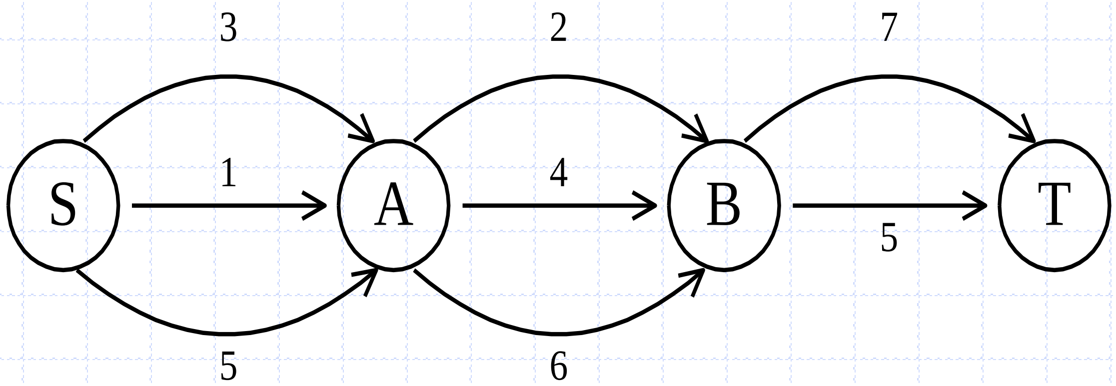
Multistage graph

The shortest path

- To find a shortest path in a multi-stage graph



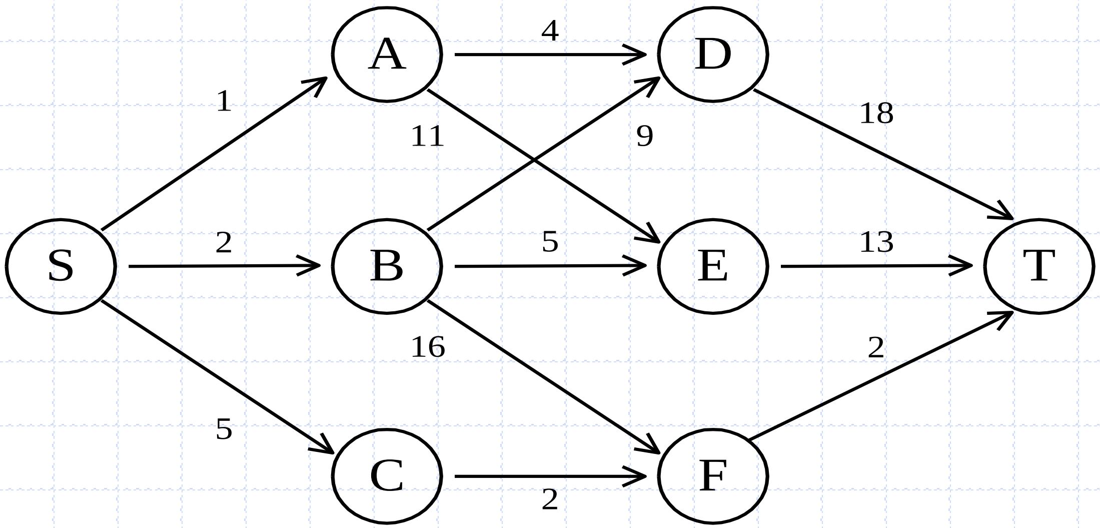
- Apply the greedy method :

  the shortest path from S to T :

  1 + 2 + 5 = 8

The shortest path in multistage graphs

- e.g.

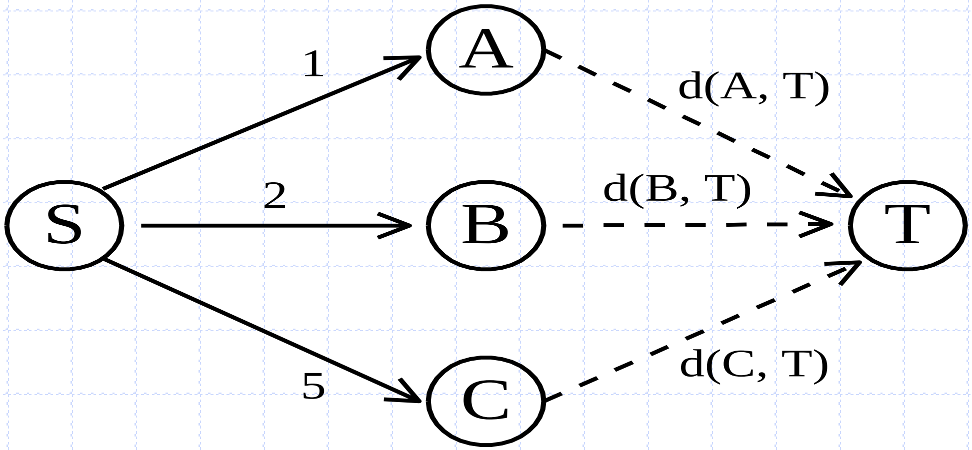


- The greedy method can not be applied to this case: (S, A, D, T) 1+4+18 = 23.
- The real shortest path is:

