Quanser NI-ELVIS Trainer (QNET) Series:

QNET Experiment #04: Inverted Pendulum Control

Rotary Pendulum (ROTPEN) Inverted Pendulum Trainer

Student Manual

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1. Laboratory Objectives

The inverted pendulum is a classic experiment used to teach dynamics and control systems. In this laboratory, the pendulum dynamics are derived using Lagrangian equations and an introduction to nonlinear control is made.

There are two control challenges: designing a balance controller and designing a swing-up control. After manually initializing the pendulum in the upright vertical position, the balance controller moves the rotary arm to keep the pendulum in this upright position. It is designed using the Linear-Quadratic Regulator technique on a linearized model of the rotary pendulum system.

The swing-up controller drives the pendulum from its suspended downward position to the vertical upright position, where the balance controller can then be used to balance the link. The pendulum equation of motion is derived using Lagrangian principles and the pendulum moment of inertia is identified experimentally to obtain a model that represents the pendulum more accurately. The swing-up controller is designed using the pendulum model and a Lyapunov function. Lyapunov functions are commonly used in control theory and design and it will be introduced to design the nonlinear swing-up control.

2. References

[1] *QNET-ROTPEN User Manual.*
[2] *NI-ELVIS User Manual.*
[3] *QNET Experiment #03: ROTPEN Gantry Control*

3. ROTPEN Plant Presentation

3.1. Component Nomenclature

As a quick nomenclature, Table 1, below, provides a list of the principal elements composing the Rotary Pendulum (ROTPEN) Trainer system. Every element is located and identified, through a unique identification (ID) number, on the ROTPEN plant represented in Figure 1, below.

ID #	Description	ID #	Description
1	DC Motor	**3**	Arm
2	Motor/Arm Encoder	**4**	Pendulum

Table 1 ROTPEN Component Nomenclature

Figure 1 ROTPEN System

3.2. ROTPEN Plant Description

The QNET-ROTPEN Trainer system consists of a 24-Volt DC motor that is coupled with an encoder and is mounted vertically in the metal chamber. The L-shaped arm, or hub, is connected to the motor shaft and pivots between ±180 degrees. At the end of the arm, there is a suspended pendulum attached. The pendulum angle is measured by an encoder.

4. Pre-Lab Assignments

This section must be read, understood, and performed before you go to the laboratory session.

The first section, Section 4.1, summarizes the control design method using the linear-quadratic regulator technique to construct the balance control. Section 4.2 is the first pre-lab exercise and involves modeling the open-loop pendulum using Lagrangian. The second pre-lab assignment, in Section 4.3, develops the equations needed to experimentally identify the pendulum inertia. Lastly, the last pre-lab exercise in Section 4.4 is designing the swing-up control.

4.1. Balance Control Design

Section 4.1.1 discusses the model of the inverted pendulum and the resulting linear state-space representation of the device. The design of a controller that balances an inverted pendulum is summarized in Section 4.1.2.

4.1.1. Open-Loop Modeling

As already discussed in the gantry experiment, ROTPEN Laboratory #3, the ROTPEN plant is free to move in two rotary directions. Thus it is a two degree of freedom, or 2 DOF, system. As described in Figure 2, the arm rotates about the Y0 axis and its angle is denoted by the symbol θ while the pendulum attached to the arm rotates about its pivot and its angle is called α. The shaft of the DC motor is connected to the arm pivot and the input voltage of the motor is the control variable.

