Basic Kalman Filter Theory

Technical Note

Document TBD
Version: Draft
Authors: Mark Pedley and Michael Stanley
Date: September 2014

Table of Contents

Glossary

A_k The linear prediction or state matrix at sample k.

$$x_k = A_k x_{k-1} + w_k$$

$$\hat{x}^{-}_k = A_k \hat{x}^{+}_{k-1}$$

C_k The measurement matrix relating x_k to z_k at sample k.

$$z_k = C_k x_k + v_k$$

$E[]$ Expectation operator

K_k The Kalman filter gain at sample k

P^{-}_k The *a priori* covariance matrix of the linear prediction (*a priori*) error $\hat{x}^{-}_{\varepsilon,k}$ at sample k.

$$P^{-}_k = E\left[\hat{x}^{-}_{\varepsilon,k} \hat{x}^{-\,T}_{\varepsilon,k}\right]$$

P^{+}_k The *a posteriori* covariance matrix of the Kalman (*a posteriori*) error $\hat{x}^{+}_{\varepsilon,k}$ at sample k.

$$P^{+}_k = E\left[\hat{x}^{+}_{\varepsilon,k} \hat{x}^{+\,T}_{\varepsilon,k}\right]$$

$Q_{w,k}$ The covariance matrix of the additive noise w_k on the process x_k

$$Q_{w,k} = E\left[w_k w_k^{T}\right]$$

$Q_{v,k}$ The covariance matrix of the additive noise v_k on the measured process z_k

$$Q_{v,k} = E\left[v_k v_k^{T}\right]$$

$V[]$ Variance operator

v_k The additive noise on the measured process z_k at sample k

w_k The additive noise on the process of interest x_k at sample k

x_k The state vector at time sample k of the process of interest x_k

$$x_k = A_k x_{k-1} + w_k$$

$\hat{x}^{-}_k$ The linear prediction (*a priori*) estimate of the process x_k at sample k.

$$\hat{x}^{-}_k = A_k \hat{x}^{+}_{k-1}$$

$\hat{x}^{+}_k$ The Kalman filter (*a posteriori*) estimate of the process x_k at sample k.

$$\hat{x}^{+}_k = \left(I - K_k C_k\right)\hat{x}^{-}_k + K_k z_k = \left(I - K_k C_k\right)A_k \hat{x}^{+}_{k-1} + K_k z_k$$

$\hat{x}^{-}_{\varepsilon,k}$	The error in the linear prediction (a priori) estimate of the process x_k. $\hat{x}^{-}_{\varepsilon,k} = \hat{x}^{-}_{k} - x_k$
$\hat{x}^{+}_{\varepsilon,k}$	The error in the *a posteriori* Kalman filter estimate of the process x_k. $\hat{x}^{+}_{\varepsilon,k} = \hat{x}^{+}_{k} - x_k$
z_k	The measured process at sample k. $z_k = C_k x_k + v_k$
$\delta_{k,j}$	The Kronecker delta function. $\delta_{k,j} = 1$ for $k = j$ and zero otherwise.

1 Introduction

This document describes the assumptions underlying the basic Kalman filter and derives the standard Kalman equations. It is intended as a primer that should be read before tackling the documentation for the more specialized indirect complementary Kalman filter used for the fusion of accelerometer, magnetometer and gyroscope data.

Section 2 derives some mathematical results used in the derivation. The derivation itself is in section 3.

2 Mathematical Lemmas

2.1 Lemma 1

The trace of the sum of two matrices equals the sum of the individual traces.

Proof

$$tr(A + B) = \sum_{i=0}^{N-1} A_{ii} + B_{ii} = \sum_{i=0}^{N-1} A_{ii} + \sum_{i=0}^{N-1} B_{ii} = tr(A) + tr(B)$$

Eq 2.1.1

2.2 Lemma 2

The derivative with respect to A of the trace of the matrix product $C = AB$ equals B^T.

