Practical - I

LIMITS AND CONTINUITY

1) 1) $$\lim_{x \to a} \left[\frac{\sqrt{a+2x} - \sqrt{3}\,x}{\sqrt{3a+x} - 2\sqrt{x}} \right]$$

$$\lim_{x \to a} \left[\frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}} \times \frac{\sqrt{a+2x} + \sqrt{3x}}{\sqrt{a+2x} + \sqrt{3x}} \times \frac{\sqrt{3a} + x + 2}{\sqrt{2a} + x + 2} \right]$$

$$\lim_{x \to a} \frac{(a+2x-3x)(\sqrt{3a+x} + 2\sqrt{x})}{(3a+x-4x)(\sqrt{a+2x} + \sqrt{3x})}$$

$$\lim_{x \to a} \frac{(a-x)(\sqrt{3a+x} + 2\sqrt{x})}{(3a-3x)(\sqrt{a+2x} + \sqrt{3x})}$$

$$\frac{1}{3} \lim_{x \to a} \frac{\cancel{(a-x)}(\sqrt{3a+x} + 2\sqrt{x})}{\cancel{(a-x)}(\sqrt{a+2x} + \sqrt{3x})}$$

$$\lim_{x \to a} \frac{\sqrt{3a+a} + 2\sqrt{a}}{\sqrt{a+2a} + \sqrt{3}\,a}$$

$$\lim_{x \to a} \frac{\sqrt{4a} + 2\sqrt{a}}{\sqrt{3a} + \sqrt{3a}}$$

$$\cdot \quad \frac{1}{3} \frac{2\sqrt{a} + 2\sqrt{a}}{\sqrt{3}\,a + \sqrt{3a}}$$

$$\frac{1}{3} \times \frac{\cancel{4}^{2}\sqrt{a}}{\cancel{2}_{1}\sqrt{3a}}$$

$$= \frac{2}{3\sqrt{3}}$$

2) $$\lim_{y \to 0} \left[\frac{\sqrt{a+y} - \sqrt{a}}{y\sqrt{a+y}} \right]$$

$$\lim_{y \to 0} \left(\frac{\sqrt{a+y} - \sqrt{a}}{y\sqrt{a+y}} \times \frac{\sqrt{a+y} + \sqrt{a}}{\sqrt{a+y} + \sqrt{a}} \right)$$

$$\lim_{y \to 0} \frac{a + y - a}{y\sqrt{a+y}\,(\sqrt{a+y} + \sqrt{a})}$$

$$\frac{\cancel{y}}{\cancel{y}(\sqrt{a+y}\,(\sqrt{a+y}) + \sqrt{a})}$$

$$\frac{1}{\sqrt{a+0}\,(\sqrt{a+0} + \sqrt{a})}$$

$$\cancel{\frac{1}{\sqrt{a}\,(\sqrt{a} + \sqrt{a})}}$$

$$\frac{1}{\sqrt{a}\,(2\sqrt{a})} = \frac{1}{2a}$$

3) $\lim_{x \to \pi/6} \frac{\cos x - \sqrt{3}\sin x}{\pi - 6x}$

$$x - \frac{\pi}{6} = h$$

$$x = h + x/6 \quad \text{where } h \to 0$$

$$\lim_{h \to 0} \frac{\cos(h + x/6) - \sqrt{3}(h + x\sqrt{6})}{x - 6(h + x/6)}$$

$$\lim_{h \to 0} \frac{(\cos h . \cos x/6 - \sin h . \sin x/6) - \sqrt{3} (\sin h . \cos x/6 + \cos h . \sin x/6)}{-6h}$$

$$\lim_{h \to 0} \frac{\left[\left(\cos h \frac{\sqrt{3}}{2} - \left(\sin h \frac{1}{2}\right)\right] - \sqrt{3} \left(\sin h \frac{\sqrt{3}}{2} + \cos h . \frac{1}{2}\right)}{-6h}$$

$$\lim_{x \to 0} \frac{\left(\cos \frac{\sqrt{3} h}{2} - \sin \frac{h}{2}\right) - \left(\sin \frac{3h}{2} + \cos \frac{\sqrt{3}}{2} h\right)}{-6h}$$

$$\lim_{h \to 0} \frac{\cancel{-}\frac{\sin 4h}{2}}{\cancel{-}6h}$$

$$\lim_{h \to 0} \frac{\sin \cancel{4}h}{\frac{\cancel{12}h}{3}}$$

$$\frac{1}{3} \times \frac{\sin h}{h}$$

$$\frac{1}{3} \times 1$$

$$= \frac{1}{3} \quad //$$

4) $$\lim_{x \to \infty} \left[\frac{\sqrt{x^2+5} - \sqrt{x^2-3}}{\sqrt{x^2+3} - \sqrt{x^2+1}} \right]$$

$$\lim_{x \to \infty} \frac{\sqrt{x^2+5} - \sqrt{x^2-3}}{\sqrt{x^2+3} - \sqrt{x^2+1}} \times \frac{\sqrt{x^2+5} + \sqrt{x^2-3}}{\sqrt{x^2+5} + \sqrt{x^2+3}} \times \frac{\sqrt{x^2+3} + \sqrt{x^2+1}}{\sqrt{x^2+3} + \sqrt{x^2+1}}$$

$$\lim_{x \to \infty} \frac{(x^2+5-x^2+3)\,(\sqrt{x^2+3} + \sqrt{x^2+1})}{(x^2+3-x^2-1)\,(\sqrt{x^2+5} + \sqrt{x^2-3}}$$

$$\lim_{x \to \infty} \frac{8 \; (\sqrt{x^2+3}) + \sqrt{x^2+1})}{2 \; (\sqrt{x^2+5} + \sqrt{x^2-3})}$$

$$\lim_{x \to \infty} 4 \left(\frac{\sqrt{x^2+3} + \sqrt{x^2+1}}{\sqrt{x^2+5} + \sqrt{x^2-3}} \right)$$

$$\lim_{x \to \infty} \frac{4\,\left(\sqrt{x^2(1+3/x^2)} + \sqrt{x^2(1+1/x^2)}\right)}{\sqrt{x^2(1+5/x^2)} + \sqrt{x^2(1-3/x^2)}}$$

$$= \frac{4\,\sqrt{x^2(1+3/x^2)} + \sqrt{x^2(1+1/x^2)}}{\sqrt{x^2(1+5/x^2)} + \sqrt{x^2(1-3/x^2}}$$

$= 4$ //

Q.2

1) $f(x) = \dfrac{\sin 2x}{\sqrt{1-\cos 2x}}$ for $0 < x \leq \pi/2$
$\quad = \dfrac{\cos x}{\pi - 2x}$ for $\pi/2 < x < \pi$ } at $x = \pi/2$

-> $\because f(\pi/2) = \dfrac{\sin 2(\pi/2)}{\sqrt{1-\cos 2(\pi/2)}}$

① $f(\pi/2) = 0$

$\therefore$ f at $x = \dfrac{\pi}{2}$ define

2) $\lim\limits_{x \to \pi/2^+} f(x) = \lim\limits_{x \to \pi/2^+} \dfrac{\cos x}{\pi - 2x}$

a) put $x - \pi/2 = h$

$\therefore x = \pi/2 + h$

as $x \to \cancel{\pi/2}$ $h \to 0^+$

$$\lim_{h \to 0} \frac{\cos\left(h + \frac{\pi}{2}\right)}{\pi - 2\left(h + \frac{\pi}{2}\right)}$$

$$\lim_{h \to 0^+} \frac{\cos\left(h + \frac{\pi}{2}\right)}{\pi - \cancel{2}\left(\frac{2h + \pi}{\cancel{2}}\right)} = \cancel{\pi} - 2h - \cancel{\pi}$$

$$\lim_{h \to 0^+} \frac{\cos\left(h + \frac{\pi}{2}\right)}{-2h}$$

$$\lim_{h \to 0^+} \frac{\cos h \cdot \cos\frac{\pi}{2} - \sin h \sin\frac{\pi}{2}}{-2h}$$

$$\lim_{h \to 0^+} \frac{\cos h \cdot 0 - \sin h}{-2h}$$

$$\lim_{h \to 0^+} \frac{\cancel{-}\sin h}{\cancel{-}2h}$$

$$= \frac{1}{2}$$

b) $$\lim_{x \to \pi/2^-} f(x) = \lim_{x \to \pi/2^-} \frac{\sin 2x}{\sqrt{1 - \cos 2x}}$$

$$\lim_{x \to \pi/2^-} \frac{2\sin x \cos x}{\sqrt{2}\ \sin x}$$

$$= \frac{2}{\sqrt{2}} \lim_{x \to \pi/2^-} \cos x$$

$$= 0$$

$\therefore$ LHL $\neq$ RHL

$\therefore$ f in not continue at $x = \pi/2$

(3) $f(x) = \dfrac{x^2 - 9}{x - 3}$ $\quad 9 < x < 3$

$= x + 3$ $\quad 3 \leq x < 6$

$= \dfrac{x^2 - 9}{x + 9}$ $\quad 6 \leq x < 9$

} at $x = 3$ and $x = 6$

-> at $x = 3$

① $f(3) = \dfrac{x^2 - 9}{x - 3} = 0$

$\therefore$ f at $x = 3$ define.

② $\lim_{x \to 3^+} 4f(x) = \lim_{x \to 3^+}$

① $f(3) = x + 3 \cancel{= 3 + 3} = 6$

f is defined at $x = 3$

② $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x + 3) = 6$

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \frac{x^2-9}{x-3} = \frac{(\cancel{x-3})(x+3)}{\cancel{x-3}}$

$\therefore$ LHL = RHL

$\therefore$ f is continous at $x=3$.

for $x=6$

① $f(6) = \frac{x^2-9}{x+3} = \frac{36-9}{6+3} = \frac{\cancel{27}^{3}}{\cancel{9}_{1}}$

$= 3$

② $\lim_{x \to 6^+} \frac{(x-3)(\cancel{x+3})}{(\cancel{x+3})}$

$\lim_{x \to 6^+} (x-3) = 6-3 = 3$

$\lim_{x \to 6^-} x+3 = 3+6 = 9$

$\therefore$ LHL $\neq$ RHL

$\therefore$ f is not continous at $x=6$

Q.3 find K

i) $f(x) = \dfrac{1-\cos 4x}{x^2}$ $x < 0$ $\Big\}$ at $x = 0$

$= K$ $x = 0$

→ ∵ f is continous at $x = 0$

$\lim\limits_{x \to 0} f(x) = f(0)$

$1 - \cos 2\theta = 2\sin^2 \theta/2$

$\lim\limits_{x \to 0} \dfrac{2\sin^2 2x}{x^2} = K$

$2\lim\limits_{x \to 0} \left(\dfrac{2\sin^2 2x}{x} \right)^2 = K$

$2(2)^2 = K$

$\therefore K = 8$

ii) $F(x) = \dfrac{\sqrt{3} - \tan x}{\pi - 3x}$ $\pi \neq \dfrac{\pi}{3}$ $\Big\}$ at $x = \dfrac{\pi}{3}$

