Linear Programming: The Simplex Method

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April 3, 2024

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    - advanced state of the software,
    - guaranteed convergence to a global minimum,
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- A new class of methods known as interior-point methods has proved to be faster for some linear programming problems.
- But the continued importance of the simplex method is assured for the foreseeable future.

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- Figure below depicts a linear program in two-dimensional space, in which the contours of the objective function are indicated by dotted lines.

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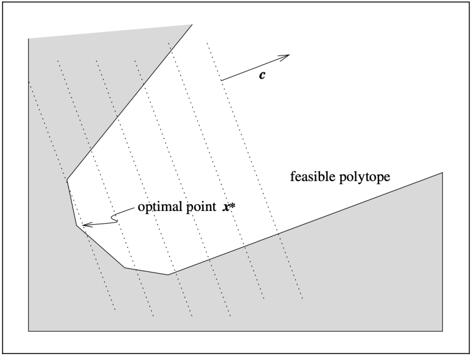

Figure: A linear program in two dimensions with solution at x^*

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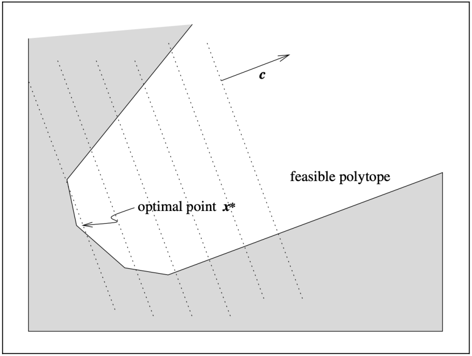

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- The solution in this case is unique-a single vertex.

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- or if the objective function is unbounded below on the feasible region (the unbounded case)

Standard Form of Linear Programs

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where

- c and x are vectors in $\mathbb{R}^n$,
- b is a vector in $\mathbb{R}^m$ and A is an $m \times n$ matrix

Transforming to Standard Form

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- $$\min c^T x, \qquad \text{subject to } Ax + z = b,\ z \geq 0, \tag{3}$$
- Still not all variables (x) are constrained to be non-negative as in the standard form.

Transforming to Standard Form

- It is dealt by splitting x into non-negative and non-positive parts.

$$x = x^+ - x^-, \; x^+ = \max(x, 0) \geq 0 \text{ and } x^- = \max(-x, 0)$$

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- Now the above considered problem can be written as:

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- The above system is now in the standard form.

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- We subtract the variables from the left hand side, they are known as surplus variables.
- We add the variables to the left hand side, they are known as deficit variables.
- By simply negating c "maximise" objective $\max c^T x$ can be converted to "minimise" form $\min \; -c^T x$.

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- Otherwise, the system $Ax = b$ contains redundant rows, or is infeasible, or defines a unique point.
- When $m \geq n$, factorisations such as the QR or LU factorisation can be used to transform the system $Ax = b$ to one with a coefficient matrix of full row rank.

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- The LICQ condition is not required to be enforced here as the KKT results continue to hold for dependent constraints provided they are linear, as is the case here.

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- Where $\lambda \in \mathbb{R}^m$ is the multiplier vector for the equality constraints $Ax = b$.
- While $s \in \mathbb{R}^n$ is the multiplier vector for the bound constraints $x \geq 0$.
- Using the definition we can write the Lagrangian function:

$$\mathscr{L}(x, \lambda, s) = c^T x - \lambda^T (Ax - b) - s^T x. \tag{4}$$

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- The first-order necessary conditions for x^* to be a solution of the linear programming problem (1) are, if there exists λ and s such that:

$$A^T\lambda + s = c, \tag{5}$$
$$Ax = b, \tag{6}$$
$$x \geq 0, \tag{7}$$
$$s \geq 0, \tag{8}$$
$$x_i s_i = 0, \ \ i = 1, 2, \ldots, n. \tag{9}$$

- The last condition, which is the complementarity condition, which says that at-least either one of x_i or s_i is zero, can be wrtitten alternatively as

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- The first order KKT conditions for optimality for LPP is indeed sufficient.
- Let $\bar{x}$ be any other feasible point, so that $A\bar{x} = b$ and $\bar{x} \geq 0$.

$$\begin{aligned} c^T \bar{x} &= (A^T\lambda^* + s^*)^T \bar{x} \\ &= b^T\lambda^* + \bar{x}^T s^* \\ &\geq b^T\lambda^* = c^T x^* \end{aligned}$$

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otherwise the inequality is strict.
- When $s_i^* > 0$ then we must have $\bar{x}_i = 0$ for all solutions $\bar{x}$ of the LPP.