Linear Programming: The Simplex Method

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- But linear programming is appealing,
    - advanced state of the software,
    - guaranteed convergence to a global minimum,
    - uncertainty in the model makes a linear model more appropriate than an overly complex non-linear model.

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- Non-linear programming may replace linear programming as the method of choice in some applications as the non-linear software improves.
- A new class of methods known as interior-point methods has proved to be faster for some linear programming problems.
- But the continued importance of the simplex method is assured for the foreseeable future.

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- Owing to the linearity of the objective function its contours are planar.
- Figure below depicts a linear program in two-dimensional space, in which the contours of the objective function are indicated by dotted lines.

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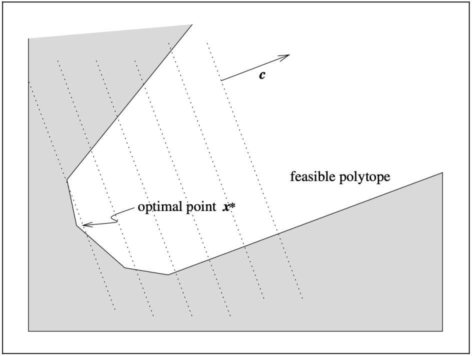

Figure: A linear program in two dimensions with solution at x^*

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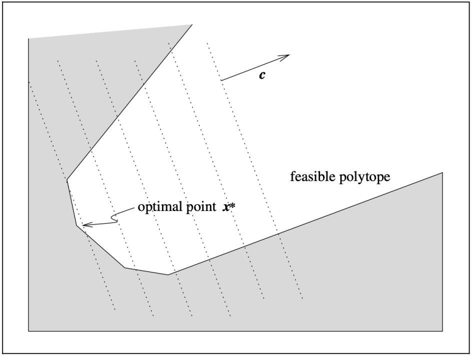

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- The solution in this case is unique-a single vertex.

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- or if the objective function is unbounded below on the feasible region (the unbounded case)

Standard Form of Linear Programs

Linear programs are usually stated and analysed in the following standard form:

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where

- c and x are vectors in $\mathbb{R}^n$,
- b is a vector in $\mathbb{R}^m$ and A is an $m \times n$ matrix

Transforming to Standard Form

- Consider the form:

$$\min c^T x, \qquad \text{subject to } Ax \leq b \tag{2}$$

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- Still not all variables (x) are constrained to be non-negative as in the standard form.

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- It is dealt by splitting x into non-negative and non-positive parts.

$$x = x^{+} - x^{-}, \; x^{+} = \max(x, 0) \geq 0 \text{ and } x^{-} = \max(-x, 0)$$

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- Now the above considered problem can be written as:

$$\min \begin{bmatrix} c \\ -c \\ 0 \end{bmatrix} \begin{bmatrix} x^{+} \\ x^{-} \\ z \end{bmatrix} \text{ s.t. } \begin{bmatrix} A & -A & I \end{bmatrix} \begin{bmatrix} x^{+} \\ x^{-} \\ z \end{bmatrix} = b, \begin{bmatrix} x^{+} \\ x^{-} \\ z \end{bmatrix} \geq 0,$$

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- The above system is now in the standard form.

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- We subtract the variables from the left hand side, they are known as surplus variables.
- We add the variables to the left hand side, they are known as deficit variables.
- By simply negating c "maximise" objective max $c^T x$ can be converted to "minimise" form min $-c^T x$.

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- Otherwise, the system $Ax = b$ contains redundant rows, or is infeasible, or defines a unique point.
- When $m \geq n$, factorisations such as the QR or LU factorisation can be used to transform the system $Ax = b$ to one with a coefficient matrix of full row rank.

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- The LICQ condition is not required to be enforced here as the KKT results continue to hold for dependent constraints provided they are linear, as is the case here.