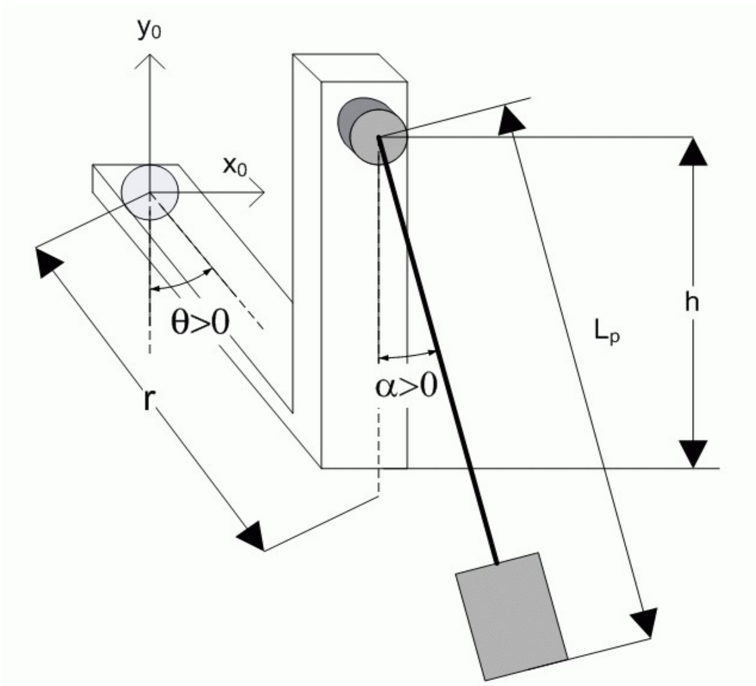

Figure 2 Rotary Pendulum System

In the inverted pendulum experiment, the pendulum angle, α, is defined to be positive when the it rotates counter-clockwise. That is, as the arm moves in the positive clockwise direction, the *inverted* pendulum moves clockwise (i.e. the *suspended* pendulum moves counter-clockwise) and that is defined as $\alpha>0$. Recall that in the gantry device, when the arm rotates in the positive clockwise direction the pendulum moves clockwise, which in turn is defined as being positive.

The nonlinear dynamics between the angle of the arm, θ, the angle of the pendulum, α, and the torque applied at the arm pivot, τ_{output}, are

$$
\begin{aligned}
\frac{d^2}{dt^2}\theta(t) = &-\frac{M_p^{\,2}\, g\, l_p^{\,2}\, r \cos(\theta(t))\, \alpha(t)}{(M_p\, r^2 \sin(\theta(t))^2 - J_{eq} - M_p\, r^2)\, J_p - M_p\, l_p^{\,2}\, J_{eq}} \\
&- \frac{J_p\, M_p\, r^2 \cos(\theta(t)) \sin(\theta(t)) \left(\frac{d}{dt}\theta(t)\right)^2}{(M_p\, r^2 \sin(\theta(t))^2 - J_{eq} - M_p\, r^2)\, J_p - M_p\, l_p^{\,2}\, J_{eq}} \\
&- \frac{J_p\, \tau_{output} + M_p\, l_p^{\,2}\, \tau_{output}}{(M_p\, r^2 \sin(\theta(t))^2 - J_{eq} - M_p\, r^2)\, J_p - M_p\, l_p^{\,2}\, J_{eq}} \\
\frac{d^2}{dt^2}\alpha(t) = &-\frac{l_p\, M_p\, (-J_{eq}\, g + M_p\, r^2 \sin(\theta(t))^2\, g - M_p\, r^2\, g)\, \alpha(t)}{(M_p\, r^2 \sin(\theta(t))^2 - J_{eq} - M_p\, r^2)\, J_p - M_p\, l_p^{\,2}\, J_{eq}} \\
&- \frac{l_p\, M_p\, r \sin(\theta(t))\, J_{eq} \left(\frac{d}{dt}\theta(t)\right)^2}{(M_p\, r^2 \sin(\theta(t))^2 - J_{eq} - M_p\, r^2)\, J_p - M_p\, l_p^{\,2}\, J_{eq}} \\
&+ \frac{l_p\, M_p\, r\, \tau_{output} \cos(\theta(t))}{(M_p\, r^2 \sin(\theta(t))^2 - J_{eq} - M_p\, r^2)\, J_p - M_p\, l_p^{\,2}\, J_{eq}}
\end{aligned}
\tag{1}
$$

where the torque generated at the arm pivot by the motor voltage V_m is

$$
\tau_{output} = \frac{K_t \left(V_m - K_m \left(\frac{d}{dt}\theta(t)\right)\right)}{R_m}. \tag{2}
$$

The ROTPEN model parameters used in [1] and [2] are defined in Table 2.

Symbol	*Description*	*Value*	*Unit*
M_p	Mass of the pendulum assembly (weight and link combined).	0.027	kg
l_p	Length of pendulum center of mass from pivot.		m
L_p	Total length of pendulum.	0.191	m
r	Length of arm pivot to pendulum pivot.	0.08260	m

Symbol	Description	Value	Unit
J_m	Motor shaft moment of inertia.	3.00E-005	kg·m^2
M_{arm}	Mass of arm.	0.028	kg
g	Gravitational acceleration constant.	9.810	m/s^2
J_{eq}	Equivalent moment of inertia about motor shaft pivot axis.	1.23E-004	kg·m^2
J_p	Pendulum moment of inertia about its pivot axis.		kg·m^2
B_{eq}	Arm viscous damping.	0.000	N·m/(rad/s)
B_p	Pendulum viscous damping.	0.000	N·m/(rad/s)
R_m	Motor armature resistance.	3.30	Ω
K_t	Motor torque constant.	0.02797	N·m
K_m	Motor back-electromotive force constant.	0.02797	V/(rad/s)