Proof

$$\frac{\partial\{tr(C)\}}{\partial A} = \frac{\partial\{tr(AB)\}}{\partial A} = \left(\left(\frac{\partial tr(AB)}{\partial A_{0,0}}\right)\left(\frac{\partial tr(AB)}{\partial A_{0,1}}\right) \dots \left(\frac{\partial tr(AB)}{\partial A_{0,N-1}}\right)\left(\frac{\partial tr(AB)}{\partial A_{1,0}}\right)\left(\frac{\partial tr(AB)}{\partial A_{1,1}}\right) \dots \left(\frac{\partial tr(AB)}{\partial A_{1,N-1}}\right) \dots \dots \dots \dots \left(\frac{\partial tr(AB)}{\partial A_{M-1,0}}\right)\left(\frac{\partial tr(AB)}{\partial A_{M-1,1}}\right)\right.$$

Eq 2.2.1

Assuming that the matrix A has dimensions MxN and the matrix B has dimensions NxM, then $C = AB$ has dimensions MxM.

The element C_{ij} of matrix C has value:

$$C_{ij} = \sum_{k=0}^{N-1} A_{ik}B_{kj} \Rightarrow tr(C) = tr(AB) = \sum_{i=0}^{M-1} C_{ii} = \sum_{i=0}^{M-1}\sum_{k=0}^{N-1} A_{ik}B_{ki}$$

Eq 2.2.2

Substituting gives:

$$\frac{\partial\{tr(AB)\}}{\partial A} = \left(\left(\frac{\partial \sum_{i=0}^{M-1}\sum_{k=0}^{N-1} A_{ik}B_{ki}}{\partial A_{0,0}}\right)\left(\frac{\partial \sum_{i=0}^{M-1}\sum_{k=0}^{N-1} A_{ik}B_{ki}}{\partial A_{0,1}}\right) \dots \left(\frac{\partial \sum_{i=0}^{M-1}\sum_{k=0}^{N-1} A_{ik}B_{ki}}{\partial A_{0,N-1}}\right)\left(\frac{\partial \sum_{i=0}^{M-1}\sum_{k=0}^{N-1} A_{ik}B_{ki}}{\partial A_{1,0}}\right)\left(\frac{\partial \sum_{i=0}^{M-1}\sum_{k=0}^{N-1} A_{ik}B_{ki}}{\partial A_{1,1}}\right) \dots \left(\frac{\partial \sum_{i=0}^{M-1}\sum_{k=0}^{N-1} A_{ik}B_{ki}}{\partial A_{1,N-1}}\right) \dots \dots \dots\right.$$

Eq 2.2.3

By inspection:

$$\left(\frac{\partial \sum_{i=0}^{M-1}\sum_{k=0}^{N-1} A_{ik}B_{ki}}{\partial A_{lm}}\right) = B_{ml}$$

Eq 2.2.4

Substituting back gives:

$$\frac{\partial\{tr(AB)\}}{\partial A} = \left(B_{0,0}\, B_{1,0}\, \dots\, B_{N-1,0}\, B_{0,1}\, B_{1,1}\, \dots\, B_{N-1,1}\, \dots \dots \dots \dots\, B_{0,M-1}\, B_{1,M-1}\, \dots\, B_{N-1,M-1}\right) = B^T$$

Eq 2.2.5

2.3 Lemma 3

The derivative with respect to A of the trace of the matrix product ABA^T equals $A(B + B^T)$.