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- Where $\lambda \in \mathbb{R}^m$ is the multiplier vector for the equality constraints $Ax = b$.
- While $s \in \mathbb{R}^n$ is the multiplier vector for the bound constraints $x \geq 0$.
- Using the definition we can write the Lagrangian function:

$$\mathscr{L}(x, \lambda, s) = c^T x - \lambda^T (Ax - b) - s^T x. \tag{4}$$

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- The first-order necessary conditions for x^* to be a solution of the linear programming problem (1) are, if there exists λ and s such that:

$$A^T\lambda + s = c, \tag{5}$$
$$Ax = b, \tag{6}$$
$$x \geq 0, \tag{7}$$
$$s \geq 0, \tag{8}$$
$$x_i s_i = 0, \;\; i = 1, 2, \ldots, n. \tag{9}$$

- The last condition, which is the complementarity condition, which says that at-least either one of x_i or s_i is zero, can be written alternatively as

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Sufficiency of Optimality Conditions

- Let (x^*, λ^*, s^*) denote a vector triple that satisfy the KKT conditions, then

$$c^T x^* = (A^T \lambda^* + s^*)^T x^* = (A x^*)^T \lambda^* = b^T \lambda^* \tag{10}$$

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- The first order KKT conditions for optimality for LPP is indeed sufficient.
- Let $\bar{x}$ be any other feasible point, so that $A\bar{x} = b$ and $\bar{x} \geq 0$.

$$\begin{aligned} c^T \bar{x} &= (A^T\lambda^* + s^*)^T \bar{x} \\ &= b^T \lambda^* + \bar{x}^T s^* \\ &\geq b^T \lambda^* = c^T x^* \end{aligned}$$

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  otherwise the inequality is strict.
- When $s_i^* > 0$ then we must have $\bar{x}_i = 0$ for all solutions $\bar{x}$ of the LPP.

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- Applies to convex problems.

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- Note that $\mathscr{L}(., \lambda)$ is convex for any $\lambda \geq 0$.

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The Dual Problem:

- The dual objective function $q : \mathbb{R}^m \to \mathbb{R}$ is given as:

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- With domain $\mathscr{D} = \{\lambda : q(\lambda) > -\infty\}$

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- If λ_1 is fixed then $\mathscr{L}$ is a convex function of $(x_1, x_2)^T$.
- The infimum with respect to $(x_1, x_2)^T$ is achieved when the partial derivatives with respect to x_1 and x_2 are zero, i.e.

$$x_1 - \lambda_1 = 0, \quad x_2 = 0$$

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- Which clearly has the solution $\lambda_1 = 1$.

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$$\max b^T\lambda, \quad \text{subject to } A^T\lambda \leq c. \tag{14}$$

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- Use $x \in \mathbb{R}^n$ to denote the Lagrange multipliers for the constraints $A^T\lambda \leq c$, the Lagrangian function is

$$\bar{\mathscr{L}}(\lambda, x) = -b^T\lambda - x^T(c - A^T\lambda).$$

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$$\begin{aligned} b^T\bar{\lambda} &= (x^*)^T A^T\bar{\lambda} = (x^*)^T(A^T\bar{\lambda} - c) + c^T x^* \\ &\leq c^T x^* \qquad (A^T\bar{\lambda} - c \leq 0 \text{ and } x^* \geq 0) \\ &= b^T\lambda^* \end{aligned}$$

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- Thus, the dual objective function $b^T\lambda$ is a lower bound on the primal objective function $c^T x$.

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Duality is important in the theory of LP (and convex opt. in general) and in primal-dual algorithms; also, the dual may be easier to solve than the primal.

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- Reformulation by adding slack, surplus, and artificial variables can also bring out the full row rank.

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A vector x is a basic feasible point if it is feasible and if there exists a subset $\mathscr{B}$ of the index set 1,2,...,n such that:

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3. If LPP is feasible and bounded, then it has an optimal solution.

VERTICES OF THE FEASIBLE POLYTOPE

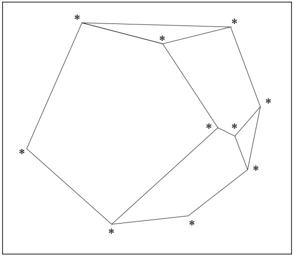

Figure: Vertices of a three-dimensional polytope (indicated by *)

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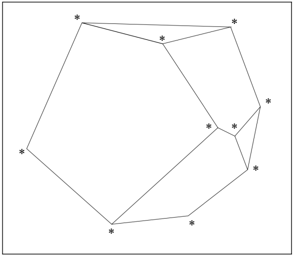

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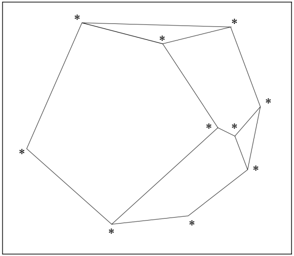

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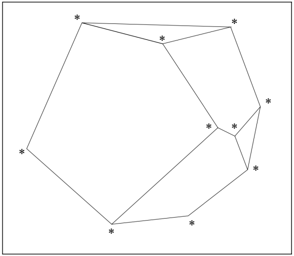

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Degeneracy

A basis $\mathscr{B}$ is said to be degenerate if $x_i = 0$ for some $i \in \mathscr{B}$, where x is the basic feasible solution corresponding to $\mathscr{B}$. A linear program is said to be degenerate if it has at least one degenerate basis.