Table 2 ROTPEN Model Nomenclature

The pendulum center of mass, l_p, is not given in Table 2 since it was calculated in the previous experiment, *ROTPEN Laboratory #3 – Gantry*. The moment of inertia parameter, J_p, is not given because it will be determined experimentally in this laboratory. However, the J_p that was calculated in *ROTPEN Laboratory #3 – Gantry* is still used in this experiment for comparison purposes. The viscous damping parameters of the pendulum, B_p, and of the arm, B_{eq}, are regarded as being negligible in this laboratory.

Similarly in *ROTPEN Laboratory #3*, the linear equations of motion of the system are found by linearizing the nonlinear equations of motions, or EOMs, presented in [1] about the operation point $\alpha = \pi$ and solving for the acceleration of the terms θ and α. For the state

$$x = [x_1, x_2, x_3, x_4]^T \qquad [3]$$

where

$$x_1 = \theta, \quad x_2 = \alpha, \quad x_3 = \frac{\partial}{\partial t}\theta, \text{ and } \quad x_4 = \frac{\partial}{\partial t}\alpha \qquad [4]$$

the linear state-space representation of the *ROTPEN Inverted Pendulum* is

$$\frac{d}{dt}\mathrm{x}(t) = A\,\mathrm{x}(t) + B\,\mathrm{u}(x)$$
$$\mathrm{y}(t) = C\,\mathrm{x}(t) + \mathrm{D}\,\mathrm{u}(x) \qquad [5]$$

where $u(x) = V_m$ and the A, B, C, and D matrices are

State-Space Matrix	Expression
A	$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{r M_p^2 l_p^2 g}{J_p J_{eq} + M_p l_p^2 J_{eq} + J_p M_p r^2} & -\dfrac{K_t K_m (J_p + M_p l_p^2)}{(J_p J_{eq} + M_p l_p^2 J_{eq} + J_p M_p r^2) R_m} & 0 \\ 0 & -\dfrac{M_p l_p g (J_{eq} + M_p r^2)}{J_p J_{eq} + M_p l_p^2 J_{eq} + J_p M_p r^2} & \dfrac{M_p l_p K_t r K_m}{(J_p J_{eq} + M_p l_p^2 J_{eq} + J_p M_p r^2) R_m} & 0 \end{bmatrix}$
B	$\begin{bmatrix} 0 \\ 0 \\ \dfrac{K_t (J_p + M_p l_p^2)}{(J_p J_{eq} + M_p l_p^2 J_{eq} + J_p M_p r^2) R_m} \\ -\dfrac{M_p l_p K_t r}{(J_p J_{eq} + M_p l_p^2 J_{eq} + J_p M_p r^2) R_m} \end{bmatrix}$
C	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
D	$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Table 3 Linear State-Space Matrices

4.1.2. LQR Control Design

The problem of balancing an inverted pendulum is like balancing a vertical stick with your hand by moving it back and forth. Thus by supplying the appropriate linear force, the stick can be kept more-or-less vertical. In this case, the pendulum is being balanced by applying torque to the arm. The balance controller supplies a motor voltage that applies a torque to the pendulum pivot and the amount of voltage supplied depends on the angular position and speed of both the arm and the pendulum.

Recall that the *linear quadratic regulator* problem is: given a plant model

$$\frac{d}{dt}\mathrm{x}(t) = A\,\mathrm{x}(t) + B\,\mathrm{u}(t) \tag{6}$$

find a control input *u* that minimizes the cost function

$$J = \int_0^\infty \mathrm{x}(t)^T Q\,\mathrm{x}(t) + \mathrm{u}(t)^T R\,\mathrm{u}(t)\,dt \tag{7}$$

where Q is an $n \times n$ positive semidefinite weighing matrix and R is an $r \times r$ positive definite symmetric matrix. That is, find a control gain *K* in the state feedback control law

$$u = K\,x \tag{8}$$

such that the quadratic cost function *J* is minimized.

The *Q* and *R* matrices set by the user affects the optimal control gain that is generated to minimize *J*. The closed-loop control performance is affected by changing the *Q* and *R* weighing matrices. Generally, the control input *u* will work harder and therefore a larger gain, *K*, will be generated if *Q* is made larger. Likewise, a larger gain will be computed by the LQR algorithm if the *R* weighing matrix is made smaller.

