Prime-Factor Bipartite Graphs: Uniqueness, Isomorphism, and an Arithmetic Approach to Graph Theory

Chan Ju

Abstract

We discuss perspectives on prime-factor bipartite graphs. First, we illustrate the concept and construct examples of bipartite graphs into two parts of (1) a finite set of primes and (2) their squarefree products. We then prove an isomorphism theorem for these graphs, showing how renaming primes induces bijections preserving adjacency. In addition, we discuss the uniqueness of the prime-factor bipartite structure, and give a lemma on bit-size constraints, i.e., the "primorial" bound.

1 Introduction and Basic Examples

A *prime-factor bipartite graph* is constructed by starting with:

$$R = \{\text{some finite set of distinct primes}\}, \quad B = \{\text{squarefree products of these primes}\}.$$

We draw an edge (r, b) precisely when $r \mid b$. Because b is squarefree, each integer in B is formed by multiplying a *subset* of primes from R. The fundamental theorem of arithmetic guarantees that every integer has a unique prime factorization, ensuring that each node in B has a well-defined (and unique) set of neighbors in R.

1.1 Example A: Smallest Possible Products

Example 1 (Example A)**.**

Graph 1:

$$R = \{2, 3\}, \quad B = \{6\} \quad (\textit{since } 6 = 2 \times 3).$$

Edges: prime 2 *connects to* 6*, and prime* 3 *connects to* 6*.*

Graph 2:

$$R = \{5, 7\}, \quad B = \{35\} \quad (\textit{since } 35 = 5 \times 7).$$

Edges: prime 5 *connects to* 35*, and prime* 7 *connects to* 35*.*

Clearly, these two bipartite graphs are isomorphic*. An explicit isomorphism is:*

$$2 \mapsto 5, \quad 3 \mapsto 7, \quad 6 \mapsto 35.$$

In other words, renaming primes (and their corresponding products) preserves the adjacency structure.

1.2 Example B: Single-Primes-as-Products

Example 2 (Example B).

Graph 1:

$$R = \{2, 5\}, \quad B = \{2, 5\}.$$

Edges arise exactly when a prime divides the integer on the other side:

1. *Prime* 2 *divides the integer* 2 *but not* 5.
2. *Prime* 5 *divides the integer* 5 *but not* 2.

Graph 2 (hypothetical):

$$R' = \{p, q\} \quad (\textit{two distinct primes in place of } 2, 5),$$

$$B' = \{p, q\}.$$

Here, prime p *divides the integer* p *only, and prime* q *divides the integer* q *only.*

By mapping

$$2 \mapsto p, \quad 5 \mapsto q, \quad (\textit{integer } 2) \mapsto p, \quad (\textit{integer } 5) \mapsto q,$$

we preserve adjacency. Hence the graphs are isomorphic under suitable prime and product relabeling.

1.3 Example C: A Swap of Primes

Example 3 (Example C).

Graph 1:

$$R_1 = \{2, 3, 5\}, \quad B_1 = \{6, 15\},$$

where

$$6 = 2 \times 3, \quad 15 = 3 \times 5.$$

Edges:

$$2 \textit{ divides } 6 \ (\textit{not } 15), \quad 3 \textit{ divides } 6 \textit{ and } 15, \quad 5 \textit{ divides } 15 \ (\textit{not } 6).$$

Hence the out-degrees of $\{2, 3, 5\}$ *in* G_1 *are* $(1, 2, 1)$.

Graph 2:

$$R_2 = \{2, 3, 5\}, \quad B_2 = \{6, 10\},$$

where

$$6 = 2 \times 3, \quad 10 = 2 \times 5.$$

Edges:

$$2 \textit{ divides } 6 \textit{ and } 10, \quad 3 \textit{ divides } 6 \textit{ only}, \quad 5 \textit{ divides } 10 \textit{ only}.$$

Hence the out-degrees of $\{2, 3, 5\}$ *in* G_2 *are* $(2, 1, 1)$.

At first glance, these look different if we keep the primes "the same." But isomorphism only requires some *bijection preserving adjacency. Indeed, define*

$$2 \mapsto 3, \quad 3 \mapsto 2, \quad 5 \mapsto 5, \quad 6 \mapsto 6, \quad 10 \mapsto 15.$$

Checking adjacency quickly verifies that this preserves edges. The out-degree mismatch disappears once we swap labels $2 \leftrightarrow 3$. Thus, these graphs are also isomorphic.

Remark 1. *These simple examples already highlight the key point: whenever we can relabel the primes in one graph to match those in another, while sending each integer to the product of its relabeled primes, the two bipartite graphs become the "same" in terms of adjacency.*

2 Uniqueness of Prime-Factor Bipartite Graphs

The next theorem shows that if we fix a finite set of distinct primes and a finite collection of their squarefree products, there is exactly *one* bipartite adjacency structure realizing "prime | integer." In other words, changing adjacency would violate unique factorization.

Theorem 1 (Uniqueness of Prime-Factor Bipartite Graph)**.** *Let*

$$R = \{r_1, r_2, \dots, r_x\}$$

be a finite set of distinct primes, and let

$$B = \{b_1, b_2, \dots, b_y\}$$

be a finite set of squarefree integers each formed as a product of primes from R. Construct a bipartite graph

$$G = (R \cup B, E)$$

by placing an edge (r_i, b_j) if and only if $r_i \mid b_j$. Then this bipartite adjacency structure is uniquely determined by R and the integer labels in B.

Proof.

1. **Factorization determines edges.** For each $b \in B$, the prime factors of b (all of which lie in R) dictate which primes connect to b. If

$$b \;=\; \prod_{r \in S_b} r$$

for some $S_b \subseteq R$, then the edges from b go exactly to the primes in S_b. Unique factorization of b ensures no ambiguity in these edges.