Proof

$$\frac{\partial\{tr(ABA^T)\}}{\partial A} = \left(\left(\frac{\partial tr(ABA^T)}{\partial A_{0,0}} \right) \left(\frac{\partial tr(ABA^T)}{\partial A_{0,1}} \right) \ \ldots \ \left(\frac{\partial tr(ABA^T)}{\partial A_{0,N-1}} \right) \left(\frac{\partial tr(ABA^T)}{\partial A_{1,0}} \right) \left(\frac{\partial tr(ABA^T)}{\partial A_{1,1}} \right) \ \ldots \ \left(\frac{\partial tr(ABA^T)}{\partial A_{1,N-1}} \right) \ \ldots \ \ldots \ \ldots \ \ldots \ \left(\frac{\partial tr(ABA^T)}{\partial A_{M-1,0}} \right) \left(\frac{\partial tr(AB}{\partial A_{M-1}} \right.$$

Eq 2.3.1

If the matrix A has dimensions MxN then the matrix B must be square with dimensions NxN for the product ABA^T to exist. The product ABA^T is always square with dimensions MxM.

The element C_{ij} of the matrix $C = AB$ has value:

$$C_{ij} = \sum_{k=0}^{N-1} A_{ik} B_{kj}$$

Eq 2.3.2

The element D_{il} of matrix $D = ABA^T = CA^T$ has value:

$$D_{il} = \sum_{j=0}^{N-1} C_{ij} A_{lj} = \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} A_{ik} B_{kj} A_{lj}$$

Eq 2.3.3

The trace of matrix $\boldsymbol{D}$ has value:

$$tr(D) = \sum_{i=0}^{N-1} D_{ii} = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} A_{ik} B_{kj} A_{ij}$$

Eq 2.3.4

The derivative of $tr(D)$ with respect to A_{lm} is then:

$$\left(\frac{\partial tr(D)}{\partial A_{lm}} \right) = \left(\frac{\partial \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} A_{ik} B_{kj} A_{ij}}{\partial A_{lm}} \right) = \left(\frac{\partial \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} A_{lk} B_{kj} A_{lj}}{\partial A_{lm}} \right)$$

Eq 2.3.5

$$= \sum_{j=0}^{N-1} A_{lj} B_{mj} + \sum_{j=0}^{N-1} A_{lj} B_{jm} = \left(AB^T \right)_{lm} + (AB)_{lm}$$

Eq 2.3.6

$$\Rightarrow \frac{\partial\{tr(ABA^T)\}}{\partial A} = A(B + B^T)$$

Eq 2.3.7

If B is also symmetric then:

$$\frac{\partial\{tr(ABA^T)\}}{\partial A} = 2AB \ if \ B = B^T$$

Eq 2.3.8

3 Kalman Filter Derivation

3.1 Process Model

The Kalman filter models the vector process of interest x_k as linear and recursive:

$$x_k = A_k x_{k-1} + w_k \quad \text{Eq 3.1.1}$$

If x_k has N degrees of freedom then A_k is an NxN linear prediction matrix (possibly time varying but assumed known) and w_k is an $Nx1$ noise vector.

The process x_k is assumed to be not directly measurable and must be estimated from a process z_k which can be measured. z_k is modeled as being linearly related to x_k with additive noise v_k.

$$z_k = C_k x_k + v_k \quad \text{Eq 3.1.2}$$

z_k is an $Nx1$ vector, C_k is an NxN matrix (possibly time varying but assumed known) and v_k is an $Nx1$ noise vector.

The noise vectors w_k and v_k are assumed to be zero mean white processes:

$$E\left[w_k\right] = 0 \quad \text{Eq 3.1.3}$$

$$E\left[v_k\right] = 0 \quad \text{Eq 3.1.4}$$

$$cov\{w_k, w_j\} = E\left[w_k w_j^T\right] = Q_{w,k}\delta_{kj} \quad \text{Eq 3.1.5}$$

$$cov\{v_k, v_j\} = E\left[v_k v_j^T\right] = Q_{v,k}\delta_{kj} \quad \text{Eq 3.1.6}$$

By definition, covariance matrices are symmetric.

$$Q_{w,k}^T = \left\{E\left[w_k w_k^T\right]\right\}^T = E\left[\left(w_k w_k^T\right)^T\right] = E\left[w_k w_j^T\right] = Q_{w,k} \quad \text{Eq 3.1.7}$$

