R_1 $\quad x + 2y + z = 2$

$\therefore x + 2\left(\dfrac{2-3t}{5}\right) + t = 2 \Rightarrow \boxed{x = \dfrac{6+t}{5}}$ $\quad$ {where t is parameter}

hence the sol^n set is, $x = \dfrac{6+t}{5}$, $y = \dfrac{2-3t}{5}$, $z = t$.

(2) $3x+y+2z=3$

$2x-3y-z=-3$

$x+2y+z=4$

Sol^n Matrix form is

$$\begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$$

Its augmented matrix is

$$(A|B) = \left[\begin{array}{ccc|c} 3 & 1 & 2 & 3 \\ 2 & -3 & -1 & -3 \\ 1 & 2 & 1 & 4 \end{array}\right]$$

R_{13}

$$= \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 2 & -3 & -1 & -3 \\ 3 & 1 & 2 & 3 \end{array}\right]$$

$R_2-2R_1,\ R_3-3R_1$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -7 & -3 & -11 \\ 0 & -5 & -1 & -9 \end{array}\right]$$

R_3-R_2

$$= \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -7 & -3 & -11 \\ 0 & 2 & 2 & 2 \end{array}\right]$$

here $\rho(A) = \rho(A|B) = 3$

= no. of unknowns

$\therefore$ System have unique sol^n

from R_1, R_2 & R_3

$x+2y+z=4$ —(1)

$-7y-3z=-11$ —(2)

$2y+2z=2 \Rightarrow y+z=1 \times (7)$

$\Rightarrow 7y+7z=7$

$-7y-3z=-11$

$4z=-4$

$\Rightarrow \boxed{z=-1}$

$\Rightarrow \cancel{z=1}$

$\cancel{-5y-3z=-11}$

$\cancel{-5y=}$

put in (2)

$-7y+3=-11$

$-7y=-14 \Rightarrow \boxed{y=2}$

put in (1)

$x+4-1=4$

$x+3=4 \Rightarrow \boxed{x=1}$

$\therefore$ Sol^n is

$\boxed{x=1,\ y=2,\ z=-1}$

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(3) $2x-3y+7z=5$
$3x+y-3z=13$
$2x+19y-47z=32$

Sol) matrix form is,

$$\begin{bmatrix} 2 & -3 & 7 \\ 3 & 1 & -3 \\ 2 & 19 & -47 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 5 \\ 13 \\ 32 \end{bmatrix}$$

∴ Augmented matrix is

$$(A|B)=\left[\begin{array}{ccc|c} 2 & -3 & 7 & 5 \\ 3 & 1 & -3 & 13 \\ 2 & 19 & -47 & 32 \end{array}\right] \overset{R_1-R_2}{=} \left[\begin{array}{ccc|c} -1 & -4 & 10 & -8 \\ 3 & 1 & -3 & 13 \\ 2 & 19 & -47 & 32 \end{array}\right]$$

$R_1 \to (-1)R_1$

$$=\left[\begin{array}{ccc|c} 1 & 4 & -10 & 8 \\ 3 & 1 & -3 & 13 \\ 2 & 19 & -47 & 32 \end{array}\right] \overset{R_2-3R_1,\ R_3-2R_1}{=} \left[\begin{array}{ccc|c} 1 & 4 & -10 & 8 \\ 0 & -11 & 27 & -11 \\ 0 & 11 & -27 & 16 \end{array}\right]$$

R_3+R_2

$$=\left[\begin{array}{ccc|c} 1 & 4 & -10 & 8 \\ 0 & -11 & 27 & -11 \\ 0 & 0 & 0 & 5 \end{array}\right].$$

here $\rho(A)=2$, $\rho(A|B)=3$

∴ $\rho(A) \neq \rho(A|B)$

hence system is inconsistent

i.e. There is no solution.

* Non-homogeneous system of eqⁿs containing unknowns:-

① For what values of k the set of eqⁿs.

$$2x - 3y + 6z - 5t = 3$$
$$y - 4z + t = 1$$
$$4x - 5y + 8z - 9t = k$$

has i> No solⁿ ii> an infinite number of solⁿ.

