

Basic Kalman Filter Theory

Technical Note

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Authors: Mark Pedley and Michael Stanley

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Glossary

A_k	<p>The linear prediction or state matrix at sample k.</p> $x_k = A_k x_{k-1} + w_k$ $\hat{x}_k^- = A_k \hat{x}_{k-1}^+$
C_k	<p>The measurement matrix relating x_k to z_k at sample k.</p> $z_k = C_k x_k + v_k$
$E[]$	Expectation operator
K_k	The Kalman filter gain at sample k
P_k^-	<p>The <i>a priori</i> covariance matrix of the linear prediction (<i>a priori</i>) error $\hat{x}_{\varepsilon,k}^-$ at sample k.</p> $P_k^- = E \left[\hat{x}_{\varepsilon,k}^- \hat{x}_{\varepsilon,k}^{-T} \right]$
P_k^+	<p>The <i>a posteriori</i> covariance matrix of the Kalman (<i>a posteriori</i>) error $\hat{x}_{\varepsilon,k}^+$ at sample k.</p> $P_k^+ = E \left[\hat{x}_{\varepsilon,k}^+ \hat{x}_{\varepsilon,k}^{+T} \right]$
$Q_{w,k}$	<p>The covariance matrix of the additive noise w_k on the process x_k</p> $Q_{w,k} = E \left[w_k w_k^T \right]$
$Q_{v,k}$	<p>The covariance matrix of the additive noise v_k on the measured process z_k</p> $Q_{v,k} = E \left[v_k v_k^T \right]$
$V[]$	Variance operator
v_k	The additive noise on the measured process z_k at sample k
w_k	The additive noise on the process of interest x_k at sample k
x_k	<p>The state vector at time sample k of the process of interest x_k</p> $x_k = A_k x_{k-1} + w_k$
\hat{x}_k^-	<p>The linear prediction (<i>a priori</i>) estimate of the process x_k at sample k.</p> $\hat{x}_k^- = A_k \hat{x}_{k-1}^+$
\hat{x}_k^+	<p>The Kalman filter (<i>a posteriori</i>) estimate of the process x_k at sample k.</p> $\hat{x}_k^+ = (I - K_k C_k) \hat{x}_k^- + K_k z_k = (I - K_k C_k) A_k \hat{x}_{k-1}^+ + K_k z_k$

$\hat{x}_{\varepsilon,k}^-$

The error in the linear prediction (a priori) estimate of the process x_k .

$$\hat{x}_{\varepsilon,k}^- = \hat{x}_k^- - x_k$$

 $\hat{x}_{\varepsilon,k}^+$

The error in the *a posteriori* Kalman filter estimate of the process x_k .

$$\hat{x}_{\varepsilon,k}^+ = \hat{x}_k^+ - x_k$$

 z_k

The measured process at sample k .

$$z_k = C_k x_k + v_k$$

 $\delta_{k,j}$

The Kronecker delta function. $\delta_{k,j} = 1$ for $k = j$ and zero otherwise.

1 Introduction

This document describes the assumptions underlying the basic Kalman filter and derives the standard Kalman equations. It is intended as a primer that should be read before tackling the documentation for the more specialized indirect complementary Kalman filter used for the fusion of accelerometer, magnetometer and gyroscope data.

Section 2 derives some mathematical results used in the derivation. The derivation itself is in section 3.

2 Mathematical Lemmas

2.1 Lemma 1

The trace of the sum of two matrices equals the sum of the individual traces.

Proof

$$tr(A + B) = \sum_{i=0}^{N-1} A_{ii} + B_{ii} = \sum_{i=0}^{N-1} A_{ii} + \sum_{i=0}^{N-1} B_{ii} = tr(A) + tr(B)$$

Eq 2.1.1

2.2 Lemma 2

The derivative with respect to A of the trace of the matrix product $C = AB$ equals B^T .

Proof

$$\frac{\partial\{tr(C)\}}{\partial A} = \frac{\partial\{tr(AB)\}}{\partial A} = \left(\left(\frac{\partial tr(AB)}{\partial A_{0,0}} \right) \left(\frac{\partial tr(AB)}{\partial A_{0,1}} \right) \dots \left(\frac{\partial tr(AB)}{\partial A_{0,N-1}} \right) \left(\frac{\partial tr(AB)}{\partial A_{1,0}} \right) \left(\frac{\partial tr(AB)}{\partial A_{1,1}} \right) \dots \left(\frac{\partial tr(AB)}{\partial A_{1,N-1}} \right) \dots \dots \dots \left(\frac{\partial tr(AB)}{\partial A_{M-1,0}} \right) \left(\frac{\partial tr(AB)}{\partial A_{M-1,1}} \right) \dots \right)$$

Eq 2.2.1

Assuming that the matrix A has dimensions $M \times N$ and the matrix B has dimensions $N \times M$, then $C = AB$ has dimensions $M \times M$.

