Knapsack problem

There are two versions of the problem:

1. "0-1 knapsack problem" and
2. "Fractional knapsack problem"

1. Items are indivisible; you either take an item or not. Solved with *dynamic programming*
2. Items are divisible: you can take any fraction of an item. Solved with a *greedy algorithm*.
   - We have already seen this version

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0-1 Knapsack problem

- Given a knapsack with maximum capacity W, and a set S consisting of n items
- Each item i has some weight w_i and benefit value b_i (all w_i , b_i and W are integer values)
- Problem: How to pack the knapsack to achieve maximum total value of packed items?

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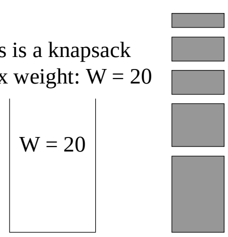
0-1 Knapsack problem: a picture

This is a knapsack
Max weight: W = 20



Items	Weight w_i	Benefit value b_i
	2	3
	3	4
	4	5
	5	8
	9	10

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0-1 Knapsack problem

- Problem, in other words, is to find

$$\max \sum_{i \in T} b_i \text{ subject to } \sum_{i \in T} w_i \leq W$$

- The problem is called a "*0-1*" problem, because each item must be entirely accepted or rejected.
- In the "*Fractional Knapsack Problem*," we can take fractions of items.

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0-1 Knapsack problem: brute-force approach

Let's first solve this problem with a straightforward algorithm

- Since there are n items, there are 2^n possible combinations of items.
- We go through all combinations and find the one with maximum value and with total weight less or equal to W
- Running time will be $O(2^n)$

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0-1 Knapsack problem: brute-force approach

- Can we do better?
- Yes, with an algorithm based on dynamic programming
- We need to carefully identify the subproblems

Let's try this:
If items are labeled *1..n*, then a subproblem would be to find an optimal solution for
S_k = *{items labeled 1, 2, .. k}*

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Defining a Subproblem

If items are labeled *1..n*, then a subproblem would be to find an optimal solution for S_k = *{items labeled 1, 2, .. k}*

- This is a reasonable subproblem definition.
- The question is: can we describe the final solution (S_n) in terms of subproblems (S_k)?
- Unfortunately, we can't do that.

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Defining a Subproblem

$w_1=2$ $b_1=3$	$w_2=4$ $b_2=5$	$w_3=5$ $b_3=8$	$w_4=3$ $b_4=4$	

?

Max weight: W = 20
For S_4:
Total weight: 14;
Maximum benefit: 20

Item #	Weight w_i	Benefit b_i
1	2	3
2	3	4
3	4	5
4	5	8
5	9	10

S_4 S_5

$w_1=2$ $b_1=3$	$w_2=4$ $b_2=5$	$w_3=5$ $b_3=8$	$w_5=9$ $b_5=10$

For S_5:
Total weight: 20
Maximum benefit: 26

Solution for S_4 is not part of the solution for S_5!!!

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Defining a Subproblem (continued)

- As we have seen, the solution for S_4 is not part of the solution for S_5
- So our definition of a subproblem is flawed and we need another one!
- Let's add another parameter: *w*, which will represent the exact weight for each subset of items
- The subproblem then will be to compute *B[k,w]*

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Recursive Formula for subproblems

Recursive formula for subproblems:

$$B[k,w] = \begin{cases} B[k-1,w] & \text{if } w_k > w \\ \max\{B[k-1,w], B[k-1,w-w_k]+b_k\} & \text{else} \end{cases}$$

It means, that the best subset of S_k that has total weight *w* is:

1) the best subset of S_{k-1} that has total weight *w*, **or**
2) the best subset of S_{k-1} that has total weight w-w_k plus the item *k*

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Recursive Formula

$$B[k,w] = \begin{cases} B[k-1,w] & \text{if } w_k > w \\ \max\{B[k-1,w], B[k-1,w-w_k]+b_k\} & \text{else} \end{cases}$$

- The best subset of S_k that has the total weight *w*, either contains item *k* or not.
- First case: $w_k > w$. Item *k* can't be part of the solution, since if it was, the total weight would be > *w*, which is unacceptable.
- Second case: $w_k \leq w$. Then the item *k* can be in the solution, and we choose the case with greater value.

