Department of Mathematics, IIT Madras

MA-5895-Numerical Optimization

Assignment 1

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Q. 1 Suppose that $f(x) = x^T Q x$, where Q is an $n \times n$ symmetric positive semidefinite matrix and $x \in \mathbb{R}^n$. Show that $f(x)$ is convex on the domain $\mathbb{R}^n$.

Hint: It may be convenient to prove the following equivalent inequality:

$$f(y + \alpha(x - y)) - \alpha f(x) - (1 - \alpha) f(y) \leq 0,$$

for all $\alpha \in [0, 1]$ and all $x, y \in \mathbb{R}^n$.

Q. 2 Consider the function $f(x_1, x_2) = (x_1 + x_2)^2$. At the point $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we consider the search direction $p = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Show that p is a descent direction and find all minimizers of the univariate problem

$$\min_{\alpha > 0} f(x + \alpha p)$$

Q. 3 Suppose A is an $m \times n$ real matrix with $\text{rank}(A) = n$ and $b \in \mathbb{R}^{m \times 1}$. Let $f(x) = \|Ax - b\|$. Show that the minimizer of the unconstrained optimization problem with f as a objective function is

$$x^* = (A^T A)^{-1} A^T b$$

Moreover, explain why it is the unique strict global minimum of this optimization problem.

Q. 4 Program the Steepest descent and Newton algorithms using the backtracking line search, algorithm. Use them to minimize the Rosenbrock function:

$$f(x) = 100(x_2 - x_1)^2 + (1 - x_1)^2 \tag{1}$$

To start with set the initial step length $\alpha_0 = 1$ and print the step length used by the each of the methods at each iteration. First try the initial point $x_0 = (1.2, 1.2)^T$, and then the more difficult starting point $x_0 = (-1.2, 1)^T$. Trace the track followed by each of the algorithms step by step to the final point on a contour plot of the function in (1).

Moreover, comment about your experience with regards to both the algorithms, in the sense how does their performance relate to the analysis carried out in the class.