

Algorithms For Unconstrained Minimisation An Overview

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January 31, 2024



Overview

- All algorithms for unconstrained minimization require the user to supply a starting point(an initial guess denoted by x_0).
- x_0 is chosen using some insight about the problem at hand.
- Beginning at x_0 , optimization algorithms generate a sequence of iterates $\{x_k\}_{k=0}^{\infty}$ that terminate when either no more progress could be made or when the solution is approximated to some desired accuracy.

Stopping Criteria

can't we use $\|x_k - x^*\|$ or $\|f(x_k) - f(x^*)\|$??

In practice given a small $\epsilon > 0$ (Tolerance) s.t.

- $\|\nabla f(x_k)\| < \epsilon$.
- $\|x_k - x_{k-1}\| < \epsilon$ or $\|x_k - x_{k-1}\| < \epsilon \|x_{k-1}\|$.
- $|f(x_k) - f(x_{k-1})| < \epsilon$ or $|f(x_k) - f(x_{k-1})| < \epsilon |f(x_{k-1})|$.

Overview

- In order to move from one iterate x_k to the next i.e. x_{k+1} , the algorithms may use just the information about the function f at x_k or it may use some or all information at previous iterates $(x_0, x_1, \dots, x_{k-1})$.
- At the new iterate x_{k+1} desirably the function value is lesser than that at x_k (monotone algorithms).
- There do exist non-monotone algorithms, but even for them at some $m > 0$, $f(x_k) < f(x_{k+m})$.
- There are two fundamental strategies (families of procedures) to move from the point x_k to x_{k+1} :

1. Line Search Methods

2. Trust Region Methods

Line Search Strategy

- The algorithm chooses a direction p_k and searches along this direction from the current iterate x_k for a new iterate with a lower function value.
- The distance to move along p_k can be found by approximately solving the following one-dimensional minimization problem to find a step length α :

$$\min_{\alpha > 0} f(x_k + \alpha p_k)$$

- Exact solution would give the maximum benefit of moving along p_k .
- But, an exact minimisation may be expensive and is usually unnecessary.
- Instead, a limited number of trial step lengths are generated by the algorithm until it finds an approximation of the minimum (loosely).
- At the new point a new step-direction and step length are calculated.

Line Search Strategy

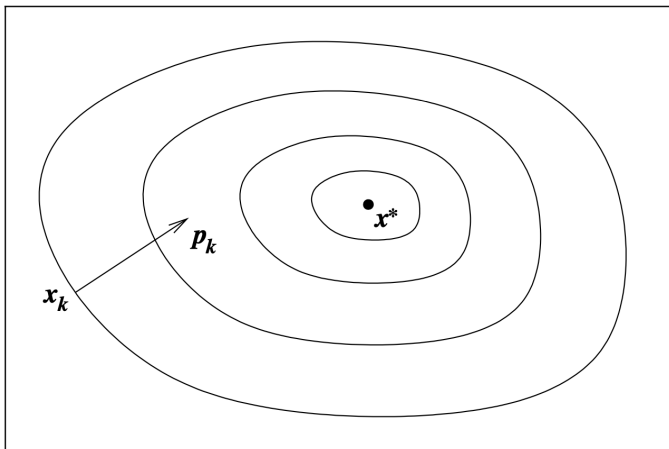


Figure: Line Search Search Direction.

Trust Region Strategy

- The information gathered about f is used to construct a **model function m_k** .
- The behaviour near the current point x_k is similar to that of the actual objective function f near x_k .
- As the model m_k may not be a good approximation of f when x is far from x_k , the search for a minimizer of m_k is restricted to a small region (**trust region**) around x_k .
- The candidate step **p is found by approximately solving** the following sub-problem

$$\min_p m_k(x_k + p)$$

where **$x_k + p$ lies inside the trust region**.

- If the candidate step doesn't procedure sufficient reduction then the trust region radius is deemed to be too large.

Trust Region Strategy

- The trust region is shrunk and the process is resumed.
- The region usually is a ball defined by

$$\|p\|_2 \leq \Delta,$$

where $\Delta > 0$ is called the trust-region radius.

- The model m_k is usually defined by a quadratic function of the form:

$$m_k(x_k + p) = f_k + p^T \nabla f_k + \frac{1}{2} p^T B_k p,$$

f_k and ∇f_k are the functional values and gradient of f at x_k .

- m_k is in agreement with f at x_k upto first order.
- The matrix B_k is either the Hessian or some approximation of it.

Trust Region Strategy

Example

Suppose that the objective is given by

$$f(x) = 10(x_2 - x_1^2)^2 + (1 - x_1)^2.$$

At the point $x_k = (0, 1)$ its gradient and Hessian are:

$$\nabla f_k = \begin{bmatrix} -2 \\ 20 \end{bmatrix}, \quad \nabla^2 f_k = \begin{bmatrix} -38 & 0 \\ 0 & 20 \end{bmatrix}$$

Trust Region Strategy

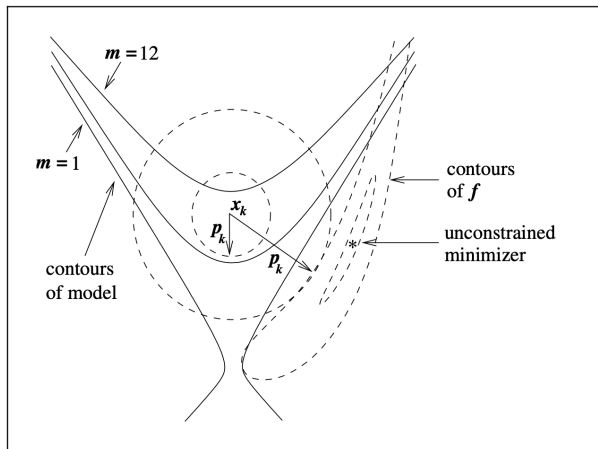


Figure: Two possible trust regions (circles) and their corresponding steps p_k . The solid lines are contours of the model function m_k .

Trust Region Strategy

- Each time the size of the trust region is decreased after failure of a candidate iterate, the step from x_k to the new candidate will be shorter, and it usually points in a different direction from the previous candidate.
- The trust-region strategy differs in this respect from line search, which stays with a single search direction.
- The line search and trust-region approaches differ in the order in which they choose the direction and distance of the move to the next iterate.
- Line search starts by fixing the direction p_k and then identifying an appropriate distance, namely the step length α_k .
- In trust region, a maximum distance the trust-region radius Δ_k is chosen and then a direction and step that attain the best improvement possible subject to this distance constraint is sought.
- If this step proves to be unsatisfactory, we reduce the distance measure Δ_k and try again.