$= K$ $x = \dfrac{\pi}{3}$

→ $x = \pi/3 + h$ $x \to \pi/3$

$x = \pi/3 + h$ $x \to \pi/3$

$h \to 0$ $h = x - x/3$

$$F(\pi/3+h) = \frac{\sqrt{3} - \tan(\pi/3 + h)}{\pi - 3(\pi/3 + h)}$$

$$\# \quad \tan(A-B) = \frac{\tan A - \tan}{1 + \tan A \cdot \tan}$$

$$= \frac{\sqrt{3} - \tan(\pi/3 + h)}{\pi - 3(\pi/3 + h)}$$

$$= \frac{\dfrac{\sqrt{3} - \tan \pi/3 + \tan h}{1 - \tan \pi/3 \cdot \tan h}}{\cancel{\pi} - \cancel{\pi} - 3h}$$

$$= \frac{\dfrac{\sqrt{3} - \tan \pi/3 \cdot \tan h - \tan \pi/3 + \tan h}{1 - \tan \pi/3 \cdot \tan h}}{-3h}$$

$$= \frac{\dfrac{\cancel{\sqrt{3}} - \sqrt{3} \cdot \tan h - \sqrt{3} \cdot \tan h}{1 - \sqrt{3} \cdot \tan h}}{-3h}$$

$$= \frac{\dfrac{-4 \tan h}{1 - \sqrt{3} \tan h}}{-3h}$$

$$= \frac{-4 \tan h}{-3h\,(1 - \sqrt{3} \tan h)}$$

$$= \frac{4}{3} \lim_{h \to 0} \frac{\tan h}{h} \lim_{h \to 0} \frac{1}{1 - \sqrt{3} \tan h}$$

$$= \frac{4}{3} \cdot \frac{1}{1 - \sqrt{3}(0)}$$

$$= \frac{4}{3}$$

Q4

① $f(x) = \dfrac{1 - \cos 3x}{x \tan x} \quad x \neq 0$
$\quad\;\; = 9 \quad x = 0$ } at $x = 0$

$\rightarrow \; f(x) = \dfrac{1 - \cos 3x}{x \tan x}$

$$= \frac{2 \sin^2 3/2 \, x}{x \tan x}$$

Multiply x^2 in N and D

$$= \frac{\dfrac{2 \sin^2 3x/2}{x^2} \times x^2}{x \cdot \dfrac{\tan x}{x^2} \times x^2}$$

$$= 2 \lim_{x \to 0} \left(\frac{3}{2}\right)^2$$

$$= \cancel{3}\cancel{/2} \times \left(\frac{9}{\cancel{4}\,4}\right) = \frac{9}{\cancel{4}\,2} \cancel{/2}$$

$$= \cancel{2}^{1} \times \left(\frac{9}{\cancel{4}_2}\right) = \frac{9}{2}$$ //

$\lim_{x \to 0} f(x) = \frac{9}{2}$ $\quad g = f(0)$

$\therefore$ f is not continous at $x = 0$

Redefine function

$$f(x) = \begin{cases} \frac{1 - \cos 3x}{x \cdot \tan x} & x \neq 0 \\ \frac{9}{2} & x = 0 \end{cases}$$

Now $\lim_{x \to 0} f(x) = f(0)$

$\therefore$ f has removable dis contunities at $x = 0$

(11) $$\left. \begin{aligned} f(x) &= \frac{(e^{3x} - 1)\sin x^\circ}{x^2} & x \neq 0 \\ &= \frac{\pi}{60} & x = 0 \end{aligned} \right\} \text{ at } x = 0$$

$$\rightarrow \lim_{x \to 0} \frac{(e^{3x} - 1)\sin\left(\frac{\pi x}{180}\right)}{x^2}$$

~~3d~~

$$= 3\lim_{x\to 0} \frac{e^{3x}-1}{3x} \lim_{x\to 0} \frac{\sin\left(\frac{\pi x}{180}\right)}{x}$$

$$3\log e \frac{\pi}{180} = \frac{\pi}{60} = f(0)$$

f is continous at $x=0$

Q.5

If $f(x) = \frac{e^{x^2} - \cos x}{x^2}$ $x \neq 0$

is continous at $x=0$

Sol^n

$\therefore$ f is continous at $x=0$

$$\lim_{x\to 0} f(x) = f(0)$$

$$\frac{e^{x^2} - \cos x}{x^2} = f(0)$$

$$= \frac{e^{x^2} - \cos x - 1 + 1}{x^2}$$

$$= \frac{(e^{x^2} - 1) + (1 - \cos x)}{x^2}$$

$$= \frac{e^{x^2}-1}{x^2} + \lim_{x \to 0} \frac{2\sin^2 x/2}{x^2}$$

$$= \log e + 2\left(\frac{\sin x/2}{x}\right)^2$$

Multiply with 2 on Num and Denominator

$$= 1 + \cancel{2}^{1} \times \frac{1}{\cancel{4}\,2}$$

$$= \frac{3}{2}$$

$$\therefore f(0) = 3/2$$

Q.6 $f(x) = \dfrac{\sqrt{2} - \sqrt{1-\sin x}}{\cos^2 x} \quad x \neq \dfrac{\pi}{2}$

is continous at $x = \dfrac{\pi}{2}$

$\rightarrow$

$$= \frac{\sqrt{2} - \sqrt{1+\sin x}}{\cos^2 x} \times \frac{\sqrt{2} + \sqrt{1+\sin x}}{\sqrt{2} + \sqrt{1+\sin x}}$$

$$= \frac{2-1+\sin x}{\cos^2(\sqrt{2}+\sqrt{1-\sin x})}$$

$$= \frac{1+\sin x}{1-\sin^2 x(\sqrt{2}+\sqrt{1+\sin x})}$$

$$= \frac{1}{(1-\sin x)(\sqrt{2}+\sqrt{1+\sin x}}$$

$$= \frac{1}{2(\sqrt{2}+\sqrt{2})}$$

$$= \frac{1}{2 \times 2\sqrt{2}}$$

$$= \frac{1}{4\sqrt{2}}$$

$$\therefore f\left(\frac{\pi}{2}\right) = \frac{1}{4\sqrt{2}}$$

PRACTICAL - 2

Topic : Derivative.

Q.1 show that the following function defined from $\mathbb{R}$ to $\mathbb{R}$ are differentiale

1) $\cot x$ 2) $\operatorname{cosec} x$ 3) $\sec x$

Q.2 If $f(x) = 4x + 1 \quad x \leq 2$

$= x^2 + 5 \quad x > 0$ at $x = 2$ then

find f is differentiable or not

Q.3. If $f(x) = 4x + 7 \quad x < 3$

$= x^3 + 3x + x \geq 3$ at $x = 3$ then

find f is differentiable or not

Q.4 If $f(x) = 8x - 5 \quad x \leq 2$

$= 3x^2 - 4x + 7 \quad x < 2$ at $x = 2$ then

find f is differentiable or not ?

Soln.

Q.1

1) $\cot x$

$f(x) = \cot x$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{\cot x - \cot a}{x - a}$$

$$= \lim_{x \to a} \frac{1/\tan x - 1/\tan a}{x - a}$$

$$= \lim_{x \to 0} \frac{\tan a - \tan x}{(x-a) \tan x \tan a}$$

put $x-a=h$ $\quad x=a+h$ $\quad$ as $x\to a,\ h\to 0$

$$Df(h) = \lim_{h\to 0} \frac{\tan a - \tan(a+h)}{(a+h-a)\tan(a+h)\tan a}$$

$$= \lim_{h\to 0} \frac{\tan a - \tan(a+h)}{h\times \tan(a+h)\tan a}$$

formula: $\tan(A-B) = \dfrac{\tan A - \tan B}{1+\tan A\cdot\tan B}$

$$\tan A - \tan B = \tan(A-B)(1+\tan A\cdot\tan B)$$

$$= \lim_{h\to 0} \frac{\tan(a-a-h) - (1+\tan a + \tan(a+h))}{h\times \tan(a+h)\tan a}$$

$$= \lim_{h\to 0} -\frac{\tan h}{h} \times \frac{1+\tan a\tan(a+h)}{\tan(a+h)\tan a}$$

$$= -1 \times \frac{1+\tan^2 a}{\tan^2 a}$$

$$= \frac{-\sec^2 a}{\tan^2 a} = \frac{-1}{\cos^2 a} \times \frac{\cos^2 a}{\sin^2 a}$$

$$= -\operatorname{cosec}^2 a$$

$$Df(a) = -\cos^2 a$$

$\therefore$ f is differentiable $\forall a \in \mathbb{R}$

2) $\operatorname{cosec} x$

$$f(x) = \operatorname{cosec} x$$

$$Df(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

$$= \lim_{x\to a} \frac{\operatorname{cosec} x - \operatorname{cosec} a}{x-a}$$

$$= \lim_{x\to a} \frac{1/\sin x - 1/\sin a}{x-a}$$

$$= \lim_{x\to a} \frac{\sin a - \sin x}{(x-a)\sin a\sin x}$$

Put $x-a=h \quad x=a+h \quad$ as $x \to a \quad h \to 0$

$$Df(h) = \lim_{h \to 0} \frac{\sin a - \sin(a+h)}{(a+h-a)\sin a \cdot \sin(a+h)}$$

$$Df(h) = \lim_{h \to 0} \frac{\sin a - \sin(a+h)}{(a+h-a)\sin a \cdot \sin(a+h)}$$

$$\because \sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

$$= \lim_{h \to 0} \frac{2\cos\left(\frac{a+a+h}{2}\right)\sin\left(\frac{a-a-h}{2}\right)}{h \times \sin a \cdot \sin(a+h)}$$

$$= \lim_{h \to 0} \frac{-\sin h/2}{h/2} \times \frac{1}{2} \times \frac{2\cos\left(\frac{2a+h}{2}\right)}{\sin a \sin(a+h/2)}$$

$$= \frac{-1/2 \times 2\cos\left(\frac{2a}{2}\right)}{\sin(a+0)}$$

$$= \frac{-\cos a}{\sin^2 a} = -\cot a \operatorname{cosec} a$$

3) $\sec x$

$f(x) = \sec x$

$$Df(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x-a}$$

$$= \lim_{x \to a} \frac{\sec(x) - \sec(a)}{x-a}$$

RHD:

$$Df(2^+) = \lim_{x \to 2^+} \frac{x^2+5-9}{x-2}$$

$$= \lim_{x \to 2^+} \frac{x^2-4}{x-2}$$

$$= \lim_{x \to 2^+} \frac{(x+2)(x-2)}{x-2}$$

$$= 2+2 = 4$$

$Df(2^+) = 4$ RHD = LHD

f is differentiable at $x = 2$

$$= \lim_{x \to a} \frac{\frac{1}{\cos x} - \frac{1}{\cos a}}{x - a}$$

$$= \lim_{x \to a} \frac{\cos a - \cos x}{(x-a)\cos a \cos x}$$

Put $x - a = h$

$x = a + h$

as $x \to a$, $h \to 0$

$$Df(h) = \lim_{h \to 0} \frac{\cos a - \cos(a+h)}{h \times \cos a \cos(a+h)}$$

$$\because -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

$$= \lim_{h \to 0} \frac{-2\sin\left(\frac{a+a+h}{2}\right)\sin\left(\frac{a-a-h}{2}\right)}{h \times \cos a \cos(a+h)}$$