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0-1 Knapsack Algorithm

```
for w = 0 to W
   B[0,w] = 0
for i = 1 to n
   B[i,0] = 0
for i = 1 to n
   for w = 0 to W
        if wi <= w // item i can be part of the solution
                if bi + B[i-1,w-wi] > B[i-1,w]
                        B[i,w] = bi + B[i-1,w- wi]
                else
                        B[i,w] = B[i-1,w]
        else B[i,w] = B[i-1,w]  // wi > w
```

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Running time

```
for w = 0 to W
    B[0,w] = 0
for i = 1 to n
    B[i,0] = 0
for i = 1 to n
    for w = 0 to W
        < the rest of the code >
```

O(W)

Repeat *n* times

O(W)

What is the running time of this algorithm?

O(n*W)

Remember that the brute-force algorithm takes $O(2^n)$

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Example

Let's run our algorithm on the following data:

n = 4 (# of elements)
W = 5 (max weight)
Elements (weight, benefit):
(2,3), (3,4), (4,5), (5,6)

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Example (2)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1						
2						
3						
4						

```
for w = 0 to W
    B[0,w] = 0
```

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Example (3)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0					
2	0					
3	0					
4	0					

```
for i = 1 to n
    B[i,0] = 0
```

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Example (4)

Items:
$\boxed{\text{1: (2,3)}}$
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0				
2	0					
3	0					
4	0					

i=1
b_i=3
w_i=2
w=1
w-w_i =-1

if w_i <= w // item i can be part of the solution
  if b_i + B[i-1,w-w_i] > B[i-1,w]
    B[i,w] = b_i + B[i-1,w- w_i]
  else
    B[i,w] = B[i-1,w]
else **B[i,w] = B[i-1,w]** // w_i > w

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Example (5)

Items:
$\boxed{\text{1: (2,3)}}$
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	**3**			
2	0					
3	0					
4	0					

i=1
b_i=3
w_i=2
w=2
w-w_i =0

if w_i <= w // item i can be part of the solution
  if b_i + B[i-1,w-w_i] > B[i-1,w]
    B[i,w] = b_i + B[i-1,w- w_i]
  else
    B[i,w] = B[i-1,w]
else B[i,w] = B[i-1,w] // w_i > w

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Example (6)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	**3**		
2	0					
3	0					
4	0					

i=1
$b_i=3$
$w_i=2$
w=3
$w-w_i=1$

if w_i <= w // item i can be part of the solution
if b_i + B[i-1,w-w_i] > B[i-1,w]
B[i,w] = b_i + B[i-1,w- w_i]
else
B[i,w] = B[i-1,w]
else B[i,w] = B[i-1,w] // w_i > w

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Example (7)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	**3**	
2	0					
3	0					
4	0					

i=1
$b_i=3$
$w_i=2$
w=4
$w-w_i=2$

if w_i <= w // item i can be part of the solution
if b_i + B[i-1,w-w_i] > B[i-1,w]
B[i,w] = b_i + B[i-1,w- w_i]
else
B[i,w] = B[i-1,w]
else B[i,w] = B[i-1,w] // w_i > w

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Example (8)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	**3**
2	0					
3	0					
4	0					

i=1
$b_i=3$
$w_i=2$
w=5
$w-w_i=3$

if w_i <= w // item i can be part of the solution
if b_i + B[i-1,w-w_i] > B[i-1,w]
B[i,w] = b_i + B[i-1,w- w_i]
else
B[i,w] = B[i-1,w]
else B[i,w] = B[i-1,w] // w_i > w

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Example (9)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	**0**				
3	0					
4	0					

i=2
$b_i=4$
$w_i=3$
w=1
$w-w_i=-2$

if w_i <= w // item i can be part of the solution
if b_i + B[i-1,w-w_i] > B[i-1,w]
B[i,w] = b_i + B[i-1,w- w_i]
else
B[i,w] = B[i-1,w]
else **B[i,w] = B[i-1,w]** // w_i > w

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Example (10)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	**3**			
3	0					
4	0					

i=2
$b_i=4$
$w_i=3$
w=2
$w-w_i=-1$

if w_i <= w // item i can be part of the solution
if b_i + B[i-1,w-w_i] > B[i-1,w]
B[i,w] = b_i + B[i-1,w- w_i]
else
B[i,w] = B[i-1,w]
else **B[i,w] = B[i-1,w]** // w_i > w