Solⁿ matrix form of given system is

$$\begin{bmatrix} 2 & -3 & 6 & -5 \\ 0 & 1 & -4 & 1 \\ 4 & -5 & 8 & -9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ k \end{bmatrix}$$

i.e. $AX = B$

consider augmented matrix

$$(A|B) = \left[\begin{array}{cccc|c} 2 & -3 & 6 & -5 & 3 \\ 0 & 1 & -4 & 1 & 1 \\ 4 & -5 & 8 & -9 & k \end{array}\right]$$

$R_3 - 2R_1$

$$= \left[\begin{array}{cccc|c} 2 & -3 & 6 & -5 & 3 \\ 0 & 1 & -4 & 1 & 1 \\ 0 & 1 & -4 & 1 & k-6 \end{array}\right]$$

now $R_3 - R_2$

$$= \left[\begin{array}{cccc|c} 2 & -3 & 6 & -5 & 3 \\ 0 & 1 & -4 & 1 & 1 \\ 0 & 0 & 0 & 0 & k-7 \end{array}\right]$$

from above matrix form
we can say that
$\rho(A) = 2$ & $\rho(A|B) = 3$ if $k-7 \neq 0$

$\therefore \rho(A) \neq \rho(A|B)$ if $k \neq 7$

i.e. There is no solⁿ for given system if $\underline{\underline{k \neq 7}}$

we can say that,

$\rho(A) = \rho(A|B) = 2$ if $k-7 = 0$

i.e. $\rho(A) = \rho(A|B) = 2 < 4$ (no. of unknowns)
for $\underline{\underline{k=7}}$

$\therefore$ system has infinite no. of solⁿs. for $\underline{\underline{k=7}}$

---

(2) For what values of λ & μ, the system of eqⁿ,

$x+y+z = 6$
$x+2y+3z = 10$
$x+2y+\lambda z = \mu$

have.
i) No solⁿ ii) unique solⁿ iii) an infinite no. of solⁿ.

Solⁿ: matrix form of above system is,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

i.e. $AX = B$

③ Use matrix method to determine values of λ for which the eqⁿs.

$x+2y+z=3$

$x+y+z=\lambda$

$3x+y+3z=\lambda^2$

are consistent and solve them for these values of λ

Solⁿ Aug. matrix is

$R_2-R_1,\ R_3-3R_1$

$$(A|B)=\left[\begin{array}{ccc|c}1&2&1&3\\1&1&1&\lambda\\3&1&3&\lambda^2\end{array}\right]=\left[\begin{array}{ccc|c}1&2&1&3\\0&-1&0&\lambda-3\\0&-5&0&\lambda^2-9\end{array}\right]$$

R_3-5R_2

$$=\left[\begin{array}{ccc|c}1&2&1&3\\0&-1&0&\lambda-3\\0&0&0&\lambda^2-5\lambda+6\end{array}\right] \quad — \quad (*)$$

For consistency $\rho(A)=\rho(A|B)$

It is possible if $\lambda^2-5\lambda+6=0$

$\therefore (\lambda-2)(\lambda-3)=0 \Rightarrow \lambda=2,\ \lambda=3$

for $\lambda=2$ eqⁿ (*) becomes

$$(A|B)=\left[\begin{array}{ccc|c}1&2&1&3\\0&-1&0&-1\\0&0&0&0\end{array}\right]$$

$\therefore \rho(A)=\rho(A|B)=2<3$

$\therefore$ system has infinite no. of solⁿ

let $z=t,\ y=1,\ x=1-t$

For $\lambda=3$

$$A|B=\left[\begin{array}{ccc|c}1&2&1&3\\0&-1&0&0\\0&0&0&0\end{array}\right]$$

let $z=t,\ y=0,\ x=3-t$

Augmented matrix is,

$$(A|B) = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$R_2 - R_1,\ R_3 - R_1$ $\qquad\qquad\qquad$ $R_3 - R_2$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right]$$

i) for no solⁿ, $\rho(A) \neq \rho(A|B)$

it is possible when $\lambda - 3 = 0$ & $\mu - 10 \neq 0$

i.e $\lambda = 3$ & $\mu \neq 10$

in this case $\rho(A) = 2$ & $\rho(A|B) = 3$

i.e $\rho(A) \neq \rho(A|B)$

∴ System has no solⁿ if $\underline{\underline{\lambda = 3}}$ & $\underline{\underline{\mu \neq 10}}$

ii) for unique solⁿ, $\rho(A) = \rho(A|B) =$ no. of unknowns (3)

it is possible if $\lambda - 3 \neq 0$ & for any choice of $\mu - 10$

i.e. $\lambda \neq 3$ & for any value of μ.