$$= \lim_{h \to 0} \frac{-2\sin\left(\frac{2a+h}{2}\right)\sin -\frac{h}{2}}{\cos a \cos(a+h) \times -\frac{h}{2}} \times -\frac{1}{2}$$

$$= -\frac{1}{2} \times \frac{-2\sin\left(\frac{2a+0}{2}\right)}{\cos a \cos(a+0)}$$

$$= -\frac{1}{2} \times -2 \frac{\sin a}{\cos a \times \cos a}$$

$= \tan a \sec a$

Q.2

LHD

$$Df(2^-) = \lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2}$$

$$= \lim_{x \to 2^-} \frac{4x+1-(4\times2+1)}{x-2}$$

$$= \lim_{x \to 2^-} \frac{4x+1-\cancel{(4\times2+1)}9}{x-2}$$

$$= \lim_{x \to 2^-} \frac{4x-8}{x-2}$$

$$= \lim_{x \to 2^-} \frac{4(x-2)}{(x-2)} = 4$$

$$Df(2^-) = 4$$

RHD:

$$Df(2^+) = \lim_{x \to 2^+} \frac{x^2+5-9}{x-2}$$

$$= \lim_{x \to 2^+} \frac{x^2-4}{x-2}$$

$$= \lim_{x \to 2^+} \frac{(x+2)\cancel{(x-2)}}{\cancel{(x-2)}}$$

$$= 2+2 = 4$$

$Df(2^+) = 4$ RHD = LHD

f is differentiable at $x=2$

→ Q.3

RHD:

$$Df(3^+) = \lim_{x \to 3^+} \frac{f(x)-f(3)}{x-3}$$

$$= \lim_{x \to 3^+} \frac{x^2+3x+1-(3^2+3\times3+1)}{x-3}$$

$$= \lim_{x \to 3^+} \frac{x^2+3x+1-19}{x-3}$$

$$= \lim_{x \to 3^+} \frac{x^2+6x-3x-18}{x-3}$$

$$= \lim_{x \to 3^+} \frac{x(x+6)-3(x+6)}{x-3}$$

$$= \lim_{x \to 3^+} \frac{(x+6)\cancel{(x-3)}}{\cancel{(x-3)}} = 3+6 = 9$$

$Df(3^+) = 9$

LHD $= Df(3^-)$

$$= \lim_{x \to 3^-} \frac{f(x) - f(3)}{x-3}$$

$$= \lim_{x \to 3^-} \frac{4x+2-14}{x-3}$$

$$= \lim_{x \to 3^-} \frac{4x-12}{x-3}$$

$$= \lim_{x \to 3^-} \frac{4(x-3)}{(x-3)}$$

$Df(3^-) = 4$

RHD $\neq$ LHD

f is not differentiable at $x = 3$

→Q.4

$f(2) = 8 \times 2 - 5 = 16 - 5 = 11$

RHD

$$Df(2^+) = \lim_{x \to 2^+} \frac{f(x) - f(2)}{x-2}$$

$$= \lim_{x \to 2^+} \frac{3x^2-4x+7-11}{x-2}$$

$$= \lim_{x \to 2^+} \frac{3x^2-4x-4}{4-2}$$

$$= \lim_{x \to 2^+} \frac{3x^2-6x+2x-4}{4-2}$$

$$= \lim_{x \to 2^+} \frac{3x(x-2)+2(x-2)}{x-2}$$

$$= \lim_{x \to 2^+} \frac{(3x+2)\cancel{(x-2)}}{\cancel{(x-2)}}$$

$$= 3 \times 2 + 2 = 8$$

$$Df(2^+) = 8$$

LHD:

$$Df(2^-) = \lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2}$$

$$= \lim_{x \to 2^-} \frac{8x - 5 - 11}{x - 2}$$

$$= \lim_{x \to 2^-} \frac{8x - 16}{x - 2}$$

$$= \lim_{x \to 2^-} \frac{8\cancel{(x-2)}}{\cancel{(x-2)}}$$

$$= 8$$

$$Df(2^-) = 8$$

LHD = RHD

f is differentiable at $x = 2$

PRACTICAL - 3

TOPIC: Application Of Derivative.

1) Find the intervals in which function is increasing or decreasing

(i) $f(x) = x^3 - 5x - 11$

(ii) $f(x) = x^2 - 4x$

(iii) $f(x) = 2x^3 + x^2 - 20x + 4$

(iv) $f(x) = x^3 - 27x + 5$

(v) $f(x) = 69 - 24x - 9x^2 + 2x^3$

2) Find the intervals in which function is concave upwords

(i) $y = 3x^2 - 2x^3$

(ii) $y = x^4 - 6x^3 + 12x^2 + 5x + 7$

(iii) $y = x^3 - 27x + 5$

(iv) $y = 69 - 24x - 9x^2 + 2x^3$

(v) $y = 2x^3 + x^2 - 20x + 4$

Sol^n

Q.1

i) $f(x) = x^3 - 5x - 11$

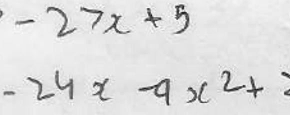
$\therefore f'(x) = 3x^2 - 5$

$\because$ f is increasing iff $f(x) > 0$

$3x^2 - 5 > 0$

$3(x^2 - 5/3) > 0$

$(x - \sqrt{5/3})(x + \sqrt{5/3}) > 0$



$x \in (-\infty, -\sqrt{5/3}) \cup (\sqrt{5/3}, \infty)$

and f is decreasing iff $f'(x) < 0$

$\therefore 3x^2 - 5 < 0$

$\therefore 3(x^2 - 5/3) < 0$

$\therefore (x - \sqrt{5}/3)(x + \sqrt{5/3}) < 0$

$+ \quad - \quad +$

$-\sqrt{5}/3 \qquad \sqrt{5}/3$ $\qquad x \in (-\sqrt{5}/3, \sqrt{5}/3)$

2) $f(x) = x^2 - 4x$

$f'(x) = 2x - 4$

$\because f(x)$ is increasing iff $f'(x) > 0$

$\therefore 2x - 4 > 0$

$\therefore 2(x-2) > 0$

$\therefore x - 2 > 0$

$x \in (2, \infty)$

and f is decreasing iff $f'(x) < 0$

$\therefore 2x - 4 < 0$

$\therefore 2(x-2) < 0$

$\therefore x - 2 < 0$

$x \in (-\infty, 2)$

3) $f(x) = 2x^3 + x^2 - 20x + 4$

$\therefore f'(x) = 6x^2 + 2x - 20$

$\because f$ is increasing iff $f'(x) > 0$

$\therefore 6x^2 + 2x - 20 > 0$

$2(3x^2 + x - 10) > 0$

$3x^2 + x - 10 > 0$

$3x^2 + 6x - 5x - 10 > 0$

$3x(x+2) - 5(x+2) > 0$

$(x+2)(3x-5) > 0$

$+ \quad - \quad +$

$2 \qquad 5/3$ $\qquad x \in (-\infty, -2) \cup (5/3, \infty)$

and f is decreasing iff $f'(x) < 0$

$\therefore 6x^2 + 2x - 20 < 0$

$\therefore 2(3x^2 + x - 10) < 0$

$\therefore 3x^2 + x - 10 < 0$

$\therefore 3x^2 + 6x - 5x - 10 < 0$

$\therefore 3x(x+2) - 5(x+2) < 0$

$\therefore (x+2)(3x-5) < 0$

$+ \quad - \quad +$

$-2 \quad 5/3$

$x \in (-2, 5/3)$

4) $f(x) = x^3 - 27x + 5$

$f(x) = 3x^2 - 27$

$\because f$ is increasing if $f'(x) > 0$

$\therefore 3(x^2 - 9) > 0$

$\therefore (x-3)(x+3) > 0$

$+ \quad - \quad +$

$-3 \quad 3$

$\therefore x \in (-\infty, -3) \cup (3, \infty)$

and f is decreasing iff $f'(x) < 0$

$\therefore 3x^2 - 27 < 0$

$\therefore 3(x^2 - 9) < 0$

$\therefore (x-3)(x+3) < 0$

$+ \quad - \quad +$

$-3 \quad 3$

$\therefore x \in (-3, 3)$

5) $f(x) = 2x^3 - 9x^2 - 24x + 64$

$f'(x) = 6x^2 - 18x - 24$

$\because f$ is increasing iff $f'(x) > 0$

$\therefore 6x^2 - 18x - 24 > 0$

$\therefore 6(x^2 - 3x - 4) > 0$

$\therefore \cancel{6}x^2 - 3x - 4 > 0$

$\therefore x^2 - 4x + x - 4 > 0$

$\therefore x(x-4) + 1(x-4) > 0$

$(x-4)\cdot(x+1)>0$

$+ \quad - \quad +$

$-1 \quad 4$

$x \in (-\infty, -1) \cup (4, \infty)$

and f is decreasing iff $f'(x)<0$

$\therefore 6x^2-18x-24<0$

$\therefore 6(x^2-3x-4)<0$

$\therefore x^2-4x+x-4<0$

$x(x-4)+1(x-4)$

$(x-4)(x+1)<0$

$- \quad +$

$-1 \quad 4$

$\therefore x \in (-1, 4)$

Q.2

1) $y=3x^2-2x^3$

$\therefore f(x)=3x^2-2x^3$

$\therefore f'(x)=6x-6x^2$

$\therefore f''(x)=6-12x$

f is concave upward iff $f''(x)>0$

$\therefore (6-12x)>0$

$\therefore 12(6/12-x)>0$

$x-1/2>0$

$x>1/2$

$\therefore f''(x)>0$

$\therefore x \in (1/2, \infty)$

2) $y = x^4 - 6x^3 + 12x^2 + 5x + 7$

$f'(x) = 4x^2 - 18x^2 + 24x + 5$

$f''(x) = 12x^2 - 36x + 24$

f is concave upward if $f''(x) > 0$

$\therefore 12x^2 - 36x + 24 > 0$

$12(x^2 - 3x + 2) > 0$

$x^2 - 3x + 2 > 0$

$x(x-2) - 1(x-2) > 0$

$(x-2)(x-1) > 0$

$+ \quad - \quad +$; 1, 2 $\qquad x \in (-\infty, 1) \cup (2, \infty)$

3) $y = x^3 - 27x + 3$

$f'(x) = 3x^2 - 27$

$f''(x) = 6x$

f is concave upward iff $f''(x) > 0$

$\therefore 6x > 0$

$\therefore x > 0$

$\therefore x \in (0, \infty)$

4) $y = 64 - 24x - 9x^2 + 2x^3$

$f(x) = 2x^3 - 9x^2 - 24x + 64$

$f'(x) = 6x^2 - 18x - 24$

$f''(x) = 12x - 18$

f is concave upward iff $f''(x) > 0$

$\therefore 12x - 18 > 0$

$\therefore 12(x - 18/12) > 0$

$\therefore x - 3/2 > 0$

$\therefore x > 3/2$

$\therefore x \in (3/2, \infty)$

5) $y = 2x^3 + x^2 - 20x + 4$

$f(x) = 2x^3 - x^2 - 20x + 4$

$f'(x) = 6x^2 + 2x - 20$

$f''(x) = 12x + 2$

f is concave upward iff $f''(x) > 0$

$\therefore f''(x) > 0$

$\therefore 12x + 2 > 0$

$\therefore 12(x + 2/12) > 0$

$\therefore x + 1/6 > 0$

$\therefore x > -1/6$

$\because f''(x) > 0$

$\therefore$ There exist ~~no~~ interval

$x \in (-1/6, \infty)$

A+
21/12/19

PRACTICAL - 4

Topic :- Application of Derivative & Newton's Method.