  (S, C, F, T) 5+2+2 = 9.

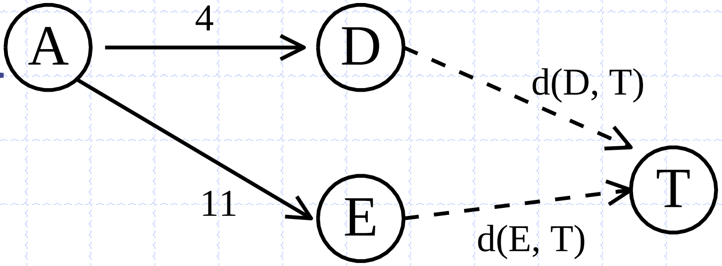
Dynamic programming approach

- Dynamic programming approach (forward approach):



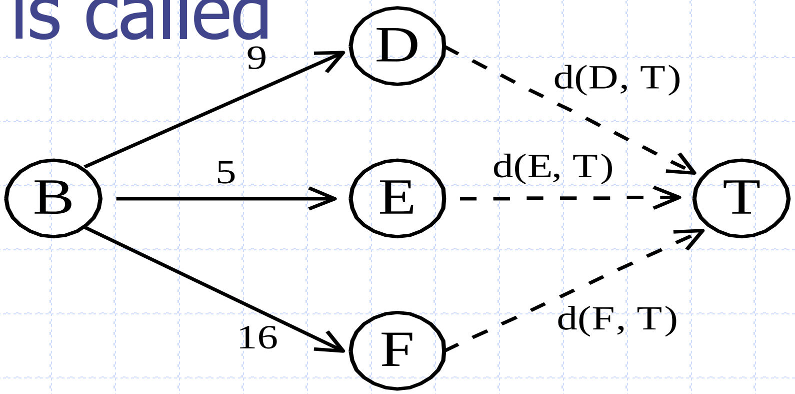
- d(S, T) = min{1+d(A, T), 2+d(B, T), 5+d(C, T)}

- d(A,T) = min{4+d(D,T), 11+d(E,T)}
= min{4+18, 11+13} = 22.



Dynamic programming

- $d(B, T) = \min\{9+d(D, T), 5+d(E, T), 16+d(F, T)\}$
  $= \min\{9+18, 5+13, 16+2\} = 18.$
- $d(C, T) = \min\{ 2+d(F, T) \} = 2+2 = 4$
- $d(S, T) = \min\{1+d(A, T), 2+d(B, T), 5+d(C, T)\}$
  $= \min\{1+22, 2+18, 5+4\} = 9.$
- The above way of reasoning is called backward reasoning.



Backward approach (forward reasoning)

- $d(S, A) = 1$
  $d(S, B) = 2$
  $d(S, C) = 5$
- $d(S,D)=\min\{d(S, A)+d(A, D), d(S, B)+d(B, D)\}$
  $= \min\{ 1+4, 2+9 \} = 5$
  $d(S,E)=\min\{d(S, A)+d(A, E), d(S, B)+d(B, E)\}$
  $= \min\{ 1+11, 2+5 \} = 7$
  $d(S,F)=\min\{d(S, A)+d(A, F), d(S, B)+d(B, F)\}$
  $= \min\{ 2+16, 5+2 \} = 7$

- $d(S,T) = \min\{d(S, D)+d(D, T), d(S,E)+ d(E,T), d(S, F)+d(F, T)\}$

  $= \min\{ 5+18, 7+13, 7+2 \}$

  $= 9$

Principle of optimality

- Principle of optimality: Suppose that in solving a problem, we have to make a sequence of decisions $D_1, D_2, \ldots, D_n$. If this sequence is optimal, then the last k decisions, $1 < k < n$ must be optimal.
- e.g. the shortest path problem

  If $i, i_1, i_2, \ldots, j$ is a shortest path from i to j, then $i_1, i_2, \ldots, j$ must be a shortest path from i_1 to j
- In summary, if a problem can be described by a multistage graph, then it can be solved by dynamic programming.