The element C_{ij} of matrix C has value:

$$C_{ij} = \sum_{k=0}^{N-1} A_{ik} B_{kj} \Rightarrow tr(C) = tr(AB) = \sum_{i=0}^{M-1} C_{ii} = \sum_{i=0}^{M-1} \sum_{k=0}^{N-1} A_{ik} B_{ki}$$

Eq 2.2.2

Substituting gives:

$$\frac{\partial\{tr(AB)\}}{\partial A} = \left(\left(\frac{\partial \sum_{i=0}^{M-1} \sum_{k=0}^{N-1} A_{ik} B_{ki}}{\partial A_{0,0}} \right) \left(\frac{\partial \sum_{i=0}^{M-1} \sum_{k=0}^{N-1} A_{ik} B_{ki}}{\partial A_{0,1}} \right) \dots \left(\frac{\partial \sum_{i=0}^{M-1} \sum_{k=0}^{N-1} A_{ik} B_{ki}}{\partial A_{0,N-1}} \right) \left(\frac{\partial \sum_{i=0}^{M-1} \sum_{k=0}^{N-1} A_{ik} B_{ki}}{\partial A_{1,0}} \right) \left(\frac{\partial \sum_{i=0}^{M-1} \sum_{k=0}^{N-1} A_{ik} B_{ki}}{\partial A_{1,1}} \right) \dots \left(\frac{\partial \sum_{i=0}^{M-1} \sum_{k=0}^{N-1} A_{ik} B_{ki}}{\partial A_{1,N-1}} \right) \dots \dots \dots \right)$$

Eq 2.2.3

By inspection:

$$\left(\frac{\partial \sum_{i=0}^{M-1} \sum_{k=0}^{N-1} A_{ik} B_{ki}}{\partial A_{lm}} \right) = B_{ml}$$

Eq 2.2.4

Substituting back gives:

$$\frac{\partial\{tr(AB)\}}{\partial A} = \left(B_{0,0} \ B_{0,1} \ \dots \ B_{0,N-1} \ B_{1,0} \ B_{1,1} \ \dots \ B_{1,N-1} \ \dots \ \dots \ \dots \ B_{M-1,0} \ B_{M-1,1} \ \dots \ B_{M-1,N-1} \right) = B^T$$

Eq 2.2.5

2.3 Lemma 3

The derivative with respect to A of the trace of the matrix product ABA^T equals $A(B + B^T)$.

Proof

$$\frac{\partial \{tr(ABA^T)\}}{\partial A} = \left(\left(\frac{\partial tr(ABA^T)}{\partial A_{0,0}} \right) \left(\frac{\partial tr(ABA^T)}{\partial A_{0,1}} \right) \cdots \left(\frac{\partial tr(ABA^T)}{\partial A_{0,N-1}} \right) \left(\frac{\partial tr(ABA^T)}{\partial A_{1,0}} \right) \left(\frac{\partial tr(ABA^T)}{\partial A_{1,1}} \right) \cdots \left(\frac{\partial tr(ABA^T)}{\partial A_{1,N-1}} \right) \cdots \cdots \cdots \left(\frac{\partial tr(ABA^T)}{\partial A_{M-1,0}} \right) \left(\frac{\partial tr(ABA^T)}{\partial A_{M-1,1}} \right) \cdots \left(\frac{\partial tr(ABA^T)}{\partial A_{M-1,N-1}} \right) \right)$$

Eq 2.3.1

If the matrix A has dimensions $M \times N$ then the matrix B must be square with dimensions $N \times N$ for the product ABA^T to exist. The product ABA^T is always square with dimensions $M \times M$.