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Example (11)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	**4**		
3	0					
4	0					

i=2
$b_i=4$
$w_i=3$
w=3
$w-w_i=0$

if w_i <= w // item i can be part of the solution
if b_i + B[i-1,w-w_i] > B[i-1,w]
B[i,w] = b_i + B[i-1,w- w_i]
else
B[i,w] = B[i-1,w]
else B[i,w] = B[i-1,w] // w_i > w

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Example (12)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	**4**	
3	0					
4	0					

$i=2$
$b_i=4$
$w_i=3$
$w=4$
$w-w_i=1$

if w_i <= w // item i can be part of the solution
  if b_i + B[i-1,w-w_i] > B[i-1,w]
    B[i,w] = b_i + B[i-1,w- w_i]
  else
    B[i,w] = B[i-1,w]
else B[i,w] = B[i-1,w] // w_i > w

Example (13)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	**7**
3	0					
4	0					

$i=2$
$b_i=4$
$w_i=3$
$w=5$
$w-w_i=2$

if w_i <= w // item i can be part of the solution
  if b_i + B[i-1,w-w_i] > B[i-1,w]
    B[i,w] = b_i + B[i-1,w- w_i]
  else
    B[i,w] = B[i-1,w]
else B[i,w] = B[i-1,w] // w_i > w

Example (14)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	**0**	**3**	**4**		
4	0					

$i=3$
$b_i=5$
$w_i=4$
$w= 1..3$

if w_i <= w // item i can be part of the solution
  if b_i + B[i-1,w-w_i] > B[i-1,w]
    B[i,w] = b_i + B[i-1,w- w_i]
  else
    B[i,w] = B[i-1,w]
else **B[i,w] = B[i-1,w]** // w_i > w

Example (15)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	**5**	
4	0					

$i=3$
$b_i=5$
$w_i=4$
$w= 4$
$w- w_i=0$

if w_i <= w // item i can be part of the solution
  if b_i + B[i-1,w-w_i] > B[i-1,w]
    B[i,w] = b_i + B[i-1,w- w_i]
  else
    B[i,w] = B[i-1,w]
else B[i,w] = B[i-1,w] // w_i > w

Example (16)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	5	**7**
4	0					

$i=3$
$b_i=5$
$w_i=4$
$w= 5$
$w- w_i=1$

if w_i <= w // item i can be part of the solution
  if b_i + B[i-1,w-w_i] > B[i-1,w]
    B[i,w] = b_i + B[i-1,w- w_i]
  else
    B[i,w] = B[i-1,w]
else B[i,w] = B[i-1,w] // w_i > w

Example (17)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	5	7
4	0	**0**	**3**	**4**	**5**	

$i=4$
$b_i=6$
$w_i=5$
$w= 1..4$

if w_i <= w // item i can be part of the solution
  if b_i + B[i-1,w-w_i] > B[i-1,w]
    B[i,w] = b_i + B[i-1,w- w_i]
  else
    B[i,w] = B[i-1,w]
else **B[i,w] = B[i-1,w]** // w_i > w

Example (18)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	5	7
4	0	0	3	4	5	**7**

i=4
b_i=6
w_i=5
w= 5
w- w_i=0

if w_i <= w // item i can be part of the solution
if b_i + B[i-1,w-w_i] > B[i-1,w]
B[i,w] = b_i + B[i-1,w- w_i]
else
B[i,w] = B[i-1,w]
else B[i,w] = B[i-1,w] // w_i > w

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Comments

- This algorithm only finds the max possible value that can be carried in the knapsack
  - I.e., the value in B[n,W]
- To know the items that make this maximum value, an addition to this algorithm is necessary.

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How to find actual Knapsack Items

- All of the information we need is in the table.
- $B[n,W]$ is the maximal value of items that can be placed in the Knapsack.
- Let i=n and k=W

if $B[i,k] \neq B[i-1,k]$ then
mark the i^{th} item as in the knapsack
$i = i-1$, $k = k-w_i$
else
$i = i-1$ // Assume the i^{th} item is <u>not</u> in the knapsack
// Could it be in the optimally packed knapsack?