Q.1 Find maximum & minimum values of following functions

i) $f(x) = x^2 + \frac{16}{x^2}$

ii) $f(x) = 3 - \sqrt{x^3} + 3x^5$

iii) $f(x) = x^3 - 3x^2 + 1$ in $[-1/2, 4]$

iv) $f(x) = 2x^3 - 3x^2 - 12x + 1$ in $[-2, 3]$

Q.2 Find the root of the following equation by Newton's Method (Take 4 iteration only). correct upto 4 decimal

i) $f(x) = x^3 - 3x^2 - 55x + 9.5$ (take $x_0 = 0$)

ii) $f(x) = x^3 - 4x - 9$ in $[2, 3]$

iii) $f(x) = x^3 - 1.8x^2 - 10x + 17$ in $[1, 2]$

Soln

Q.1

i) $f(x) = x^2 + \frac{16}{x^2}$

$f'(x) = 2x - \frac{32}{x^3}$

Now consider

$f'(x) = 0$

$\therefore 2x - \frac{3x}{x^3} = 0$

$2x - \frac{32}{x^3}$

$$\cancel{2x}\; x^4 = \frac{32}{2}$$

$$x^4 = 16$$

$$x = \pm 2$$

$$f''(x) = 2 + \frac{96}{x^4}$$

$$f''(2) = 2 + \frac{96}{2^4} = 2 + \frac{96}{16} = 2 + 6 = 8 > 0$$

$\therefore$ f has minimum value at $x = 0$

$$f(2) = 2^2 + \frac{16}{2^2}$$

$$= 4 + \frac{16}{4}$$

$$= 4 + 4 = 8$$

$$f''(-2) = 2 + \frac{96}{-2^4}$$

$$= 2 + \frac{96}{16} = 2 + 6 = 8 > 0$$

$\therefore$ f has minimum value at $x = 2$

$\therefore$ function reaches minimum value at $x = 2$, $x = -2$

ii) $f(x) = 3 - 5x^3 + 3x^5$

$f'(x) = -15x^2 + 15x^4$

Consider,

$$f'(x) = 0$$

$$-15x^2 + 15x^4 = 0$$

$$15x^4 = 15x^2$$

$x^2 = 1$

$x = \pm 1$

$f''(x) = -30x + 60x^3$

$f''(1) = -30 + 60$

$= 30 > 0$

$\therefore f$ has minimum at $x = 1$

$f(1) = 3 - 5(1)^3 + 3(1)^5$

$= 6 - 5$

$= 1$

$f''(-1) = -30(-1) + 60(-1)^3$

$= 30 - 60$

$= -30 < 0$

$\therefore f$ has maximum at $x = -1$

$\therefore f(-1) = 3 - 5(-1)^3 + 3(-1)^5$

$= 3 + 5 - 3$

$= 8 - 3 = 5$

$\therefore f$ has the maximum value 5 at $x = 1$ and has minimum value 1 at $x = 1$

iii) $f(x) = x^3 - 3x^2 + 1$ in $[\frac{-1}{2}, 4)$

$f'(x) = 3x^2 - 6x$

Consider, $f'(x) = 0$

$\therefore 3x^2 - 6x = 0$

$3x(x - 2) = 0$

$3x = 0$ or $x - 2 = 0$

$x = 0$ or $x = 2$

$f''(x) = 6x - 6$

$f''(0) = 6(0) - 6$

$= -6 < 0$

$\therefore f$ has maximum Value at $x = 0$

$\therefore f(0) = (0)^3 - 3(0)^2 + 1$

$= 1$

$f''(2) = 6(2) - 6$

$= 12 - 6$

$= 6 > 0$

$\therefore$ f has minimum value at $x = 2$

$f(2) = (2)^3 - 3(2)^2 + 1$

$= 8 - 3(4) + 1$

$= 9 - 12 = -3$

$\therefore$ f has maximum value 1 or $x = 0$ and f has minimum value -3 at $x = 2$

iv) $f(x) = 2x^3 - 3x^2 - 12x + 1$ in $[-2, 3]$

$f'(x) = 6x^2 - 6x - 12$

Consider,

$f'(x) = 0$

$\therefore 6x^2 - 6x - 12 = 0$

$6(x^2 - x - 2) = 0$

$x^2 - x - 2 = 0$

$x^2 + x - 2x - 2 = 0$

$x(x+1) - 2(x+1) = 0$

$(x-2)(x+1) = 0$

$\therefore x = 2$ or $x = -1$

$\therefore f''(x) = 12x - 6$

$f''(2) = 12(2) - 6$

$= 24 - 6 = 18 > 0$

$\therefore$ f has minimum value at $x = 2$

$f(2) = 2(2)^3 - 3(2)^2 - 12(2) + 1$

$= 2(8) - 3(4) - 12(2) + 1$

$= 16 - 12 - 24 + 1$

$= -19$

$f''(-1) = 12(-1) - 6$

$= -12 - 6 = -18 < 0$

$\therefore$ f has maximum Value at $x = -1$

$\therefore f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 1$

$= -2 - 3 + 12 + 1 = 8$

$\therefore$ f has maximum value 8 at $x = -1$ and
f has minimum value -19 at $x = 2$

Q.2

i) $f(x) = x^3 - 3x^2 - 55x + 9.5$ (take $x_0 = 0$)

$f'(x) = 3x^2 - 6x - 55$

by Newton's Method

$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$f'(x_0) = 0 - 0 - 55$

$= -55$

$\cancel{f(x_0)} =$

$f(x_0) = 0 - 0 - 0 + 9.5$

$= 9.5$

$x_1 = 0 + \dfrac{9.5}{55}$

$\therefore x_1 = 0.1727$

$\therefore f(x_1) = (0.1727)^3 - 3(0.1727)^2 - 55(0.1727) + 9.5$

$= 0.0051 - 0.0895 - 9.4985 + 9.5$

$= -0.0828$

$f'(x_1) = 3(0.1727)^2 - 6(0.1727) - 55$

$= 0.0895 - 1.0362 - 55$

$= -55.9467$

$\therefore x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$= 0.1727 - \dfrac{0.0828}{55.9467}$

$= 0.1712$

$f(x_2) = (0.1712)^3 - 3(0.1712)^2 - 55(0.1712) + 9.5$

$= 0.0050 - 0.0879 - 9.416 + 9.5$

$= 0.0010$

$f'(x_2) = 3(0.1712)^2 - 6(0.1712) - 55$

$= 0.0879 - 1.0272 - 55$

$= -55.9393$

$\therefore x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$

$= 0.1712 + \dfrac{0.0011}{55.9393}$

$= 0.1712$

$\therefore$ The root of the equation is 0.1712

ii) $f(x) = x^3 - 4x - 9 \qquad [2, 3)$

$f'(x) = 3x^2 - 4$

$f(2) = 2^3 - 4(2) - 9$

$= 8 - 8 - 9$

$= -9$

$f(3) = 3^3 - 4(3) - 9$

$= 27 - 12 - 9$

$= 6$

Let $x_0 = 3$ be the initial approximation,

$\therefore$ By Newton's Method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 3 - \frac{6}{23}$$

$$= 2.7391$$

$$f(x_1) = (2.7391)^3 - 4(2.7391) - 9$$

$$= 20.5528 - 10.9568 - 9$$

$$= 0.\cancel{596}5941$$

$$f''(x_1) = 3(2.7391)^2 - 4 = 18.5080$$

$$\therefore x_2 = 2.734 - \frac{0.5941}{18.5080}$$

$$x_2 = 2.7070$$

for x_3

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_2 = 2.7070$$

$$f(x_2) = (2.7070)^3 - 4(2.7070) - 9 = 0.0084$$

$$f'(x_2) = 3(2.7070)^2 - 4 = 17.9835$$

$$\therefore x_3 = 2.7070 - \frac{0.0084}{17.9835}$$

$$x_3 = 2.7065$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$\cancel{f}(x_3) = 2.7065$$

$$f(x_3) = (2.7065)^3 - 4(2.7065) - 9 = -0.0005$$

$$f'(x_3) = 3(2.7065)^2 - 4 = 55.4764$$

$$\therefore x_4 = 2.7065 - \left(-\frac{0.0005}{55.4764}\right)$$

$= 2.7065$

iii) $f(x) = x^3 - 1.8x^2 - 10x + 17 \quad [1, 2]$

$f'(x) = 3x^2 - 3.6x - 10$

$f(1) = (1)^3 - 1.8(1)^2 - 10(1) + 17 = 1 - 1.8 - 10 + 17 = 7.8$

$f(2) = (2)^3 - 1.8(2)^2 - 10(2) + 17 = 8 - 7.2 - 20 + 17 = -2.2$

let $x_0 = 2$, be the initial approximation

By newtons method.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$x_0 = 2$

$f(x_0) = -2.2$

$f'(x_0) = 3(2)^2 - 3.6(2) - 10 = -5.2$

$\therefore x_1 = 2 - \frac{(-2.2)}{(-5.2)}$

$x_1 = 2 - \frac{2.2}{5.2} = 1.5769$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$x_1 = 1.5769$

$f(x_1) = (1.5769)^3 - 1.8(1.5769)^2 - 10(1.5769) + 17 = 0.7258$

$f'(x_1) = 3(1.5769)^2 - 3.6(1.5769) - 10 = -8.2520$

$x_2 = 1.5769 - \frac{0.7256}{(-8.2520)} = 1.5769 + \frac{0.7256}{8.2520}$

$\therefore x_2 = 1.6588$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$x_2 = 1.6588$

$f(x_2) = (1.6588)^3 - 1.8(1.6588)^2 - 10(1.6588) + 17 = 0.0234$

$f'(x_2) = 3(1.6588)^2 - 3.6(1.6588) - 10 = -7.7168$

$$x_3 = 1.6588 - \frac{0.0234}{-7.7168}$$

$$x_3 = 1.6588 + \frac{0.0234}{7.7168} = 1.6618$$

$$\cancel{f}\, x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$x_3 = 1.6618$