3.2 Derivation

The objective of the Kalman filter is to compute an unbiased *a posterori* estimate $\hat{x}_k^+$ of the underlying process x_k from i) extrapolation from the previous iteration's *a posteriori* estimate $\hat{x}_{k-1}^+$ and ii) from the current measurement z_k:

$$\hat{x}_k^+ = K_k^{'} \hat{x}_{k-1}^+ + K_k z_k \quad \text{Eq 3.2.1}$$

The time-varying Kalman gain matrices $K_k^{'}$ and K_k define the relative weightings given to the previous iteration's Kalman filter estimate $\hat{x}^{+}_{k-1}$ and to the current measurement z_k. If the measurements z_k have low noise then a higher weighting will be given to the term $K_k z_k$ compared to the extrapolated component $K'_k \hat{x}^{+}_{k-1}$ and vice versa. The Kalman filter is therefore a time varying recursive filter.

Unbiased estimate constraint (determines K'_k)

For $\hat{x}^{+}_{k}$ to be an unbiased estimate of x_k, the expectation value of the *a posteriori* Kalman filter error $\hat{x}^{+}_{\varepsilon,k}$ must be zero:

$$E\left[\hat{x}^{+}_{\varepsilon,k}\right] = E\left[\hat{x}^{+}_{k} - x_k\right] = 0 \qquad \text{Eq 3.2.2}$$

Subtracting x_k from equation 3.2.1 gives:

$$\hat{x}^{+}_{\varepsilon,k} = \hat{x}^{+}_{k} - x_k = K'_k \hat{x}^{+}_{k-1} + K_k z_k - x_k \qquad \text{Eq 3.2.3}$$

Substituting equation 3.1.2 for z_k gives:

$$\hat{x}^{+}_{\varepsilon,k} = K'_k \hat{x}^{+}_{k-1} + K_k\left(C_k x_k + v_k\right) - x_k \qquad \text{Eq 3.2.4}$$

Substituting for x_k from equation 3.1.1 and re-arranging gives:

$$\hat{x}^{+}_{\varepsilon,k} = K'_k\left(\hat{x}^{+}_{\varepsilon,k-1} + x_{k-1}\right) + K_k\left\{C_k\left(A_k x_{k-1} + w_k\right) + v_k\right\} - \left(A_k x_{k-1} + w_k\right) \qquad \text{Eq 3.2.5}$$

$$= K'_k \hat{x}^{+}_{\varepsilon,k-1} + \left(K_k C_k A_k - A_k + K'_k\right)x_{k-1} + \left(K_k C_k - I\right)w_k + K_k v_k \qquad \text{Eq 3.2.6}$$

Taking the expectation value of equation 3.2.6 and applying the unbiased estimate constraint gives:

$$E\left[\hat{x}^{+}_{\varepsilon,k}\right] = E\left[K'_k \hat{x}^{+}_{\varepsilon,k-1}\right] + E\left[\left(K_k C_k A_k - A_k + K'_k\right)x_{k-1}\right] + E\left[\left(K_k C_k - I\right)w_k\right] + E\left[K_k v_k\right] = 0 \qquad \text{Eq 3.2.7}$$

Since the noise vectors w_k and v_k are zero mean and uncorrelated with the Kalman matrices for the same iteration, it follows that:

$$E\left[\left(K_k C_k - I\right)w_k\right] = E\left[K_k v_k\right] = 0 \qquad \text{Eq 3.2.8}$$

With the additional assumption that the process x_{k-1} is independent of the Kalman matrices at iteration k:

$$E\left[\left(K_k C_k A_k - A_k + K'_k\right)x_{k-1}\right] = \left(K_k C_k A_k - A_k + K'_k\right)E\left[x_{k-1}\right] = 0 \qquad \text{Eq 3.2.9}$$