The element C_{ij} of the matrix $C = AB$ has value:

$$C_{ij} = \sum_{k=0}^{N-1} A_{ik} B_{kj}$$

Eq 2.3.2

The element D_{il} of matrix $D = ABA^T = CA^T$ has value:

$$D_{il} = \sum_{j=0}^{N-1} C_{ij} A_{lj} = \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} A_{ik} B_{kj} A_{lj}$$

Eq 2.3.3

The trace of matrix D has value:

$$tr(D) = \sum_{i=0}^{N-1} D_{ii} = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} A_{ik} B_{kj} A_{ij}$$

Eq 2.3.4

The derivative of $tr(D)$ with respect to A_{lm} is then:

$$\left(\frac{\partial tr(D)}{\partial A_{lm}} \right) = \left(\frac{\partial \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} A_{ik} B_{kj} A_{ij}}{\partial A_{lm}} \right) = \left(\frac{\partial \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} A_{lk} B_{kj} A_{lj}}{\partial A_{lm}} \right)$$

Eq 2.3.5

$$= \sum_{j=0}^{N-1} A_{lj} B_{mj} + \sum_{j=0}^{N-1} A_{lj} B_{jm} = (AB^T)_{lm} + (AB)_{lm}$$

Eq 2.3.6

$$\Rightarrow \frac{\partial \{tr(ABA^T)\}}{\partial A} = A(B + B^T)$$

Eq 2.3.7

If B is also symmetric then:

$$\frac{\partial \{tr(ABA^T)\}}{\partial A} = 2AB \text{ if } B = B^T$$

Eq 2.3.8

3 Kalman Filter Derivation

3.1 Process Model

The Kalman filter models the vector process of interest x_k as linear and recursive:

$$x_k = A_k x_{k-1} + w_k \quad \text{Eq 3.1.1}$$

If x_k has N degrees of freedom then A_k is an $N \times N$ linear prediction matrix (possibly time varying but assumed known) and w_k is an $N \times 1$ noise vector.

The process x_k is assumed to be not directly measurable and must be estimated from a process z_k which can be measured. z_k is modeled as being linearly related to x_k with additive noise v_k .

$$z_k = C_k x_k + v_k \quad \text{Eq 3.1.2}$$

z_k is an $N \times 1$ vector, C_k is an $N \times N$ matrix (possibly time varying but assumed known) and v_k is an $N \times 1$ noise vector.

The noise vectors w_k and v_k are assumed to be zero mean white processes:

$$E[w_k] = 0 \quad \text{Eq 3.1.3}$$

$$E[v_k] = 0 \quad \text{Eq 3.1.4}$$

$$\text{cov}\{w_k, w_j\} = E[w_k w_j^T] = Q_{w,k} \delta_{kj} \quad \text{Eq 3.1.5}$$

$$\text{cov}\{v_k, v_j\} = E[v_k v_j^T] = Q_{v,k} \delta_{kj} \quad \text{Eq 3.1.6}$$

By definition, covariance matrices are symmetric.

$$Q_{w,k}^T = \{E[w_k w_k^T]\}^T = E[(w_k w_k^T)^T] = E[w_k w_k^T] = Q_{w,k} \quad \text{Eq 3.1.7}$$

3.2 Derivation

The objective of the Kalman filter is to compute an unbiased *a posteriori* estimate

\hat{x}_k^+ of the underlying process x_k from i) extrapolation from the previous iteration's *a posteriori* estimate \hat{x}_{k-1}^+ and ii) from the current measurement z_k :

$$\hat{x}_k^+ = K_k' \hat{x}_{k-1}^+ + K_k z_k \quad \text{Eq 3.2.1}$$

The time-varying Kalman gain matrices K_k' and K_k define the relative weightings given to the previous

iteration's Kalman filter estimate \hat{x}_{k-1}^+ and to the current measurement z_k . If the measurements z_k have low noise then a higher weighting will be given to the term $K_k z_k$ compared to the extrapolated component $K_k' \hat{x}_{k-1}^+$ and vice versa. The Kalman filter is therefore a time varying recursive filter.

Unbiased estimate constraint (determines K_k')

For \hat{x}_k to be an unbiased estimate of x_k , the expectation value of the *a posteriori* Kalman filter error $x_{\varepsilon,k}^+$ must be zero:

$$E[x_{\varepsilon,k}^+] = E[\hat{x}_k^+ - x_k] = 0 \quad \text{Eq 3.2.2}$$

Subtracting x_k from equation 3.2.1 gives:

$$x_{\varepsilon,k}^+ = \hat{x}_k^+ - x_k = K_k' \hat{x}_{k-1}^+ + K_k z_k - x_k \quad \text{Eq 3.2.3}$$

Substituting equation 3.1.2 for z_k gives:

$$x_{\varepsilon,k}^+ = K_k' \hat{x}_{k-1}^+ + K_k (C_k x_k + v_k) - x_k \quad \text{Eq 3.2.4}$$