2. **No alternative adjacency.** Suppose there were a different bipartite graph on the same nodes with a different edge pattern. Then at least one $b \in B$ would be adjacent to a prime r not in its factorization (or fail to be adjacent to a prime actually in its factorization), violating the fundamental theorem of arithmetic.

3. **Independence of naming or ordering.** Reordering or renaming the primes $r_i \in R$ does not change whether $r_i \mid b_j$. Likewise for the elements of B. Thus, once R and the integer labels in B are fixed, there is exactly one way to realize the bipartite graph structure.

□

3 Isomorphism Criteria via Prime Relabeling

We now show precisely when two prime-factor bipartite graphs can be considered the "same" in a graph-theoretic sense.

Theorem 2 (Isomorphism Theorem for Prime-Factor Bipartite Graphs)**.** *Let*

$$G_1 \;=\; (R_1 \cup B_1, E_1) \quad \textit{and} \quad G_2 \;=\; (R_2 \cup B_2, E_2)$$

be bipartite graphs such that each $b \in B_i$ is a squarefree integer whose prime factors are all in R_i. Then G_1 and G_2 are isomorphic if and only if there is a bijection

$$\phi : R_1 \to R_2$$

such that for each $b \in B_1$, its image $\psi(b) \in B_2$ is precisely the product of primes $\phi(r)$ for $r \mid b$. In other words, ψ on the nodes of B_1 follows from relabeling each prime factor r by $\phi(r)$.

Proof. **($\Longleftarrow$) Sufficiency.** Suppose we have a bijection $\phi : R_1 \to R_2$, and define

$$\psi : B_1 \to B_2$$

by:

$$\psi\Big(\prod_{r|b} r\Big) \;=\; \prod_{r|b} \phi(r).$$

Since the integers in B_1 are squarefree, each prime $r \mid b$ appears exactly once. We must check adjacency preservation:

$$(r, b) \in E_1 \quad \Longleftrightarrow \quad r \mid b \quad \Longleftrightarrow \quad \phi(r) \mid \psi(b) \quad \Longleftrightarrow \quad (\phi(r), \psi(b)) \in E_2.$$

Thus $(r, b) \in E_1$ if and only if $(\phi(r), \psi(b)) \in E_2$. Therefore, (ϕ, ψ) is an isomorphism from G_1 to G_2.

($\Longrightarrow$) Necessity. If G_1 and G_2 are isomorphic, there exist bijections

$$\phi : R_1 \to R_2, \quad \psi : B_1 \to B_2$$

preserving edges. In particular, for each $b \in B_1$ and prime $r \in R_1$, we have

$$(r, b) \in E_1 \iff (\phi(r), \psi(b)) \in E_2.$$

Since $(r, b) \in E_1$ means exactly $r \mid b$, the prime $\phi(r)$ must divide $\psi(b)$ if and only if r divides b. By unique factorization of $\psi(b)$ in B_2, it follows that $\psi(b)$ is exactly the product of primes $\{\phi(r) : r \mid b\}$. Hence ψ has the required form. $\square$

Key observation: *Once you fix how primes in R_1 are sent to those in R_2, the images of the integers in B_1 are forced.* Thus, isomorphism is determined entirely by whether a permutation of primes in R_1 can match the "subset factorization patterns" in B_2.

4 A Lemma on Bit-Count Constraints (Primorial Bound)

Beyond isomorphism, a natural question is how large these squarefree products can get if we try to store them as integers in a computer. One common bound uses the idea of the *primorial* of x primes.

Lemma 3 (Bit-Count Constraint for Storing Prime Products)**.** *Let $\{r_1, r_2, \ldots, r_x\}$ be the first x primes, and define the* primorial

$$r_x^{\#} = r_1 \times r_2 \times \cdots \times r_x.$$

Suppose every product (squarefree integer) in B is stored as a q-bit integer. Then to ensure all possible products of primes in R fit into q bits, we require

$$r_x^{\#} \leq 2^q.$$

Sketch. The largest product among all subsets of $\{r_1, \ldots, r_x\}$ is the full product $r_x^{\#}$. If we want to store that product in q bits, we need $r_x^{\#} \leq 2^q$. If $r_x^{\#}$ exceeds 2^q, then there exists a subset product (namely, the product of all x primes) too large for q-bit representation. □

Remark 2. *This consideration highlights a space–time tradeoff. Storing a bipartite graph by storing each node of B as a single integer is very compact, but factorizing an integer to discover its adjacency in R can be time-consuming. Conversely, explicitly listing edges can be more direct but may require more space.*

5 Why the Isomorphism Works (One-to-One and Onto)

One crucial part of the isomorphism argument is understanding that the prime map ϕ must be:

- *Injective (one-to-one):* No two distinct primes in R_1 can map to the same prime in R_2. Otherwise, unique factorization on the ψ-image of B_1 would fail.
- *Surjective (onto):* Every prime in R_2 must appear as an image of some prime in R_1; otherwise, we would not be able to realize all adjacencies in G_2.

Hence ϕ is a bijection from R_1 to R_2. Once ϕ is fixed, each integer in B_1 *must* map to the corresponding product of the images of its prime factors; this ensures adjacency is preserved in both directions.

In conclusion, prime-factor bipartite graphs demonstrate an elegant dynamic between arithmetic and graph theory. Unique factorization enforces a rigid structure on adjacency, and isomorphisms arise precisely by permuting the underlying primes. This viewpoint can simplify proofs of isomorphism (or non-isomorphism) in certain algebraic or combinatorial contexts: if we can match prime factors, we match the graphs; if we cannot, said graphs differ definitively.