Dynamic programming

- Forward approach and backward approach:
  - Note that if the recurrence relations are formulated using the forward approach then the relations are solved backwards . i.e., beginning with the last decision
  - On the other hand if the relations are formulated using the backward approach, they are solved forwards.
- To solve a problem by using dynamic programming:
  - Find out the recurrence relations.
  - Represent the problem by a multistage graph.

The resource allocation problem

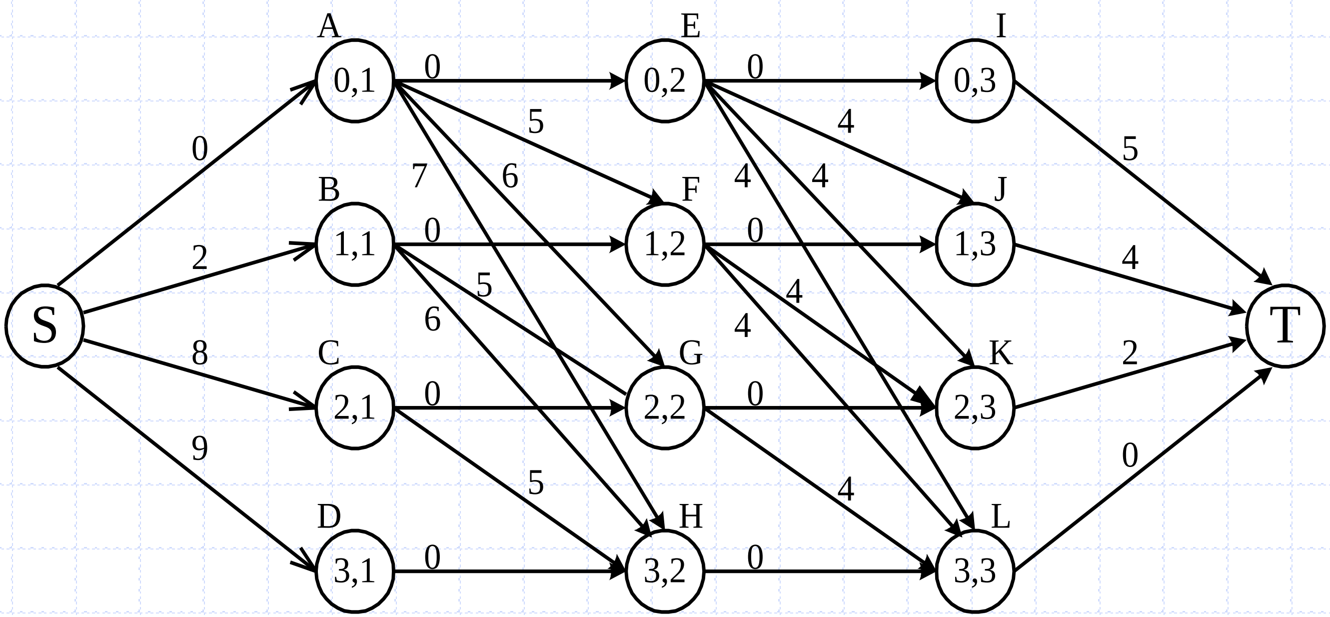
- m resources, n projects

  profit p(i, j) : j resources are allocated to project i.

  maximize the total profit.

Resource / Project	1	2	3
1	2	8	9
2	5	6	7
3	4	4	4
4	2	4	5

The multistage graph solution



- The resource allocation problem can be described as a multistage graph.
- (i, j) : i resources allocated to projects 1, 2, …, j

e.g. node H=(3, 2) : 3 resources allocated to projects 1, 2.

- Find the longest path from S to T :

  (S, C, H, L, T), 8+5+0+0=13

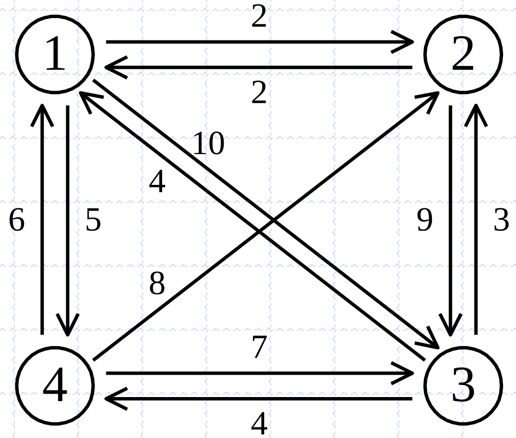
  2 resources allocated to project 1.

