Constrained Optimization

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March 13, 2024

Motivation

Manufacturing

- Suppose we have m different materials; we have s_i units of each material i in stock.
- We can manufacture k different products; product j gives us profit p_j and uses c_{ij} amount of material i to make.
- To maximize profits, we can solve the following optimization problem for the total amount x_j we should manufacture of each item j:

$$\begin{aligned} \max_{x \in \mathbb{R}^n} \sum_{j=1}^{k} p_j x_j \\ \text{such that } x_j \geq 0 \ \forall \ j \in \{1, 2, \ldots, k\} \\ \sum_{j=1}^{k} c_{ij} x_j \leq s_i, \ \forall \ i \in \{1, 2, \ldots, m\} \end{aligned} \tag{1}$$

- The first constraint ensures that we do not make negative numbers of any product,
- and the second ensures that we do not use more than our stock of each material.

Constrained Problem

A general formulation of these problems is:

$$\min_{x\in\mathbb{R}^n} f(x) \qquad \text{subject to} \qquad \begin{cases} c_i(x) & = 0, \quad i \in \mathscr{E} \\ c_j(x) & \geq 0, \quad j \in \mathscr{I} \end{cases} \tag{2}$$

f and c_i are scalar valued functions of the vector of unknowns x and $\mathscr{E}$ and $\mathscr{I}$ are set of indices.

- x is a vector of variables, also called unknown or parameters;
- f is the objective function, a function of x that we want to optimise (minimise or maximise);
- c is the vector function of constraints that must be satisfied by the unknowns x.
- c_i, $i \in \mathscr{E}$ are the equality constraints.
- c_i, $i \in \mathscr{I}$ are the inequality constraints.

Compact form of Constrained Problem

Definition

Define the feasible set Ω to be the set of points x that satisfy the constraints; that is,

$$\Omega = \{x \mid c_i(x) = 0, \quad i \in \mathcal{E}; \quad c_i(x) \geq 0, \quad i \in \mathcal{I}\}, \tag{3}$$

Now (2) can be rewritten more compactly as:

Constrained Problem

$$\min_{x \in \Omega} f(x). \tag{4}$$

Characterizations of the Solutions

- For the unconstrained optimization problems the solution point x^* was characterised in the following way:
- Necessary conditions: Local minima of unconstrained problems have

  $$\nabla f(x^*) = 0$$

  and,

  $$\nabla^2 f(x^*) \quad \text{is positive semidefinite}$$

- Sufficient conditions: Any point x^* at which $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*)$ is positive definite is a strong local minimiser of f.

LOCAL AND GLOBAL SOLUTIONS

- We have seen already that global solutions are difficult to find even when there are no constraints.
- The situation may improve when we add constraints.
- The feasible set might exclude many of the local minima.
- It might be comparatively easy to pick the global minimum from those that remain.

LOCAL AND GLOBAL SOLUTIONS

- Consider the problem

$$\min_{x\in\mathbb{R}^n} ||x||_2^2, \quad \text{subject to } ||x||_2^2 \geq 1. \tag{5}$$

- Without the constraint, this is a convex quadratic problem with unique minimiser $x = 0$.
- When the constraint is added, any vector x with $||x|| = 1$ solves the problem.
- There are infinitely many such vectors (hence, infinitely many local minima) whenever $n \geq 2$

LOCAL AND GLOBAL SOLUTIONS

- Addition of a constraint produces a large number of local solutions that do not form a connected set.
- Consider

$$\min_{x \in \mathbb{R}^2} (x_2 + 100)^2 + 0.01x_1^2, \quad \text{subject to } x_2 - \cos x_1 \geq 0, \tag{6}$$

- Without the constraint, the problem has the unique solution $(-100, 0)$.
- With the constraint there are local solutions near the points

$$(x_1, x_2) = (k\pi, -1), \quad \text{for } k = \pm 1, \pm 3, \pm 5, \ldots$$

LOCAL AND GLOBAL SOLUTIONS

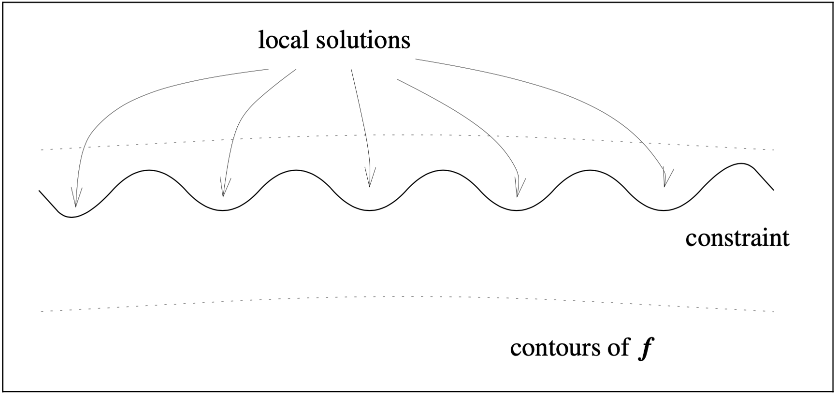

Figure Constrained problem with many isolated local solutions.

LOCAL AND GLOBAL SOLUTIONS

- Local and global solutions are defined in a very similar fashion as they were for the unconstrained case.
- The new caveat that comes into action in the definitions for the constrained case is the inclusion of constraints leading to a restriction imposed via a feasible set (space).

Definition

A vector x^* is a local solution of the constrained minimisation problem (4) if $x^* \in \Omega$ and there exists a neighbourhood $\mathscr{N}$ of x^* such that

$$f(x^*) \leq f(x) \quad \text{for all } x \in \Omega \cap \mathscr{N}$$

LOCAL AND GLOBAL SOLUTIONS

Definition

A vector x^* is called a strict local solution (also called a strong local solution) if $x^* \in \Omega$ and there is a neighbourhood $\mathcal{N}$ of x^* such that

$$f(x^*) < f(x) \quad \text{for all } x \in \mathcal{N} \cap \Omega \quad \text{with } x \neq x^*$$

Definition

A point x^* is an isolated local solution if $x^* \in \Omega$ and there is a neighbourhood $\mathcal{N}$ of x^* such that x^* is the only local minimiser in $N \cap \Omega$.