Substituting for x_k from equation 3.1.1 and re-arranging gives:

$$x_{\varepsilon,k}^+ = K_k' (\hat{x}_{\varepsilon,k-1}^+ + x_{k-1}) + K_k \{C_k (A_k x_{k-1} + w_k) + v_k\} - (A_k x_{k-1} + w_k) \quad \text{Eq 3.2.5}$$

$$= K_k' \hat{x}_{\varepsilon,k-1}^+ + (K_k C_k A_k - A_k + K_k') x_{k-1} + (K_k C_k - I) w_k + K_k v_k \quad \text{Eq 3.2.6}$$

Taking the expectation value of equation 3.2.6 and applying the unbiased estimate constraint gives:

$$E[x_{\varepsilon,k}^+] = E[K_k' \hat{x}_{\varepsilon,k-1}^+] + E[(K_k C_k A_k - A_k + K_k') x_{k-1}] + E[(K_k C_k - I) w_k] + E[K_k v_k] = 0 \quad \text{Eq 3.2.7}$$

Since the noise vectors w_k and v_k are zero mean and uncorrelated with the Kalman matrices for the same iteration, it follows that:

$$E[(K_k C_k - I) w_k] = E[K_k v_k] = 0 \quad \text{Eq 3.2.8}$$

With the additional assumption that the process x_{k-1} is independent of the Kalman matrices at iteration k :

$$E[(K_k C_k A_k - A_k + K_k') x_{k-1}] = (K_k C_k A_k - A_k + K_k') E[x_{k-1}] = 0 \quad \text{Eq 3.2.9}$$

Since x_k is not, in general, a zero mean process:

$$K_k C_k A_k - A_k + K'_k = 0 \Rightarrow K'_k = A_k - K_k C_k A_k = (I - K_k C_k) A_k \quad \text{Eq 3.2.10}$$

Eliminating K'_k in equation 3.2.1 gives:

$$\hat{x}_k^+ = (I - K_k C_k) \hat{x}_{k-1}^+ + K_k z_k \quad \text{Eq 3.2.11}$$

A priori estimate

The *a priori* Kalman filter estimate \hat{x}_k^- is defined as resulting from the application of the linear prediction matrix A_k to the previous iteration's *a posteriori* estimate \hat{x}_{k-1}^+ :

$$\hat{x}_k^- = A_k \hat{x}_{k-1}^+ \quad \text{Kalman equation 1} \quad \text{Eq 3.2.12}$$

Definition of a posteriori estimate

Substituting the *a priori* estimate \hat{x}_k^- into equation 3.2.11 gives:

$$\hat{x}_k^+ = (I - K_k C_k) \hat{x}_k^- + K_k z_k \quad \text{Kalman equation 4} \quad \text{Eq 3.2.13}$$

An equivalent form is:

$$\hat{x}_k^+ = \hat{x}_k^- + K_k (z_k - C_k \hat{x}_k^-) \quad \text{Eq 3.2.14}$$

P_k^- **as a function of** P_{k-1}^+

The *a priori* and *a posteriori* error covariance matrices P_k^- and P_k^+ are defined as:

$$P_k^- = \text{cov}\{\hat{x}_{\varepsilon,k}^-, \hat{x}_{\varepsilon,k}^-\} = E \left[\hat{x}_{\varepsilon,k}^- \hat{x}_{\varepsilon,k}^{-T} \right] = E \left[\left(\hat{x}_k^- - x_k \right) \left(\hat{x}_k^- - x_k \right)^T \right] \quad \text{Eq 3.2.15}$$

$$P_k^+ = \text{cov}\{\hat{x}_{\varepsilon,k}^+, \hat{x}_{\varepsilon,k}^+\} = E \left[\hat{x}_{\varepsilon,k}^+ \hat{x}_{\varepsilon,k}^{+T} \right] = E \left[\left(\hat{x}_k^+ - x_k \right) \left(\hat{x}_k^+ - x_k \right)^T \right] \quad \text{Eq 3.2.16}$$