Since x_k is not, in general, a zero mean process:

$$K_kC_kA_k - A_k + K_k^{'} = 0 \Rightarrow K_k^{'} = A_k - K_kC_kA_k = \left(I - K_kC_k\right)A_k \qquad \text{Eq 3.2.10}$$

Eliminating $K_k^{'}$ in equation 3.2.1 gives:

$$\hat{x}_k^{+} = \left(I - K_kC_k\right)A_k\hat{x}_{k-1}^{+} + K_kz_k \qquad \text{Eq 3.2.11}$$

A priori estimate

The *a priori* Kalman filter estimate $\hat{x}_k^{-}$ is defined as resulting from the application of the linear prediction matrix A_k to the previous iteration's *a posteriori* estimate $\hat{x}_{k-1}^{+}$:

$$\hat{x}_k^{-} = A_k\hat{x}_{k-1}^{+} \qquad \textbf{Kalman equation 1} \qquad \text{Eq 3.2.12}$$

Definition of a posteriori estimate

Substituting the a priori estimate $\hat{x}_k^{-}$ into equation 3.2.11 gives:

$$\hat{x}_k^{+} = \left(I - K_kC_k\right)\hat{x}_k^{-} + K_kz_k \qquad \textbf{Kalman equation 4} \qquad \text{Eq 3.2.13}$$

An equivalent form is:

$$\hat{x}_k^{+} = \hat{x}_k^{-} + K_k\left(z_k - C_k\hat{x}_k^{-}\right) \qquad \text{Eq 3.2.14}$$

P_k^{-} as a function of P_{k-1}^{+}

The *a priori and a posteriori* error covariance matrices P_k^{-} and P_k^{+} are defined as:

$$P_k^{-} = cov\{\hat{x}_{\varepsilon,k}^{-}, \hat{x}_{\varepsilon,k}^{-}\} = E\left[\hat{x}_{\varepsilon,k}^{-}\hat{x}_{\varepsilon,k}^{-\,T}\right] = E\left[\left(\hat{x}_k^{-} - x_k\right)\left(\hat{x}_k^{-} - x_k\right)^T\right] \qquad \text{Eq 3.2.15}$$

$$P_k^{+} = cov\{\hat{x}_{\varepsilon,k}^{+}, \hat{x}_{\varepsilon,k}^{+}\} = E\left[\hat{x}_{\varepsilon,k}^{+}\hat{x}_{\varepsilon,k}^{+\,T}\right] = E\left[\left(\hat{x}_k^{+} - x_k\right)\left(\hat{x}_k^{+} - x_k\right)^T\right] \qquad \text{Eq 3.2.16}$$

Substituting the definitions of $\hat{x}_k^{-}$ and x_k into equation 3.2.15 gives:

$$P_k^- = E\left[\left(A_k\hat{x}_{k-1}^+ - A_k x_{k-1} - w_k\right)\left(A_k\hat{x}_{k-1}^+ - A_k x_{k-1} - w_k\right)^T\right]$$ Eq 3.2.17

$$= E\left[\left\{A_k\left(\hat{x}_{k-1}^+ - x_{k-1}\right) - w_k\right\}\left\{A_k\left(\hat{x}_{k-1}^+ - x_{k-1}\right) - w_k\right\}^T\right]$$ Eq 3.2.18

$$= A_k E[\left(\hat{x}_{k-1}^+ - x_{k-1}\right)\left(\hat{x}_{k-1}^+ - x_{k-1}\right)^T]A_k^T + Q_{w,k}$$ Eq 3.2.19

$$\Rightarrow P_k^- = A_k P_{k-1}^+ A_k^T + Q_{w,k}$$ **Kalman equation 2** Eq 3.2.20

Minimum error covariance constraint (determines K_k)

The Kalman gain matrix K_k minimizes the *a posteriori* error $\hat{x}_{\varepsilon,k}^+$ variance via the trace of the *a posteriori* error covariance matrix P_k^+:

$$E\left[\hat{x}_{\varepsilon,k}^{+\,T}\hat{x}_{\varepsilon,k}^+\right] = tr\left(P_k^+\right)$$ Eq 3.2.21

