Fundamentals of Algorithms for Non-linear Constrained Optimisation

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Non-linear Constrained Problem

Consider the general constrained optimization problem:

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{subject to} \quad \begin{cases} c_i(x) & = 0, \quad i \in \mathscr{E} \\ c_i(x) & \geq 0, \quad i \in \mathscr{I} \end{cases} \tag{1}$$

- f is the objective function.
- $c_i : G \subset \mathbb{R}^n \to \mathbb{R}$ smooth,
- $\mathscr{I}$ and $\mathscr{E}$ are the finite index sets of inequality and equality constraints.
- We focus on fundamental concepts and building blocks that are common to more than one algorithm.

Special cases (for which specialized algorithms exist):

- *Linear programming (LP):* f, all c_i linear; solved by simplex & interior-point methods.
- *Quadratic programming (QP):* f quadratic, all c_i linear.
- *Linearly constrained optimization:* all c_i linear.
- *Bound-constrained optimization:* constraints are only of the form $x_i \geq l_i$ or $x_i \leq u_i$.
- *Convex programming:* f convex, equality c_i linear, inequality c_i concave.

Categorization of algorithms

Quadratic programming:

- for solving quadratic programming problems.
- its particular characteristics can be exploited by efficient algorithms,
- quadratic programming sub-problems need to be solved by sequential quadratic programming methods and certain interior-point methods for non-linear programming.
- Consist of active set, interior-point, and gradient projection methods.

Categorisation of algorithms

penalty and augmented Lagrangian methods

- Combining the objective function and constraints into a penalty function $\phi(x;\mu)$, attack problem (1) by solving a sequence of unconstrained problems.
- μ is called the penalty parameter $\mu > 0$.
- e.g. if only equality constraints exist:

$$\phi(x;\mu) = f(x) + \frac{\mu}{2}\sum_{i\in\mathscr{E}} c_i(x)^2$$

- Minimise this unconstrained function, for a series of increasing values of μ, until the solution of the constrained optimisation problem is identified to sufficient accuracy.