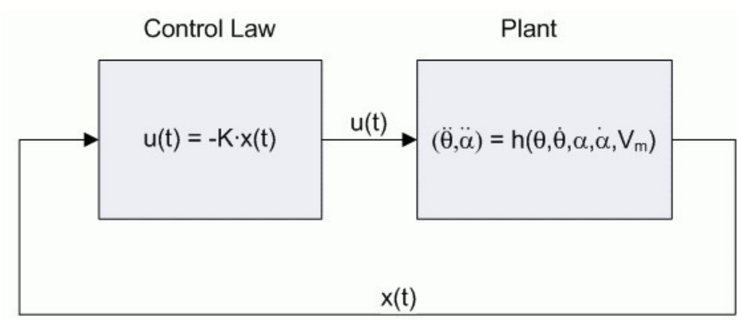

Figure 3 Closed-Loop Control System

The closed-loop system that balances the pendulum is shown in Figure 3. The controller computes a voltage V_m depending on the position and velocity of the arm and pendulum angles. The box labeled *Plant* shown in Figure 3 represents the nonlinear dynamics given in [1] and [2]. Similarly to the gantry experiment, the LQR gain *K* is automatically generated in the LabView Virtual Instrument by tuning the *Q* and *R* matrix.

4.1.3. Inverted Pendulum Control Specifications

Design an LQR control, that is tune the *Q* weighing matrix, such that the closed-loop response meets the following specifications:

(1) **Arm Regulation**: $|\theta(t)| < 30°$
(2) **Pendulum Regulation**: $|\alpha(t)| < 1.5°$
(3) **Control input limit**: $V_m < 10$ V

Thus the control should regulate the arm about zero degrees within 30° as it balances the pendulum without angle α going beyond 1.5°. The arm angle is re-defined to zero degrees, $\theta = 0°$, when the balance controller is activated. Additionally, the control input must be kept under the voltage range of the motor, 10 Volts.

4.2. Pre-Lab Assignment #1: Open-loop Modeling of the Pendulum

In Reference [3], the full model representing the two degrees-of-freedom motion of the gantry was developed using Lagrange. The following exercises deals instead with modeling only the pendulum shown in Figure 4 and assuming that the torque at the pendulum pivot, which is not directly actuated, is a control variable. Later, the dynamics between the input voltage of the DC motor and the torque applied to the pendulum pivot will be expressed. The Lagrange method will be used to find the *nonlinear* equations of motion of the pendulum. Thus the kinematics, potential energy, and kinetic energy are first calculated and the equations of motion are found using Euler-Lagrange.

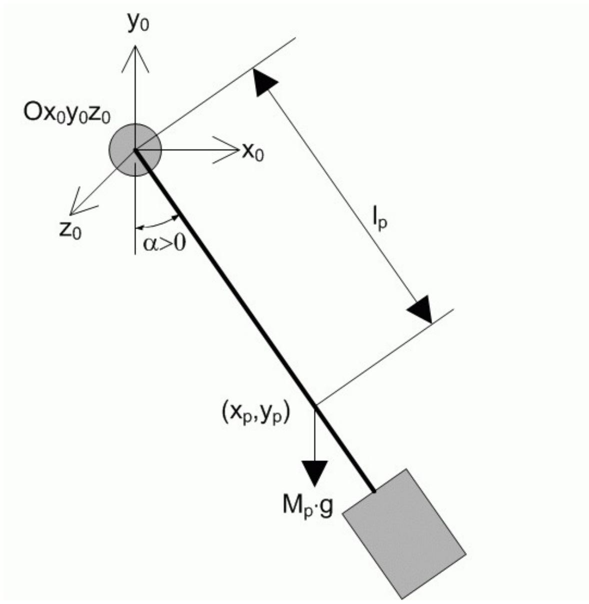

Figure 4 Free body diagram of pendulum considered a single rigid body.

4.2.1. Exercise: Kinematics

Figure 4 is the pendulum of the ROTPEN system when being considered as a single rigid object. It rotates about the axis z_0, at an angle α that is positive, by convention, when the pendulum moves in the counter-clockwise fashion. Further, $\alpha = 0$ when the pendulum is in the vertical downward position. Find the forward kinematics of the center of mass, or CM, of the pendulum with respect to the base frame $o_0x_0y_0z_0$, as shown in Figure 4 More specifically, express the position, x_p and y_p, of the CM and the velocity, xd_p and yd_p, of the pendulum CM in terms of the angle α.