Substituting equation 2.1.2 for z_k into equation 3.2.11 gives a relation between the *a posteriori* and *a priori* errors:

$$\hat{x}_k^+ = \hat{x}_{\varepsilon,k}^+ + x_k = \left(I - K_k C_k\right)\hat{x}_k^- + K_k z_k = \left(I - K_k C_k\right)\left(\hat{x}_{\varepsilon,k}^- + x_k\right) + K_k\left(C_k x_k + v_k\right)$$ Eq 3.2.22

$$\Rightarrow \hat{x}_{\varepsilon,k}^+ + x_k = \left(I - K_k C_k\right)\hat{x}_{\varepsilon,k}^- + x_k - K_k C_k x_k + K_k\left(C_k x_k + v_k\right)$$ Eq 3.2.23

$$\Rightarrow \hat{x}_{\varepsilon,k}^+ = \left(I - K_k C_k\right)\hat{x}_{\varepsilon,k}^- + K_k v_k$$ Eq 3.2.24

Substituting this result into the definition of the *a posteriori* covariance matrix P_k^+ gives:

$$P_k^+ = E\left[\left\{\left(I - K_k C_k\right)\hat{x}_{\varepsilon,k}^- + K_k v_k\right\}\left\{\left(I - K_k C_k\right)\hat{x}_{\varepsilon,k}^- + K_k v_k\right\}^T\right]$$ Eq 3.2.25

$$= \left(I - K_k C_k\right)E\left[\hat{x}_{\varepsilon,k}^-\hat{x}_{\varepsilon,k}^{-\,T}\right]\left(I - K_k C_k\right)^T + K_k E\left[v_k v_k^T\right]K_k^T$$ Eq 3.2.26

$$= \left(I - K_k C_k\right)P_k^-\left(I - K_k C_k\right)^T + K_k Q_{v,k} K_k^T$$ Eq 3.2.27

$$= P_k^- - P_k^- C_k^T K_k^T - K_k C_k P_k^- + K_k C_k P_k^- C_k^T K_k^T + K_k Q_{v,k} K_k^T$$ Eq 3.2.28

The Kalman filter gain K_k is that which minimizes the trace of the *a posteriori* error covariance matrix P_k^+:

$$\frac{\partial}{\partial K_k} tr\left(P_k^+\right) = \frac{\partial}{\partial K_k}\left\{tr\left(P_k^-\right) - tr\left(P_k^- C_k^T K_k^T\right) - tr\left(K_k C_k P_k^-\right) + tr\left(K_k C_k P_k^- C_k^T K_k^T\right) + tr\left(K_k Q_{v,k} K_k^T\right)\right\} = 0$$

Eq 3.2.29

The term $tr\left(P_k^-\right)$ has no dependence on K_k giving:

$$\frac{\partial\left\{tr(P_k^-)\right\}}{\partial K_k} = \frac{\partial\left\{tr(A_k P_{k-1}^+ A_k^T + Q_{w,k})\right\}}{\partial K_k} = 0$$

Eq 3.2.30

Since the trace of a transposed matrix equals the trace of the original matrix and using equation 2.2.5 gives:

$$\frac{\partial\left\{tr(P_k^- C_k^T K_k^T)\right\}}{\partial K_k} = \frac{\partial\left\{tr(K_k C_k P_k^-)\right\}}{\partial K_k} = \left(C_k P_k^-\right)^T = P_k^- C_k^T$$

Eq 3.2.31

The third term can be simplified using equations 2.3.7 and 2.3.8 exploiting the fact that the covariance matrix is symmetric:

$$\frac{\partial\left\{tr(K_k C_k P_k^- C_k^T K_k^T)\right\}}{\partial K_k} = K_k\left\{C_k P_k^- C_k^T + \left(C_k P_k^- C_k^T\right)^T\right\} = 2K_k C_k P_k^- C_k^T$$

