TANGENT CONE AND CONSTRAINT QUALIFICATIONS

Saurav Samantaray

5th july 2016

1 TANGENT CONE

- We determined whether or not it was possible to take a feasible descent step away from a given feasible point x;
- by examining the first derivatives of f and;
- the constraint functions c_i.
- The first-order Taylor series expansion of these functions about x was used to form an approximate problem in which both objective and constraints are linear.
- Makes sense if the linearised approximation captures the essential geometric features of the feasible set near the point x in question.
- Assumptions about the nature of the constraints c_i that are active at x are needed to be made to ensure that the linearised approximation is similar to the feasible set, near x.
- Given a feasible point x, $\{z_k\}$ is called a feasible sequence approaching x, if $z_k \in \Omega$ for all k, sufficiently large and $z_k \to x$.

Definition (Cone)

A cone is a set $\mathscr{F}$ with the property that for all $x \in \mathscr{F}$ we have

$$x \in \mathscr{F} \implies \alpha x \in \mathscr{F}, \text{ for all } \alpha > 0.$$

Example

The set $\mathscr{F} \subset \mathbb{R}^2$ defined by

$$\{(x_1, x_2)^T | x_1 > 0, x_2 \geq 0\}$$

is a cone in $\mathbb{R}^2$.

Definition

The vector d is said to be a tangent (or tangent vector) to Ω at a point x if there are a feasible sequence $\{z_k\}$ approaching x and a sequence of positive scalars $\{t_k\}$ with $t_k \to 0$ such that

$$\lim_{k\to\infty} \frac{z_k - x}{t_k} = d. \tag{1}$$

The set of all tangents to Ω at x^* is called the tangent cone and is denoted by $T_\Omega(x^*)$.

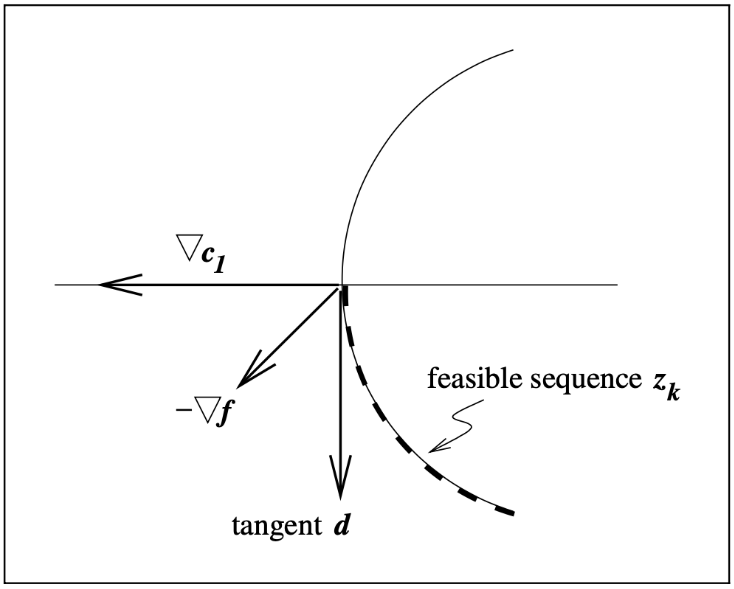

Figure 1: Constraint normal, objective gradient, and feasible sequence

Definition (Linearised Feasible Direction)

Given a feasible point x and the active constraint set $\mathscr{A}(x)$, the set of linearised feasible directions $\mathscr{F}(x)$ is

$$\mathscr{F}(x) = \left\{ d \middle| \begin{array}{ll} d^T \nabla c_i(x) = 0, & \text{for all } i \in \mathscr{E} \\ d^T \nabla c_i(x) \geq 0, & \text{for all } i \in \mathscr{A}(x) \cap \mathscr{I} \end{array} \right\} \tag{2}$$

- $\mathscr{F}(x)$ is also a cone.
- The definition of tangent cone does not explicitly depend on the constraints c_i it depends on the geometry of Ω.
- The linearised feasible direction set does, however, depend on the definition of the constraint functions c_i , $i \in \mathscr{E} \cup \mathscr{I}$.

Tangent Cone and Feasible Direction for One Equality Constraint

- Consider the problem with one equality constraint.
- The objective function $f(x) = x_1 + x_2$, $\mathscr{E} = \{1\}$, $\mathscr{I} = \phi$
- $c_1(x) = x_1^2 + x_2^2 - 2$
- The feasible set for this problem is the circle of radius $\sqrt{2}$ centered at the origin.

- Consider the non-optimal point $x = (\sqrt{2}, 0)^T$.
- The figure also shows a feasible sequence approaching x.

$$z_k = \begin{bmatrix} -\sqrt{2 - 1/k^2} \\ -1/k \end{bmatrix}$$

- Choose $t_k = ||z_k - x||$, to get $d = (0, -1)^T$ is a tangent.

