TANGENT CONE AND CONSTRAINT QUALIFICATIONS

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1 TANGENT CONE

- We determined whether or not it was possible to take a feasible descent step away from a given feasible point x;
- by examining the first derivatives of f and;
- the constraint functions c_i.
- The first-order Taylor series expansion of these functions about x was used to form an approximate problem in which both objective and constraints are linear.
- Makes sense if the linearised approximation captures the essential geometric features of the feasible set near the point x in question.
- Assumptions about the nature of the constraints c_i that are active at x are needed to be made to ensure that the linearised approximation is similar to the feasible set, near x.
- Given a feasible point x, $\{z_k\}$ is called a feasible sequence approaching x, if $z_k \in \Omega$ for all k, sufficiently large and $z_k \to x$.

Definition (Cone)

A cone is a set $\mathscr{F}$ with the property that for all $x \in \mathscr{F}$ we have

$$x \in \mathscr{F} \implies \alpha x \in \mathscr{F}, \text{ for all } \alpha > 0.$$

Example

The set $\mathscr{F} \subset \mathbb{R}^2$ defined by

$$\{(x_1, x_2)^T | x_1 > 0, x_2 \geq 0\}$$

is a cone in $\mathbb{R}^2$.

Definition

The vector d is said to be a tangent (or tangent vector) to Ω at a point x if there are a feasible sequence $\{z_k\}$ approaching x and a sequence of positive scalars $\{t_k\}$ with $t_k \to 0$ such that

$$\lim_{k\to\infty} \frac{z_k - x}{t_k} = d. \tag{1}$$

The set of all tangents to Ω at x^* is called the tangent cone and is denoted by $T_\Omega(x^*)$.

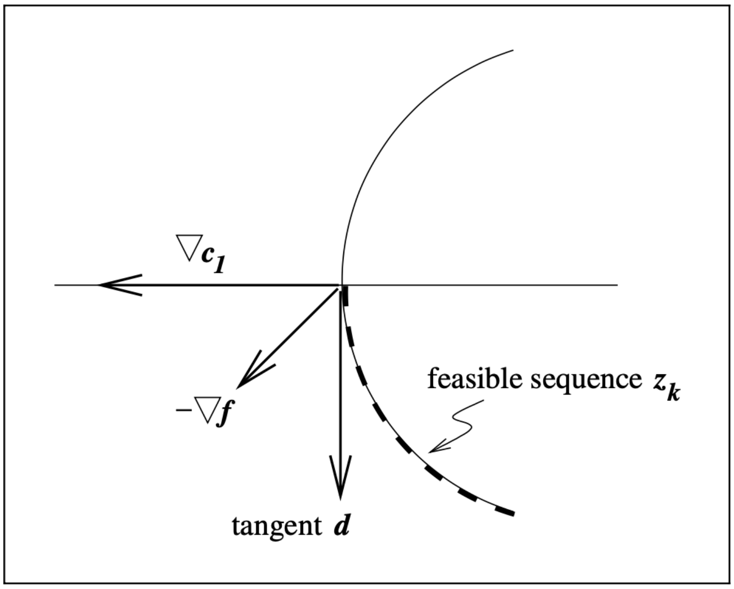

Figure 1: Constraint normal, objective gradient, and feasible sequence

Definition (Linearised Feasible Direction)

Given a feasible point x and the active constraint set $\mathscr{A}(x)$, the set of linearised feasible directions $\mathscr{F}(x)$ is

$$\mathscr{F}(x) = \left\{ d| \begin{array}{ll} d^T \nabla c_i(x) = 0, & \text{for all } i \in \mathscr{E} \\ d^T \nabla c_i(x) \geq 0, & \text{for all } i \in \mathscr{A}(x) \cap \mathscr{I} \end{array} \right\} \tag{2}$$

- $\mathscr{F}(x)$ is also a cone.
- The definition of tangent cone does not explicitly depend on the constraints c_i it depends on the geometry of Ω.
- The linearised feasible direction set does, however, depend on the definition of the constraint functions c_i , $i \in \mathscr{E} \cup \mathscr{I}$.

1.1 Tangent Cone and Feasible Direction for One Equality Constraint

- Consider the problem with one equality constraint.
- The objective function $f(x) = x_1 + x_2$, $\mathscr{E} = \{1\}$, $\mathscr{I} = \phi$
- $c_1(x) = x_1^2 + x_2^2 - 2$
- The feasible set for this problem is the circle of radius $\sqrt{2}$ centered at the origin.
- Consider the non-optimal point $x = (\sqrt{2}, 0)^T$.
- The figure also shows a feasible sequence approaching x.

$$z_k = \begin{bmatrix} -\sqrt{2 - 1/k^2} \\ -1/k \end{bmatrix}$$

- Choose $t_k = ||z_k - x||$, to get $d = (0, -1)^T$ is a tangent.
- f increases as we move along z_k, i.e. $f(z_{k+1}) > f(z_k)$ for all $k = 2, 3, \ldots$.
- $f(z_k) < f(x)$ for $k = 2, 3, \ldots$, so x cannot be a minimiser.