Eq 3.2.32

The final term can be simplified also using equations 2.3.7 and 2.3.8 to give:

$$\frac{\partial\left\{tr(K_k Q_{v,k} K_k^T)\right\}}{\partial K_k} = 2K_k Q_{v,k}$$

Eq 3.2.33

Substituting back into equation 2.2.29 gives the optimal Kalman filter gain matrix K_k:

$$-\ 2P_k^- C_k^T + 2K_k C_k P_k^- C_k^T + 2K_k Q_{v,k} = 0$$ Eq 3.2.34

$$\Rightarrow K_k\left(C_k P_k^- C_k^T + Q_{v,k}\right) = P_k^- C_k^T$$ Eq 3.2.35

$$\Rightarrow K_k = P_k^- C_k^T\left(C_k P_k^- C_k^T + Q_{v,k}\right)^{-1}$$ **Kalman equation 3** Eq 3.2.36

P_k^+ as a function of P_k^-

Rearranging equation 3.2.35 gives:

$$K_k Q_{v,k} = P_k^- C_k^T - K_k C_k P_k^- C_k^T$$ Eq 3.2.37

Substituting equation 3.2.37 into equation 3.2.27 gives:

$$P_k^+ = \left(I - K_k C_k\right) P_k^- \left(I - C_k^T K_k^T\right) + \left(I - K_k C_k\right) P_k^- C_k^T K_k^T \quad \text{Eq 3.2.38}$$

$$\Rightarrow P_k^+ = \left(I - K_k C_k\right) P_k^- \quad \textbf{Kalman equation 5} \quad \text{Eq 3.2.39}$$

This completes the derivation of the standard Kalman filter equations.

3.3 Standard Kalman Equations

Kalman equation 1

The linear prediction (*a priori*) estimate $\hat{x}_k^-$ is made by applying the linear prediction matrix A_k to the previous sample's Kalman (*a posteriori*) filter estimate $\hat{x}_{k-1}^+$.

$$\hat{x}_k^- = A_k \hat{x}_{k-1}^+ \quad \text{Eq 3.3.1}$$

Kalman equation 2

The *a priori* (linear extrapolation) error covariance matrix P_k^- is then updated using the model matrix A_k and the noise matrix $Q_{w,k}$.

$$P_k^- = A_k P_{k-1}^+ A_k^T + Q_{w,k} \quad \text{Eq 3.3.2}$$

Kalman equations 2 and 5 can be combined to give a recursive update of P_k^- without explicit calculation of the *a posteriori* error covariance matrix P_k^+ in Kalman equation 5:

$$P_k^- = A_k \left(I - K_{k-1} C_{k-1}\right) P_{k-1}^- A_k^T + Q_{w,k} \quad \text{Eq 3.3.3}$$

Kalman equation 3

The Kalman filter gain matrix K_k is updated:

$$K_k = P_k^- C_k^T \left(C_k P_k^- C_k^T + Q_{v,k}\right)^{-1} \quad \text{Eq 3.3.4}$$

Kalman equation 4

The Kalman filter (*a posteriori*) estimate $\hat{x}_k^+$ is computed from the current *a priori* estimate $\hat{x}_k^-$ and the current measurement z_k:

$$\hat{x}_k^+ = \hat{x}_k^- + K_k \left(z_k - C_k \hat{x}_k^-\right) = \left(I - K_k C_k\right) \hat{x}_k^- + K_k z_k \quad \text{Eq 3.3.5}$$

Kalman equation 5

The *a posteriori* Kalman error covariance matrix P_k^+ is updated ready for the next iteration. This equation can be skipped if P_k^- is updated recursively in terms of itself as in equation 3.3.3.

$$P_k^+ = \left(I - K_k C_k\right) P_k^- \qquad \text{Eq 3.3.6}$$