∴ System has unique solⁿ if $\underline{\underline{\lambda \neq 3}}$.

iii) For infinite no. of solⁿ, $\rho(A) = \rho(A|B) < 3$

it is possible if $\lambda - 3 = 0$ & $\mu - 10 = 0$

i.e $\lambda = 3$ & $\mu = 10$

∴ $\rho(A) = \rho(A|B) = 2 < 3$.

∴ System has infinite no. of solⁿs if $\underline{\underline{\lambda = 3}}$ & $\underline{\underline{\mu = 10}}$

Condition for consistency of homogeneous Equations:-

Consider a system of m linear equations in n unknowns $x_1, x_2, \dots x_n$.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0$$

The matrix form of above system is

$$AX = Z$$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where A is coefficient matrix. Z is null or zero matrix.
X is matrix of unknowns.

The augmented matrix is given by.

$$(A|Z) = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & 0 \\ a_{21} & a_{22} & \dots & a_{2n} & 0 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & 0 \end{array}\right]$$

Note: In homogeneous system of eqn $\rho(A) = \rho(A|Z)$.
∴ Homogeneous system of eqn is always consistent.

Depending on rank of matrix of system of eqn with n unknowns there are following cases.

Case I If $\rho(A) = \rho(A|Z) = n$ (no. of unknowns)

then system has unique soln and ~~wh~~ it is called trivial or zero soln.

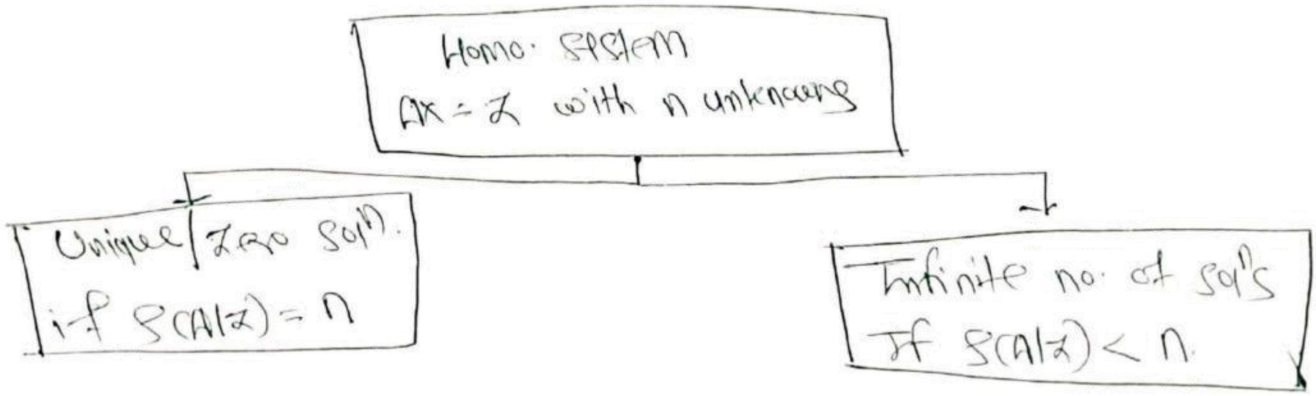
i.e. $x_1 = x_2 = \dots = x_n = 0$.

Case II : If $\rho(A) = \rho(A|Z) = r < n$

then system has infinite no of solutions called as non trivial soln.

In this case there are $(n-r)$ ~~un~~ free unknowns. denote them as a parameters and write remaining unknowns in terms of that parameters. which gives infinite no. of solutions.