$f(x_3) = (1.6618)^3 - 1.8(1.6618)^2 - 10(1.6618) + 17 = 0.0003$

$f'(x_3) = 3(1.6618)^2 - 3.6(1.6618) - 10 = -7.6977$

$$x_4 = 1.6618 + \frac{0.0003}{7.6977}$$

$x_4 = 1.6618$

PRACTICAL - 5

Integration

1) $\displaystyle\int \frac{dx}{\sqrt{x^2+2x-3}}$

2) $\displaystyle\int (4e^{3x}+1)\,dx$

3) $\displaystyle\int (2x^2-3\sin x+5\sqrt{x})\,dx$

4) $\displaystyle\int x^3+\frac{3x+4}{\sqrt{x}}\,dx$

5) $\displaystyle\int t^7\sin(2t^4)\,dx$

6) $\displaystyle\int \sqrt{x}\,(x^2-1)\,dx$

7) $\displaystyle\int \frac{1}{x^3}\sin\left(\frac{1}{x^2}\right)dx$

8) $\displaystyle\int \frac{\cos x}{\sqrt[3]{\sin^2 x}}$

9) $\displaystyle\int e^{\cos^2 x}\sin 2x\,dx$

10) $\displaystyle\int \frac{x^2-2x}{(x^3-3x^2+1)}\,dx.$

Soln

1) $\int \frac{dx}{\sqrt{x^2+2x-3}}$

$I = \int \frac{dx}{\sqrt{x^2+2x-3}}$

$= \int \frac{dx}{\sqrt{(x+1)^2-(2)^2}}$

comparing with $\int \frac{dx}{\sqrt{x^2-a^2}}$ $= x^2 = (x+1)^2$

$I = \log|x+\sqrt{x^2-a^2}| + C$

$= \log|x+1+\sqrt{(x+1)^2-4}| + C$ //

2) $\int (4e^{3x}+1)dx$

$I = \int (4e^{3x}+1)dx$

$= \int 4e^{3x}dx + \int 1 dx$

$= \frac{4e^{3x}}{3} + x + C$

//

3) $\int (2x^2 - 3\sin x + 5\sqrt{x})dx$

$I = \int (2x^2 - 3\sin x + 5\sqrt{x})dx$

$= 2\int x^2 dx - 3\int \sin x\, dx + 5\int \sqrt{x}\, dx$

$= \frac{2}{3}x^3 - 3\cos x + \frac{5x^{3/2}}{3/2} + C$

$= \frac{2}{3}x^3 - 3\cos x + \frac{10}{3}x^{3/2} + C$ //

4) $\int \frac{x^3+3x+4}{\sqrt{x}} \, dx$

$$I = \int \frac{x^3+3x+4}{\sqrt{x}} \, dx$$

$$\int \frac{x^3+3x+4}{\sqrt{x}} \, dx$$

$$\int \left(\frac{x^3}{x^{1/2}} + \frac{3x}{x^{1/2}} + \frac{4}{x^{1/2}} \right) dx$$

$$\int x^{5/2} \, dx + 3 \int x^{-1/2} \, dx + 4 \int x^{-1/2} \, dx$$

$$= \frac{2}{7} x^{7/2} + 2x^{3/2} + 8\sqrt{x} + C$$

5) $\int t^7 \sin(2t^4) dx$

$$I = \int t^7 \sin(2t^4) dx$$

let $t^4 = x$

$4t^3 \cdot dt = dx$

$$= \frac{1}{4} \int 4t^3 . t^4 \sin(2t^4) dt$$

$$= \frac{1}{4} (x . \sin(2x) dx$$

$$= \frac{1}{4} \left[x \int \sin 2x - \int \left[\int \sin 2x \cdot \frac{d}{dx} x \right] \right.$$

$$= \frac{1}{4} \left[-x \frac{\cos 2x}{2} + 1 \int \cos 2x - 1 \right]$$

$$= \frac{1}{4} \left[-\frac{\cos 2x}{2} + \frac{1}{4} \sin 2x \right] + C$$

$$= -\frac{1}{8} x \cos 2x + \frac{1}{16} \sin 2x + C$$

$$= \frac{-1}{8} t^4 \cos 2(t^4) + \frac{1}{16} \sin(2t^4) + c$$

6) $\int \sqrt{x}\,(x^2 - 1)\,dx$

$$I = \int \sqrt{x}\,(x^2 - 1)\,dx$$

$$\int (\sqrt{x}\, x^2 - \sqrt{x})\,dx$$

$$\int (x^{5/2} - \sqrt{x})\,dx$$

$$= \frac{2}{7} x^{7/2} - \frac{2}{3} x^{3/2} + c$$

7) $\int \frac{1}{x^3} \sin\left(\frac{1}{x^2}\right) dx$

$$I = \int \frac{1}{x^3} \sin\left(\frac{1}{x^2}\right) dx$$

let $\frac{1}{x^2} = t$

$$x^{-2} = t$$

$$\frac{-2}{x^3}\, dx = dt$$

$$I = \frac{-1}{2} \int \frac{-2}{x^3} \sin\left(\frac{1}{x^2}\right) dx$$

$$= \frac{-1}{2} \int \sin t$$

$$= \frac{-1}{2} (-\cos t) + c$$

$= \frac{1}{2} \cos t + c$

Resubstitution $\quad t = 1/x^2$

$$I = \frac{1}{2} \cos\left(\frac{1}{x^2}\right) + c$$

8) $\displaystyle\int \frac{\cos x}{\sqrt[3]{\sin^2 x}}\, dx$

$$I = \int \frac{\cos x}{3\sqrt{\sin^2 x}}$$

let $\sin x = t$

$\cos x\, dx = dt$

$$I = \int \frac{dt}{\sqrt[3]{t^2}}$$

$$= \int \frac{dt}{t^{2/3}}$$

$$= \int t^{-2/3}\, dt$$

$$= 3t^{1/3} + c$$

$$= 3(\sin x)^{1/3} + c$$

$$= 3\sqrt[3]{\sin x} + c$$

9) $\int e^{\cos^2 x} \sin 2x \, dx$

$I = \int e^{\cos^2 x} \sin 2x \, dx$

Let $\cos^2 x = t$

$-2 \cos x \cdot \sin x \, dx = dt$

$-\sin 2x \, dx = dt$

$I = \int -\sin 2x \, e^{\cos^2 x} \cdot dx$

$= -\int e^t dt$

$= e^t + c$

Resubstituting $t = \cos^2 x$

$I = -e^{\cos^2 x} + c$ //

10) $\int \frac{x^2 - 2x}{(x^3 - 3x^2 + 1)} dx$

$I = \int \frac{x^2 \cancel{*} - 2x}{(x^3 - 3x^2 + 1} dx$

let $x^3 - 3x^2 + 1 = t$

$3(x^2 - 2x) dx = dt$

$(x^2 - 2x) dx = dt/3$

$I = \int 1/t \, dt$

$= \frac{1}{3} \int \frac{dt}{t}$

$= \frac{1}{3} \log t + c$

$= \frac{1}{3} \log (x^3 - 3x^2 + 1) + c$ //

04/01/2020

Practical - 6

Application of Integration and Numerical Integration

Q.1 Find the length of the following curve.

1. $x = t\sin t$, $y = 1-\cos t$ $t \in (0, 2\pi)$

$x = t\sin t$ $\quad\quad$ $y = t\cos t$

$\frac{dx}{dt} = 1-\cos t$ $\quad\quad$ $\frac{dy}{dx} = \sin t$

$$L = \int_0^{2\pi} \sqrt{(1-\cos t + \sin t)^2 + (\sin t)^2}$$

$$= \int_0^{2\pi} \sqrt{1-2\cos t+1}\ dt$$

$$= \int_0^{2\pi} \sqrt{2(1-\cos t)}\ dt$$

$$= \int_0^{2\pi} \sqrt{4\sin^2 t/2}\ dt$$

$$= 2\int_0^{2\pi} \sin t/2\ dt$$

$$= 2\left[-\cos t/2\right]_0^{2\pi} = 2\left[-\cos 2\pi + \cos 0\right]$$

$$= 4$$

2) $y = \sqrt{4 - x^2}$, $x \in [-2, 2]$

$$L = \int_0^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$y = \sqrt{4 - x^2}$

$\therefore \frac{dy}{dx} = 2 \int 1 + \left(\frac{-x}{\sqrt{4-x^2}}\right)^2 dx$

$= 2 \int_0^2 \sqrt{\frac{1+x^2}{4-x^2}}\, dx$

$= 4 \int_0^2 \frac{1}{\sqrt{4-x^2}}\, dx$

$= 4 \left(\sin^{-1}(x/2)\right]_0^2 = 2\pi$

3) $y = x^{3/2}$ in $[0, 4]$

$f'(x) = 3/2 \; x^{1/2}$

$[f(x)]^2 = 9/4 \; x$

$$L = \int_0^4 \sqrt{1 + (f'(x))^2}\, dx$$

$$= \int_0^4 \sqrt{1 + \frac{9}{4}x}\, dx$$

$$= \int_0^4 \sqrt{\frac{4+9x}{4}}\, dx$$

$$= \frac{1}{2} \int_0^4 \sqrt{4 + 9x}\, dx$$

$$= \frac{1}{2}\int_0^4 \frac{(4+9x)^{1/2}}{1/2+1}\, dx$$

$$= \frac{1}{27}\left[[4+0]^{3/2} - [4+36]^{3/2}\right]$$

$$= \frac{1}{27}\left[(4)^{3/2} - (40)^{3/2}\right]$$

4. $x = 3\sin t \quad y = 3\cos t$

$$\frac{dx}{dt} = 3\cos t \qquad \frac{dy}{dt} = -3\sin t$$

$$l = \int_0^{2\pi} \sqrt{(3\cos t)^2 + (-3\sin t)^2}\, dt$$

$$l = \int_0^{2\pi} \sqrt{9(\sin^2 t + \cos t)}\, dt$$

$$= \int_0^{2\pi} \sqrt{9}\, dt = 3\int_0^{2\pi} dt$$

$$= 3[t]_0^{2\pi}$$

$$= 3(2\pi - 0)$$

$$= 6\pi$$

5) $x = \frac{1}{6} y^3 + \frac{1}{2y}$, $y \in [1, 2]$

$$\frac{dx}{dy} = \frac{y^2}{2} - \frac{1}{2y^2} = \frac{y^4 - 1}{2y^2}$$

$$= \int_1^2 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_1^2 \sqrt{\frac{(y^4+1)^2}{(2y^2)}}\, dy$$

$$= \int_1^2 \frac{y^4+1}{2y^2}\, dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3} \quad \frac{-y^{-2}}{1}\right]_1^2$$

$$= \frac{1}{2}\left[\frac{8}{3} - \frac{1}{2} - \frac{1}{3} + 1\right]$$

$$= \frac{1}{2}\left[\frac{7}{3} - \frac{1}{2}\right] = \frac{17}{12}$$

2 Using Simpson's Rule solve the following.

① $\int_0^2 e^{x^2}\, dx$ with $h = 4$

$a = 0$ $b = 2$, $h = 4$

$\cancel{y = \frac{2-0}{4}}$

x	0	0.5	1	1.5	2
y	1	1.2840	2.7182	9.4877	54.5981
	y_0	y_1	y_2	y_3	y_4

By Simpson's Rule

$$\int_0^2 e^{x^2}\,dx = \frac{0.5}{3}\left[(1+54.5981)+4(1.2840+9.4877)+2(2.7182+54.5981)\right]$$

$$= \frac{0.5}{3}\left[55.5481 + 43.0868 + 114.63266\right]$$

$$= 1.1279$$

(2) $\int_0^4 x^2\,dx$

$$L = \frac{4-0}{4} = 1$$

(3) $\int_0^{\pi/3} \sqrt{\sin x}\,dx \quad n=6$

x	0	$\pi/18$	$2\pi/18$	$3\pi/18$	$4\pi/18$	$5\pi/18$	$8\pi/18$
y	0	0.4166	0.58	0.70	0.8087	0.8722	0.99

$$\int_0^{\pi/3} \sqrt{\sin x} = \frac{\pi}{4} \times 12.1163$$

$$= 0.7049$$

PRACTICAL - 7

Differential Equation.