4.2.2. Exercise: Potential Energy

Express the total potential energy, to be denoted as $U_T(\alpha)$, of the rotary pendulum system. The gravitational potential energy depends on the vertical position of the pendulum center-of-mass. The potential energy expression should be 0 Joules when the pendulum is at $\alpha = 0$, the downward position, and is positive when the $\alpha > 0$. It should reach its maximum value when the pendulum is upright and perfectly vertical.

4.2.3. Exercise: Kinetic Energy

Find the total kinetic energy, T_t, of the pendulum. In this case, the system being considered is a pendulum that rotates about a fixed pivot, therefore the entire kinetic energy is rotational kinetic energy.

4.2.4. Exercise: Lagrange of System

Calculate the *Lagrangian* of the pendulum

$$L\left(\alpha, \frac{d}{dt}\alpha(t)\right) = T_t - U_t \quad [9]$$

where T_t is the total kinetic energy calculated in Exercise 4.2.3, and U_t is the total potential energy of the system calculated in Exercise 4.2.2.

4.2.5. Exercise: Euler-Lagrange Equations of Motions

The Euler-Lagrange equations of motion are calculated from the Lagrangian of a system using

$$\left(\frac{\partial^2}{\partial t\, \partial qdot_i} L\right) - \left(\frac{\partial}{\partial q_i} L\right) = Q_i \quad [10]$$

where for an *n* degree-of-freedom, or *n* DOF, structure $i = \{1,..,n\}$, q_i is a generalized coordinate, and Q_i is a generalized force.

For the 1 DOF pendulum being considered, $q_1(t) = \alpha(t)$ and the generalized force is,

$$Q_1 = \tau_{pend} - B_p \left(\frac{d}{dt}\alpha(t)\right) \quad [11]$$

where τ_{pend} is the torque applied to the pendulum pivot. The generalized force, expression

[11] above, becomes $Q_1 = \tau_{pend}$ since the viscous damping of the pendulum, B_p, is regarded as being negligible.

Calculate the nonlinear equation of motion of the pendulum using [10] on the Lagrange calculated in Exercise 4.2.4. The answer should be in the form

$$\mathrm{f}\left(\alpha, \frac{\partial^2}{\partial t^2}\alpha\right) = \tau_{pend}, \qquad [12]$$

where the function *f* represents the differential equation in terms of the position and acceleration of the pendulum angle α. Do *not* express in terms of generalized coordinates.

4.3. Pre-Lab Assignment #2: Finding Inertia of the Pendulum Experimentally

The inertia of the pendulum about its pivot point was calculated analytically using integrals in the previous gantry experiment, *ROTPEN Laboratory #3*. In this laboratory, the inertia of the pendulum is found experimentally by measuring the frequency at which the pendulum freely oscillates. The nonlinear equation of motion derived in the previous exercise is used to find a formula that relates frequency and inertia. The nonlinear equation of motion must first be linearized about a point and then solved for angle α.

4.3.1. Exercise: Linearize Nonlinear EOMs of Pendulum

The inertia is found by measuring the frequency of the pendulum when it is allowed to swing freely, or without actuation. Thus the torque at the pivot is zero, $\tau_{pend} = 0$, and the nonlinear EOM found in [12] becomes

$$f\left(\alpha, \frac{\partial^2}{\partial t^2}\alpha\right) = 0 \qquad [13]$$

where f is the differential expression in [12] that represents the motions of the pendulum.

Linearize function [13] about the operating point $\alpha = 0°$, which is the angle the pendulum will be swinging about in order to measure its frequency.

4.3.2. Exercise: Differential Equation Solution

Solve the linear differential equation found in [13] for $\alpha(t)$ given that its initial conditions are

$$\alpha(0) = \alpha_0 \text{ and } \frac{d}{dt}\alpha(0) = 0 \qquad [14]$$

The solution should be in the form

$$\alpha(t) = \alpha_0 \cos(2\pi f t), \qquad [15]$$

where f is the frequency of the pendulum in *Hertz*.