Simplex Method

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- The steps may follow an edge along which the objective function is reduced, and along which we can move infinitely far without ever reaching a vertex.

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Simplex Method

Definition

The non-basic index set $\mathscr{N}$ is the complement of $\mathscr{B}$, that is:

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- Partition the n-element vectors x, s, and c according to the index sets $\mathscr{B}$ and $\mathscr{N}$

$$\begin{aligned} x_B &= [x_i]_{i \in \mathscr{B}} & \qquad x_N &= [x_i]_{i \in \mathscr{N}} \\ s_B &= [s_i]_{i \in \mathscr{B}} & \qquad s_N &= [s_i]_{i \in \mathscr{N}} \\ c_B &= [c_i]_{i \in \mathscr{B}} & \qquad c_N &= [c_i]_{i \in \mathscr{N}} \end{aligned}$$

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- if it does, i.e.,$s_N \geq 0$, we have found an optimal (x, λ, s) and we have finished

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- This process of selecting entering and leaving indices, and performing the algebraic operations necessary to keep track of the values of the variables x, λ, and s is known as pivoting.

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- Geometrically speaking, (23) is usually a move along an edge of the feasible polytope that decreases $c^T x$.

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- In other words, the constraint $x_B^+ = x_B - B^{-1}A_q \geq 0$ holds for all positive values of x_q^+.
- In such cases, the linear program is unbounded; the simplex method has identified a ray that lies entirely within the feasible polytope along which the objective $c^T x$ decreases to $-\infty$.

Simplex Method

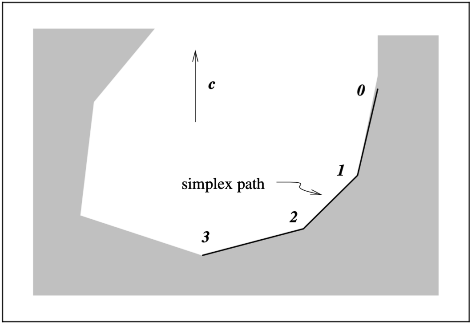

Figure: Simplex iterates for a two-dimensional problem.

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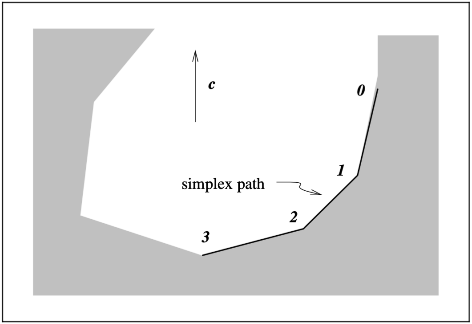

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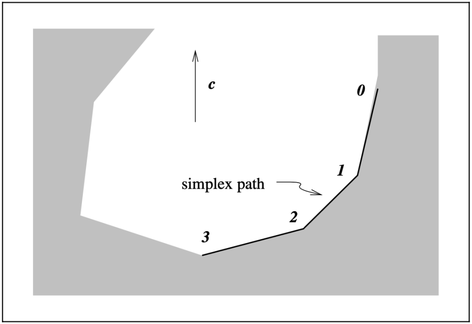

Figure: Simplex iterates for a two-dimensional problem.

- In this example, the optimal vertex x^* is found in three steps.
- If the basis $\mathscr{B}$ is non-degenerate, then its guaranteed that $x_q^+ > 0$, so it is assured to get a strict decrease in the objective function $c^T x$ at this step.

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If the problem is non-degenerate, it can be ensured to get a decrease in $c^T x$ at every step, and can therefore prove the following result concerning termination of the simplex method.

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Theorem

Provided that the linear program is non-degenerate and bounded, the simplex method terminates at a basic optimal point.

A SINGLE STEP OF THE METHOD

Procedure 13.1 (One Step of Simplex).