Q.1 Solve the following differential equation.

(i) $x\frac{dy}{dx} + \frac{1}{x}y = \frac{e^x}{x}$

$p(x) = \frac{1}{x}$ $Q(x) = \frac{e^x}{x}$

$IF = e^{\int \frac{1}{x}dx}$

$y(IF) = \int Q(x)(IF)dx + c$

$= \int \frac{e^x}{x} \cdot x \cdot dx + c$

$= \int e^x dx + c$

$xy = e^x + c$ //

Q2) $e^x \frac{dy}{dx} + 2e^x y = 1$

→ $\cancel{\frac{dy}{dx} + \frac{2e^x}{e^x}} y = \frac{1}{e^x}$

$\frac{dy}{dx} + 2y = \frac{1}{e^x}$

$\frac{dy}{dx} + 2y = e^{-x}$

$$p(x) = 2 \qquad Q(x) = e^{-x}$$

$$IF = e^{\int p(x)\,dx}$$
$$= e^{\int 2dx}$$
$$= e^{2x}$$

$$y(IF) = \int Q(x)(IF)\,dx + c$$
$$= \int e^{-x}\, e^{2x}\, dx + c$$

$$= \int e^{x} dx + c$$

$$y \cdot e^{2x} = e^{x} + c$$

3) $x\dfrac{dy}{dx} = \dfrac{\cos x}{x} - 2y$

Sol^n $x\dfrac{dy}{dx} = \dfrac{\cos x}{x} - 2y$

$$\frac{dy}{dx} + \frac{2y}{x} = \frac{\cos x}{x^2}$$

$$p(x) = 2(x) \qquad Q(x) = \frac{\cos x}{x^2}$$

$$IF = e^{\int p(x)dx}$$

$$= e^{\int 2x\,dx}$$

$$= x^2$$

$$y(IF) = \int Q(x)(IF)\,dx + c$$

$$= \int \frac{\cos x}{x^2} \cdot x^2\, dx + c$$

$$= \int \cos x + c$$

4) $x \cdot \frac{dy}{dx} + 3y = \frac{\sin x}{x^2}$

Sol^n $\frac{dy}{dx} + \frac{3y}{x} = \frac{\sin x}{x^3}$

$P(x) = 3/x \qquad Q(x) = \sin x / x^3$

$P(x) = \int 3/x \, dx$

$= x^3$

$IF = e^{\int P(x) dx}$

$= x^3$

$Y(IF) = \int Q(x)(IF) dx + c$

$= \int \frac{\sin x}{x^3} \cdot x^3 dx + c$

$= \int \sin x + c$

$x^3 y = -\cos x + c$ //

5) $e^{2x} \frac{dy}{dx} + 2e^{2x} y = 2x$

$\frac{dy}{dx} + 2y = \frac{2x}{e^2 x}$

~~Put~~ $P(x) = 2 \qquad Q(x) = \frac{2x}{e^{2x}} = 2xe^{-2x}$

$(IF) = e^{\int P(x) dx}$

$= e^{\int 2 dx}$

$= e^{2x}$

$Y(IF) = \int Q(x)(IF) dx + c$

$= \int 2xe^{-2x} \; e^{2x} + c$

$$ye^{2x} = \int 2x + c \quad = x^2 + c$$ //

6. $\sec^2 x . \tan y \; dx + \sec^2 y \tan x \; dy = 0$

$\rightarrow \sec^2 x . \tan y \; dx = -\sec^2 y \tan x \; dy$

$$\frac{\sec^2 x}{\tan x} dx = -\frac{\sec^2 x}{\tan y} dy$$

$$\int \frac{\sec^2 x}{\tan x} \; dx = -\int \frac{\sec^2 x}{\tan y} dy$$

$$\log |\tan x| = -\log |\tan y| + c$$

$$\log |\tan x + \tan y| = c$$

$$\tan x + \tan y = e^c$$ //

7. $\frac{dy}{dx} = \sin^2(x - y + 1)$

put $x - y + 1 = v$

$$x - y + 1 = v$$

$$1 - \frac{dy}{dx} = \frac{dv}{dx}$$

$$\cancel{\frac{1 - dv}{dx}} = \frac{dy}{dx}$$

$$1 - \frac{dv}{dx} = \sin^2 v$$

$$\frac{dv}{dx} = 1 - \sin^2 v$$

$$\frac{dv}{dx} = \cos^2 v$$

$$\frac{dv}{\cos^2 v} = dx$$

$$\int \sec^2 dv = \int dx$$

$$\tan . v = x + c$$

$$\tan(x - y + 1) = x + c$$

8) $\frac{dy}{dx} = \frac{2x + 3y - 1}{6x + 9y + 6}$

→ Put $2x + 3y = v$

$$2 + \frac{3dy}{dx} = \frac{dv}{dx}$$

$$\frac{dy}{dx} = \frac{1}{3}\left(\frac{dv}{dx} - 2\right)$$

$$= \frac{1}{3}\left(\frac{dv}{x} - 2\right) = \frac{1}{3}\left(\frac{v-1}{v+2}\right)$$

$$\frac{dv}{dx} = \frac{v-1}{v+2} + 2$$

$$\frac{dv}{dx} = \frac{v - 1 + 2v + 4}{v+2}$$

$$= \frac{3v + 3}{v+2}$$

$$= \frac{3(v+1)}{v+2}$$

$$\int \frac{v+2}{v+1}\, dv = 3\, dx$$

$$\int \frac{v+1}{v}\, dx + \int \frac{1}{v+1}\, dv - 3dx$$

$$v + \log|v| = 3x + c$$

$$2x + 3y + \log|2x+3y+1] = 3x + c$$

$$3y + x - \log|2x+3+1| + c$$

19/01/2020

PRACTICAL - 8

Euler's Method

1) $\frac{dy}{dx} = y + e^x - 2 \quad y(0) = 2 \quad h = 0.5$ find $Y(2)$

$f(x) = y + e^x - 2 \quad x_0 = 0$

$y_0 = 2 \quad h = 0.5$

$y(2) = ?$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	2	1	2.5
1	0.5	2.5	2.1487	3.5743
2	1	3.5743	4.2925	5.7205
3	1.5	5.7205	8.2021	9.8215
4	2	9.8215		

$Y(2) = 9.8215$

2) $\frac{dy}{dx} = 1 + y^2 \quad y(0) = 0 \quad , h = 0.2 \qquad$ find $y(1)$

$x_0 = 0 \quad , \quad y_0 = 0$

$h = 0.2$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	0	1	0.2
1	0.2	0.2	1.04	0.408
2	0.4	0.408	1.1664	0.6412
3	0.~~8~~6	0.6412	1.4111	0.9234
4	0.8	0.9234	1.8526	1.2939
5	~~0~~1	1.2939		

$y(1) = 1.2939$

3) $\frac{dy}{dx} = \sqrt{\frac{x}{y}}$ $y(0) = 1$, $h = 0.2$ find $y(1)$

$x_0 = 0$ $y_0 = 1$ $h = 0.2$

n	x_n	y_n	$f(x_n\ y_n)$	y_{n+1}
0	0	1	0	1
1	0.2	1	0.4472	1.0894
2	0.4	1.0894	0.6059	1.2105
3	0.6	1.2105	0.7040	1.3513
4	0.8	1.3513	0.7644	1.5051
5	1	1.5051		

$y(1) = 1.5051$

4) $\frac{dy}{dx} = 3x^2+1$ $y(1) = 2$ find $y(2)$

$h = 0.5$ & $h = 0.25$

$y_0 = 2$ $x_0 = 1$

* $h = 0.5$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	1	2	4	4
1	1.5	4	7.75	7.875
2	2	7.875		

$y(2) = 7.875$

* $h = 0.25$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	1	2	4	3
1	1.25	3	5.6875	4.4218
2	1.50	4.4218	59.6569	19.3360
3	1.75	19.3360	1122.6429	299.9966
4	2	299.9966		

$y(2) = 299.99.60$

5) $\frac{dy}{dx} = \sqrt{xy} + 2$

$y(1) = 1$ ~~find h = 0.2~~ find $y(1.2)$

$y_0 = 1$ $x_0 = 1$ $h = 0.2$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	1	1	3	3.6
1	1.2	3.6		

11/01/2020

PRACTICAL - 9

Limits and Partial Order Derivative

Q.1

(i) $$\lim_{(x,y)\to(-4,1)} \frac{x^3-3y+y^2-1}{xy+5}$$

At $(-4,-1)$, Denominator $\neq 0$

$\therefore$ By applying limit

$$= \frac{(-4)^3-3(-1)+(-1)^2-1}{-4(-1)+5}$$

$$= \frac{-64+3+1-1}{4+5}$$

$$= \frac{-61}{9}$$

(ii) $$\lim_{(x,y)\to(2,0)} \frac{(y+1)(x^2+y^2-4x)}{x+3y}$$

At $(2,0)$, Denominator $\neq 0$

$\therefore$ By applying limit,

$$= \frac{(0+1)((2)^2+0-(2)))}{2+0}$$

$$= \frac{1(4+0-8)}{2}$$

$$= \frac{-4}{2}$$

$$= -2$$

ii) $\lim\limits_{(x,y,z)\to(1,1,1)} \dfrac{x^2-y^2z^2}{x^3-x^2yz}$

At $(1,1,1)$, Denominator $= 0$

$\therefore \lim\limits_{(x,y,z)\to(1,1,1)} \dfrac{x^2-y^2z^2}{x^3-x^2yz}$

$= \lim\limits_{(x,y,z)\to(1,1,1)} \dfrac{\cancel{(x-yz)}(x+yz)}{x^2\cancel{(x-yz)}}$