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Finding the Items

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	5	7
4	0	0	3	4	5	7

i=4
k= 5
b_i=6
w_i=5
$B[i,k]$ = 7
$B[i-1,k]$ =7

i=n, k=W
while i,k > 0
if $B[i,k] \neq B[i-1,k]$ then
mark the i^{th} item as in the knapsack
$i = i-1$, $k = k-w_i$
else
$i = i-1$

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Finding the Items (2)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	5	7
4	0	0	3	4	5	7

i=4
k= 5
b_i=6
w_i=5
$B[i,k]$ = 7
$B[i-1,k]$ =7

i=n, k=W
while i,k > 0
if $B[i,k] \neq B[i-1,k]$ then
mark the i^{th} item as in the knapsack
$i = i-1$, $k = k-w_i$
else
i = i–1

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Finding the Items (3)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	5	7
4	0	0	3	4	5	7

i=3
k= 5
b_i=6
w_i=4
$B[i,k]$ = 7
$B[i-1,k]$ =7

i=n, k=W
while i,k > 0
if $B[i,k] \neq B[i-1,k]$ then
mark the i^{th} item as in the knapsack
$i = i-1$, $k = k-w_i$
else
i = i–1

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Finding the Items (4)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	5	7
4	0	0	3	4	5	7

i=2
k= 5
b_i=4
w_i=3
$B[i,k]$ = 7
$B[i–1,k]$ =3
k – w_i=2

```
i=n, k=W
while i,k > 0
    if B[i,k] ≠ B[i–1,k] then
        mark the ith item as in the knapsack
        i = i–1, k = k-wi
    else
        i = i–1
```


Finding the Items (5)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	5	7
4	0	0	3	4	5	7

i=1
k= 2
b_i=3
w_i=2
$B[i,k]$ = 3
$B[i–1,k]$ =0
k – w_i=0

```
i=n, k=W
while i,k > 0
    if B[i,k] ≠ B[i–1,k] then
        mark the ith item as in the knapsack
        i = i–1, k = k-wi
    else
        i = i–1
```


Finding the Items (6)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	5	7
4	0	0	3	4	5	7

i=0
k= 0

The optimal knapsack should contain {1, 2}

```
i=n, k=W
while i,k > 0
    if B[i,k] ≠ B[i–1,k] then
        mark the nth item as in the knapsack
        i = i–1, k = k-wi
    else
        i = i–1
```


Finding the Items (7)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	5	7
4	0	0	3	4	5	7

The optimal knapsack should contain {1, 2}

```
i=n, k=W
while i,k > 0
    if B[i,k] ≠ B[i–1,k] then
        mark the nth item as in the knapsack
        i = i–1, k = k-wi
    else
        i = i–1
```


Review: The Knapsack Problem And Optimal Substructure

- Both variations exhibit optimal substructure
- To show this for the 0-1 problem, consider the most valuable load weighing at most W pounds
  - *If we remove item j from the load, what do we know about the remaining load?*
  - A: remainder must be the most valuable load weighing at most W - w_j that thief could take, excluding item j

Solving The Knapsack Problem

- The optimal solution to the fractional knapsack problem can be found with a greedy algorithm
  - *Do you recall how?*
  - Greedy strategy: take in order of dollars/pound
- The optimal solution to the 0-1 problem cannot be found with the same greedy strategy
  - Example: 3 items weighing 10, 20, and 30 pounds, knapsack can hold 50 pounds
    - *Suppose item 2 is worth $100. Assign values to the other items so that the greedy strategy will fail*

The Knapsack Problem: Greedy Vs. Dynamic

- The fractional problem can be solved greedily
- The 0-1 problem can be solved with a dynamic programming approach

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Memoization

- *Memoization* is another way to deal with overlapping subproblems in dynamic programming
  - » After computing the solution to a subproblem, store it in a table
  - » Subsequent calls just do a table lookup
- With memoization, we implement the algorithm recursively:
  - » If we encounter a subproblem we have seen, we look up the answer
  - » If not, compute the solution and add it to the list of subproblems we have seen.
- Must useful when the algorithm is easiest to implement recursively
  - » Especially if we do not need solutions to all subproblems.

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Conclusion

- Dynamic programming is a useful technique of solving certain kind of problems
- When the solution can be recursively described in terms of partial solutions, we can store these partial solutions and re-use them as necessary (memoization)
- Running time of dynamic programming algorithm vs. naïve algorithm:
  - » 0-1 Knapsack problem: **O(W*n)** vs. **O(2^n)**

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