Given $\mathcal{B}, \mathcal{N}, x_B = B^{-1}b \geq 0, x_N = 0$;
Solve $B^T\lambda = c_B$ for λ,
Compute $s_N = c_N - N^T\lambda$; (* pricing *)
if $s_N \geq 0$
    stop; (* optimal point found *)
Select $q \in \mathcal{N}$ with $s_q < 0$ as the entering index;
Solve $Bd = A_q$ for d;
if $d \leq 0$
    stop; (* problem is unbounded *)
Calculate $x_q^+ = \min_{i \mid d_i > 0} (x_B)_i / d_i$, and use p to denote the minimizing i;
Update $x_B^+ = x_B - dx_q^+, x_N^+ = (0, \ldots, 0, x_q^+, 0, \ldots, 0)^T$;
Change $\mathcal{B}$ by adding q and removing the basic variable corresponding to column p of B.

Example

- Consider the problem

$$\begin{aligned} \min \;\; -3x_1 - 2x_2 \quad & \text{subject to} \\ x_1 + x_2 + x_3 &= 5, \\ 2x_1 + (1/2)x_2 + x_4 &= 8, \\ x &\geq 0. \end{aligned}$$

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- $$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1/2 & 0 & 1 \end{bmatrix}$$
- $c = \begin{bmatrix} -3 & -2 & 0 & 0 \end{bmatrix}^T \qquad b = \begin{bmatrix} 5 & 8 \end{bmatrix}^T$
- The constraints that we have are:

$$Ax = b \quad \text{and} \quad x \geq 0$$

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- Lets start with the basis $\mathscr{B} = \{3, 4\}$ and $\mathscr{N} = \{1, 2\}$

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$$x_B = B^{-1}b \implies \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$
- $c_B = \begin{bmatrix} 0 & 0 \end{bmatrix}^T \qquad B^T\lambda = c_B$
$$\lambda = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$$
- $$N = \begin{bmatrix} 1 & 1 \\ 2 & 1/2 \end{bmatrix}$$

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- $c_N = \begin{bmatrix} -3 & -2 \end{bmatrix}^T$
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- $d > 0$
- $$x_q^+ = \min_{i|d_i>0} \frac{(x_B)_i}{d_i} = \min \left\{ \frac{5}{1}, \frac{8}{2} \right\} = 4$$

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- With an objective value of $c^T x = -41/3$, $s_N \geq 0$, the optimality test is satisfied, and we terminate.

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- Handling of degenerate bases and degenerate steps, in which it is not possible to choose a positive value of x_q^+ without violating feasibility.

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- It is less expensive to update the factorisation than to calculate it afresh at each iteration.

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- More often than not its difficulty is equivalent to that of actually solving a linear program.
- A two-phase approach is commonly used to deal with this difficulty in practical implementations.

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- In Phase II, a second linear program similar to the original LP is solved, with the Phase-I solution as a starting point.
- The solution of the original LP can be extracted easily from the solution of the Phase-II problem.

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- where $z \in \mathbb{R}^m$, $e = (1, 1, \dots, 1)^T$, and E is a diagonal matrix whose diagonal elements are:

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- The point (x, z) defined by

$$x = 0, \qquad z_j = |b_j|, \qquad j = 1, 2, \ldots, m, \tag{28}$$

is a basic feasible point for (27).

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- If there exists a vector $(\tilde{x}, \tilde{z})$ that is feasible for (27) such that $e^T \tilde{z} = 0, \implies \tilde{z} = 0$.
- Therefore $A\tilde{x} = b$ and $\tilde{x} \geq 0$, so $\tilde{x}$ is feasible for the original LP.

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- In Phase I, the simplex method is applied to (27) from the initial point (28).
- The objective function is bounded below by 0.
- So the simplex method will terminate at an optimal point.
- If $e^T z$ is positive at this solution, conclude by the argument above that the original LP is infeasible.
- Otherwise, the method identifies a point $(\tilde{x}, \tilde{z})$ with $e^T \tilde{z} = 0$, which is also a basic feasible point for the Phase-II problem.

Phase II

- The Phase II problem is given as:

$$\min c^T x \quad \text{subject to} \quad Ax + z = b, x \geq 0, \quad x \geq 0, \; 0 \geq z \geq 0. \tag{29}$$

- The objective function of (29) is same as the original LP.
- Upper bounds of 0 have been imposed on z from Phase I.
- The original LP is equivalent to (29), because any solution (and indeed any feasible point) must have $z = 0$.
- Need to retain the artificial variables z in Phase II, since some components of z may still be present in the optimal basis from Phase I that are used as the initial basis for (29).
- Though of course the values z_j of these components must be zero.