4.3.3. Relating Pendulum Inertia and Frequency

Solving the frequency expression in [15] for the moment of inertia of the pendulum, J_p, should yield the equation

$$J_p = \frac{1}{4} \frac{M_p\, g\, l_p}{\pi^2 f^2}, \qquad [16]$$

where M_p is the mass of the pendulum assembly, l_p is the center of mass of the pendulum system, g is the gravitational acceleration, and f is the frequency of the pendulum. Expression [16] will be used in the in-lab session to find the pendulum moment of inertia in terms of the frequency measured when the pendulum is allowed to swing freely after a perturbation. The frequency can be measured using

$$f = \frac{n_{cyc}}{t_1 - t_0},\qquad [17]$$

where n_{cyc} is the number of cycles within the time duration t_1-t_0, t_0 is the time when the first cycle begins, and t_1 is the time of the last cycle.

4.4. Pre-Lab Assignment #3: Swing-Up Control Design

The controller using the Linear-Quadratic Regulator technique in Section 4.1 balances the pendulum in the upright vertical position after it is *manually* rotated within a certain range about its upright vertical angle. In this section, a controller is designed to automatically swing the pendulum in the upwards vertical position. Once the pendulum is within the range of the balance controller, it kicks-in and balances the pendulum. The closed-loop system that uses the swing-up controller and the balance controller is depicted in Figure 5.

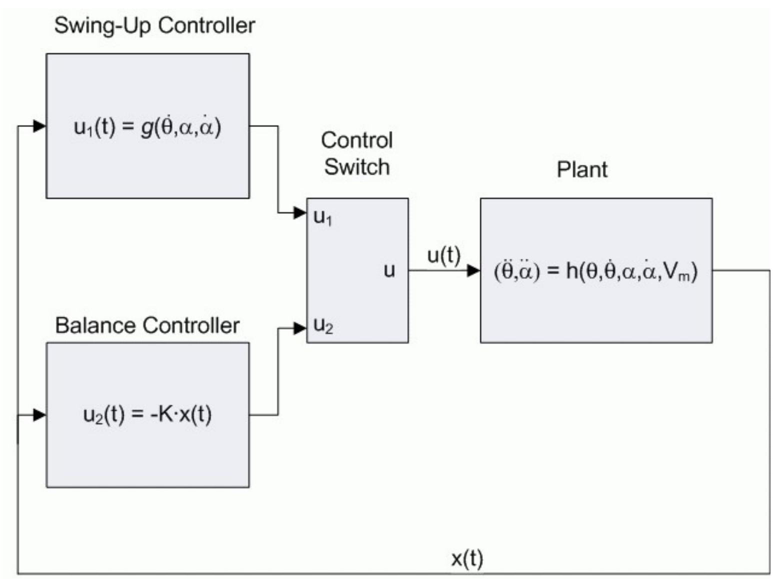

Figure 5 Swing-Up/Balance Closed-Loop System

The swing-up controller computes the torque that needs to be applied at the base of arm such that the pendulum can be rotated upwards. It is a nonlinear control that uses the pendulum energy to self-erect the pendulum. The swing-up controller will be designed using a Lyapunov function. Lyapunov functions are often used to study the stability properties of systems and can be used to design controllers.

4.4.1. Exercise: Re-defining System Dynamics

The controller that will be designed attempts to minimize an expression that is a function of the system's total energy. In order to rotate the pendulum into its upwards vertical position, the total energy of the pendulum and its dynamics must be redefined in terms of the angle

$$\alpha_{up} = \alpha - \pi \,, \qquad [18]$$

resulting in the system shown in Figure 6. Thus angle zero is defined to be when the pendulum is vertically upright. The translational acceleration of the pendulum pivot is denoted by the variable u and is m/s^2.

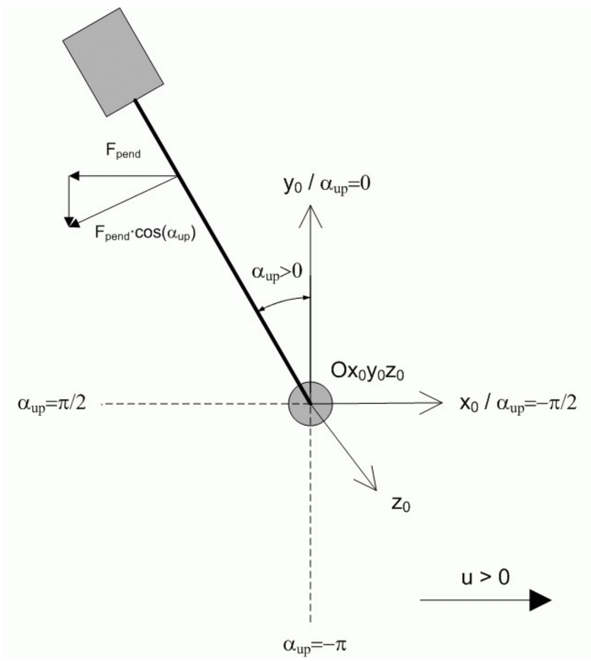

Figure 6 New Angle Definition

Re-define the nonlinear pendulum equations of motion found in Exercise 4.2.5 in terms of α_{up},

