Linear Programming: The Simplex Method

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Linear programming and simplex method

- Today, linear programming and the simplex method continue to hold sway as the most widely used of all optimisation tools.
- The technique is to formulate linear models and solve them with simplex-based software.
- Often, the situations they model are actually non-linear.
- But linear programming is appealing,
    - advanced state of the software,
    - guaranteed convergence to a global minimum,
    - uncertainty in the model makes a linear model more appropriate than an overly complex non-linear model.

Non-linear Programming Might be the Future !!!

- Non-linear programming may replace linear programming as the method of choice in some applications as the non-linear software improves.
- A new class of methods known as interior-point methods has proved to be faster for some linear programming problems.
- But the continued importance of the simplex method is assured for the foreseeable future.

LINEAR PROGRAMMING

Linear programs have:

- linear objective function;
- linear constraints;
- which may include both equalities and inequalities.
- The feasible set is a polytope, a convex, connected set with flat, polygonal faces.
- Owing to the linearity of the objective function its contours are planar.
- Figure below depicts a linear program in two-dimensional space, in which the contours of the objective function are indicated by dotted lines.

LINEAR PROGRAMMING

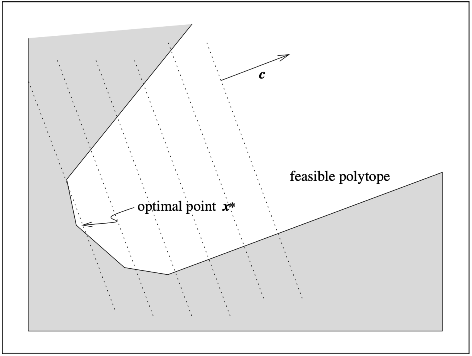

Figure: A linear program in two dimensions with solution at x^*

- The solution in this case is unique-a single vertex.

Solution to Linear Programs

- The solution to a linear program could be non-unique as well.
- It could be an entire edge instead of just one vertex.
- In higher dimensions, the set of optimal points can be a single vertex, an edge or face, or even the entire feasible set.
- The problem has no solution if the feasible set is empty (infeasible case);
- or if the objective function is unbounded below on the feasible region (the unbounded case)

Standard Form of Linear Programs

Linear programs are usually stated and analysed in the following standard form:

Linear Program

$$\min\ c^T x, \qquad \text{subject to } Ax = b,\ x \geq 0, \tag{1}$$

where

- c and x are vectors in $\mathbb{R}^n$,
- b is a vector in $\mathbb{R}^m$ and A is an $m \times n$ matrix

Transforming to Standard Form

- Consider the form:

$$\min\ c^T x, \qquad \text{subject to } Ax \leq b \tag{2}$$

  without any bounds on x.
- By introducing a vector of slack variables z the inequality constraints can be converted to equalities.
- $$\min\ c^T x, \qquad \text{subject to } Ax + z = b,\ z \geq 0, \tag{3}$$
- Still not all variables (x) are constrained to be non-negative as in the standard form.

Transforming to Standard Form

- It is dealt by splitting x into non-negative and non-positive parts.

$$x = x^+ - x^-, \; x^+ = \max(x, 0) \geq 0 \text{ and } x^- = \max(-x, 0)$$

- Now the above considered problem can be written as:

$$\min \begin{bmatrix} c \\ -c \\ 0 \end{bmatrix} \begin{bmatrix} x^+ \\ x^- \\ z \end{bmatrix} \text{ s.t. } \begin{bmatrix} A & -A & I \end{bmatrix} \begin{bmatrix} x^+ \\ x^- \\ z \end{bmatrix} = b, \begin{bmatrix} x^+ \\ x^- \\ z \end{bmatrix} \geq 0,$$

- The above system is now in the standard form.

Transforming to Standard Form

- Inequality constraints of the form $x \leq u$ and $Ax \geq b$ can be converted to equality constraints by adding or subtracting slack variables.
- $$x \leq u \Leftrightarrow \; x + w = u, \;\; w \geq 0,$$ $$Ax \geq b \Leftrightarrow \; Ax - y = b \;\, y \geq 0$$
- We subtract the variables from the left hand side, they are known as surplus variables.
- We add the variables to the left hand side, they are known as deficit variables.
- By simply negating c "maximise" objective max $c^T x$ can be converted to "minimise" form min $-c^T x$.

LINEAR PROGRAMMING

- The linear program is said to be infeasible if the feasible set is empty.
- The problem is considered to be unbounded if the objective function is unbounded below on the feasible region.
- That is, there is a sequence of points x_k in the feasible region such that $c^T x_K \downarrow -\infty$.
- Unbounded problems have no solution.
- For the standard formulation , we will assume throughout that $m < n$.
- Otherwise, the system $Ax = b$ contains redundant rows, or is infeasible, or defines a unique point.
- When $m \geq n$, factorisations such as the QR or LU factorisation can be used to transform the system $Ax = b$ to one with a coefficient matrix of full row rank.

OPTIMALITY CONDITIONS

- Optimality conditions can be derived from the first-order conditions, the Karush–Kuhn–Tucker (KKT) conditions.
- Convexity of the problem ensures that these conditions are sufficient for a global minimum.
- Do not need to refer to the second-order conditions, which are not informative because the Hessian of the Lagrangian is zero.
- The LICQ condition is not required to be enforced here as the KKT results continue to hold for dependent constraints provided they are linear, as is the case here.

OPTIMALITY CONDITIONS

- The Lagrange multipliers for linear problems are partitioned into two vectors λ and s.
- Where $\lambda \in \mathbb{R}^m$ is the multiplier vector for the equality constraints $Ax = b$.
- While $s \in \mathbb{R}^n$ is the multiplier vector for the bound constraints $x \geq 0$.
- Using the definition we can write the Lagrangian function:

$$\mathscr{L}(x, \lambda, s) = c^T x - \lambda^T (Ax - b) - s^T x. \tag{4}$$

OPTIMALITY CONDITIONS

- The first-order necessary conditions for x^* to be a solution of the linear programming problem (1) are, if there exists λ and s such that:

$$A^T\lambda + s = c, \tag{5}$$
$$Ax = b, \tag{6}$$
$$x \geq 0, \tag{7}$$
$$s \geq 0, \tag{8}$$
$$x_i s_i = 0, \ \ i = 1, 2, \ldots, n. \tag{9}$$

- The last condition, which is the complementarity condition, which says that at-least either one of x_i or s_i is zero, can be wrtitten alternatively as

$$x^T s = 0$$

OPTIMALITY CONDITIONS

- Let (x^*, λ^*, s^*) denote a vector triple that satisfy the KKT conditions, then

$$c^T x^* = (A^T \lambda^* + s^*)^T x^* = (Ax^*)^T \lambda^* = b^T \lambda^* \qquad (10)$$

- The first order KKT conditions for optimality for LPP is indeed sufficient.
- Let $\bar{x}$ be any other feasible point, so that $A\bar{x} = b$ and $\bar{x} \geq 0$.

$$\begin{aligned} c^T \bar{x} &= (A^T \lambda^* + s^*)^T \bar{x} \\ &= b^T \lambda^* + \bar{x}^T s^* \\ &\geq b^T \lambda^* = c^T x^* \end{aligned}$$

OPTIMALITY CONDITIONS

- The above inequality tells that no other feasible point can have a lower objective value than $c^T x^*$.
- To say more the feasible point $\bar{x}$ is optimal if and only if

$$\bar{x}^T s^* = 0$$

  otherwise the inequality is strict.
- When $s_i^* > 0$ then we must have $\bar{x}_i = 0$ for all solutions $\bar{x}$ of the LPP.