Flowchart



Some Examples :-

1) Examine for non-trivial solⁿ. the following set of equations & solve them.

$$4x - y + 2z + t = 0$$
$$2x + 3y - z - 2t = 0$$
$$7y - 4z - 5t = 0$$
$$2x - 11y + 7z + 8t = 0$$

Solⁿ: The matrix form of above system can be written as $AX = Z$

Its aug matrix is

$$(A|Z) = \left[\begin{array}{cccc|c} 4 & -1 & 2 & 1 & 0 \\ 2 & 3 & -1 & -2 & 0 \\ 0 & 7 & -4 & -5 & 0 \\ 2 & -11 & 7 & 8 & 0 \end{array}\right]$$

R_{12}

$$= \left[\begin{array}{cccc|c} 2 & 3 & -1 & -2 & 0 \\ 4 & -1 & 2 & 1 & 0 \\ 0 & 7 & -4 & -5 & 0 \\ 2 & -11 & 7 & 8 & 0 \end{array}\right]$$

$R_2 - 2R_1,\ R_4 - R_1$

$$= \left[\begin{array}{cccc|c} 2 & 3 & -1 & -2 & 0 \\ 0 & -7 & 4 & 5 & 0 \\ 0 & 7 & -4 & -5 & 0 \\ 0 & -14 & 8 & 10 & 0 \end{array}\right]$$

$R_3 + R_2,\ R_4 - 2R_2$

$$= \left[\begin{array}{cccc|c} 2 & 3 & -1 & -2 & 0 \\ 0 & -7 & 4 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

put $z = a,\ t = b$

$$\therefore x = \frac{-5b - a}{14},\quad y = \frac{4b + 5a}{7}$$

(2) Solve the following system of eq^n.

$$x+2y+3z=0$$
$$2x+3y+z=0$$
$$4x+5y+4z=0$$
$$x+y-2z=0$$

Sol^n : Ans unique. Sol^n

③ for different values of k, discuss the nature of solution of following equations

$x+2y-z=0$

$3x+(k+7)y-3z=0$

$2x+4y+(k-3)z=0$

Sol^n The matrix form of system is,

$AX=Z$.

Aug. matrix is,

$$(A|Z)=\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 3 & k+7 & -3 & 0 \\ 2 & 4 & k-3 & 0 \end{array}\right]$$

$R_2-3R_1,\ R_3-2R_1$

$$=\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & k+1 & 0 & 0 \\ 0 & -2 & k-1 & 0 \end{array}\right] \quad \frac{R_2}{k+1} \quad =\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & k-1 & 0 \end{array}\right]$$

R_3+2R_2

$$=\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & k-1 & 0 \end{array}\right]$$

for $k\neq 1$, $\rho(A)=\rho(A|Z)=3=$ no. of unknowns.

$\therefore$ system has unique sol^n. i.e. trivial sol^n.

$\therefore$ for $k=1$, $x=y=z=0$.

for $k=1$, $\rho(A)=\rho(A|Z)=2 <$ no. of unknowns.

$\therefore$ System has non-trivial soln.

$\therefore$ <u>for $k=1$</u>

$$(A|Z)= \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore$ From R_1 & R_2

$x+2y-z=0$ — ①

$y=0$

put $z=t$ & $y=0$ in ①

$x=t$

$\therefore$ <u>for $k=1$</u>.

$$\boxed{x=t,\ y=0,\ z=t}$$

Linear Dependent and independent vectors.

we define n dimensional vector as an ordered set of n elements $x_1, x_2, \ldots x_n$ and is denoted by

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} \text{ or } X = [x_1, x_2 \cdots x_n]$$

The elements $x_1, x_2 \ldots x_n$ are called components of X.

Linearly Dependent vectors:

let $X_1, X_2, \ldots X_n$ be n vectors

if there exists n scalars $c_1, c_2, \ldots c_n$ not all zero s.t.

$c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0$

∴ These vectors are called as Linearly Dependent (LD) vectors.