Substituting the definitions of \hat{x}_k^- and x_k into equation 3.2.15 gives:

$$P_k^- = E \left[\left(A_k \hat{x}_{k-1}^+ - A_k x_{k-1} - w_k \right) \left(A_k \hat{x}_{k-1}^+ - A_k x_{k-1} - w_k \right)^T \right] \quad \text{Eq 3.2.17}$$

$$= E \left[\left\{ A_k \left(\hat{x}_{k-1}^+ - x_{k-1} \right) - w_k \right\} \left\{ A_k \left(\hat{x}_{k-1}^+ - x_{k-1} \right) - w_k \right\}^T \right] \quad \text{Eq 3.2.18}$$

$$= A_k E \left[\left(\hat{x}_{k-1}^+ - x_{k-1} \right) \left(\hat{x}_{k-1}^+ - x_{k-1} \right)^T \right] A_k^T + Q_{w,k} \quad \text{Eq 3.2.19}$$

$$\Rightarrow P_k^- = A_k P_{k-1}^+ A_k^T + Q_{w,k} \quad \text{Kalman equation 2} \quad \text{Eq 3.2.20}$$

Minimum error covariance constraint (determines K_k)

The Kalman gain matrix K_k minimizes the *a posteriori* error $\hat{x}_{\varepsilon,k}^+$ variance via the trace of the *a posteriori* error covariance matrix P_k^+ :

$$E \left[\begin{matrix} \hat{x}_{\varepsilon,k}^+ \\ \hat{x}_{\varepsilon,k}^+ \end{matrix} \right] = \text{tr}(P_k^+) \quad \text{Eq 3.2.21}$$

Substituting equation 2.1.2 for z_k into equation 3.2.11 gives a relation between the *a posteriori* and *a priori* errors:

$$\hat{x}_k^+ = \hat{x}_{\varepsilon,k}^+ + x_k = (I - K_k C_k) \hat{x}_k^- + K_k z_k = (I - K_k C_k) (\hat{x}_{\varepsilon,k}^- + x_k) + K_k (C_k x_k + v_k) \quad \text{Eq 3.2.22}$$

$$\Rightarrow \hat{x}_{\varepsilon,k}^+ + x_k = (I - K_k C_k) \hat{x}_{\varepsilon,k}^- + x_k - K_k C_k x_k + K_k (C_k x_k + v_k) \quad \text{Eq 3.2.23}$$

$$\Rightarrow \hat{x}_{\varepsilon,k}^+ = (I - K_k C_k) \hat{x}_{\varepsilon,k}^- + K_k v_k \quad \text{Eq 3.2.24}$$

Substituting this result into the definition of the *a posteriori* covariance matrix P_k^+ gives:

$$P_k^+ = E \left[\left\{ (I - K_k C_k) \hat{x}_{\varepsilon,k}^- + K_k v_k \right\} \left\{ (I - K_k C_k) \hat{x}_{\varepsilon,k}^- + K_k v_k \right\}^T \right] \quad \text{Eq 3.2.25}$$

$$= (I - K_k C_k) E \left[\begin{matrix} \hat{x}_{\varepsilon,k}^- \\ \hat{x}_{\varepsilon,k}^- \end{matrix} \right] (I - K_k C_k)^T + K_k E \left[v_k v_k^T \right] K_k^T \quad \text{Eq 3.2.26}$$

$$= (I - K_k C_k) P_k^- (I - K_k C_k)^T + K_k Q_{v,k} K_k^T \quad \text{Eq 3.2.27}$$

$$= P_k^- - P_k^- C_k^T K_k^T - K_k C_k P_k^- + K_k C_k P_k^- C_k^T K_k^T + K_k Q_{v,k} K_k^T \quad \text{Eq 3.2.28}$$

The Kalman filter gain K_k is that which minimizes the trace of the *a posteriori* error covariance matrix P_k^+ :

$$\frac{\partial}{\partial K_k} \text{tr}(P_k^+) = \frac{\partial}{\partial K_k} \left\{ \text{tr}(P_k^-) - \text{tr}(P_k^- C_k^T K_k^T) - \text{tr}(K_k C_k P_k^-) + \text{tr}(K_k C_k P_k^- C_k^T K_k^T) + \text{tr}(K_k Q_{v,k} K_k^T) \right\} = 0 \quad \text{Eq 3.2.29}$$

The term $\text{tr}(P_k^-)$ has no dependence on K_k giving:

$$\frac{\partial \{ \text{tr}(P_k^-) \}}{\partial K_k} = \frac{\partial \{ \text{tr}(A_k P_{k-1}^+ A_k^T + Q_{w,k}) \}}{\partial K_k} = 0 \quad \text{Eq 3.2.30}$$