$= \lim\limits_{(x,y,z)\to(1,1,1)} \dfrac{x+yz}{x^2}$

On Applying limit

$= \dfrac{1+1(1)}{(1)^2}$

$= 2$

Q.2

(i) $f(x,y) = xye^{x^2+y^2 \cancel{+y^2}}$

$\therefore f_x = \dfrac{\partial}{\partial x}(f(x,y))$

$= \dfrac{\partial}{\partial x}(xye^{x^2+y^2 \cancel{+y^2}})$

$= ye^{x^2+y^2 \cancel{+y^2}}(2x)$

$\therefore f_x = 2xye^{x^2+y^2} \cancel{+y^2}$

$f_y = \frac{\partial}{\partial y}(f(x,y))$

$= \frac{\partial}{\partial y}(xye^{x^2+y^2} \cancel{+y^2})$

$= xe^{x^2+y^2} \cancel{+} (2y)$

$\therefore f_y = 2yxe^{x^2+y^2}$

(ii) $f(x,y) = e^x \cos y$

$f_x = \frac{\partial}{\partial x}(f(x,y))$

$= \frac{\partial}{\partial x}(e^x \cos y)$

$\therefore f_x = e^x \cos y$

$f_y = \frac{\partial}{\partial y}(f(x,y))$

$= \frac{\partial}{\partial y}(e^x \cos y)$

$f_y = -e^x \sin y$

(iii) $f(x,y) = x^3y^2 - 3x^2y + y^3 + 1$

$f_x = \frac{\partial}{\partial x}(f(x,y))$

$$= \frac{\partial}{\partial x}(x^3y^2 - 3x^2y + y^3 + 1)$$

$$fx = 3x^2y^2 - 6xy$$

$$fy = \frac{\partial}{\partial y}(f(x,y))$$

$$= \frac{\partial}{\partial y}(x^3y^2 - 3x^2y + y^3 + 1)$$

$$\therefore fy = 2x^3y - 3x^2 + 3y^2$$

Q.3

(i) $f(x,y) = \frac{2x}{1+y^2}$

$$fx = \frac{\partial}{\partial x}\left(\frac{2x}{1+y^2}\right)$$

$$= \frac{1+y^2 \frac{\partial}{\partial x}(2x) - 2x\frac{\partial}{\partial x}(1+y^2)}{(1+y^2)^2}$$

$$= \frac{2+2y^2 - 0}{(1+y^2)^2}$$

$$\cancel{= \frac{2(1+y^2)}{(1+y^2)(1+y)^2}}$$

$$= \frac{2}{1+y^2}$$

At (0,0)

$$= \frac{2}{1+0}$$

$$= 2$$

$$f_y = \frac{\partial}{\partial y}\left(\frac{2x}{1+y^2}\right)$$

$$= \frac{1+y^2 \frac{\partial}{\partial x}(2x) - 2x \frac{\partial}{\partial x}(1+y^2)}{(1+y^2)^2}$$

$$= \frac{1+y^2(0) - 2x(2y)}{(1+y^2)^2}$$

$$= \frac{-4xy}{(1+y^2)^2}$$

At (0,0)

$$= \frac{-4(0)(0)}{(1+0)^2}$$

$$= 0$$

Q-4

(i) $$f(x,y) = \frac{y^2-xy}{x^2}$$

$$f_x = \frac{x^2 \frac{d}{dx}(y^2-xy) - (y^2-xy)\frac{d}{dx}(x^2)}{(x^2)^2}$$

$$= \frac{x^2(-y) - (y^2-xy)(2x)}{x^4}$$

$$= \frac{-x^2y - 2x(y^2-xy)}{x^4}$$

82 $f_y = \frac{2y - x}{x^2}$

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{-x^2y - 2x(y^2 - xy)}{x^4}\right)$$

$$= \frac{x^4\left(\frac{\partial}{\partial x}(-x^2y - 2xy^2 + 2x^2y)\right) - (-x^2y - 2xy^2 + 2x^2y)\frac{\partial}{\partial x}(x^4)}{(x^4)^2}$$

$$= \frac{x^4(-2xy - 2y^2 + 4xy) - 4x^3(-x^2y - 2xy + 2x^2y)}{x^6} \quad \text{— (i)}$$

$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{2y - x}{x^2}\right)$$

$$= \frac{2 - 0}{x^2} = \frac{2}{x^2} \quad \text{— (ii)}$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{-x^2y - 2xy^2 + 2x^2y}{x^4}\right)$$

$$= \frac{-x^2 - 4xy + 2x^2}{x^4} \quad \text{— (iii)}$$

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{2y - x}{x^2}\right)$$

$$= \frac{x^2\frac{\partial}{\partial x}(2y - x) - (2y - x)\frac{\partial}{\partial x}(x^2)}{(x^2)^2} \quad \text{— (iv)}$$

∴ from (iii) and (iv)

$f_{xy} = f_{yx}$

(ii) $F(x,y) = x^3 + 3x^2y^2 - \log(x^2+1)$

$f_x = \frac{\partial}{\partial x}(x^3 + 3x^2y^2 - \log(x^2+1))$ $\quad f_y = \frac{\partial}{\partial y}(x^3+3x^2y^2 - \log(x^2+1))$

$= 3x^2 + 6xy^2 - \frac{2x}{x^2+1}$ $\quad = 0 + 6x^2y - 0$

$= 6x^2y$

$f_{xx} = 6x + 6y^2 - \left(\frac{x^2+1 \frac{\partial(2x)}{\partial x} - 2x \frac{\partial(x^2+1)}{\partial x}}{(x^2+1)^2}\right)$

$= 6x + 6y^2 - \left(\frac{2(x^2+1) - 4x^2}{(x^2+1)^2}\right)$ — (I)

$f_{yy} = \frac{\partial}{\partial y}(6x^2y)$

$= 6x^2$

— (II)

$f_{xy} = \frac{\partial}{\partial y}\left(3x^2 + 6xy^2 - \frac{2x}{x^2+1}\right)$

$= 0 + 12xy - 0$

$= 12xy$ — (III)

$f_{yx} = \frac{\partial}{\partial x}(6x^2y)$

$= 12xy$ — (IV)

from (III) and (IV)

$\therefore f_{xy} = f_{yx}$

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(iii) $f(x,y) = \sin(xy) + e^{x+y}$

$f_x = y\cos(xy) + e^{x+y}(1)$ $\quad f_y = x\cos(xy) + e^{x+y}(1)$

$= y\cos(xy) + e^{x+y}$ $\quad \cancel{f_x} = x\cos(xy) + e^{x+y}$

$\therefore f_{xx} = \frac{\partial}{\partial x}(y\cos(xy) + e^{x+y})$

$= -y\sin(xy) \cdot (y) + e^{xy}(1)$

$= -y^2\sin(xy) + e^{x+y}$ — (i)

$\therefore f_{yy} = \frac{\partial}{\partial y}(x\cos(xy) + e^{x+y})$

$= -x\sin(xy)(x) + e^{x+y}(1)$

$= -x^2\sin(xy) + e^{x+y}$ — (ii)

$\therefore f_{xy} = \frac{\partial}{\partial y}(y\cos(xy) + e^{x+y})$

$= -y^2\sin(xy) + \cos(xy) + e^{x+y}$ — (iii)

$\therefore f_{yx} = \frac{\partial}{\partial x}(x\cos(xy) + e^{x+y})$

$= -x^2\sin(xy) + \cos(xy) + e^{x+y}$ — (iv)

$\therefore$ from (iii) and (iv)

$f_{xy} \neq f_{yx}$

Q·5

(i) $f(x,y) = \sqrt{x^2+y^2}$ at $(1,1)$

$\rightarrow$ $f(1,1) = \sqrt{(1)^2+(1)^2} = \sqrt{2}$

$f_x = \frac{1}{2\sqrt{x^2+y^2}}(2x)$ $\qquad$ $f_y = \frac{1}{2\sqrt{x^2+y^2}}(2y)$

$= \frac{x}{\sqrt{x^2+y^2}}$ $\qquad$ $= \frac{y}{\sqrt{x^2+y^2}}$

f_x at $(1,1) = \frac{1}{\sqrt{2}}$ $\qquad$ f_y at $(1,1) = \frac{1}{\sqrt{2}}$

$\therefore L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$= \sqrt{2} + \frac{1}{\sqrt{2}}(x-1) + \frac{1}{\sqrt{2}}(y-1)$

$= \sqrt{2} + \frac{1}{\sqrt{2}}(x-1+y-1)$

$= \sqrt{2} + \frac{1}{\sqrt{2}}(x+y-2)$

$= \cancel{\sqrt{2}} + \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y \cancel{-\frac{2}{\sqrt{2}}}$

$= \frac{x+y}{\sqrt{2}}$

(ii) $f(x,y) = 1-x+y\sin x$ at $(\pi/2, 0)$

$f(\pi/2, 0) = 1-\pi/2+0 = 1-\pi/2$

$f_x = 0-1+y\cos x$ $\qquad$ $f_y = 0-0+\sin x$

fx at $(\pi/2, 0) = -1 + 0$ $\quad$ fy at $(\pi/2, 0) = \sin \pi/2$

$= -1$ $\quad$ $= 1$

$L(x,y) = f(a,b) + fx(a,b)(x-a) + fy(a,b)(y-b)$

$= 1 - \pi/2 + (-1)(x - \pi/2) + 1(y - 0)$

$= 1 - \cancel{\pi/2} - x + \cancel{\pi/2} + y$

$= 1 - x + y$

(iii) $f(x,y) = \log x + \log y$ at $(1,1)$

$f(1,1) = \log(1) + \log(1) = 0$

$fx = \frac{1}{x} + 0$ $\quad$ $fy = 0 + \frac{1}{y}$

fx at $(1,1) = 1$ $\quad$ fy at $(1,1) = 1$

$\therefore L(x,y) = f(a,b) + f(a,b)(x-a) + f(a,b)(y-b)$

$= 0 + 1(x-1) + 1(y-1)$

$= x - 1 + y - 1$

$= x + y - 2$

25/9/2020

PRACTICAL - 10

Q.1)

i) $f(x,y) = x + 2y - 2$ $a = (1, -1)$ $u = 3i - j$

→ Here, $U = 3i - j$ is not a unit vector

$|\bar{u}| = \sqrt{(3)^2 + (-1)^2} = \sqrt{9+1} = \sqrt{10}$

Unit vector along u is $\frac{u}{|u|} = \frac{1}{\sqrt{10}}(3, -1)$

$= \left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$

$f(a+hu) = f(1,-1) + h\left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$

$f(a) = f(1,-1) = (1) + 2(-1) - 3 = 1 - 2 = -4$

$f(a+hu) = f(1,-1) + h\left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$

$= f\left(1 + \frac{3}{\sqrt{10}}\right), \left(-1 - \frac{h}{\sqrt{10}}\right)$

$f(a+hu) = \left(1 + \frac{3}{\sqrt{10}}\right) + 2\left(-1 - \frac{h}{\sqrt{10}}\right) - 3$

$= 1 + \frac{3}{\sqrt{10}} - 2 - \frac{2h}{\sqrt{10}} - 3$

$f(a+hu) = -4 + \frac{h}{\sqrt{10}}$

$$D_u f(a) = \lim_{h \to 0} \frac{f(a+hu) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{-4} + \cancel{h}/\sqrt{10} + \cancel{4}}{\cancel{h}}$$

$$D_u f(a) = \frac{1}{\sqrt{10}}$$

ii) $f(x) = y^2 - 4x + 1 \quad a = (3, 4) . \quad u = i + 5j$

Here, $u = i + 5j$ is not a unit vector

$|\bar{u}| = \sqrt{(1)^2 + (5)^2} = \sqrt{26}$

Unit vector along u is $\frac{u}{|u|} = \frac{1}{\sqrt{26}} (1, 5)$

$$= \left(\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}} \right)$$

$f(a) = f(3,4) = (4)^2 - 4(3) + 1 = 5$

$f(a+hu) = f(3, 4) + h \left(\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}} \right)$

$$= f\left(3 + \frac{h}{\sqrt{26}}, 4 + \frac{5h}{\sqrt{26}} \right)$$

$f_{x,y}(a+hu) = \left(4 + \frac{5h}{\sqrt{26}} \right)^2 - 4 \left(3 + \frac{h}{\sqrt{20}} \right) + 1$

$$= 16 + \frac{25h^2}{26} + \frac{40h}{\sqrt{26}} - 12 - \frac{4h}{\sqrt{26}} + 1$$

$$= \frac{25h^2}{26} + \frac{40h}{\sqrt{26}} - \frac{4h}{\sqrt{26}} + 5$$

$$= \frac{25h^2}{26} + \frac{40h - 4h}{\sqrt{26}} + 5$$

$$= \frac{25h^2}{26} + \frac{36h}{\sqrt{26}} + 5$$

$$D_u f(a) = \lim_{h \to 0} \frac{\frac{25h^2}{26} + \frac{36h}{\sqrt{26}} \cancel{+5} \cancel{-5}}{h\left(\frac{25h}{26} + \frac{36}{\sqrt{26}}\right)}$$

$$D_u f(a) = \frac{25h}{26} + \frac{36}{\sqrt{26}}$$

(iii) $2x + 3y \quad a = (1, 2), \; u = (3i + 4j)$

Here $u = 3i + 4j$ is not a unit vector

$$|\vec{u}| = \sqrt{(3)^2 + (4)^2} = \sqrt{25} = 5$$

Unit vector along u is $\frac{u}{|u|} = \frac{1}{5}(3, 4)$

$$= \left(\frac{3}{5}, \frac{4}{5}\right)$$

$$f(a) = f(1, 2) = 2(1) + 3(2) = 8$$

$$f(a + hu) = f(1, 2) + h\left(\frac{3}{5}, \frac{4}{5}\right)$$

$$= f\left(1 + \frac{3h}{5}, \frac{2+4}{5}\right)$$

$$f(a + hu) = 2\left(1 + \frac{3h}{5}\right) + 3\left(2 + \frac{4h}{5}\right)$$

$$f(1,-1) = \left(\frac{1}{2}, \tan^{-1}(1)(-2)\right)$$
$$= \left(\frac{1}{2}, \frac{\pi}{\cancel{4}_2}(\cancel{-2})\right)$$
$$= \left(\frac{1}{2}, -\frac{\pi}{2}\right)$$