If all scalars $c_1, c_2 \ldots c_n$ are zero.

then vectors are called Linearly independent (LI) vectors:

Note: while checking LD & LI of given vectors create system of eqn in which unknowns are scalars of given vector which is homo. system of eqn

Then solve that system by matrix method and find values of scalars.

If all scalars are zero then system is LI.

If all scalar are non-zero system is LD.

(*) Some Examples

(1) Examine for Linear Dependence or independence of vectors $(2,-1,3,2)$, $(1,3,4,2)$ $(3,-5,2,2)$ if dependent find relation betn them. (A)

Soln Let given vectors be X_1, X_2, X_3 respectively.

consider matrix eqn. $c_1X_1+c_2X_2+c_3X_3 = 0$ ——(*)

where c_1, c_2, c_3 are Scalars.

$\therefore c_1(2,-1,3,2)+c_2(1,3,4,2)+c_3(3,-5,2,2)=0$

$\therefore 2c_1+c_2+3c_3=0$

$-c_1+3c_2+-5c_3=0$

$3c_1+4c_2+2c_3=0$

$2c_1+2c_2+2c_3=0$

which is homo. system of eqn. whose matrix form is

$$\begin{bmatrix} 2 & 1 & 3 \\ -1 & 3 & -5 \\ 3 & 4 & 2 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e. $AX = Z$

Its augmented matrix is,

$$(A|Z) = \left[\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ -1 & 3 & -5 & 0 \\ 3 & 4 & 2 & 0 \\ 2 & 2 & 2 & 0 \end{array}\right]$$

$R_{\cancel{1}}+R_2$ $\qquad$ $R_2+R_1,\ R_3-3R_1,\ R_4-2R_1$

$$(A|Z)=\left[\begin{array}{ccc|c}1&4&-2&0\\-1&3&-5&0\\3&4&2&0\\2&2&2&0\end{array}\right]=\left[\begin{array}{ccc|c}1&4&-2&0\\0&7&-7&0\\0&-8&8&0\\0&-6&6&0\end{array}\right]$$

$\frac{1}{7}R_2,\ \frac{1}{8}R_3,\ \frac{1}{6}R_4$ $\qquad$ $R_3+R_2,\ R_4+R_2$

$$=\left[\begin{array}{ccc|c}1&4&-2&0\\0&1&-1&0\\0&-1&1&0\\0&-1&1&0\end{array}\right]=\left[\begin{array}{ccc|c}1&4&-2&0\\0&1&-1&0\\0&0&0&0\\0&0&0&0\end{array}\right]$$

$\rho(A)=\rho(A|Z)=2<$ no. of unknowns (3)

$\therefore$ System has infinite no. of sol^n s. (i.e. non-trivial sol^n)

from R_1 & R_2

~~x+4y~~ $c_1+4c_2-2c_3=0$ — ①

$c_2-c_3=0$ — ②

put $\boxed{c_3=t}$ in ② $\Rightarrow$ $\boxed{c_2=t}$

eq^n ① becomes $c_1+4t-2t=0$

$c_1+2t=0 \Rightarrow \boxed{c_1=-2t}$

put all values in eq^n (*)

$-2t\,x_1+t\,x_2+t\,x_3=0$

$$\boxed{\therefore\ 2x_1=x_2+x_3}$$

which is required relation bet^n x_1, x_2, x_3.

——— x ———

② $X_1 = (1, -1, 1),\ X_2 = (2, 1, 1),\ X_3 = (3, 0, 2)$ Linear

$\underline{\text{Sol}^n}$ $\underline{\text{ans}}$ $X_1 + X_2 = X_3$

Linear Transformation :-

The transformation $X' = AX$

i.e. $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

is called linear transformation from (x, y, z) to (x', y', z') in three dimension.

The relation $Y = AX$ express $y_1, y_2, \ldots y_n$ in terms of $x_1, x_2, \ldots x_n$.

It may be noted that every square matrix defines linear transformation.

A is called as matrix of transformation.

Note:

① If $|A| = 0$ then Transformation is called singular transformation.

② If $|A| \neq 0$ then it is called regular transformation.

Some Examples :-

① Given the transformation $Y = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.
Find the co-ordinates (x_1, x_2, x_3) in X corresponding to $(1, 2, -1)$ in Y.