- f increases as we move along z_k, i.e. $f(z_{k+1}) > f(z_k)$ for all $k = 2, 3, \ldots$.
- $f(z_k) < f(x)$ for $k = 2, 3, \ldots$, so x cannot be a minimiser.
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- f increases as we move along z_k, i.e. $f(z_{k+1}) > f(z_k)$ for all $k = 2, 3, \ldots$.
- $f(z_k) < f(x)$ for $k = 2, 3, \ldots$, so x cannot be a minimiser.
- Another feasible sequence is one that approaches $x = (-\sqrt{2}, 0)^T$ from the opposite direction.
- $$z_k = \begin{bmatrix} -\sqrt{2 - 1/k^2} \\ 1/k \end{bmatrix}$$
- f decreases along this sequence.
- The tangents corresponding to this sequence are $d = (0, \alpha)^T$.
- In summary, the tangent cone at $x = (-\sqrt{2}, 0)^T$ is $\{(0, d_2)^T | d_2 \in \mathbb{R}\}$.
- For the set of linearised feasible directions $\mathscr{F}(x)$, $d = (d_1, d_2)^T \in \mathscr{F}(x)$ if

$$0 = \nabla c_1(x)^T d = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}^T \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = -2\sqrt{2} d_1$$

- $\mathscr{F}(x) = \{(0, d_2)^T | d_2 \in \mathbb{R}\}$.
- In this case $T_\Omega(x) = \mathscr{F}(x)$.
- Suppose that the feasible set is defined instead by the formula

$$\Omega = \{x | c_1(x) = 0\}, \qquad \text{where } c_1(x) = (x_1^2 + x_2^2 - 2)^2 = 0$$

- Ω is geometrically the same, but with a different algebraic specification.
- Then d belongs to the linearised feasible set if:

$$0 = \nabla c_1(x)^T d = \begin{bmatrix} 4(x_1^2 + x_2^2 - 2)x_1 \\ 4(x_1^2 + x_2^2 - 2)x_2 \end{bmatrix}^T \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

- which is true for all $(d_1, d_2)^T$.
- $\mathscr{F}(x) = \mathbb{R}^2$.
- So for this algebraic specification of Ω, the tangent cone and linearised feasible sets differ.

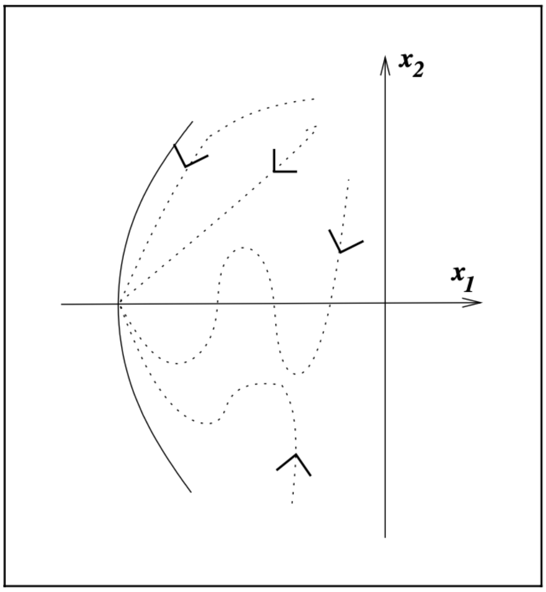

Figure 2: Feasible sequences converging to a particular feasible point for the region defined by $x_1^2 + x_2^2 \leq 2$

Tangent Cone and Feasible Direction for One In-Equality Constraint

- The solution $x = (-1, -1)^T$ is the same as in the equality-constrained case.
- But, there is a much more extensive collection of feasible sequences that converge to any given feasible point.
- From the point $x = (-\sqrt{2}, 0)^T$, all the feasible sequences defined above for the equality-constrained problem are still feasible.
- There are also infinitely many feasible sequences that converge to x, along a straight line from the interior of the circle.
- $$z_k = (-\sqrt{2}, 0)^T + (1/k)w,$$
  where w is any vector whose first component is positive ($w_1 > 0$).
- z_k is feasible provided that $||z_k|| \leq \sqrt{2}$ i.e.
  $$(-\sqrt{2} + w_1/k)^2 + (w_2/k)^2 \leq 2,$$
- Which is true when $k \geq (w_1^2 + w_2^2)/(2\sqrt{2}w_1)$
- we can also define an infinite variety of sequences that approach x along a curve from the interior of the circle.
- To summarize, the tangent cone to this set at $(-\sqrt{2}, 0)^T$ is $\{(w_1, w_2)^T | w_1 \geq 0\}$.
- For the feasibility set $\mathscr{F}(x)$ let us consider:
  $$0 \leq \nabla c_1(x)^T d = \begin{bmatrix} -2x_1 \\ -2x_2 \end{bmatrix}^T \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = 2\sqrt{2}d_1$$
- Hence, we obtain $\mathscr{F}(x) = T_\Omega(x)$ for this particular algebraic specification of the feasible set.

2 Constraint qualifications

- Constraint qualifications are conditions under which the linearised feasible set $\mathscr{F}(x)$ is similar to the tangent cone $T_\Omega(x)$.
- Most constraint qualifications ensure that these two sets are identical.
- These conditions ensure that the $\mathscr{F}(x)$, which is constructed by linearising the algebraic description of the set Ω at x, captures the essential geometric features of the set Ω in the vicinity of x, as represented by T_Ω.

Definition

Given the point x and the active set $\mathscr{A}(x)$, we say that the linear independence constraint qualification (LICQ) holds if the set of active constraint gradients $\{\nabla c_i(x) | i \in \mathscr{A}(x)\}$ is linearly independent.

In general, if LICQ holds, none of the active constraint gradients can be zero.