(ii) $f(x,y,z) = xyz - e^{x+y+z}$, $a = (1,-1,0)$

$f_x = yz - e^{x+y+z}$

$f_y = xz - e^{x+y+z}$

$f_z = xy - e^{x+y+z}$

$\nabla f(x,y,z) = f_x, f_y, f_z$

$= yz - e^{x+y+z}, xz - e^{x+y+z}, xy - e^{x+y+z}$

$f(1,-1,0) = \left((-1)(0) - e^{1+(-1)+0}, (1)(0) - e^{1+(-1)+0}, (1)(-1), e^{1+(-1)+0}\right)$

$= (0 - e^0, 0 - e^0, -1 - e^0)$
$= (\cancel{-4}\,{-1}, -1, 2)$

Q.3

(i) $x^2 \cos y + e^{xy} = 2$ at $(1,0)$

$f_x = \cos y \; 2x + e^{xy}\, y$

$f_y = x^2(-\sin y) + e^{xy}\, x$

$(x_0, y_0) = (1, 0)$ $\therefore x_0 = 1,\ y_0 = 0$

eq^n of tangent

$f_x(x - x_0) + f_y(y - y_0) = 0$

$f_x(x_0, y_0) = \cancel{(1)} \cos 0 \; 2(1) + e^0 \, 0$
$= 1(2) + 0$
$= 2$

$f_y(x_0, y_0) = (1)^2(-\sin 0) + e^0 . 1$
$= 0 + 1.1$
$= 1$

$2(x-1) + 1(y-0) = 0$
$2x - 2 + y = 0$
$2x + y - 2 = 0$ → It is the required eq^n of tangent.

eq^n of Normal.

$= ax + by + c = 0$
$= bx + ay + d = 0$

$1(1) + 2(y) + d = 0$
$1 + 2y + d = 0$ at $(1, 0)$
$1 + 2(0) + d = 0$
$d + 1 = 0$

$\therefore d = -1$

(ii) $x^2 + y^2 - 2x + 3y + 2 = 0$ at $(2, -2)$

$f_x = 2x + 0 - 2 + 0 + 0$

$= 2x - 2$

$f_y = 0 + 2y - 0 + 3 + 0$

$= 2y + 3$

$(x_0, y_0) = (2, -2)$ $\therefore x_0 = 2,\ y_0 = -2$

$f_x(x_0, y_0) = 2(2) - 2 = 2$

$f_y(x_0, y_0) = 2(-2) + 3 = -1$

eq^n of tangent

$f_x(x - x_0) + f_y(y - y_0) = 0$

$2(x-2) + (-1(y+2)) = 0$

$2x - 2 - y - 2 = 0$

$2x - y - 4 = 0$ → It is required eq^n of tangent.

eq^n of Normal

$= ax + by + c = 0$

$bx + ay + d = 0$

$= -1(x) + 2(y) + d = 0$

$-x + 2y + d = 0$ at $(2, -2)$

$-2 + 2(-2) + d = 0$

$-2 - 4 + d = 0$

$-6 + d = 0$

$\therefore d = 6$

Q.4.

(i) $x^2 - 2yz + 3y + xz = 7$ at $(2, 1, 0)$

$$f_x = 2x - 0 + 0 + z$$
$$fx = 2x + z$$
$$fy = 0 - 2z + 3 + 0$$
$$= 2z + 3$$
$$f_z = 0 - 2y + 0 + x$$
$$= -2y + x$$

$(x_0, y_0, z_0) = (2, 1, 0)$ $\therefore$ $x_0 = 2$, $y_0 = 1$, $z_0 = 0$

$$fx(x_0, y_0, z_0) = 2(2) + 0 = 4$$
$$fy(x_0, y_0, z_0) = 2(0) + 3 = 3$$
$$f_z(x_0, y_0, z_0) = -2(1) + 2 = 0$$

eq^n of tangent

$$fx(x - x_0) + fy(y - y_0) + f_z(z - z_0) = 0$$
$$= 4(x - 2) + 3(y - 1) + 0(z - 0) = 0$$
$$= 4x - 8 + 3y - 3 = 0$$
$$4x + 3y - 11 = 0$$ → this required eq^n of tangent.

Eq^n of normal at $(4, 3, -11)$

$$\frac{x - x_0}{f_x} = \frac{y - y_0}{fy} = \frac{z - z_0}{f_z}$$

$$= \frac{x - 2}{4} = \frac{y - 1}{3} = \frac{z + 11}{0}$$

//

ii) $3xyz - x - y + z = -4$ at $(1, -1, 2)$.

$3xyz - x - y + z + 4 = 0$ at $(1, -1, 2)$

$f_x = 3yz - 1 - 0 + 0 + 0$
$= 3yz - 1$

$f_y = 3xz - 0 - 1 + 0 + 0$
$= 3xz - 1$

$f_z = 3xy - 0 - 0 + 1 + 0$
$= 3xy + 1$

$(x_0, y_0, z_0) = (1, -1, 2) \therefore x_0 = 1,\ y_0 = -1,\ z_0 = 2$

$f_x(x_0, y_0, z_0) = 3(-1)(2) - 1 = -7$
$f_y(x_0, y_0, z_0) = 3(+1)(2) - 1 = 5$
$f_z(x_0, y_0, z_0) = 3(1)(-1) + 1 = -2$

eq^n of tangent

$-7(x-1) + 5(y+1) - 2(z-2) = 0$
$-7x + 7 + 5y + 5 - 2z + 4 = 0$
$-7x + 5y - 2z + 16 = 0$ → This is required eq^n of tangent.

Eq^n of normal at $(-7, 5, -2)$

$$\frac{x - x_0}{f_x} = \frac{y - y_0}{f_y} = \frac{z - z_0}{f_z}$$

$$= \frac{x-1}{-7} = \frac{y+1}{5} = \frac{z-2}{-2}$$

//

Q.5

(i) $f(x,y) = 3x^2 + y^2 - 3xy + 6x - 4y$

$f_x = 6x + 0 - 3y + 6 - 0$

$= 6x - 3y + 6$

$f_y = 0 + 2y - 3x + 0 - 4$

$= 2y - 3x - 4$

$f_x = 0$

$6x - 3y + 6 = 0$

$3(2x - y + 2) = 0$

$2x - y + 2 = 0$

$2x - y = -2 \rightarrow (1)$

$f_y = 0$

$2y - 3x - 4 = 0$

$2y - 3x = 4 \rightarrow (2)$

Multiply eq^n 1 with 2

$\therefore 4x - \cancel{2y} = -4$

$\underline{\cancel{2y} - 3x = 4}$

$x = 0$

Substitute value of x in eq^n (1)

$2(0) - y = -2$

$\cancel{-}y = \cancel{-}2 \quad \therefore y = 2$

$\therefore$ Critical points are $(0, 2)$

$r = f_{xx} = 6$

$t = f_{yy} = 2$

$s = f_{xy} = -3$

Here $r > 0$

$= rt - s^2$

$= 6(2) - (-3)^2$

$= 12 - 9$

$= 3 > 0$

$\therefore$ f has maximum at $(0, 2)$

$3x^2 + y^2 - 3xy + 6x - 4y$ at $(0, 2)$

$3x^2 + y^2 - 3xy + 6x - 4$ at $(0, 2)$

$3(0)^2 + (2)^2 - 3(0)(2) + 6(0) - 4(2)$

$0 + 4 - 0 + 0 - 8$

$= -4$ //

(ii) $f(x, y) = 2x^4 + 3x^2y - y^2$

$fy = 8x^3 + 6xy$

$fy = 3x^2 - 2y$

$fx = 0$

$8x^3 + 6xy = 0$

$2x(4x^2 + 3y) = 0$

$4x^2 + 3y = 0 \rightarrow (1)$

$fy = 0$

$3x^2 - 2y = 0 \rightarrow (2)$

Multiply eq^n (1) with 3

(2) with 4

$\cancel{12x^2} + 9y = 0$

$-\cancel{12x^2} \mp 8y = 0$

$17y = 0$

$y = 0$ //

Substitute value of y in eq^n (1)

$4x^2 + 3(0) = 0$

$4x^2 = 0$

$x = 0$

Critical point is $(0, 0)$

$r = f_{xx} = 24x^2 + 6x$

$t = f_{yy} = 0 - 2 = -2$

$s = f_{xy} = 6x - 0 = 6x = 6(0) = 0$

r at $(0, 0)$

$= 24(0) + 6(0) = 0$

$\therefore r = 0$

$f(x, y)$ at $(0, 0)$

$2(0)^4 + 3(0)^2(0) - (0)^2$

$= 0 + 0 - 0$

$= 0$

$rt - s^2 = 0(-2) - s)^2$

$= 0 - 0 = 0$

$r = 0$ & $rt - s^2 = 0$

(nothing to say)

(iii) $f(x, y) = x^2 - y^2 + 2x + 8y - 70$

$f_x = 2x + 2$

$f_y = -2y + 8$

$f_x = 0 \quad \therefore 2x + 2 = 0$

$x = \frac{-2}{2} \quad \therefore x = 1$

$f_y = 0 \quad -2y + 8 = 0$

$y = -\frac{8}{-2} \quad 4$

$\therefore y = 4$

$\therefore$ critical point is $(-1, 4)$

$r = f_{xx} = 2$

$t = f_{yy} = -2$

$s = f_{xy} = 0$

$r > 0$

$$rt - s^2 = 2(-2) - (0)^2$$
$$= -4 - 0$$
$$= -4 < 0$$

$f(x, y)$ at $(-1, 4)$

$$(-1)^2 - (4)^2 + 2(-1) + 8(4) - 70$$
$$= 1 + 16 - 2 + 32 - 70$$
$$= 17 + 30 - 70$$
$$= 37 - 70$$
$$= 33$$