Sol^n Given that $Y = AX$.

i.e. $AX = Y \Rightarrow$

$$\therefore \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

which is non-homo. system. to solve it Augmented matrix is.

$$(A|B) = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 0 & -2 & -1 \end{array}\right]$$

$$\overset{R_{12}}{=} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 2 & 1 & 1 & 1 \\ 1 & 0 & -2 & -1 \end{array}\right] \overset{R_2 - 2R_1,\ R_3 - R_1}{=} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & -1 & -3 & -3 \\ 0 & -1 & -4 & -3 \end{array}\right]$$

$$\overset{R_3 + -R_2}{=} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & -1 & -3 & -3 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$\therefore \rho(A) = \rho(A|B) = 3 =$ no. of unknown

$\therefore$ System is consistent with unique sol^n

$\therefore x + y + 2z = 2$
$-y - 3z = -3$
$-z = 0 \Rightarrow z = 0$

$\boxed{\therefore y = 3}$

$x + 3 = 2 \Rightarrow x = -1$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}$$

hence $(-1, 3, 0)$ corresponding to $(1, 2, -1)$ in Y

) Find linear operator in two dimensions which maps the vectors (1,1), and (3,-2) into the vectors (2,1) and (1,2) respectively.

Sol^n: w.k.T. the linear T. is $Y = AX$.

i.e. $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

given that (1,1) maps to (2,1)

$\therefore \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\Rightarrow a_1 + b_1 = 2 \quad — (1)$

$a_2 + b_2 = 1 \quad — (2)$

given that (3,-2) maps to (1,2)

$\therefore \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

$\Rightarrow \therefore 3a_1 - 2b_1 = 1 \quad — (3)$

$3a_2 - 2b_2 = 2 \quad — (4)$

multiply eq^n (1) by 2 & add to eq^n (3)

$3a_1 - 2b_1 = 1$

$2a_1 + 2b_1 = 4$

$5a_1 = 5$

$\Rightarrow \boxed{a_1 = 1} \quad \boxed{b_1 = 1}$

$2a_2 + 2b_2 = 2$

$3a_2 - 2b_2 = 2$

$5a_2 = 4 \Rightarrow a_2 = \frac{4}{5}$

$b_2 = 1 - \frac{4}{5} = \frac{1}{5}$

$\therefore A = \begin{bmatrix} 1 & 1 \\ \frac{4}{5} & \frac{1}{5} \end{bmatrix}$ is the required operator

(3) S.T. the transformations

$y_1 = x_1 + x_2 + x_3$

$y_2 = 2x_1 + 3x_2 + 4x_3$

$y_3 = x_1 - x_2 + x_3$

is regular. also find the co-ordinates (x_1, x_2, x_3) corresponding to $(6, 20, 2)$ in Y. (ans 1, 2, 3)

5) Orthogonal Transformation and orthogonal matrix :-

Defn. The transformation $Y = AX$ is said to orthogonal Transformation if matrix of transformation A is orthogonal matrix.

Orthogonal matrix :-

The matrix A is said to be orthogonal if $AA^T = I$

where A^T is transpose matrix of matrix A.

i.e. $\boxed{A^T = A^{-1}}$

OR $\boxed{A^{-1} = A^T}$ in orthogonal matrix.

*** Some examples:**

① Show that $A = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$ is an orthogonal and find its inverse.

Sol^n : W.K.T. A is orthogonal if $AA^T = I$.

let $AA^T = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$

$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 & 0 \\ 0 & 1 & 0 \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & 0 & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

$\therefore$ A is orthogonal matrix.
w.k.T. in orthogonal matrix $A^{-1} = A^{T}$

$$\therefore A^{-1} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

---

② Determine the values of a, b, c when

$$\begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$$ is orthogonal.

Sol^n Let A be the given matrix, which is ortho.