  1 resource allocated to project 2.

  0 resource allocated to projects 3, 4.

The traveling salesperson (TSP) problem

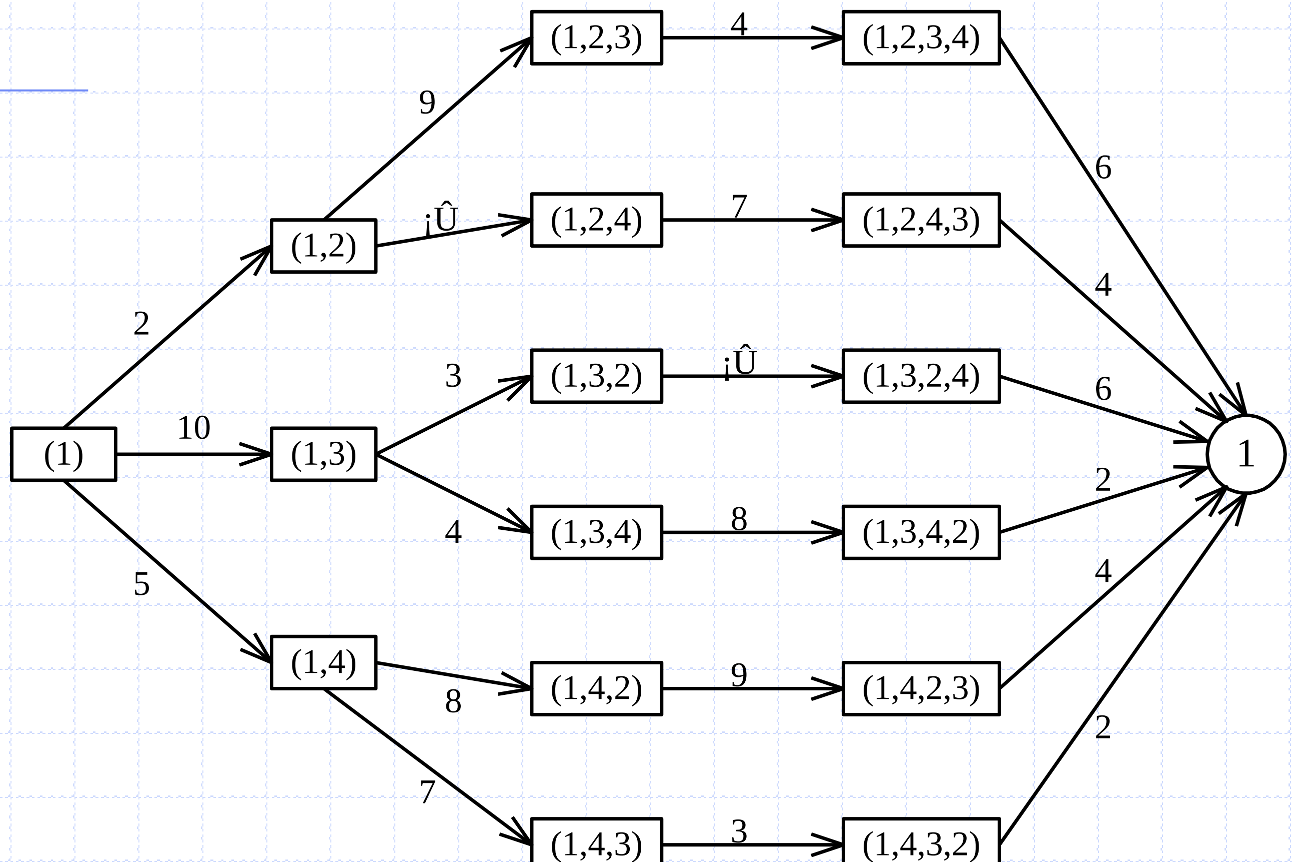
- e.g. a directed graph :



- Cost matrix:

$$
\begin{array}{c c}
 & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \left[\begin{array}{cccc} \infty & 2 & 10 & 5 \\ 2 & \infty & 9 & \infty \\ 4 & 3 & \infty & 4 \\ 6 & 8 & 7 & \infty \end{array}\right]
\end{array}
$$

The multistage graph solution



- A multistage graph can describe all possible tours of a directed graph.
- Find the shortest path:(1, 4, 3, 2, 1)   5+7+3+2=17