$$f\left(\alpha_{up}, \frac{\partial^2}{\partial t^2}\alpha_{up}\right) = \tau_{pend}(\alpha_{up}, u) \qquad [19]$$

and for the Lagrange calculated in Exercise 4.2.4, express energy E with respect to the upright angle

$$E(\alpha_{up}) = L(\alpha_{up} + \pi) \,. \qquad [20]$$

Given the pendulum is not moving, the pendulum energy should be zero when it is vertically upright, thus $E(0) = 0\ J$, and should be negative when in the vertically down position, more specifically

$$\mathrm{E}(-\pi) = -2\, M_p\, g\, l_p\ . \qquad [21]$$

4.4.2. Exercise: Actuator Dynamics

The swing-up controller that will be designed generates an acceleration at which the pendulum pivot should be moving at, denoted as u in Figure 6. The pendulum pivot acceleration however is *not* directly controllable, the input voltage of the DC motor voltage of the ROTPEN system is the input that is controlled by the computer. The dynamics between the acceleration of the pendulum pivot, u, and the input motor voltage, V_m, is required to supply the acceleration that is commanded by the swing-up control.

The dynamics between the torque applied at the arm by the motor, which is already given in [2], is

$$\tau_{output} = \frac{K_t\left(V_m - K_m\left(\frac{d}{dt}\theta(t)\right)\right)}{R_m} \qquad [22]$$

The torque applied to the arm moves the pendulum pivot, situated at the end of arm, at an acceleration u, thus

$$\tau_{ouput} = M_{arm}\, u\, r \qquad [23]$$

where M_{arm} is the mass of the arm and r is the length between the arm pivot and pendulum pivot. These parameters are both defined in Table 2. As shown above in Figure 6, the resulting torque applied on the pivot of the pendulum from acceleration u is

$$\tau_{pend}(\alpha_{up}, u) = l_p\, F_{pend} \cos(\alpha_{up}), \qquad [24]$$

where l_p is the length between the pendulum CM and its axis of rotation and

$$F_{pend} = -M_p\, u. \qquad [25]$$

As depicted in Figure 6, the force acting on the pendulum due to the pivot acceleration is defined as being negative in the x_0 direction for a positive u going along the x_0 axis.

The swing-up controller computes a desired acceleration, u, and a voltage must be given that can achieve that acceleration. Express the input DC motor voltage in terms of u using the above equations.

4.4.3. Exercise: Lyapunov Function

The goal of the self-erecting control is for $\alpha_{up}(t)$ to converge to zero, or $\alpha_{up}(t) \rightarrow 0$ in a finite time t. Instead of dealing with the angle directly, the controller will be designed to stabilize the energy of the pendulum using expression [20]. The idea is that if $E \rightarrow 0$ *J* then $\alpha_{up}(t) \rightarrow 0$. Thus the controller will be designed to regulate the energy such that $E \rightarrow 0$ *J*.

The swing-up control computes a pivot acceleration that is required to bring E down to zero and self-erect the pendulum. The control will be designed using the following Lyapunov function

$$\mathrm{V}(E) = \frac{1}{2}\,\mathrm{E}(\alpha_{up})^2 , \qquad [26]$$

where $E(\alpha_{up})$ was found in Exercise 4.4.1. By *Lyapunov's stability theorem*, the equilibrium point $E(\alpha_{up})=0$ is stable if the following properties hold

(1) $\mathrm{V}(0) = 0$

(2) $0 < \mathrm{V}(E)$ for all values of $E(\alpha_{up}) \neq 0$ [27]

(3) $\frac{\partial}{\partial t}\mathrm{V}(E) \leq 0$ for all values of $E(\alpha_{up})$.

The equilibrium point $E(\alpha_{up}) = 0$ is stable if the time derivative of function *V(E)* is negative or zero for all values of $E(\alpha_{up})$. The function *V(E)* approaches zero when its time derivative is negative (i.e. its a decreasing function) and that implies its variable, $E(\alpha_{up})$, converges to zero as well. According to the energy expression in [20], this means the upright angle converges to zero as well.