$\therefore AA^T = I.$

$$\therefore \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}\begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 2b^2+c^2 & 2b^2-c^2 & -2b^2+c^2 \\ 2b^2-c^2 & a^2+b^2+c^2 & a^2-b^2-c^2 \\ -2b^2+c^2 & a^2-b^2-c^2 & a^2+b^2+c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore 2b^2 + c^2 = 1$
$2b^2 - c^2 = 0$

$4b^2 = 1 \Rightarrow b^2 = \frac{1}{4}$

$\boxed{\therefore b = \pm \frac{1}{2}}$

$a^2 + b^2 + c^2 = 1$
$a^2 - b^2 - c^2 = 0$

$2a^2 = 1 \Rightarrow a^2 = \frac{1}{2}$

$\boxed{a = \pm \frac{1}{\sqrt{2}}}$

~~$2b^2 + c^2 = 1 \Rightarrow c^2 = \frac{1}{2}$~~
~~$-2b^2 \mp c^2 = 0$~~
~~$0 \quad 2c^2 = 1$~~

$2\left(\frac{1}{4}\right) - c^2 = 0$
$\frac{1}{2} - c^2 = 0$
$c^2 = \frac{1}{2} \Rightarrow \boxed{c = \pm \frac{1}{\sqrt{2}}}$

— x —

Applications to Problems in engineering :

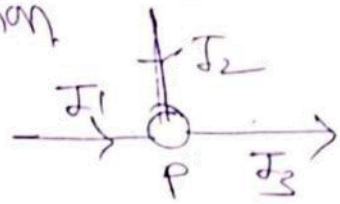
we study applications in linear electric circuit. In an electric circuit current flow is governed by following laws.

(*) ohms law - The voltage drop V across resistance R is $\boxed{V = IR}$ where I is current flow.

(*) kirchoff's law of current :- (KCL)

The current flow into node is equal to current flow out of node. The law can also be stated as the algebraic sum of the current entering any node is zero

Node is a point at which two or more ~~elents~~ elements the circuit have a common connection



By KCL $I_1 + I_2 = I_3$

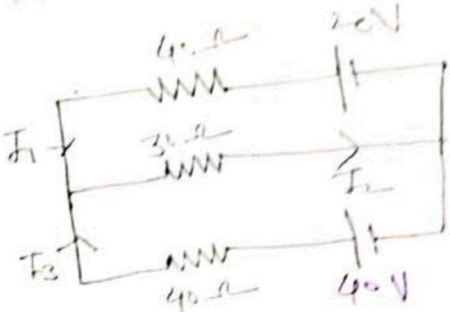
or $I_1 + I_2 - I_3 = 0$

(*) kirchoff's voltage law :- (KVL)

The algebraic sum of voltage drops around a closed loop is equal to the total voltage source in the loop.

⊛ Some Examples

① Determine the currents in the network given in figure bellow.



Sol^n Applying KCL & KVL we get.

$$I_1 + I_3 = I_2$$

$$\therefore I_1 - I_2 + I_3 = 0 \quad -①$$

$$40 I_1 + 30 I_2 = 20 \Rightarrow 4I_1 + 3I_2 = 2 \quad -②$$

$$40 I_3 + 30 I_2 = 40 \Rightarrow 3I_2 + 4I_3 = 4 \quad -③$$

Matrix form of eq^n ① ② & ③

$$\begin{bmatrix} 1 & -1 & 1 \\ 4 & 3 & 0 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}$$

which is non homo. system of eq^n.

Aug. matrix is

$$(A|B) = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 4 & 3 & 0 & 2 \\ 0 & 3 & 4 & 4 \end{array}\right] \overset{R_2 - 4R_1}{=} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 7 & -4 & 2 \\ 0 & 3 & 4 & 4 \end{array}\right]$$

$$\overset{R_3 + R_2}{=} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 7 & -4 & 2 \\ 0 & 10 & 0 & 6 \end{array}\right] \quad \therefore \rho(A) = \rho(A|B) = 3$$

Unique sol^n.

$$I_1 - I_2 + I_3 = 0$$

$$7I_2 - 4I_3 = 2$$

$$10 I_2 = 6 \Rightarrow \boxed{I_2 = \frac{6}{10}} \quad \boxed{I_1 = \frac{1}{20},\ I_3 = \frac{11}{20}}$$