Line Search Methods

Saurav Samantaray

Department of Mathematics

Indian Institute of Technology Madras

January 26, 2024

Line Search Method

- In each iteration of a line search method a search direction p_k is computed, and
- then its decided how far to move along that direction.
- An iteration is given by

  $$x_{k+1} = x_k + \alpha_k p_k \tag{1}$$

  where $\alpha_k > 0$ (scalar) called the step length.

The success of a line search method depends on effective choices of both:

1. the direction p_k
2. the step length α_k

The Steepest Descent Direction

- The steepest descent direction $-\nabla f_k$ is the most obvious choice for search direction for a line search method.
- Among all Directions once could from x_k, along $-\nabla f_k$, f decreases most rapidly.

Justification

- Consider any search direction p and step-length α, we have

$$f(x_k+\alpha p) = f(x_k)+\alpha p^T\nabla f_k+\frac{1}{2}\alpha^2 p^T\nabla^2 f(x_k+tp)p \text{ for some } \ t \in (0,$$

- Let $\alpha << 1$ (small) and we consider the first-order approximation of f at $x_k + \alpha p$ around x_k as:

$$f(x_k + \alpha p) \approx f(x_k) + \alpha p^T \nabla f_k$$

- Change in f moving from x_k to $x_k + \alpha p$ is

$$f(x_k + \alpha p) - f(x_k)$$

The Steepest Descent Direction

- As the distance moved in the direction is α, therefore, the rate of change of f along the direction p at x_k is

$$\frac{f(x_k + \alpha p) - f(x_k)}{\alpha}$$

- which is coefficient of α, i.e.

$$p^T \nabla f_k$$

- This implies smaller the above value is, more descent can be achieved.
- Hence, the unit direction p of most rapid decrease is the solution to the problem

$$\min_p p^T \nabla f_k, \qquad \text{subject to } ||p|| = 1.$$

The Steepest Descent Direction

- $$p^T \nabla f_k = ||p||\ ||\nabla f_k||\ \cos\theta = ||\nabla f_k||\ \cos\theta$$
  where θ is the angle between p and ∇f_k.
- The minimiser is attained when $\cos\theta = -1$ and
  $$p = -\frac{\nabla f_k}{||\nabla f_k||}$$

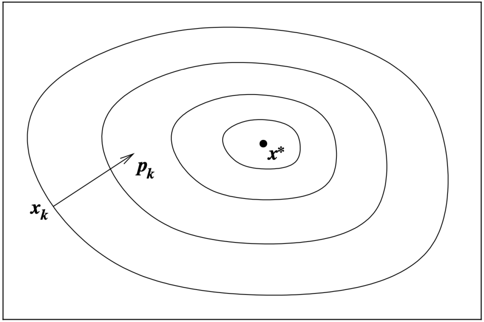

Figure illustrates this direction is orthogonal to the contours of the function

The Steepest Descent Direction

- At every step (iteration) in the steepest descent method the search direction is chosen along

$$p = -\nabla f_k$$

- α_k can be chosen in a variety of ways.
- One advantage of this method is it requires only the calculation of gradient (∇f_k), but not second derivatives.
- Line search methods may use search directions other than the steepest descent direction.

Descent Direction

Any direction that makes an angle of strictly less than $\frac{\pi}{2}$ radians with ∇f_k is guaranteed to produce a decrease in f, provided the step length is sufficiently small and is called a descent direction.