The derivative of *V(E)* is given by

$$\frac{\partial}{\partial t}\mathrm{V}(E) = \mathrm{E}(\alpha_{up})\left(\frac{\partial}{\partial t}\mathrm{E}(\alpha_{up})\right). \qquad [28]$$

Compute the Lyapunov derivative in [28]. First, calculate the time derivative of $E(\alpha_{up})$

$$\frac{\partial}{\partial t}\mathrm{E}(\alpha_{up}) , \qquad [29]$$

and make the corresponding substitutions using the re-defined dynamics in [19] that introduces the pivot acceleration control variable *u*. When expressing the Lyapunov derivative leave $E(\alpha_{up})$ as a variable in [28] and do *not* substitute the expression of $E(\alpha_{up})$ in [20].

4.4.4. Exercise: Swing-Up Control Design

The preceding calculations should yield a Lyapunov derivative in the form

$$\frac{\partial}{\partial t} \mathrm{V}(E) = -\mathrm{E}(\alpha_{up})\, \mathrm{g}(\alpha_{up})\, u \tag{30}$$

where $g(\alpha_{up})$ is real valued function that can be either negative or positive depending on the pendulum angle. The swing-up controller is an expression u that guarantees [30] will be negative or zero, $V\_dot(E) \leq 0$, for all values of $E(\alpha_{up})$.

For example, determine if $V\_dot(E) \leq 0$ for the simple proportional controller $u = \mu$ where $\mu \geq 0$ is a user-defined control gain. Substituting the control u inside [30] gives

$$\frac{\partial}{\partial t} \mathrm{V}(E) = -\mu\, \mathrm{E}(\alpha_{up})\, \mathrm{g}(\alpha_{up}) \tag{31}$$

The equilibrium point $E(\alpha_{up}) = 0$ is shown as being *unstable* using the control $u = \mu$ because $V\_dot(E)$ is *not* negative for all values of $E(\alpha_{up})$. Since either the function $g(\alpha_{up})$ or $E(\alpha_{up})$ can be negative, [31] can become positive which means $V(E)$ would not be a decreasing function and, as a result, $E(\alpha_{up})$ is not guaranteed to approach zero. In conclusion, this control design is not suitable for swinging up the pendulum because there is *no* guarantee the proper acceleration u will be generated such that α_{up} will converge towards zero.

Determine and explain if $E(\alpha_{up}) = 0$ is stable using the following controllers and the *V_dot*(*E*) calculated in Exercise 4.4.3

1) $u = \mu \cdot E(\alpha_{up})$, where $\mu \geq 0$ is a user-defined control gain.

2) $u = \mu \cdot E(\alpha_{up}) \cdot \cos(\alpha_{up})$

3) $u = \mu \cdot E(\alpha_{up}) \cdot (d\alpha_{up}(t)/dt) \cdot \cos(\alpha_{up})$

4) $u = \mu \cdot sgn(E(\alpha_{up}) \cdot (d\alpha_{up}(t)/dt) \cdot \cos(\alpha_{up}))$, where *sgn()* represents the *signum* function.

4.4.5. Controller Implementation

The controller that is implemented in LabView is

$$u = sat_{u_{max}}\left(\mu \operatorname{sgn}\left(\mathrm{E}(\alpha_{up}) \left(\frac{d}{dt} \alpha_{up}(t) \right) \cos(\alpha_{up}) \right) \right) \qquad [32]$$

where *sat()* is the *saturation* function and u_{max} represents the maximum acceleration of the pendulum pivot. The *signum* function makes for a control with the largest variance and overall tends to perform very well. However, the problem with using a *signum* function is the switching is high-frequency and can cause the voltage of the motor to chatter. In

LabView, a smooth approximation of the *signum* function to help prevent motor damage.

Given that the maximum motor input voltage is V_m = 10V and neglecting the motor back-electromotive force constant, K_m = 0, calculate the maximum acceleration of the pendulum pivot u_{max} using the equations supplied in Section 4.4.2.

The control gain, μ, is an acceleration and it basically changes the amount of torque the motor outputs. The maximum acceleration, u_{max}, is the maximum value that the control gain can be set.

5. In-Lab Session

5.1. System Hardware Configuration

This in-lab session is performed using the NI-ELVIS system equipped with a QNET-ROTPEN board and the Quanser Virtual Instrument (VI) controller file *QNET_ROTPEN_Lab_04_Inv_Pend_Control.vi*. Please refer to Reference [2] for the setup and wiring information required to carry out the present control laboratory. Reference [2] also provides the specifications and a description of the main components composing your system.