Since the trace of a transposed matrix equals the trace of the original matrix and using equation 2.2.5 gives:

$$\frac{\partial \{ \text{tr}(P_k^- C_k^T K_k^T) \}}{\partial K_k} = \frac{\partial \{ \text{tr}(K_k C_k P_k^-) \}}{\partial K_k} = (C_k P_k^-)^T = P_k^- C_k^T \quad \text{Eq 3.2.31}$$

The third term can be simplified using equations 2.3.7 and 2.3.8 exploiting the fact that the covariance matrix is symmetric:

$$\frac{\partial \{ \text{tr}(K_k C_k P_k^- C_k^T K_k^T) \}}{\partial K_k} = K_k \left\{ C_k P_k^- C_k^T + (C_k P_k^- C_k^T)^T \right\} = 2 K_k C_k P_k^- C_k^T \quad \text{Eq 3.2.32}$$

The final term can be simplified also using equations 2.3.7 and 2.3.8 to give:

$$\frac{\partial \{ \text{tr}(K_k Q_{v,k} K_k^T) \}}{\partial K_k} = 2 K_k Q_{v,k} \quad \text{Eq 3.2.33}$$

Substituting back into equation 2.2.29 gives the optimal Kalman filter gain matrix K_k :

$$- 2 P_k^- C_k^T + 2 K_k C_k P_k^- C_k^T + 2 K_k Q_{v,k} = 0 \quad \text{Eq 3.2.34}$$

$$\Rightarrow K_k (C_k P_k^- C_k^T + Q_{v,k}) = P_k^- C_k^T \quad \text{Eq 3.2.35}$$

$$\Rightarrow K_k = P_k^- C_k^T (C_k P_k^- C_k^T + Q_{v,k})^{-1} \quad \text{Kalman equation 3} \quad \text{Eq 3.2.36}$$

P_k^+ **as a function of** P_k^-

Rearranging equation 3.2.35 gives:

$$K_k Q_{v,k} = P_k^- C_k^T - K_k C_k P_k^- C_k^T \quad \text{Eq 3.2.37}$$

Substituting equation 3.2.37 into equation 3.2.27 gives:

$$P_k^+ = (I - K_k C_k) P_k^- (I - C_k^T K_k^T) + (I - K_k C_k) P_k^- C_k^T K_k^T \quad \text{Eq 3.2.38}$$

$$\Rightarrow P_k^+ = (I - K_k C_k) P_k^- \quad \text{Kalman equation 5} \quad \text{Eq 3.2.39}$$

This completes the derivation of the standard Kalman filter equations.

3.3 Standard Kalman Equations

Kalman equation 1

The linear prediction (*a priori*) estimate \hat{x}_k^- is made by applying the linear prediction matrix A_k to the previous sample's Kalman (*a posteriori*) filter estimate \hat{x}_{k-1}^+ .

$$\hat{x}_k^- = A_k \hat{x}_{k-1}^+ \quad \text{Eq 3.3.1}$$

Kalman equation 2

The *a priori* (linear extrapolation) error covariance matrix P_k^- is then updated using the model matrix A_k and the noise matrix $Q_{w,k}$.

$$P_k^- = A_k P_{k-1}^+ A_k^T + Q_{w,k} \quad \text{Eq 3.3.2}$$

Kalman equations 2 and 5 can be combined to give a recursive update of P_k^- without explicit calculation of the *a posteriori* error covariance matrix P_k^+ in Kalman equation 5:

$$P_k^- = A_k (I - K_{k-1} C_{k-1}) P_{k-1}^- A_k^T + Q_{w,k} \quad \text{Eq 3.3.3}$$

Kalman equation 3

The Kalman filter gain matrix K_k is updated:

$$K_k = P_k^- C_k^T (C_k P_k^- C_k^T + Q_{v,k})^{-1} \quad \text{Eq 3.3.4}$$

Kalman equation 4

The Kalman filter (*a posteriori*) estimate \hat{x}_k^+ is computed from the current *a priori* estimate \hat{x}_k^- and the current measurement z_k :

$$\hat{x}_k^+ = \hat{x}_k^- + K_k (z_k - C_k \hat{x}_k^-) = (I - K_k C_k) \hat{x}_k^- + K_k z_k \quad \text{Eq 3.3.5}$$

Kalman equation 5

The *a posteriori* Kalman error covariance matrix P_k^+ is updated ready for the next iteration. This equation can be skipped if P_k^- is updated recursively in terms of itself as in equation 3.3.3.

$$P_k^+ = (I - K_k C_k) P_k^- \quad \text{Eq 3.3.6}$$

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